Polyhedral Neighborhoods vs Tubular Neighborhoods: New Insights for the Fractal Zeta Functions

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Abstract

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Contents

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1 Introduction

Fractal zeta functions are classically often obtained ans of tube formulas. However, when it comes to fractals obtained by means of nonlinear and noncontractive iterated function system (i.f.s.), as is the case for the Weierstrass Curve (see [Dav18]), difficulties arise. Indeed, we cannot obtain the expression of the volume of the general tubular neigborhood of the m^{th} prefractal approximation – and, hence, of the associated geometric tube zeta function. Note that, in addition, nonlinearity makes the geometry awfully complicated; in particular, one cannot obtain the exact values of the underlying elementary lengths and angles. As a consequence, we only dispose of an approximate expression of the tube zeta function. In practise, since the approximation only concerns the coefficients in factor of the residues, and since we dispose of sharp estimates of those coefficients, we can obtain the exact values of the poles – and, hence, of the Complex Dimensions associated to each prefractal approximation. However, a natural question is wether the Complex Dimensions of the fractal are the same as those of the prefractal approximations.

In [DL22a], we gave a partial anwser, by introducing the concept of *Weierstrass Iterated Fractal Drums* (in short, Weierstrass IFDs), by analogy with the *relative fractal drums* involved, for instance, in the case of the Cantor Staircase, in [LRŽ17], Section 5.5.4, and in [LRŽ18]. Weierstrass IFDs simply consist in a sequence of polygonal domains which contain the Weierstrass Curve $\Gamma_{\mathcal{W}}$, and are sufficiently close to $\Gamma_{\mathcal{W}}$. As is first explained in [DL22a], we could expect the Complex Dimensions of the IFD to be the same as the Comple Dimensions of the Weierstrass Curve $\Gamma_{\mathcal{W}}$. An interesting argument arises if we consider the fractal Complex Dimensions as dynamical quantities, which evolve with the scales: to each prefractal approximation $\Gamma_{\mathcal{W}_m}$ of $\Gamma_{\mathcal{W}}$, we can associate specific values of the Complex Dimensions, as is proved in [DL22d]. In this light, it is natural to define the Complex Dimensions of the Weierstrass Curve as the set of the Complex Dimensions of the Weierstrass IFD.

In [DL22b], we have introduced a specific polyhedral measure, by means of a polygonal neighborhood of the Weierstrass Curve $\Gamma_{\mathcal{W}}$. It so happens that the sequence of polygonal neighborhoods and the sequence of tubular neighborhoods associated with the Weierstrass IFD are nested: given any integer $m \in \mathbb{N}$, there exists $m_1 \in \mathbb{N}$ such that, for all $k \geqslant m_1$, the polygonal neighborhood of the m^{th} prefractal approximation contains, but for a finite number of wedges, the $(m+k, \varepsilon_{m+k}^{m+k})$–neighborhood. On the other hand, there also exists $m_2 \in \mathbb{N}$ such that, for all $k \geqslant m_1$, the polygonal neighborhood of the m^{th} tubular neighborhood contains the $(m+k, \varepsilon_{m+k}^{m+k})$ polygonal neighborhood. In light of this, it was natural to consider the fractal zeta functions associated with the sequence of polygonal neighborhoods, instead of the fractal zeta functions associated with the sequence of tubular neighborhoods. For this sequence, as will be shown, in the present paper, not only do we dispose of the exact expression of the polygonal neighborhood, at a given step of the prefractal approximation, but we also have the recurrence relation between consecutive fractal zeta functions. This, in particular, enables us to prove that the limit fractal zeta function – the one associated with the limit fractal object – has the same (possible) poles as the fractal zeta function at a given step of the prefractal approximation, and, hence, that the Complex Dimensions of the fractal are exactly the same as the Complex Dimensions of any sufficiently good prefractal approximation.

We point out that our result is stronger than the one obtained in [DL22c], where the values of the possible Complex Dimensions of the Weierstrass IFD included -2. While such a result can seem intriguing, it simply comes from the fact that tubular neighborhoods involve wedges, whose contribution leads to the occurence of -2 as a possible pole of the associated local fractal effective tube zeta function. Indeed, as we will see, polyhedral neighborhoods, which are the natural neighborhoods adapted to fractal curves, do not lead to this value as a possible pole of the associated local fractal effective polyhedral zeta function. More precisely, since the volume of the polyhedral neighborhood of each prefractal approximation can be expressed by means of the Weierstrass function, the possible Complex Dimensions are exactly the same as the Complex Dimensions of the Weierstrass function

itself, as obtained in [DL22d].

Our main results in the present setting can be found in the following places:

i. In Definition 3.4, on page 23, where we introduce the sequence of polyhedral neighborhoods.

ii. In Property 3.11, on page 26, where we show that the polyhedral and tubular neighborhoods are nested.

iii. In Property 3.13, on page 32, where we give the exact expression of the m^{th}–polygonal neighborhood.

iv. In Theorem 3.16, on page 40, where we introduce *the local and global effective polyhedral zeta function.*

v. In Theorem 4.1, on page 48, where we give a new proof of the computation of the box–counting dimension (or, equivalently, of the Minkowski dimension) of the Weierstrass Curve $\Gamma_{\mathcal{W}}$, by simply using the covers of $\Gamma_{\mathcal{W}}$ by polyhedral neighborhoods.

2 Geometry of the Weierstrass Curve

We begin by reviewing the main geometric properties of the Weierstrass Curve (and of the associated IFD), which will be key to our work in the rest of this paper.

Henceforth, we place ourselves in the Euclidean plane, equipped with a direct orthonormal frame. The usual Cartesian coordinates are denoted by (x, y). The horizontal and vertical axes will be respectively referred to as $(x'x)$ and $(y'y)$.

Notation 1 (**Set of all Natural Numbers, and Intervals**)**.**

As in Bourbaki [Bou04] (Appendix E. 143), we denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of all natural numbers, and set $\mathbb{N}^\star = \mathbb{N} \setminus \{0\}$.

Given a, b with $-\infty \leqslant a \leqslant b \leqslant \infty$, $]a, b[= (a, b)$ denotes an open interval, while, for example, $]a, b] = (a, b]$ denotes a half-open, half-closed interval.

Notation 2 (**Wave Inequality Symbol**)**.**

Given two positive numbers a and b, we will use the notation $a \lesssim b$ when there exists a strictly positive constant C such that $a \leqslant C\, b$.

Notation 3 (**Weierstrass Parameters**)**.**

In the sequel, λ and N_b are two real numbers such that

$$0 < \lambda < 1 \quad , \quad N_b \in \mathbb{N}^\star \quad \text{and} \quad \lambda\, N_b > 1 \cdot \qquad (\clubsuit) \tag{$\mathcal{R}\,1$}$$

Note that this implies that $N_b > 1$; i.e., $N_b \geqslant 2$, if $N_b \in \mathbb{N}^\star$, as will be the case in this paper.

As explained in [Dav19], we deliberately made the choice to introduce the notation N_b which replaces the initial number b, in so far as, in Hardy's paper [Har16] (in contrast to Weierstrass' original article [Wei75]), b is any positive real number satisfying $\lambda\, b > 1$, whereas we deal here with the specific case of a natural integer, which accounts for the natural notation N_b.

Definition 2.1 (**Weierstrass Function, Weierstrass Curve**).

We consider the *Weierstrass function* $\mathcal{W}$ (also called, in short, the $\mathcal{W}$-*function*) defined, for any real number x, by

$$\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right) \cdot \tag{$\mathcal{R}\,2$}$$

We call the associated graph the *Weierstrass Curve*, and denote it by $\Gamma_{\mathcal{W}}$.

Due to the one–periodicity of the Weierstrass function (since $N_b \in \mathbb{N}^\star$), from now on, and without loss of generality, we restrict our study to the interval $[0,1[\,=[0,1)$. Note that $\mathcal{W}$ is continuous, and hence, bounded on all of $\mathbb{R}$. In particular, $\Gamma_{\mathcal{W}}$ is a (nonempty) compact subset of $\mathbb{R}^2$.

Property 2.1 (**Scaling Properties of the Weierstrass Function, and Consequences [DL22c]**).

Since, for any real number x, $\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right)$, *one also has*

$$\mathcal{W}(N_b\, x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^{n+1}\, x\right) = \frac{1}{\lambda} \sum_{n=1}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right) = \frac{1}{\lambda}\,\left(\mathcal{W}(x) - \cos\left(2\,\pi\, x\right)\right) ,$$

which yields, for any strictly positive integer m *and any* j *in* $\{0, \ldots, \#V_m\}$,

$$\mathcal{W}\left(\frac{j}{(N_b-1)\, N_b^m}\right) = \lambda\, \mathcal{W}\left(\frac{j}{(N_b-1)\, N_b^{m-1}}\right) + \cos\left(\frac{2\,\pi\, j}{(N_b-1)\, N_b^m}\right) .$$

By induction, one then obtains that

$$\mathcal{W}\left(\frac{j}{(N_b-1)\, N_b^m}\right) = \lambda^m\, \mathcal{W}\left(\frac{j}{(N_b-1)}\right) + \sum_{k=0}^{m-1} \lambda^k \cos\left(\frac{2\,\pi\, N_b^k\, j}{(N_b-1)\, N_b^m}\right) .$$

Definition 2.2 (**Weierstrass Complexified Function**).

We introduce the *Weierstrass Complexified function* $\mathcal{W}_{comp}$, defined, for any real number x, by

$$\mathcal{W}_{comp}(x) = \sum_{n=0}^{\infty} \lambda^n \, e^{2\,i\,\pi\,N_b^n\,x} \, .$$

Clearly, $\mathcal{W}_{comp}$ is also a continuous and 1-periodic function on $\mathbb{R}$.

Notation 4 (**Logarithm**)**.**

Given $y > 0$, $\ln y$ denotes the natural logarithm of y, while, given $a > 0$, $a \neq 1$, $\ln_a y = \dfrac{\ln y}{\ln a}$ denotes the logarithm of y in base a; so that, in particular, $\ln = \ln_e$.

Notation 5 (**Minkowski Dimension and Hölder Exponent**)**.**

For the parameters λ and N_b satisfying condition (♣) (see Notation 3, on page 3), we denote by

$$D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln N_b} = 2 - \ln_{N_b} \frac{1}{\lambda} \in \,]1, 2[$$

the box–counting dimension (or Minkowski dimension) of the Weierstrass Curve $\Gamma_{\mathcal{W}}$, which happens to be equal to its Hausdorff dimension [KMPY84], [BBR14], [She18], [Kel17]. We point out that the results in our previous paper [DL22c], combined with those obtained in the present paper in Section 4 below, will also provide a direct geometric proof of the fact that $D_{\mathcal{W}}$, the Minkowski dimension (or box–counting dimension) of $\Gamma_{\mathcal{W}}$, exists and takes the above values, as well as of the fact that $\mathcal{W}$ is Hölder continuous with *optimal* Hölder exponent

$$2 - D_{\mathcal{W}} = -\frac{\ln \lambda}{\ln N_b} = \ln_{N_b} \frac{1}{\lambda} \, .$$

Convention (**The Weierstrass Curve as a Cyclic Curve**)**.**

In the sequel, we identify the points $(0, \mathcal{W}(0))$ and $(1, \mathcal{W}(1)) = (1, \mathcal{W}(0))$. This is justified by the fact that the Weierstrass function $\mathcal{W}$ is 1–periodic, since N_b is an integer.

Remark 2.1. The above convention makes sense, because, in addition to the periodicity property of the $\mathcal{W}$-function, the points $(0, \mathcal{W}(0))$ and $(1, \mathcal{W}(1))$ have the same vertical coordinate.

Property 2.2 (**Symmetry with Respect to the Vertical Line $x = \dfrac{1}{2}$**)**.**

Since, for any $x \in [0, 1]$*,*

$$\mathcal{W}(1 - x) = \sum_{n=0}^{\infty} \lambda^n \, \cos\left(2\,\pi\,N_b^n - 2\,\pi\,N_b^n\,x\right) = \mathcal{W}(x) \, ,$$

the Weierstrass Curve is symmetric with respect to the vertical straight line $x = \dfrac{1}{2}$*.*

Proposition 2.3 (**Nonlinear and Noncontractive Iterated Function System (IFS)**).

Following our previous work [Dav18]*, we approximate the restriction* $\Gamma_{\mathcal{W}}$ *to* $[0,1[\times\mathbb{R}$*, of the Weierstrass Curve, by a sequence of finite graphs, built via an iterative process. For this purpose, we use the nonlinear iterated function system (IFS) consisting of a finite family of* C^{∞} *bijective maps from* $\mathbb{R}^2$ *to* $\mathbb{R}^2$ *and denoted by*

$$\mathcal{T}_{\mathcal{W}} = \left\{T_0, \ldots, T_{N_b-1}\right\},$$

where, for any integer i *belonging to* $\{0, \ldots, N_b - 1\}$ *and any point* (x, y) *of* $\mathbb{R}^2$*,*

$$T_i(x,y) = \left(\frac{x+i}{N_b}, \lambda\, y + \cos\left(2\,\pi\,\left(\frac{x+i}{N_b}\right)\right)\right).$$

Note that unlike in the classical situation, these maps are not contractions. Nevertheless, $\Gamma_{\mathcal{W}}$ *can be recovered from this IFS in the usual way, as we next explain.*

Property 2.4 (**Attractor of the IFS [Dav18], [Dav19]**).

The Weierstrass Curve $\Gamma_{\mathcal{W}}$ *is the attractor of the IFS* $\mathcal{T}_{\mathcal{W}}$*, and hence, is the unique nonempty compact subset* $\mathcal{K}$ *of* $\mathbb{R}^2$ *satisfying* $\mathcal{K} = \bigcup_{i=0}^{N_b-1} T_i(\mathcal{K})$*; in particular, we have that* $\Gamma_{\mathcal{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}})$*.*

Notation 6 (**Fixed Points**).

For any integer i belonging to $\{0, \ldots, N_b - 1\}$, we denote by

$$P_i = (x_i, y_i) = \left(\frac{i}{N_b - 1}, \frac{1}{1-\lambda}\,\cos\left(\frac{2\,\pi\,i}{N_b - 1}\right)\right)$$

the unique fixed point of the map T_i; see [Dav19].

Definition 2.3 (**Sets of Vertices, Prefractals**).

We denote by V_0 the ordered set (according to increasing abscissae) of the points

$$\left\{P_0, \ldots, P_{N_b-1}\right\}.$$

The set of points V_0 – where, for any integer i in $\{0, \ldots, N_b - 2\}$, the point P_i is linked to the point P_{i+1} – constitutes an oriented finite graph, ordered according to increasing abscissae, which we will denote by $\Gamma_{\mathcal{W}_0}$. Then, V_0 is called *the set of vertices* of the graph $\Gamma_{\mathcal{W}_0}$.

For any nonnegative integer m, i.e., for $m \in \mathbb{N}$, we set $V_m = \bigcup_{i=0}^{N_b-1} T_i\,(V_{m-1})$.

The set of points V_m, where two consecutive points are linked, is an oriented finite graph, ordered according to increasing abscissae, called the $\boldsymbol{m^{th}}$ **order** $\boldsymbol{\mathcal{W}}$**-prefractal**. Then, V_m is called *the set of vertices* of the prefractal $\Gamma_{\mathcal{W}_m}$; see Figure 2, on page 10.

Property 2.5 (**Density of the Set $V^\star = \bigcup_{n\in\mathbb{N}} V_n$ in the Weierstrass Curve [DL22c]**).

The set $V^\star = \bigcup_{n\in\mathbb{N}} V_n$ is dense in the Weierstrass Curve $\Gamma_{\mathcal{W}}$.

Definition 2.4 (**Adjacent Vertices, Edge Relation**).

For any $m \in \mathbb{N}$, the prefractal graph $\Gamma_{\mathcal{W}_m}$ is equipped with an edge relation $\underset{m}{\sim}$, as follows: two vertices X and Y of $\Gamma_{\mathcal{W}_m}$ (i.e., two points belonging to V_m) will be said to be *adjacent* (i.e., *neighboring* or *junction points*) if and only if the line segment $[X, Y]$ is an edge of $\Gamma_{\mathcal{W}_m}$; we then write $X \underset{m}{\sim} Y$. Note that this edge relation depends on m, which means that points adjacent in V_m might not remain adjacent in V_{m+1}.

We refer to part *iv.* of Property 2.6, along with Figure 1, for the definition of the polygons $\mathcal{P}_{m,k}$ and $\mathcal{Q}_{m,k}$ associated with the Weierstrass Curve.

Property 2.6. [Dav18] *For any $m \in \mathbb{N}$, the following statements hold*:

i. $V_m \subset V_{m+1}$.

ii. $\#V_m = (N_b - 1)\, N_b^m + 1$, *where $\#V_m$ denotes the number of elements in the finite set V_m.*

iii. The prefractal graph $\Gamma_{\mathcal{W}_m}$ has exactly $(N_b - 1)\, N_b^m$ edges.

iv. The consecutive vertices of the prefractal graph $\Gamma_{\mathcal{W}_m}$ are the vertices of N_b^m simple nonregular polygons $\mathcal{P}_{m,k}$ with N_b sides. For any strictly positive integer m, the junction point between two consecutive polygons $\mathcal{P}_{m,k}$ and $\mathcal{P}_{m,k+1}$ is the point

$$\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}\right)\right) \quad , \quad 1 \leqslant k \leqslant N_b^m - 1\,.$$

Hence, the total number of junction points is $N_b^m - 1$. For instance, in the case $N_b = 3$, the polygons are all triangles; see Figure 1, on page 9.

We call extreme vertices *of the polygon $\mathcal{P}_{m,k}$ the junction points*

$$\mathcal{V}_{initial}\left(\mathcal{P}_{m,k}\right) = \left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}\right)\right) \quad , \quad 0 \leqslant k \leqslant N_b^m - 1\,,$$

and

$$\mathcal{V}_{end}\left(\mathcal{P}_{m,k}\right) = \left(\frac{(N_b-1)\,(k+1)}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,(k+1)}{(N_b-1)\,N_b^m}\right)\right) \quad , \quad 0 \leqslant k \leqslant N_b^m - 2\,.$$

In the sequel, we will denote by $\mathcal{P}_0$ ***the initial polygon****, whose vertices are the fixed points of the maps T_i, $0 \leqslant i \leqslant N_b - 1$, introduced in Notation 6, on page 6 and Definition 2.3, on page 6,*

i.e., $\{P_0, \ldots, P_{N_b-1}\}$*; see, again, Figure 1, on page 9.*

In the same way, the consecutive vertices of the prefractal graph $\Gamma_{\mathcal{W}_m}$*, distinct from the fixed points* P_0 *and* P_{N_b-1} *(see Notation 6, on page 6), are also the vertices of* $N_b^m - 1$ *simple nonregular polygons* $\mathcal{Q}_{m,j}$*, for* $1 \leqslant j \leqslant N_b^m - 2$*, again with* N_b *sides. For any integer* j *such that* $1 \leqslant j \leqslant N_b^m - 2$*, one obtains each polygon* $\mathcal{Q}_{m,j}$ *by connecting the point number* j *(i.e., with the notation of Property 2.6, on page 10, the vertex* $M_{j,m}$*) to the point number* $j+1$ *(i.e., the vertex* $M_{j+1,m}$*) if* $j \equiv i \bmod N_b$*, for* $1 \leqslant i \leqslant N_b - 1$*, and the point number* j *to the point number* $j - N_b + 1$ *if* $j \equiv 0 \bmod N_b$*.*

As previously, we call extreme vertices *of the polygon* $\mathcal{Q}_{m,k}$ *the junction points*

$$\mathcal{V}_{initial}\left(\mathcal{Q}_{m,k}\right) = \left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}\right)\right) \quad , \quad 1 \leqslant k \leqslant N_b^m - 1\,,$$

and

$$\mathcal{V}_{end}\left(\mathcal{Q}_{m,k}\right) = \left(\frac{(N_b-1)\,(k+1)}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,(k+1)}{(N_b-1)\,N_b^m}\right)\right) \quad , \quad 1 \leqslant k \leqslant N_b^m - 2\,.$$

Definition 2.5 (**Polygonal Sets**)**.**

For any $m \in \mathbb{N}$, we introduce the following *polygonal sets*

$$\mathcal{P}_m = \left\{\mathcal{P}_{m,k}\,,\, 0 \leqslant k \leqslant N_b^m - 1\right\} \quad \text{and} \quad \mathcal{Q}_m = \left\{\mathcal{Q}_{m,k}\,,\, 0 \leqslant k \leqslant N_b^m - 2\right\}\,.$$

