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# Polyhedral Neighborhoods vs Tubular Neighborhoods:

## New Insights for Fractal Zeta Functions

### and Complex Dimensions

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#### Abstract

We provide exact expressions for the volume of *polyhedral neighborhoods* for the sequence of pre-fractal graphs which converge to the Weierstrass Curve, called *Weierstrass Iterated Fractal Drums* (in short, Weierstrass IFDs), associated with a suitable (and geometrically meaningful) sequence of values of a parameter  $\epsilon$  tending to zero, also known as *the cohomology infinitesimal*, due to its connections with fractal cohomology, as developed in an earlier work of the authors. We then introduce the associated *local* and *global polyhedral fractal zeta functions*, and prove that the poles of the global polyhedral fractal zeta function are exactly the same as the Complex Dimensions of the Weierstrass function itself. Contrary to the set of possible Complex Dimensions usually (and classically) obtained by means of tubular neighborhoods, as in an earlier work of ours about the Weierstrass IFD, not only is our result more precise (we obtain the set of *actual* or *exact Complex Dimensions*, instead of the set of *possible Complex Dimensions*), but it also enables us to build a bridge with fractal cohomology, where, for any nonnegative integer  $m$ , the  $m^{\text{th}}$  cohomology group is constituted of continuous functions which possess a generalized Taylor expansion, with fractional derivatives of orders the underlying – and actual – Complex Dimensions.

Then, the aforementioned exact expressions of the polyhedral neighborhoods also enables us to revisit the computation of the box-counting (or Minkowski) dimension of the Weierstrass Curve, in a fully rigorous manner, and thereby to provide a complete and geometric proof of part of Mandelbrot's conjecture about the fractal dimension of the Weierstrass Curve.

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**Keywords:** Weierstrass Curve, iterated fractal drum (IFD), fractal zeta functions, Complex Dimen-

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sions of an IFD, box-counting (or Minkowski) dimension, cohomology infinitesimal, intrinsic scale, Hodge Diamond Star relation, polyhedral measure, tubular neighborhoods, polyhedral (or polygonal) neighborhoods, effective local and global polyhedral zeta functions.

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## 1 Introduction

Fractal zeta functions are classically often obtained in terms of tube formulas (see [LvF13], [LRŽ17], [LRŽ18], [Lap19]). However, when it comes to fractals defined by means of a nonlinear and noncontractive iterated function system (i.f.s.), as is the case for the Weierstrass Curve (see [Dav18]), difficulties arise. Indeed, we cannot obtain an explicit expression for the volume of the general tubular neighborhood of the  $m^{\text{th}}$  prefractal approximation – and, hence, of the associated geometric tube zeta function. Note that, in addition, nonlinearity makes the geometry awfully complicated; in particular, one cannot obtain the exact values of the underlying elementary lengths and angles. As a consequence, we only dispose of an approximate expression for the tube zeta function. In practice, since the approximation only concerns the coefficients in factor of the residues, and since we dispose of sharp estimates for those coefficients, we can obtain the exact values of the poles – and, hence, of the Complex Dimensions associated to each prefractal approximation. However, a natural question is whether the Complex Dimensions of the fractal itself are the same as those of the prefractal approximations, at least for sufficiently large values of these approximations.

In [DL22a], we gave a partial answer to the latter question, by introducing the concept of *Weierstrass Iterated Fractal Drums* (in short, Weierstrass IFDs), by analogy with the *relative fractal drums* involved, for instance, in the case of the Cantor Staircase, in [LRŽ17], Section 5.5.4, and in [LRŽ18]. Weierstrass IFDs simply consist in a sequence of polygonal domains which contain the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ , and are sufficiently close to  $\Gamma_{\mathcal{W}}$ . As is first explained in [DL22c] and [DL22a], we could expect the Complex Dimensions of the IFD to be the same as the Complex Dimensions of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ . An interesting argument arises if we consider the fractal Complex Dimensions as dynamical quantities, which evolve with the scales: to each prefractal approximation  $\Gamma_{\mathcal{W}_m}$  of  $\Gamma_{\mathcal{W}}$ , we can associate specific values of the Complex Dimensions, as is shown in [DL22d]. In this light, it is natural to define the Complex Dimensions of the Weierstrass Curve as the set of the Complex Dimensions of the Weierstrass IFD.

In [DL22b], we have introduced a specific polyhedral measure, by means of polygonal neighborhoods of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ . It so happens that the sequence of polygonal neighborhoods and the sequence of tubular neighborhoods associated with the Weierstrass IFD are nested: given any integer  $m \in \mathbb{N}$ , there exists  $k_1 \in \mathbb{N}$  such that, for all  $k \geq k_1$ , the polygonal neighborhood of the  $m^{\text{th}}$  prefractal approximation contains, but for a finite number of wedges, the  $(m+k, \varepsilon_{m+k}^{m+k})$ -neighborhood.

On the other hand, there also exists  $k_2 \in \mathbb{N}$  such that, for all  $k \geq k_1$ , the polygonal neighborhood of the  $m^{\text{th}}$  tubular neighborhood contains the  $(m+k, \varepsilon_{m+k}^{m+k})$  polygonal neighborhood. Here, the sequence  $(\varepsilon_m^m)_{m \in \mathbb{N}}$  is the so-called *cohomology infinitesimal* introduced in [DL22c] and [DL22d] and expressed in terms of the geometry of the Weierstrass Curve – where  $\varepsilon_m^m = (\varepsilon_m)^m$ , with  $\varepsilon_m > 0$ ,  $\varepsilon_m^m \xrightarrow{m \rightarrow \infty} 0$  and  $\varepsilon_m \xrightarrow{m \rightarrow \infty} L$ , for some  $L > 0$ .

In light of this fact, it was natural to consider the fractal zeta functions associated with the sequence of polygonal neighborhoods, instead of the fractal zeta functions associated with the sequence of tubular neighborhoods. For this sequence, as will be shown, in the present paper, not only do we dispose of the exact expression for the volume (i.e., area) of the polygonal neighborhood, at a given step of the prefractal approximation, but we also have the recurrence relation between consecutive fractal zeta functions. This, in particular, enables us to obtain the poles of the limit fractal zeta function – the one associated with the limit fractal object – and, hence, the Complex Dimensions of the fractal.

More specifically, a key outcome of our study is that the Complex Dimensions of the Weierstrass IFD are all simple, *exact* and given by the following doubly infinite sequence:

$$D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) + i\ell \mathbf{p},$$

where  $k = 0, 1, 2, \dots$  and  $\ell \in \mathbb{Z}$  is arbitrary,  $D_{\mathcal{W}}$  is the Minkowski (or, equivalently box) dimension of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  and where  $\mathbf{p}$  denotes the oscillatory period of the associated IFD. The codimension  $2 - D_{\mathcal{W}}$  is the sharp Hölder exponent of the Weierstrass function  $\mathcal{W}$ .

We point out that our result is stronger than the one obtained in [DL22c], where the values of the possible Complex Dimensions of the Weierstrass IFD – obtained via fractal zeta functions first defined in terms of tubular neighborhoods – included, in addition to the aforementioned values, the values  $-2, 0$  and  $1 - 2k + i\ell \mathbf{p}$ , with  $k = 0, 1, 2, \dots$  and  $\ell \in \mathbb{Z}$  arbitrary. While such a result can seem intriguing, it simply comes from the fact that tubular neighborhoods involve wedges, whose contribution leads to the occurrence of  $-2$  as a possible pole of the associated local fractal effective tube zeta function. Indeed, as we will see, polyhedral neighborhoods, which are the natural neighborhoods adapted to fractal curves, do not lead to this value as a possible pole of the associated local fractal effective polyhedral zeta function. More precisely, since the volume of the polyhedral neighborhood of each prefractal approximation can be expressed by means of values of the Weierstrass function, the possible Complex Dimensions are exactly the same as the Complex Dimensions of the Weierstrass function itself, as obtained in [DL22d].

Our main results in the present setting can be found in the following places:

- i.* In Definition 3.4, on page 24 and Property 3.4, on page 24, where we introduce and study the sequence of polyhedral neighborhoods and show that it converges to the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ .
- ii.* In Theorem 3.9, on page 27, and Corollary 3.10, on page 28, where we show that our polyhedral measure is a quasi self-similar measure. This result extends the classical notion of self-similar measure.
- iii.* In Property 4.1, on page 28, where we show that the polyhedral and tubular neighborhoods are nested.
- iv.* In Theorem 4.3, on page 34, where we obtain recurrence relations satisfied by the coefficients of the exact fractal (or Complex Dimensions) expansion for the complexified and the ordinary

Weierstrass functions.

- v. In Property 4.4, on page 37, where we show that the (complex) coefficients associated with the Complex Dimensions series expansion of the Weierstrass function satisfy a *Hodge Diamond Star* relation, reminiscent of Poincaré duality (see [DL22d]).
- vi. In Theorem 4.5, on page 38, where we obtain an exact fractal power series expansion for the volume of suitable  $m^{\text{th}}$  polyhedral neighborhoods of the Weierstrass Curve.
- vii. In Theorem 4.7, on page 54, where we introduce *the local and global effective polyhedral zeta function*, obtain fractal power series expansions and recurrence relations for the local zeta functions, and show that the global zeta function is well defined, meromorphic in all of  $\mathbb{C}$ , and is given by an explicit fractal power series, evaluated along the cohomology infinitesimal.
- viii. In Theorem 4.9, on page 60, where we obtain the exact Complex Dimensions of the Weierstrass Curve (or of the Weierstrass IFD), precisely of the same form as the possible Complex Dimensions occurring in the fractal cohomology theory developed in [DL22d].
- ix. In Theorem 5.1, on page 64, where we introduce a new and complete proof of the computation of the box-counting dimension (or, equivalently, of the Minkowski dimension) of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ , by simply using the covers of  $\Gamma_{\mathcal{W}}$  by polyhedral neighborhoods – therefore fully establishing part of Mandelbrot’s conjecture [Man82] about the fractal dimension of  $\Gamma_{\mathcal{W}}$ .

## 2 Geometry of the Weierstrass Curve

We begin by reviewing the main geometric properties of the Weierstrass Curve (and of the associated IFD), which will be needed in this paper.

Henceforth, we place ourselves in the Euclidean plane, equipped with a direct orthonormal frame. The usual Cartesian coordinates are denoted by  $(x, y)$ . The horizontal and vertical axes will be respectively referred to as  $(x'x)$  and  $(y'y)$ .

### Notation 1 (Set of all Natural Numbers, and Intervals).

As in Bourbaki [Bou04] (Appendix E. 143), we denote by  $\mathbb{N} = \{0, 1, 2, \dots\}$  the set of all natural numbers, and set  $\mathbb{N}^{\star} = \mathbb{N} \setminus \{0\}$ .

Given  $a, b$  with  $-\infty \leq a \leq b \leq \infty$ ,  $]a, b[ = (a, b)$  denotes an open interval, while, for example,  $]a, b] = (a, b]$  denotes a half-open, half-closed interval.

### Notation 2 (Wave Inequality Symbol).

Given two positive numbers  $a$  and  $b$ , we will use the notation  $a \lesssim b$  when there exists a strictly positive constant  $C$  such that  $a \leq Cb$ .

**Notation 3 (Weierstrass Parameters).**

In the sequel,  $\lambda$  and  $N_b$  are two real numbers such that

$$0 < \lambda < 1 \quad , \quad N_b \in \mathbb{N}^* \quad \text{and} \quad \lambda N_b > 1 \quad . \quad (\clubsuit) \quad (\mathcal{R}1)$$

Note that this implies that  $N_b > 1$ ; i.e.,  $N_b \geq 2$ , if  $N_b \in \mathbb{N}^*$ , as will be the case in this paper.

As explained in [Dav19], we deliberately made the choice to introduce the notation  $N_b$  which replaces the initial number  $b$ , in so far as, in Hardy's paper [Har16] (in contrast to Weierstrass' original article [Wei75]),  $b$  is any positive real number satisfying  $\lambda b > 1$ , whereas we deal here with the specific case of a natural integer, which accounts for the natural notation  $N_b$ .

**Definition 2.1 (Weierstrass Function, Weierstrass Curve).**

We consider the *Weierstrass function*  $\mathcal{W}$  (also called, in short, the *W-function*) defined, for any real number  $x$ , by

$$\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi N_b^n x) \quad . \quad (\mathcal{R}2)$$

We call the associated graph the *Weierstrass Curve*, and denote it by  $\Gamma_{\mathcal{W}}$ .

Due to the one-periodicity of the Weierstrass function (since  $N_b \in \mathbb{N}^*$ ), from now on, and without loss of generality, we restrict our study to the interval  $[0, 1[ = [0, 1)$ . Note that  $\mathcal{W}$  is continuous, and hence, bounded on all of  $\mathbb{R}$ . In particular,  $\Gamma_{\mathcal{W}}$  – which is also the graph of  $\mathcal{W}$  restricted to  $[0, 1[$  – is a (nonempty) compact subset of  $\mathbb{R}^2$ .

**Definition 2.2 (Weierstrass Complexified Function).**

We introduce the *Weierstrass Complexified function*  $\mathcal{W}_{comp}$ , defined, for any real number  $x$ , by

$$\mathcal{W}_{comp}(x) = \sum_{n=0}^{\infty} \lambda^n e^{2i\pi N_b^n x} \quad .$$

Clearly,  $\mathcal{W}_{comp}$  is also a continuous and 1-periodic function on  $\mathbb{R}$  and  $\mathcal{W}$  is the real part of  $\mathcal{W}_{comp}$ .

**Notation 4 (Logarithm).**

Given  $y > 0$ ,  $\ln y$  denotes the natural logarithm of  $y$ , while, given  $a > 1$ ,  $\ln_a y = \frac{\ln y}{\ln a}$  denotes the logarithm of  $y$  in base  $a$ ; so that, in particular,  $\ln = \ln_e$ .

**Notation 5 (Minkowski Dimension and Hölder Exponent).**

For the parameters  $\lambda$  and  $N_b$  satisfying condition  $(\clubsuit)$  (see Notation 3, on page 5), we denote by

$$D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln N_b} = 2 - \ln_{N_b} \frac{1}{\lambda} \in ]1, 2[$$

the box-counting dimension (or Minkowski dimension) of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ , which happens to be equal to its Hausdorff dimension [KMPY84], [BBR14], [She18], [Kel17]. We point out that the results in our previous paper [DL22c], combined with those obtained in the present paper in Section 5 below, will provide a direct geometric and fully rigorous proof of the fact that  $D_{\mathcal{W}}$ , the Minkowski dimension (or box-counting dimension) of  $\Gamma_{\mathcal{W}}$ , exists and takes the above values, as well as of the fact that  $\mathcal{W}$  is Hölder continuous with *optimal* Hölder exponent

$$2 - D_{\mathcal{W}} = -\frac{\ln \lambda}{\ln N_b} = \ln_{N_b} \frac{1}{\lambda}.$$

**Convention (The Weierstrass Curve as a Cyclic Curve).**

In the sequel, we identify the points  $(0, \mathcal{W}(0))$  and  $(1, \mathcal{W}(1)) = (1, \mathcal{W}(0))$ . This is justified by the fact that the Weierstrass function  $\mathcal{W}$  is 1-periodic, since  $N_b$  is an integer.

*Remark 2.1.* The above convention makes sense, because, in addition to the periodicity property of the  $\mathcal{W}$ -function, the points  $(0, \mathcal{W}(0))$  and  $(1, \mathcal{W}(1))$  have the same vertical coordinate.

**Property 2.1 (Symmetry with Respect to the Vertical Line  $x = \frac{1}{2}$ ).**

Since, for any  $x \in [0, 1]$ ,

$$\mathcal{W}(1 - x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi N_b^n - 2\pi N_b^n x) = \mathcal{W}(x),$$

the Weierstrass Curve is symmetric with respect to the vertical straight line  $x = \frac{1}{2}$ .

**Proposition 2.2 (Nonlinear and Noncontractive Iterated Function System (IFS)).**

Following our previous work [Dav18], we approximate the restriction  $\Gamma_{\mathcal{W}}$  to  $[0, 1] \times \mathbb{R}$ , of the Weierstrass Curve, by a sequence of finite graphs, built via an iterative process. For this purpose, we use the nonlinear (and noncontractive) iterated function system (IFS) consisting of a finite family of  $C^\infty$  bijective maps from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  and denoted by

$$\mathcal{T}_{\mathcal{W}} = \{T_0, \dots, T_{N_b-1}\},$$

where, for any integer  $i$  belonging to  $\{0, \dots, N_b - 1\}$  and any point  $(x, y)$  of  $\mathbb{R}^2$ ,

$$T_i(x, y) = \left( \frac{x+i}{N_b}, \lambda y + \cos\left(2\pi \left(\frac{x+i}{N_b}\right)\right) \right).$$

Note that unlike in the classical situation, these maps are not contractions. Nevertheless,  $\Gamma_{\mathcal{W}}$  can be recovered from this IFS in the usual way, as we next explain.

**Property 2.3 (Attractor of the IFS [Dav18], [Dav19]).**

The Weierstrass Curve  $\Gamma_{\mathcal{W}}$  is the attractor of the IFS  $\mathcal{T}_{\mathcal{W}}$ , and hence, is the unique nonempty compact subset  $\mathcal{K}$  of  $\mathbb{R}^2$  satisfying  $\mathcal{K} = \bigcup_{i=0}^{N_b-1} T_i(\mathcal{K})$ ; in particular, we have that  $\Gamma_{\mathcal{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}})$ .

**Notation 6 (Fixed Points).**

For any integer  $i$  belonging to  $\{0, \dots, N_b - 1\}$ , we denote by

$$P_i = (x_i, y_i) = \left( \frac{i}{N_b - 1}, \frac{1}{1 - \lambda} \cos\left(\frac{2\pi i}{N_b - 1}\right) \right)$$

the unique fixed point of the map  $T_i$ ; see [Dav19].

**Definition 2.3 (Sets of Vertices, Prefractals).**

We denote by  $V_0$  the ordered set (according to increasing abscissae) of the points

$$\{P_0, \dots, P_{N_b-1}\}.$$

The set of points  $V_0$  – where, for any integer  $i$  in  $\{0, \dots, N_b - 2\}$ , the point  $P_i$  is linked to the point  $P_{i+1}$  – constitutes an oriented finite graph, ordered according to increasing abscissae, which we will denote by  $\Gamma_{\mathcal{W}_0}$ . Then,  $V_0$  is called *the set of vertices* of the graph  $\Gamma_{\mathcal{W}_0}$ .

For any positive integer  $m$ , i.e., for  $m \in \mathbb{N}^*$ , we set  $V_m = \bigcup_{i=0}^{N_b-1} T_i(V_{m-1})$ .

The set of points  $V_m$ , where two consecutive points are linked, is an oriented finite graph, ordered according to increasing abscissae, called the  **$m^{\text{th}}$  order  $\mathcal{W}$ -prefractal**. Then,  $V_m$  is called *the set of vertices* of the prefractal  $\Gamma_{\mathcal{W}_m}$ ; see Figure 2, on page 11. We call *Weierstrass Iterated Fractal Drums (IFD)* the sequence of prefractal graphs which converge to the Weierstrass Curve.

**Property 2.4 (Density of the Set  $V^* = \bigcup_{n \in \mathbb{N}} V_n$  in the Weierstrass Curve [DL22c]).**

The set  $V^* = \bigcup_{n \in \mathbb{N}} V_n$  is dense in the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ .

**Definition 2.4 (Adjacent Vertices, Edge Relation).**

For any  $m \in \mathbb{N}$ , the prefractal graph  $\Gamma_{\mathcal{W}_m}$  is equipped with an edge relation  $\sim_m$ , as follows: two vertices  $X$  and  $Y$  of  $\Gamma_{\mathcal{W}_m}$  (i.e., two points belonging to  $V_m$ ) are said to be *adjacent* (i.e., *neighboring* or *junction points*) if and only if the line segment  $[X, Y]$  is an edge of  $\Gamma_{\mathcal{W}_m}$ ; we then write  $X \sim_m Y$ . Note that this edge relation depends on  $m$ , which means that points adjacent in  $V_m$  might not remain adjacent in  $V_{m+1}$ .



**Property 2.5 (Scaling Properties of the Weierstrass Function, and Consequences [DL22c]).**

Since, for any real number  $x$ ,  $\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi N_b^n x)$ , one also has

$$\mathcal{W}(N_b x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi N_b^{n+1} x) = \frac{1}{\lambda} \sum_{n=1}^{\infty} \lambda^n \cos(2\pi N_b^n x) = \frac{1}{\lambda} (\mathcal{W}(x) - \cos(2\pi x)),$$

which yields, for any strictly positive integer  $m$  and any  $j$  in  $\{0, \dots, \#V_m\}$ ,

$$\mathcal{W}\left(\frac{j}{(N_b - 1) N_b^m}\right) = \lambda \mathcal{W}\left(\frac{j}{(N_b - 1) N_b^{m-1}}\right) + \cos\left(\frac{2\pi j}{(N_b - 1) N_b^m}\right).$$

By induction, one then obtains that

$$\mathcal{W}\left(\frac{j}{(N_b - 1) N_b^m}\right) = \lambda^m \mathcal{W}\left(\frac{j}{(N_b - 1)}\right) + \sum_{k=0}^{m-1} \lambda^k \cos\left(\frac{2\pi N_b^k j}{(N_b - 1) N_b^m}\right).$$

We refer to part *iv.* of Property 2.6, on page 8, along with Figure 1, on page 10, for the definition of the polygons  $\mathcal{P}_{m,k}$  and  $\mathcal{Q}_{m,k}$  associated with the Weierstrass Curve.

**Property 2.6.** [Dav18] For any  $m \in \mathbb{N}$ , the following statements hold:

*i.*  $V_m \subset V_{m+1}$ .

*ii.*  $\#V_m = (N_b - 1) N_b^m + 1$ , where  $\#V_m$  denotes the number of elements in the finite set  $V_m$ .

*iii.* The prefractal graph  $\Gamma_{\mathcal{W}_m}$  has exactly  $(N_b - 1) N_b^m$  edges.

*iv.* The consecutive vertices of the prefractal graph  $\Gamma_{\mathcal{W}_m}$  are the vertices of  $N_b^m$  simple nonregular polygons  $\mathcal{P}_{m,k}$  with  $N_b$  sides. For any strictly positive integer  $m$ , the junction point between two consecutive polygons  $\mathcal{P}_{m,k}$  and  $\mathcal{P}_{m,k+1}$  is the point

$$\left( \frac{(N_b - 1) k}{(N_b - 1) N_b^m}, \mathcal{W}\left(\frac{(N_b - 1) k}{(N_b - 1) N_b^m}\right) \right), \quad 1 \leq k \leq N_b^m - 1.$$

Hence, the total number of junction points is  $N_b^m - 1$ . For instance, in the case  $N_b = 3$ , the polygons are all triangles; see Figure 1, on page 10.

We call extreme vertices of the polygon  $\mathcal{P}_{m,k}$  the junction points

$$\mathcal{V}_{\text{initial}}(\mathcal{P}_{m,k}) = \left( \frac{(N_b - 1) k}{(N_b - 1) N_b^m}, \mathcal{W}\left(\frac{(N_b - 1) k}{(N_b - 1) N_b^m}\right) \right), \quad 0 \leq k \leq N_b^m - 1,$$

and

$$\mathcal{V}_{\text{end}}(\mathcal{P}_{m,k}) = \left( \frac{(N_b - 1) (k + 1)}{(N_b - 1) N_b^m}, \mathcal{W}\left(\frac{(N_b - 1) (k + 1)}{(N_b - 1) N_b^m}\right) \right), \quad 0 \leq k \leq N_b^m - 2.$$

In the sequel, we will denote by  $\mathcal{P}_0$  **the initial polygon**, whose vertices are the fixed points of the maps  $T_i$ ,  $0 \leq i \leq N_b - 1$ , introduced in Notation 6, on page 7 and Definition 2.3, on page 7, i.e.,  $\{P_0, \dots, P_{N_b-1}\}$ ; see, again, Figure 1, on page 10.

In the same way, the consecutive vertices of the prefractal graph  $\Gamma_{\mathcal{W}_m}$ , distinct from the fixed points  $P_0$  and  $P_{N_b-1}$  (see Notation 6, on page 7), are also the vertices of  $N_b^m - 1$  simple nonregular polygons  $\mathcal{Q}_{m,j}$ , for  $1 \leq j \leq N_b^m - 2$ , again with  $N_b$  sides. For any integer  $j$  such that  $1 \leq j \leq N_b^m - 2$ , one obtains each polygon  $\mathcal{Q}_{m,j}$  by connecting the point number  $j$  (i.e., with the notation of Definition 2.6, on page 11, the vertex  $M_{j,m}$ ) to the point number  $j + 1$  (i.e., the vertex  $M_{j+1,m}$ ) if  $j \equiv i \pmod{N_b}$ , for  $1 \leq i \leq N_b - 1$ , and the point number  $j$  to the point number  $j - N_b + 1$  if  $j \equiv 0 \pmod{N_b}$ .

As previously, we call extreme vertices of the polygon  $\mathcal{Q}_{m,k}$  the points

$$\mathcal{V}_{initial}(\mathcal{Q}_{m,k}) = \left( \frac{(N_b - 1)k}{(N_b - 1)N_b^m}, \mathcal{W} \left( \frac{(N_b - 1)k}{(N_b - 1)N_b^m} \right) \right), \quad 1 \leq k \leq N_b^m - 1,$$

and

$$\mathcal{V}_{end}(\mathcal{Q}_{m,k}) = \left( \frac{(N_b - 1)(k + 1)}{(N_b - 1)N_b^m}, \mathcal{W} \left( \frac{(N_b - 1)(k + 1)}{(N_b - 1)N_b^m} \right) \right), \quad 1 \leq k \leq N_b^m - 2.$$

**Definition 2.5 (Polygonal Sets).**

For any  $m \in \mathbb{N}$ , we introduce the following *polygonal sets*

$$\mathcal{P}_m = \{\mathcal{P}_{m,k}, 0 \leq k \leq N_b^m - 1\} \quad \text{and} \quad \mathcal{Q}_m = \{\mathcal{Q}_{m,k}, 0 \leq k \leq N_b^m - 2\}.$$

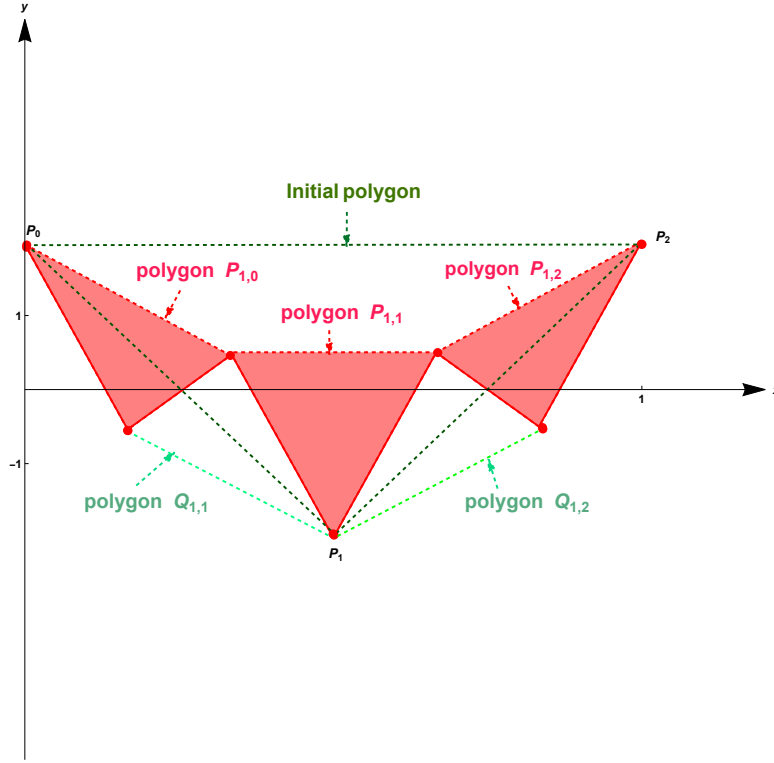


Figure 1: The initial polygon  $\mathcal{P}_0$ , and the respective polygons  $\mathcal{P}_{0,1}$ ,  $\mathcal{P}_{1,1}$ ,  $\mathcal{P}_{1,2}$ ,  $\mathcal{Q}_{1,1}$ ,  $\mathcal{Q}_{1,2}$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ . (See also Figure 2, on page 11.)

