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# LIOUVILLE PROPERTY FOR GROUPS AND CONFORMAL DIMENSION

NICOLÁS MATTE BON, VOLODYMYR NEKRASHEVYCH, AND TIANYI ZHENG

ABSTRACT. Conformal dimension is a fundamental invariant of metric spaces, particularly suited to the study of self-similar spaces, such as spaces with an expanding self-covering (e.g. Julia sets of complex rational functions). The dynamics of these systems are encoded by the associated iterated monodromy groups, which are examples of contracting self-similar groups. Their amenability is a well-known open question. We show that if  $G$  is an iterated monodromy group, and if the (Alfhors-regular) conformal dimension of the underlying space is strictly less than 2, then every symmetric random walk with finite second moment on  $G$  has the Liouville property. As a corollary, every such group is amenable. This criterion applies to all examples of contracting groups previously known to be amenable, and to many new ones. In particular, it implies that for every post-critically finite complex rational function  $f$  whose Julia set is not the whole sphere, the iterated monodromy group of  $f$  is amenable.

## 1. INTRODUCTION

Conformal dimension was introduced in the late 1980s by P. Pansu in the study of quasi-isometries of hyperbolic spaces and related constructions (see, for example [40]). The conformal dimension of a metric space is the infimum of the Hausdorff dimensions of spaces quasi-symmetric to it. Since quasi-isometries of hyperbolic spaces correspond to quasi-symmetries of their boundaries, the conformal dimension is a natural invariant of hyperbolic groups and their boundaries. Since its definition, the conformal (and related Ahlfors-regular conformal) dimension became an important invariant in geometric group theory and dynamical systems. See [31] for a survey of its properties and applications.

Ahlfors-regular conformal dimension is especially useful and natural for self-similar metric spaces. Boundaries of hyperbolic groups are examples of such self-similar spaces. Another natural class of examples is metric spaces together with an expanding (branched) covering map  $f : \mathcal{J} \rightarrow \mathcal{J}$  (for example complex rational functions in restriction to their Julia set). Their quasi-conformal geometry is the subject of the works [19–21] by P. Haïssinsky and K. Pilgrim.

Expanding self-coverings are encoded by the associated *iterated monodromy groups*, introduced in the early 2000s, see [34, 37]. They are examples of *contracting self-similar groups*. The self-covering is reconstructed from the iterated monodromy group as their *limit dynamical system*. The potential importance of the conformal dimension to the study of contracting self-similar groups was observed in [20].

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The amenability of iterated monodromy groups has been a well-known open question since their introduction. In fact, amenability of the iterated monodromy group of the complex polynomial  $z^2 - 1$  (also known as the *Basilica group*) was asked even before the iterated monodromy groups were defined (see [18]).

Amenability of the Basilica group was proved by L. Bartholdi and B. Virág in [9]. Their proof was later generalized to include iterated monodromy groups of all post-critically finite complex polynomials in [7] (and more generally, all groups generated by *bounded automata*). Their methods were extended to larger classes of self-similar groups in [1, 2].

Amenability was proved in these papers by constructing a self-similar random walk step distribution  $\mu$  on the group and using the self-similarity to prove that the asymptotic entropy (equivalently, speed) of the  $\mu$ -random walk is zero (see also [26]). By the results of V. Kaimanovich and A. Vershik [27] and Y. Derriennic [12] this implies the triviality of the Poisson boundary of the corresponding random walk, also known as the *Liouville property* that all bounded  $\mu$ -harmonic functions are constant, which in turn implies amenability of the group. Self-similarity is a very restrictive condition in this case, and it is not clear how to construct such self-similar random walks in general (it seems that a finitely supported self-similar step distribution does not exist in most cases), so the corresponding groups have to be embedded into special “mother groups” for which such a self-similar random walk can be constructed. For groups generated by bounded automata, this self-similarity restriction was removed in [1], by showing that every symmetric finitely supported measure generates a Liouville random walk.

A different approach to prove the amenability of groups is based on the notion of extensively amenable actions, first introduced by K. Juschenko and N. Monod [24]. This method was used in [25] to show amenability of a class of iterated monodromy groups. This approach allows one to prove the amenability of iterated monodromy groups of complex rational functions with “crochet” Julia sets, see [39]. The framework of extensive amenability is not based on the study of the Liouville property for random walks, but random walks are implicitly present in it, as one of the main conditions used to prove amenability is recurrence of the induced random walk on an orbit of some action of the group.

We show in our paper how conformal geometry can be used to prove the Liouville property for symmetric random walks under suitable moment conditions on many contracting groups (equivalently, iterated monodromy groups of expanding covering maps).

Namely, we prove the following.

**Theorem 1.1.** *Let  $(G, \mathfrak{B})$  be a finitely generated contracting self-similar group, and suppose that its limit space has Ahlfors-regular conformal dimension  $\alpha$ , with  $\alpha < 2$ . Then for every symmetric measure  $\mu$  on  $G$  having finite  $\beta$ -moment for some  $\beta > \alpha$ , the  $\mu$ -random walk is Liouville. In particular, any such group is amenable.*

**Corollary 1.2.** *Let  $f$  be a post-critically finite complex rational function such that its Julia set is not the whole sphere. Then the iterated monodromy group of  $f$  is amenable.*

Our results cover many new examples of amenable groups. Namely, all previously known contracting amenable groups had limit spaces of conformal dimension equal to 1 (this is true for any contracting group which can be generated by an automaton of polynomial activity in the sense of S. Sidki [44], see Theorem 6.1). In this case, Theorem 1.1 implies that any random walk on them generated by symmetric measures of finite

$(1 + \varepsilon)$ -moment is Liouville, encompassing various previous results for some more restricted classes of groups and measures. Beyond this case, Theorem 1.1 can be applied to contracting groups whose amenability was an open question until now, in particular to groups whose limit space is homeomorphic to the Sierpiński carpet. As shown in [21], there exist hyperbolic rational maps with Sierpiński carpet Julia sets whose conformal dimensions are arbitrarily close to 2 (the corresponding groups must therefore be generated by automata of exponential activity).

It is known that for symmetric random walks with finite second moment, sufficiently slow decay of return probability, namely  $\mu^{(2n)}(id)$  decaying slower than  $\exp(-n^{1/2})$  implies that  $\mu$ -random walk is Liouville, see [41]. However this result is not applicable in the study of contracting self-similar groups: first it is not clear for what subclass of groups one can expect such a slow decay, secondly, even in the case of bounded automata groups, it is challenging to prove lower estimates for return probability. Indeed, to the best of our knowledge, for all contracting groups whose amenability has been proven so far and those covered by Theorem 1.1, the best known lower bound for  $\mu^{(2n)}(id)$  comes from entropy, that is,  $\mu^{(2n)}(id) \geq \exp(-H(X_{2n}))$ . It is an open problem to find methods to establish sharper lower estimates for return probability, even in examples such as the Basilica group, and a tightly related problem is to find contracting groups that are amenable but not Liouville for symmetric simple random walks (see § 6.4 for an example of non-Liouville contracting group).