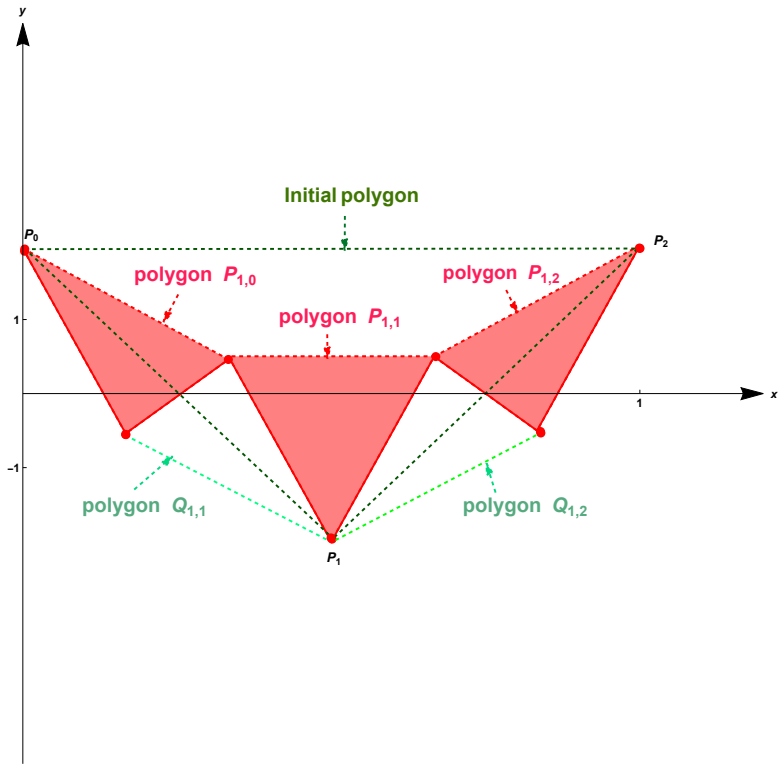

Figure 1: **The initial polygon $\mathcal{P}_0$, and the respective polygons $\mathcal{P}_{0,1}$, $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, $\mathcal{Q}_{1,1}$, $\mathcal{Q}_{1,2}$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$. (See also Figure 2, on page 10.)**

Notation 7. For any $m \in \mathbb{N}$, we denote by:

ii. $X \in \mathcal{P}_m$ (resp., $X \in \mathcal{Q}_m$) a vertex of a polygon $\mathcal{P}_{m,k}$, with $0 \leqslant k \leqslant N_b^m - 1$ (resp., a vertex of a polygon $\mathcal{Q}_{m,k}$, with $1 \leqslant k \leqslant N_b^m - 2$).

ii. $\mathcal{P}_m \bigcup \mathcal{Q}_m$ the reunion of the polygonal sets $\mathcal{P}_m$ and $\mathcal{Q}_m$, which consists in the set of all the vertices of the polygons $\mathcal{P}_{m,k}$, with $0 \leqslant k \leqslant N_b^m - 1$, along with the vertices of the polygons $\mathcal{Q}_{m,k}$, with $1 \leqslant k \leqslant N_b^m - 2$. In particular, $X \in \mathcal{P}_m \bigcup \mathcal{Q}_m$ simply denotes a vertex in $\mathcal{P}_m$ or $\mathcal{Q}_m$.

iii. $\mathcal{P}_m \bigcap \mathcal{Q}_m$ the intersection of the polygonal sets $\mathcal{P}_m$ and $\mathcal{Q}_m$, which consists in the set of all the vertices of both a polygon $\mathcal{P}_{m,k}$, with $0 \leqslant k \leqslant N_b^m - 1$, and a polygon $\mathcal{Q}_{m,k'}$, with $1 \leqslant' k \leqslant N_b^m - 2$.

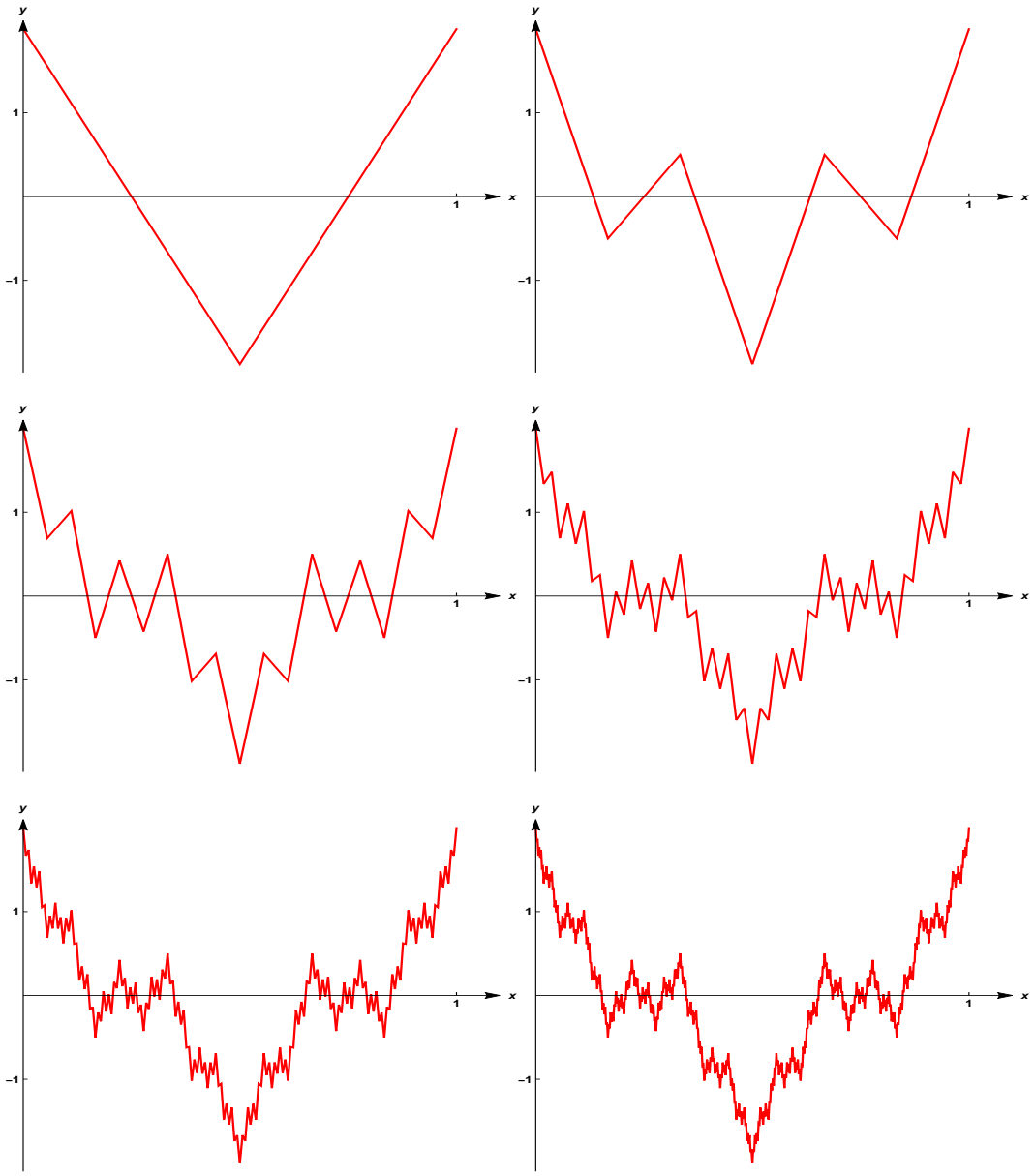

Figure 2: **The prefractal graphs $\Gamma_{\mathcal{W}_0}$, $\Gamma_{\mathcal{W}_1}$, $\Gamma_{\mathcal{W}_2}$, $\Gamma_{\mathcal{W}_3}$, $\Gamma_{\mathcal{W}_4}$, $\Gamma_{\mathcal{W}_5}$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$. For example, $\Gamma_{\mathcal{W}_1}$ is on the right side of the top row, while $\Gamma_{\mathcal{W}_4}$ is on the left side of the bottom row.**

Definition 2.6 (**Vertices of the Prefractals, Elementary Lengths, Heights and Angles**).

Given a strictly positive integer m, we denote by $\left(M_{j,m}\right)_{0 \leqslant j \leqslant (N_b-1)\, N_b^m}$ *the set of vertices* of the prefractal graph $\Gamma_{\mathcal{W}_m}$. One thus has, for any integer j in $\left\{0, \ldots, (N_b-1)\, N_b^m\right\}$:

$$M_{j,m} = \left(\frac{j}{(N_b-1)\, N_b^m}, \mathcal{W}\left(\frac{j}{(N_b-1)\, N_b^m} \right) \right).$$

We also introduce, for any integer j in $\left\{0, \ldots, (N_b-1)\, N_b^m - 1\right\}$:

i. the elementary horizontal lengths:

$$L_m = \frac{j}{(N_b - 1)\, N_b^m}\,; \tag{$\mathcal{R}\,3$}$$

ii. the elementary lengths:

$$\ell_{j,j+1,m} = d\big(M_{j,m}, M_{j+1,m}\big) = \sqrt{L_m^2 + h_{j,j+1,m}^2}\,,$$

where $h_{j,j+1,m}$ is defined in *iii.* just below.

iii. the elementary heights:

$$h_{j,j+1,m} = \left|\mathcal{W}\left(\frac{j+1}{(N_b - 1)\, N_b^m}\right) - \mathcal{W}\left(\frac{j}{(N_b - 1)\, N_b^m}\right)\right|\,;$$

iv. the minimal height:

$$h_m^{inf} = \inf_{0 \leqslant j \leqslant (N_b - 1)\, N_b^m - 1} h_{j,j+1,m}\,, \tag{$\mathcal{R}\,4$}$$

along with the the maximal height:

$$h_m = \sup_{0 \leqslant j \leqslant (N_b - 1)\, N_b^m - 1} h_{j,j+1,m}\,, \tag{$\mathcal{R}\,5$}$$

v. the geometric angles:

$$\theta_{j-1,j,m} = \big((y'y), (\widehat{M_{j-1,m}M_{j,m}})\big) \quad , \quad \theta_{j,j+1,m} = \big((y'y), (\widehat{M_{j,m}M_{j+1,m}})\big)\,,$$

which yield **the value of the geometric angle between consecutive edges**, namely, $\big[M_{j-1,m}\, M_{j,m}, M_{j,m}\, M_{j+1,m}\big]$:

$$\theta_{j-1,j,m} + \theta_{j,j+1,m} = \arctan \frac{L_m}{\big|h_{j-1,j,m}\big|} + \arctan \frac{L_m}{\big|h_{j,j+1,m}\big|}\,.$$

Property 2.7. *For the geometric angle $\theta_{j-1,j,m}$, $0 \leqslant j \leqslant (N_b - 1)\, N_b^m$, $m \in \mathbb{N}$, we have the following relation:*

$$\tan \theta_{j-1,j,m} = \frac{h_{j-1,j,m}}{L_m}\,.$$

Property 2.8 (A Consequence of the Symmetry with Respect to the Vertical Line $x = \frac{1}{2}$).

For any strictly positive integer m and any j in $\{0, \ldots, \#V_m\}$, we have that

$$\mathcal{W}\left(\frac{j}{(N_b - 1)\, N_b^m}\right) = \mathcal{W}\left(\frac{(N_b - 1)\, N_b^m - j}{(N_b - 1)\, N_b^m}\right)\,,$$

which means that the points

$$\left(\frac{(N_b-1)\,N_b^m-j}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{(N_b-1)\,N_b^m-j}{(N_b-1)\,N_b^m}\right)\right) \quad \textit{and} \quad \left(\frac{j}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right)$$

are symmetric with respect to the vertical line $x = \dfrac{1}{2}$*; see Figure 3, on page 12.*

Definition 2.7 (**Left–Side and Right–Side Vertices**).

Given natural integers m, k such that $0 \leqslant k \leqslant N_b^m - 1$, and a polygon $\mathcal{P}_{m,k}$, we define:

i. The set of its *left–side vertices* as the set of the first $\left[\dfrac{N_b-1}{2}\right]$ vertices, where $[y]$ denotes the integer part of the real number y.

ii. The set of its *right–side vertices* as the set of the last $\left[\dfrac{N_b-1}{2}\right]$ vertices.

When the integer N_b is odd, we define the bottom vertex as the $\left(\dfrac{N_b-1}{2}\right)^{th}$ one; see Figure 4, on page 13.

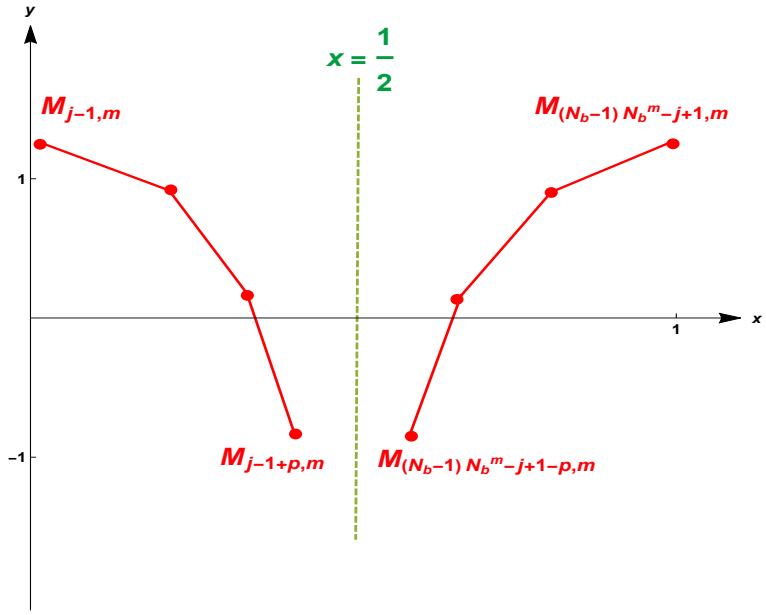

Figure 3: **Symmetric points with respect to the vertical line** $x = \dfrac{1}{2}$**.**

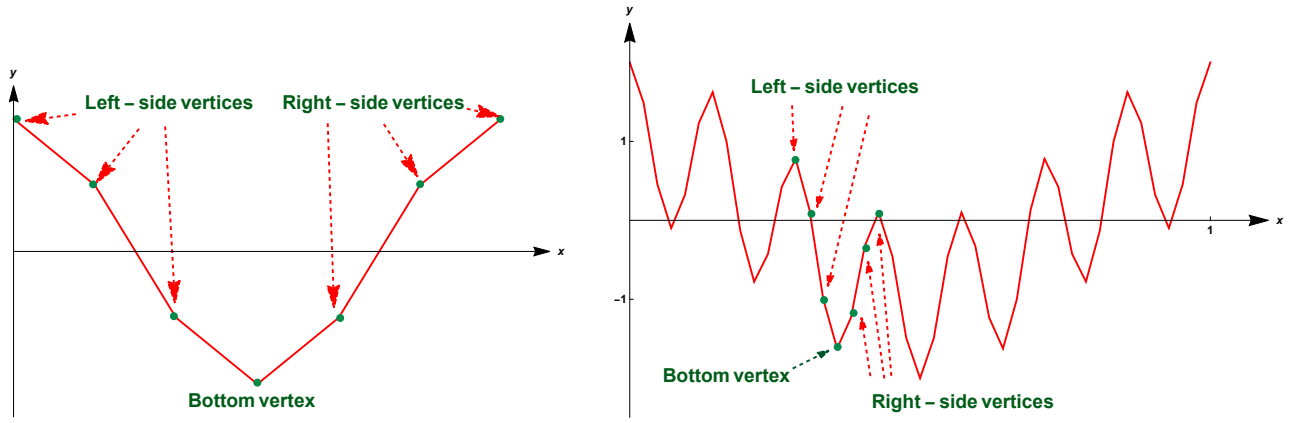

Figure 4: **The Left and Right–Side Vertices.**

Property 2.9 ([DL22c]).

For any integer j in $\{0, \ldots, N_b - 1\}$:

$$\mathcal{W}\left(\frac{j}{N_b - 1}\right) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, \frac{j}{(N_b - 1)}\right) = \sum_{n=0}^{\infty} \lambda^n \cos\left(\frac{2\,\pi\, j}{N_b - 1}\right) = \frac{1}{1-\lambda} \cos\left(\frac{2\,\pi\, j}{N_b - 1}\right).$$

Property 2.10 ([DL22c]).

For $0 \leqslant j \leqslant \dfrac{(N_b - 1)}{2}$ (resp., for $\dfrac{(N_b - 1)}{2} \leqslant j \leqslant N_b - 1$), we have that

$$\mathcal{W}\left(\frac{j+1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \leqslant 0 \quad \left(\textit{resp.,}\ \mathcal{W}\left(\frac{j+1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \geqslant 0\right).$$

Notation 8 (**Signum Function**).

The *signum function* of a real number x is defined by

$$\text{sgn}\,(x) = \begin{cases} -1, & \text{if} \quad x < 0, \\ 0, & \text{if} \quad x = 0, \\ +1, & \text{if} \quad x > 0\,. \end{cases}$$

Property 2.11 ([DL22c]).

Given any strictly positive integer m, we have the following properties:

i. For any j in $\{0, \ldots, \#V_m\}$, the point

$$\left(\frac{j}{(N_b - 1)\, N_b^m}, \mathcal{W}\left(\frac{j}{(N_b - 1)\, N_b^m}\right)\right)$$

is the image of the point

$$\left(\frac{j}{(N_b-1)\,N_b^{m-1}}-i,\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}}-i\right)\right)=\left(\frac{j-i\,(N_b-1)\,N_b^{m-1}}{(N_b-1)\,N_b^{m-1}},\mathcal{W}\left(\frac{j-i\,(N_b-1)\,N_b^{m-1}}{(N_b-1)\,N_b^{m-1}}\right)\right)$$

under the map T_i, where $i \in \{0,\ldots,N_b-1\}$ is arbitrary.

Consequently, for $0 \leqslant j \leqslant N_b-1$, ***the j^{th} vertex of the polygon*** *$\mathcal{P}_{m,k}$, $0 \leqslant k \leqslant N_b^m-1$, i.e., the point*

$$\left(\frac{(N_b-1)\,k+j}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{(N_b-1)\,k+j}{(N_b-1)\,N_b^m}\right)\right),$$

is the image of the point

$$\left(\frac{(N_b-1)\left(k-i\,(N_b-1)\,N_b^{m-1}\right)+j}{(N_b-1)\,N_b^{m-1}},\mathcal{W}\left(\frac{(N_b-1)\left(k-i\,(N_b-1)\,N_b^{m-1}\right)+j}{(N_b-1)\,N_b^{m-1}}\right)\right)$$

under the map T_i, where $i \in \{0,\ldots,N_b-1\}$ is again arbitrary. It is also ***the j^{th} vertex of the polygon*** *$\mathcal{P}_{m-1,k-i\,(N_b-1)\,N_b^{m-1}}$. Therefore, there is an exact correspondance between vertices of the polygons at consecutive steps $m-1$, m.*

ii. Given j in $\{0,\ldots,N_b-2\}$ and k in $\left\{0,\ldots,N_b^m-1\right\}$, we have that

$$sgn\left(\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right)-\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right)=sgn\left(\mathcal{W}\left(\frac{j+1}{N_b-1}\right)-\mathcal{W}\left(\frac{j}{N_b-1}\right)\right).$$

Proof.

i. Given $m \in \mathbb{N}^\star$, let us consider $i \in \{0,\ldots,N_b-1\}$. The image of the point

$$\left(\frac{j}{(N_b-1)\,N_b^{m-1}}-i,\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}}-i\right)\right)$$

under the map T_i is obtained by applying the analytic expression given in Property 2.3, on page 6, to the coordinates of this point, which, thanks to Property 2.1, on page 4 above, yields the expected result, namely,

$$\left(\begin{array}{c} \dfrac{j}{(N_b-1)\,N_b^m}, \lambda \underbrace{\mathcal{W}\left(\dfrac{j}{(N_b-1)\,N_b^{m-1}} - i\right)}_{\mathcal{W}\left(\dfrac{j}{(N_b-1)\,N_b^{m-1}}\right)} + \cos\dfrac{2\,\pi\,j}{(N_b-1)\,N_b^m} \\ \text{(by 1–periodicity)} \end{array} \right) = \left(\dfrac{j}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\dfrac{j}{(N_b-1)\,N_b^m}\right)\right).$$

ii. See [DL22c].

□

Property 2.12 (**Lower Bound and Upper Bound for the Elementary Heights [DL22c]**).

For any strictly positive integer m and any j in $\{0,\ldots,(N_b-1)\,N_b^m\}$, we have the following estimates, where L_m is the elementary horizontal length introduced in part i. of Definition 2.6, on page 10:

$$C_{inf}\,L_m^{2-D_{\mathcal{W}}} \leqslant \underbrace{\left|\mathcal{W}\left((j+1)\,L_m\right) - \mathcal{W}\left(j\,L_m\right)\right|}_{h_{j,j+1,m}} \leqslant C_{sup}\,L_m^{2-D_{\mathcal{W}}} \quad , \quad m \in \mathbb{N},\, 0 \leqslant j \leqslant (N_b-1)\,N_b^m, \quad (✠)$$

where the finite and positive constants C_{inf} and C_{sup} are given by

$$C_{inf} = (N_b-1)^{2-D_{\mathcal{W}}} \min_{0\leqslant j\leqslant N_b-1} \left|\mathcal{W}\left(\frac{j+1}{N_b-1}\right) - \mathcal{W}\left(\frac{j}{N_b-1}\right)\right|$$

and

$$C_{sup} = (N_b-1)^{2-D_{\mathcal{W}}} \left(\max_{0\leqslant j\leqslant N_b-1} \left|\mathcal{W}\left(\frac{j+1}{N_b-1}\right) - \mathcal{W}\left(\frac{j}{N_b-1}\right)\right| + \frac{2\,\pi}{(N_b-1)\,(\lambda\,N_b-1)}\right).$$

One should note, in addition, that these constants C_{inf} and C_{sup} depend on the initial polygon $\mathcal{P}_0$.