**Notation 7.** For any  $m \in \mathbb{N}$ , we denote by:

- ii.*  $X \in \mathcal{P}_m$  (resp.,  $X \in \mathcal{Q}_m$ ) a vertex of a polygon  $\mathcal{P}_{m,k}$ , with  $0 \leq k \leq N_b^m - 1$  (resp., a vertex of a polygon  $\mathcal{Q}_{m,k}$ , with  $1 \leq k \leq N_b^m - 2$ ).
- ii.*  $\mathcal{P}_m \cup \mathcal{Q}_m$  the reunion of the polygonal sets  $\mathcal{P}_m$  and  $\mathcal{Q}_m$ , which consists in the set of all the vertices of the polygons  $\mathcal{P}_{m,k}$ , with  $0 \leq k \leq N_b^m - 1$ , along with the vertices of the polygons  $\mathcal{Q}_{m,k}$ , with  $1 \leq k \leq N_b^m - 2$ . In particular,  $X \in \mathcal{P}_m \cup \mathcal{Q}_m$  simply denotes a vertex in  $\mathcal{P}_m$  or  $\mathcal{Q}_m$ .
- iii.*  $\mathcal{P}_m \cap \mathcal{Q}_m$  the intersection of the polygonal sets  $\mathcal{P}_m$  and  $\mathcal{Q}_m$ , which consists in the set of all the vertices of both a polygon  $\mathcal{P}_{m,k}$ , with  $0 \leq k \leq N_b^m - 1$ , and a polygon  $\mathcal{Q}_{m,k'}$ , with  $1 \leq k' \leq N_b^m - 2$ .

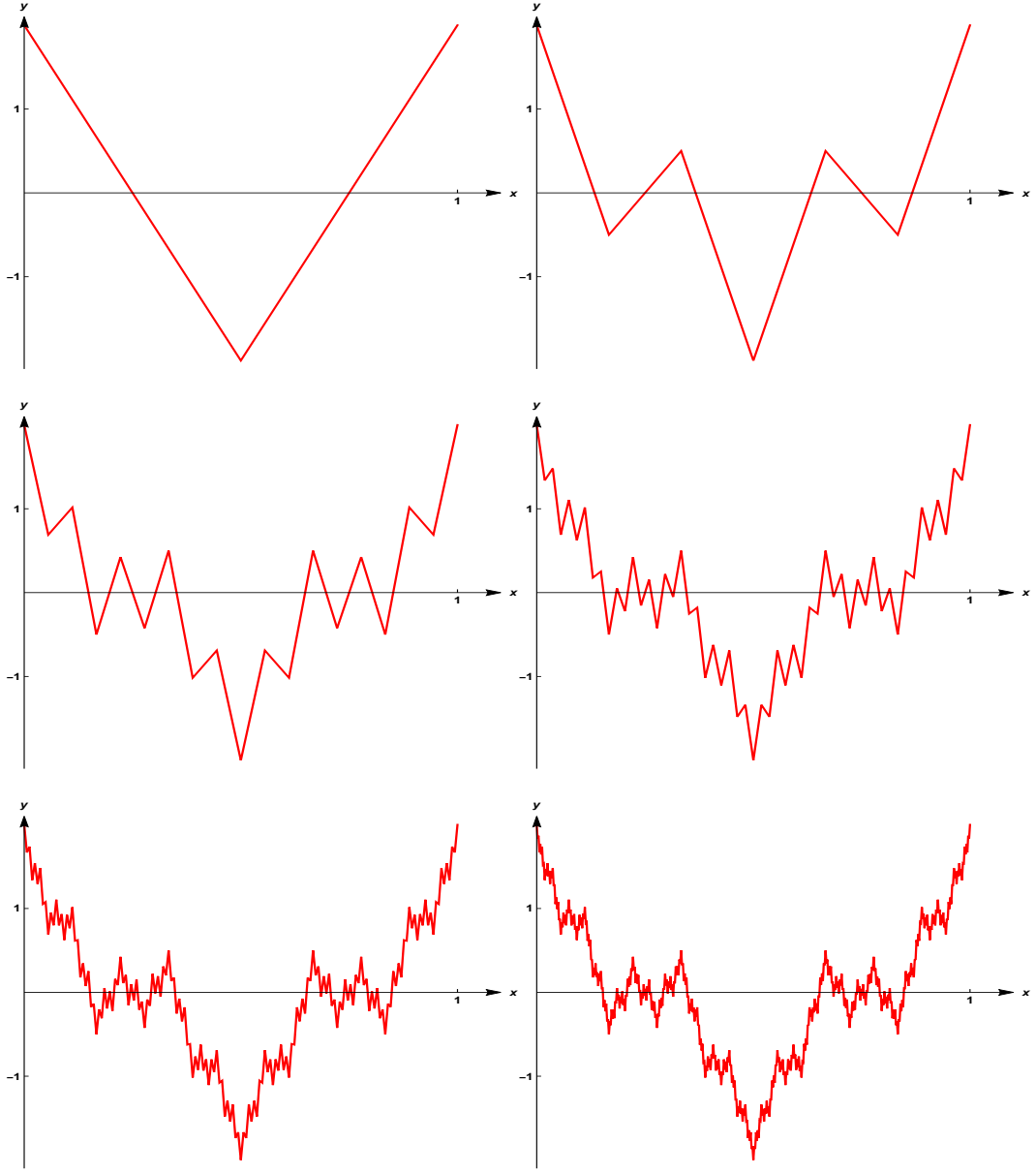


Figure 2: The prefractal graphs  $\Gamma_{\mathcal{W}_0}, \Gamma_{\mathcal{W}_1}, \Gamma_{\mathcal{W}_2}, \Gamma_{\mathcal{W}_3}, \Gamma_{\mathcal{W}_4}, \Gamma_{\mathcal{W}_5}$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ . For example,  $\Gamma_{\mathcal{W}_1}$  is on the right side of the top row, while  $\Gamma_{\mathcal{W}_4}$  is on the left side of the bottom row.

**Definition 2.6 (Vertices of the Prefractals, Elementary Lengths, Heights and Angles).**

Given a strictly positive integer  $m$ , we denote by  $(M_{j,m})_{0 \leq j \leq (N_b-1)N_b^m}$  the set of vertices of the prefractal graph  $\Gamma_{\mathcal{W}_m}$ . One thus has, for any integer  $j$  in  $\{0, \dots, (N_b-1)N_b^m\}$ :

$$M_{j,m} = \left( \frac{j}{(N_b-1)N_b^m}, \mathcal{W} \left( \frac{j}{(N_b-1)N_b^m} \right) \right).$$

We also introduce, for any integer  $j$  in  $\{0, \dots, (N_b - 1) N_b^m - 1\}$ :

i. the elementary horizontal lengths:

$$L_m = \frac{j}{(N_b - 1) N_b^m}; \quad (\mathcal{R}3)$$

ii. the elementary lengths:

$$l_{j,j+1,m} = d(M_{j,m}, M_{j+1,m}) = \sqrt{L_m^2 + h_{j,j+1,m}^2},$$

where  $h_{j,j+1,m}$  is defined in iii. just below.

iii. the elementary heights:

$$h_{j,j+1,m} = \left| \mathcal{W} \left( \frac{j+1}{(N_b - 1) N_b^m} \right) - \mathcal{W} \left( \frac{j}{(N_b - 1) N_b^m} \right) \right|;$$

iv. the minimal height:

$$h_m^{inf} = \inf_{0 \leq j \leq (N_b - 1) N_b^m - 1} h_{j,j+1,m}, \quad (\mathcal{R}4)$$

along with the maximal height:

$$h_m = \sup_{0 \leq j \leq (N_b - 1) N_b^m - 1} h_{j,j+1,m}, \quad (\mathcal{R}5)$$

v. the geometric angles:

$$\theta_{j-1,j,m} = ((y'y), (\widehat{M_{j-1,m} M_{j,m}})) \quad , \quad \theta_{j,j+1,m} = ((y'y), (\widehat{M_{j,m} M_{j+1,m}})),$$

which yield **the following value of the geometric angle between consecutive edges**, namely,  $[M_{j-1,m} M_{j,m}, M_{j,m} M_{j+1,m}]$ , with  $\arctan = \tan^{-1}$ :

$$\theta_{j-1,j,m} + \theta_{j,j+1,m} = \arctan \frac{L_m}{|h_{j-1,j,m}|} + \arctan \frac{L_m}{|h_{j,j+1,m}|}.$$

**Property 2.7.** For the geometric angle  $\theta_{j-1,j,m}$ ,  $0 \leq j \leq (N_b - 1) N_b^m$ ,  $m \in \mathbb{N}$ , we have the following relation:

$$\tan \theta_{j-1,j,m} = \frac{h_{j-1,j,m}}{L_m}.$$

**Property 2.8 (A Consequence of the Symmetry with Respect to the Vertical Line  $x = \frac{1}{2}$ ).**

For any strictly positive integer  $m$  and any  $j$  in  $\{0, \dots, \#V_m\}$ , we have that

$$\mathcal{W} \left( \frac{j}{(N_b - 1) N_b^m} \right) = \mathcal{W} \left( \frac{(N_b - 1) N_b^m - j}{(N_b - 1) N_b^m} \right),$$

which means that the points

$$\left( \frac{(N_b - 1)N_b^m - j}{(N_b - 1)N_b^m}, \mathcal{W}\left(\frac{(N_b - 1)N_b^m - j}{(N_b - 1)N_b^m}\right) \right) \quad \text{and} \quad \left( \frac{j}{(N_b - 1)N_b^m}, \mathcal{W}\left(\frac{j}{(N_b - 1)N_b^m}\right) \right)$$

are symmetric with respect to the vertical line  $x = \frac{1}{2}$ ; see Figure 3, on page 13.

**Definition 2.7 (Left-Side and Right-Side Vertices).**

Given natural integers  $m, k$  such that  $0 \leq k \leq N_b^m - 1$ , and a polygon  $\mathcal{P}_{m,k}$ , we define:

- i. The set of its *left-side vertices* as the set of the first  $\left\lfloor \frac{N_b - 1}{2} \right\rfloor$  vertices, where  $\lfloor y \rfloor$  denotes the integer part of the real number  $y$ .
- ii. The set of its *right-side vertices* as the set of the last  $\left\lfloor \frac{N_b - 1}{2} \right\rfloor$  vertices.

When the integer  $N_b$  is odd, we define the bottom vertex as the  $\left(\frac{N_b - 1}{2}\right)^{th}$  one; see Figure 4, on page 14.

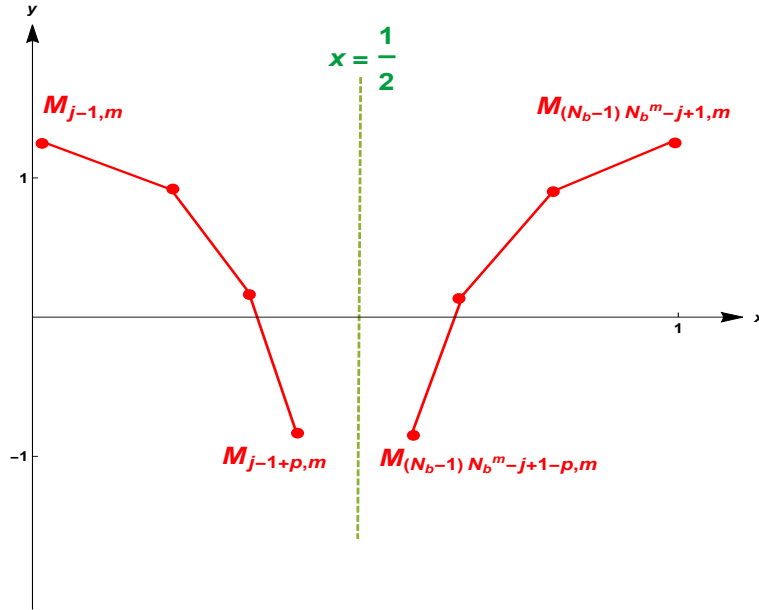


Figure 3: Symmetric points with respect to the vertical line  $x = \frac{1}{2}$ .

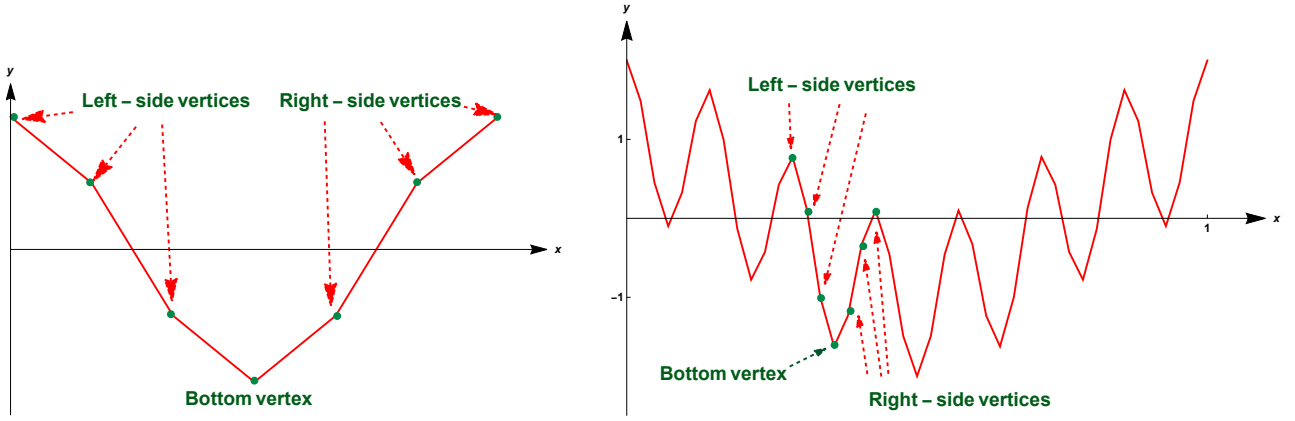


Figure 4: The Left and Right-Side Vertices, in the case when  $N_b = 7$ .

**Property 2.9** ([DL22c]).

For any integer  $j$  in  $\{0, \dots, N_b - 1\}$ :

$$\mathcal{W}\left(\frac{j}{N_b - 1}\right) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\pi N_b^n \frac{j}{N_b - 1}\right) = \sum_{n=0}^{\infty} \lambda^n \cos\left(\frac{2\pi j}{N_b - 1}\right) = \frac{1}{1 - \lambda} \cos\left(\frac{2\pi j}{N_b - 1}\right).$$

**Property 2.10** ([DL22c]).

For  $0 \leq j \leq \frac{(N_b - 1)}{2}$  (resp., for  $\frac{(N_b - 1)}{2} \leq j \leq N_b - 1$ ), we have that

$$\mathcal{W}\left(\frac{j+1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \leq 0 \quad \left(\text{resp., } \mathcal{W}\left(\frac{j+1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \geq 0\right).$$

**Notation 8 (Signum Function).**

The *signum function* of a real number  $x$  is defined by

$$\text{sgn}(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ +1, & \text{if } x > 0. \end{cases}$$

**Property 2.11** ([DL22c]).

Given any strictly positive integer  $m$ , we have the following properties:

i. For any  $j$  in  $\{0, \dots, \#V_m\}$ , the point

$$\left(\frac{j}{(N_b - 1) N_b^m}, \mathcal{W}\left(\frac{j}{(N_b - 1) N_b^m}\right)\right)$$

is the image of the point

$$\left( \frac{j}{(N_b - 1) N_b^{m-1}} - i, \mathcal{W} \left( \frac{j}{(N_b - 1) N_b^{m-1}} - i \right) \right) = \left( \frac{j - i(N_b - 1) N_b^{m-1}}{(N_b - 1) N_b^{m-1}}, \mathcal{W} \left( \frac{j - i(N_b - 1) N_b^{m-1}}{(N_b - 1) N_b^{m-1}} \right) \right)$$

under the map  $T_i$ , where  $i \in \{0, \dots, N_b - 1\}$  is arbitrary.

Consequently, for  $0 \leq j \leq N_b - 1$ , **the  $j^{\text{th}}$  vertex of the polygon  $\mathcal{P}_{m,k}$** ,  $0 \leq k \leq N_b^m - 1$ , i.e., the point

$$\left( \frac{(N_b - 1)k + j}{(N_b - 1) N_b^m}, \mathcal{W} \left( \frac{(N_b - 1)k + j}{(N_b - 1) N_b^m} \right) \right),$$

is the image of the point

$$\left( \frac{(N_b - 1)(k - i(N_b - 1) N_b^{m-1}) + j}{(N_b - 1) N_b^{m-1}}, \mathcal{W} \left( \frac{(N_b - 1)(k - i(N_b - 1) N_b^{m-1}) + j}{(N_b - 1) N_b^{m-1}} \right) \right)$$

under the map  $T_i$ , where  $i \in \{0, \dots, N_b - 1\}$  is again arbitrary. This latter point is also **the  $j^{\text{th}}$  vertex of the polygon  $\mathcal{P}_{m-1, k-i(N_b-1)N_b^{m-1}}$** . Therefore, there is an exact correspondance between vertices of the polygons at consecutive steps  $m - 1$ ,  $m$ .

ii. Given  $j$  in  $\{0, \dots, N_b - 2\}$  and  $k$  in  $\{0, \dots, N_b^m - 1\}$ , we have that

$$\text{sgn} \left( \mathcal{W} \left( \frac{k(N_b - 1) + j + 1}{(N_b - 1) N_b^m} \right) - \mathcal{W} \left( \frac{k(N_b - 1) + j}{(N_b - 1) N_b^m} \right) \right) = \text{sgn} \left( \mathcal{W} \left( \frac{j + 1}{N_b - 1} \right) - \mathcal{W} \left( \frac{j}{N_b - 1} \right) \right).$$

*Proof.*

i. Given  $m \in \mathbb{N}^*$ , let us consider  $i \in \{0, \dots, N_b - 1\}$ . The image of the point

$$\left( \frac{j}{(N_b - 1) N_b^{m-1}} - i, \mathcal{W} \left( \frac{j}{(N_b - 1) N_b^{m-1}} - i \right) \right)$$

under the map  $T_i$  is obtained by applying the analytic expression given in Property 2.2, on page 6, to the coordinates of this point, which, thanks to Property 2.5, on page 8 above, yields the expected result, namely,



$$\left( \begin{array}{c} \frac{j}{(N_b - 1) N_b^m}, \lambda \underbrace{\mathcal{W}\left(\frac{j}{(N_b - 1) N_b^{m-1}} - i\right)}_{\mathcal{W}\left(\frac{j}{(N_b - 1) N_b^{m-1}}\right)} + \cos \frac{2\pi j}{(N_b - 1) N_b^m} \\ \text{(by 1-periodicity)} \end{array} \right) = \left( \frac{j}{(N_b - 1) N_b^m}, \mathcal{W}\left(\frac{j}{(N_b - 1) N_b^m}\right) \right).$$

ii. See [DL22c]. □

**Property 2.12 (Lower Bound and Upper Bound for the Elementary Heights [DL22c]).**

For any strictly positive integer  $m$  and any  $j$  in  $\{0, \dots, (N_b - 1) N_b^m\}$ , we have the following estimates, where  $L_m$  is the elementary horizontal length introduced in part i. of Definition 2.6, on page 11:

$$C_{inf} L_m^{2-D_W} \leq \underbrace{|\mathcal{W}((j+1)L_m) - \mathcal{W}(jL_m)|}_{h_{j,j+1,m}} \leq C_{sup} L_m^{2-D_W} \quad , \quad m \in \mathbb{N}, 0 \leq j \leq (N_b - 1) N_b^m, \quad (\star)$$

where the finite and positive constants  $C_{inf}$  and  $C_{sup}$  are given by

$$C_{inf} = (N_b - 1)^{2-D_W} \min_{0 \leq j \leq N_b - 1, \mathcal{W}\left(\frac{j+1}{N_b-1}\right) \neq \mathcal{W}\left(\frac{j}{N_b-1}\right)} \left| \mathcal{W}\left(\frac{j+1}{N_b-1}\right) - \mathcal{W}\left(\frac{j}{N_b-1}\right) \right|$$

and

$$C_{sup} = (N_b - 1)^{2-D_W} \left( \max_{0 \leq j \leq N_b - 1} \left| \mathcal{W}\left(\frac{j+1}{N_b-1}\right) - \mathcal{W}\left(\frac{j}{N_b-1}\right) \right| + \frac{2\pi}{(N_b - 1)(\lambda N_b - 1)} \right).$$

One should note, in addition, that these constants  $C_{inf}$  and  $C_{sup}$  depend on the initial polygon  $\mathcal{P}_0$ , but are independent of  $m \in \mathbb{N}$  sufficiently large.

As a consequence, we also have that

$$C_{inf} L_m^{2-D_W} \leq h_m^{inf} \leq C_{sup} L_m^{2-D_W} \quad \text{and} \quad C_{inf} L_m^{2-D_W} \leq h_m \leq C_{sup} L_m^{2-D_W},$$

where  $h_m^{inf}$  and  $h_m$  respectively denote the minimal and maximal heights introduced in part iv. of Definition 2.6, on page 11.

**Theorem 2.13 (Sharp Local Discrete Reverse Hölder Properties of the Weierstrass Function [DL22c]).**

For any natural integer  $m$  (i.e., for any  $m \in \mathbb{N}$ ), let us consider a pair of real numbers  $(x, x')$  such that

$$x = \frac{(N_b - 1)k + j}{(N_b - 1)N_b^m} = ((N_b - 1)k + j) L_m \quad , \quad x' = \frac{(N_b - 1)k + j + \ell}{(N_b - 1)N_b^m} = ((N_b - 1)k + j + \ell) L_m ,$$

where  $0 \leq k \leq N_b - 1^m - 1$ . We then have the following (discrete, local) reverse-Hölder inequality, with sharp Hölder exponent  $-\frac{\ln \lambda}{\ln N_b} = 2 - D_{\mathcal{W}}$ :

$$C_{inf} |x' - x|^{2-D_{\mathcal{W}}} \leq |\mathcal{W}(x') - \mathcal{W}(x)| ,$$

where  $(x, \mathcal{W}(x))$  and  $(x', \mathcal{W}(x'))$  are adjacent vertices of the same  $m^{\text{th}}$  prefractal approximation,  $\Gamma_{\mathcal{W}_m}$ , with  $m \in \mathbb{N}$  arbitrary. Here,  $C_{inf}$  is given as in Property 2.12, on page 16 just above.

**Corollary 2.14 (Optimal Hölder Exponent for the Weierstrass Function (see [DL22c]).**

The local reverse Hölder property of Theorem 2.13, on page 17 just above – in conjunction with the Hölder condition satisfied by the Weierstrass function (see also [Zyg02], Chapter II, Theorem 4.9, page 47) – shows that the Codimension  $2 - D_{\mathcal{W}} = -\frac{\ln \lambda}{\ln N_b} \in ]0, 1[$  is the best (i.e., optimal) Hölder exponent for the Weierstrass function (as was originally shown, by a completely different method, by G. H. Hardy in [Har16]).

Note that, as a consequence, since the Hölder exponent is strictly smaller than one, it follows that the Weierstrass function  $\mathcal{W}$  is nowhere differentiable.

**Corollary 2.15 (of Property 2.12 (see [DL22c]).**

Thanks to Property 2.12, on page 16, one may now write, for any strictly positive integer  $m$  and any integer  $j$  in  $\{0, \dots, (N_b - 1)N_b^m - 1\}$ , and with  $C_{inf}$  and  $C_{supf}$  defined as in Property 2.12, on page 16:

i. for the elementary heights:

$$h_{j-1,j,m} = L_m^{2-D_{\mathcal{W}}} \mathcal{O}(1) ; \tag{R6}$$

ii. for the elementary quotients:

$$\frac{h_{j-1,j,m}}{L_m} = L_m^{1-D_{\mathcal{W}}} \mathcal{O}(1) , \tag{R7}$$

and where

$$0 < C_{inf} \leq \mathcal{O}(1) \leq C_{sup} < \infty.$$

Note that, thanks to Property 2.12, on page 16,  $\mathcal{O}(1)$  may depend on  $m$ , but is bounded away from 0 and  $\infty$ .

**Corollary 2.16 (Nonincreasing Sequence of Geometric Angles (Coming from Property 2.11; see [DL22c])).**

For the **geometric angles**  $\theta_{j-1,j,m}$ ,  $0 \leq j \leq (N_b - 1)N_b^m$ ,  $m \in \mathbb{N}$ , introduced in part v. of Definition 2.6, on page 11, we have the following result:

$$\tan \theta_{j-1,j,m} = \frac{L_m}{h_{j-1,j,m}}(N_b - 1) > \tan \theta_{j-1,j,m+1},$$

which yields

$$\theta_{j-1,j,m} > \theta_{j-1,j,m+1} \quad \text{and} \quad \theta_{j-1,j,m+1} \lesssim L_m^{D_{\mathcal{W}}-1}.$$

**Corollary 2.17 (Local Extrema (Coming from Property 2.11, on page 14; see [DL22c])).**

i. The set of local maxima of the Weierstrass function on the interval  $[0, 1]$  is given by

$$\left\{ \left( \frac{(N_b - 1)k}{N_b^m}, \mathcal{W} \left( \frac{(N_b - 1)k}{N_b^m} \right) \right) : 0 \leq k \leq N_b^m - 1, m \in \mathbb{N} \right\},$$

and corresponds to the extreme vertices of the polygons  $\mathcal{P}_{m,k}$  and  $\mathcal{Q}_{m,k}$  (i.e., they are the vertices connecting consecutive polygons; see part iv. of Property 2.6, on page 8).

ii. For odd values of  $N_b$ , the set of local minima of the Weierstrass function on the interval  $[0, 1]$  is given by

$$\left\{ \left( \frac{(N_b - 1)k + \frac{N_b - 1}{2}}{(N_b - 1)N_b^m}, \mathcal{W} \left( \frac{(N_b - 1)k + \frac{N_b - 1}{2}}{(N_b - 1)N_b^m} \right) \right) : 0 \leq k \leq N_b^m - 1, m \in \mathbb{N} \right\},$$

and corresponds to the bottom vertices of the polygons  $\mathcal{P}_{m,k}$  and  $\mathcal{Q}_{m,k}$  at a given step  $m$ ; see also part iv. of Property 2.6, on page 8.