**1.1. On the proof.** Let us give some further insight on Theorem 1.1 and compare it with previous work. Recall that contracting group  $G$  is generated by a finite state-automata over a finite alphabet  $X$ , and has a natural action on the associated rooted tree  $X^*$  whose vertex set consists of finite words. For every integer  $n$ , the action of an element  $g \in G$  on  $X^*$  is determined by two pieces of information: the permutation induced on the  $n$ -th level of the tree, and associated collection of sections  $(g|_v)_{v \in X^n}$ , which describe the action of  $g$  on the subtrees rooted at level  $n$ . In order to show vanishing of the asymptotic entropy of the random walk measure  $\mu$ , one is immediately led to study the distribution of sections  $\mathbf{g}_t|_v$  for a sample  $\mathbf{g}_t$  of the  $\mu$ -random walk. More precisely, it is enough to show that the rate of growth of the sum of word lengths of all sections  $\mathbf{g}_t|_v$  at level  $n$  is bounded by some constant which tends to 0 as  $n$  tends to infinity (Proposition 5.4). This criterion was the starting point, in various different formulations, of the previous results establishing the Liouville property for random walk on automata groups [1, 2, 7, 9]. All these results were dealing with groups generated by some explicit automaton of polynomial (in fact, bounded or linear) activity. This means that for every element  $s$  in a generating set of the group, the number of vertices  $v \in X^n$  such that the sections  $s|_v$  is non-trivial is bounded by a polynomial in  $n$ . The behaviour of the process  $\mathbf{g}_t|_v$  was analysed by explicit computations based on the form of the automaton. In order to control the behaviour of the section, one needs to first understand the behaviour of the process  $\mathbf{g}_t(v)$ , which performs a random walk on the finite Schreier graph associated with the  $n$ -th level of the tree. A crucial feature in all previous examples is the fact that the sections  $\mathbf{g}_t|_v$  of the random walk can be essentially controlled by studying the visits of  $\mathbf{g}_t(v)$  to some uniformly finite subset consisting of “singular points”. In the simplest case of automata groups of bounded activity, the singular points are just the vertices  $w$  such that  $s|_w$  is nontrivial for some generator  $s$  (for groups of polynomial activity, they can be defined inductively on the activity degree). In this situation, the length of the  $\mathbf{g}_t|_v$  can be estimated by the number of *traverses* up to time  $t$  of the walk  $(\mathbf{g}_i(v))$  between certain pairs of distinct singular points. The expected number

of traverses can be estimated by analysing the Schreier graphs explicitly and computing the  $\ell^2$ -capacity (or effective conductance) between singular points. This is precisely the method used in [1] to show that all symmetric finitely supported measures on automata groups of bounded activity generate a Liouville random walk. The finiteness of the singular points can equivalently be interpreted in terms of germs of the action on the boundary of the tree  $X^\omega$ , and is crucial also to apply the framework of extensive amenability [3, 25, 39], which relies in a similar way on recurrence of the Schreier graphs to control visits to the singular points, and deduce amenability.

For iterated monodromy groups, the finiteness condition of singular points seems to be related to the existence of countable separating subsets in the limit space. This is a restrictive condition limited to examples with limit space of conformal dimension 1, see [39].

For a group satisfying the assumption in Theorem 1.1, the relevant set of singularities is intrinsically infinite (corresponding to a Cantor subset of the boundary of the tree). An automaton generating the group will typically have exponential activity and might be quite complicated. Our proof of Theorem 1.1 is no longer based on the combinatorial analysis of some explicit set of generators, but instead on the study of contracting groups through the theory of limit spaces. It has three ingredients. The first ingredient, purely geometric, is Theorem 2.17, which estimates the total length of  $n$ -th level sections of a word  $g = s_t \cdots s_1$  in terms of the paths that the word describes on the corresponding Schreier graph. It says that in order for the total length of sections to be large, these paths must traverse many times between certain regions which approximate some disjoint parts of the limit space (which roughly speaking correspond to different *tiles*). The precise identifications of these regions requires the framework of *contracting models* developed in [36, 38]. The second ingredient is Theorem 5.6, where the assumption that conformal dimension of the limit space is less than 2 is used to show that the  $\ell^2$ -capacity between such disjoint regions must tend to 0 as  $n \rightarrow \infty$ . This is reminiscent of the results of M. Carrasco [11] and J. Kigami [28], which relate the conformal dimension of a metric space to potential theory on graphs that approximate the space (although our proof of Theorem 5.6 is self-contained). These two ingredients put us in position to apply a traverse-counting argument to show vanishing of random walk asymptotic entropy, at least for symmetric finitely supported measures.

The third ingredient, which allows to extend the result to measures under a moment condition, is the study of another numeric invariant for contracting self-similar groups: *critical exponent*. For every  $p \in (0, \infty]$  we define the  $\ell^p$ -contraction coefficient  $\eta_p$  of a finitely generated contracting group  $(G, \mathfrak{B})$ . It is the exponential rate of decay of the  $\ell^p$ -norm of the vector of lengths of the sections  $g|_v$  for the vertices  $v$  of the  $n$ th level of the tree. The condition for a self-similar group to be contracting is equivalent to the condition  $\eta_\infty < 1$ . The function  $p \mapsto \eta_p$  is continuous and non-increasing. The critical exponent  $p_c(G, \mathfrak{B})$  is the infimum of the values  $p$  for which  $\eta_p < 1$ . It is a measure of the complexity of the group. For example, the action on the tree of every element  $g \in G$  of length  $n$  can be explicitly described with  $O(n^{p_c+\epsilon})$  bits of information. In particular,  $p_c < 1$  implies sub-exponential volume growth. In our setting, such complexity bounds can be used to control contribution to entropy from long jumps of the random walk.

In analogy to critical constant for recurrence introduced in [14], for a finitely generated group  $G$  equipped with word metric, we define the (symmetric) *critical constant for the Liouville property*, denoted by  $\text{Cr}_{\text{Liouv}}(G)$ , to be the infimum of  $p$ , such that all symmetric random walks with finite  $p$ -moment are Liouville. For a contracting group  $(G, \mathfrak{B})$  we

denote by  $\text{ARdim}(G, \mathfrak{B})$  the Ahlfors-regular conformal dimension of its limit space. Between these critical quantities and conformal dimension, we show the following inequality.

**Theorem 1.3.** *Let  $(G, \mathfrak{B})$  be a finitely generated contracting self-similar group such that  $\text{ARdim}(G, \mathfrak{B}) < 2$ , then*

$$C_{\text{Liouv}}(G) \leq p_c(G, \mathfrak{B}) \leq \text{ARdim}(G, \mathfrak{B}).$$

Then Theorem 1.1 above is a consequence of this inequality. The second inequality is not sharp in general, see the discussion after Theorem 4.7; we do not know of an example of a contracting group for which the first inequality is strict.