As a consequence, we also have that

$$C_{inf}\,L_m^{2-D_{\mathcal{W}}} \leqslant h_m^{inf} \leqslant C_{sup}\,L_m^{2-D_{\mathcal{W}}} \quad \text{and} \quad C_{inf}\,L_m^{2-D_{\mathcal{W}}} \leqslant h_m \leqslant C_{sup}\,L_m^{2-D_{\mathcal{W}}},$$

where h_m^{inf} and h_m respectively denote the minimal and maximal heights introduced in part iv. of Definition 2.6, on page 10.

Theorem 2.13 (**Sharp Local Discrete Reverse Hölder Properties of the Weierstrass Function [DL22c]**).

For any natural integer m (i.e., for any $m \in \mathbb{N}$), let us consider a pair of real numbers (x, x') such that

$$x = \frac{(N_b - 1)\, k + j}{(N_b - 1)\, N_b^m} = ((N_b - 1)\, k + j)\; L_m \quad , \quad x' = \frac{(N_b - 1)\, k + j + \ell}{(N_b - 1)\, N_b^m} = ((N_b - 1)\, k + j + \ell)\; L_m \, ,$$

where $0 \leqslant k \leqslant N_b - 1^m - 1$. We then have the following (discrete, local) reverse-Hölder inequality, *with sharp Hölder exponent $-\dfrac{\ln \lambda}{\ln N_b} = 2 - D_{\mathcal{W}}$:*

$$C_{inf}\, |x' - x|^{2 - D_{\mathcal{W}}} \leqslant \left| \mathcal{W}(x') - \mathcal{W}(x) \right| \, ,$$

where $(x, \mathcal{W}(x))$ and $\left(x', \mathcal{W}(x')\right)$ are adjacent vertices of the same m^{th} prefractal approximation, $\Gamma_{\mathcal{W}_m}$, with $m \in \mathbb{N}$ arbitrary. Here, C_{inf} is given as in Property 2.12, on page 15 just above.

Corollary 2.14 (**Optimal Hölder Exponent for the Weierstrass Function (see [DL22c])**).

The local reverse Hölder property of Theorem 2.13, on page 16 just above – in conjunction with the Hölder condition satisfied by the Weierstrass function (see also [Zyg02], Chapter II, Theorem 4.9, page 47) – shows that the Codimension $2 - D_{\mathcal{W}} = -\dfrac{\ln \lambda}{\ln N_b} \in]0, 1[$ is the best (i.e., optimal) Hölder exponent for the Weierstrass function (as was originally shown, by a completely different method, by G. H. Hardy in [Har16]).

Note that, as a consequence, since the Hölder exponent is strictly smaller than one, it follows that the Weierstrass function $\mathcal{W}$ is nowhere differentiable.

Corollary 2.15 (**of Property 2.12 (see [DL22c])**).

Thanks to Property 2.12, on page 15, one may now write, for any strictly positive integer m and any integer j in $\left\{0, \ldots, (N_b - 1)\, N_b^m - 1\right\}$, and with C_{inf} and C_{supf} defined as in Property 2.12, on page 15:

i. <u>for the elementary heights:</u>

$$h_{j-1,j,m} = L_m^{2 - D_{\mathcal{W}}}\, \mathcal{O}(1) \; ; \tag{$\mathcal{R}$ 6}$$

ii. *<u>for the elementary quotients:</u>*

$$\frac{h_{j-1,j,m}}{L_m} = L_m^{1 - D_{\mathcal{W}}}\, \mathcal{O}(1) \quad , \tag{$\mathcal{R}$ 7}$$

and where

$$0 < C_{inf} \leqslant \mathcal{O}(1) \leqslant C_{sup} < \infty \, .$$

Corollary 2.16 (**Nonincreasing Sequence of Geometric Angles** (**Coming from Property** 2.11; see [DL22c])).

For the ***geometric angles*** $\theta_{j-1,j,m}$, $0 \leqslant j \leqslant (N_b - 1)\, N_b^m$, $m \in \mathbb{N}$, *introduced in part v. of Definition 2.6, on page 10, we have the following result:*

$$\tan\theta_{j-1,j,m} = \frac{L_m}{h_{j-1,j,m}}(N_b - 1) > \tan\theta_{j-1,j,m+1}\,,$$

which yields

$$\theta_{j-1,j,m} > \theta_{j-1,j,m+1} \quad \textit{and} \quad \theta_{j-1,j,m+1} \lesssim L_m^{D_{\mathcal{W}}-1}\,.$$

Corollary 2.17 (**Local Extrema** (**Coming from Property** 2.11, on page 13; see [DL22c])).

i. The set of local maxima of the Weierstrass function on the interval $[0,1]$ *is given by*

$$\left\{\left(\frac{(N_b-1)\,k}{N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k}{N_b^m}\right)\right) : 0 \leqslant k \leqslant N_b^m - 1,\, m \in \mathbb{N}\right\},$$

and corresponds to the extreme vertices of the polygons $\mathcal{P}_{m,k}$ *and* $\mathcal{Q}_{m,k}$ *(see Property 2.6, on page 7) at a given step* m *(i.e., they are the vertices connecting consecutive polygons; see part iv. of Property 2.6, on page 7).*

ii. For odd values of N_b, *the set of local minima of the Weierstrass function on the interval* $[0,1]$ *is given by*

$$\left\{\left(\frac{(N_b-1)\,k + \frac{N_b-1}{2}}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k + \frac{N_b-1}{2}}{(N_b-1)\,N_b^m}\right)\right) : 0 \leqslant k \leqslant N_b^m - 1,\, m \in \mathbb{N}\right\},$$

and corresponds to the bottom vertices of the polygons $\mathcal{P}_{m,k}$ *and* $\mathcal{Q}_{m,k}$ *at a given step* m*; see also part iv. of Property 2.6, on page 7.*

Property 2.18 (**Existence of Reentrant Angles [DL22c]**).

i. The initial polygon $\mathcal{P}_0$, *admits* ***reentrant interior angles***, *at a vertex* P_j, *with* $0 < j \leqslant N_b - 1$, *in the sense that, with the right-hand rule, according to which angles are measured in a counterclockwise direction* $\left(\widehat{(P_jP_{j+1})}, \widehat{(P_jP_{j-1})}\right) > \pi$, *in the case when*

$$0 < j \leqslant \frac{N_b - 3}{4} \quad \textit{or} \quad \frac{3\,N_b - 1}{4} \leqslant j < N_b - 1$$

(see Figure 5, on page 18), which does not occur for values of $N_b < 7$.

The number of reentrant angles is then equal to $2\left[\frac{N_b - 3}{4}\right]$.

ii. At a given step $m \in \mathbb{N}^\star$*, with the above convention, a polygon* $\mathcal{P}_{m,k}$ *admits reentrant interior angles in the sole cases when* $N_b \geqslant 7$*, at vertices* M_{k+j}*,* $1 \leqslant k \leqslant N_b^m$*,* $0 < j \leqslant N_b - 1$*, as well as in the case when*

$$0 < j \leqslant \frac{N_b - 3}{4} \quad \textit{or} \quad \frac{3\,N_b - 1}{4} \leqslant j < N_b - 1\,.$$

The number of reentrant angles is then equal to $2\,N_b^m \left[\dfrac{N_b - 3}{4}\right]$.

Remark 2.2. Note that due to the respective definitions of the polygons $\mathcal{P}_{m,k}$ and $\mathcal{Q}_{m,k}$, the existence of reentrant interior angles for $\mathcal{P}_{m,k}$ at a vertex M_{k+j}, $1 \leqslant k \leqslant N_b^m$, $0 < j \leqslant N_b - 1$, also results in the existence of reentrant interior angles for $\mathcal{Q}_{m,k}$ at the vertices M_{k+j-1}, $1 \leqslant k \leqslant N_b^m$, $1 < j \leqslant N_b - 1$ and M_{k+j+1}, $1 \leqslant k \leqslant N_b^m$, $0 < j \leqslant N_b - 2$.

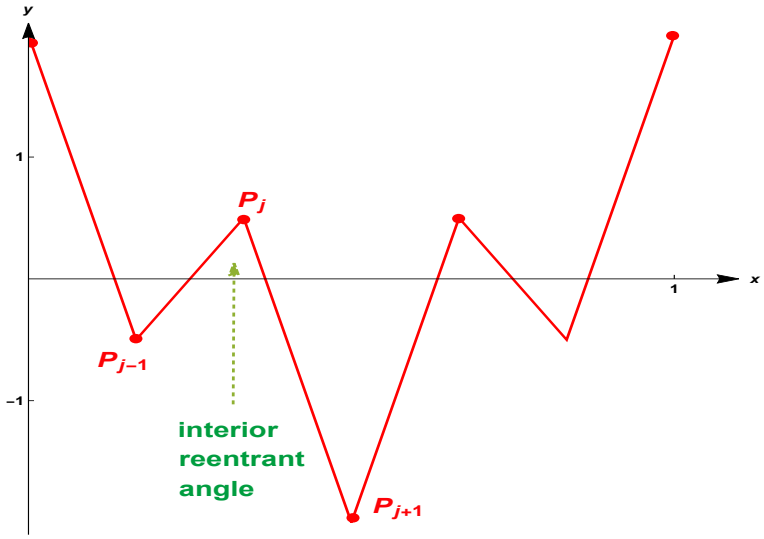

Figure 5: **An interior reentrant angle. Here,** $N_b = 7$ **and** $\lambda = \dfrac{1}{2}$**.**

3 Polyhedral Measure, Polyhedral Neighborhoods

Our results on fractal cohomology obtained in [DL22d] (see also [DL22a]) have highlighted the role played by specific threshold values for a sequence of geometric invariants at any step $m \in \mathbb{N}$ of the prefractal graph approximation; namely, the sequence of the m^{th} *cohomology infinitesimals* introduced in Definition 3.1, on page 19 just below, which also plays a major part in studying the polyhedral neighborhoods associated with the sequence of prefractal approximations of the Weierstrass Curve (or, equivalently, of the Weierstrass IFD).

Definition 3.1 (**m^{th} Cohomology Infinitesimal [DL22c], [DL22d]**).

From now on, given any $m \in \mathbb{N}$, we will call m^{th} *cohomology infinitesimal* the number $\varepsilon_m^m > 0$ which, modulo a multiplicative constant equal to $\dfrac{1}{N_b - 1}$, i.e., $\varepsilon_m^m = \dfrac{1}{N_b - 1}\,\dfrac{1}{N_b^m}$ (recall that $N_b > 1$), stands as the elementary horizontal length introduced in part *i.* of Definition 2.6, on page 10, i.e.,

$$\frac{1}{N_b^m}\,.$$

Observe that, clearly, ε_m itself – and not just ε_m^m – depends on m; hence, we should really write $\varepsilon_m^m = (\varepsilon_m)^m$, for all $m \in \mathbb{N}$.

In addition, since $N_b > 1$, ε_m^m satisfies the following asymptotic behavior,

$$\varepsilon_m^m \to 0\,,\ \text{as } m \to \infty,$$

which, naturally, results in the fact that the larger m, the smaller ε_m^m. It is for this reason that we call ε_m^m – or rather, the *infinitesimal sequence* $\left(\varepsilon_m^m\right)_{m=0}^{\infty}$ of positive numbers tending to zero as $m \to \infty$, with $\varepsilon_m^m = (\varepsilon_m)^m$, for each $m \in \mathbb{N}$ – an *infinitesimal*. Note that this m^{th} cohomology infinitesimal is the one naturally associated to the scaling relation of Property 2.1, on page 4.

In the sequel, it is also useful to keep in mind that the sequence of positive numbers $(\varepsilon_m)_{m=0}^{\infty}$ itself satisfies

$$\varepsilon_m \sim \frac{1}{N_b}\,,\ \text{as } m \to \infty\ ;$$

i.e., $\varepsilon_m \to \dfrac{1}{N_b}$, as $m \to \infty$. In particular, $\varepsilon_m \not\to 0$, as $m \to \infty$, but, instead, ε_m tends to a strictly positive and finite limit.

Remark 3.1 (**Addressing Numerical Estimates**).

From a practical point of view, an important question is the value of the ratio

$$\frac{\text{Cohomology infinitesimal}}{\text{Maximal height}} = \frac{\varepsilon_m^m}{h_m}\ ;$$

see relation ($\mathcal{R}5$), on page 11.

Thanks to the estimates given in relation ($\mathcal{R}7$), on page 16, we have that

$$\frac{\varepsilon_m^m}{h_m} = L_m^{1-D_{\mathcal{W}}}\,\mathcal{O}(1) = \varepsilon_m^{m\,(1-D_{\mathcal{W}})}\,\mathcal{O}(1)\ ,$$

with

$$0 < C_{inf} \leqslant \mathcal{O}(1) \leqslant C_{sup}\,.$$

Given $q \in \mathbb{N}^\star$, we then have

$$\frac{1}{10^q}\,C_{inf} \leqslant \frac{\varepsilon_m^m}{h_m} \leqslant \frac{1}{10^q}\,C_{sup}$$

when

$$\frac{C_{inf}}{10^q} \leqslant e^{(1-D_{\mathcal{W}})\,\ln L_m} \leqslant \frac{C_{sup}}{10^q}\,,$$

or, equivalently, when

$$-\frac{1}{\ln N_b}\,\ln\left((N_b-1)\left(\frac{C_{sup}}{10^q}\right)^{\frac{1}{1-D_{\mathcal{W}}}}\right) \leqslant m \leqslant -\frac{1}{\ln N_b}\,\ln\left((N_b-1)\left(\frac{C_{inf}}{10^q}\right)^{\frac{1}{1-D_{\mathcal{W}}}}\right).$$

Numerical values for $N_b = 3$ and $\lambda = \dfrac{1}{2}$ yield:

i. For $q = 1$: $2 \leqslant m \leqslant 3$.

ii. For $q = 2$: $7 \leqslant m \leqslant 9$.

iii. For $q = 3$: $13 \leqslant m \leqslant 15$.

Hence, when m increases, the ratio $\dfrac{\varepsilon_m^m}{h_m}$ decreases, and tends to 0. This numerical – but very practical and explicit argument – also accounts for our forthcoming use of neighborhoods, of width equal to the cohomology infinitesimal.

We refer to part *iv.* of Property 2.6, on page 7 above, along with Figure 1, on page 9 for the definition of the polygons $\mathcal{P}_{m,j}$ (resp., $\mathcal{Q}_{m,j}$) associated with the Weierstrass Curve in the next definition, as well as throughout the rest of this section. See also Definition 2.5, on page 8 where the polygonal families are introduced.

Definition 3.2 (Power of a Vertex of the Prefractal Graph $\Gamma_{\mathcal{W}_m}$, $m \in \mathbb{N}^\star$, with Respect to the Polygonal Families $\mathcal{P}_m$ and $\mathcal{Q}_m$).

Given a strictly positive integer m, a vertex X of the prefractal graph $\Gamma_{\mathcal{W}_m}$ will be said:

i. of power one relative to the polygonal family $\mathcal{P}_m$ if X belongs to (to be understood in the sense that X is a vertex of) one and only one N_b–gon $\mathcal{P}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 1$;

ii. of power $\dfrac{1}{2}$ *relative to the polygonal family* $\mathcal{P}_m$ if X is a common vertex to two consecutive N_b–gons $\mathcal{P}_{m,j}$ and $\mathcal{P}_{m,j+1}$, $0 \leqslant j \leqslant N_b^m - 2$;

iii. of power zero relative to the polygonal family $\mathcal{P}_m$ if X does not belong to (to be understood in the sense that X is not a vertex of) any N_b–gon $\mathcal{P}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 1$.

Similarly, given $m \in \mathbb{N}$, a vertex X of the prefractal graph $\Gamma_{\mathcal{W}_m}$ is said:

i. of power one relative to the polygonal family $\mathcal{Q}_m$ if X belongs to (as above, to be understood in the sense that X is a vertex of) one and only one N_b–gon $\mathcal{Q}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 2$;

ii. of power $\frac{1}{2}$ *relative to the polygonal family* $\mathcal{Q}_m$ if X is a common vertex to two consecutive N_b–gons $\mathcal{Q}_{m,j}$ and $\mathcal{Q}_{m,j+1}$, $0 \leqslant j \leqslant N_b^m - 3$;

iii. of power zero relative to the polygonal family $\mathcal{Q}_m$ if X does not belong to (as previously, to be understood in the sense that X is not a vertex of) any N_b–gon $\mathcal{Q}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 2$.

Notation 9. In the sequel, given a strictly positive integer m, the *power of a vertex X of the prefractal graph $\Gamma_{\mathcal{W}_m}$ relative to the polygonal families $\mathcal{P}_m$ and $\mathcal{Q}_m$* will be respectively denoted by

$$p(X, \mathcal{P}_m) \quad \text{and} \quad p(X, \mathcal{Q}_m)\,.$$

Notation 10 (Lebesgue Measure on $\mathbb{R}^2$).

In the sequel, we denote by $\mu_{\mathcal{L}}$ the Lebesgue measure on $\mathbb{R}^2$.

Notation 11. For any $m \in \mathbb{N}$, and any vertex X of V_m, we set:

$$\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) = \begin{cases} \dfrac{1}{N_b}\, p(X, \mathcal{P}_m) \displaystyle\sum_{0 \leqslant j \leqslant N_b^m - 1, X \text{ vertex of } \mathcal{P}_{m,j}} \mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right), \text{ if } X \notin \mathcal{Q}_m\,, \\ \dfrac{1}{N_b}\, p(X, \mathcal{Q}_m) \displaystyle\sum_{1 \leqslant j \leqslant N_b^m - 2, X \text{ vertex of } \mathcal{P}_{m,j}} \mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j}\right), \text{ if } X \notin \mathcal{P}_m\,, \\ \dfrac{1}{2\,N_b} \left\{ p(X, \mathcal{P}_m) \displaystyle\sum_{0 \leqslant j \leqslant N_b^m - 1, X \text{ vertex of } \mathcal{P}_{m,j}} \mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right) + p(X, \mathcal{Q}_m) \displaystyle\sum_{1 \leqslant j \leqslant N_b^m - 2, X \text{ vertex of } \mathcal{P}_{m,j}} \mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j}\right) \right\}, \\ \text{if } X \in \mathcal{P}_m \cap \mathcal{Q}_m\,. \end{cases}$$

Property 3.1. *For any $m \in \mathbb{N}$, and any pair (X, Y) of adjacent vertices of V_m belonging to the same polygon $\mathcal{P}_{m,j}$, with $0 \leqslant j \leqslant N_b^m - 1$ (resp., $\mathcal{Q}_{m,j}$, with $0 \leqslant j \leqslant N_b^m - 2$), we have that*

$$d_{eucl}(X, Y) = \sqrt{h_{jm}^2 + L_m^2} > |h_{jm}|\,,$$

which, due to the inequality given in Property 2.12, page 15, ensures that

$$\frac{1}{d_{eucl}(X, Y)} < \frac{1}{|h_{jm}|} \lesssim L_m^{D_{\mathcal{W}} - 2} \lesssim N_b^{(2 - D_{\mathcal{W}})\, m}\,.$$

At the same time, we also have that

$$d_{eucl}(X, Y) \lesssim h_m \lesssim L_m^{2 - D_{\mathcal{W}}} \lesssim N_b^{(D_{\mathcal{W}} - 2)\, m}\,.$$

Proof. This follows at once from Property 2.12, on page 15.

□

Corollary 3.2. *For any $m \in \mathbb{N}$, any natural integer j of $\{0, \ldots, N_b^m - 1\}$, and any pair of points (X, Y) of $\mathcal{P}_{m,j}$ or of $\mathcal{Q}_{m,j}$, we have that*

$$\frac{1}{d_{eucl}(X,Y)} \lesssim L_m^{D_{\mathcal{W}}-2} \lesssim N_b^{(2-D_{\mathcal{W}})\,m}$$

and

$$d_{eucl}(X,Y) \lesssim h_m \lesssim L_m^{2-D_{\mathcal{W}}} \lesssim N_b^{(D_{\mathcal{W}}-2)\,m}.$$

Property 3.3. *For any $m \in \mathbb{N}$, and any vertex X of V_m:*

$$\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \lesssim h_m L_m \lesssim L_m^{3-D_{\mathcal{W}}} \lesssim N_b^{(D_{\mathcal{W}}-3)\,m}$$

and

$$\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \lesssim h_m L_m \lesssim L_m^{3-D_{\mathcal{W}}} \lesssim N_b^{(D_{\mathcal{W}}-3)\,m}.$$

Proof. This also directly follows from Property 2.12, page 15.