**Property 2.18 (Existence of Reentrant Angles [DL22c]).**

i. The initial polygon  $\mathcal{P}_0$ , admits **reentrant interior angles**, at a vertex  $P_j$ , with  $0 < j \leq N_b - 1$ , in the sense that, with the right-hand rule, according to which angles are measured in a counter-clockwise direction  $((P_j P_{j+1}), (P_j P_{j-1})) > \pi$ , in the case when

$$0 < j \leq \frac{N_b - 3}{4} \quad \text{or} \quad \frac{3N_b - 1}{4} \leq j < N_b - 1$$

(see Figure 5, on page 19), which does not occur for values of  $N_b < 7$ .

The number of reentrant angles is then equal to  $2 \left\lceil \frac{N_b - 3}{4} \right\rceil$ .

ii. At a given step  $m \in \mathbb{N}^*$ , with the above convention, a polygon  $\mathcal{P}_{m,k}$  admits reentrant interior angles in the sole cases when  $N_b \geq 7$ , at vertices  $M_{k+j}$ ,  $1 \leq k \leq N_b^m$ ,  $0 < j \leq N_b - 1$ , as well as in the case when

$$0 < j \leq \frac{N_b - 3}{4} \quad \text{or} \quad \frac{3N_b - 1}{4} \leq j < N_b - 1.$$

The number of reentrant angles is then equal to  $2N_b^m \left\lceil \frac{N_b - 3}{4} \right\rceil$ , where  $[y]$  denotes the integer part of the real number  $y$ .

*Remark 2.2.* Note that, still when  $N_b \geq 7$ , due to the respective definitions of the polygons  $\mathcal{P}_{m,k}$  and  $\mathcal{Q}_{m,k}$ , the existence of reentrant interior angles for  $\mathcal{P}_{m,k}$  at a vertex  $M_{k+j}$ ,  $1 \leq k \leq N_b^m$ ,  $0 < j \leq N_b - 1$ , also results in the existence of reentrant interior angles for  $\mathcal{Q}_{m,k}$  at the vertices  $M_{k+j-1}$ ,  $1 \leq k \leq N_b^m$ ,  $1 < j \leq N_b - 1$  and  $M_{k+j+1}$ ,  $1 \leq k \leq N_b^m$ ,  $0 < j \leq N_b - 2$ .

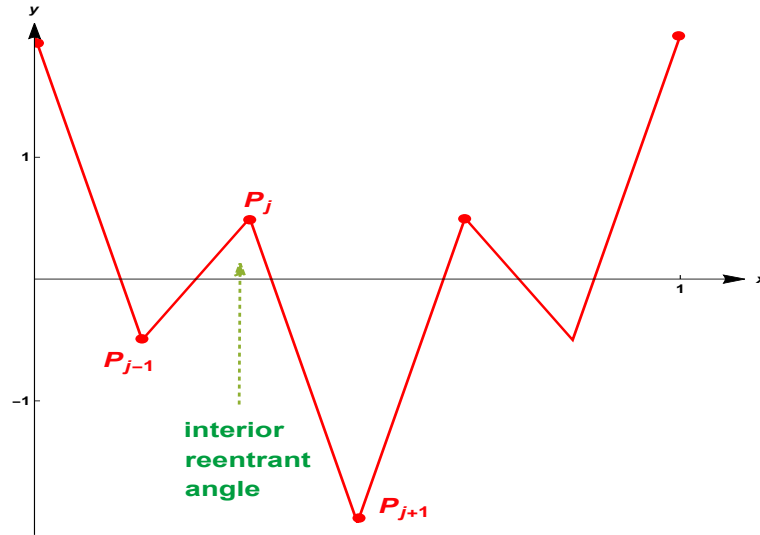


Figure 5: An interior reentrant angle. Here,  $N_b = 7$  and  $\lambda = \frac{1}{2}$ .

### 3 Polyhedral Measure

Our results on fractal cohomology obtained in [DL22d] (see also [DL22c], [DL22a]) have highlighted the role played by specific threshold values for a sequence of geometric invariants at any step  $m \in \mathbb{N}$  of the prefractal graph approximation; namely, the sequence of the  $m^{\text{th}}$  *cohomology infinitesimals* introduced in Definition 3.1, on page 20 just below, which also plays a major part in studying the polyhedral neighborhoods associated with the sequence of prefractal approximations of the Weierstrass Curve (or, equivalently, of the Weierstrass IFD). Those polyhedral neighborhoods have enabled us in [DL22b] to construct a specific polyhedral measure. We hereafter recall the main corresponding results.

**Definition 3.1** ( $m^{\text{th}}$  *Cohomology Infinitesimal* [DL22c], [DL22d] and  $m^{\text{th}}$  *Intrinsic Cohomology Infinitesimal*).

From now on, given any  $m \in \mathbb{N}$ , we will call  $m^{\text{th}}$  *cohomology infinitesimal* the number  $\varepsilon_m^m > 0$  which, modulo a multiplicative constant equal to  $\frac{1}{N_b - 1}$ , i.e.,  $\varepsilon_m^m = \frac{1}{N_b - 1} \frac{1}{N_b^m}$  (recall that  $N_b > 1$ ), stands as the elementary horizontal length introduced in part *i.* of Definition 2.6, on page 11; i.e.,

$$\frac{1}{N_b^m}.$$

Observe that, clearly,  $\varepsilon_m$  itself – and not just  $\varepsilon_m^m$  – depends on  $m$ ; hence, we should really write  $\varepsilon_m^m = (\varepsilon_m)^m$ , for all  $m \in \mathbb{N}$ .

In addition, since  $N_b > 1$ ,  $\varepsilon_m^m$  satisfies the following asymptotic behavior,

$$\varepsilon_m^m \rightarrow 0, \text{ as } m \rightarrow \infty,$$

which, naturally, results in the fact that the larger  $m$ , the smaller  $\varepsilon_m^m$ . It is for this reason that we call  $\varepsilon_m^m$  – or rather, the *infinitesimal sequence*  $(\varepsilon_m^m)_{m=0}^{\infty}$  of positive numbers tending to zero as  $m \rightarrow \infty$ , with  $\varepsilon_m^m = (\varepsilon_m)^m$ , for each  $m \in \mathbb{N}$  – an *infinitesimal*. Note that this  $m^{\text{th}}$  cohomology infinitesimal is the one naturally associated to the scaling relation of Property 2.5, on page 8.

In the sequel, it is also useful to keep in mind that the sequence of positive numbers  $(\varepsilon_m)_{m=0}^{\infty}$  itself satisfies

$$\varepsilon_m \sim \frac{1}{N_b}, \text{ as } m \rightarrow \infty ;$$

i.e.,  $\varepsilon_m \rightarrow \frac{1}{N_b}$ , as  $m \rightarrow \infty$ . In particular,  $\varepsilon_m \not\rightarrow 0$ , as  $m \rightarrow \infty$ , but, instead,  $\varepsilon_m$  tends to a strictly positive and finite limit.

We also introduce, given any  $m \in \mathbb{N}$ , the  $m^{\text{th}}$  *intrinsic cohomology infinitesimal*, denoted by  $\varepsilon^m > 0$ , such that

$$\varepsilon^m = \frac{1}{N_b^m},$$

where

$$\varepsilon = \frac{1}{N_b}.$$

We call  $\varepsilon$  *the intrinsic scale*, or *intrinsic subdivision scale*.

Note that

$$\varepsilon_m^m = \frac{\varepsilon^m}{N_b - 1}.$$

and that the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m$  is asymptotic (when  $m$  tends to  $\infty$ ) and up to a positive multiplicative constant, equal to the  $m^{\text{th}}$  cohomology infinitesimal  $\varepsilon_m^m$ .

*Remark 3.1.* We point out that the choice of the  $m^{\text{th}}$  intrinsic cohomology infinitesimal – instead of the  $m^{\text{th}}$  cohomology infinitesimal, as is done in [DL22c], [DL22d] – will considerably simplify the computation of the polyhedral fractal volumes, the polyhedral global zeta functions, and hence also, of the (intrinsic) Complex Dimensions of the Weierstrass Curve.

*Remark 3.2 (Addressing Numerical Estimates).*

From a practical point of view, an important question is the value of the ratio

$$\frac{\text{Cohomology infinitesimal}}{\text{Maximal height}} = \frac{\varepsilon_m^m}{h_m};$$

see relation (R5), on page 12.

Thanks to the estimates given in relation (R7), on page 17, we have that

$$\frac{\varepsilon_m^m}{h_m} = L_m^{1-D_W} \mathcal{O}(1) = \varepsilon_m^{m(1-D_W)} \mathcal{O}(1),$$

with

$$0 < C_{inf} \leq \mathcal{O}(1) \leq C_{sup} < \infty,$$

where  $C_{inf}$  and  $C_{sup}$  are given in Property 2.12, on page 16.

Given  $q \in \mathbb{N}^*$ , we then have

$$\frac{1}{10^q} C_{inf} \leq \frac{\varepsilon_m^m}{h_m} \leq \frac{1}{10^q} C_{sup}$$

when

$$\frac{C_{inf}}{10^q} \leq e^{(1-D_W) \ln L_m} \leq \frac{C_{sup}}{10^q},$$

or, equivalently, when

$$-\frac{1}{\ln N_b} \ln \left( (N_b - 1) \left( \frac{C_{sup}}{10^q} \right)^{\frac{1}{1-D_W}} \right) \leq m \leq -\frac{1}{\ln N_b} \ln \left( (N_b - 1) \left( \frac{C_{inf}}{10^q} \right)^{\frac{1}{1-D_W}} \right).$$

Numerical values for  $N_b = 3$  and  $\lambda = \frac{1}{2}$  yield:

- i.* For  $q = 1$ :  $2 \leq m \leq 3$ .
- ii.* For  $q = 2$ :  $7 \leq m \leq 9$ .
- iii.* For  $q = 3$ :  $13 \leq m \leq 15$ .

Hence, when  $m$  increases, the ratio  $\frac{\varepsilon_m^m}{h_m}$  decreases, and tends to 0. This numerical – but very practical and explicit argument – also accounts for our forthcoming use of the sequence of polyhedral neighborhoods, of width equal to the cohomology infinitesimal.

We refer to part *iv.* of Property 2.6, on page 8 above, along with Figure 1, on page 10 for the definition of the polygons  $\mathcal{P}_{m,j}$  (resp.,  $\mathcal{Q}_{m,j}$ ) associated with the Weierstrass Curve in the next definition, as well as throughout the rest of this section. See also Definition 2.5, on page 9 where the polygonal families are introduced.

**Definition 3.2 (Power of a Vertex of the Prefractal Graph  $\Gamma_{\mathcal{W}_m}$ ,  $m \in \mathbb{N}^*$ , with Respect to the Polygonal Families  $\mathcal{P}_m$  and  $\mathcal{Q}_m$ ).**

Given a strictly positive integer  $m$ , a vertex  $X$  of the prefractal graph  $\Gamma_{\mathcal{W}_m}$  will be said to be:

- i.* of power one relative to the polygonal family  $\mathcal{P}_m$  if  $X$  belongs to (to be understood in the sense that  $X$  is a vertex of) one and only one  $N_b$ -gon  $\mathcal{P}_{m,j}$ ,  $0 \leq j \leq N_b^m - 1$ ;
- ii.* of power  $\frac{1}{2}$  relative to the polygonal family  $\mathcal{P}_m$  if  $X$  is a common vertex to two consecutive  $N_b$ -gons  $\mathcal{P}_{m,j}$  and  $\mathcal{P}_{m,j+1}$ ,  $0 \leq j \leq N_b^m - 2$ ;
- iii.* of power zero relative to the polygonal family  $\mathcal{P}_m$  if  $X$  does not belong to (to be understood in the sense that  $X$  is not a vertex of) any  $N_b$ -gon  $\mathcal{P}_{m,j}$ ,  $0 \leq j \leq N_b^m - 1$ .

Similarly, given  $m \in \mathbb{N}$ , a vertex  $X$  of the prefractal graph  $\Gamma_{\mathcal{W}_m}$  is said to be:

- i.* of power one relative to the polygonal family  $\mathcal{Q}_m$  if  $X$  belongs to (as above, to be understood in the sense that  $X$  is a vertex of) one and only one  $N_b$ -gon  $\mathcal{Q}_{m,j}$ ,  $0 \leq j \leq N_b^m - 2$ ;
- ii.* of power  $\frac{1}{2}$  relative to the polygonal family  $\mathcal{Q}_m$  if  $X$  is a common vertex to two consecutive  $N_b$ -gons  $\mathcal{Q}_{m,j}$  and  $\mathcal{Q}_{m,j+1}$ ,  $0 \leq j \leq N_b^m - 3$ ;
- iii.* of power zero relative to the polygonal family  $\mathcal{Q}_m$  if  $X$  does not belong to (as previously, to be understood in the sense that  $X$  is not a vertex of) any  $N_b$ -gon  $\mathcal{Q}_{m,j}$ ,  $0 \leq j \leq N_b^m - 2$ .

**Notation 9.** In the sequel, given a strictly positive integer  $m$ , the *power of a vertex  $X$  of the prefractal graph  $\Gamma_{\mathcal{W}_m}$  relative to the polygonal families  $\mathcal{P}_m$  and  $\mathcal{Q}_m$*  will be respectively denoted by

$$p(X, \mathcal{P}_m) \quad \text{and} \quad p(X, \mathcal{Q}_m).$$

**Notation 10 (Lebesgue Measure on  $\mathbb{R}^2$ ).**

In the sequel, we denote by  $\mu_{\mathcal{L}}$  the Lebesgue measure on  $\mathbb{R}^2$ .

**Notation 11.** For any  $m \in \mathbb{N}$ , and any vertex  $X$  of  $V_m$ , we set:

$$\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) = \begin{cases} \frac{1}{N_b} p(X, \mathcal{P}_m) \sum_{0 \leq j \leq N_b^m - 1, X \text{ vertex of } \mathcal{P}_{m,j}} \mu_{\mathcal{L}}(\mathcal{P}_{m,j}), & \text{if } X \notin \mathcal{Q}_m, \\ \frac{1}{N_b} p(X, \mathcal{Q}_m) \sum_{1 \leq j \leq N_b^m - 2, X \text{ vertex of } \mathcal{Q}_{m,j}} \mu_{\mathcal{L}}(\mathcal{Q}_{m,j}), & \text{if } X \notin \mathcal{P}_m, \\ \frac{1}{2N_b} \left\{ p(X, \mathcal{P}_m) \sum_{0 \leq j \leq N_b^m - 1, X \text{ vertex of } \mathcal{P}_{m,j}} \mu_{\mathcal{L}}(\mathcal{P}_{m,j}) + p(X, \mathcal{Q}_m) \sum_{1 \leq j \leq N_b^m - 2, X \text{ vertex of } \mathcal{Q}_{m,j}} \mu_{\mathcal{L}}(\mathcal{Q}_{m,j}) \right\}, & \\ \text{if } X \in \mathcal{P}_m \cap \mathcal{Q}_m. \end{cases}$$

**Property 3.1.** For any  $m \in \mathbb{N}$ , and any pair  $(X, Y)$  of adjacent vertices of  $V_m$  belonging to the same polygon  $\mathcal{P}_{m,j}$ , with  $0 \leq j \leq N_b^m - 1$  (resp.,  $\mathcal{Q}_{m,j}$ , with  $0 \leq j \leq N_b^m - 2$ ), we have that (with  $d_{Eucl}$  denoting the Euclidean distance on  $\mathbb{R}^2$ ),

$$d_{Eucl}(X, Y) = \sqrt{h_{jm}^2 + L_m^2} > |h_{jm}|,$$

which, due to the inequality given in Property 2.12, page 16, ensures that

$$\frac{1}{d_{Eucl}(X, Y)} < \frac{1}{|h_{jm}|} \lesssim L_m^{D_W - 2} \lesssim N_b^{(2 - D_W)m}.$$

At the same time, we also have that

$$d_{Eucl}(X, Y) \lesssim h_m \lesssim L_m^{2 - D_W} \lesssim N_b^{(D_W - 2)m}.$$

*Proof.* This follows at once from Property 2.12, on page 16. □

**Corollary 3.2.** For any  $m \in \mathbb{N}$ , any natural integer  $j$  of  $\{0, \dots, N_b^m - 1\}$ , and any pair of points  $(X, Y)$  of  $\mathcal{P}_{m,j}$  or of  $\mathcal{Q}_{m,j}$ , we have that

$$\frac{1}{d_{Eucl}(X, Y)} \lesssim L_m^{D_W - 2} \lesssim N_b^{(2 - D_W)m}$$

and

$$d_{Eucl}(X, Y) \lesssim h_m \lesssim L_m^{2 - D_W} \lesssim N_b^{(D_W - 2)m}.$$

**Property 3.3.** For any  $m \in \mathbb{N}$ , and any vertex  $X$  of  $V_m$ :

$$\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \lesssim h_m L_m \lesssim L_m^{3 - D_W} \lesssim N_b^{(D_W - 3)m}$$

and

$$\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \lesssim h_m L_m \lesssim L_m^{3 - D_W} \lesssim N_b^{(D_W - 3)m}.$$



*Proof.* This also directly follows from Property 2.12, on page 16. □

**Definition 3.3 (Trace of a Polygon on the Weierstrass Curve [DL22b]).**

Given  $m \in \mathbb{N}$ , and  $0 \leq j \leq N_b^m - 1$  (resp.,  $0 \leq j \leq N_b^m - 2$ ), of extreme vertices  $\mathcal{V}_{initial}(\mathcal{P}_{m,j}) \in V_m$  and  $\mathcal{V}_{end}(\mathcal{P}_{m,j}) \in V_m$  (resp.,  $\mathcal{V}_{initial}(\mathcal{Q}_{m,j}) \in V_m$  and  $\mathcal{V}_{end}(\mathcal{Q}_{m,j}) \in V_m$ ; see Property 2.6, on page 8), we define *the trace* of the polygon  $\mathcal{P}_{m,j}$  (resp.,  $\mathcal{Q}_{m,j}$ ) on the Weierstrass Curve as the set  $tr_{\gamma_{\mathcal{W}}}(\mathcal{P}_{m,j})$  (resp.,  $tr_{\gamma_{\mathcal{W}}}(\mathcal{Q}_{m,j})$ ) of points  $\{\mathcal{V}_{initial}(\mathcal{P}_{m,j}), M_{\star}, \mathcal{V}_{end}(\mathcal{P}_{m,j})\}$  (resp.,  $\{\mathcal{V}_{initial}(\mathcal{Q}_{m,j}), M_{\star}, \mathcal{V}_{end}(\mathcal{Q}_{m,j})\}$ ), where we denote by  $M_{\star}$  any point of the Weierstrass Curve strictly located between  $\mathcal{V}_{initial}(\mathcal{P}_{m,j})$  and  $\mathcal{V}_{end}(\mathcal{P}_{m,j})$  (resp.,  $\mathcal{V}_{initial}(\mathcal{Q}_{m,j})$  and  $\mathcal{V}_{end}(\mathcal{Q}_{m,j})$ ).

**Definition 3.4 (Sequence of Domains Delimited by the Weierstrass IFD – Polyhedral Neighborhood of the Weierstrass Curve [DL22b]).**

We introduce *the sequence of domains delimited by the Weierstrass IFD*, or *polyhedral neighborhood of the Weierstrass Curve* as the sequence  $(\mathcal{D}(\Gamma_{\mathcal{W}_m}))_{m \in \mathbb{N}}$  of open, connected polygonal sets  $(\mathcal{P}_m \cup \mathcal{Q}_m)_{m \in \mathbb{N}}$ , where, for each  $m \in \mathbb{N}$ ,  $\mathcal{P}_m$  and  $\mathcal{Q}_m$  respectively denote the polygonal sets introduced in Definition 2.5, on page 9.

Given  $m \in \mathbb{N}$ , we call  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  the  $m^{\text{th}}$  *polyhedral neighborhood* (of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ ).

**Property 3.4 (Domain Delimited by the Weierstrass IFD).**

We call domain, delimited by the Weierstrass IFD, the set which is equal to the following limit,

$$\mathcal{D}(\Gamma_{\mathcal{W}}) = \lim_{m \rightarrow \infty} \mathcal{D}(\Gamma_{\mathcal{W}_m}),$$

where the convergence is interpreted in the sense of the Hausdorff metric on  $\mathbb{R}^2$ ; see [DL22b]. In fact, we have that

$$\mathcal{D}(\Gamma_{\mathcal{W}}) = \Gamma_{\mathcal{W}}.$$

**Notation 12 (Minimal and Maximal Values of the Weierstrass Function  $\mathcal{W}$  on  $[0, 1]$ ).**

We set

$$m_{\mathcal{W}} = \min_{t \in [0,1]} \mathcal{W}(t) = -\frac{1}{1-\lambda} \quad , \quad M_{\mathcal{W}} = \max_{t \in [0,1]} \mathcal{W}(t) = \frac{1}{1-\lambda}.$$

**Notation 13.** Henceforth, for a given  $m \in \mathbb{N}$ , the notation  $\sum_{X \in \mathcal{P}_m \cup \mathcal{Q}_m}$  means that the associated finite sum runs through all of the vertices of the polygons belonging to the sets  $\mathcal{P}_m$  or  $\mathcal{Q}_m$  introduced in Definition 2.5, on page 9; see also Notation 7, on page 10 following that definition.

**Property 3.5 ([DL22b]).**

Given a continuous function  $u$  on  $[0, 1] \times [m_{\mathcal{W}}, M_{\mathcal{W}}]$ , we have that, for any  $m \in \mathbb{N}$ , and any vertex  $X$  of  $V_m$ :

$$\left| \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) u(X) \right| \leq \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \left( \max_{[0,1] \times [m_{\mathcal{W}}, M_{\mathcal{W}}]} |u| \right) \leq N_b^{-(3-D_{\mathcal{W}})m}.$$

Consequently, with the notation of Definition 3.1, on page 20, we have that

$$\varepsilon_m^{m(D_{\mathcal{W}}-2)} \left| \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) u(X) \right| \leq \varepsilon_m^{-m}.$$

Since the sequence  $\left( \sum_{X \in \mathcal{P}_m \cup \mathcal{Q}_m} \varepsilon^{-m} \right)_{m \in \mathbb{N}}$  is a positive and increasing sequence (the number of vertices involved increases as  $m$  increases), this ensures the existence of the finite limit

$$\lim_{m \rightarrow \infty} \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \cup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) u(X),$$

where we have used Notation 13, on page 24.

**Property 3.6 (Polyhedral Measure on the Weierstrass IFD [DL22b]).**

We introduce the polyhedral measure on the Weierstrass IFD, denoted by  $\mu$ , such that for any continuous function  $u$  on the Weierstrass Curve, with the use of Notation 11, on page 23 and Notation 13, on page 24,

$$\int_{\Gamma_{\mathcal{W}}} u d\mu = \lim_{m \rightarrow \infty} \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \cup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) u(X), \quad (\star)$$

which, thanks to Property 3.4, on page 24, can also be understood in the following way,

$$\int_{\Gamma_{\mathcal{W}}} u d\mu = \int_{\mathcal{D}(\Gamma_{\mathcal{W}})} u d\mu.$$

*Remark 3.3.* In a sense, our polyhedral measure can be seen as a measure which is an extension of the Riemann integral, where the step functions are replaced by upper and lower affine functions which approximate the Weierstrass Curve.

**Theorem 3.7 ([DL22b]).**

The polyhedral measure  $\mu$  is well defined, positive, as well as a bounded, nonzero, Borel measure on  $\mathcal{D}(\Gamma_{\mathcal{W}})$ . The associated total mass is given by

$$\mu(\mathcal{D}(\Gamma_{\mathcal{W}})) = \lim_{m \rightarrow \infty} \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \cup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m), \quad (\star\star)$$

and satisfies the following estimate:

$$0 < \mu(\mathcal{D}(\Gamma_{\mathcal{W}})) \leq \frac{2}{N_b} (N_b - 1)^2 C_{sup}. \quad (\star\star\star)$$

Furthermore, the support of  $\mu$  coincides with the entire curve:

$$\text{supp } \mu = \mathcal{D}(\Gamma_{\mathcal{W}}) = \Gamma_{\mathcal{W}}.$$

In addition,  $\mu$  is the weak limit as  $m \rightarrow \infty$  of the following discrete measures (or Dirac Combs), given, for each  $m \in \mathbb{N}$ , by

$$\mu_m = \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \cup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \delta_X,$$

where  $\varepsilon_m^m$  denotes the cohomology infinitesimal introduced in Definition 3.1, on page 20,  $\delta_X$  is the Dirac measure concentrated at  $X$ , and we have used Notation 11, on page 23 for  $\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m)$ , along with Notation 13, on page 24.

We next establish a useful new result concerning the polyhedral measure, according to which it is a quasi self-similar measure; see Property 3.8, on page 26 and Theorem 3.9, on page 27.

**Property 3.8 (The Quasi Self-Similar Sequence of Discrete Polyhedral Measures).**

The sequence of discrete polyhedral measures  $(\mu_m)_{m \in \mathbb{N}}$  introduced in Theorem 3.7, on page 25 just above, satisfies the following recurrence relation, for all  $m \in \mathbb{N}^*$ ,

$$\mu_m = N_b^{D_{\mathcal{W}}-2} \sum_{T_j \in \mathcal{T}_{\mathcal{W}}} \mu_{m+1} \circ T_j^{-1}, \quad (\spadesuit)$$

where for  $\mathcal{T}_{\mathcal{W}} = \{T_0, \dots, T_{N_b-1}\}$  is the nonlinear iterated function system (IFS) introduced in Proposition 2.2, on page 6.

Note that relation  $(\spadesuit)$  can be viewed as a generalization of classical self-similar measures, as considered in [Hut81], page 714.