**1.2. Organization of the article.** Section 2 of the paper is an overview of the theory of self-similar contracting groups and their limit spaces. In Section 3 we describe the natural class of pairwise (weakly) quasi-symmetric metrics on the limit space. Similarly to the case of boundaries of hyperbolic groups, examples of such metrics can be defined as the visual metrics on the boundary of a natural Gromov-hyperbolic graph (the *self-similarity complex* of the group). The Ahlfors-regular conformal dimension  $\text{ARdim}(G, \mathfrak{B})$  is defined as the infimum of the Hausdorff dimensions of metrics weakly quasisymmetric to a visual metric, see [20]. If the group is *self-replicating* (which is automatically satisfied, for example, for the iterated monodromy groups of postcritically finite rational functions), then any weak quasisymmetry is actually a quasisymmetry (we do not know if this is true in general).

Section 4 discusses the *critical exponent*  $p_c(G, \mathfrak{B})$  for contracting self-similar groups as mentioned above. We prove that the critical exponent  $p_c(G, \mathfrak{B})$  of a contracting group is not larger than the conformal dimension  $\text{ARdim}(G, \mathfrak{B})$ , see Theorem 4.7. Lower estimates of for  $p_c(G, \mathfrak{B})$ , and hence also for  $\text{ARdim}(G, \mathfrak{B})$ , can be obtained from exhibiting a carefully chosen sequence of group elements. In Theorem 4.11, we show a relation between  $\ell^p$ -contraction and spectral radius of the linear operator  $T_p$ , where  $T_p$  is the Thurston linear transform adapted to  $\ell^p$  calculations.

Section 5 contains proofs of statements on Liouville property. It starts with a brief review of preliminaries on random walk and electric network theory, bounded harmonic functions and the entropy criterion. We then show a direct bound on  $\ell^q$ -capacity on the Schreier graphs of action of a contracting group  $(G, \mathfrak{B})$  on levels of the tree, which is valid for  $q > \text{ARdim}(G, \mathfrak{B})$  and for all symmetric measures which admit a finite moment of order  $p > p_c(G, \mathfrak{B})$ , see Proposition 5.5, from which estimates on capacity stated in Theorem 5.6 follow. We then give a proof of Liouville property for the finite support case based on traverse counting and  $\ell^2$ -capacity estimates. For the general case under finite  $p$ -moment,  $p > p_c(G, \mathfrak{B})$ , we first apply a geometric argument similar to the proof of Theorem 2.17, then random walk entropy is bounded by contributions from traverses and long jumps separately. The former is controlled by  $\ell^2$ -capacity estimates in the same way as in the finite supported case; and the latter is controlled by  $\ell^p$ -contraction.

Section 6 contains several examples of applications of the main theorem. The first class of examples are groups generated by automata of *polynomial activity growth*, as defined by S. Sidki in [44]. It was shown in [39] that limit spaces of all contracting self-similar groups in this class have Ahlfors-regular dimension 1. Amenability of such groups also follows (as explained in [39]) from the theory of extensively amenable actions, developed in [23, 25] (the Liouville property for all groups in this class seems to be new even for finitely supported symmetric measures).

The second class of examples is iterated monodromy groups of post-critically finite complex rational functions. If  $f$  is a post-critically finite rational function, then the limit space of its iterated monodromy group  $\text{IMG}(f)$  is canonically homeomorphic to the Julia

set of  $f$ . Moreover, the spherical metric restricted to the Julia set, seen as a metric on the limit space of the group, is quasi-conformal and Ahlfors-regular. In particular, the Hausdorff dimension of the Julia set is an upper bound of the conformal dimension of the limit space of the iterated monodromy group. It follows that if the Julia set of the function is not the whole sphere, then the iterated monodromy group is Liouville (for every symmetric measure with finite second moment) and thus amenable.

We provide an explicit example of the iterated monodromy group of a rational function whose Julia set is homeomorphic to the Sierpiński carpet (and thus can not be generated by an automaton of polynomial activity growth).

We also describe explicitly a contracting self-similar group whose limit space coincides with the classical Sierpiński carpet (i.e., quasi-symmetric to it).

Finally, we give an example of a self-similar contracting group, which is not Liouville for any finitely supported measure.

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## 2. SELF-SIMILAR GROUPS AND THEIR LIMIT SPACES

### 2.1. Basic definitions.

**Definition 2.1.** An action of a group  $G$  on the set  $X^*$  of finite words over a finite alphabet  $X$  is said to be *self-similar* if such that for every  $g \in G$  and  $v \in X^*$  there exists an element  $g|_v \in G$  such that

$$g(vw) = g(v)g|_v(w)$$

for all  $w \in X^*$ .

Consider a faithful self-similar action of  $G$  on  $X^*$ . The *biset of the action* is the set  $\mathfrak{B}$  of transformations of  $X^*$  of the form

$$x \cdot g : v \mapsto xg(v)$$

for  $x \in X$  and  $g \in G$ . Since the action is faithful, the set  $\mathfrak{B}$  is in a natural bijection with the set  $X \times G$ .

The set  $\mathfrak{B}$  is closed under pre- and post-compositions with elements of  $G$ , hence it is a *biset*, i.e., a set with commuting left and right actions of  $G$ . They are given by the rules

$$(x \cdot g) \cdot h = x \cdot gh, \quad h \cdot (x \cdot g) = h(x) \cdot h|_{xg}.$$

The right action of  $G$  on the biset  $\mathfrak{B}$  is free and has  $|X|$  orbits, where  $X \cdot \{1\} \subset \mathfrak{B}$  (naturally identified with  $X$ ) intersects each orbit exactly once.

In general, a *self-similar group*  $(G, \mathfrak{B})$  is a group  $G$  and a biset  $\mathfrak{B}$  such that the right action is free and has finitely many orbits. A subset  $X \subset \mathfrak{B}$  is called a *basis* of the biset if it intersects each orbit exactly once.

If  $\mathfrak{B}_1, \mathfrak{B}_2$  are two  $G$ -bisets, then the biset  $\mathfrak{B}_1 \otimes \mathfrak{B}_2$  is quotient of  $\mathfrak{B}_1 \times \mathfrak{B}_2$  by the identification  $(x \cdot g, y) \sim (x, g \cdot y)$  and with the action on the quotient induced by the actions  $g \cdot (x, y) \cdot h = (g \cdot x, y \cdot h)$ . We denote the image of the element  $(x, y) \in \mathfrak{B}_1 \times \mathfrak{B}_2$  in  $\mathfrak{B}_1 \otimes \mathfrak{B}_2$  by  $x \otimes y$  or just by  $xy$ .

This operation is associative up to isomorphisms of bisets. In particular, for every biset  $\mathfrak{B}$  we get bisets  $\mathfrak{B}^{\otimes n}$  for all  $n \geq 0$ . Here  $\mathfrak{B}^{\otimes 0}$  is defined as  $G$  with the natural left and right  $G$ -actions.