□

Definition 3.3 (**Trace of a Polygon on the Weierstrass Curve**)**.**

Given $m \in \mathbb{N}$, and $0 \leqslant j \leqslant N_b^m - 1$ (resp., $0 \leqslant j \leqslant N_b^m - 2$), of extreme vertices $\mathcal{V}_{initial}(\mathcal{P}_{m,j}) \in V_m$ and $\mathcal{V}_{end}(\mathcal{P}_{m,j}) \in V_m$ (resp., $\mathcal{V}_{initial}(\mathcal{Q}_{m,j}) \in V_m$ and $\mathcal{V}_{end}(\mathcal{Q}_{m,j}) \in V_m$; see Definition 2.6, on page 7), we define *the trace* of the polygon $\mathcal{P}_{m,j}$ (resp., $\mathcal{Q}_{m,j}$) on the Weierstrass Curve as the set $tr_{\gamma_{\mathcal{W}}}(\mathcal{P}_{m,j})$ (resp., $tr_{\gamma_{\mathcal{W}}}(\mathcal{Q}_{m,j})$) of points $\{\mathcal{V}_{initial}(\mathcal{P}_{m,j}), M_\star, \mathcal{V}_{end}(\mathcal{P}_{m,j})\}$ (resp., $\{\mathcal{V}_{initial}(\mathcal{Q}_{m,j}), M_\star, \mathcal{V}_{end}(\mathcal{Q}_{m,j})\}$), where we denote by $M_\star$ any point of the Weierstrass Curve strictly located between $\mathcal{V}_{initial}(\mathcal{P}_{m,j})$ and $\mathcal{V}_{end}(\mathcal{P}_{m,j})$ (resp., $\mathcal{V}_{initial}(\mathcal{Q}_{m,j})$ and $\mathcal{V}_{end}(\mathcal{Q}_{m,j})$).

Definition 3.4 (**Sequence of Domains Delimited by the Weierstrass IFD – Polyhedral Neighborhood of the Weierstrass Curve**)**.**

We introduce *the sequence of domains delimited by the Weierstrass IFD*, or *polygonal neighborhood of the Weierstrass Curve* as the sequence $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$ of open, connected polygonal sets $(\mathcal{P}_m \cup \mathcal{Q}_m)_{m\in\mathbb{N}}$, where, for each $m \in \mathbb{N}$, $\mathcal{P}_m$ and $\mathcal{Q}_m$ respectively denote the polygonal sets introduced in Definition 2.5, on page 8.

Given $\in \mathbb{N}$, we call $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ the m^{th} *polyhedral neighborhood* (of the Weierstrass Curve $\Gamma_{\mathcal{W}}$).

Property 3.4 (**Domain Delimited by the Weierstrass IFD**).

We call domain, delimited by the Weierstrass IFD*, the set, which is equal to the following limit,*

$$\mathcal{D}\left(\Gamma_{\mathcal{W}}\right) = \lim_{m\to\infty} \mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right) ,$$

where the convergence is interpreted in the sense of the Hausdorff metric on $\mathbb{R}^2$*; see [DL22b]. In fact, we have that*

$$\mathcal{D}\left(\Gamma_{\mathcal{W}}\right) = \Gamma_{\mathcal{W}} .$$

Notation 12 (**Minimal and Maximal Values of the Weierstrass Function** $\mathcal{W}$ **on** $[0,1]$).

We set

$$m_{\mathcal{W}} = \min_{t\in[0,1]} \mathcal{W}(t) = -\frac{1}{1-\lambda} \quad , \quad M_{\mathcal{W}} = \max_{t\in[0,1]} \mathcal{W}(t) = \frac{1}{1-\lambda} .$$

Notation 13. Henceforth, for a given $m \in \mathbb{N}$, the notation $\displaystyle\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m}$ means that the associated finite sum runs through all of the vertices of the polygons belonging to the sets $\mathcal{P}_m$ and $\mathcal{Q}_m$ introduced in Definition 2.5, on page 8; see also Notation 7, on page 9 following that definition.

Property 3.5. *Given a continuous function* u *on* $[0,1]\times[m_{\mathcal{W}}, M_{\mathcal{W}}]$*, we have that, for any* $m \in \mathbb{N}$*, and any vertex* X *of* V_m*:*

$$\left|\mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right) u\left(X\right)\right| \lesssim \mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\left(\max_{[0,1]\times[m_{\mathcal{W}},M_{\mathcal{W}}]} |u|\right) \lesssim N_b^{-(3-D_{\mathcal{W}})\,m} .$$

Consequently, with the notation of Definition 3.1, on page 19, we have that

$$\varepsilon^{m\,(D_{\mathcal{W}}-2)}\left|\mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right) u\left(X\right)\right| \lesssim \varepsilon_m^{-m} .$$

Since the sequence $\left(\displaystyle\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m} \varepsilon^{-m}\right)_{m\in\mathbb{N}}$ *is a positive and increasing sequence (the number of vertices involved increases as* m *increases), this ensures the existence of the finite limit*

$$\lim_{m\to\infty} \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m} \mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right) u\left(X\right) ,$$

where we have used Notation 13, on page 23.

Property 3.6 (**Polyhedral Measure on the Weierstrass IFD [DL22b]**).

We introduce the polyhedral measure on the Weierstrass IFD*, denoted by* μ*, such that for any continuous function* u *on the Weierstrass Curve, with the use of Notation 11, on page 21 and 13, on page 23,*

$$\int_{\Gamma_{\mathcal{W}}} u\, d\mu = \lim_{m\to\infty} \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m} \mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m)\, u(X)\,, \quad (\star)$$

which, thanks to Property 3.4, on page 23, can also be understood in the following way,

$$\int_{\Gamma_{\mathcal{W}}} u\, d\mu = \int_{\mathcal{D}(\Gamma_{\mathcal{W}})} u\, d\mu\,.$$

Remark 3.2. In a sense, our polyhedral measure can be seen as a measure which is an extension of the Riemann integral, where the step functions are replaced by upper and lower affine functions which approximate the Weierstrass Curve.

Theorem 3.7 ([DL22b])**.**

The polyhedral measure μ *is well defined, positive, as well as a bounded, nonzero, Borel measure on* $\mathcal{D}(\Gamma_{\mathcal{W}})$*. The associated total mass is given by*

$$\mu(\mathcal{D}(\Gamma_{\mathcal{W}})) = \lim_{m\to\infty} \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m} \mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m)\,, \quad (\star\star)$$

and satisfies the following estimate:

$$\mu(\mathcal{D}(\Gamma_{\mathcal{W}})) \leqslant \frac{2}{N_b}(N_b-1)^2 C_{sup}\,. \quad (\star\star\star)$$

Furthermore, the support of μ *coincides with the entire curve:*

$$supp\,\mu = \mathcal{D}(\Gamma_{\mathcal{W}}) = \Gamma_{\mathcal{W}}\,.$$

In addition, μ *is the weak limit as* $m\to\infty$ *of* the following discrete measures *(or Dirac Combs), given, for each* $m\in\mathbb{N}$*, by*

$$\mu_m = \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m} \mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m)\,\delta_X\,,$$

where ε_m^m *denotes the cohomology infinitesimal introduced in Definition 3.1, on page 19,* δ_X *is the Dirac measure concentrated at* X*, and we have used Notation 11, on page 21 for* $\mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m)$*, along with Notation 13, on page 23.*

Property 3.8 (**The Quasi Self–Similar Sequence of Discrete Polyhedral Measures**)**.**

The sequence of discrete polyhedral measures $(\mu_m)_{m\in\mathbb{N}}$ *introduced in Theorem 3.7, on page 24 just above, satisfies the following recurrence relation, for all* $m\in\mathbb{N}^\star$*,*

$$\mu_m = N_b^{D_{\mathcal{W}}-2} \sum_{T_j\in\mathcal{T}_{\mathcal{W}}} \mu_{m+1}\circ T_j^{-1}\,, \qquad (\spadesuit)$$

where for $\mathcal{T}_{\mathcal{W}} = \left\{T_0, \ldots, T_{N_b-1}\right\}$ *is the nonlinear iterated function system (IFS) introduced in Proposition 2.3, on page 6.*

Note that relation (♠) *can be viewed as a generalization of classical self–similar measures, as considered in [Hut81], page* 714.

Proof. First, we can note that, for $m \in \mathbb{N}^{\star}$,

$$\varepsilon_{m+1}^{m+1} = \frac{1}{N_b}\, \varepsilon_m^m\,,$$

which ensures that

$$\varepsilon_{m+1}^{(m+1)\,(D_{\mathcal{W}}-2)} = \frac{1}{N_b^{D_{\mathcal{W}}-2}}\, \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} = N_b^{2-D_{\mathcal{W}}}\, \varepsilon_m^{m\,(D_{\mathcal{W}}-2)}\,.$$

We then simply use the result of Property 2.11, on page 13, according to which, for $0 \leqslant j \leqslant N_b - 1$, the j^{th} vertex of the polygon $\mathcal{P}_{m+1,k}$, $0 \leqslant k \leqslant N_b^m - 1$, is the image of the the j^{th} vertex of the polygon $\mathcal{P}_{m,k-i\,(N_b-1)\,N_b^m}$ under the bijective map T_j, where $0 \leqslant j \leqslant N_b - 1$ is arbitrary. Therefore, there is an exact correspondance between polygons at consecutive steps m, $m+1$: indeed, polygons at the $(m+1)^{th}$ step of the prefractal approximation process are obtained by applying each (bijective) map T_j, for $0 \leqslant j \leqslant N_b - 1$, to the polygons at the m^{th} step of the prefractal approximation process:

$$\bigcup_{j=0}^{N_b-1} \left\{T_j^{-1}(X)\,,\, X \in V_{m+1}\right\} = V_m\,,$$

which can equivalently be written as

$$\bigcup_{j=0}^{N_b-1} \left\{T_j^{-1}(X)\,,\, X \in \mathcal{P}_{m+1} \bigcup \mathcal{Q}_{m+1}\right\} = \left\{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m\right\}.$$

We can then deduce that

$$\sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\left(X, \mathcal{P}_m, \mathcal{Q}_{m+}\right) \delta_X = \sum_{T_j \in \mathcal{T}_{\mathcal{W}}} \sum_{X \in \mathcal{P}_{m+1} \bigcup \mathcal{Q}_{m+1}} \mu^{\mathcal{L}}\left(X, T_j^{-1}\left(\mathcal{P}_{m+1}\right), T_j^{-1}\left(\mathcal{Q}_{m+1}\right)\right) \delta_X\,,$$

which yields the desired result.

□

Theorem 3.9 (**The Quasi Self–Similar Polyhedral Measure**)**.**

The polyhedral measure μ *satisfies the following* quasi self–similarity relation*,*

$$\mu = N_b^{D_{\mathcal{W}}-2} \sum_{T_j \in \mathcal{T}_{\mathcal{W}}} \mu \circ T_j^{-1}\,, \qquad (\spadesuit\spadesuit)$$

where $\mathcal{T}_{\mathcal{W}} = \left\{T_0, \ldots, T_{N_b-1}\right\}$ *is the nonlinear iterated function system (IFS) introduced in Proposition 2.3, on page 6.*

Proof. As in the proof of Property 3.8, on page 24, we use the fact that, for all $m \in \mathbb{N}^{\star}$,

$$\bigcup_{j=0}^{N_b-1} \left\{ T_j^{-1}(X)\,,\, X \in \mathcal{P}_m \bigcup \mathcal{Q}_m \right\} = \left\{ X \in \mathcal{P}_{m-1} \bigcup \mathcal{Q}_{m-1} \right\}.$$

We then have that, for any continuous function u on the Weierstrass Curve,

$$\begin{aligned}
\sum_{T_j \in \mathcal{T}_{\mathcal{W}}} \int_{\Gamma_{\mathcal{W}}} u\, d\left(\mu \circ T_j^{-1}\right) &= \sum_{T_j \in \mathcal{T}_{\mathcal{W}}} \lim_{m\to\infty} \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\left(T_j^{-1}(X), \mathcal{P}_m, \mathcal{Q}_m\right) u\left(T_j^{-1}(X)\right) \\
&= \lim_{m\to\infty} \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_{m-1} \bigcup \mathcal{Q}_{m-1}} \mu^{\mathcal{L}}\left(X, \mathcal{P}_{m-1}, \mathcal{Q}_{m-1}\right) u(X) \\
&= \lim_{m\to\infty} N_b^{D_{\mathcal{W}}-2}\, \varepsilon_{m-1}^{(m-1)(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_{m-1} \bigcup \mathcal{Q}_{m-1}} \mu^{\mathcal{L}}\left(X, \mathcal{P}_{m-1}, \mathcal{Q}_{m-1}\right) u(X) \\
&= N_b^{2-D_{\mathcal{W}}} \lim_{m\to\infty} \varepsilon_{m-1}^{(m-1)(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_{m-1} \bigcup \mathcal{Q}_{m-1}} \mu^{\mathcal{L}}\left(X, \mathcal{P}_{m-1}, \mathcal{Q}_{m-1}\right) u(X) \\
&= N_b^{2-D_{\mathcal{W}}} \int_{\Gamma_{\mathcal{W}}} u\, d\mu\,,
\end{aligned}$$

from which we deduce, as desired, that

$$\mu = N_b^{D_{\mathcal{W}}-2} \sum_{T_j \in \mathcal{T}_{\mathcal{W}}} \mu \circ T_j^{-1}\,.$$

□

Corollary 3.10. *The polyhedral measure is the unique fixed point, with respect to the Kantorovich distance, of the map Φ which, to any probability measure ν associates*

$$\Phi(\nu) = N_b^{D_{\mathcal{W}}-2} \sum_{T_j \in \mathcal{T}_{\mathcal{W}}} \nu \circ T_j^{-1}\,.$$

Definition 3.5 (**(m, ε_m^m)–Neighborhood** [DL22c])**.**

Given $m \in \mathbb{N}$ sufficiently large (so that ε_m^m be a sufficiently small positive number), we define the (m, ε_m^m)–neighborhood of the m^{th} prefractal approximation $\Gamma_{\mathcal{W}_m}$ as follows:

$$\mathcal{D}\left(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m\right) = \left\{ M = (x,y) \in \mathbb{R}^2\,,\, d\left(M, \Gamma_{\mathcal{W}_m}\right) \leqslant \varepsilon_m^m \right\}.$$

Property 3.11 (**The Nested Neighborhoods**)**.**

i. Given $m \in \mathbb{N}$ sufficiently large, there exists $m_1 \in \mathbb{N}$ such that, for all $k \geqslant m_1$, the polygonal neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ introduced in Definition 3.4, on page 23 contains, but for a finite number of wedges,

the $(m+k, \varepsilon_{m+k}^{m+k})$*–neighborhood; see Figures 6–9, on pages 29–30.*

ii. Given $m \in \mathbb{N}$ *sufficiently large, there exists* $m_2 \in \mathbb{N}$ *such that, for all* $k \geqslant m_2$*, the* (m, ε_m^m)*–neighborhood contains the polygonal neighborhood* $\mathcal{D}\left(\Gamma_{\mathcal{W}_{m+k}}\right)$ *introduced in Definition 3.4, on page 23; see Figure 10, on page 31 and Figure 11, on page 32.*

iii. Given $m \in \mathbb{N}$ *sufficiently large, there exists* $m_3 \in \mathbb{N}$ *such that, for all* $k \geqslant m_3$*, the polygonal neighborhood* $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ *introduced in Definition 3.4, on page 23 contains the polygonal neighborhood* $\mathcal{D}\left(\Gamma_{\mathcal{W}_{m+k}}\right)$*.*

Proof.

i. At a given step $m \geqslant 0$, between two adjacent vertices $M_{i,m}$ and $M_{i+1,m}$ of V_m, there are $N_b - 1$ consecutive vertices of $V_{m+1} \setminus V_m$, $\left(M_{j+1,m+1}, \ldots, M_{j+N_b-2,m+1}\right) \in V_{m+1}^{N_b-1}$ such that

$$M_{i,m} = M_{j,m+1} \quad \text{and} \quad M_{i+1,m} = M_{j+N_b,m+1}\,.$$

Thanks to Property 2.11, on page 13, we obtain an exact correspondance between vertices of the polygons at the step $m+1$, and at the initial step $m = 0$. Since reentrant angles occur when $N_b \geqslant 7$ (see Property 2.18, on page 17), we can restrict ourselves to the cases $N_b \leqslant 6$ (in the case of reentrant angles, the following arguments can be suitably adjusted). We then simply have to consider the $\left[\dfrac{N_b-2}{2}\right]$ vertices $M_{j+k,m+1}$, with $1 \leqslant k \leqslant \left[\dfrac{N_b-2}{2}\right]$ (the same arguments holds for the vertices $M_{j+N_b-k,m+1}$). Also, again thanks to Property 2.11, on page 13, given j in $\{0, \ldots, N_b-2\}$ and k in $\left\{0, \ldots, N_b^{m+1}-1\right\}$, we have that

$$\operatorname{sgn}\left(\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^{m+1}}\right) - \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^{m+1}}\right)\right) = \operatorname{sgn}\left(\mathcal{W}\left(\frac{j+1}{N_b-1}\right) - \mathcal{W}\left(\frac{j}{N_b-1}\right)\right),$$

i.e., equivalently,

$$\operatorname{sgn}\left(\mathcal{W}\left(\left(k\,(N_b-1)+j+1\right)L_{m+1}\right) - \mathcal{W}\left(\left(k\,(N_b-1)+j\right)L_{m+1}\right)\right) = \operatorname{sgn}\left(\mathcal{W}\left(\frac{j+1}{N_b-1}\right) - \mathcal{W}\left(\frac{j}{N_b-1}\right)\right).$$

Due to the symmetry of the initial polygon $\mathcal{P}_0$ (or, equivalently, of the initial prefractal graph $\Gamma_{\mathcal{W}_0}$) with respect to the vertical line $x = \dfrac{1}{2}$ (see Property 2.2, on page 5), this means that we can restrict ourselves to the case when

$$\mathcal{W}\left(j\,L_{m+1}\right) \geqslant \mathcal{W}\left((j+1)\,L_{m+1}\right) \geqslant \ldots \geqslant \mathcal{W}\left(\left(j + \left[\frac{N_b-2}{2}\right]\right)L_{m+1}\right)$$

and

$$\underbrace{\mathcal{W}\left(j\,L_{m+1}\right)}_{\mathcal{W}\left(i\,L_m\right)} \geqslant \underbrace{\mathcal{W}\left((j+N_b)\;L_{m+1}\right)}_{\mathcal{W}\left((i+1)\,L_m\right)}.$$

Thanks to Property 2.12, on page 15, we deduce, by the triangle inequality, for $1 \leqslant k \leqslant \left[\frac{N_b - 2}{2}\right]$, that

$$\left| \mathcal{W}\left((j+k)\, L_{m+1}\right) - \underbrace{\mathcal{W}\left(j\, L_{m+1}\right)}_{\mathcal{W}\left(i\, L_m\right)} \right| \leqslant \left[\frac{N_b - 2}{2}\right] C_{sup}\, L_{m+1}^{2-D_{\mathcal{W}}} .$$

Since

$$L_{m+1} = \frac{L_m}{N_b} ,$$

we then obtain that

$$\left| \mathcal{W}\left((j+k)\, L_{m+1}\right) - \underbrace{\mathcal{W}\left(j\, L_{m+1}\right)}_{\mathcal{W}\left(i\, L_m\right)} \right| \leqslant \left[\frac{N_b - 2}{2}\right] N_b^{D_{\mathcal{W}}-2}\, C_{sup}\, L_m^{2-D_{\mathcal{W}}} .$$

Recall now (see Property 2.12, on page 15) that

$$C_{inf} = (N_b - 1)^{2-D_{\mathcal{W}}} \min_{0 \leqslant j \leqslant N_b - 1} \left| \mathcal{W}\left(\frac{j+1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \right|$$

and

$$C_{sup} = (N_b - 1)^{2-D_{\mathcal{W}}} \left(\max_{0 \leqslant j \leqslant N_b - 1} \left| \mathcal{W}\left(\frac{j+1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \right| + \frac{2\,\pi}{(N_b - 1)\,(\lambda\, N_b - 1)} \right) .$$

Here, we have that

$$\mathcal{W}\left(\frac{j}{N_b - 1}\right) = \frac{1}{1-\lambda} \cos \frac{2\,\pi\, j}{N_b - 1} .$$

This ensures that

$$\left| \mathcal{W}\left(\frac{j+1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \right| \leqslant \frac{2\,\pi}{N_b - 1} \frac{1}{1-\lambda} .$$

We can check numerically that

$$\left[\frac{N_b - 2}{2}\right] N_b^{D_{\mathcal{W}}-2}\, C_{sup} \leqslant C_{inf} ,$$

from which we immediately deduce that for, $1 \leqslant k \leqslant \left[\frac{N_b - 2}{2}\right]$,

$$\left| \mathcal{W}\left((j+k)\, L_{m+1}\right) - \underbrace{\mathcal{W}\left(j\, L_{m+1}\right)}_{\mathcal{W}\left(i\, L_m\right)} \right| \leqslant \left| \mathcal{W}\left((i+1)\, L_m\right) - \mathcal{W}\left(i\, L_m\right) \right| .$$

For $1 \leqslant k \leqslant \left[\frac{N_b - 2}{2}\right]$, the vertices $M_{j+k,m+1}$ are then strictly between the vertices $M_{i,m}$ and $M_{i+1,m}$. As is explained previously, we can show, in a similar way, that for $1 \leqslant k \leqslant \left[\frac{N_b - 2}{2}\right]$, the vertices $M_{j+N_b-k,m+1}$ are also strictly between the vertices $M_{i,m}$ and $M_{i+1,m}$.