*Proof.* First, we can note that, for  $m \in \mathbb{N}^*$ ,

$$\varepsilon_{m+1}^{m+1} = \frac{1}{N_b} \varepsilon_m^m,$$

which ensures that

$$\varepsilon_{m+1}^{(m+1)(D_{\mathcal{W}}-2)} = \frac{1}{N_b^{D_{\mathcal{W}}-2}} \varepsilon_m^{m(D_{\mathcal{W}}-2)} = N_b^{2-D_{\mathcal{W}}} \varepsilon_m^{m(D_{\mathcal{W}}-2)}.$$

We then simply use the result of Property 2.11, on page 14, according to which, for  $0 \leq j \leq N_b - 1$ , the  $j^{\text{th}}$  vertex of the polygon  $\mathcal{P}_{m+1,k}$ ,  $0 \leq k \leq N_b^m - 1$ , is the image of the  $j^{\text{th}}$  vertex of the polygon  $\mathcal{P}_{m,k-i(N_b-1)N_b^m}$  under the bijective map  $T_j$ , where  $0 \leq j \leq N_b - 1$  is arbitrary. Therefore, there is an exact correspondance between polygons at consecutive steps  $m$ ,  $m + 1$ : indeed, polygons at the  $(m + 1)^{\text{th}}$  step of the prefractal approximation process are obtained by applying each (bijective) map  $T_j$ , for  $0 \leq j \leq N_b - 1$ , to the polygons at the  $m^{\text{th}}$  step of the prefractal approximation process:

$$\bigcup_{j=0}^{N_b-1} \{T_j^{-1}(X), X \in V_{m+1}\} = V_m,$$

which can equivalently be written as

$$\bigcup_{j=0}^{N_b-1} \{T_j^{-1}(X), X \in \mathcal{P}_{m+1} \cup \mathcal{Q}_{m+1}\} = \{X \in \mathcal{P}_m \cup \mathcal{Q}_m\}.$$

We can then deduce that

$$\sum_{X \in \mathcal{P}_m \cup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \delta_X = \sum_{T_j \in \mathcal{T}_W} \sum_{X \in \mathcal{P}_{m+1} \cup \mathcal{Q}_{m+1}} \mu^{\mathcal{L}}(X, T_j^{-1}(\mathcal{P}_{m+1}), T_j^{-1}(\mathcal{Q}_{m+1})) \delta_X,$$

which yields the desired result.  $\square$

### Theorem 3.9 (The Quasi Self-Similar Polyhedral Measure).

The polyhedral measure  $\mu$  satisfies the following quasi self-similarity relation,

$$\mu = N_b^{D_W-2} \sum_{T_j \in \mathcal{T}_W} \mu \circ T_j^{-1}, \quad (\spadesuit\spadesuit)$$

where  $\mathcal{T}_W = \{T_0, \dots, T_{N_b-1}\}$  is the nonlinear iterated function system (IFS) introduced in Proposition 2.2, on page 6.

*Proof.* As in the proof of Property 3.8, on page 26, we use the fact that, for all  $m \in \mathbb{N}^*$ ,

$$\bigcup_{j=0}^{N_b-1} \{T_j^{-1}(X), X \in \mathcal{P}_m \cup \mathcal{Q}_m\} = \{X \in \mathcal{P}_{m-1} \cup \mathcal{Q}_{m-1}\}.$$

We then have that, for any continuous function  $u$  on the Weierstrass Curve,

$$\begin{aligned} \sum_{T_j \in \mathcal{T}_W} \int_{\Gamma_W} u d(\mu \circ T_j^{-1}) &= \sum_{T_j \in \mathcal{T}_W} \lim_{m \rightarrow \infty} \varepsilon_m^{m(D_W-2)} \sum_{X \in \mathcal{P}_m \cup \mathcal{Q}_m} \mu^{\mathcal{L}}(T_j^{-1}(X), \mathcal{P}_m, \mathcal{Q}_m) u(T_j^{-1}(X)) \\ &= \lim_{m \rightarrow \infty} \varepsilon_m^{m(D_W-2)} \sum_{X \in \mathcal{P}_{m-1} \cup \mathcal{Q}_{m-1}} \mu^{\mathcal{L}}(X, \mathcal{P}_{m-1}, \mathcal{Q}_{m-1}) u(X) \\ &= \lim_{m \rightarrow \infty} N_b^{D_W-2} \varepsilon_{m-1}^{(m-1)(D_W-2)} \sum_{X \in \mathcal{P}_{m-1} \cup \mathcal{Q}_{m-1}} \mu^{\mathcal{L}}(X, \mathcal{P}_{m-1}, \mathcal{Q}_{m-1}) u(X) \\ &= N_b^{2-D_W} \lim_{m \rightarrow \infty} \varepsilon_{m-1}^{(m-1)(D_W-2)} \sum_{X \in \mathcal{P}_{m-1} \cup \mathcal{Q}_{m-1}} \mu^{\mathcal{L}}(X, \mathcal{P}_{m-1}, \mathcal{Q}_{m-1}) u(X) \\ &= N_b^{2-D_W} \int_{\Gamma_W} u d\mu, \end{aligned}$$

from which we deduce, as desired, that

$$\mu = N_b^{D_{\mathcal{W}}-2} \sum_{T_j \in \mathcal{T}_{\mathcal{W}}} \mu \circ T_j^{-1}.$$

□

**Corollary 3.10.** *The polyhedral measure is the unique fixed point, with respect to the Kantorovich distance, of the map  $\Phi$  which, to any Borel probability measure  $\nu$  associates*

$$\Phi(\nu) = N_b^{D_{\mathcal{W}}-2} \sum_{T_j \in \mathcal{T}_{\mathcal{W}}} \nu \circ T_j^{-1}.$$

*Remark 3.4.* The results of Theorem 3.9, on page 28 and of Corollary 3.10, on page 28 just above, foreshadow and announce the recurrence relation satisfied by the sequence of effective polyhedral volumes, and established in the proof of Theorem 4.7, on page 54.

## 4 Polyhedral Neighborhoods

**Definition 4.1** ( *$(m, \varepsilon_m^m)$ -Neighborhood [DL22c]*).

Given  $m \in \mathbb{N}$  sufficiently large (so that  $\varepsilon_m^m$  be a sufficiently small positive number), we define the  $(m, \varepsilon_m^m)$ -neighborhood of the  $m^{\text{th}}$  prefractal approximation  $\Gamma_{\mathcal{W}_m}$  as follows:

$$\mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m) = \{M = (x, y) \in \mathbb{R}^2, d(M, \Gamma_{\mathcal{W}_m}) \leq \varepsilon_m^m\},$$

where  $d(M, \Gamma_{\mathcal{W}_m}) = \min \{d_{\text{Eucl}}(M, P), P \in \Gamma_{\mathcal{W}_m}\}$ .

**Property 4.1** (*The Nested Neighborhoods*).

- i.* Given  $m \in \mathbb{N}$  sufficiently large, there exists  $k_1 \in \mathbb{N}$  such that, for all  $k \geq k_1$ , the polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  introduced in Definition 3.4, on page 24 contains, but for a finite number of wedges, the  $(m+k, \varepsilon_{m+k}^{m+k})$ -neighborhood; see Figures 6–9, on pages 31–32.
- ii.* Given  $m \in \mathbb{N}$  sufficiently large, there exists  $k_2 \in \mathbb{N}$  such that, for all  $k \geq k_2$ , the  $(m, \varepsilon_m^m)$ -neighborhood contains the polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_{m+k}})$  introduced in Definition 3.4, on page 24; see Figure 10, on page 33 and Figure 11, on page 34.
- iii.* Given  $m \in \mathbb{N}$  sufficiently large, there exists  $k_3 \in \mathbb{N}$  such that, for all  $k \geq k_3$ , the polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  introduced in Definition 3.4, on page 24 contains the polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_{m+k}})$ .

*Proof.*

*i.* At a given step  $m \geq 0$ , between two adjacent vertices  $M_{i,m}$  and  $M_{i+1,m}$  of  $V_m$ , there are  $N_b - 1$  consecutive vertices of  $V_{m+1} \setminus V_m$ ,  $(M_{j+1,m+1}, \dots, M_{j+N_b-2,m+1}) \in V_{m+1}^{N_b-1}$  such that

$$M_{i,m} = M_{j,m+1} \quad \text{and} \quad M_{i+1,m} = M_{j+N_b,m+1}.$$

Thanks to Property 2.11, on page 14, we obtain an exact correspondance between vertices of the polygons at the step  $m + 1$ , and at the initial step  $m = 0$ . Since reentrant angles occur when  $N_b \geq 7$  (see Property 2.18, on page 18), we can restrict ourselves to the cases  $N_b \leq 6$  (in the case of reentrant angles, the following arguments can be suitably adjusted). We then simply have to consider the  $\left\lceil \frac{N_b - 2}{2} \right\rceil$  vertices  $M_{j+k,m+1}$ , with  $1 \leq k \leq \left\lceil \frac{N_b - 2}{2} \right\rceil$  (the same arguments hold for the vertices  $M_{j+N_b-k,m+1}$ ). Also, again thanks to Property 2.11, on page 14, given  $j$  in  $\{0, \dots, N_b - 2\}$  and  $k$  in  $\{0, \dots, N_b^{m+1} - 1\}$ , we have that

$$\text{sgn} \left( \mathcal{W} \left( \frac{k(N_b - 1) + j + 1}{(N_b - 1)N_b^{m+1}} \right) - \mathcal{W} \left( \frac{k(N_b - 1) + j}{(N_b - 1)N_b^{m+1}} \right) \right) = \text{sgn} \left( \mathcal{W} \left( \frac{j + 1}{N_b - 1} \right) - \mathcal{W} \left( \frac{j}{N_b - 1} \right) \right);$$

i.e., equivalently,

$$\text{sgn} (\mathcal{W}((k(N_b - 1) + j + 1)L_{m+1}) - \mathcal{W}((k(N_b - 1) + j)L_{m+1})) = \text{sgn} \left( \mathcal{W} \left( \frac{j + 1}{N_b - 1} \right) - \mathcal{W} \left( \frac{j}{N_b - 1} \right) \right),$$

where  $\text{sgn}$  has been defined in Notation 8, on page 14.

Due to the symmetry of the initial polygon  $\mathcal{P}_0$  (or, equivalently, of the initial prefractal graph  $\Gamma_{\mathcal{W}_0}$ ) with respect to the vertical line  $x = \frac{1}{2}$  (see Property 2.1, on page 6), this means that we can restrict ourselves to the case when

$$\mathcal{W}(jL_{m+1}) \geq \mathcal{W}((j + 1)L_{m+1}) \geq \dots \geq \mathcal{W} \left( \left( j + \left\lceil \frac{N_b - 2}{2} \right\rceil \right) L_{m+1} \right)$$

and

$$\frac{\mathcal{W}(jL_{m+1})}{\mathcal{W}(iL_m)} \geq \frac{\mathcal{W}((j + N_b)L_{m+1})}{\mathcal{W}((i + 1)L_m)}.$$

Thanks to Property 2.12, on page 16, we deduce, by the triangle inequality, for  $1 \leq k \leq \left\lceil \frac{N_b - 2}{2} \right\rceil$ , that

$$\left| \frac{\mathcal{W}((j + k)L_{m+1}) - \mathcal{W}(jL_{m+1})}{\mathcal{W}(iL_m)} \right| \leq \left\lceil \frac{N_b - 2}{2} \right\rceil C_{sup} L_{m+1}^{2-D_{\mathcal{W}}}.$$

Since

$$L_{m+1} = \frac{L_m}{N_b},$$

we then obtain that

$$\left| \frac{\mathcal{W}((j+k)L_{m+1}) - \mathcal{W}(jL_{m+1})}{\mathcal{W}(iL_m)} \right| \leq \left[ \frac{N_b - 2}{2} \right] N_b^{D_{\mathcal{W}} - 2} C_{sup} L_m^{2 - D_{\mathcal{W}}}.$$

Recall now (see Property 2.12, on page 16) that

$$C_{inf} = (N_b - 1)^{2 - D_{\mathcal{W}}} \min_{0 \leq j \leq N_b - 1} \left| \mathcal{W}\left(\frac{j+1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \right|$$

and

$$C_{sup} = (N_b - 1)^{2 - D_{\mathcal{W}}} \left( \max_{0 \leq j \leq N_b - 1} \left| \mathcal{W}\left(\frac{j+1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \right| + \frac{2\pi}{(N_b - 1)(\lambda N_b - 1)} \right).$$

Here, we have that

$$\mathcal{W}\left(\frac{j}{N_b - 1}\right) = \frac{1}{1 - \lambda} \cos \frac{2\pi j}{N_b - 1}.$$

This ensures that

$$\left| \mathcal{W}\left(\frac{j+1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \right| \leq \frac{2\pi}{N_b - 1} \frac{1}{1 - \lambda}.$$

We can check numerically that

$$\left[ \frac{N_b - 2}{2} \right] N_b^{D_{\mathcal{W}} - 2} C_{sup} \leq C_{inf},$$

from which we immediately deduce that for,  $1 \leq k \leq \left[ \frac{N_b - 2}{2} \right]$ ,

$$\left| \frac{\mathcal{W}((j+k)L_{m+1}) - \mathcal{W}(jL_{m+1})}{\mathcal{W}(iL_m)} \right| \leq |\mathcal{W}((i+1)L_m) - \mathcal{W}(iL_m)|.$$

For  $1 \leq k \leq \left[ \frac{N_b - 2}{2} \right]$ , the vertices  $M_{j+k, m+1}$  are then strictly between the vertices  $M_{i, m}$  and  $M_{i+1, m}$ .

As was explained previously, we can show, in a similar way, that for  $1 \leq k \leq \left[ \frac{N_b - 2}{2} \right]$ , the vertices  $M_{j+N_b-k, m+1}$  are also strictly between the vertices  $M_{i, m}$  and  $M_{i+1, m}$ .

By induction, we then obtain that, given four consecutive adjacent vertices  $M_{i, m}$ ,  $M_{i+1, m}$ ,  $M_{i+3, m}$  and  $M_{i+4, m}$  of  $V_m$ , with  $1 \leq i \leq \#V_m - 5$  and  $k \in \mathbb{N}$ , the vertices of  $V_{m+k} \setminus V_m$  located between  $M_{i, m}$  and  $M_{i+4, m}$  can be all contained in the simple and convex polygon  $M_{i, m}M_{i+1, m}M_{i+3, m}M_{i+4, m}$ , which coincides with the union of two consecutive polygons  $\mathcal{P}_{m, j}$  and  $\mathcal{Q}_{m, j}$ . Thus, there exists  $m_0 \in \mathbb{N}$  such that, for all  $k \geq m_0$ , the  $(m+k, \varepsilon_{m+k}^{m+k})$ -neighborhood

$$\mathcal{D}\left(\Gamma_{\mathcal{W}_{m+k}}, \varepsilon_{m+k}^{m+k}\right) = \left\{ M = (x, y) \in \mathbb{R}^2, d(M, \Gamma_{\mathcal{W}_{m+k}}) \leq \varepsilon_{m+k}^{m+k} \right\},$$

from which we remove the wedges associated to the vertices  $M_{i, m}$ ,  $M_{i+1, m}$ ,  $M_{i+3, m}$  and  $M_{i+4, m}$  (see [DL22c]), can be fully contained in the polygon  $M_{i, m}M_{i+1, m}M_{i+3, m}M_{i+4, m}$ . Hence, there exists  $k_1 \in \mathbb{N}$  such that, for all  $k \geq k_1$ , the  $(m, \varepsilon_m^m)$ -neighborhood but for a finite number of wedges, the  $(m+k, \varepsilon_{m+k}^{m+k})$ -neighborhood  $\mathcal{D}\left(\Gamma_{\mathcal{W}_{m+k}}, \varepsilon_{m+k}^{m+k}\right)$ , can be fully contained in the polygonal domain  $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ .

*ii.* This latter result has been obtained in [DL22b]. It comes from the fact that, in the sense of the Hausdorff metric on  $\mathbb{R}^2$ ,

$$\lim_{m \rightarrow \infty} \mathcal{D}(\Gamma_{\mathcal{W}_m}) = \Gamma_{\mathcal{W}}.$$

*iii.* This is an immediate consequence of *i.* and *ii.*

□

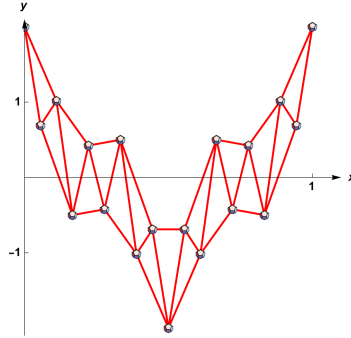


Figure 6: **The polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_2})$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .**

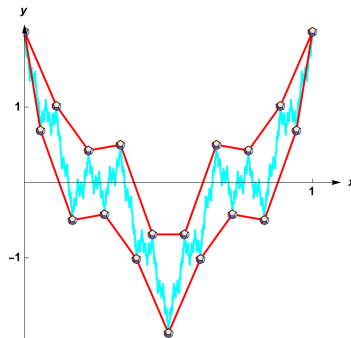


Figure 7: **The exterior boundary of the polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_2})$  (in red), and the tubular neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_7}, \varepsilon_7^7)$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .**



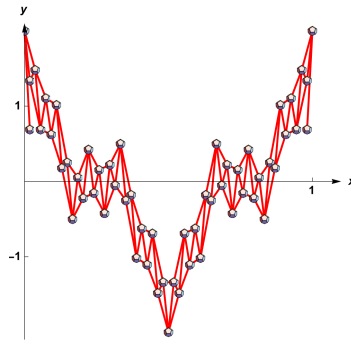


Figure 8: The polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_3})$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .

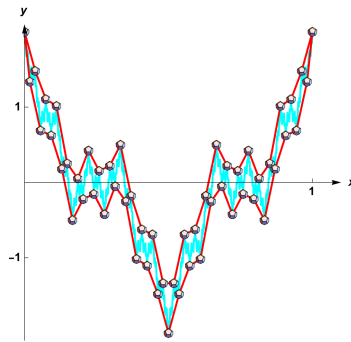


Figure 9: The exterior boundary of the polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_3})$  (in red), and the tubular neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_7}, \varepsilon_7^7)$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .

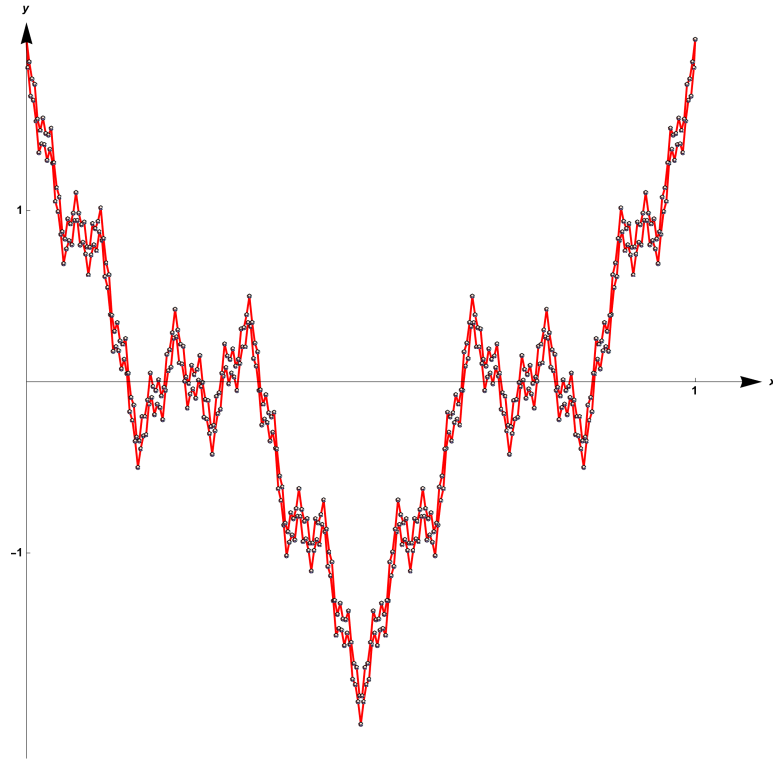


Figure 10: The polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_5})$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .

**Corollary 4.2** (of Property 4.1, given on page 28).

We immediately deduce from Property 4.1 that there exists  $m_0 \in \mathbb{N}$  such that, for all  $k \geq m_0$ , the polyhedral neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  introduced in Definition 3.4, on page 24 contains the vertices of the prefractal approximation  $\Gamma_{\mathcal{W}_{m+k}}$  to the the Weierstrass Curve; i.e., the set of vertices  $V_{m+k}$ . In particular, the density Property 2.4, on page 7, also ensures that the Weierstrass Curve is contained in  $\Gamma_{\mathcal{W}_m}$ :

$$\forall m \geq m_0 : \Gamma_{\mathcal{W}} \subset \mathcal{D}(\Gamma_{\mathcal{W}_m}) .$$

Next, we give the Complex Dimensions series expansion of the Weierstrass function, required to obtain a key exact fractal formula for the volume (i.e., area) of the polyhedral neighborhoods of the prefractal approximations. Much of the remainder of this paper will rely on this result.

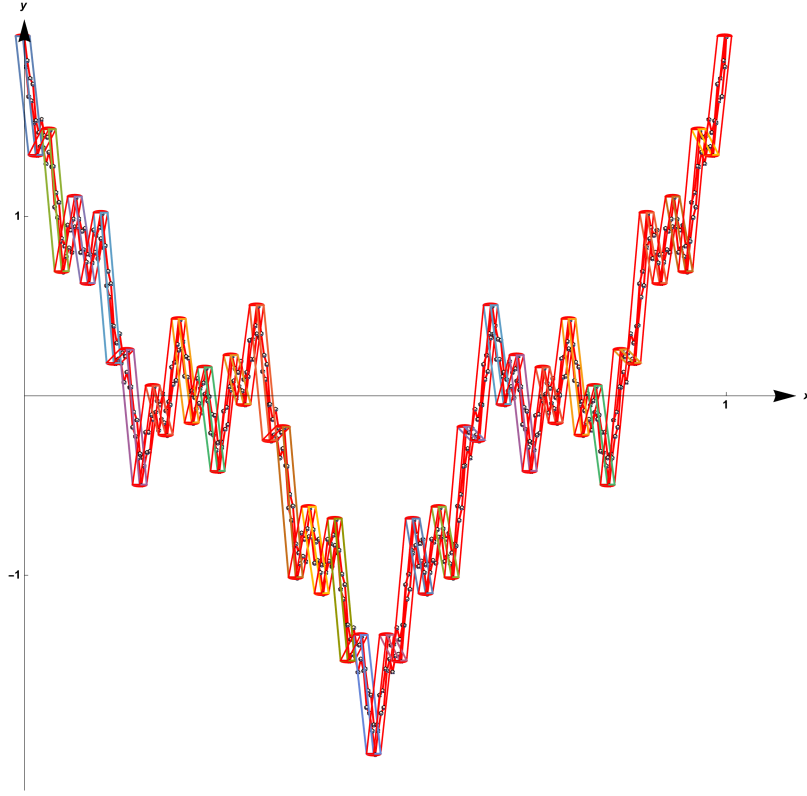


Figure 11: The tubular neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_3}, \varepsilon_3^3)$  and the polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_5})$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ . The size of the vertex points of  $\mathcal{D}(\Gamma_{\mathcal{W}_5})$  has intentionally been magnified, in order to obtain an illustrative and understandable figure.

**Property 4.3 (Complex Dimensions Series Expansion of the Weierstrass Complexified Function  $\mathcal{W}_{comp}$  [DL22d], and of the Weierstrass function  $\mathcal{W}$ ).**

For any sufficiently large positive integer  $m$  and any  $j$  in  $\{0, \dots, \#V_m - 1\}$ , we have the following exact expansion, indexed by the Complex Codimensions  $k(D_{\mathcal{W}} - 2) + i k \ell_{k,j,m} \mathbf{p}$ , with  $0 \leq k \leq m$ ,

$$\begin{aligned}
 \mathcal{W}_{comp}(j \varepsilon_m^m) &= \mathcal{W}_{comp}\left(\frac{j \varepsilon^m}{N_b - 1}\right) \\
 &= \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}_{comp}\left(\frac{j}{N_b - 1}\right) + \sum_{k=0}^{m-1} c_{k,j,m} \varepsilon^{k(2-D_{\mathcal{W}})} \varepsilon^{i \ell_{k,j,m} \mathbf{p}} \quad (\mathcal{R}8) \\
 &= \sum_{k=0}^m c_{k,j,m} \varepsilon^{k(2-D_{\mathcal{W}})} \varepsilon^{i \ell_{k,j,m} \mathbf{p}},
 \end{aligned}$$

where, for  $0 \leq k \leq m$ ,  $\varepsilon^k$  is the  $k^{\text{th}}$  intrinsic cohomology infinitesimal, introduced in Definition 3.1, on page 20, with  $\mathbf{p} = \frac{2\pi}{\ln N_b}$  denoting the oscillatory period of the Weierstrass Curve, as introduced in [DL22c] and where:

- i.  $\ell_{k,j,m} \in \mathbb{Z}$  is arbitrary.
- ii.  $c_{m,j,m} = \mathcal{W}_{comp}\left(\frac{j}{N_b - 1}\right)$  and, for  $0 \leq k \leq m - 1$ ,  $c_{k,j,m} \in \mathbb{C}$  is given by

$$c_{k,j,m} = \exp\left(\frac{2i\pi}{N_b-1} j \varepsilon^{m-k}\right). \quad (\diamond\diamond) \quad (\mathcal{R}9)$$

For any  $m \in \mathbb{N}$ , the complex numbers  $\{c_{0,j,m+1}, \dots, c_{m+1,j,m+1}\}$  and the integers  $\{\ell_{0,j,+1}, \dots, \ell_{m+1,j,m+1}\}$  respectively satisfy the following recurrence relations:

$$c_{m+1,j,m+1} = \mathcal{W}\left(\frac{j}{N_b-1}\right) = c_{m,j,m} \quad (\mathcal{R}10)$$

and

$$\forall k \in \{1, \dots, m\} : \quad c_{k,j,m+1} = c_{k,j,m} \quad , \quad \ell_{k,j,m+1} = \ell_{k,j,m} \quad , \quad (\mathcal{R}11)$$

In addition, since relation (R8) is valid for any  $m \in \mathbb{N}^*$  (and since, clearly, relation (R9) implies that the coefficients  $c_{k,j,m}$  are nonzero for  $0 \leq k \leq m$ ), we deduce that the associated Complex Dimensions (i.e., in fact, the Complex Dimensions associated with the Weierstrass function) are

$$D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) + i\ell_{k,j,m} \mathbf{P}$$

where  $0 \leq j \leq \#V_m - 1$ ,  $0 \leq k \leq m$  and  $\ell_{k,j,m} \in \mathbb{Z}$  is arbitrary.