If  $X \subset \mathfrak{B}$  is a basis of a biset, then every element of  $\mathfrak{B}$  is written in a unique way as  $x \cdot g$  for  $x \in X$  and  $g \in G$ . For every  $g \in G$  and every  $x \in G$  there exist unique  $y \in X$  and  $h \in G$  such that

$$g \cdot x = y \cdot h.$$

It is easy to see that the map  $x \mapsto y$  is a permutation of  $X$  (since the left  $G$ -action permutes the right  $G$ -orbits). We get the *associated action* of  $G$  on  $X$ , and denote  $y = g(x)$ ,  $h = g|_x$ . The map  $g \mapsto \sigma_g(g|_x)_{x \in X} \in \text{Symm}(X) \times G^X$ , where  $\sigma_g \in \text{Symm}(X)$  is the image of  $g$  under the associated action on  $X$ , is called the *wreath recursion*. It is a homomorphism from  $G$  to the wreath product  $\text{Symm}(X) \times G^X$ .

Changing the basis  $X$  corresponds to composing the wreath recursion with an inner automorphism of the wreath product.

If  $X$  is a basis of  $\mathfrak{B}$ , then  $X^{\otimes n}$  is a basis of  $\mathfrak{B}^{\otimes n}$ . We identify the set  $X^{\otimes n}$  with  $X^n$  in the natural way.

In particular, for every  $v \in X^n \subset \mathfrak{B}^{\otimes n}$  and for every  $g \in G$  there exist unique  $u \in X^n$  and  $g|_v \in G$  such that

$$g \cdot v = u \cdot g|_v.$$

The map  $v \mapsto u$  is a permutation of  $X^n$  (conjugate to the permutation induced by  $g$  on the set of right orbits of  $\mathfrak{B}^{\otimes n}$ ), so we denote  $u = g(v)$ .

We get hence the *associated action* of  $G$  on  $X^*$ . The associated action on  $X^*$  does not depend, up to conjugacy on the choice of the basis  $X$  (since it is conjugate to the left action of  $G$  on  $\bigcup_{n \geq 0} \mathfrak{B}^{\otimes n}/G$ ). The *faithful quotient* of a self-similar group  $(G, \mathfrak{B})$  is the quotient of  $G$  by the kernel of the associated action.

**2.2. Contracting groups and limit space.** Let  $(G, \mathfrak{B})$  be self-similar group. Let  $X \subset \mathfrak{B}$  be a basis. The self-similar group is said to be *contracting* if there exists a finite set  $\mathcal{N} \subset G$  such that for every  $g \in G$  there exists  $n$  such that

$$g|_v \in \mathcal{N}$$

for all  $v \in X^m$  for all  $m \geq n$ . The smallest set  $\mathcal{N}$  satisfying this condition is called the *nucleus* of the group. It is shown in [34, Corollary 2.11.7] that the property to be contracting does not depend on the choice of the basis  $X$ . The nucleus, however, depends on  $X$ .

**Definition 2.2.** Let  $(G, \mathfrak{B})$  be a contracting group. Choose a basis  $X \subset \mathfrak{B}$ , and let  $X^{-\omega}$  be the space of left-infinite sequences  $\dots x_2 x_1$  with the direct product topology (where  $X$  is discrete). The *limit space*  $\mathcal{J}_G$  is the quotient of  $X^{-\omega}$  by the *asymptotic equivalence relation*, identifying two sequences  $\dots x_2 x_1, \dots y_2 y_1$  if and only if there exists a sequence  $g_n$  of elements of the nucleus  $\mathcal{N}$  such that  $g_n(x_n \dots x_2 x_1) = y_n \dots y_2 y_1$ .

It is shown in [34] that the asymptotic equivalence relation is transitive, and that the limit space is a metrizable compact space of finite topological dimension.

It follows from the definition that the asymptotic equivalence relation is invariant under the shift  $\dots x_2 x_1 \mapsto \dots x_3 x_2$ . Consequently, the shift induces a continuous map  $s : \mathcal{J}_G \rightarrow \mathcal{J}_G$ , which we call the *limit dynamical system*. It is shown in [34] that the limit dynamical system does not depend, up to topological conjugacy, on the choice of the basis  $X$ .

**Definition 2.3.** Let  $(G, \mathfrak{B})$  and  $X$  be as above. The *limit  $G$ -space*  $\mathcal{X}_G$  is the quotient of the direct product  $X^{-\omega} \times G$  (where  $G$  is discrete) by the equivalence relation identifying

$\dots x_2 x_1 \cdot g$  and  $\dots y_2 y_1 \cdot h$  if and only if there exists a sequence  $g_n$  of elements of the nucleus such that

$$g_n \cdot x_n \dots x_2 x_1 \cdot g = y_n \dots y_2 y_1 \cdot h$$

in  $\mathfrak{B}^{\otimes n}$  for all  $n$ .

The equality in the definition of  $\mathcal{X}_G$  is equivalent to the equalities

$$g_n(x_n \dots x_2 x_1) = y_n \dots y_2 y_1, \quad g_n|_{x_n \dots x_2 x_1} g = h.$$

We have a natural right action of  $G$  on  $\mathcal{X}_G$  induced by the action  $(\dots x_2 x_1 \cdot g) \cdot h = \dots x_2 x_1 \cdot gh$  on  $X^{-\omega} \times G$ . One can show that the space  $\mathcal{X}_G$  and the action do not depend (up to topological conjugacy) on the choice of the basis  $X$ .

**Definition 2.4.** We denote by  $\mathcal{T}$  the image of  $X^{-\omega}$  in  $\mathcal{X}_G$ .

The action of  $G$  on  $\mathcal{X}_G$  is proper and co-compact with the quotient  $\mathcal{X}_G/G$  naturally homeomorphic to the limit space  $\mathcal{J}_G$ . The set  $\mathcal{T}$  is compact and intersects every  $G$ -orbit.

**Definition 2.5.** We say that a self-similar group  $(G, \mathfrak{B})$  is *self-replicating* if the left action of  $G$  on  $\mathfrak{B}$  is transitive.

It is shown in [34, Theorem 3.5.1] that if a contracting group  $(G, \mathfrak{B})$  is self-replicating, then the spaces  $\mathcal{J}_G$  and  $\mathcal{X}_G$  are connected and locally connected.

**Definition 2.6.** For  $v \in X^*$ , we denote by  $\mathcal{T}_v$  the image of the set  $X^{-\omega}v$  of sequences ending with  $v$  in the limit space  $\mathcal{J}_G$ . We call it a *tile of  $n$ th level* if  $n = |v|$ .

For every point  $\xi \in \mathcal{J}_G$  and every  $n \geq 0$ , the union of the tiles of the  $n$ th level containing  $\xi$  is a neighborhood of  $\xi$ .

**Proposition 2.7.** Two tiles  $\mathcal{T}_v, \mathcal{T}_u$  of the same level intersect if and only if there exists an element  $g$  of the nucleus such that  $g(v) = u$ .

**2.3. Contracting models.** Let  $(G, \mathfrak{B})$  be a contracting self-similar group, and let  $\mathcal{X}$  be a metric space on which  $G$  acts properly and co-compactly by isometries.

Then  $\mathcal{X} \otimes \mathfrak{B}$  is defined as the quotient of  $\mathcal{X} \times \mathfrak{B}$  (where  $\mathfrak{B}$  is discrete) by the action  $(\xi, x) \mapsto (\xi \cdot g^{-1}, g \cdot x)$ . We denote the orbit of  $(\xi, x)$  by  $\xi \otimes x$ . We have then  $\xi \otimes g \cdot x = \xi \cdot g \otimes x$  for all  $\xi \in \mathcal{X}, x \in \mathfrak{B}$ , and  $g \in G$ .