By induction, we then obtain that, given four consecutive adjacent vertices $M_{i,m}$, $M_{i+1,m}$, $M_{i+3,m}$ and $M_{i+4,m}$ of V_m, with $1 \leqslant i \leqslant \#V_m - 5$ and $k \in \mathbb{N}$, the vertices of $V_{m+k} \setminus V_m$ located between $M_{i,m}$ and $M_{i+4,m}$ can be all contained in the simple and convex polygon $M_{i,m}M_{i+1,m}M_{i+3,m}M_{i+4,m}$, which coincides with the union of two consecutive polygons $\mathcal{P}_{m,j}$ and $\mathcal{Q}_{m,j}$. Thus, there exists $m_0 \in \mathbb{N}$ such that, for all $k \geqslant m_0$, the $(m+k, \varepsilon_{m+k}^{m+k})$-neighborhood

$$\mathcal{D}\left(\Gamma_{\mathcal{W}_{m+k}}, \varepsilon_{m+k}^{m+k}\right) = \left\{M = (x,y) \in \mathbb{R}^2\,,\, d\left(M, \Gamma_{\mathcal{W}_{m+k}}\right) \leqslant \varepsilon_{m+k}^{m+k}\right\}\,,$$

from which we remove the wedges associated to the vertices $M_{i,m}$, $M_{i+1,m}$, $M_{i+3,m}$ and $M_{i+4,m}$ (see [DL22c]), can be fully contained in the polygon $M_{i,m}M_{i+1,m}M_{i+3,m}M_{i+4,m}$. Hence, there exists $m_1 \in \mathbb{N}$ such that, for all $k \geqslant m_1$, the (m, ε_m^m)-neighborhood but for a finite number of wedges, the $(m+k, \epsilon_{m+k})$-neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_{m+k}}, \varepsilon_{m+k}^{m+k}\right)$, can be fully contained in the polygonal domain $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$.

ii. This latter result has been obtained in [DL22b]. It comes from the fact that, in the sense of the Hausdorff metric on $\mathbb{R}^2$,

$$\lim_{m\to\infty} \mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right) = \Gamma_{\mathcal{W}}\,.$$

iii. This is an immediate consequence of *i.* and *ii.*

□

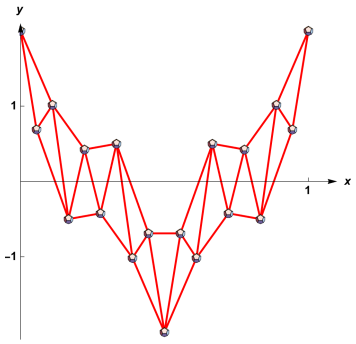

Figure 6: **The polygonal neigborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_2}\right)$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**

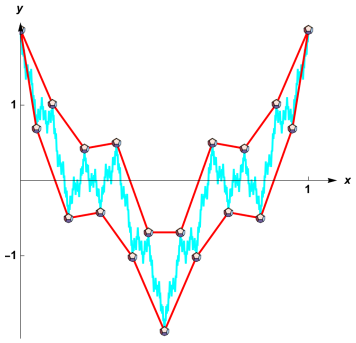

Figure 7: **The exterior boundary of the polygonal neigborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_2}\right)$ (in red), and the tubular neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_7}, \varepsilon_7^7\right)$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**

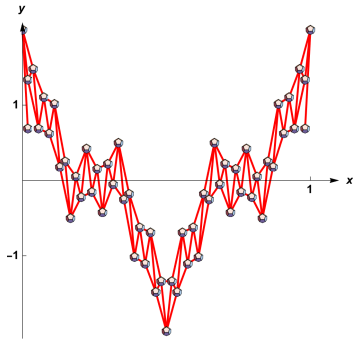

Figure 8: **The polygonal neigborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_3}\right)$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**

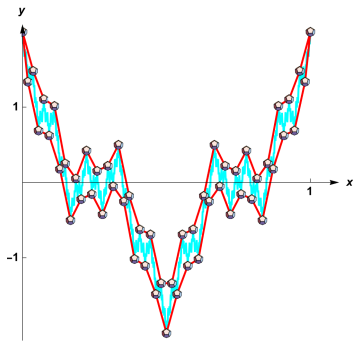

Figure 9: **The exterior boundary of the polygonal neigborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_3}\right)$ (in red), and the tubular neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_7}, \varepsilon_7^7\right)$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**

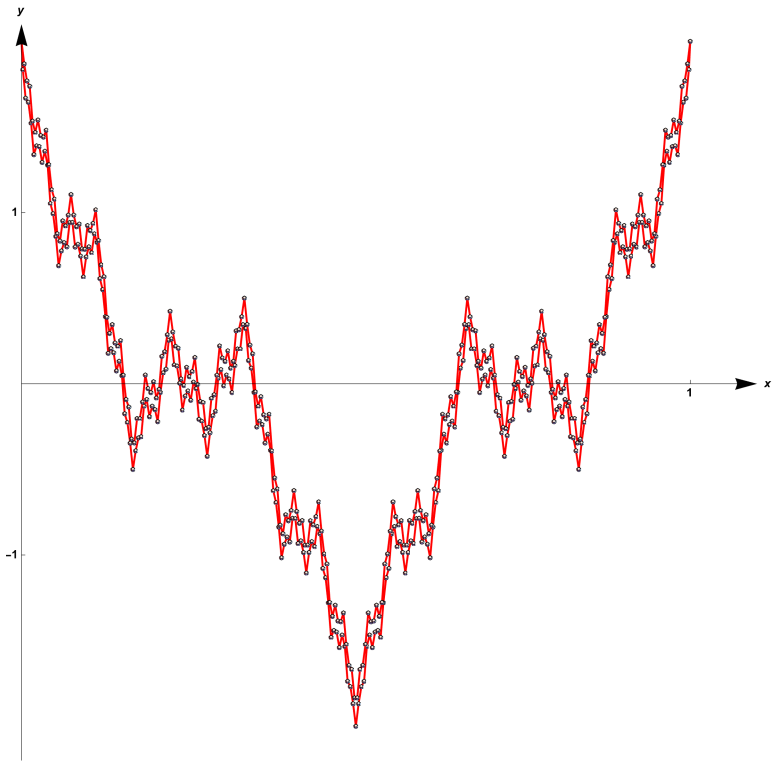

Figure 10: **The polygonal neigborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_5}\right)$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**

Corollary 3.12 (**(of Property 3.11, given on page 26)**).

We immediately deduce from Property 3.11 that there exists $m_0 \in \mathbb{N}$ such that, for all $k \geqslant m_0$, the polyhedral neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ introduced in Definition 3.4, on page 23 contains the vertices of the prefractal approximation $\Gamma_{\mathcal{W}_{m+k}}$ to the the Weierstrass Curve; i.e., the set of vertices V_{m+k}. In particular, the density property 2.5, on page 7, also ensures that the Weierstrass Curve is contained in $\Gamma_{\mathcal{W}_m}$:

$$\forall\, m \geqslant m_0 \;:\quad \Gamma_{\mathcal{W}} \subset \mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right).$$

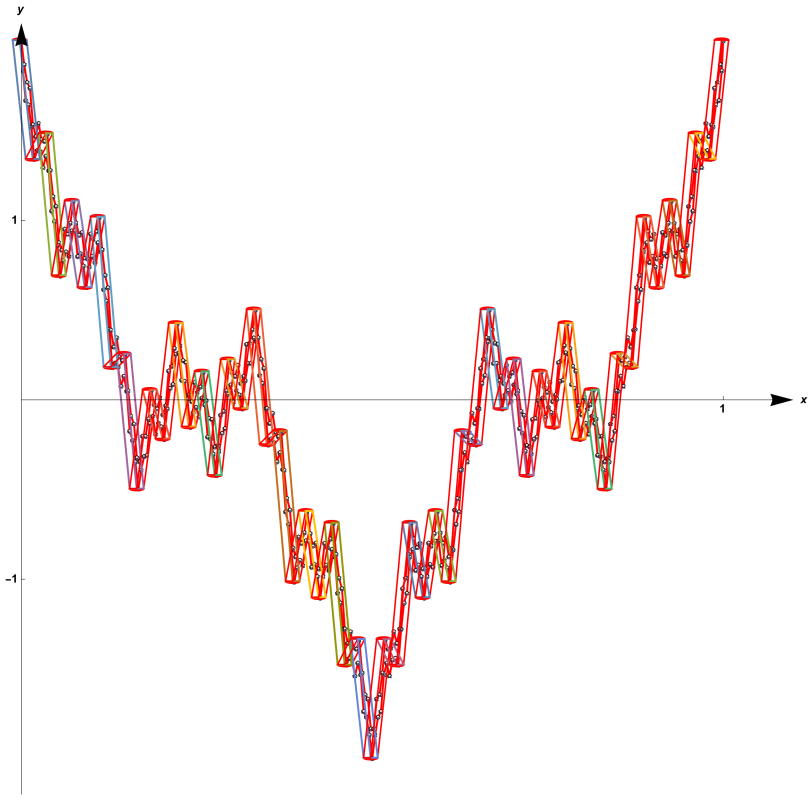

Figure 11: **The tubular neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_3}, \varepsilon_3^3\right)$ and the polygonal neigborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_5}\right)$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.** The size of the vertex points of $\mathcal{D}\left(\Gamma_{\mathcal{W}_5}\right)$ has intentionally been magnified, in order to obtain an illustrative and understandable figure.

Property 3.13 (Exact Expression of the Volume of the m^{th}–Polyhedral Neighborhood (or m^{th} Natural Polyhedral Volume)).

In the case when $N_b = 3$, given $m \in \mathbb{N}^\star$, the volume (or two–dimensional Lebesgue measure) $\mathcal{V}_m(\varepsilon_m^m)$ of the m^{th}–polygonal neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$, or m^{th} natural polyhedral volume*, is given by*

$$\begin{aligned}
\mathcal{V}_m(\varepsilon_m^m) &= \mu_{\mathcal{L}}\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right) = \frac{\varepsilon_m^m}{2}\left(\mathcal{W}(0) + \mathcal{W}\left(2\,\varepsilon_m^m\right) - 2\,\mathcal{W}\left(\varepsilon_m^m\right)\right) \\
&= \frac{\varepsilon_m^m\,\mathcal{W}(0)}{2} + \frac{(N_b-1)^{2-D_{\mathcal{W}}}\,\varepsilon_m^m}{2}\,\varepsilon_m^{m\,(2-D_{\mathcal{W}})}\left(\mathcal{W}\left(\frac{2}{(N_b-1)}\right) - 2\,\mathcal{W}\left(\frac{1}{(N_b-1)}\right)\right) \\
&\quad + \frac{(N_b-1)^{2-D_{\mathcal{W}}}\,\varepsilon_m^m}{2}\sum_{k=0}^{m-1} N_b^{(m-k)\,(2-D_{\mathcal{W}})}\,\varepsilon_m^{m\,k\,(2-D_{\mathcal{W}})}\,\mathcal{R}e\left(e^{\frac{4\,i\,\pi\,N_b^k\,\varepsilon_m^m}{(N_b-1)}} - 2\,e^{\frac{2\,i\,\pi\,N_b^k\,\varepsilon_m^m}{(N_b-1)}}\right),
\end{aligned} \tag{$\mathcal{R}$ 8}$$

where ε_m^m is the m^{th} cohomology infinitesimal introduced in Definition 3.1, on page 19.

If we introduce, for any integer k in $\{0, \ldots, m-1\}$, the coefficients $c_{k,1} \in \mathbb{C}$, $c_{k,2} \in \mathbb{C}$, along with the integers $\ell_{k,1} \in \mathbb{Z}$ and $\ell_{k,2} \in \mathbb{Z}$ such that

$$c_{m,k,1}\,\varepsilon_m^{i\,m\,\ell_{k,1}\,\mathbf{p}} = c_{m,k,1}\,e^{i\,m\,\ell_{k,1}\,\mathbf{p}\,\ln\varepsilon_m^m} = N_b^{(m-k)\,(2-D_{\mathcal{W}})}\,e^{\frac{2\,i\,\pi\,N_b^k\,\varepsilon_m^m}{(N_b-1)}}$$

and

$$c_{m,k,2}\,\varepsilon_m^{i\,m\,\ell_{k,2}\,\mathbf{p}} = c_{m,k,2}\,e^{i\,m\,\ell_{k,2}\,\mathbf{p}\,\ln\varepsilon_m^m} = N_b^{(m-k)\,(2-D_{\mathcal{W}})}\,e^{\frac{2\,i\,\pi\,N_b^k\,\varepsilon_m^m}{(N_b-1)}}\,,$$

we can also obtain the following expression for $\mathcal{V}_m(\varepsilon_m^m)$*:*

$$\begin{aligned}\mathcal{V}_m(\varepsilon_m^m) \;=\; & \frac{\varepsilon_m^m\,\mathcal{W}(0)}{2} + \frac{(N_b-1)^{2-D_{\mathcal{W}}}\,\varepsilon_m^m}{2}\,\varepsilon_m^{m\,(2-D_{\mathcal{W}})}\left(\mathcal{W}\left(\frac{2}{(N_b-1)}\right) - 2\,\mathcal{W}\left(\frac{1}{(N_b-1)}\right)\right)\\ & + \frac{(N_b-1)^{2-D_{\mathcal{W}}}\,\varepsilon_m^m}{2}\,\sum_{k=0}^{m-1}\varepsilon_m^{m\,k\,(2-D_{\mathcal{W}})}\,\mathcal{R}e\left(c_{m,k,2}\,\varepsilon_m^{i\,m\,\ell_{k,2}\,\mathbf{p}} - 2\,c_{m,k,1}\,\varepsilon_m^{i\,m\,\ell_{k,1}\,\mathbf{p}}\right).\end{aligned}$$

Moreover, the sequence of natural polyhedral volumes $\left(\mathcal{V}_m(\varepsilon_m^m)\right)_{m\,\mathbb{N}^\star}$ *satisfies the following recurrence relation, for all* $m \in \mathbb{N}^\star$*,*

$$\mathcal{V}_{m+1}(\varepsilon_{m+1}^{m+1}) = \frac{\lambda}{N_b}\,\mathcal{V}_m(N_b\,\varepsilon_{m+1}^{m+1}) + \frac{1}{N_b}\,\frac{N_b\,\varepsilon_{m+1}^{m+1}}{2}\,(1-\lambda) + \frac{N_b\,\varepsilon_{m+1}^{m+1}}{2}\left(\cos\left(4\,\pi\,N_b\,\varepsilon_{m+1}^{m+1}\right) - 2\,\cos\left(2\,\pi\,N_b\,\varepsilon_{m+1}^{m+1}\right)\right).$$

Note that the remaining cases, i.e., when $N_b \geqslant 4$*, can be obtained in a similar manner. For the sake of concision, we do not include the associated (and heavy) computations associated with these cases.*

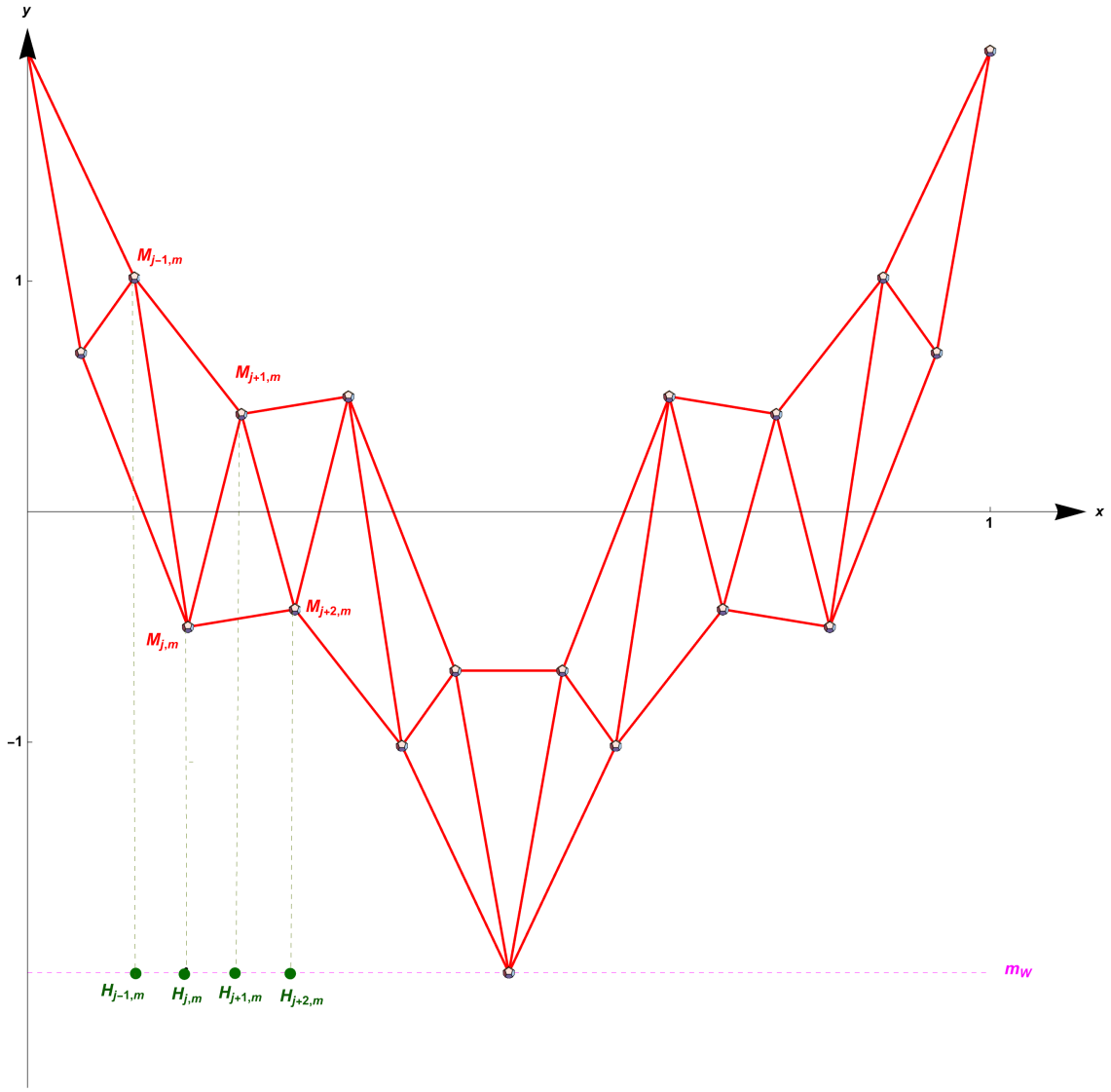

Figure 12: **The orthogonal projections $H_{j,m}$ of the vertices $M_{j,m}$, for $0 \leqslant j \leqslant \#V_m$, onto the horizontal line $y = m_{\mathcal{W}}$.**

Proof. First, note that since $N_b = 3$, the polygons involved are triangles.

Given $m \in \mathbb{N}^\star$, and $0 \leqslant j \leqslant \#V_m$, we introduce the (vertical) orthogonal projection $H_{j,m}$ of the vertex $M_{j,m}$ onto the horizontal line $y = m_{\mathcal{W}}$; see Figure 12, on page 34.