This immediately ensures, for the Weierstrass function (i.e., the real part of the Weierstrass complexified function  $\mathcal{W}_{comp}$ ), that, for any strictly positive integer  $m$  and for any  $j$  in  $\{0, \dots, \#V_m\}$ ,

$$\begin{aligned} \mathcal{W}(j \varepsilon_m^m) &= \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}_{comp}\left(\frac{j}{N_b-1}\right) + \sum_{k=0}^{m-1} \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e\left(c_{k,j,m} \varepsilon_k^{i\ell_{k,j,m} \mathbf{P}}\right) \\ &= \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}_{comp}\left(\frac{j}{N_b-1}\right) + \frac{1}{2} \sum_{k=0}^{m-1} \varepsilon^{k(2-D_{\mathcal{W}})} \left(c_{k,j,m} \varepsilon_k^{i\ell_{k,j,m} \mathbf{P}} + \overline{c_{k,j,m}} \varepsilon_k^{-i\ell_{k,j,m} \mathbf{P}}\right) \\ &= \frac{1}{2} \sum_{k=0}^m \varepsilon^{k(2-D_{\mathcal{W}})} \left(c_{k,j,m} \varepsilon_k^{i\ell_{k,j,m} \mathbf{P}} + \overline{c_{k,j,m}} \varepsilon_k^{-i\ell_{k,j,m} \mathbf{P}}\right). \end{aligned} \quad (\mathcal{R}12)$$

*Proof.* Thanks to Property 2.5, on page 8 (the Scaling Property of the Weierstrass function  $\mathcal{W}$ ), for any strictly positive integer  $m$  and any  $j$  in  $\{0, \dots, \#V_m - 1\}$ , we then have that

$$\mathcal{W}\left(\frac{j}{(N_b-1)N_b^m}\right) = \lambda^m \mathcal{W}\left(\frac{j}{N_b-1}\right) + \sum_{k=0}^{m-1} \lambda^k \cos\left(\frac{2\pi N_b^k j}{(N_b-1)N_b^m}\right).$$

By using the Minkowski Dimension  $D_{\mathcal{W}}$  introduced in Notation 5, on page 5, this latter expression can be rewritten as

$$\begin{aligned} \mathcal{W}(j \varepsilon_m^m) &= N_b^{m(D_{\mathcal{W}}-2)} \mathcal{W}\left(\frac{j}{N_b-1}\right) + \sum_{k=0}^{m-1} N_b^{k(D_{\mathcal{W}}-2)} \cos\left(\frac{2\pi}{N_b-1} j \varepsilon^{m-k}\right) \\ &= \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}\left(\frac{j}{N_b-1}\right) + \sum_{k=0}^{m-1} \varepsilon^{k(2-D_{\mathcal{W}})} \cos\left(\frac{2\pi}{N_b-1} j \varepsilon^{m-k}\right) \\ &= \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}\left(\frac{j}{N_b-1}\right) + \sum_{k=0}^{m-1} \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e\left(\exp\left(\frac{2i\pi}{N_b-1} j \varepsilon^{m-k}\right)\right). \end{aligned}$$

We must then show that there exist integers  $\{\ell_{0,j,m}, \dots, \ell_{m-1,j,m}\}$  and complex numbers  $\{c_{0,j,m}, \dots, c_{m,j,m}\}$  such that, for any integer  $k$  in  $\{0, \dots, m\}$ ,

$$\exp\left(\frac{2i\pi}{N_b-1} j \varepsilon^{m-k}\right) = c_{k,j,m} \varepsilon^{i\ell_{k,j,m} \mathbf{P}} = c_{k,j,m} \exp(i\ell_{k,j,m} \mathbf{P} \ln \varepsilon) = c_{k,j,m} \exp(-2i\pi \ell_{k,j,m}).$$

We obviously obtain that  $\ell_{k,j,m} \in \mathbb{Z}$  is arbitrary,  $c_{m,j,m} = \mathcal{W}_{comp}\left(\frac{j}{N_b-1}\right)$  and that, for  $0 \leq k \leq m-1$ ,

$$c_{k,j,m} = \exp\left(\frac{2i\pi}{N_b-1} j \varepsilon^{m-k}\right).$$

We note that, since, for any positive integer  $m$  and any  $j$  in  $\{0, \dots, \#V_m - 1\}$  (see Property 2.5, on page 8),

$$\mathcal{W}_{comp}\left(\frac{j}{(N_b-1)N_b^{m+1}}\right) = \lambda \mathcal{W}\left(\frac{j}{(N_b-1)N_b^m}\right) + \exp\left(\frac{2\pi j}{(N_b-1)N_b^{m+1}}\right),$$

this ensures that

$$\begin{aligned} \mathcal{W}_{comp}\left(\frac{j}{(N_b-1)N_b^{m+1}}\right) &= \lambda \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}\left(\frac{j}{N_b-1}\right) + \lambda \sum_{k=0}^{m-1} \varepsilon^{k(2-D_{\mathcal{W}})} \exp\left(\frac{2i\pi}{N_b-1} j \varepsilon^{m-k}\right) \\ &\quad + \exp\left(\frac{2\pi j}{(N_b-1)N_b^{m+1}}\right) \\ &= \varepsilon^{2-D_{\mathcal{W}}} \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}\left(\frac{j}{N_b-1}\right) + \varepsilon^{2-D_{\mathcal{W}}} \sum_{k=0}^{m-1} \varepsilon^{k(2-D_{\mathcal{W}})} \exp\left(\frac{2i\pi}{N_b-1} j \varepsilon^{m-k}\right) \\ &\quad + \exp\left(\frac{2\pi j}{(N_b-1)N_b^{m+1}}\right) \\ &= \varepsilon^{(m+1)(2-D_{\mathcal{W}})} \mathcal{W}\left(\frac{j}{N_b-1}\right) + \sum_{k=0}^{m-1} \varepsilon^{(k+1)(2-D_{\mathcal{W}})} \exp\left(\frac{2i\pi}{N_b-1} j \varepsilon^{m+1-k-1}\right) \\ &\quad + \exp\left(\frac{2\pi j}{(N_b-1)N_b^{m+1}}\right) \\ &= \varepsilon^{(m+1)(2-D_{\mathcal{W}})} \mathcal{W}\left(\frac{j}{N_b-1}\right) + \sum_{k=1}^m \varepsilon^{k(2-D_{\mathcal{W}})} \left(\exp\left(\frac{2i\pi}{N_b-1} j \varepsilon^{m-k}\right)\right) \\ &\quad + \exp\left(\frac{2\pi j}{(N_b-1)N_b^{m+1}}\right) \\ &= \varepsilon^{(m+1)(2-D_{\mathcal{W}})} \mathcal{W}\left(\frac{j}{N_b-1}\right) + \sum_{k=0}^m \varepsilon^{k(2-D_{\mathcal{W}})} \left(\exp\left(\frac{2i\pi}{N_b-1} j \varepsilon^{m-k}\right)\right) \\ &\quad + \exp\left(\frac{2\pi j}{(N_b-1)N_b^{m+1}}\right) \end{aligned}$$

At the same time, we have that

$$\mathcal{W}_{comp} \left( \frac{j}{(N_b - 1) N_b^{m+1}} \right) = \sum_{k=0}^{m+1} c_{k,j,m+1} \varepsilon^{(k+1)(2-D_{\mathcal{W}})} \varepsilon^{i \ell_{k+1,j,m+1} \mathbf{P}},$$

We thus deduce that the stated recurrence relations  $(\mathcal{R}10)$ , on page 35, hold:

$$c_{m+1,j,m+1} = \mathcal{W} \left( \frac{j}{N_b - 1} \right) = c_{m,j,m}$$

and that, for  $1 \leq k \leq m + 1$ ,

$$c_{k,j,m+1} = c_{k,j,m} \quad , \quad \ell_{k,j,m+1} = \ell_{k-1,j,m} .$$

□

*Remark 4.1.* We note that the results obtained in Property 4.3 will play a key role in the remainder of this paper. In particular, relations  $(\mathcal{R}10)$  and  $(\mathcal{R}11)$ , on page 35 above, will be the key to obtain the recurrence relations satisfied by the local polyhedral effective fractal zeta functions and thereby, to establish the existence of the global polyhedral effective fractal zeta function.

#### Property 4.4 (A Hodge Diamond Star Relation).

For any  $m \in \mathbb{N}^*$ , any  $k$  in  $\{1, \dots, m\}$  and any  $j$  in  $\{0, \dots, \#V_m - 1\}$ , we have the following Hodge Diamond Star relation

$$c_{k,(N_b-1)N_b^{m-1}-j,m} = \overline{c_{k,j,m}} , \tag{\mathcal{R}13}$$

which is directly connected to the symmetry with respect to the vertical line  $x = \frac{1}{2}$ , stated in Property 2.8, on page 12, since the points

$$\left( \frac{(N_b - 1) N_b^m - j}{(N_b - 1) N_b^m}, \mathcal{W} \left( \frac{(N_b - 1) N_b^m - j}{(N_b - 1) N_b^m} \right) \right) \quad \text{and} \quad \left( \frac{j}{(N_b - 1) N_b^m}, \mathcal{W} \left( \frac{j}{(N_b - 1) N_b^m} \right) \right)$$

are symmetric with respect to the vertical line  $x = \frac{1}{2}$ ; see Figure 3, on page 13. It is also reminiscent of Poincaré duality (see our previous work [DL22d]).

*Proof.* We have that, for  $1 \leq k \leq m - 1$ ,

$$\begin{aligned} c_{k,(N_b-1)N_b^{m-1}-j,m} &= \exp \left( \frac{2i\pi}{N_b - 1} \left( (N_b - 1) N_b^{m-1} - j \right) \varepsilon^{m-k} \right) \\ &= \exp \left( 2i\pi N_b^{m-1} \varepsilon^{m-k} \right) \exp \left( -\frac{2i\pi j}{N_b - 1} \varepsilon^{m-k} \right) \\ &= \underbrace{\exp \left( 2i\pi N_b^{m-1} N_b^{k-m} \right)}_1 \exp \left( -\frac{2i\pi j}{N_b - 1} \varepsilon^{m-k} \right) \\ &= \overline{c_{k,j,m}} . \end{aligned}$$

We note that this relation is directly connected to the symmetry with respect to the vertical line  $x = \frac{1}{2}$ , given in Property 2.8, on page 12, since the points

$$\left( \frac{(N_b - 1) N_b^m - j}{(N_b - 1) N_b^m}, \mathcal{W} \left( \frac{(N_b - 1) N_b^m - j}{(N_b - 1) N_b^m} \right) \right) \quad \text{and} \quad \left( \frac{j}{(N_b - 1) N_b^m}, \mathcal{W} \left( \frac{j}{(N_b - 1) N_b^m} \right) \right)$$

are symmetric with respect to the vertical line  $x = \frac{1}{2}$ . □

**Definition 4.2 (The Weierstrass Complex Curve).**

We place ourselves in the Riemann sphere (or complex projective line)  $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$ . We define the *Weierstrass Curve* as the graph in  $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$  associated with the Weierstrass complexified function  $\mathcal{W}_{comp}$ ; i.e., the set denoted by  $\Gamma_{\mathcal{W},comp}$  and such that

$$\Gamma_{\mathcal{W},comp} = \{(x, \mathcal{W}_{comp}(x)) , x \in [0, 1]\} = \Gamma_{\mathcal{W}} + i \Gamma_{\mathcal{W},\mathcal{I}} ,$$

where  $\Gamma_{\mathcal{W},\mathcal{I}}$  is the graph (in  $\mathbb{R}^2$ ) associated with the imaginary part  $\mathcal{W}_{\mathcal{I}}$  of the Weierstrass complexified function  $\mathcal{W}_{comp}$  and defined, for any real number  $x$ , by

$$\mathcal{W}_{\mathcal{I}}(x) = \sum_{n=0}^{\infty} \lambda^n \sin(2 \pi N_b^n x) . \tag{R 14}$$

**Theorem 4.5 (Exact Expression for the Volume of the  $m^{th}$  Polyhedral Neighborhood (or  $m^{th}$  Natural Polyhedral Volume)).**

Given  $m \in \mathbb{N}^*$  sufficiently large, the volume (or two-dimensional Lebesgue measure)  $\mathcal{V}_m(\varepsilon_m^m)$  of the  $m^{th}$  polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$ , or  $m^{th}$  natural polyhedral volume, is given by

$$\begin{aligned} \mathcal{V}_m(\varepsilon_m^m) &= \mu_{\mathcal{L}}(\mathcal{D}(\Gamma_{\mathcal{W}_m})) \\ &= \varepsilon^m \sum_{j=0}^{N_b^m - 1} \sum_{q=0}^{N_b} \alpha_q(N_b) \mathcal{W} \left( \frac{(N_b - 1) j + q}{(N_b - 1) N_b^m} \right) \\ &= \varepsilon^m \sum_{j=0}^{N_b^m - 1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e \left( c_{m,k,(N_b-1)j+q} \varepsilon^{i k \ell_{k,(N_b-1)j+q,m} \mathbf{P}} \right) , \end{aligned} \tag{R 15}$$

where  $\varepsilon^m = (N_b - 1) \varepsilon_m^m$  is the  $m^{th}$  intrinsic cohomology infinitesimal introduced in Definition 3.1, on page 20, while the real numbers  $\alpha_q(N_b)$ , with  $0 \leq q \leq N_b$ , are coefficients which depend on  $N_b$  and where the complex numbers  $c_{m,k,(N_b-1)j+q} \in \mathbb{C}$  have been introduced in part ii. of Property 4.3, on page 34, while the integers  $\ell_{m,k,(N_b-1)j+q} \in \mathbb{Z}$  are arbitrary (see again Property 4.3, on page 34), with

a. When the integer  $N_b$  is odd:

$$\alpha_0(N_b) = \alpha_{N_b-1}(N_b) = -\alpha_{\frac{N_b-1}{2}}(N_b) = -\alpha_{\frac{N_b-1}{2}+N_b-1}(N_b) = \frac{N_b - 2}{2(N_b - 1)}$$

and for  $1 \leq q \leq N_b - 2$ ,

$$\alpha_{\frac{N_b-1}{2}+q}(N_b) = -\alpha_q(N_b) = -\alpha_{N_b^m, (N_b-1)N_b^m - (N_b-1)+q}(N_b) = \frac{1}{N_b - 1},$$

along with

$$\alpha_{N_b^m, (N_b-1)N_b^m - (N_b-1)}(N_b) = \alpha_{N_b^m, 1}(N_b) = \frac{N_b - 2}{2(N_b - 1)}.$$

b. When the integer  $N_b$  is even:

$$\alpha_0(N_b) = \alpha_{N_b-1}(N_b) = -\alpha_{\frac{N_b}{2}}(N_b) = -\alpha_{\frac{N_b}{2}+N_b-1}(N_b) = \frac{N_b - 2}{2(N_b - 1)}$$

and for  $1 \leq q \leq N_b - 2$ ,

$$\alpha_{\frac{N_b}{2}+q}(N_b) = -\alpha_q(N_b) = -\alpha_{N_b^m, (N_b-1)N_b^m - (N_b-1)+q}(N_b) = -\frac{1}{N_b - 1},$$

along with

$$\alpha_{N_b^m, (N_b-1)N_b^m - (N_b-1)}(N_b) = \alpha_{N_b^m, 1}(N_b) = \frac{N_b - 2}{2(N_b - 1)}.$$

Given the form of the expression in relation (R15) just above, it is natural to introduce, for any  $m \in \mathbb{N}^*$  sufficiently large, the associated  $m^{\text{th}}$  complex natural polyhedral volume  $\mathcal{V}_{m, \text{comp}}(\varepsilon_m^m)$ , such that

$$\mathcal{V}_{m, \text{comp}}(\varepsilon_m^m) = \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m c_{m, k, (N_b-1)j+q} \varepsilon^{k(2-D_{\mathcal{W}})+ik \ell_{k, (N_b-1)j+q, m}} \mathbf{P}. \quad (\mathcal{R} 16)$$

By considering, in the same manner, given  $m \in \mathbb{N}^*$  sufficiently large, the volume (or two-dimensional Lebesgue measure)  $\mathcal{V}_{\mathcal{I}, m}(\varepsilon_m^m)$  of the  $m^{\text{th}}$  polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}, \mathcal{I}, m})$ , associated with the  $m^{\text{th}}$  prefractional approximation to the graph  $\Gamma_{\mathcal{W}, \mathcal{I}}$  of the imaginary part  $\mathcal{W}_{\mathcal{I}}$  of the Weierstrass complexified function  $\mathcal{W}_{\text{comp}}$  (see Definition 4.2, on page 38 above), we note that the complex volume  $\mathcal{V}_{m, \text{comp}}(\varepsilon_m^m)$  can be envisioned as the volume associated with the Weierstrass complex curve  $\Gamma_{\mathcal{W}, \text{comp}}$  (see again Definition 4.2, on page 38) in the Riemann sphere (or complex projective line)  $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$ .

Observe that much as for its Euclidean counterpart obtained in our earlier work, [DL22c], the  $m^{\text{th}}$  polyhedral fractal formula obtained in relation (R15) of Theorem 4.5, on page 38, is expressed at the  $m^{\text{th}}$  cohomology infinitesimal  $\varepsilon_m^m$ , for any  $m \in \mathbb{N}^*$  sufficiently large (instead of being expressed at any  $\varepsilon > 0$ ).



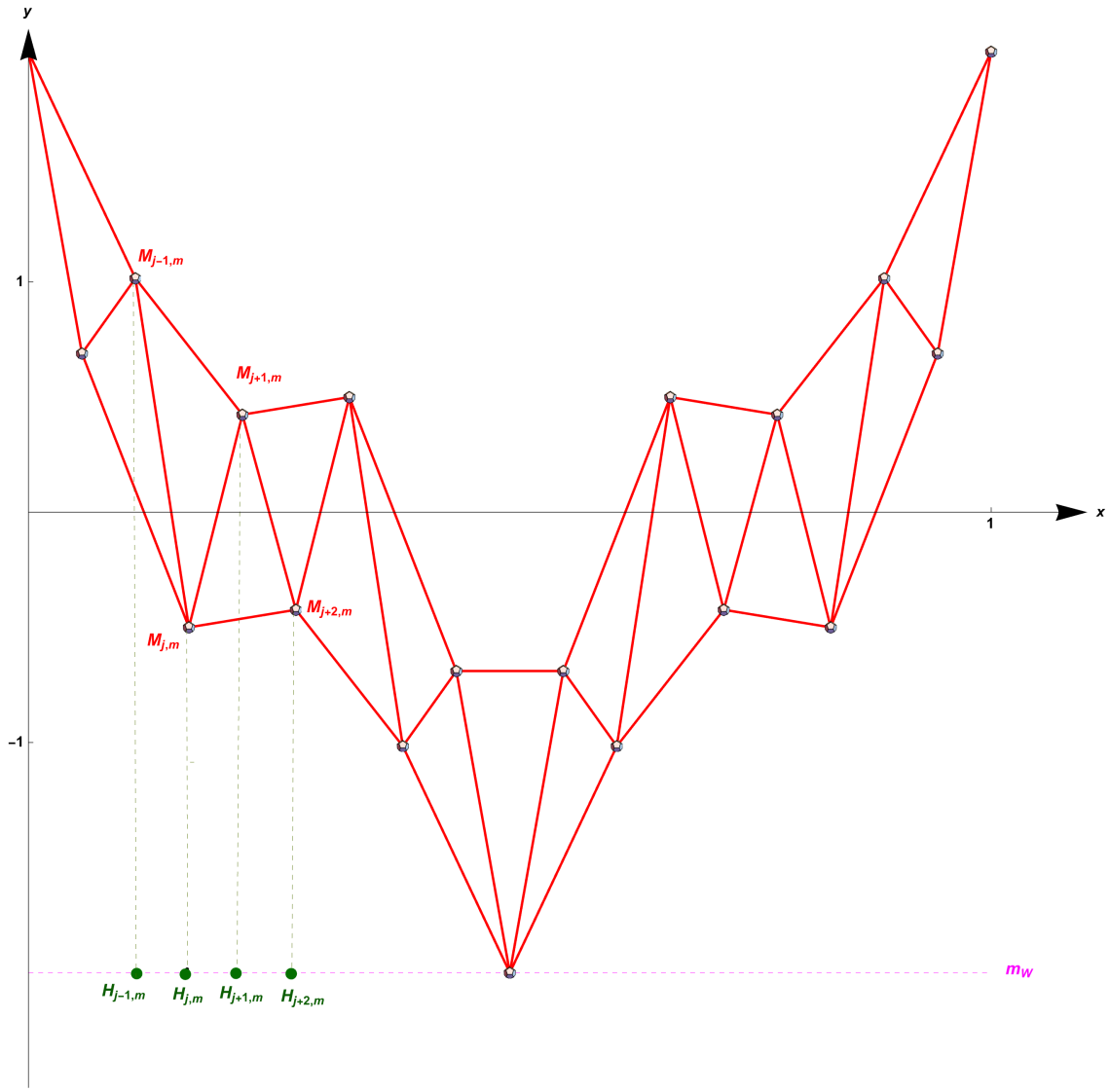


Figure 12: The orthogonal projections  $H_{j,m}$  of the vertices  $M_{j,m}$ , for  $0 \leq j \leq \#V_m - 1$ , onto the horizontal line  $y = m_{\mathcal{W}}$ .

*Proof.*

*i.* The special case  $N_b = 3$

First, note that since  $N_b = 3$ , the polygons involved are triangles.

Given  $m \in \mathbb{N}^*$  sufficiently large, and  $0 \leq j \leq \#V_m$ , we introduce the (vertical) orthogonal projection  $H_{j,m}$  of the vertex  $M_{j,m}$  onto the horizontal line  $y = m_{\mathcal{W}}$ , where  $m_{\mathcal{W}}$  is introduced in Notation 12, on page 24; see Figure 12, on page 40.

We note that, for  $0 \leq j \leq N_b^m - 1$ , with the notation of Property 2.6, on page 11, the two-dimensional Lebesgue measure (i.e., area) of the polygon  $\mathcal{P}_{m,j+1}$ , with (consecutive) vertices

$$M_{(N_b-1)j,m} \quad , \quad M_{(N_b-1)j+1,m} \quad \text{and} \quad M_{(N_b-1)j+2,m} \quad ,$$

is obtained by subtracting from the area of the trapezoid

$$H_{(N_b-1)j,m} M_{(N_b-1)j,m} M_{(N_b-1)j+2,m} H_{(N_b-1)j+2,m}$$

the area of the trapezoid

$$H_{(N_b-1)j,m} M_{(N_b-1)j,m} M_{(N_b-1)j+1,m} H_{(N_b-1)j+1,m}$$

and the area of the trapezoid

$$H_{(N_b-1)j+1,m} M_{(N_b-1)j+1,m} M_{(N_b-1)j+2,m} H_{(N_b-1)j+2,m} ;$$

i.e.,

$$\begin{aligned} \mu_{\mathcal{L}}(\mathcal{P}_{m,j}) &= \frac{2\varepsilon_m^m}{2} \left( m_{\mathcal{W}} + \mathcal{W} \left( \frac{(N_b-1)j}{(N_b-1)N_b^m} \right) + m_{\mathcal{W}} + \mathcal{W} \left( \frac{(N_b-1)j+2}{(N_b-1)N_b^m} \right) \right) \\ &\quad - \frac{\varepsilon_m^m}{2} \left( m_{\mathcal{W}} + \mathcal{W} \left( \frac{(N_b-1)j}{(N_b-1)N_b^m} \right) + m_{\mathcal{W}} + \mathcal{W} \left( \frac{(N_b-1)j+1}{(N_b-1)N_b^m} \right) \right) \\ &\quad - \frac{\varepsilon_m^m}{2} \left( m_{\mathcal{W}} + \mathcal{W} \left( \frac{(N_b-1)j+1}{(N_b-1)N_b^m} \right) + m_{\mathcal{W}} + \mathcal{W} \left( \frac{(N_b-1)j+2}{(N_b-1)N_b^m} \right) \right) \\ &= \frac{\varepsilon_m^m}{2} \left( \mathcal{W} \left( \frac{(N_b-1)j}{(N_b-1)N_b^m} \right) + \mathcal{W} \left( \frac{(N_b-1)j+2}{(N_b-1)N_b^m} \right) - 2 \mathcal{W} \left( \frac{(N_b-1)j+1}{(N_b-1)N_b^m} \right) \right) \end{aligned}$$

Similarly, for  $0 \leq j \leq N_b^m - 1$ , still with the notation of Definition 2.6, on page 11, the area of the polygon (triangle)  $\mathcal{Q}_{m,j+1}$ , with (consecutive) vertices

$$M_{(N_b-1)j+1,m} \quad , \quad M_{(N_b-1)j+2,m} \quad \text{and} \quad M_{(N_b-1)j+3,m} ,$$

is obtained by adding the area of the trapezoid

$$H_{(N_b-1)j+1,m} M_{(N_b-1)j+1,m} M_{(N_b-1)j+2,m} H_{(N_b-1)j+2,m}$$

to the area of the trapezoid

$$H_{(N_b-1)j+2,m} M_{(N_b-1)j+2,m} M_{(N_b-1)j+3,m} H_{(N_b-1)j+3,m}$$

and then, by subtracting from this result the area of the trapezoid

$$H_{(N_b-1)j+1,m} M_{(N_b-1)j+1,m} M_{(N_b-1)j+3,m} H_{(N_b-1)j+3,m} ;$$

i.e.,

$$\begin{aligned}
\mu_{\mathcal{L}}(\mathcal{Q}_{m,j+1}) &= \frac{\varepsilon_m^m}{2} \left( m_{\mathcal{W}} + \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + m_{\mathcal{W}} + \mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) + m_{\mathcal{W}} + \mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) + m_{\mathcal{W}} + \mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) \right) \\
&\quad - \frac{2\varepsilon_m^m}{2} \left( m_{\mathcal{W}} + \mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) + m_{\mathcal{W}} + \mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) \right) \\
&= \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) - \mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) - 2\mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) \right).
\end{aligned}$$

We thus have that

$$\begin{aligned}
\mathcal{V}_m(\varepsilon_m^m) &= \sum_{j=0}^{N_b^m-2} (\mu_{\mathcal{L}}(\mathcal{P}_{m,j+1}) + \mu_{\mathcal{L}}(\mathcal{Q}_{m,j+1})) + \mu_{\mathcal{L}}(\mathcal{P}_{m,N_b^m}) \\
&= \sum_{j=0}^{N_b^m-2} \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) - 2\mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) \right) \\
&\quad + \sum_{j=0}^{N_b^m-2} \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) - \mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) - 2\mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) \right) \\
&\quad \quad \quad + 2\frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)N_b^m-2}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{((N_b-1)(N_b^m))}{(N_b-1)N_b^m}\right) \right) \\
&\quad \quad \quad - \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)N_b^m-2}{(N_b-1)N_b^m}\right) + 2\mathcal{W}\left(\frac{(N_b-1)N_b^m-1}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)N_b^m}{(N_b-1)N_b^m}\right) \right) \\
&= \sum_{j=0}^{N_b^m-2} \frac{\varepsilon_m^m}{2} \left( 2\mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + 2\mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) \right) \\
&\quad - \sum_{j=0}^{N_b^m-2} \frac{\varepsilon_m^m}{2} \left( 3\mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) + 2\mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) \right) \\
&\quad \quad \quad + \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)N_b^m-2}{(N_b-1)N_b^m}\right) - 2\mathcal{W}\left(\frac{(N_b-1)N_b^m-1}{(N_b-1)N_b^m}\right) + 3\mathcal{W}(1) \right). \\
&= \frac{1}{N_b-1} \sum_{j=0}^{N_b^m-2} \frac{\varepsilon_m^m}{2} \left( 2\mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + 2\mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) \right) \\
&\quad - \frac{1}{N_b-1} \sum_{j=0}^{N_b^m-2} \frac{\varepsilon_m^m}{2} \left( 3\mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) + 2\mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) \right) \\
&\quad \quad \quad + \frac{1}{N_b-1} \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)N_b^m-2}{(N_b-1)N_b^m}\right) - 2\mathcal{W}\left(\frac{(N_b-1)N_b^m-1}{(N_b-1)N_b^m}\right) + 3\mathcal{W}(1) \right),
\end{aligned}$$

since

$$\varepsilon_m^m = \frac{1}{N_b - 1} \varepsilon^m.$$

For the sake of concision, we will write it in the following form

$$\mathcal{V}_m(\varepsilon_m^m) = \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \mathcal{W} \left( \frac{(N_b - 1)j + q}{(N_b - 1)N_b^m} \right),$$

where, for  $0 \leq j \leq N_b^m - 2$ ,

$$\alpha_0(N_b) = \frac{1}{N_b - 1} \quad , \quad \alpha_1(N_b) = \frac{1}{N_b - 1} \quad , \quad \alpha_2(N_b) = -\frac{3}{2(N_b - 1)} \quad , \quad \alpha_3(N_b) = -\frac{1}{N_b - 1}$$

and where

$$\alpha_{N_b^m-1,0}(N_b) = \frac{1}{2(N_b - 1)} \quad , \quad \alpha_{N_b^m-1,1}(N_b) = -\frac{1}{N_b - 1} \quad , \quad \alpha_{N_b^m-1,2}(N_b) = \frac{3}{2(N_b - 1)} \text{quad}, \quad \alpha_{N_b^m-1,4}(N_b) = 0$$

Hence, by using the respective expressions for the  $m^{\text{th}}$  cohomology infinitesimal  $\varepsilon_m^m$  and for the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m = (N_b - 1)\varepsilon_m^m$  given in Definition 3.1, on page 20, we have that

$$\begin{aligned} \mathcal{V}_m(\varepsilon_m^m) &= \mu_{\mathcal{L}}(\mathcal{D}(\Gamma_{\mathcal{W}_m})) \\ &= \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e \left( c_{m,k,(N_b-1)j+q} \varepsilon^{ik \ell_{k,(N_b-1)j+q,m} \mathbf{P}} \right), \end{aligned}$$

where the complex coefficients  $c_{k,(N_b-1)j+q,m}$  are given in part *ii.* of Property 4.3, on page 34, with  $\ell_{m,k,(N_b-1)j+q} \in \mathbb{Z}$  arbitrary (see part *i.* of Property 4.3, on page 34).