Let  $X$  be a basis of  $\mathfrak{B}$ . Assume that  $I : \mathcal{X} \otimes \mathfrak{B} \rightarrow \mathcal{X}$  is a  $G$ -equivariant map. Then  $I$  is uniquely determined by the maps  $I_x(\xi) = I(\xi \otimes x)$ , which satisfy the following condition:

$$(1) \quad I_x(\xi \cdot g) = I_{g(x)}(\xi) \cdot g|_x,$$

for all  $x \in X, g \in G$ , and  $\xi \in \mathcal{X}$ , where  $g(x) \in X$  and  $g|_x \in G$ , as usual, are given by  $g \cdot x = g(x) \cdot g|_x$ .

Conversely, every collection of maps  $I_x : \mathcal{X} \rightarrow \mathcal{X}$  satisfying (1) defines an equivariant map  $I : \mathcal{X} \otimes \mathfrak{B} \rightarrow \mathcal{X}$  by the formula  $I(\xi \otimes x \cdot g) = I_x(\xi) \cdot g$ .

Every equivariant map  $I : \mathcal{X} \otimes \mathfrak{B} \rightarrow \mathcal{X}$  induces an equivariant map  $I_n : \mathcal{X} \otimes \mathfrak{B}^{\otimes(n+1)} \rightarrow \mathcal{X} \otimes \mathfrak{B}^{\otimes n}$  by the rule

$$I_n(\xi \otimes x \otimes v) = I(\xi \otimes x) \otimes v$$

for  $\xi \in \mathcal{X}, x \in \mathfrak{B}, v \in \mathfrak{B}^{\otimes n}$ . According to this, we will sometimes denote  $I$  by  $I_0$ .

We also denote by  $I^n : \mathcal{X} \otimes \mathfrak{B}^{\otimes n} \rightarrow \mathcal{X}$  the composition  $I_{n-1} \circ I_{n-2} \circ \dots \circ I_1 \circ I$ .

**Definition 2.8.** A  $G$ -equivariant map  $I : \mathcal{X} \otimes \mathfrak{B} \rightarrow \mathcal{X}$  is said to be *contracting* if there exist  $n \geq 1$  and  $\lambda \in (0, 1)$  such that  $d(I(\xi_1 \otimes v), I(\xi_2 \otimes v)) \leq \lambda d(\xi_1, \xi_2)$  for all  $\xi_1, \xi_2 \in \mathcal{X}$  and  $v \in \mathfrak{B}^{\otimes n}$ .

A  $G$ -equivariant map  $I : \mathcal{X} \otimes \mathfrak{B} \rightarrow \mathcal{X}$  is contracting if and only if there exist  $C > 0$  and  $\lambda \in (0, 1)$  such that

$$d(I^n(\xi_1 \otimes v), I^n(\xi_2 \otimes v)) \leq C \lambda^n d(\xi_1, \xi_2)$$

for all  $n, v \in \mathfrak{B}^{\otimes n}, \xi_1, \xi_2 \in \mathcal{X}$ .

The natural map  $\mathcal{X}_G \otimes \mathfrak{B} \rightarrow \mathcal{X}_G$  mapping  $\xi \otimes x$ , where  $\xi$  is represented by  $\dots \otimes x_2 \otimes x_1 \in \Omega$ , to the point of  $\mathcal{X}_G$  represented by  $\dots \otimes x_2 \otimes x_1 \otimes x$ , is a  $G$ -equivariant homeomorphism  $\mathcal{X}_G \otimes \mathfrak{B} \rightarrow \mathcal{X}_G$ . Since it is “identical,” we will denote the image of  $\xi \otimes x$  just by  $\xi \otimes x$ , thus identifying  $\mathcal{X}_G \otimes \mathfrak{B}$  with  $\mathcal{X}_G$ .

It is proved in [38, Theorem 2.10] that the natural homeomorphism  $\mathcal{X}_G \otimes \mathfrak{B} \rightarrow \mathcal{X}_G$  is contracting in the sense of Definition 2.8 (with respect to a natural distance on  $\mathcal{X}_G$ ) provided  $G$  is finitely generated.

The following theorem is proved in [36].

**Theorem 2.9.** *Let  $I : \mathcal{X} \otimes \mathfrak{B} \rightarrow \mathcal{X}$  be a contracting  $G$ -equivariant map. Then the inverse limit of the sequence of  $G$ -spaces  $\mathcal{X} \otimes \mathfrak{B}^{\otimes n}$  and maps  $I_n$  is  $G$ -equivariantly homeomorphic to the limit  $G$ -space  $\mathcal{X}_G$ . The map on the limit induced by the maps  $I_n$  is conjugated by the homeomorphism with the natural homeomorphism  $\mathcal{X}_G \otimes \mathfrak{B} \rightarrow \mathcal{X}_G$ .*

**Definition 2.10.** Let  $(G, \mathfrak{B})$  be a contracting self-similar group. A *contracting model* of  $(G, \mathfrak{B})$  is a proper co-compact action of  $G$  on a metric space  $\mathcal{X}$  by isometries together with a contracting equivariant map  $I : \mathcal{X} \otimes \mathfrak{B} \rightarrow \mathcal{X}$ .

It is proved in [36] (see also [38, Theorem 3.1]) that every finitely generated contracting group has a contracting model  $I : \mathcal{X} \otimes \mathfrak{B} \rightarrow \mathcal{X}$  where  $\mathcal{X}$  is a connected simplicial complex and the maps  $I_x$  are piecewise affine.

We will need the following simple lemma.

**Lemma 2.11.** *Suppose that  $G$  is finitely generated, and let  $l(g)$  denote the length of an element  $g \in G$  with respect to a fixed finite generating set of  $G$ . Choose a basis  $\mathsf{X} \subset \mathfrak{B}$ .*

*Let  $\mathcal{X}$  be a metric space on which  $G$  acts by isometries from the right so that the action is co-compact and proper. Let  $I : \mathcal{X} \otimes \mathfrak{B} \rightarrow \mathcal{X}$  be a  $G$ -equivariant map. Let  $\mathcal{T}$  be a compact subset of  $\mathcal{X}$  such that  $I^n(\xi \otimes v) \in \mathcal{T}$  for every  $\xi \in \mathcal{T}$  and  $v \in \mathsf{X}^*$ .*

*Then for every  $\xi \in \mathcal{X}$  there exists a constant  $C > 0$  such that*

$$l(g|_v) \leq Cd(I^n(\xi \otimes v), I^n(\xi \cdot g \otimes v)) + C$$

for all  $n \geq 1, g \in G, \xi \in \mathcal{T}$ , and  $v \in \mathsf{X}^n$ .