We note that, for $1 \leqslant j \leqslant \#V_m - 2$, the two–dimensional Lebesgue measure (i.e., area) of the polygon (triangle) $\mathcal{Q}_{m,j}$, with bottom vertices $M_{j,m}$ and $M_{j+2,m}$, is obtained by adding the area of the trapezoid $H_{j,m}M_{j,m}M_{j+1,m}H_{j+1,m}$ to the area of the trapezoid $H_{j+1,m}M_{j+1,m}M_{j+2,m}H_{j+2,m}$, and then, by substracting to this result the area of the trapezoid $H_{j,m}M_{j,m}M_{j+2,m}H_{j+2,m}$; i.e.,

$$\begin{aligned}
\mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j}\right) &= \frac{\varepsilon_m^m}{2}\left(\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)+\mathcal{W}\left(\frac{j+1}{(N_b-1)\,N_b^m}\right)+\mathcal{W}\left(\frac{j+1}{(N_b-1)\,N_b^m}\right)+\mathcal{W}\left(\frac{j+2}{(N_b-1)\,N_b^m}\right)\right)\\
&\qquad -\frac{2\,\varepsilon_m^m}{2}\left(\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)-2\,\mathcal{W}\left(\frac{j+2}{(N_b-1)\,N_b^m}\right)\right)\\
&= \frac{\varepsilon_m^m}{2}\left(2\,\mathcal{W}\left(\frac{j+1}{(N_b-1)\,N_b^m}\right)-\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)-\mathcal{W}\left(\frac{j+2}{(N_b-1)\,N_b^m}\right)\right).
\end{aligned}$$

Similarly, for $1 \leqslant j \leqslant \#V_m - 1$, the area of the polygon (triangle) $\mathcal{P}_{m,j}$, with top vertices $M_{j-1,m}$ and $M_{j+1,m}$ is obtained by substracting to the area of the trapezoid $H_{j-1,m}M_{j-1,m}M_{j+1,m}H_{j+1,m}$ the area of the trapezoid $H_{j-1,m}M_{j-1,m}M_{j,m}H_{j,m}$ and the area of the trapezoid $H_{j,m}M_{j,m}M_{j+1,m}H_{j+1,m}$; i.e.,

$$\begin{aligned}
\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right) &= \frac{2\,\varepsilon_m^m}{2}\left(\mathcal{W}\left(\frac{j-1}{(N_b-1)\,N_b^m}\right)+\mathcal{W}\left(\frac{j+1}{(N_b-1)\,N_b^m}\right)\right)\\
&\qquad -\frac{\varepsilon_m^m}{2}\left(\mathcal{W}\left(\frac{j-1}{(N_b-1)\,N_b^m}\right)+\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)+\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)+\mathcal{W}\left(\frac{j+1}{(N_b-1)\,N_b^m}\right)\right)\\
&= \frac{\varepsilon_m^m}{2}\left(\mathcal{W}\left(\frac{j-1}{(N_b-1)\,N_b^m}\right)+\mathcal{W}\left(\frac{j+1}{(N_b-1)\,N_b^m}\right)-2\,\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right).
\end{aligned}$$

We then have that

$$
\begin{aligned}
\mathcal{V}_m(\varepsilon_m^m) &= \sum_{j=1}^{\#V_m-3} \left(\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right) + \mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j}\right)\right) + \mu_{\mathcal{L}}\left(\mathcal{P}_{m,N_b^m}\right) \\
&= \sum_{j=1}^{(N_b-1)\,N_b^m-2} \left(\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right) + \mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j}\right)\right) + \mu_{\mathcal{L}}\left(\mathcal{P}_{m,(N_b-1)\,N_b^m}\right) \\
&= \sum_{j=1}^{(N_b-1)\,N_b^m-2} \frac{\varepsilon_m^m}{2}\left(3\,\mathcal{W}\left(\frac{j+1}{(N_b-1)\,N_b^m}\right) - 3\,\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right) \\
&+ \sum_{j=1}^{(N_b-1)\,N_b^m-2} \frac{\varepsilon_m^m}{2}\left(-\mathcal{W}\left(\frac{j+2}{(N_b-1)\,N_b^m}\right) + \mathcal{W}\left(\frac{j-1}{(N_b-1)\,N_b^m}\right)\right) + \mu_{\mathcal{L}}\left(\mathcal{P}_{m,(N_b-1)\,N_b^m}\right) \\
&= \frac{\varepsilon_m^m}{2}\left(3\,\mathcal{W}\left(\frac{(N_b-1)\,N_b^m-1}{(N_b-1)\,N_b^m}\right) - 3\,\mathcal{W}\left(\frac{1}{(N_b-1)\,N_b^m}\right)\right) \\
&\quad + \frac{\varepsilon_m^m}{2}\left(-\mathcal{W}(1) - \mathcal{W}\left(\frac{(N_b-1)\,N_b^m-1}{(N_b-1)\,N_b^m}\right) - \mathcal{W}\left(\frac{(N_b-1)\,N_b^m-2}{(N_b-1)\,N_b^m}\right)\right) \\
&\quad + \frac{\varepsilon_m^m}{2}\left(\mathcal{W}\left(\frac{1}{(N_b-1)\,N_b^m}\right) + \mathcal{W}\left(\frac{2}{(N_b-1)\,N_b^m}\right) + \mathcal{W}(0)\right) \\
&\quad + \varepsilon_m^m\left(\mathcal{W}\left(\frac{(N_b-1)\,N_b^m-1}{(N_b-1)\,N_b^m}\right) - \mathcal{W}\left(\frac{1}{(N_b-1)\,N_b^m}\right)\right) \\
&\quad + \frac{\varepsilon_m^m}{2}\left(-\mathcal{W}\left(\frac{(N_b-1)\,N_b^m-2}{(N_b-1)\,N_b^m}\right) + \mathcal{W}\left(\frac{2}{(N_b-1)\,N_b^m}\right)\right) + \mu_{\mathcal{L}}\left(\mathcal{P}_{m,(N_b-1)\,N_b^m-1}\right).
\end{aligned}
$$

Note that, thanks to the symmetry with respect to the vertical line $x = \frac{1}{2}$ (see Property 2.2, on page 5, along with Property 2.8, on page 11), we have that

$$
\mathcal{W}\left(\frac{(N_b-1)\,N_b^m-1}{(N_b-1)\,N_b^m}\right) = \mathcal{W}\left(\frac{1}{(N_b-1)\,N_b^m}\right) \quad \text{and} \quad \mathcal{W}\left(\frac{(N_b-1)\,N_b^m-2}{(N_b-1)\,N_b^m}\right) = \mathcal{W}\left(\frac{2}{(N_b-1)\,N_b^m}\right).
$$

We then simply obtain that

$$\begin{aligned}
\mathcal{V}_m(\varepsilon_m^m) &= \mu_{\mathcal{L}}\left(\mathcal{P}_{m,(N_b-1)\,N_b^m-1}\right)\\
&= \frac{\varepsilon_m^m}{2}\left(\mathcal{W}\left(\frac{(N_b-1)\,N_b^m-2}{(N_b-1)\,N_b^m}\right)+\mathcal{W}(1)-2\,\mathcal{W}\left(\frac{(N_b-1)\,N_b^m-1}{(N_b-1)\,N_b^m}\right)\right)\\
&= \frac{\varepsilon_m^m}{2}\left(\mathcal{W}\left(\frac{2}{(N_b-1)\,N_b^m}\right)+\underbrace{\mathcal{W}(1)}_{\mathcal{W}(0)}-2\,\mathcal{W}\left(\frac{1}{(N_b-1)\,N_b^m}\right)\right) \quad \text{(by symmetry)}\\
&= \frac{\varepsilon_m^m}{2}\left(\mathcal{W}(0)+\mathcal{W}\left(\frac{2}{(N_b-1)\,N_b^m}\right)-2\,\mathcal{W}\left(\frac{1}{(N_b-1)\,N_b^m}\right)\right)\\
&= \frac{\varepsilon_m^m}{2}\left(\mathcal{W}(0)+\mathcal{W}\left(2\,\varepsilon_m^m\right)-2\,\mathcal{W}\left(\varepsilon_m^m\right)\right).
\end{aligned}$$

We then have that

$$\begin{aligned}
\mathcal{V}_m(\varepsilon_m^m) &= \frac{\varepsilon_m^m\,\mathcal{W}(0)}{2}+\frac{(N_b-1)^{2-D_{\mathcal{W}}}\,\varepsilon_m^m}{2}\,\varepsilon^{m\,(2-D_{\mathcal{W}})}\left(\mathcal{W}\left(\frac{2}{(N_b-1)}\right)-2\,\mathcal{W}\left(\frac{1}{(N_b-1)}\right)\right)\\
&\quad+\frac{(N_b-1)^{2-D_{\mathcal{W}}}\,\varepsilon_m^m}{2}\sum_{k=0}^{m-1}\varepsilon_k^{k\,(2-D_{\mathcal{W}})}\left(\cos\left(\frac{2\,\pi\,N_b^k}{(N_b-1)\,N_b^m}\right)-2\,\cos\left(\frac{4\,\pi\,N_b^k}{(N_b-1)\,N_b^m}\right)\right)\\
&= \frac{\varepsilon_m^m\,\mathcal{W}(0)}{2}+\frac{(N_b-1)^{2-D_{\mathcal{W}}}\,\varepsilon_m^m}{2}\,\varepsilon_m^{m\,(2-D_{\mathcal{W}})}\left(\mathcal{W}\left(\frac{2}{(N_b-1)}\right)-2\,\mathcal{W}\left(\frac{1}{(N_b-1)}\right)\right)\\
&\quad+\frac{(N_b-1)^{2-D_{\mathcal{W}}}\,\varepsilon_m^m}{2}\sum_{k=0}^{m-1}\varepsilon_k^{k\,(2-D_{\mathcal{W}})}\,\mathcal{R}e\left(e^{\frac{4\,i\,\pi\,\varepsilon_{m-k}^{m-k}}{(N_b-1)}}-2\,e^{\frac{2\,i\,\pi\,\varepsilon_{m-k}^{m-k}}{(N_b-1)}}\right).
\end{aligned}$$

Since, for $0 \leqslant k \leqslant m_1$,

$$\varepsilon_k^k = N_b^{m-k}\,\varepsilon_m^m \quad \text{and} \quad \varepsilon_{m-k}^{m-k} = N_b^k\,\varepsilon_m^m,$$

we deduce that

$$\begin{aligned}
\mathcal{V}_m(\varepsilon_m^m) &= \frac{\varepsilon_m^m\,\mathcal{W}(0)}{2}+\frac{(N_b-1)^{2-D_{\mathcal{W}}}\,\varepsilon_m^m}{2}\,\varepsilon_m^{m\,(2-D_{\mathcal{W}})}\left(\mathcal{W}\left(\frac{2}{(N_b-1)}\right)-2\,\mathcal{W}\left(\frac{1}{(N_b-1)}\right)\right)\\
&\quad+\frac{(N_b-1)^{2-D_{\mathcal{W}}}\,\varepsilon_m^m}{2}\sum_{k=0}^{m-1}N_b^{(m-k)\,(2-D_{\mathcal{W}})}\,\varepsilon_m^{m\,k\,(2-D_{\mathcal{W}})}\,\mathcal{R}e\left(e^{\frac{4\,i\,\pi\,N_b^k\,\varepsilon_m^m}{(N_b-1)}}-2\,e^{\frac{2\,i\,\pi\,N_b^k\,\varepsilon_m^m}{(N_b-1)}}\right).
\end{aligned}$$

Let us recall that

$$\varepsilon_m^m = \frac{1}{N_b-1}\,\frac{1}{N_b^m} \quad \text{and} \quad \mathbf{p} = \frac{2\,\pi}{\ln N_b}.$$

We then have that

$$\mathbf{p}\,\ln\varepsilon_m^m = \left(\frac{2\,\pi}{\ln N_b}\right)\left(-\ln(N_b-1) - m\,\ln N_b\right) = -2\,\pi\,\ln(N_b-1) - 2\,\pi\,m$$

and

$$e^{i\,m\,\ell_{k,1}\,\mathbf{p}\,\ln\varepsilon_m^m} = e^{-2\,i\,m\,\pi\,\ln(N_b-1)}\,.$$

We then introduce, for any integer k in $\{0,\ldots,m-1\}$, the (bounded away from 0 and ∞)) coefficients $c_{m,k,1}\in\mathbb{C}$, $c_{m,k,2}\in\mathbb{C}$, along with the integers $\ell_{k,1}\in\mathbb{Z}$ and $\ell_{k,2}\in\mathbb{Z}$ such that

$$c_{m,k,1}\,\varepsilon_m^{k\,m\,(2-D_{\mathcal{W}})+i\,m\,\ell_{k,1}\,\mathbf{p}} = c_{m,k,1}\,\varepsilon_m^{k\,m\,(2-D_{\mathcal{W}})}\,e^{i\,m\,\ell_{k,1}\,\mathbf{p}\,\ln\varepsilon_m^m} = \varepsilon_k^{k\,(2-D_{\mathcal{W}})}\,e^{\frac{4\,i\,\pi\,N_b^k\,\varepsilon_m^m}{(N_b-1)}}$$

and

$$c_{m,k,2}\,\varepsilon_m^{k\,m\,(2-D_{\mathcal{W}})+i\,m\,\ell_{k,2}\,\mathbf{p}} = c_{m,k,2}\,\varepsilon_m^{k\,m\,(2-D_{\mathcal{W}})}\,e^{i\,m\,\ell_{k,1}\,\mathbf{p}\,\ln\varepsilon_m^m} = \varepsilon_k^{k\,(2-D_{\mathcal{W}})}\,e^{\frac{2\,i\,\pi\,N_b^k\,\varepsilon_m^m}{(N_b-1)}}\,.$$

This enables us to write

$$\begin{aligned}\mathcal{V}_m(\varepsilon_m^m) \;=\; & \frac{\varepsilon_m^m\,\mathcal{W}(0)}{2} + \frac{(N_b-1)^{2-D_{\mathcal{W}}}\,\varepsilon_m^m}{2}\,\varepsilon_m^{m\,(2-D_{\mathcal{W}})}\left(\mathcal{W}\left(\frac{2}{(N_b-1)}\right) - 2\,\mathcal{W}\left(\frac{1}{(N_b-1)}\right)\right)\\ & + \frac{(N_b-1)^{2-D_{\mathcal{W}}}\,\varepsilon_m^m}{2}\sum_{k=0}^{m-1}\varepsilon_m^{m\,k\,(2-D_{\mathcal{W}})}\,\mathcal{R}e\left(c_{m,k,2}\,\varepsilon_m^{i\,m\,\ell_{k,2}\,\mathbf{p}} - 2\,c_{m,k,1}\,\varepsilon_m^{i\,m\,\ell_{k,1}\,\mathbf{p}}\right),\end{aligned}$$

as claimed.

□

Definition 3.6 (***m^{th}* Tubular Volume** [DL22c])**.**

Given $m\in\mathbb{N}$ sufficiently large (so that ε_m^m be a sufficiently small positive number), we call m^{th} *tubular volume* the two–dimensional Lebesgue measure $\mathcal{V}_m^{tube}\left(\varepsilon_m^m\right)$ of the (m,ε_m^m)–neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_m},\varepsilon_m^m\right)$ of the m^{th} prefractal approximation $\Gamma_{\mathcal{W}_m}$ introduced in Definition 3.5, on page 26.

As was already encountered in [DL22c], in our present context, when it comes to obtaining the associated fractal zeta function, we cannot, as in the case of an arbitrary bounded subset of $\mathbb{R}^2$ (see [LRŽ17], Definition 2.2.8, page 118), directly use an integral formula of the form

$$\tilde{\zeta}_m(s) = \int_0^{\varepsilon_m^m} t^{s-3}\,\mathcal{V}_m(t)\,dt = \int_0^{\varepsilon_m^m} t^{s-2}\,\mathcal{V}_m(t)\,\frac{dt}{t}\,, \tag{$\mathcal{R}$ 9}$$

since the volume formulas can only be expressed in an explicit way at a cohomology infinitesimal.

However, and as was already done in [DL22c], in the case of ordinary tubular neighborhoods, if, instead of considering the volume function at a given step m_0, we consider the more general volume functions *for all integers* $m\geqslant m_0$, we can bypass this difficulty by means of Riemann sums, as explained in Lemma 3.14, on page 39 just below.

Lemma 3.14 (**Natural Polyhedral Volume Extension Formula**).

Given $m \geqslant 1$ sufficiently large, and an arbitrary nonnegative integer $k_m \geqslant m$, let us consider the following nonuniform partition $\mathcal{P}_{m,k}$ of the interval $\left[0, \varepsilon_m^m\right]$,

$$\left[0, \varepsilon_m^m\right] = \left[0, \varepsilon_{m\,k_m}^{m\,k_m}\right] \bigcup \left\{ \bigcup_{m+k_m+p\,=\,m\,k_m-2}^{m+k_m+p=m+k_m} \left[\varepsilon_{m+k_m+p+1}^{m+k_m+p+1}, \varepsilon_{m+k_m+p}^{m+k_m+p}\right] \right\} \bigcup \left[\varepsilon_{m+k_m}^{m+k_m}, \varepsilon_m^m\right],$$

where the second union is taken over all $p \in \mathbb{N}$ satisfying the indicated relations and equalities.

Note that the mesh of the partition (i.e., the length of the largest subinterval) is

$$\varepsilon_m^m - \varepsilon_{m+k_m}^{m+k_m}, \quad \textit{with} \quad \lim_{k\to\infty} \left(\varepsilon_m^m - \varepsilon_{m+k_m}^{m+k_m}\right) = 0\,.$$

We set

$$\sigma_{0,m\,k_m} = \varepsilon_{m\,k_m}^{m\,k_m} \quad , \quad \sigma_{m,m+k_m} = \varepsilon_m^m - \varepsilon_{m+k_m}^{m+k_m}\,, \tag{$\mathcal{R}$ 10}$$

and, for all $p \in \{0, \ldots, m\,k - m - k_m - 2\}$,

$$\sigma_p = \varepsilon_{m+k_m+p}^{m+k_m+p} - \varepsilon_{m+k_m+p+1}^{m+k_m+p+1}\,. \tag{$\mathcal{R}$ 11}$$

We can then introduce the Riemann sum

$$\begin{aligned} \mathcal{R}iemann\,(k,m,\mathcal{V}) \;=\; & \sigma_0 \left(\varepsilon_{m\,m}^{m\,k_m}\right)^{s-3} \mathcal{V}_{m\,k_m}\left(\varepsilon_{m\,k}^{m\,k_m}\right) + \sigma_{m+k_m}\left(\varepsilon_m^m\right)^{s-3} \mathcal{V}_{m\,k_m}\left(\varepsilon_m^m\right) \\ & + \sum_{0\leqslant p\leqslant m\,k-m-k-2} \sigma_p \left(\varepsilon_{m+k_m+p}^{m+k+p}\right)^{s-3} \mathcal{V}_{m+k_m+p+1}\left(\varepsilon_{m+k_m+p}^{m+k_m+p}\right). \end{aligned} \tag{$\mathcal{R}$ 12}$$

We then have that

$$\lim_{k_m\to\infty} \mathcal{R}iemann\,(k_m,m,\mathcal{V}) = \int_0^{\varepsilon_m^m} t^{s-2}\,\tilde{\mathcal{V}}_m(t)\,\frac{dt}{t}\,, \tag{$\mathcal{R}$ 13}$$

where, for all sufficiently large $n \in \mathbb{N}^\star$, $\tilde{\mathcal{V}}_n$ is the continuous function defined for all $t \in \left[0, \varepsilon_n^n\right]$ by substituting t for ε_n^n in the expression for $\mathcal{V}_n\left(\varepsilon_n^n\right)$.

As is explained in [DL22c], one can think of $\tilde{\mathcal{V}}_n = \tilde{\mathcal{V}}_{n,}(t)$ as being the effective polyhedral volume *of the n^{th} prefractal approximation to the Weierstrass Curve.*

Notation 14 (**Natural Polyhedral Volume Extension – Natural Volume Extension**).

For the sake of simplicity, given $m \in \mathbb{N}$, we will from now on call *the m^{th} natural polyhedral volume extension*, the volume extension function $\tilde{\mathcal{V}}_m$ associated with the m^{th} natural polyhedral volume $\mathcal{V}_m$ introduced in Property 3.13, on page 32. Alternatively, $\tilde{\mathcal{V}}_m$ will be called *the m^{th} effective polyhedral volume.*

In the same way, as is done in [DL22c], and given $m \in \mathbb{N}$, we call *the m^{th} natural volume extension*, the volume extension function $\tilde{\mathcal{V}}_m^{tube}$ associated with the *the m^{th} tubular volume* $\mathcal{V}_m^{tube}$. Alternatively, $\tilde{\mathcal{V}}_m^{tube}$ will be called *the m^{th} effective tubular volume.*

Property 3.15 (**Complex Dimensions Series Expansion of the Weierstrass Complexified Function $\mathcal{W}_{comp}$ [DL22d]**).

We then have, for any strictly positive integer m and any j in $\{0, \cdots, \#V_m\}$, the following exact *expansion, indexed by the Complex Codimensions $k\,(D_{\mathcal{W}} - 2) + i\,k\,\ell_k\,\mathbf{p}$, with $0 \leqslant k \leqslant m$,*

$$\mathcal{W}_{comp}\left(j\,\varepsilon_m^m\right) \;=\; (N_b-1)^{2-D_{\mathcal{W}}}\,\varepsilon_m^{m\,(2-D_{\mathcal{W}})}\,\mathcal{W}_{comp}\left(\frac{j}{N_b-1}\right) + (N_b-1)^{2-D_{\mathcal{W}}}\,\sum_{k=0}^{m-1} c_{k,m}\,\varepsilon_k^{k\,(2-D_{\mathcal{W}})}\,\varepsilon_k^{i\,k\,\ell_k\,\mathbf{p}}\,, \tag{$\mathcal{R}$ 14}$$

where, for $0 \leqslant k \leqslant m$, ε_k^k is the k^{th} cohomollogy infinitesimal, introduced in Definition 3.1, on page 19 and where, for any integer k in $\{0, \cdots, m-1\}$, $c_k \in \mathbb{C}$ and $\ell_k \in \mathbb{Z}$ are such that

$$(N_b-1)^{2-D_{\mathcal{W}}}\,e^{\,i\,\frac{2\,\pi}{N_b-1}\,j\,\varepsilon_m^{m-k}} = c_{k,m}\,\varepsilon_k^{i\,k\,\ell_k\,\mathbf{p}}\,, \quad (\diamond)$$

with $\mathbf{p} = \dfrac{2\,\pi}{\ln N_b}$ denoting the oscillatory period of the Weierstrass Curve, as introduced in [DL22c].