#### ii. The general case $N_b \geq 4$

As for the general case, when  $N_b \geq 4$ , it can be obtained in a similar manner.

$\rightsquigarrow$  For instance, when  $N_b = 4$ , still with the notation of Definition 2.6, on page 11, the polygons  $\mathcal{P}_{m,j}$  involved are quadrilaterals, while the polygons  $\mathcal{Q}_{m,j}$  are triangles (see Figure 13 just above). We note that, for  $0 \leq j \leq N_b^m - 1$ , the two-dimensional Lebesgue measure (i.e., area) of the quadrilateral  $\mathcal{P}_{m,j+1}$ , with (consecutive) vertices

$$M_{(N_b-1)j,m} \quad , \quad M_{(N_b-1)j+1,m} \quad , \quad M_{(N_b-1)j+2,m} \quad \text{and} \quad M_{(N_b-1)j+3,m} \quad ,$$

is obtained by subtracting from the area of the trapezoid

$$H_{(N_b-1)j,m} M_{(N_b-1)j,m} M_{(N_b-1)j+3,m} H_{(N_b-1)j+3,m} \quad ,$$

the respective areas of the following three trapezoids

$$H_{(N_b-1)j,m} M_{(N_b-1)j,m} M_{(N_b-1)j+1,m} H_{(N_b-1)j+1,m} \quad ,$$

$$H_{(N_b-1)j+1,m} M_{(N_b-1)j+1,m} M_{(N_b-1)j+2,m} H_{(N_b-1)j+2,m}$$

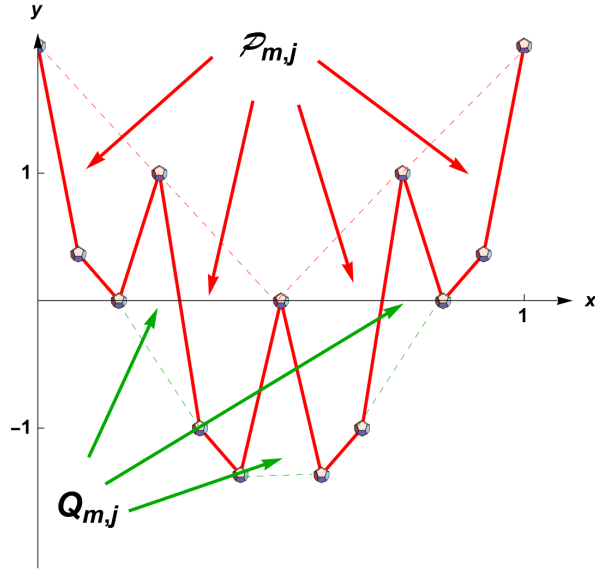


Figure 13: The polygons  $\mathcal{P}_{m,j}$  and  $\mathcal{P}_{m,j}$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 4$ .

and

$$H_{(N_b-1)j+2,m} M_{(N_b-1)j+2,m} M_{(N_b-1)j+3,m} H_{(N_b-1)j+3,m};$$

i.e.,

$$\begin{aligned} \mu_{\mathcal{L}}(\mathcal{P}_{m,j+1}) &= \frac{3\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) \right) \\ &\quad - \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) \right) \\ &\quad - \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) \right) \\ &\quad - \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) \right) \\ &= \frac{\varepsilon_m^m}{2} \left( 2\mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + 2\mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) \right) \\ &\quad - \frac{\varepsilon_m^m}{2} \left( 2\mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) + 2\mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) \right). \end{aligned}$$

For  $0 \leq j \leq N_b^m - 2$ , the area of the triangles  $\mathcal{Q}_{m,j+1}$  are obtained in the same manner as in the case when  $N_b = 3$ ; i.e., in the context of  $N_b = 4$ ; i.e.,

$$\begin{aligned}
\mu_{\mathcal{L}}(\mathcal{Q}_{m,j+1}) &= \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+4}{(N_b-1)N_b^m}\right) \right) \\
&\quad - \frac{2\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+4}{(N_b-1)N_b^m}\right) \right) \\
&= \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) - \mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) - \mathcal{W}\left(\frac{(N_b-1)j+4}{(N_b-1)N_b^m}\right) \right).
\end{aligned}$$

We thus have that

$$\begin{aligned}
\mathcal{V}_m(\varepsilon_m^m) &= \sum_{j=0}^{N_b^m-2} (\mu_{\mathcal{L}}(\mathcal{P}_{m,j+1}) + \mu_{\mathcal{L}}(\mathcal{Q}_{m,j+1})) + \mu_{\mathcal{L}}(\mathcal{P}_{m,N_b^m}) \\
&= \sum_{j=0}^{N_b^m-2} \frac{\varepsilon_m^m}{2} \left( 2\mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) - 2\mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) - 2\mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) + 2\mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) \right) \\
&\quad + \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) - \mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) - \mathcal{W}\left(\frac{(N_b-1)j+4}{(N_b-1)N_b^m}\right) \right) \\
&\quad + \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)(N_b^m-1)}{(N_b-1)N_b^m}\right) - 2\mathcal{W}\left(\frac{(N_b-1)(N_b^m-1)+1}{(N_b-1)N_b^m}\right) \right. \\
&\quad \left. - 2\mathcal{W}\left(\frac{(N_b-1)(N_b^m-1)+2}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)(N_b^m-1)+3}{(N_b-1)N_b^m}\right) \right) \\
&= \sum_{j=0}^{N_b^m-2} \frac{\varepsilon_m^m}{2} \left( 2\mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) - \mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) - 3\mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) \right) \\
&\quad + \sum_{j=0}^{N_b^m-2} \frac{\varepsilon_m^m}{2} \left( 3\mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) - \mathcal{W}\left(\frac{(N_b-1)j+4}{(N_b-1)N_b^m}\right) \right) \\
&\quad + \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)(N_b^m-1)}{(N_b-1)N_b^m}\right) - 2\mathcal{W}\left(\frac{(N_b-1)(N_b^m-1)+1}{(N_b-1)N_b^m}\right) \right. \\
&\quad \left. - 2\mathcal{W}\left(\frac{(N_b-1)(N_b^m-1)+2}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)(N_b^m-1)+3}{(N_b-1)N_b^m}\right) \right).
\end{aligned}$$

As previously, taking into account the fact that

$$\varepsilon_m^m = \frac{1}{N_b-1} \varepsilon^m,$$

we write  $\mathcal{V}_m(\varepsilon_m^m)$  in the following form,

$$\mathcal{V}_m(\varepsilon_m^m) = \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \mathcal{W}\left(\frac{(N_b-1)j+q}{(N_b-1)N_b^m}\right),$$

where, for  $0 \leq j \leq N_b^m - 2$ ,

$$\alpha_0(N_b) = \frac{1}{N_b-1}, \quad \alpha_1(N_b) = -\frac{1}{2(N_b-1)}, \quad \alpha_2(N_b) = -\frac{3}{2(N_b-1)},$$

$$\alpha_3(N_b) = \frac{3}{2(N_b - 1)} \quad , \quad \alpha_4(N_b) = -\frac{1}{2(N_b - 1)}$$

and

$$\alpha_{N_b^m-1,0}(N_b) = \frac{1}{2(N_b - 1)} \quad , \quad \alpha_{N_b^m-1,1}(N_b) = -\frac{1}{N_b - 1} \quad , \quad \alpha_{N_b^m-1,2}(N_b) = -\frac{1}{N_b - 1} \quad ,$$

$$\alpha_{N_b^m-1,3}(N_b) = \frac{1}{2(N_b - 1)} \quad , \quad \alpha_{N_b^m-1,4}(N_b) = 0 \quad .$$

Hence, by using the respective expressions of the  $m^{\text{th}}$  cohomology infinitesimal  $\varepsilon_m^m$  and of the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m = (N_b - 1)\varepsilon_m^m$  given in Definition 3.1, on page 20, we have that

$$\begin{aligned} \mathcal{V}_m(\varepsilon_m^m) &= \mu_{\mathcal{L}}(\mathcal{D}(\Gamma_{\mathcal{W}_m})) \\ &= \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e \left( c_{m,k,(N_b-1)j+q} \varepsilon^{ik \ell_{k,(N_b-1)j+q,m} \mathbf{P}} \right) , \end{aligned}$$

where the complex coefficients  $c_{k,(N_b-1)j+q,m}$  are given in part *ii.* of Property 4.3, on page 34, with  $\ell_{m,k,(N_b-1)j+q} \in \mathbb{Z}$  arbitrary (see part *i.* of Property 4.3, on page 34).

$\rightsquigarrow$  When  $N_b = 5$ , the polygons involved are pentagons and quadrilaterals. We note that, for  $0 \leq j \leq N_b^m - 1$ , the two-dimensional Lebesgue measure (i.e., area) of a pentagon  $\mathcal{P}_{m,j+1}$ , (consecutive) vertices

$$M_{(N_b-1)j,m} \quad , \quad M_{(N_b-1)j+1,m} \quad , \quad M_{(N_b-1)j+2,m} \quad , \quad M_{(N_b-1)j+3,m} \quad \text{and} \quad M_{(N_b-1)j+4,m} \quad ,$$

is obtained by subtracting from the area of the trapezoid

$$H_{(N_b-1)j,m} M_{(N_b-1)j,m} M_{(N_b-1)j+3,m} H_{(N_b-1)j+4,m}$$

the respective areas of the following four trapezoids

$$H_{(N_b-1)j,m} M_{(N_b-1)j,m} M_{(N_b-1)j+1,m} H_{(N_b-1)j+1,m} \quad , \quad H_{(N_b-1)j+1,m} M_{(N_b-1)j+1,m} M_{(N_b-1)j+2,m} H_{(N_b-1)j+2,m} \quad ,$$

$$H_{(N_b-1)j+2,m} M_{(N_b-1)j+2,m} M_{(N_b-1)j+3,m} H_{(N_b-1)j+3,m} \quad , \quad H_{(N_b-1)j+3,m} M_{(N_b-1)j+3,m} M_{(N_b-1)j+4,m} H_{(N_b-1)j+4,m} \quad ;$$

i.e.,

$$\begin{aligned}
\mu_{\mathcal{L}}(\mathcal{P}_{m,j+1}) &= \frac{4\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) \right) \\
&\quad - \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) \right) \\
&\quad - \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) \right) \\
&\quad - \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) \right) \\
&\quad - \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+4}{(N_b-1)N_b^m}\right) \right) \\
&= \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(3\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) - 2\mathcal{W}\left(\frac{(N_b-1)j+1}{(N_b-1)N_b^m}\right) - 2\mathcal{W}\left(\frac{(N_b-1)j+2}{(N_b-1)N_b^m}\right) \right) \\
&= \frac{\varepsilon_m^m}{2} \left( 3\mathcal{W}\left(\frac{(N_b-1)j+3}{(N_b-1)N_b^m}\right) - 2\mathcal{W}\left(\frac{(N_b-1)j+4}{(N_b-1)N_b^m}\right) \right).
\end{aligned}$$

As previously, for  $0 \leq j \leq N_b^m - 2$ , the area of the quadrilaterals are obtained similarly as in the case when  $N_b = 4$ .

$\rightsquigarrow$  More generally, given  $N_b \geq 6$ , the polygons  $\mathcal{P}_{m,j}$  involved are  $N_b$ -gons, while the polygons  $\mathcal{Q}_{m,j}$  are  $N_{b'}$ -gons, with  $N_{b'} \leq N_b$ . The cases  $N_b$  odd and  $N_b$  even have to be studied separately.

a. When  $N_b$  is an odd number:

For  $0 \leq j \leq N_b^m - 1$ , the two-dimensional Lebesgue measure (i.e., area) of the  $N_b$ -gon  $\mathcal{P}_{m,j+1}$ , with (consecutive) vertices

$$M_{(N_b-1)j,m}, \dots, M_{(N_b-1)j+N_b-1,m}$$

is obtained by subtracting from the area of the trapezoid

$$H_{(N_b-1)j,m}M_{(N_b-1)j,m}M_{(N_b-1)j+N_b-1,m}H_{(N_b-1)j+N_b-1,m}$$

the respective areas of the following  $N_b - 1$  trapezoids

$$H_{(N_b-1)j,m}M_{(N_b-1)j,m}M_{(N_b-1)j+1,m}H_{(N_b-1)j+1,m}$$

$\vdots$

$$H_{(N_b-1)j+N_b-2,m}M_{(N_b-1)j+N_b-2,m}M_{(N_b-1)j+N_b-1,m}H_{(N_b-1)j+N_b-1,m};$$

i.e.,



$$\begin{aligned}
\mu_{\mathcal{L}}(\mathcal{P}_{m,j+1}) &= \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+N_b-1}{(N_b-1)N_b^m}\right) \right) \\
&\quad - \frac{\varepsilon_m^m}{2} \sum_{q=0}^{N_b-2} \left( \mathcal{W}\left(\frac{(N_b-1)j+q}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+q+1}{(N_b-1)N_b^m}\right) \right) \\
&= \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+N_b-1}{(N_b-1)N_b^m}\right) \right) \\
&\quad - \frac{\varepsilon_m^m}{2} \sum_{q=1}^{N_b-2} 2 \mathcal{W}\left(\frac{(N_b-1)j+q}{(N_b-1)N_b^m}\right) - \frac{\varepsilon_m^m}{2} \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right).
\end{aligned}$$

We note that, for  $0 \leq j \leq N_b^m - 1$ , the two-dimensional Lebesgue measure (i.e., area) of the non necessarily convex  $N_b$ -gon  $\mathcal{Q}_{m,j+1}$ , with (consecutive) vertices

$$M_{(N_b-1)j+\frac{N_b-1}{2},m}, \dots, M_{(N_b-1)j+\frac{N_b-1}{2}+N_b-1,m}$$

is obtained by subtracting the area of the trapezoid

$$H_{(N_b-1)j+\frac{N_b-1}{2},m} M_{(N_b-1)j+\frac{N_b-1}{2},m} M_{(N_b-1)j+\frac{N_b-1}{2}+N_b-1,m} H_{(N_b-1)j+\frac{N_b-1}{2}+N_b-1,m},$$

from the sum of the respective areas of the trapezoids

$$H_{(N_b-1)j+\frac{N_b-1}{2},m} M_{(N_b-1)j+\frac{N_b-1}{2},m} M_{(N_b-1)j+\frac{N_b-1}{2}+1,m} H_{(N_b-1)j+\frac{N_b-1}{2}+1,m}$$

⋮

$$H_{(N_b-1)j+\frac{N_b-1}{2}+N_b-2,m} M_{(N_b-1)j+\frac{N_b-1}{2}+N_b-2,m} M_{(N_b-1)j+\frac{N_b-1}{2}+N_b-1,m} H_{(N_b-1)j+\frac{N_b-1}{2}+N_b-1,m};$$

i.e.,

$$\begin{aligned}
\mu_{\mathcal{L}}(\mathcal{Q}_{m,j+1}) &= \frac{\varepsilon_m^m}{2} \sum_{q=0}^{N_b-2} \left( \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+q}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+q+1}{(N_b-1)N_b^m}\right) \right) \\
&\quad - \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+N_b-1}{(N_b-1)N_b^m}\right) \right) \\
&= \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+1}{(N_b-1)N_b^m}\right) + \frac{\varepsilon_m^m}{2} \sum_{q=1}^{N_b-1} 2 \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+q}{(N_b-1)N_b^m}\right) \\
&\quad - \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+N_b-1}{(N_b-1)N_b^m}\right) \right).
\end{aligned}$$

We thus have that

$$\begin{aligned}
\mathcal{V}_m(\varepsilon_m^m) &= \sum_{j=0}^{N_b^m-2} (\mu_{\mathcal{L}}(\mathcal{P}_{m,j+1}) + \mu_{\mathcal{L}}(\mathcal{Q}_{m,j+1})) + \mu_{\mathcal{L}}(\mathcal{P}_{m,N_b^m}) \\
&= \sum_{j=0}^{N_b^m-2} \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+N_b-1}{(N_b-1)N_b^m}\right) \right) \\
&\quad - \sum_{j=0}^{N_b^m-2} \frac{\varepsilon_m^m}{2} \sum_{q=1}^{N_b-2} 2\mathcal{W}\left(\frac{(N_b-1)j+q}{(N_b-1)N_b^m}\right) - \frac{\varepsilon_m^m}{2} \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) \\
&\quad + \sum_{j=0}^{N_b^m-2} \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+1}{(N_b-1)N_b^m}\right) + \frac{\varepsilon_m^m}{2} \sum_{q=1}^{N_b-1} 2\mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+q}{(N_b-1)N_b^m}\right) \\
&\quad - \sum_{j=0}^{N_b^m-2} \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+N_b-1}{(N_b-1)N_b^m}\right) \right) \\
&\quad + \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)N_b^m-(N_b-1)}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)N_b^m}{(N_b-1)N_b^m}\right) \right) \\
&\quad - \frac{\varepsilon_m^m}{2} \sum_{q=0}^{N_b-2} \left( \mathcal{W}\left(\frac{(N_b-1)N_b^m-(N_b-1)+q}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)N_b^m-(N_b-1)+q+1}{(N_b-1)N_b^m}\right) \right).
\end{aligned}$$

As previously, taking into account the fact that

$$\varepsilon_m^m = \frac{1}{N_b-1} \varepsilon^m,$$

we write  $\mathcal{V}_m(\varepsilon_m^m)$  in the following form

$$\mathcal{V}_m(\varepsilon_m^m) = \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \mathcal{W}\left(\frac{(N_b-1)j+q}{(N_b-1)N_b^m}\right),$$

where

$$\alpha_0(N_b) = \frac{N_b-1}{2(N_b-1)} - \frac{1}{2(N_b-1)} = \frac{N_b-2}{2(N_b-1)}$$

and for  $1 \leq q \leq N_b-2$ ,

$$\alpha_{\frac{N_b-1}{2}+q}(N_b) = \frac{2}{2(N_b-1)} = \frac{1}{N_b-1}, \quad \alpha_q(N_b) = -\frac{2}{2(N_b-1)} = -\frac{1}{N_b-1},$$

$$\alpha_{N_b^m, (N_b-1)N_b^m-(N_b-1)+q}(N_b) = -\frac{2}{2(N_b-1)} = -\frac{1}{N_b-1},$$

along with

$$\alpha_{\frac{N_b-1}{2}}(N_b) = \frac{1}{2(N_b-1)} - \frac{(N_b-1)}{2(N_b-1)} = -\frac{(N_b-2)}{2(N_b-1)},$$

$$\begin{aligned}\alpha_{\frac{N_b-1}{2}+N_b-1}(N_b) &= \frac{1}{2(N_b-1)} - \frac{(N_b-1)}{2(N_b-1)} = -\frac{(N_b-2)}{2(N_b-1)}, \\ \alpha_{N_b-1}(N_b) &= -\frac{1}{2(N_b-1)} + \frac{N_b-1}{2(N_b-1)} = \frac{N_b-2}{2(N_b-1)} = \alpha_0(N_b), \\ \alpha_{N_b^m, (N_b-1)N_b^m - (N_b-1)}(N_b) &= \alpha_{N_b^m, 1}(N_b) = \frac{N_b-1}{2(N_b-1)} = \frac{1}{N_b-1}.\end{aligned}$$

Hence, by using the respective expressions of the  $m^{\text{th}}$  cohomology infinitesimal  $\varepsilon_m^m$  and of the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m = (N_b-1)\varepsilon_m^m$  given in Definition 3.1, on page 20, we have that

$$\begin{aligned}\mathcal{V}_m(\varepsilon_m^m) &= \mu_{\mathcal{L}}(\mathcal{D}(\Gamma_{\mathcal{W}_m})) \\ &= \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e \left( c_{m,k,(N_b-1)j+q} \varepsilon^{ik \ell_{k,(N_b-1)j+q,m} \mathbf{P}} \right),\end{aligned}$$

where the complex coefficients  $c_{k,(N_b-1)j+q,m}$  are given in part *ii.* of Property 4.3, on page 34, with  $\ell_{m,k,(N_b-1)j+q} \in \mathbb{Z}$  arbitrary (see part *i.* of Property 4.3, on page 34).