It follows directly from co-compactness of the action and Definition 2.8 that a compact set  $\mathcal{T}$  satisfying the conditions of the lemma exists for every contracting equivariant map  $I : \mathcal{X} \otimes \mathfrak{B} \rightarrow \mathcal{X}$  and every basis  $\mathsf{X}$  of  $\mathfrak{B}$ .

*Proof.* Since the action of  $G$  on  $\mathcal{X}$  is isometric, proper, and co-compact, there exists  $C$  such that  $l(g) \leq Cd(\xi_1, \xi_2 \cdot g) + C$  for all  $\xi_1, \xi_2 \in \mathcal{T}$ .

It follows that  $l(g|_v) \leq Cd(I^n(\xi \otimes v), I^n(\xi \otimes g(v)) \cdot g|_v) + C = Cd(I^n(\xi \otimes v), I^n(\xi \cdot g \otimes v)) + C$ .  $\square$

#### 2.4. Iterated monodromy groups.

**Definition 2.12.** A *virtual endomorphism* of a topological space  $\mathcal{J}$  is a finite degree covering map  $f : \mathcal{J}_1 \rightarrow \mathcal{J}$  together with a continuous map  $\phi : \mathcal{J}_1 \rightarrow \mathcal{J}$ .

Let  $f, \phi : \mathcal{J}_1 \rightarrow \mathcal{J}$  be a virtual endomorphism. Suppose that  $\mathcal{J}$  is path connected. Consider the fundamental group  $G = \pi_1(\mathcal{J}, t)$ . The *associated biset* is the set  $\mathfrak{B}$  of pairs  $(z, [\delta])$ , where  $z \in f^{-1}(t)$ , and  $[\delta]$  is the homotopy class of a path  $\delta$  in  $\mathcal{J}$  connecting  $t$  to  $\phi(z)$ . The action of  $G$  on  $\mathfrak{B}$  is given by the formulas

$$(z, [\delta]) \cdot [\gamma] = (z, [\delta\gamma]), \quad [\gamma](z, [\delta]) = (y, [\phi(\gamma_z)\delta]),$$

where  $\gamma$  is a loop based at  $t$ ,  $\gamma_z$  is the lift of  $\gamma$  by  $f$  starting in  $z$ ,  $y$  is the end of  $\gamma_z$ . We multiply paths as functions: in a product  $\alpha\beta$  the path  $\beta$  is passed before  $\alpha$ .

The *iterated monodromy group* of the virtual endomorphism is the faithful quotient of the self-similar group  $(G, \mathfrak{B})$ .

By the definition of the associated biset, a basis of  $\mathfrak{B}$  is a collection

$$\{(z_1, [\ell_1]), (z_2, [\ell_2]), \dots, (z_n, [\ell_n])\},$$

where  $\{z_1, z_2, \dots, z_n\} = f^{-1}(t)$ , and  $\ell_i$  is a path from  $t$  to  $\phi(z_i)$ . Let us denote  $x_i = (z_i, [\ell_i])$ , then the action of the iterated monodromy group on the tree  $\mathbf{X}^*$  for  $\mathbf{X} = \{x_1, x_2, \dots, x_n\}$  is given by the recurrent formula

$$(2) \quad [\gamma](x_i w) = x_j [\phi(\ell_j^{-1} \gamma_i \ell_i)](w),$$

where  $\gamma_i$  is the unique lift of  $\gamma$  by  $f$  starting in  $z_i$ , and  $j$  is such that  $z_j$  is the end of  $\gamma_i$ .

A particular case of this situation is a *partial self-covering*, i.e., a finite degree covering map  $f : \mathcal{J}_1 \rightarrow \mathcal{J}$  where  $\mathcal{J}_1 \subset \mathcal{J}$ . It is a virtual endomorphism with the map  $\phi : \mathcal{J}_1 \rightarrow \mathcal{J}$  equal to the identical embedding. Then the *iterated monodromy group* of  $f$  is the iterated monodromy group of the virtual endomorphism  $f, \phi : \mathcal{J}_1 \rightarrow \mathcal{J}$ . In this case, it is natural to omit  $\phi$  (as it is the identity map) in the formula (2).

An important class of examples consists of *post-critically finite complex rational functions*. Let  $f(z) \in \mathbb{C}(z)$  be a rational function seen as a self-map of the Riemannian sphere  $\mathbb{P}\mathbb{C}^1$ . Let  $P_f$  be the union of the forward orbits of the *critical values* of  $f$  (i.e., values at the critical points of the map  $f : \mathbb{P}\mathbb{C}^1 \rightarrow \mathbb{P}\mathbb{C}^1$ ). We say that  $f$  is *post-critically finite* if the set  $P_f$  is finite.

Then  $f$  defines a partial self-covering  $f : \mathbb{P}\mathbb{C}^1 \setminus f^{-1}(P_f) \rightarrow \mathbb{P}\mathbb{C}^1 \setminus P_f$ . The iterated monodromy group of this partial self-covering is, by definition, the iterated monodromy group of  $f$ .

The following theorem is proved in [34, Theorem 6.4.4].

**Theorem 2.13.** *Let  $f$  be a post-critically finite complex rational function. Then its iterated monodromy group  $G = \text{IMG}(f)$  is contracting and its limit dynamical system  $\mathfrak{s} : \mathcal{J}_G \rightarrow \mathcal{J}_G$  is topologically conjugate to the restriction of  $f$  to its Julia set.*

In fact, Theorem 2.13 is a corollary of Theorem 2.9 and of the Schwarz-Pick theorem.

**2.5. One-dimensional limit spaces.** The following theorem is proved in [38].

**Theorem 2.14.** *Suppose that the limit space  $\mathcal{J}_G$  (equivalently the  $\mathcal{X}_G$ ) of a contracting finitely generated group  $(G, \mathfrak{B})$  has topological dimension 1. Then there is  $n_0$  and a contracting model  $I : \Delta_0 \otimes \mathfrak{B}^{\otimes n_0} \rightarrow \Delta_0$ , where  $\Delta_0$  is a locally finite connected graph on which  $G$  acts properly and co-compactly by automorphisms.*

Below we suppose to be in the situation of Theorem 2.14. Denote  $\Delta_n = \Delta_0 \otimes \mathfrak{B}^{\otimes n}$ . Since the action of  $G$  on  $\Delta_0$  is by automorphisms of graphs, the spaces  $\Delta_n$  are also graphs.

Denote by  $\Gamma_n$  the quotient  $\Delta_n/G$  and by  $\pi_n : \Delta_n \rightarrow \Gamma_n$  the quotient projection. Then  $\Gamma_n$  is a finite graph. The maps  $I_n : \Delta_{n+1} \rightarrow \Delta_n$  induce continuous maps  $\phi_n : \Gamma_{n+1} \rightarrow \Gamma_n$ . We may assume that in the original action of  $G$  on  $\Delta_0$ , no pair of neighbouring vertices

of  $\Delta_0$  belong to the same  $G$ -orbit (otherwise pass to the barycentric subdivision of  $\Delta_0$ ). In particular,  $\Gamma_0$  has no loops.