Since relation ($\mathcal{R}$14) is valid for any $m \in \mathbb{N}^\star$, we note that the associated Complex Dimensions (i.e., in fact, the Complex Dimensions associated with the Weierstrass function) are $D_{\mathcal{W}} - k\,(2 - D_{\mathcal{W}}) + i\,\ell_k\,\mathbf{p}$, where $k \in \mathbb{N}$ is arbitrary.

Theorem 3.16 (**Local and Global Effective Polyhedral Zeta Functions**).

In the case when $N_b = 3$, given $m \in \mathbb{N}$, we introduce the m^{th}–local effective polyhedral zeta function $\tilde{\zeta}_m^e$*, such that,*

$$\begin{aligned}
\tilde{\zeta}_m^e(s) \;&=\; \int_0^{\varepsilon_m^m} t^{s-3}\,\tilde{\mathcal{V}}_m(t)\,dt \quad (\textit{for } \mathcal{R}e\,(s) > D_{\mathcal{W}}) \\
&=\; \mathcal{W}(0)\int_0^{\varepsilon_m^m}\frac{t^{s-2}}{2}\,dt + \int_0^{\varepsilon_m^m} t^{s-2}\,\left(\mathcal{W}\,(2\,t) - 2\,\mathcal{W}\,(t)\right)\,dt \quad (\textit{for all } s \in \mathbb{C}) \\
&=\; \frac{\varepsilon_m^{m\,(s-1)}\,\mathcal{W}(0)}{2\,(s-1)} + \int_0^{\varepsilon_m^m} t^{s-2}\,\left(\mathcal{W}\,(2\,t) - 2\,\mathcal{W}\,(t)\right)\,dt \quad (\textit{for all } s \in \mathbb{C})\,,
\end{aligned}$$

For all $s \in \mathbb{C}$, we can also write explicitly $\tilde{\zeta}_m^e(s)$ in the following form:

$$\begin{aligned}
\tilde{\zeta}_m^e(s) \;=\;& \frac{\varepsilon_m^{m\,(s-1)}}{2\,(s-1)}\,\mathcal{W}(0) + (N_b-1)^{2-D_{\mathcal{W}}}\,\left(\mathcal{W}\left(\frac{1}{(N_b-1)}\right) - 2\,\mathcal{W}\left(\frac{1}{(N_b-1)}\right)\right)\frac{\varepsilon_m^{m\,(s+1-D_{\mathcal{W}})}}{s+1-D_{\mathcal{W}}} \\
&+(N_b-1)^{2-D_{\mathcal{W}}}\,\sum_{k=0}^{m-1}\mathcal{R}e\left(c_{k,2}\,\frac{\varepsilon_m^{m\,(s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,2}\,\mathbf{p}}}{s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,2}\,\mathbf{p}} - 2\,c_{k,1}\,\frac{\varepsilon_m^{m\,(s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,1}\,\mathbf{p})}}{s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,1}\,\mathbf{p})}\right),
\end{aligned}$$

where, for any integer k in $\{0, \ldots, m-1\}$, the coefficients $c_{k,1} \in \mathbb{C}$, $c_{k,2} \in \mathbb{C}$, along with the integers $\ell_{k,1} \in \mathbb{Z}$ and $\ell_{k,2} \in \mathbb{Z}$, have been introduced in relation ($\mathcal{R}$3.13), on page 32, and in relation ($\mathcal{R}$3.13), on page 33.

The associated sequence $\left(\widetilde{\zeta}^{e}_{m}\right)_{m\in\mathbb{N}}$ *satisfies the following recurrence relation, for all values of the positive integer* m *sufficiently large, for all* $s\in\mathbb{C}$*,*

$$\widetilde{\zeta}^{e}_{m+1}(s) \;=\; N_b^{3-s}\,\widetilde{\zeta}^{e}_{m}(s)+\frac{1}{2}\,(1-\lambda)\,N_b^{3-s}\,\frac{\varepsilon_m^{m\,(s-1)}}{s-1}+N_b^{3-s}\int_0^{\varepsilon_m^m} t^{s-2}\,\frac{1}{2\,N_b}\,\mathcal{R}e\left(e^{i\,4\,\pi\,t}-2\,e^{i\,2\,\pi\,t}\right)dt\,. \quad (\mathcal{E}_m)$$

(*R* 15)

This ensures the existence of the limit fractal zeta function $\widetilde{\zeta}^{e}_{\mathcal{W}}$*, i.e., the fractal zeta function associated with the Weierstrass Curve* $\Gamma_{\mathcal{W}}$ *(or, rather, with the Weierstrass IFD), or* global effective polyhedral zeta function*, given by*

$$\widetilde{\zeta}^{e}_{\mathcal{W}}=\lim_{m\to\infty}\widetilde{\zeta}^{e}_{m}\,,$$

where the convergence is locally uniform on $\mathbb{C}$*, along with the existence of an integer* $m_0\in\mathbb{N}$ *such that the poles of* $\widetilde{\zeta}^{e}_{\mathcal{W}}$ *are the same as the poles of the* m_0^{th} *fractal effective polyhedral zeta function* $\widetilde{\zeta}^{e}_{m_0}$*. More precisely, we have the following explicit formula, for the limit effective fractal zeta function* $\widetilde{\zeta}^{e}_{\mathcal{W}}$*, whih is valid for all* $s\in\mathbb{C}$*:*

$$\begin{aligned}\widetilde{\zeta}^{e}_{\mathcal{W}}(s) \;=\;& \sum_{m=m_0}^{\infty}\frac{\varepsilon_m^{m\,(s-1)}}{2\,(s-1)}\,\mathcal{W}(0)+(N_b-1)^{2-D_{\mathcal{W}}}\left(\mathcal{W}\left(\frac{1}{(N_b-1)}\right)-2\,\mathcal{W}\left(\frac{1}{(N_b-1)}\right)\right)\sum_{m=m_0}^{\infty}\frac{\varepsilon_m^{m\,(s+1-D_{\mathcal{W}})}}{s+1-D_{\mathcal{W}}}\\ &+(N_b-1)^{2-D_{\mathcal{W}}}\sum_{m=m_0}^{\infty}\sum_{k=0}^{m-1}\mathcal{R}e\left(c_{k,2}\,\frac{\varepsilon_m^{m\,(s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,2}\,\mathbf{p}}}{s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,2}\,\mathbf{p}}-2\,c_{k,1}\,\frac{\varepsilon_m^{m\,(s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,1}\,\mathbf{p}}}{s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,1}\,\mathbf{p}}\right).\end{aligned}$$

(*R* 16)

As was mentioned in the introduction, we note that our result is stronger than the one previously obtained in [DL22c], where the values of the possible Complex Dimensions of the Weierstrass IFD included -2*. As we can see in relation (*$\mathcal{R}$*16) just above, the poles of the limit effective fractal zeta function* $\widetilde{\zeta}^{e}_{\mathcal{W}}$ *are exactly the same as the Complex Dimensions of the Weierstrass function itself; see Property 3.15, on page 40.*

As was explained previously in Property 3.13, on page 32, the remaining cases, i.e., when $N_b\geqslant 4$*, can be obtained in a similar manner, with entirely analogous conclusions. For the sake of concision, we do not include the associated (and heavy) computations associated with these cases.*

Proof.

i. We first prove the recurrence relation.

We restrict ourselves to sufficienly large values of $m\in\mathbb{N}$, i.e., $m\geqslant m_0$, for some suitable integer $m_0\in\mathbb{N}$.

We then have that, for $\mathcal{R}e\,(s)>D_{\mathcal{W}}$,

$$\widetilde{\zeta}^{e}_{m+1}(s) \;=\; \int_0^{\varepsilon_{m+1}^{m+1}} t^{s-3}\,\mathcal{V}_{m+1}(t)\,dt\,.$$

Let us now note that

$$\mathcal{V}_m\left(\varepsilon_m^m\right) \;=\; \frac{\varepsilon_m^m}{2}\,\left(\mathcal{W}(0)+\mathcal{W}\left(2\,\varepsilon_m^m\right)-2\,\mathcal{W}\left(\varepsilon_m^m\right)\right)$$

and

$$\mathcal{V}_{m+1}\left(\varepsilon_{m+1}^{m+1}\right) \;=\; \frac{\varepsilon_{m+1}^{m+1}}{2}\,\left(\mathcal{W}(0)+\mathcal{W}\left(2\,\varepsilon_{m+1}^{m+1}\right)-2\,\mathcal{W}\left(\varepsilon_{m+1}^{m+1}\right)\right).$$

Since

$$\varepsilon_{m+1}^{m+1} = \frac{1}{N_b}\,\varepsilon_m^m$$

and thanks to the scaling relation given in Property 2.1, on page 4; namely,

$$\mathcal{W}\left(\varepsilon_{m+1}^{m+1}\right) = \mathcal{W}\left(\frac{1}{N_b}\,\varepsilon_m^m\right) = \lambda\,\mathcal{W}\left(\varepsilon_m^m\right)+\cos\left(2\,\pi\,\varepsilon_m^m\right)$$

and

$$\mathcal{W}\left(2\,\varepsilon_{m+1}^{m+1}\right) = \mathcal{W}\left(\frac{2}{N_b}\,\varepsilon_m^m\right) = \lambda\,\mathcal{W}\left(2\,\varepsilon_m^m\right)+\cos\left(4\,\pi\,\varepsilon_m^m\right),$$

we can deduce that

$$\begin{aligned}
\mathcal{V}_{m+1}(\varepsilon_{m+1}^{m+1}) \;&=\; \frac{1}{N_b}\,\frac{\varepsilon_m^m}{2}\,\left(\mathcal{W}(0)+\lambda\,\left(\mathcal{W}\left(2\,\varepsilon_m^m\right)-2\,\mathcal{W}\left(\varepsilon_m^m\right)\right)+\cos\left(4\,\pi\,\varepsilon_m^m\right)-\cos\left(2\,\pi\,\varepsilon_m^m\right)\right)\\
&=\; \frac{\lambda}{N_b}\,\mathcal{V}_m(\varepsilon_m^m)+\frac{1}{N_b}\,\frac{\varepsilon_m^m}{2}\,(1-\lambda)+\frac{\varepsilon_m^m}{2}\,\left(\cos\left(4\,\pi\,\varepsilon_m^m\right)-2\,\cos\left(2\,\pi\,\varepsilon_m^m\right)\right)\\
&=\; \frac{\lambda}{N_b}\,\mathcal{V}_m(N_b\,\varepsilon_{m+1}^{m+1})+\frac{1}{N_b}\,\frac{N_b\,\varepsilon_{m+1}^{m+1}}{2}\,(1-\lambda)+\frac{N_b\,\varepsilon_{m+1}^{m+1}}{2}\,\left(\cos\left(4\,\pi\,N_b\,\varepsilon_{m+1}^{m+1}\right)-2\,\cos\left(2\,\pi\,N_b\,\varepsilon_{m+1}^{m+1}\right)\right).
\end{aligned}$$

We then obtain that

$$\begin{aligned}
\tilde{\zeta}_{m+1}^e(s) \;&=\; \int_0^{\varepsilon_{m+1}^{m+1}} t^{s-3}\,\tilde{\mathcal{V}}_{m+1}(t)\,dt\\
&=\; \int_0^{\frac{1}{N_b}\,\varepsilon_m^m} t^{s-3}\,\left(\frac{\lambda}{N_b}\,\tilde{\mathcal{V}}_m(N_b\,t)+\frac{1}{N_b}\,\frac{N_b\,t}{2}\,(1-\lambda)+\frac{N_b\,t}{2}\,\left(\cos\left(4\,\pi\,N_b\,t\right)-2\,\cos\left(2\,\pi\,N_b\,t\right)\right)\right)dt\\
&=\; \int_0^{\varepsilon_m^m} N_b^{3-s}\,t^{s-3}\,\left(\tilde{\mathcal{V}}_m(t)+\frac{t}{2}\,(1-\lambda)+\frac{t}{2\,N_b}\,\left(\cos\left(4\,\pi\,t\right)-2\,\cos\left(2\,\pi\,t\right)\right)\right)dt\\
&=\; N_b^{3-s}\,\tilde{\zeta}_m(s)+N_b^{3-s}\,\int_0^{\varepsilon_m^m} t^{s-2}\,\left(\frac{1}{2}\,(1-\lambda)+\frac{1}{2\,N_b}\,\left(\cos\left(4\,\pi\,t\right)-\cos\left(2\,\pi\,t\right)\right)\right)dt\\
&=\; N_b^{3-s}\,\tilde{\zeta}_m^e(s)+N_b^{3-s}\,\int_0^{\varepsilon_m^m} t^{s-2}\,\left(\frac{1}{2}\,(1-\lambda)+\frac{1}{2\,N_b}\,\mathcal{R}e\,\left(e^{i\,4\,\pi\,t}-2\,e^{i\,2\,\pi\,t}\right)\right)dt\\
&=\; N_b^{3-s}\,\tilde{\zeta}_m^e(s)+\frac{1}{2}\,(1-\lambda)\,N_b^{3-s}\,\frac{\varepsilon_m^{m\,(s-1)}}{s-1}+N_b^{3-s}\,\int_0^{\varepsilon_m^m} t^{s-2}\,\frac{1}{2\,N_b}\,\mathcal{R}e\,\left(e^{i\,4\,\pi\,t}-2\,e^{i\,2\,\pi\,t}\right)dt,
\end{aligned}$$

where the first equality is only valid for $\mathcal{R}e\,(s) > D_{\mathcal{W}}$, while the other equalities are valid for all s in $\mathbb{C}$.

ii. We now assume that the recurrence relation $(\mathcal{E}_m)$ (see relation $(\mathcal{R}15)$, on page 41) holds for all $m \geqslant m_0$. In particular, we have that, for all $m \geqslant m_0$ and for all s in $\mathbb{C}$,

$$\begin{aligned}\widetilde{\zeta}^e_m(s) \quad = \quad & \sum_{n=m_0}^{m} \frac{\varepsilon_n^{n\,(s-1)}}{2\,(s-1)}\,\mathcal{W}(0) + (N_b-1)^{2-D_{\mathcal{W}}}\left(\mathcal{W}\left(\frac{1}{(N_b-1)}\right) - 2\,\mathcal{W}\left(\frac{1}{(N_b-1)}\right)\right)\sum_{n=m_0}^{\infty}\frac{\varepsilon_n^{n\,(s+1-D_{\mathcal{W}})}}{s+1-D_{\mathcal{W}}} \\ & + (N_b-1)^{2-D_{\mathcal{W}}}\sum_{n=m_0}^{m}\sum_{k=0}^{n-1}\mathcal{R}e\left(c_{k,2}\,\frac{\varepsilon_n^{n\,(s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,2}\,\mathbf{p}}}{s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,2}\,\mathbf{p}} - 2\,c_{k,1}\,\frac{\varepsilon_n^{n\,(s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,1}\,\mathbf{p}}}{s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,1}\,\mathbf{p}}\right)\end{aligned} \tag{$\mathcal{R}\,17$}$$

This implies, in particular, that the poles of the m^{th} effective fractal zeta function $\widetilde{\zeta}^e_m$ are a subset of the set of Complex Dimensions of the Weierstrass function itself; see Property 3.15, on page 40.

a. We denote by $\mathcal{P}\left(\widetilde{\zeta}^e_m\right) \subset \mathbb{C}$ the set of poles of the zeta function $\widetilde{\zeta}^e_m$, and by $\mathcal{P}\left(\widetilde{\zeta}^e_{m_0}\right) \subset \mathbb{C}$ the set of poles of the zeta function $\widetilde{\zeta}_{m_0}$.

We note that

$$\mathcal{P}\left(\widetilde{\zeta}^e_{m_0}\right) \subset \{s \in \mathbb{C}\,,\, \mathcal{R}e(s) < 2\} \subset \{s \in \mathbb{C}\,,\, \mathcal{R}e(s) < 3\}\,.$$

We set

$$\mathcal{U}^+ = \left(\mathbb{C} \setminus \mathcal{P}\left(\widetilde{\zeta}_{m_0}\right)\right) \cap \{s \in \mathbb{C}\,,\, \mathcal{R}e(s) < 1\}\,.$$

$$\left(\text{resp.,}\ \mathcal{U}^- = \left(\mathbb{C} \setminus \mathcal{P}\left(\widetilde{\zeta}_{m_0}\right)\right) \cap \{s \in \mathbb{C}\,,\, 1 < \mathcal{R}e(s) < 3\}\right)$$

Then, the series

$$\sum_{m \geqslant m_0}\left(N_b^{3-s}\,\widetilde{\zeta}^e_m(s) + \frac{1}{2}\,(1-\lambda)\,N_b^{3-s}\,\frac{\varepsilon_m^{m\,(s-1)}}{s-1} + N_b^{3-s}\int_0^{\varepsilon_m^m} t^{s-2}\,\frac{1}{2\,N_b}\,\mathcal{R}e\left(e^{i\,4\,\pi\,t} - 2\,e^{i\,2\,\pi\,t}\right)dt\right)$$

is (locally) normally convergent, and, hence, uniformly convergent on $\mathcal{U}^+$ (resp., on $\mathcal{U}^-$). This ensures the existence of the limit effective fractal zeta function, i.e., the fractal zeta function associated with the Weierstrass Curve $\Gamma_{\mathcal{W}}$ (or with the Weierstrass IFD), given by

$$\widetilde{\zeta}^e_{\mathcal{W}} = \lim_{m\to\infty}\widetilde{\zeta}^e_m = \sum_{m \geqslant m_0} N_b^{3-s}\,\widetilde{\zeta}_m(s) + \frac{1}{2}\,(1-\lambda)\,N_b^{3-s}\,\frac{\varepsilon_m^{m\,(s-1)}}{s-1} + N_b^{3-s}\int_0^{\varepsilon_m^m} t^{s-2}\,\frac{1}{2\,N_b}\,\mathcal{R}e\left(e^{i\,4\,\pi\,t} - 2\,e^{i\,2\,\pi\,t}\right)dt\,.$$

Here, and in the remainder of this proof, a (complex–valued) meromorphic function f is viewed as a continuous function with values in $\mathbb{P}^1(\mathbb{C})$, equipped with the chordal metric, and such that, for any pole ω of f, $f(\omega)$ takes the value ∞ (for instance, as in [LvF13], Section 3. 4 and Appendix C).