*b.* When  $N_b$  is an even number:

For  $0 \leq j \leq N_b^m - 1$ , the two-dimensional Lebesgue measure (i.e., area) of the  $N_b$ -gon  $\mathcal{P}_{m,j+1}$ , with (consecutive) vertices

$$M_{(N_b-1)j,m}, \dots, M_{(N_b-1)j+N_b-1,m}$$

is obtained exactly as in part *a.* It is the computation, for  $0 \leq j \leq N_b^m - 1$ , of the two-dimensional Lebesgue measure (i.e., area) of the nonconvex  $N_b$ -gon  $\mathcal{Q}_{m,j+1}$  which slightly changes. More precisely, for  $0 \leq j \leq N_b^m - 1$ , the area of the non necessarily convex  $N_b$ -gon  $\mathcal{Q}_{m,j+1}$  with (consecutive) vertices

$$M_{(N_b-1)j+\frac{N_b}{2},m}, \dots, M_{(N_b-1)j+\frac{N_b}{2}+N_b-1,m}$$

is obtained by subtracting the area of the trapezoid

$$H_{(N_b-1)j+\frac{N_b}{2},m} M_{(N_b-1)j+\frac{N_b}{2},m} M_{(N_b-1)j+\frac{N_b}{2}+N_b-1,m} H_{(N_b-1)j+\frac{N_b}{2}+N_b-1,m},$$

from the sum of the respective areas of the trapezoids

$$H_{(N_b-1)j+\frac{N_b}{2},m} M_{(N_b-1)j+\frac{N_b}{2},m} M_{(N_b-1)j+\frac{N_b}{2}+1,m} H_{(N_b-1)j+\frac{N_b}{2}+1,m}$$

⋮

$$H_{(N_b-1)j+\frac{N_b}{2}+N_b-2,m} M_{(N_b-1)j+\frac{N_b}{2}+N_b-2,m} M_{(N_b-1)j+\frac{N_b}{2}+N_b-1,m} H_{(N_b-1)j+\frac{N_b}{2}+N_b-1,m};$$

i.e.,

$$\begin{aligned}
\mu_{\mathcal{L}}(\mathcal{Q}_{m,j+1}) &= \frac{\varepsilon_m^m}{2} \sum_{q=0}^{N_b-2} \left( \mathcal{W}\left(\frac{(N_b-1)j + \frac{N_b}{2} + q}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j + \frac{N_b}{2} + q + 1}{(N_b-1)N_b^m}\right) \right) \\
&\quad - \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j + \frac{N_b}{2}}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j + \frac{N_b}{2} + N_b - 1}{(N_b-1)N_b^m}\right) \right) \\
&= \mathcal{W}\left(\frac{(N_b-1)j + \frac{N_b}{2} + 1}{(N_b-1)N_b^m}\right) + \frac{\varepsilon_m^m}{2} \sum_{q=1}^{N_b-1} 2 \mathcal{W}\left(\frac{(N_b-1)j + \frac{N_b}{2} + q}{(N_b-1)N_b^m}\right) \\
&\quad - \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j + \frac{N_b}{2}}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j + \frac{N_b}{2} + N_b - 1}{(N_b-1)N_b^m}\right) \right).
\end{aligned}$$

We thus have that

$$\begin{aligned}
\mathcal{V}_m(\varepsilon_m^m) &= \sum_{j=0}^{N_b^m-2} (\mu_{\mathcal{L}}(\mathcal{P}_{m,j+1}) + \mu_{\mathcal{L}}(\mathcal{Q}_{m,j+1})) + \mu_{\mathcal{L}}(\mathcal{P}_{m,N_b^m}) \\
&= \sum_{j=0}^{N_b^m-2} \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j + N_b - 1}{(N_b-1)N_b^m}\right) \right) \\
&\quad - \sum_{j=0}^{N_b^m-2} \frac{\varepsilon_m^m}{2} \sum_{q=1}^{N_b-2} 2 \mathcal{W}\left(\frac{(N_b-1)j + q}{(N_b-1)N_b^m}\right) - \frac{\varepsilon_m^m}{2} \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) \\
&\quad + \sum_{j=0}^{N_b^m-2} \mathcal{W}\left(\frac{(N_b-1)j + \frac{N_b}{2} + 1}{(N_b-1)N_b^m}\right) + \frac{\varepsilon_m^m}{2} \sum_{q=1}^{N_b-1} 2 \mathcal{W}\left(\frac{(N_b-1)j + \frac{N_b}{2} + q}{(N_b-1)N_b^m}\right) \\
&\quad - \sum_{j=0}^{N_b^m-2} \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j + \frac{N_b}{2}}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j + \frac{N_b}{2} + N_b - 1}{(N_b-1)N_b^m}\right) \right) \\
&\quad + \mu_{\mathcal{L}}(\mathcal{P}_{m,N_b^m}).
\end{aligned}$$

As previously, taking into account the fact that

$$\varepsilon_m^m = \frac{1}{N_b-1} \varepsilon^m,$$

we write  $\mathcal{V}_m(\varepsilon_m^m)$  in the following form,

$$\mathcal{V}_m(\varepsilon_m^m) = \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \mathcal{W}\left(\frac{(N_b-1)j + q}{(N_b-1)N_b^m}\right),$$

where

$$\alpha_0(N_b) = \frac{N_b-1}{2(N_b-1)} - \frac{1}{2(N_b-1)} = \frac{N_b-2}{2(N_b-1)}$$

and for  $1 \leq q \leq N_b - 2$ ,

$$\alpha_{\frac{N_b}{2}+q}(N_b) = \frac{2}{2(N_b - 1)} = \frac{1}{N_b - 1} \quad , \quad \alpha_q(N_b) = -\frac{2}{2(N_b - 1)} = -\frac{1}{N_b - 1} \quad ,$$

along with

$$\begin{aligned} \alpha_{\frac{N_b}{2}}(N_b) &= \frac{1}{2(N_b - 1)} - \frac{(N_b - 1)}{2(N_b - 1)} = -\frac{(N_b - 2)}{2(N_b - 1)} \quad , \\ \alpha_{\frac{N_b}{2}+N_b-1}(N_b) &= \frac{1}{2(N_b - 1)} - \frac{(N_b - 1)}{2(N_b - 1)} = -\frac{(N_b - 2)}{2(N_b - 1)} \quad , \\ \alpha_{N_b-1}(N_b) &= -\frac{1}{2(N_b - 1)} + \frac{N_b - 1}{2(N_b - 1)} = \frac{N_b - 2}{2(N_b - 1)} = \alpha_0(N_b) \quad . \end{aligned}$$

Hence, by using the respective expressions of the  $m^{\text{th}}$  cohomology infinitesimal  $\varepsilon_m^m$  and of the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m = (N_b - 1)\varepsilon_m^m$  given in Definition 3.1, on page 20, we have that

$$\begin{aligned} \mathcal{V}_m(\varepsilon_m^m) &= \mu_{\mathcal{L}}(\mathcal{D}(\Gamma_{\mathcal{W}_m})) \\ &= \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e \left( c_{m,k,(N_b-1)j+q} \varepsilon^{ik \ell_{k,(N_b-1)j+q,m} \mathbf{P}} \right) \quad , \end{aligned}$$

where the complex coefficients  $c_{k,(N_b-1)j+q,m}$  are given in part *ii.* of Property 4.3, on page 34, with  $\ell_{m,k,(N_b-1)j+q} \in \mathbb{Z}$  arbitrary (see part *i.* of Property 4.3, on page 34).

□

### Definition 4.3 ( $m^{\text{th}}$ Tubular Volume [DL22c]).

Given  $m \in \mathbb{N}$  sufficiently large (so that  $\varepsilon_m^m$  be a sufficiently small positive number), we call  $m^{\text{th}}$  *tubular volume* the two-dimensional Lebesgue measure  $\mathcal{V}_m^{\text{tube}}(\varepsilon_m^m)$  of the  $(m, \varepsilon_m^m)$ -neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m)$  of the  $m^{\text{th}}$  prefractal approximation  $\Gamma_{\mathcal{W}_m}$  introduced in Definition 4.1, on page 28.

There now remains to overcome the following difficulty: contrary to our previous work involving ordinary tubular neighborhoods (see [DL22c]), the formulae respectively obtained in relation (R15), on page 38 above and in relation (R18), on page 53 below, which give, for all  $m \in \mathbb{N}$ , the respective expressions for the  $m^{\text{th}}$  natural polyhedral volume  $\mathcal{V}_m(\varepsilon_m^m)$  and the  $m^{\text{th}}$  natural complex polyhedral volume  $\mathcal{V}_{m,\text{comp}}(\varepsilon_m^m)$ , do not have the same form as the expression computed in [DL22c] for the  $m^{\text{th}}$  tubular volume  $\mathcal{V}_m^{\text{tube}}$ .

In our present context, the  $m^{\text{th}}$  natural polyhedral volume  $\mathcal{V}_m(\varepsilon_m^m)$  and the  $m^{\text{th}}$  natural complex polyhedral volume  $\mathcal{V}_{m,\text{comp}}(\varepsilon_m^m)$  are expressed in function of the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m$ , but associated with a sum of  $N_b^m = \varepsilon^{-m}$  terms, themselves expressed in terms of the  $k^{\text{th}}$  intrinsic cohomology infinitesimals  $\varepsilon^k$ , for  $0 \leq k \leq m$ . Therefore,  $\mathcal{V}_m(\varepsilon_m^m)$  and  $\mathcal{V}_{m,\text{comp}}(\varepsilon_m^m)$  cannot thus be expressed in an explicit way at an arbitrary value  $\varepsilon > 0$ .

For those reasons, when it comes to obtaining the associated fractal zeta function, we cannot, as in the case of an arbitrary bounded subset of  $\mathbb{R}^2$  (see [LRŽ17], Definition 2.2.8, page 118), directly use an integral formula of the form

$$\tilde{\zeta}_m(s) = \int_0^{\varepsilon^m} t^{s-3} \mathcal{V}_m(t) dt = \int_0^{\varepsilon^m} t^{s-2} \mathcal{V}_m(t) \frac{dt}{t}. \quad (\mathcal{R}17)$$

We can bypass these difficulties by noting that, given  $m \in \mathbb{N}$  sufficiently large,

- i. the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m$  associated with the aforementioned sum of  $N_b^m = \varepsilon^{-m}$  terms has to be considered as a constant value in any computation of a fractal zeta function associated with the  $m^{\text{th}}$  polyhedral volume. This means, in particular, that we cannot substitute an integration variable for this specific  $\varepsilon^m$ .
- ii. As for the terms of the sum of  $N_b^m = \varepsilon^{-m}$  terms, themselves expressed in terms of the  $k^{\text{th}}$  intrinsic cohomology infinitesimals  $\varepsilon^k$ , for  $0 \leq k \leq m$ , we note that due to Property 2.5, on page 8 above, they can be interpreted as the expansion of the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m$  with respect to a specific polynomial basis.

**Lemma 4.6 (Natural Polyhedral Volume Extension Formula).**

Given  $m \geq 1$  sufficiently large, we set

$$\mathcal{V}_{m,\text{comp}}(\varepsilon^m) = \varepsilon^m \mathcal{V}_{\text{partial},m,\text{comp}}(\varepsilon^m), \quad (\mathcal{R}18)$$

where

$$\mathcal{V}_{\text{partial},m,\text{comp}}(\varepsilon^m) = \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^m \alpha_q(N_b) c_{m,k,(N_b-1)j+q} \varepsilon^{k(2-D_{\mathcal{W}})+ik \ell_{k,(N_b-1)j+q,m} \mathbf{P}}.$$

Here,  $\varepsilon^m = (N_b - 1) \varepsilon_m^m$  is the  $m^{\text{th}}$  intrinsic cohomology infinitesimal introduced in Definition 3.1, on page 20, while the real numbers  $\alpha_q(N_b)$ , with  $0 \leq q \leq N_b$ , have been introduced in Theorem 4.5, on page 38, and the complex numbers  $c_{m,k,(N_b-1)j+q} \in \mathbb{C}$  have been introduced in part ii. of Property 4.3, on page 34, while the integers  $\ell_{m,k,(N_b-1)j+q} \in \mathbb{Z}$  are arbitrary (see again, Property 4.3, on page 34). We have used the notation  $\mathcal{V}_{\text{partial},m,\text{comp}}(\varepsilon^m)$ , since, as was mentioned previously, the sum

$$\sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^m \alpha_q(N_b) c_{m,k,(N_b-1)j+q} \varepsilon^{k(2-D_{\mathcal{W}})+ik \ell_{k,(N_b-1)j+q,m} \mathbf{P}}$$

can be considered as a function of the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m$ .

We then introduce, for all sufficiently large  $m \in \mathbb{N}^*$ ,  $\tilde{\mathcal{V}}_{\text{partial},m,\text{comp}}$  as the continuous function defined for all  $t \in [0, \varepsilon]$  by substituting  $t$  for  $\varepsilon$  on the right-hand side of the expression for  $\mathcal{V}_{\text{partial},m,\text{comp}}(\varepsilon^m)$ , in relation (R18) jus above; i.e.,

$$\mathcal{V}_{\text{partial},m,\text{comp}}(t^m) = \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^m \alpha_q(N_b) c_{m,k,(N_b-1)j+q} t^{k(2-D_{\mathcal{W}})+ik \ell_{k,(N_b-1)j+q,m} \mathbf{P}}. \quad (\mathcal{R}19)$$

Note that, in relation (R19), the expression of  $\mathcal{V}_{\text{partial},m,\text{comp}}$  depends on  $t^m$ , and not on  $t$ . Indeed,  $t^m$  can be decomposed into a polynomial basis involving the  $k^{\text{th}}$  powers of  $t$ , with  $0 \leq k \leq m$ .

As is explained in [DL22c] (in the case of the ordinary Euclidean volume), one can think of  $\varepsilon^m \tilde{\mathcal{V}}_{\text{partial},m,\text{comp}}(t^m)$  as being the effective polyhedral volume of the  $m^{\text{th}}$  prefractal approximation to the Weierstrass Curve.

**Notation 14 (Natural Polyhedral Complex Volume Extension).**

For the sake of simplicity, given  $m \in \mathbb{N}^*$  sufficiently large, we will from now on call the  $m^{\text{th}}$  natural polyhedral complex volume extension, the volume extension function  $\tilde{\mathcal{V}}_{m,\text{comp}} = \varepsilon^m \tilde{\mathcal{V}}_{\text{partial},m,\text{comp}}$  associated in Lemma 4.6, on page 53, with the  $m^{\text{th}}$  natural polyhedral complex volume  $\mathcal{V}_{m,\text{comp}}$  introduced in Theorem 4.5, on page 38. Alternatively,  $\tilde{\mathcal{V}}_{m,\text{comp}}$  will be called the  $m^{\text{th}}$  effective polyhedral complex volume.

In the same way, as is done in [DL22c], and given  $m \in \mathbb{N}^*$  sufficiently large, we call the  $m^{\text{th}}$  natural volume extension, the volume extension function  $\tilde{\mathcal{V}}_m^{\text{tube}}$  associated with the  $m^{\text{th}}$  tubular volume  $\mathcal{V}_m^{\text{tube}}$ . Alternatively,  $\tilde{\mathcal{V}}_m^{\text{tube}}$  will be called the  $m^{\text{th}}$  effective tubular volume.

**Theorem 4.7 (Local and Global Polyhedral Effective Zeta Functions).**

Given any  $m \in \mathbb{N}^*$  sufficiently large, we introduce the  $m^{\text{th}}$  local polyhedral effective zeta function  $\tilde{\zeta}_m^e$ , such that, for  $\text{Re}(s) > D_{\mathcal{W}}$ ,

$$\tilde{\zeta}_m^e(s) = \int_0^\varepsilon t^{s-3} \varepsilon^m \tilde{\mathcal{V}}_{m,\text{comp}}(t) dt \quad (\mathcal{R} 20)$$

where  $\tilde{\mathcal{V}}_m = \varepsilon^m \tilde{\mathcal{V}}_{\text{partial},m,\text{comp}}$  is the  $m^{\text{th}}$  effective polyhedral complex volume, introduced in Notation 14, on page 54 above, with  $\varepsilon$  denoting the  $m^{\text{th}}$  intrinsic cohomology infinitesimal introduced in Definition 3.1, on page 20,.

Then, still for all  $m \in \mathbb{N}^*$  sufficiently large, we have that, for all  $s \in \mathbb{C}$ ,

$$\tilde{\zeta}_m^e(s) = \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m \frac{c_{k,(N_b-1)j+q,m} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}, \quad (\mathcal{R} 21)$$

where  $\varepsilon$  is the intrinsic scale introduced in Definition 3.1, on page 20), and where, for  $0 \leq j \leq N_b^m - 1$  and  $0 \leq q \leq N_b$ , the coefficients  $\alpha_q(N_b) \in \mathbb{R}$  have been introduced in Theorem 4.5, on page 38, while the coefficients  $c_{k,(N_b-1)j+q,m} \in \mathbb{C}$ , along with the arbitrary integers  $\ell_{k,(N_b-1)j+q,m} \in \mathbb{Z}$ , have been introduced in Property 4.3, on page 34.

More specifically, still for all  $m \in \mathbb{N}^*$  sufficiently large, the function  $\tilde{\zeta}_m^e$  is well defined and meromorphic in all of  $\mathbb{C}$ . Furthermore, its (necessarily unique) meromorphic extension (still denoted by  $\tilde{\zeta}_m^e$ ) is given, for all  $s \in \mathbb{C}$  by the expressions given in relation (R21) above.

Moreover, the associated sequence  $(\tilde{\zeta}_m^e)_{m \in \mathbb{N}}$  initially given, for  $\text{Re}(s) > D_{\mathcal{W}}$ , by the truncated Mellin transform in relation (R20) above, satisfies the following recurrence relation, for all values of the positive integer  $m$  sufficiently large, and for all  $s \in \mathbb{C}$ :

$$\begin{aligned}
\tilde{\zeta}_{m+1}^e(s) &= \varepsilon \tilde{\zeta}_m^e(s) + \varepsilon^{m+1} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \frac{c_{m+1,(N_b-1)j+q,m+1} \varepsilon^{s-2+(m+1)(2-D_{\mathcal{W}})+i\ell_{m+1,(N_b-1)j+q,m+1} \mathbf{P}}}{s+2(m+1-1)-(m+1)D_{\mathcal{W}}+i\ell_{m+1,(N_b-1)j+q,m+1} \mathbf{P}} \\
&\quad + \varepsilon^{m+1} \sum_{j=N_b^m}^{N_b^{m+1}-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m+1} \frac{c_{k,(N_b-1)j+q,m+1} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}}.
\end{aligned} \tag{R 22}$$

This ensures the existence of the limit fractal zeta function  $\tilde{\zeta}_{\mathcal{W}}^e$ , i.e., the fractal zeta function associated with the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  (or, rather, with the Weierstrass IFD), called the global polyhedral effective zeta function and given by

$$\tilde{\zeta}_{\mathcal{W}}^e = \lim_{m \rightarrow \infty} \tilde{\zeta}_m^e,$$

where the convergence is locally uniform on  $\mathbb{C}$ , along with the existence of an integer  $m_0 \in \mathbb{N}$  such that, for any integer  $m \geq m_0$ , the set of poles of  $\tilde{\zeta}_{\mathcal{W}}^e$  is a subset of the set of poles of  $\tilde{\zeta}_{m+1}^e$  – and hence also, of that of  $\tilde{\zeta}_m^e$ . More specifically,  $\tilde{\zeta}_{\mathcal{W}}^e$  is meromorphic in all of  $\mathbb{C}$  and its meromorphic extension (still denoted  $\tilde{\zeta}_{\mathcal{W}}^e$ ) is given, for all  $s \in \mathbb{C}$  by

$$\begin{aligned}
\tilde{\zeta}_{\mathcal{W}}^e(s) &= \sum_{m=m_0}^{\infty} \varepsilon^{m-m_0+1} \tilde{\zeta}_m^e(s) \\
&\quad + \sum_{m=m_0}^{\infty} \varepsilon^{m+1} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \frac{c_{m+1,(N_b-1)j+q,m+1} \varepsilon^{s-2+(m+1)(2-D_{\mathcal{W}})+i\ell_{m+1,(N_b-1)j+q,m+1} \mathbf{P}}}{s+2m-(m+1)D_{\mathcal{W}}+i\ell_{m+1,(N_b-1)j+q,m+1} \mathbf{P}} \\
&\quad + \sum_{m=m_0}^{\infty} \varepsilon^{m+1} \sum_{j=N_b^m}^{N_b^{m+1}-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m+1} \frac{c_{k,(N_b-1)j+q,m+1} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}}.
\end{aligned} \tag{R 23}$$

In other words,  $\tilde{\zeta}_{\mathcal{W}}^e$  admits a (necessarily unique) meromorphic continuation, given, for all  $s \in \mathbb{C}$ , by relation (23) just above.

As was mentioned in the introduction, we note that our result is stronger than the one previously obtained in [DL22c], where, in particular, the values of the possible Complex Dimensions of the Weierstrass IFD included  $-2$ ,  $0$  and  $1-2k$ , with  $k \in \mathbb{N}$  arbitrary. As we can see in relation (23) just above, the poles of the limit effective fractal zeta function  $\tilde{\zeta}_{\mathcal{W}}^e$  are exactly the same as the Complex Dimensions of the Weierstrass function itself; see Theorem 4.3, on page 34. Note that, in [DL22c], the Complex Dimensions are defined in terms of the volume of the tubular (rather than polyhedral) neighborhood of the Weierstrass IFD.

*Proof.*

*i.* We first give the explicit expression for the  $m^{\text{th}}$  local effective polyhedral zeta function  $\tilde{\zeta}_m^e$ .

We restrict ourselves to sufficiently large values of  $m \in \mathbb{N}$ , i.e.,  $m \geq m_0$ , for some suitable integer  $m_0 \in \mathbb{N}$ .



We have that, for  $\mathcal{R}e(s) > D_{\mathcal{W}}$ ,

$$\begin{aligned}
\tilde{\zeta}_m^e(s) &= \int_0^\varepsilon t^{s-3} \tilde{\mathcal{V}}_{m,comp}(t) dt \quad (\text{for } \mathcal{R}e(s) > D_{\mathcal{W}}) \\
&= \varepsilon^m \int_0^\varepsilon t^{s-3} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m c_{k,(N_b-1)j+q,m} t^{k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}} dt \\
&= \varepsilon^m \int_0^\varepsilon \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m c_{k,(N_b-1)j+q,m} t^{s-3+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}} dt \\
&= \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m \frac{c_{k,(N_b-1)j+q,m} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}} \\
&= \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m \frac{c_{k,(N_b-1)j+q,m} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}.
\end{aligned}$$

We then have that, for any  $m \in \mathbb{N}^*$  sufficiently large,

$$\begin{aligned}
\tilde{\zeta}_{m+1}^e(s) &= \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m+1} \frac{c_{k,(N_b-1)j+q,m+1} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}} \\
&= \varepsilon^{m+1} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m \frac{c_{k,(N_b-1)j+q,m+1} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}} \\
&+ \varepsilon^{m+1} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \frac{c_{m+1,(N_b-1)j+q,m+1} \varepsilon^{s-2+(m+1)(2-D_{\mathcal{W}})+i\ell_{m+1,(N_b-1)j+q,m+1} \mathbf{P}}}{s+2(m+1-1)-(m+1)D_{\mathcal{W}}+i\ell_{m+1,(N_b-1)j+q,m+1} \mathbf{P}} \\
&+ \varepsilon^{m+1} \sum_{j=N_b^m}^{N_b^{m+1}-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m+1} \frac{c_{k,(N_b-1)j+q,m+1} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}} \\
&= \varepsilon^{m+1} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m \frac{c_{k,(N_b-1)j+q,m} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}} \\
&+ \varepsilon^{m+1} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \frac{c_{m+1,(N_b-1)j+q,m+1} \varepsilon^{s-2+(m+1)(2-D_{\mathcal{W}})+i\ell_{m+1,(N_b-1)j+q,m+1} \mathbf{P}}}{s+2(m+1-1)-(m+1)D_{\mathcal{W}}+i\ell_{m+1,(N_b-1)j+q,m+1} \mathbf{P}} \\
&+ \varepsilon^{m+1} \sum_{j=N_b^m}^{N_b^{m+1}-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m+1} \frac{c_{k,(N_b-1)j+q,m+1} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}} \\
&= \varepsilon \tilde{\zeta}_m^e(s) + \varepsilon^{m+1} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \frac{c_{m+1,(N_b-1)j+q,m+1} \varepsilon^{s-2+(m+1)(2-D_{\mathcal{W}})+i\ell_{m+1,(N_b-1)j+q,m+1} \mathbf{P}}}{s+2m-(m+1)D_{\mathcal{W}}+i\ell_{m+1,(N_b-1)j+q,m+1} \mathbf{P}} \\
&+ \varepsilon^{m+1} \sum_{j=N_b^m}^{N_b^{m+1}-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m+1} \frac{c_{k,(N_b-1)j+q,m+1} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}} .
\end{aligned} \tag{R 24}$$

ii. For any integer  $m \geq m_0$ , we denote by  $\mathcal{P}(\tilde{\zeta}_m^e) \subset \mathbb{C}$  the set of poles of the zeta function  $\tilde{\zeta}_m^e$ .

We set

$$\mathcal{U} = \{s \in \mathbb{C}, 1 < \operatorname{Re}(s) < 3\} .$$

We note that, for all  $m \geq m_0$ , we have that

$$\mathcal{P}(\tilde{\zeta}_{m_0}^e) \subset \mathcal{P}(\tilde{\zeta}_m^e) \subset \mathcal{U} .$$

The series

$$\begin{aligned}
& \sum_{m=m_0}^{\infty} \varepsilon^{m-m_0+1} \tilde{\zeta}_m^e(s) \\
& + \sum_{m=m_0}^{\infty} \varepsilon^{m+1} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \frac{c_{m+1,(N_b-1)j+q,m+1} \varepsilon^{s-2+(m+1)(2-D_{\mathcal{W}})+i\ell_{m+1,(N_b-1)j+q,m+1} \mathbf{P}}}{s+2m-(m+1)D_{\mathcal{W}}+i\ell_{m+1,(N_b-1)j+q,m+1} \mathbf{P}} \\
& + \sum_{m=m_0}^{\infty} \varepsilon^{m+1} \sum_{j=N_b^m}^{N_b^{m+1}-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m+1} \frac{c_{k,(N_b-1)j+q,m+1} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}}
\end{aligned} \tag{R 25}$$

is (locally) normally convergent, and, hence, uniformly convergent on  $\mathcal{U}$ . This ensures the existence of the limit effective global fractal zeta function, i.e., the fractal zeta function associated with the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  (or with the Weierstrass IFD), given, for all  $s \in \mathbb{C}$ , by

$$\begin{aligned}
\tilde{\zeta}_{\mathcal{W}}^e(s) &= \lim_{m \rightarrow \infty} \tilde{\zeta}_m^e(s) \\
&= \sum_{m=m_0}^{\infty} \varepsilon^{m-m_0+1} \tilde{\zeta}_m^e(s) \\
&+ \sum_{m=m_0}^{\infty} \varepsilon^{m+1} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \frac{c_{m+1,(N_b-1)j+q,m+1} \varepsilon^{s-2+(m+1)(2-D_{\mathcal{W}})+i\ell_{m+1,(N_b-1)j+q,m+1} \mathbf{P}}}{s+2m-(m+1)D_{\mathcal{W}}+i\ell_{m+1,(N_b-1)j+q,m+1} \mathbf{P}} \\
&+ \sum_{m=m_0}^{\infty} \varepsilon^{m+1} \sum_{j=N_b^m}^{N_b^{m+1}-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m+1} \frac{c_{k,(N_b-1)j+q,m+1} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}+i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P}}.
\end{aligned}$$

Here, and in the remainder of this proof, a (complex-valued) meromorphic function  $f$  is viewed as a continuous function with values in  $\mathbb{P}^1(\mathbb{C})$ , equipped with the chordal metric, and such that, for any pole  $\omega$  of  $f$ ,  $f(\omega)$  takes the value  $\infty$  (for instance, as in [LvF13], Section 3.4 and Appendix C).

More precisely, if  $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$  denotes the Riemann sphere (or complex projective line), we can show that, for *the chordal metric*, defined, for all  $(z_1, z_2) \in (\mathbb{P}^1(\mathbb{C}))^2$  by

$$\|z_1, z_2\| = \frac{|z_1 - z_2|}{\sqrt{1 + |z_1^2|} \sqrt{1 + |z_2^2|}}, \text{ if } z_1 \neq \infty \text{ and } z_2 \neq \infty,$$

and

$$\|z_1, \infty\| = \frac{1}{\sqrt{1 + |z_1^2|}}, \text{ if } z_1 \neq \infty,$$

we have, thanks to the local uniform convergence of the series on  $\mathbb{C}$ ,

$$\lim_{m \rightarrow \infty} \left\| \tilde{\zeta}_m^e, \tilde{\zeta}_{\mathcal{W}}^e \right\| = 0.$$

Indeed, for any  $\eta > 0$  and any compact set  $\mathcal{K} \subset \mathbb{C}$ , we can choose  $m_0 \in \mathbb{N}^*$  such that, for all integers  $m \geq m_0$  and all  $s \in \mathcal{K}$ , we have that

$$\left| \tilde{\zeta}_m^e(s) - \tilde{\zeta}_{\mathcal{W}}^e(s) \right| \leq \eta,$$

and hence, still for  $m \geq m_0$  and all  $s \in \mathcal{K}$ ,

$$\left\| \tilde{\zeta}_m^e(s), \tilde{\zeta}_{\mathcal{W}}^e(s) \right\| \leq \left| \tilde{\zeta}_m^e(s) - \tilde{\zeta}_{\mathcal{W}}^e(s) \right| \leq \eta.$$

The sum of this series, i.e., the (uniform) limit fractal zeta function  $\tilde{\zeta}_{\mathcal{W}}^e$ , is holomorphic on  $\mathcal{U}$ . We can then deduce that, for all  $m \geq m_0$ , the zeta function  $\tilde{\zeta}_m^e$  is meromorphic on  $\mathbb{C}$ .

Moreover, the counterpart of the results obtained in [DL22c] for the sequence of tube zeta functions associated with the Weierstrass IFD, which admit a meromorphic continuation to all of  $\mathbb{C}$ , obviously hold for the sequence of polyhedral tube zeta functions: hence,  $\tilde{\zeta}_m^e$  is meromorphic on  $\mathbb{C}$ , with only simple poles, as specified in Theorem 4.9, on page 60 below.