**Notation 2.15.** Denote by  $U_z$  the ball of radius  $1/3$  with center in a vertex  $z$  of the graph  $\Gamma_0 = \mathcal{X}/G$ . It is a bouquet of segments of length  $1/3$ . (We use the combinatorial distance in the graphs, identifying each edge with a real segment of length 1.)

Fix a basepoint  $\xi_0 \in \Delta_0$ . Denote by  $U_{z,n}$  the set of elements  $v \in X^n$  such that  $\pi_0(I^n(\xi_0 \otimes v)) \in U_z$ .

**Definition 2.16.** Let  $s_t s_{t-1} \dots s_1$  be a word in elements  $s_i \in G$  (formally, an element of the free group with basis  $G$ ). A *traverse of level  $n$  from  $z_1$  to  $z_2$*  is a triple  $(i, j, v_1, v_2)$ , where  $1 \leq i \leq j \leq t$ ,  $v_1 \in U_{z_1,n}$ ,  $v_2 \in U_{z_2,n}$  for  $z_1 \neq z_2$  are such that  $s_k \dots s_{i+1} s_i(v_1) \notin U_{z_1,n} \cup U_{z_2,n}$  for  $i < k < j$ , and  $s_j s_{j-1} \dots s_i(v_1) = v_2$ .

In other words, a traverse is a segment of a the word  $s_t s_{t-1} \dots s_1$  corresponding to a travel from a point of  $U_{z_1,n}$  to a point of  $U_{z_2,n}$  such that the sets  $U_{z_1,n}$  and  $U_{z_2,n}$  are not touched during the travel.

We denote by  $\tau_n(s_t s_{t-1} \dots s_1)$  the total number of traverses of level  $n$  for the word  $s_t s_{t-1} \dots s_1$ .

**Theorem 2.17.** *Let  $S$  be a finite generating set for  $G$ . There exists  $C > 1$  such that for all  $n$  big enough and every element  $g = s_t s_{t-1} \dots s_1 \in G$ , where  $s_i \in S$ , we have*

$$\sum_{v \in X^n} l(g|_v) \leq C \tau_n(s_t s_{t-1} \dots s_1) + C |X^n|.$$

*Proof.* Since the  $G$  action is isometric, the quantity  $d(\xi_0 \cdot g, \xi_0 \cdot sg)$  is uniformly bounded for  $s \in S$  and  $g \in G$ . Hence there exists  $n_0$  such that  $d(I^n(\xi_0 \cdot g \otimes v), I^n(\xi_0 \cdot sg \otimes v)) < 1/3$  for every  $g \in G$ ,  $s \in S$  and  $v \in X^n$ ,  $n \geq n_0$ . Fix such a  $v$ , consider  $g = s_t \dots s_1$  as in the statement of the theorem, and set  $\xi_i = \xi_0 \cdot s_i \dots s_1$ . Then the distance between two consecutive points  $I^n(\xi_i \otimes v)$  is at most  $1/3$ . By Lemma 2.11, the length of  $g|_v$  is bounded from above by  $C_1 d(I^n(\xi_0 \otimes v), I^n(\xi_t \otimes v)) + C_1$  for some constant  $C_1 > 0$ . Let  $\delta_i$  be a geodesic curve joining  $I^n(\xi_{i-1} \otimes v)$  to  $I^n(\xi_i \otimes v)$  and  $\gamma = \delta_t \dots \delta_1$  be their concatenation (where the rightmost curve is passed first). Consider an edge  $e$  which is crossed by  $\gamma$ , meaning that there is a subcurve  $\gamma|_{[t_1, t_2]}$  such that  $\gamma(t_1), \gamma(t_2)$  are the extreme points of the edge and  $\gamma(s)$  is in the interior of  $e$  for  $s \in (t_1, t_2)$ . For every such edge crossing we can find  $i < j$  such that  $I^n(\xi_i \otimes v)$  and  $I^n(\xi_j \otimes v)$  are at distance at most  $1/3$  from the two extreme points of  $e$ , and every  $I^n(\xi_k \otimes v)$  with  $i < k < j$  is in the interior of  $e$  at distance at least  $1/3$  from the extreme points. The distance between  $d(I^n(\xi_0 \otimes v), I^n(\xi_t \otimes v))$  is not larger than the total number of edge crossings of  $\gamma$  plus 2. Now recall that the quotient projection  $\pi_0: \Delta_0 \rightarrow \Gamma_0$  maps edges homeomorphically to edges, and moreover

$$\pi_0(I^n(\xi_i \otimes v)) = \pi_0(I^n(\xi_0 \otimes s_i \dots s_1(v)) \cdot s_i \dots s_1|_v) = \pi_0(I^n(\xi_0 \otimes s_i \dots s_1(v))).$$

We deduce that to every edge-crossing of  $\gamma$  we can associate (injectively) a traverse. Thus there is a constant  $C > 0$  such that

$$l(g|_v) \leq C \tau_{n,v}(s_t \dots s_1) + C,$$

where  $\tau_{n,v}$  denotes the number traverses at level  $n$  of the form  $(i, j, v_1, v_2)$  with  $v_1 = s_i \dots s_1(v)$ . Summing over  $v$  we obtain the statement of the theorem.  $\square$

## 3. CONFORMAL DIMENSION OF CONTRACTING GROUPS

**3.1. Visual metric.** Let  $(G, \mathfrak{B})$  be a contracting group. Choose a basis  $X$  of  $\mathfrak{B}$  and let  $\mathcal{N}$  be the corresponding nucleus of  $G$ . The *self-similarity complex* is the graph with set of vertices  $X^*$  in which two vertices are connected by an edge in one of the following two situations: either they are of the form  $v, xv$  for  $v \in X^*$  and  $x \in X$  (*vertical edges*), or they are of the form  $v, g(v)$  for  $v \in X^*$  and  $g \in \mathcal{N}$  (*horizontal edges*).

The following theorem is proved in [33] (see also [34, Theorem 3.8.8]). The statement about the Gromov product follows directly from the proof.

**Theorem 3.1.** *The self-similarity complex is Gromov hyperbolic. Its boundary is homeomorphic to the limit space  $\mathcal{J}_G$ , where a point of  $\mathcal{J}_G$  represented by  $\dots x_2x_1 \in X^{-\omega}$  corresponds to the limit of the geodesic path  $(x_1, x_2x_1, x_3x_2x_1, \dots)$  in the self-similarity complex.*

*The Gromov product  $\ell(x_n \dots x_2x_1, y_m \dots y_2y_1)$  of the vertices  $x_n \dots x_2x_1$  and  $y_m \dots y_2y_1$  is equal (up to an additive constant) to the largest  $k \leq \min(n, m)$  such that there exists  $g \in \mathcal{N}$  such that  $g(x_k \dots x_2x_1) = y_k \dots y_2y_1$ .*

Define  $\ell(\dots x_2x_1, \dots y_2y_1)$  as the largest  $n$  such that  $g(x_n \dots x_2x_1) = y_n \dots y_2y_1$  for some  $g \in \mathcal{N}$ . It is the natural extension of the Gromov product to  $\mathcal{J}_G$  (its value does not depend, up to an additive constant, on the choice of the sequences  $\dots x_2x_1, \dots y_2y_1 \in X^{-\omega}$  representing the points of the limit space). Note that we define  $\ell(\xi, \xi) = \infty$  for  $\xi \in \mathcal{J}_G$ .