More precisely, if $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$ denotes the Riemann sphere (or complex projective line), we can show that, for *the chordal metric*, defined, for all $(z_1, z_2) \in \left(\mathbb{P}^1(\mathbb{C})\right)^2$ by

$$\|z_1, z_2\| = \frac{|z_1 - z_2|}{\sqrt{1+|z_1^2|}\,\sqrt{1+|z_2^2|}}\,,\ \text{if } z_1 \neq \infty \text{ and } z_2 \neq \infty$$

and

$$\|z_1, \infty\| = \frac{1}{\sqrt{1 + |z_1^2|}}\,, \text{ if } z_1 \neq \infty\,,$$

we have, thanks to the uniform convergence of the series,

$$\lim_{m\to\infty} \left\|\tilde{\zeta}_m^e, \tilde{\zeta}_{\mathcal{W}}^e\right\| = 0\,.$$

Indeed, for any $\eta > 0$, we can choose $m_0 \in \mathbb{N}^\star$ such that, for all $s \in \mathbb{P}^1(\mathbb{C})$, we have that

$$\left|\tilde{\zeta}_m^e(s) - \tilde{\zeta}_{\mathcal{W}}^e(s)\right| \leqslant \eta\,,$$

and, hence, for all $s \in \mathbb{P}^1(\mathbb{C})$,

$$\left\|\tilde{\zeta}_m(s), \tilde{\zeta}_{\mathcal{W}}(s)\right\| \leqslant \left|\tilde{\zeta}_m(s) - \tilde{\zeta}_{\mathcal{W}}(s)\right| \leqslant \eta\,.$$

The sum of this series, i.e., the (uniform) limit fractal zeta function $\tilde{\zeta}_{\mathcal{W}}$, is holomorphic on $\mathcal{U}^+$ (resp., on $\mathcal{U}^-$). We can then deduce that, for all $m \geqslant m_0$, the zeta function $\tilde{\zeta}_m^e$ is meromorphic on $\mathbb{C} \setminus \{s \in \mathbb{C}\,,\, \mathcal{R}e(s) = 1\}$, and that its poles in $\mathbb{C} \setminus \{s \in \mathbb{C}\,,\, \mathcal{R}e(s) = 1\}$ are exactly the same as the poles of $\tilde{\zeta}_{m_0}^e$. Moreover, the results obtained in [DL22a] for the sequence of tube zeta functions associated with the Weierstrass IFD, which admit a meromorphic continuation to all of $\mathbb{C}$, obviously hold for the sequence of polyhedral tube zeta functions: hence, $\tilde{\zeta}_m^e$ is meromorphic on $\mathbb{C}$, and its poles belong to $\mathcal{P}\left(\tilde{\zeta}_{m_0}\right)$. Consequently, for all $m \geqslant m_0$, the poles of $\tilde{\zeta}_m$ are simple, and are the same as the poles of $\tilde{\zeta}_{m_0}$:

$$\mathcal{P}\left(\tilde{\zeta}_m^e\right) = \mathcal{P}\left(\tilde{\zeta}_{m_0}^e\right)\,.$$

b. Let ω be a pole of $\tilde{\zeta}_{m_0}^e$. Note that, thanks to the results in part *a.*, the poles of $\tilde{\zeta}_{m_0}^e$ are a subset of the set of Complex Dimensions of the Weierstrass function itself; see Property 3.15, on page 40. More precisely, this means that ω is of the following form,

$$D_{\mathcal{W}} - k\,(2 - D_{\mathcal{W}}) + i\,\ell_k\,\mathbf{p}\,, \text{ with } \ell_k \in \mathbb{Z} \text{ and where } k \in \mathbb{N} \text{ is arbitrary.}$$

Then, for all $m \geqslant m_0$, ω is also a pole of $\tilde{\zeta}_m^e$ (by the exact m^{th} natural polyhedral volume formula and its consequences, in particular, relation ($\mathcal{R}17$), on page 43 above); i.e., for all $m \geqslant m_0$:

$$\tilde{\zeta}_m^e(\omega) = \tilde{\zeta}_{m_0}^e(\omega) = \infty\,.$$

Hence, since

$$\lim_{m\to\infty} \tilde{\zeta}_m^e(\omega) = \tilde{\zeta}_{\mathcal{W}}^e(\omega)\,,$$

it implies that

$$\tilde{\zeta}_{\mathcal{W}}^e(\omega) = \infty\,.$$

Therefore, ω is also a pole of $\tilde{\zeta}_{\mathcal{W}}^e$.

c. Now, if $\mathcal{P}\left(\tilde{\zeta}_{\mathcal{W}}^e\right) \subset \mathbb{C}$ denotes the set of poles of the limit fractal zeta function $\tilde{\zeta}_{\mathcal{W}}^e$, we similarly have that

$$\lim_{m\to\infty} \mathcal{P}\left(\tilde{\zeta}_m^e\right) = \mathcal{P}\left(\tilde{\zeta}_{m_0}^e\right) \subset \mathcal{P}\left(\tilde{\zeta}_{\mathcal{W}}^e\right)\,.$$

d. Conversely, if ω is a pole of $\widetilde{\zeta}^e_{\mathcal{W}}$ (i.e., $\widetilde{\zeta}^e_{\mathcal{W}}(\omega) = \infty$) and since, for all $m \geqslant m_0$, $\widetilde{\zeta}^e_m(\omega) = \widetilde{\zeta}^e_{m_0}(\omega)$, as well as

$$\lim_{m\to\infty} \widetilde{\zeta}^e_m(\omega) = \widetilde{\zeta}^e_{\mathcal{W}}(\omega)\,,$$

we must have that, for all $m \geqslant m_0$,

$$\widetilde{\zeta}^e_m(\omega) = \widetilde{\zeta}^e_{m_0}(\omega) = \widetilde{\zeta}^e_{\mathcal{W}}(\omega) = \infty\,.$$

Hence, ω is also a pole of $\widetilde{\zeta}^e_{m_0}$ (i.e. also, for all $m \geqslant m_0$, a pole of $\widetilde{\zeta}^e_m$).

e. Since, for all $m \geqslant m_0$, the poles of $\widetilde{\zeta}^e_m$ are known to be simple (see *a.* above), we also have to show that the poles of $\widetilde{\zeta}^e_{\mathcal{W}}$ are all simple.

For all integers $q \geqslant 1$, we have that

$$\mathrm{res}_q\left(\widetilde{\zeta}^e_m, \omega\right) = \lim_{s\to\omega} (s-\omega)^q\ \widetilde{\zeta}^e_m(\omega)$$

and

$$\begin{aligned}
\mathrm{res}_q\left(\widetilde{\zeta}^e_{\mathcal{W}}, \omega\right) &= \lim_{s\to\omega} (s-\omega)^q\ \widetilde{\zeta}^e_{\mathcal{W}}(\omega) \\
&= \lim_{s\to\omega} (s-\omega)^q \lim_{m\to\infty} \widetilde{\zeta}^e_m(s) \\
&= \lim_{s\to\omega} \lim_{m\to\infty} (s-\omega)^q\ \widetilde{\zeta}^e_m(s) \\
&= \lim_{m\to\infty} \lim_{s\to\omega} (s-\omega)^q\ \widetilde{\zeta}^e_m(s) \\
&= \lim_{m\to\infty} \mathrm{res}_q\left(\widetilde{\zeta}^e_m, \omega\right)\,,
\end{aligned}$$

where the interchange of limits is justified because of the local uniform convergence on $\mathbb{C}$ – and hence, also, on a compact neighborhood of ω.

Since, for all integers $q > 1$, because we know from the results in [DL22c] that, still for $m \geqslant m_0$, the poles of $\widetilde{\zeta}^e_m$ are simple,

$$\mathrm{res}_q\left(\widetilde{\zeta}^e_m, \omega\right) = 0\,;$$

it then follows that, for any fixed integer $q > 1$,

$$\mathrm{res}_q\left(\widetilde{\zeta}^e_{\mathcal{W}}, \omega\right) = \lim_{m\to\infty} \mathrm{res}_q\left(\widetilde{\zeta}^e_m, \omega\right) = 0\,.$$

Hence, since ω is also a pole of $\widetilde{\zeta}^e_{\mathcal{W}}$ by the previous argument, we deduce that ω must be a simple pole of $\widetilde{\zeta}^e_{\mathcal{W}}$.

Therefore, the set of poles of $\widetilde{\zeta}^e_{m_0}$ (or, equivalently, of $\widetilde{\zeta}^e_m$, for all $m \geqslant m_0$) coincides with the set of poles of $\widetilde{\zeta}^e_{\mathcal{W}}$.

Consequently, as desired, the poles of the limit effective fractal zeta function $\widetilde{\zeta}^e_{\mathcal{W}}$ are simple, and are the same as the poles of $\widetilde{\zeta}^e_{m_0}$.

f. We have that

$$\begin{aligned}
\tilde{\zeta}_m^e(s) &= \int_0^{\varepsilon_m^m} t^{s-3}\,\tilde{\mathcal{V}}_m(t)\,dt \\
&= \int_0^{\varepsilon_m^m} \frac{t^{s-2}}{2}\,\mathcal{W}(0)\,dt \\
&\quad +(N_b-1)^{2-D_{\mathcal{W}}}\left(\mathcal{W}\left(\frac{1}{(N_b-1)}\right)-2\,\mathcal{W}\left(\frac{1}{(N_b-1)}\right)\right)\int_0^{\varepsilon_m^m} t^{s-3+(2-D_{\mathcal{W}})}\,dt \\
&\quad +(N_b-1)^{2-D_{\mathcal{W}}}\sum_{k=0}^{m-1}\int_0^{\varepsilon_m^m} t^{s-2+k(2-D_{\mathcal{W}})}\,\mathcal{R}e\left(c_{k,2}\,t^{i\,\ell_{k,2}\,\mathbf{p}}-2\,c_{k,1}\,t^{i\,\ell_{k,1}\,\mathbf{p}}\right)dt \\
&= \frac{\varepsilon_m^{m\,(s-1)}}{2\,(s-1)}\,\mathcal{W}(0)+(N_b-1)^{2-D_{\mathcal{W}}}\left(\mathcal{W}\left(\frac{1}{(N_b-1)}\right)-2\,\mathcal{W}\left(\frac{1}{(N_b-1)}\right)\right)\frac{\varepsilon_m^{m\,(s-D_{\mathcal{W}})}}{s-D_{\mathcal{W}}} \\
&\quad +(N_b-1)^{2-D_{\mathcal{W}}}\sum_{k=0}^{m-1}\mathcal{R}e\left(c_{k,2}\,\frac{\varepsilon_m^{m\,(s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,2}\,\mathbf{p}}}{s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,2}\,\mathbf{p}}-2\,c_{k,1}\,\frac{\varepsilon_m^{m\,(s-2+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,1}\,\mathbf{p}}}{s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,1}\,\mathbf{p}}\right),
\end{aligned}$$

where the first equality is only valid for $\mathcal{R}e\,(s) > D_{\mathcal{W}}$, while the other ones are valid for all s in $\mathbb{C}$.

We then deduce the following explicit formula, for the limit fractal effective zeta function $\tilde{\zeta}_{\mathcal{W}}^e$, for all s in $\mathbb{C}$,

$$\begin{aligned}
\tilde{\zeta}_{\mathcal{W}}^e(s) &= \sum_{m=m_0}^{\infty}\frac{\varepsilon_m^{m\,(s-1)}}{2\,(s-1)}\,\mathcal{W}(0)+(N_b-1)^{2-D_{\mathcal{W}}}\left(\mathcal{W}\left(\frac{1}{(N_b-1)}\right)-2\,\mathcal{W}\left(\frac{1}{(N_b-1)}\right)\right)\sum_{m=m_0}^{\infty}\frac{\varepsilon_m^{m\,(s+1-D_{\mathcal{W}})}}{s+1-D_{\mathcal{W}}} \\
&\quad +(N_b-1)^{2-D_{\mathcal{W}}}\sum_{m=m_0}^{\infty}\sum_{k=0}^{m-1}\mathcal{R}e\left(c_{k,2}\,\frac{\varepsilon_m^{m\,(s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,2}\,\mathbf{p}}}{s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,2}\,\mathbf{p}}-2\,c_{k,1}\,\frac{\varepsilon_m^{m\,(s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,1}\,\mathbf{p}}}{s-1+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,1}\,\mathbf{p}}\right),
\end{aligned}$$

as desired. □

Property 3.17 (**From the m^{th} Local Effective Polyhedral Zeta Function, to the m^{th} Local Effective Tube Zeta Function**)**.**

Given $m \in \mathbb{N}$, the Lebesgue measure $\mathcal{V}_m^{tube}\left(\varepsilon_m^m\right)$ of the tubular neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m\right)$ can be connected to the Lebesgue measure $\mathcal{V}_m\left(\varepsilon_m^m\right)$ of the m^{th} polyhedral neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ introduced in Definition 3.4, on page 23, by means of the following relation:

$$\mathcal{V}_m\left(\varepsilon_m^m\right) = \mathcal{V}_m^{tube}\left(\varepsilon_m^m\right) + \mathcal{R}_m\,,$$

where

$$\mathcal{V}_m^{tube}\left(\varepsilon_m^m\right) = \mu_{\mathcal{L}}\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)\,,$$

and where the sequence of error zeta functions $\left(\mathcal{R}_m\right)_{m\geq m_0}$ *(locally) uniformly converges to 0.*

This ensures, for the associated zeta function

$$s \mapsto \int_0^{\varepsilon_m^m} t^{s-3}\,\mathcal{R}_m(t)\,dt\,,$$

that (locally, uniformly on $\mathbb{C}$)

$$\lim_{m\to\infty} \int_0^{\varepsilon_m^m} t^{s-3}\, \mathcal{R}_m(t)\, dt = 0\,.$$

Proof. This directly comes from Property 3.4, on pag 23, in conjunction with Property 3.11, on page 26. Given $\eta > 0$, there exist then $m_0 \in \mathbb{N}$ such that, for all $m \geqslant m_0$,

$$\left|\mathcal{V}_m\left(\varepsilon_m^m\right) - \mathcal{V}_m^{tube}\left(\varepsilon_m^m\right)\right| \leqslant \eta\,.$$

We then proceed as in the proof of Theorem 3.16, on page 40. Since, for admissible values of $s \in \mathbb{C}$,

$$\int_0^{\varepsilon_m^m} t^{s-3}\, \mathcal{R}_m(t)\, dt = \int_0^{\varepsilon_m^m} t^{s-3} \left(\tilde{\mathcal{V}}_m(t) - \tilde{\mathcal{V}}_m^{tube}(t)\right) dt\,,$$

we obtain the expected (locally) uniform convergence.

□

Theorem 3.18 (**Complex Dimensions of the Weierstrass Curve**)**.**

The possible Complex Dimensions of The Weierstrass Curve (or of the Weierstrass IFD) are all simple, and given as follows:

$$D_{\mathcal{W}} - k\,(2 - D_{\mathcal{W}}) + i\,\ell\,\mathbf{p} \quad , \quad \text{with } k \in \mathbb{N}\,,\ \ell \in \mathbb{Z}\,.$$

Proof.

We simply apply Property 3.16, on page 40, ensures the existence of an integer $m_0 \in \mathbb{N}$ such that the poles of the limit fractal effective zeta function $\widetilde{\zeta}_{\mathcal{W}}^e$, i.e., the fractal zeta function associated with the Weierstrass Curve $\Gamma_{\mathcal{W}}$, are the same as the poles of the fractal zeta function $\widetilde{\zeta}_{m_0}^e$.

□

Corollary 3.19. *The possible Complex Dimensions of The Weierstrass Curve (or of the Weierstrass IFD) are the same as those obtained in [DL22c] and are given by Theorem 3.18, on page 47. Furthermore, if they are actual poles, they are simple poles. More precisely, those possible Complex Dimensions are $D_{\mathcal{W}} - k\,(2 - D_{\mathcal{W}}) + i\,\ell_k\,\mathbf{p}$, with $\ell_k \in \mathbb{Z}$ and where $k \in \mathbb{N}$ is arbitrary.*

Moreover, they coincide with the actual poles of the tube fractal zeta function $\widetilde{\zeta}_m^{e,tube}$, for all $m \geqslant m_0$. In particular, we then deduce that the following possible Complex Dimensions previously obtained in [DL22c], i.e., $1 - 2\,k + i\,\ell\,\mathbf{p}$, with $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$, along with 0 and 1, are not actual Complex Dimensions.

Furthermore, the Complex Dimensions associated with $D_{\mathcal{W}}$ are actual Complex Dimensions of $\Gamma_{\mathcal{W}}$.

For the exceptional cases, we refer to [DL22c].

4 Revisiting the Computation of the Minkowski Dimension

When it comes to computing the box–counting – or Minkowski dimension – of a fractal, one usually proceeds by using a (classical) cover of the fractal under study, constituted of balls; namely, in dimension 2, disks or squares; see Definition 4.1, on page 48 below.

However, when a nonaffine, complicated fractal curve, as the Weierstrass Curve is involved, such methods are not optimal. In fact, covering such a curve with disks or squares is not natural. We hereafter propose to consider, instead, the polyhedral neighborhood introduced in Definition 3.4, on page 23.

Definition 4.1 (**Box–Counting Dimension**)**.**

As can be found, for instance, in [Fal86], we recall that the *box–counting dimension* (or *box dimension*, in short), of $\Gamma_{\mathcal{W}}$, is given by

$$D_{\mathcal{W}} = -\lim_{\delta \to 0^+} \frac{\ln N_\delta\left(\Gamma_{\mathcal{W}}\right)}{\ln \delta}\,, \quad (\diamond)$$

where $N_\delta\left(\Gamma_{\mathcal{W}}\right)$ stands for any of the following quantities:

i. the smallest number of sets (here, subsets of $\mathbb{R}^2$) of diameter at most δ that cover $\Gamma_{\mathcal{W}}$ on $[0,1[$;

ii. the smallest number of closed balls (disks, here) of radius δ that cover $\Gamma_{\mathcal{W}}$ on $[0,1[$;

iii. the smallest number of cubes (squares, here) of side δ that cover $\Gamma_{\mathcal{W}}$ on $[0,1[$;

iv. the number of δ–mesh cubes (squares, here) that intersect $\Gamma_{\mathcal{W}}$ on $[0,1[$;

v. the largest number of disjoint balls (disks, here) of radius δ with centers in $\Gamma_{\mathcal{W}}$ on $[0,1[$.

Furthermore, for the Weierstrass Curve $\Gamma_{\mathcal{W}}$, as, more generally, for any bounded subset of Euclidean space – the box–counting dimension coincides with the Minkowski dimension.

Theorem 4.1. *The box–counting dimension – or, equivalently, the Minkowski dimension – of the Weierstrass Curve* $\Gamma_{\mathcal{W}}$ *exists and is equal to* $D_{\mathcal{W}} = 2 + \dfrac{\ln \lambda}{\ln N_b}$.

Proof. We simply apply the result given in Corollary 3.12, on page 31. We then have the existence of an integer $m_0 \in \mathbb{N}$ such that

$$\forall\, m \geqslant m_0 \ : \quad \Gamma_{\mathcal{W}} \subset \mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\,.$$

In other words, for all integers $m \geqslant m_0$, the polyhedral neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ covers the Weierstrass Curve $\Gamma_{\mathcal{W}}$.

Note that we thus dispose, with the sequence of polyhedral domains $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m \geqslant m_0}$, of a nonusual (but admissible, in the sense of Definition 4.1, on page 48 above) sequence of covers. However, $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m \geqslant m_0}$ is the most natural and optimal sequence of covers of the Weierstrass Curve $\Gamma_{\mathcal{W}}$, insofar that for

all $m \geqslant m_0$, each domain $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ contains the Weierstrass Curve $\Gamma_{\mathcal{W}}$, while, at the same time, the sequence $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m \geqslant m_0}$ converges to $\Gamma_{\mathcal{W}}$. This, in particular, means that when $m \to \infty$, the two–dimensional Lebesgue measure (i.e., area) of each polygon belonging to $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ tends to 0.

Next, observe that, thanks to Remark 3.1, on page 19, we have that:

$$h_m \gg \varepsilon_m^m \, .$$

For all $m \in \mathbb{N}^\star$, each polygon belonging to $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ has a vertical height at most equal to $(N_b - 1)\, h_m$ (this directly comes from the definition of h_m; see relation (5), on page 11, and an horizontal width at most equal to $(N_b - 1)\, \varepsilon_m^m$. Note now that, thanks to Remark 3.1, on page 19, we have that:

$$h_m \gg \varepsilon_m^m \, .$$

Hence, it is the cohomology infinitesimal ε_m^m which plays the role of the elementary diameter δ in Definition 4.1, on page 48. Moreover, the area of each polygon belonging to $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ also corresponds to the two–dimensional Lebesgue measure of the reunion of $\dfrac{(N_b - 1)\, h_m}{\varepsilon_m^m}$ elementary and smaller polygons, each of diameter ε_m^m, where *the diameter* of a polygon is to be understood in the sense of the largest distance between any pair of vertices of the considered polygon.

Since $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ is itself constituted of the reunion of $(2\, N_b^m - 1) = \dfrac{2\, \varepsilon_m^{-m}}{N_b - 1} - 1$ polygons, we then obtain the box–counting dimension as

$$- \lim_{m \to \infty} \frac{\ln\left(\left(\dfrac{2\, \varepsilon_m^{-m}}{N_b - 1} - 1\right) \dfrac{(N_b - 1)\, h_m}{\varepsilon_m^m}\right)}{\ln \varepsilon_m^m} = - \lim_{m \to \infty} \frac{\ln\left(\varepsilon_m^{-2\,m}\, h_m\right)}{\ln \varepsilon_m^m} \, .$$

Thanks to the estimate given in Property 2.12, on page 15, we have that

$$C_{inf}\, \left(\varepsilon_m^m\right)^{2 - D_{\mathcal{W}}} \leqslant h_m \leqslant C_{sup}\, \left(\varepsilon_m^m\right)^{2 - D_{\mathcal{W}}} \, ,$$

which ensures that

$$C_{inf}\, \left(\varepsilon_m^m\right)^{-D_{\mathcal{W}}} \leqslant \varepsilon_m^{-2\,m}\, h_m \leqslant C_{sup}\, \left(\varepsilon_m^m\right)^{-D_{\mathcal{W}}} \, ,$$

and, consequently,

$$\ln C_{inf} - D_{\mathcal{W}}\, \ln\left(\varepsilon_m^m\right) \leqslant \ln\left(\varepsilon_m^{-2\,m}\, h_m\right) \leqslant \ln C_{sup} - D_{\mathcal{W}}\, \ln\left(\varepsilon_m^m\right) \, .$$

Since $\lim\limits_{m \to \infty} \varepsilon_m^m = 0$, we obviously have that

$$\lim_{m \to \infty} \frac{\ln C_{inf}}{\ln\left(\varepsilon_m^m\right)} = \lim_{m \to \infty} \frac{\ln C_{sup}}{\ln\left(\varepsilon_m^m\right)} = 0 \, ,$$

from which we immediately deduce that the value of the box–counting dimension of the Weierstrass Curve is exactly

$$- \lim_{m \to \infty} \frac{\ln\left(\varepsilon_m^{-2\,m}\, h_m\right)}{\ln \varepsilon_m^m} = D_{\mathcal{W}} \, .$$

This concludes the proof of the existence of the box–counting (i.e., the Minkowski) dimension and of the fact that it is given by $D_{\mathcal{W}} = 2 + \dfrac{\ln \lambda}{\ln N_b}$.

□

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