□

**Property 4.8 (From the  $m^{\text{th}}$  Local Effective Polyhedral Zeta Function to the  $m^{\text{th}}$  Local Effective Tube Zeta Function).**

Given  $m \in \mathbb{N}^*$  sufficiently large, the Lebesgue measure  $\mathcal{V}_m^{\text{tube}}(\varepsilon_m^m)$  of the tubular neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m)$  can be connected to the Lebesgue measure  $\mathcal{V}_m(\varepsilon_m^m)$  of the  $m^{\text{th}}$  polyhedral neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  introduced in Definition 3.4, on page 24, by means of the following relation:

$$\mathcal{V}_m(\varepsilon_m^m) = \mathcal{V}_m^{\text{tube}}(\varepsilon_m^m) + \mathcal{R}_m, \quad (\mathcal{R}26)$$

where  $\mathcal{R}_m$  is a suitable error term and

$$\mathcal{V}_m^{\text{tube}}(\varepsilon_m^m) = \mu_{\mathcal{L}}(\mathcal{D}(\Gamma_{\mathcal{W}_m})).$$

As is done in Lemma 4.6, on page 53, we extend relation (R26) to the corresponding effective volumes; i.e., for  $0 \leq t \leq \varepsilon_m^m$ ,

$$\tilde{\mathcal{V}}_m(t) = \tilde{\mathcal{V}}_m^{\text{tube}}(t) + \tilde{\mathcal{R}}_m(t), \quad (\mathcal{R}27)$$

where, for all sufficiently large  $n \in \mathbb{N}^*$ ,  $\tilde{\mathcal{V}}_n$  (resp.,  $\tilde{\mathcal{V}}_n^{\text{tube}}$  or  $\tilde{\mathcal{R}}_n$ ) is the continuous function defined for all  $t \in [0, \varepsilon_n^n]$  by substituting  $t$  for  $\varepsilon_n^n$  in the expression for  $\mathcal{V}_n(\varepsilon_n^n)$  (resp.,  $\mathcal{V}_n^{\text{tube}}$  or  $\mathcal{R}_n$ ), obtained in Theorem 4.5, on page 38.

This ensures, for the associated zeta function  $\mathcal{Z}_{m, \mathcal{R}}$ , given by

$$s \mapsto \int_0^\varepsilon t^{s-3} \tilde{\mathcal{R}}_m(t) dt,$$

that (locally, uniformly on  $\mathbb{C}$ )

$$\lim_{m \rightarrow \infty} \int_0^\varepsilon t^{s-3} \tilde{\mathcal{R}}_m(t) dt = 0.$$

*Proof.* This follows directly from Property 3.4, on page 24, in conjunction with Property 4.1, on page 28. Given  $\eta > 0$ , there exists  $m_0 \in \mathbb{N}$  such that, for all  $m \geq m_0$ ,

$$\left| \mathcal{V}_m(\varepsilon_m^m) - \mathcal{V}_m^{\text{tube}}(\varepsilon_m^m) \right| \leq \eta.$$

We then proceed as in the proof of Theorem 4.7, on page 54. Since, for  $\text{Re}(s) > D_{\mathcal{W}}$ ,

$$\int_0^\varepsilon t^{s-3} \widetilde{\mathcal{R}}_m(t) dt = \int_0^\varepsilon t^{s-3} (\widetilde{\mathcal{V}}_m(t) - \widetilde{\mathcal{V}}_m^{tube}(t)) dt,$$

we obtain the expected (local) uniform convergence. □

**Definition 4.4 (Complex Dimensions of the Weierstrass Curve and of its Prefractal Approximations).**

The *Complex Dimensions* of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  (or, rather, of the Weierstrass IFD) are the poles of the global polyhedral effective zeta function  $\widetilde{\zeta}_{\mathcal{W}}^e$ . To distinguish these Complex Dimensions from those defined in our earlier paper [DL22c] (via local polyhedral effective tube zeta functions, we may wish to call them *intrinsic Complex Dimensions* of  $\Gamma_{\mathcal{W}}$  (or, rather, of the associated IFD).

More specifically, the phrase “*possible* (or *potential*) Complex Dimensions” refers to those  $\omega \in \mathbb{C}$  which appear as an  $\omega$ -exponent in the fractal power series expansion (R23), on page 55; accordingly, its associated residue may vanish (i.e.,  $\text{res}(\widetilde{\zeta}_{\mathcal{W}}^e, \omega) = 0$ ) or else, it may be nonzero (i.e.,  $\text{res}(\widetilde{\zeta}_{\mathcal{W}}^e, \omega) \neq 0$ ). In the latter case,  $\omega$  is called an *exact Complex Dimension* of  $\Gamma_{\mathcal{W}}$  (or of the associated IFD).

We use an entirely similar definition for the Complex Dimensions of the prefractal approximations  $\Gamma_{\mathcal{W}_m}$ , for any  $m \in \mathbb{N}^*$  sufficiently large, except for  $\widetilde{\zeta}_m^e$  instead of  $\widetilde{\zeta}_{\mathcal{W}}^e$  and for the fractal powers series in relation (R21), on page 54, instead of in relation (R23), on page 55.

**Theorem 4.9 (Exact Complex Dimensions of the Weierstrass Curve).**

*The Complex Dimensions of the Weierstrass Curve (or of the Weierstrass IFD) are given as follows:*

$$D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) + i\ell \mathbf{p}, \quad \text{with } m \in \mathbb{N}, 0 \leq k \leq m, \quad (\mathcal{R}28)$$

where the integers  $k \in \mathbb{N}$  and  $\ell \in \mathbb{Z}$  are arbitrary (compare Property 4.3, on page 34).

*We note that they only partly coincide with the possible poles of the tube fractal zeta function  $\widetilde{\zeta}_m^{e,tube}$ . In particular, we then deduce that the following possible Complex Dimensions previously obtained in [DL22c], i.e.,  $1 - 2k + i\ell \mathbf{p}$ , with  $k \in \mathbb{N}$  and  $\ell \in \mathbb{Z}$ , along with  $-2$  and  $0$ , are not actual Complex Dimensions, in the sense introduced in the present paper.*

*Moreover, the Complex Dimensions associated with  $D_{\mathcal{W}}$ , i.e.,*

$$D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) + i\ell \mathbf{p},$$

*with  $k \in \mathbb{N}$  and  $\ell \in \mathbb{Z}$  arbitrary, are actual Complex Dimensions of  $\Gamma_{\mathcal{W}}$ ; i.e., they are exact Complex Dimensions of  $\Gamma_{\mathcal{W}}$ , in the sense of Definition 4.4, on page 60.*

*In short, all of the possible (intrinsic) Complex Dimensions of  $\Gamma_{\mathcal{W}}$  (or of the associated IFD) are given by relation (R28) above, are exact (still in the sense of Definition 4.4). There are no other Complex Dimensions.*

*For the exceptional cases, we refer to [DL22c] for a closely related discussion.*

*Proof.*

Let  $\omega$  be a pole (i.e., a possible Complex Dimension) of the global effective fractal zeta function  $\tilde{\zeta}_{\mathcal{W}}^e$ , given by relation (R23) of Theorem 4.7, on page 54.  $\omega$  is of the following form

$$\omega = \omega_{k,\ell} = D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) + i \ell_{k,(N_b-1)j+q,m+1} \mathbf{P},$$

with  $(k, j, m) \in \mathbb{N}^3$ ,  $0 \leq q \leq N_b$  and  $\ell_{k,(N_b-1)j+q,m+1} \in \mathbb{Z}$  arbitrary.

The residue of  $\tilde{\zeta}_{\mathcal{W}}^e$  at  $\omega_{k,\ell}$  is given by

$$\begin{aligned} \operatorname{res} \left( \tilde{\zeta}_{\mathcal{W}}^e, \omega_{k,\ell} \right) &= \lim_{s \rightarrow \omega_{k,\ell}} (s - \omega_{k,\ell}) \tilde{\zeta}_{\mathcal{W}}^e(s) \\ &= \varepsilon^{m+1} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1}, \end{aligned} \tag{R 29}$$

where the nonzero coefficients  $\alpha_q(N_b)$  and  $c_{k,(N_b-1)j+q,m+1}$  are respectively given in Theorem 4.5, on page 38 and in part *ii.* of Property 4.3, on page 34. This ensures that the residue of  $\tilde{\zeta}_{\mathcal{W}}^e$  at  $\omega_{k,\ell}$  is nonzero:  $\omega_{k,\ell}$  is an actual Complex Dimension of the Weierstrass Curve. □

Recall that in the standard mathematical theory of Complex Dimensions (see, e.g., [LvF13], [LRŽ17], [Lap19]), a geometric object is said to be *fractal* if it admits at least one *nonreal* Complex Dimension (defined as a pole of the associated geometric or fractal zeta function). [In the present context, the fractal zeta function is the global polyhedral effective zeta function in the case of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  (resp., the  $m^{\text{th}}$  local polyhedral effective zeta function of its  $m^{\text{th}}$  prefractal approximation, with  $m \geq m_0$ ; see Definition 4.4, on page 60 above).]

In addition, given  $d \in \mathbb{R}$ , it is *fractal in dimension  $d$*  if it has at least one *nonreal* Complex Dimension with real part  $d$ . (Note that nonreal Complex Dimensions come in complex conjugate pairs.) In particular, it is *principally fractal* if it is fractal in dimension  $D$  (here,  $D_{\mathcal{W}}$ ), the abscissa of convergence of  $\tilde{\zeta}_{\mathcal{W}}^e$  (resp., of  $\tilde{\zeta}_m^e$ , with  $m \geq m_0$ ), which is the largest possible value of  $d$ .

Accordingly, we have the following two corollaries (Corollaries 4.10 and 4.11) of Theorems 4.7, on page 54 and 4.9, on page 60.

**Corollary 4.10 (Fractality of the Weierstrass Curve).**

*The Weierstrass Curve  $\Gamma_{\mathcal{W}}$  (or the Weierstrass IFD) is fractal; in fact, it has infinitely (and countably) many Complex Dimensions given by Theorem 4.9, on page 60. In particular, it is both principally fractal (since it has infinitely many Complex Dimensions with real part  $D_{\mathcal{W}}$ , the Minkowski dimension of  $\Gamma_{\mathcal{W}}$ ) and is fractal in infinitely (and countably) many values of  $d$  (namely,  $d = D_{\mathcal{W}} - k(2 - D_{\mathcal{W}})$ , with  $k \in \mathbb{N}$  arbitrary).*

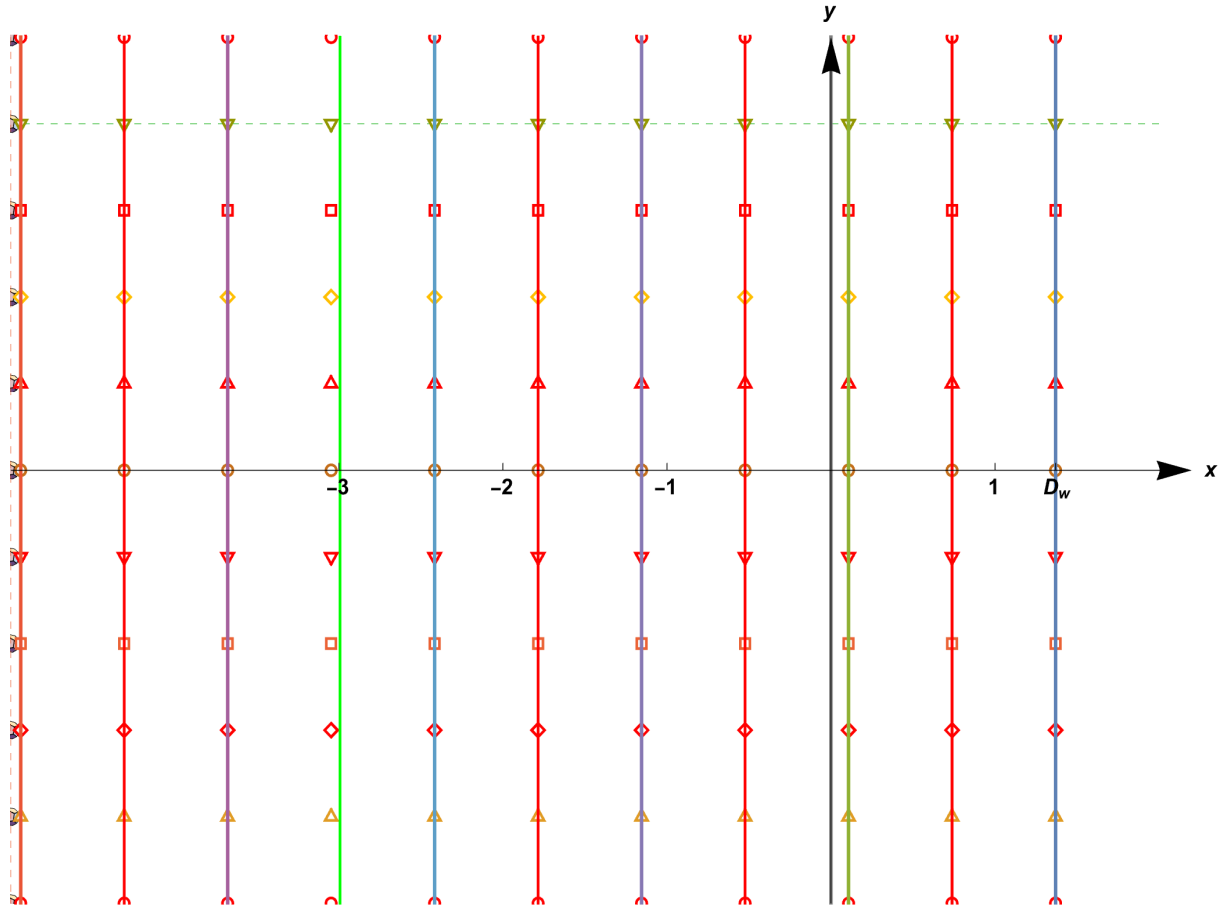


Figure 14: **The Complex Dimensions of the Weierstrass IFD.** The nonzero Complex Dimensions are periodically distributed (with the same period  $p = \frac{2\pi}{\ln N_b}$ , the oscillatory period of the Weierstrass IFD) along countably many vertical lines, with abscissae  $D_{\mathcal{W}} - (k - 1)(2 - D_{\mathcal{W}})$ , where  $k \in \mathbb{N}$  is arbitrary. For the sake of representation, there is a different color for each vertical line, and a specific symbol is used to plot the imaginary parts of the Complex Dimensions associated with a given vertical line.

**Corollary 4.11 (Fractality of the Prefractal Approximations).**

For all sufficiently large positive integers  $m$  (i.e., for all  $m \geq m_0$ ; see the proof on page 55 (of Theorem 4.7, on page 54), the  $m^{\text{th}}$  prefractal approximation  $\Gamma_{\mathcal{W}_m}$  to the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  is fractal and even, principally fractal. More specifically, it is fractal in dimension  $d$ , for precisely the following values of  $d$ :

$$d = D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}),$$

with the integer  $k$  satisfying  $0 \leq k \leq m$ .

Indeed, the Complex Dimensions of  $\Gamma_{\mathcal{W}_m}$  (for any  $m \geq m_0$ ) are all simple and exact; furthermore, they are given by

$$d = D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) + i\ell p,$$

where  $k \in \mathbb{N}$ ,  $0 \leq k \leq m$  and  $\ell \in \mathbb{Z}$  is arbitrary.

*Proof.* Corollary 4.11 follows at once from Theorem 4.9, on page 60.

The statement regarding the exactness of the Complex Dimensions in Corollary 4.10, on page 61, is proved in the same way as its counterpart in the proof of Theorem 4.9. □

*Remark 4.2.* Observe that there is an important and natural difference between  $\Gamma_{\mathcal{W}}$  and  $\Gamma_{\mathcal{W}_m}$  itself. Namely,  $\Gamma_{\mathcal{W}}$  is fractal for infinitely many values of  $d$ , whereas, for all  $m \geq m_0$ ,  $\Gamma_{\mathcal{W}_m}$  is only fractal for finitely many values of  $d$ .

*Remark 4.3 (Connections with Fractal Cohomology).*

In our previous work [DL22d], we proved that the cohomology groups associated with the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  consist of continuous functions (on  $\Gamma_{\mathcal{W}}$ ) which are expressed in terms of (finite, or countably infinite) sums indexed by the Complex Dimensions  $2 - k(2 - D_{\mathcal{W}}) - i\ell_k \mathbf{p}$ , where  $\mathbf{p}$  denotes the oscillatory period of the Weierstrass Curve, for  $k \in \mathbb{N}$  arbitrary, and where  $\ell_k \in \mathbb{Z}$  satisfies a specific condition governed by the underlying geometry of the Curve. Except for this latter and specific condition, our results in Theorem 4.9, on page 60 just above, are precisely of the same form as in [DL22d], with an additional condition expressed in terms of the integers involved in the imaginary part of the Complex Dimensions associated with the fractal cohomology group, which appears as a natural feature in the fractal cohomological context of [DL22d].

## 5 Revisiting the Computation of the Minkowski Dimension

In this section, we provide a fully rigorous proof of the fact that the Minkowski dimension – which, as is well known, coincides with the box-counting (or box) dimension  $D_{\mathcal{W}}$  of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  – is given by the expected formula, originally conjectured by Benoît Mandelbrot in [Man82] (in Chapter XI, on top of page 390). Note that B. Mandelbrot only mentioned he was talking about the *fractal dimension* and did not specify whether it was the Hausdorff or the Minkowski dimension.

In our present context (see Theorem 5.1, on page 64 below), we have that  $D_{\mathcal{W}} = 2 - \frac{\ln \frac{1}{\lambda}}{\ln N_b}$ .

It appears that earlier proofs of this fact were either not fully rigorous or complete. The methods developed in the present paper and in our earlier work, especially, [DL22c] and [DL22b], building in part upon the geometric approach of [Dav18], enable us to establish, in a new manner, Mandelbrot’s conjecture.

When it comes to computing the box-counting – or Minkowski dimension – of a fractal, one usually proceeds by using a (classical) cover of the fractal under study, constituted of balls; namely, in dimension 2, disks or squares; see Definition 5.1, on page 64 below.

However, when a non self-affine, complicated fractal curve, as the Weierstrass Curve is involved, such methods are not optimal. In fact, covering such a curve with disks or squares is not natural. We



hereafter propose to consider, instead, the polyhedral neighborhoods introduced in Definition 3.4, on page 24.

**Definition 5.1 (Box-Counting Dimension).**

As can be found, for instance, in [Fal86], we recall that the *box-counting dimension* (or *box dimension*, in short), of  $\Gamma_{\mathcal{W}}$ , is given by

$$D_{\mathcal{W}} = - \lim_{\delta \rightarrow 0^+} \frac{\ln N_{\delta}(\Gamma_{\mathcal{W}})}{\ln \delta}, \quad (\diamond)$$

where  $N_{\delta}(\Gamma_{\mathcal{W}})$  stands for any of the following quantities:

- i.* the smallest number of sets (here, subsets of  $\mathbb{R}^2$ ) of diameter at most  $\delta$  that cover  $\Gamma_{\mathcal{W}}$  on  $[0, 1[$  ;
- ii.* the smallest number of closed balls (disks, here) of radius  $\delta$  that cover  $\Gamma_{\mathcal{W}}$  on  $[0, 1[$  ;
- iii.* the smallest number of cubes (squares, here) of side  $\delta$  that cover  $\Gamma_{\mathcal{W}}$  on  $[0, 1[$ ;
- iv.* the number of  $\delta$ -mesh cubes (squares, here) that intersect  $\Gamma_{\mathcal{W}}$  on  $[0, 1[$ ;
- v.* the largest number of disjoint balls (disks, here) of radius  $\delta$  with centers in  $\Gamma_{\mathcal{W}}$  on  $[0, 1[$ .

Furthermore, for the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ , as, more generally, for any bounded subset of Euclidean space – the box-counting dimension coincides with the Minkowski dimension.

**Theorem 5.1.** *The box-counting dimension – or, equivalently, the Minkowski dimension – of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  exists and is equal to  $D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln N_b}$ .*

*Proof.* We simply apply the result given in Corollary 4.2, on page 33. We then have the existence of an integer  $m_0 \in \mathbb{N}$  such that

$$\forall m \geq m_0 : \quad \Gamma_{\mathcal{W}} \subset \mathcal{D}(\Gamma_{\mathcal{W}_m}).$$

In other words, for all integers  $m \geq m_0$ , the polyhedral neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  covers the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ .

Note that we thus dispose, with the sequence of polyhedral domains  $(\mathcal{D}(\Gamma_{\mathcal{W}_m}))_{m \geq m_0}$ , of a nonusual (but admissible, in the sense of Definition 5.1, on page 64 above) sequence of covers. However,  $(\mathcal{D}(\Gamma_{\mathcal{W}_m}))_{m \geq m_0}$  is the most natural and optimal sequence of covers of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ , insofar that for any  $m \geq m_0$ , each domain  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  contains the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ , while, at the same time, the sequence  $(\mathcal{D}(\Gamma_{\mathcal{W}_m}))_{m \geq m_0}$  converges to  $\Gamma_{\mathcal{W}}$ . This, in particular, means that when  $m \rightarrow \infty$ , the two-dimensional Lebesgue measure (i.e., area) of each polygon belonging to  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  tends to 0.

For all  $m \in \mathbb{N}^*$ , each polygon belonging to  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  has a vertical height at most equal to  $(N_b - 1)h_m$  (this directly comes from the definition of  $h_m$ ; see relation (R5), on page 12, and an horizontal width at

most equal to  $(N_b - 1)\varepsilon_m^m$ . Note now that, thanks to Remark 3.2, on page 21, we have that  $h_m \gg \varepsilon_m^m$ .

Hence, it is the cohomology infinitesimal  $\varepsilon_m^m$  which plays the role of the elementary diameter  $\delta$  in Definition 5.1, on page 64. Moreover, the area of each polygon belonging to  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  also corresponds to the area of the reunion of  $\frac{(N_b - 1)h_m}{\varepsilon_m^m}$  elementary and smaller polygons, each of diameter  $\varepsilon_m^m$ , where *the diameter* of a polygon is to be understood in the sense of the largest distance between any pair of vertices of the considered polygon.

Since  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  consists itself of the reunion of  $(2N_b^m - 1) = \frac{2\varepsilon_m^{-m}}{N_b - 1} - 1$  polygons, we then obtain the box-counting dimension as

$$-\lim_{m \rightarrow \infty} \frac{\ln \left( \left( \frac{2\varepsilon_m^{-m}}{N_b - 1} - 1 \right) \frac{(N_b - 1)h_m}{\varepsilon_m^m} \right)}{\ln \varepsilon_m^m} = -\lim_{m \rightarrow \infty} \frac{\ln (\varepsilon_m^{-2m} h_m)}{\ln \varepsilon_m^m}.$$

Thanks to the estimate given in Property 2.12, on page 16, we have that

$$C_{inf} (\varepsilon_m^m)^{2-D_{\mathcal{W}}} \leq h_m \leq C_{sup} (\varepsilon_m^m)^{2-D_{\mathcal{W}}},$$

where the constants  $C_{inf}$  and  $C_{sup}$  do not depend on  $m$ , and are such that

$$0 < C_{inf} < C_{sup} < \infty,$$

which ensures that

$$C_{inf} (\varepsilon_m^m)^{-D_{\mathcal{W}}} \leq \varepsilon_m^{-2m} h_m \leq C_{sup} (\varepsilon_m^m)^{-D_{\mathcal{W}}},$$

and, consequently,

$$\ln C_{inf} - D_{\mathcal{W}} \ln (\varepsilon_m^m) \leq \ln (\varepsilon_m^{-2m} h_m) \leq \ln C_{sup} - D_{\mathcal{W}} \ln (\varepsilon_m^m).$$

Since  $\lim_{m \rightarrow \infty} \varepsilon_m^m = 0$ , we obviously have that

$$\lim_{m \rightarrow \infty} \frac{\ln C_{inf}}{\ln (\varepsilon_m^m)} = \lim_{m \rightarrow \infty} \frac{\ln C_{sup}}{\ln (\varepsilon_m^m)} = 0,$$

from which we immediately deduce that the value of the box-counting dimension of the Weierstrass Curve is exactly

$$-\lim_{m \rightarrow \infty} \frac{\ln (\varepsilon_m^{-2m} h_m)}{\ln \varepsilon_m^m} = D_{\mathcal{W}}.$$

This concludes the proof of the existence of the box-counting (i.e., the Minkowski) dimension and of the fact that it is given by  $D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln N_b}$ . □

## 6 A Bridge to Fractal Cohomology

In [DL22d], we established the connections between the Complex Dimensions of a fractal object, and the associated fractal cohomology. In the test case of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ , we obtained a complete characterization of all the elements of the cohomology groups associated to the Curve, which

consist of continuous functions (on the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ ), expressed in terms of Taylor-like expansions, with fractional derivatives of orders the underlying – and actual – Complex Dimensions of the Weierstrass function  $\mathcal{W}$ . As powerful as this result stands, there remained an unanswered question, insofar as the Complex Dimensions involved did not comprise all the possible values obtained in [DL22c] by means of the (classical) tube zeta functions, defined via ordinary (Euclidean) neighborhoods; namely,  $-2$ ,  $0$  and  $1$ .

Our effective local and global polyhedral fractal zeta functions enables us to finally answer this remaining question: their poles, which exactly correspond to the Complex Dimensions both of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  and of the Weierstrass function  $\mathcal{W}$ , do not include the values  $-2$ ,  $0$  and  $-1 + i\ell \mathbf{p}$  where, contrary to our present results,  $\ell \in \mathbb{Z}$  is arbitrary.

More specifically, the possible Complex Dimensions are all exact and precisely of the following form:  $D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) + i\ell \mathbf{p}$ , where  $k \in \mathbb{N}$  and for a specific choice of  $\ell \in \mathbb{Z}$ . Note that the aforementioned classical approach gives a less precise result, without any condition on  $\ell \in \mathbb{Z}$ .

For this reason, we claim that the polyhedral approach, by means of polyhedral neighborhoods and polyhedral fractal zeta functions, is intrinsic, as well as appears to be more natural and better suited than the tubular (or Euclidean) approach introduced in [DL22c] to the study of a more general class of fractal curves, including the Takagi function, the Knopp functions, along with the Koch function and the Koch Curve. The connections with fractal cohomology – and the fact that we fall back on the same values of the Complex Dimensions of the fractal curve involved – open new perspectives in terms of the understanding of those everywhere singular objects. In particular, the polygons which constitute the polyhedral neighborhoods certainly play an important role in the simplicial (and natural) homology, dual to the fractal cohomology introduced in [DL22e] and [DL22d], and to be developed in a later book of the authors.

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