The following is a classical result of the theory of Gromov-hyperbolic spaces, see [16, §3].

**Theorem 3.2.** *There exists  $\alpha_0 > 0$  such that for every  $\alpha < \alpha_0$  there exists a metric  $d_\alpha$  on  $\mathcal{J}_G$  and a constant  $C > 1$  such that*

$$(3) \quad C^{-1}e^{-\alpha\ell(\xi_1, \xi_2)} \leq d_\alpha(\xi_1, \xi_2) \leq Ce^{-\alpha\ell(\xi_1, \xi_2)}$$

for all  $\xi_1, \xi_2 \in \mathcal{J}_G$ .

Note that (for a fixed value of  $\alpha$ ) the metric  $d_\alpha$  is unique up to bi-Lipschitz equivalence. We call it the *visual metric of exponent  $\alpha$* .

If  $f, g$  are real-valued functions with the same domain of definition, we write  $f \asymp g$  if there exists a constant  $C > 0$  such that  $C^{-1}f \leq g \leq Cf$ .

It follows from Proposition 2.7 that for every  $\zeta \in \mathcal{J}_G$  the set

$$T(\zeta, n) = \{\xi \in \mathcal{J}_G : \ell(\xi, \zeta) \geq n\}$$

is equal to the union of the tiles  $\mathcal{T}_v$  of  $n$ th level intersecting tiles of the  $n$ th level that contain  $\zeta$ . Note that  $T(\zeta, n)$  is a union of not more than  $|\mathcal{N}|^2$  tiles of the  $n$ th level.

Note also that if  $d_\alpha$  is a visual metric, then we have

$$(4) \quad B(\zeta, C^{-1}e^{-\alpha n}) \subset T(\zeta, n) \subset B(\zeta, Ce^{-\alpha n}),$$

where  $B(x, R)$  denotes the ball of radius  $R$  with center in  $x$ , and  $C$  satisfies (3).

Let  $\mu$  be the push-forward of the uniform Bernoulli measure on  $X^{-\omega}$  by the quotient map  $X^{-\omega} \rightarrow \mathcal{J}_G$ . Since the quotient map is at most  $|\mathcal{N}|$ -to-one, we have, for every  $v \in X^n$

$$|X|^{-n} \leq \mu(\mathcal{T}_v) \leq |\mathcal{N}| \cdot |X|^{-n}.$$

It follows that  $\mu(T(\zeta, n)) \asymp |X|^{-n}$ , so

$$\mu(B(\zeta, R)) \asymp |X|^{-\frac{\ln R}{\alpha}} = \exp\left(\frac{\ln R \ln |X|}{\alpha}\right) = R^{\frac{\ln |X|}{\alpha}}.$$

**3.2. Conformal dimension.** Let  $(G, \mathfrak{B})$  be a contracting self-similar group. Recall that  $T(\xi, n)$  for  $\xi \in \mathcal{J}_G$  denotes the union of the  $n$ th level tiles  $\mathcal{T} \cdot v$ ,  $v \in \mathfrak{B}^{\otimes n}$  intersecting the  $n$ th level tiles that contain  $\xi$ .

**Definition 3.3.** A metric on  $\mathcal{J}_G$  is said to be *quasi-conformal* if there exists  $C > 1$  such that for every  $\xi \in \mathcal{J}_G$  and every  $n \geq 1$  there exists  $r$  such that

$$B(\xi, r) \subset T(\xi, n) \subset B(\xi, Cr),$$

where the balls  $B(\xi, r)$  are defined using the metric  $d$ .

A metric  $d$  on a space  $\mathcal{X}$  is said to be *Ahlfors-regular* if there exists  $\beta > 0$  and a measure  $\mu$  on  $\mathcal{X}$  such that

$$\mu(B(x, R)) \asymp R^\beta$$

for all  $x \in \mathcal{X}$  and  $R < \text{diam}(\mathcal{X})$ . The exponent  $\beta$  is equal to the Hausdorff dimension of  $d$ .

We have seen that the visual metric  $d_\alpha$  on  $\mathcal{J}_G$  is quasi-conformal. We have also seen that it is Ahlfors-regular of Hausdorff dimension  $\frac{\ln |\mathcal{X}|}{\alpha}$ .

**Definition 3.4.** The infimum of the set of Hausdorff dimensions of Ahlfors-regular quasi-conformal metrics on  $\mathcal{J}_G$  is called the *Ahlfors-regular conformal dimension* of  $\mathcal{J}_G$  and is denoted  $\text{ARdim } \mathcal{J}_G$  or  $\text{ARdim}(G, \mathfrak{B})$ .

A map  $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  between metric spaces  $(\mathcal{X}_1, d_1)$  and  $(\mathcal{X}_2, d_2)$  is said to be *quasisymmetric* if there exists a homeomorphism  $\eta : [0, \infty) \rightarrow [0, \infty)$  such that

$$\frac{d_2(f(x), f(y))}{d_2(f(x), f(z))} \leq \eta \left( \frac{d_1(x, y)}{d_1(x, z)} \right)$$

for all  $x, y, z \in \mathcal{X}_1$ .

It is not hard to see that the inverse of every quasisymmetric homeomorphism is also quasisymmetric.

Two metrics on the same spaces are said to be quasisymmetric to each other if the identity map is quasisymmetric.

It follows directly from the definitions that any two visual metrics  $d_{\alpha_1}, d_{\alpha_2}$  on  $\mathcal{J}_G$  are quasisymmetric to each other.

**Proposition 3.5.** *Let  $(G, \mathfrak{B})$  be a self-replicating group. Then an Ahlfors-regular metric  $d$  on  $\mathcal{J}_G$  is quasi-conformal if and only if it is quasisymmetric to a visual metric.*

*Proof.* It is easy to see that every Ahlfors-regular metric on  $\mathcal{J}_G$  quasisymmetric to a visual metric is quasiconformal.

If  $d_\alpha$  is a visual metric, and  $d$  is quasi-conformal, condition (3) implies that there exists a constant  $C > 1$  such that for every  $\zeta \in \mathcal{J}_G$  and every  $r > 0$  (such that  $B_{d_\alpha}(\zeta, r) \neq \mathcal{J}_G$ ) there exists  $R_{r, \zeta} > 0$  such that

$$\overline{B}_d(\zeta, R_{r, \zeta}) \subset B_{d_\alpha}(\zeta, r)$$

and

$$\overline{B}_{d_\alpha}(\zeta, r) \subset B_d(\zeta, CR_{r, \zeta}),$$

where  $B$  and  $\overline{B}$  denote the open and the closed balls, respectively.

Let  $\xi_1, \xi_2 \in \mathcal{J}_G$  be such that  $d_\alpha(\zeta, \xi_1) \leq d_\alpha(\zeta, \xi_2)$ . Let  $d_\alpha(\zeta, \xi_2) = r$ . Then  $d(\zeta, \xi_1) < CR_{\zeta, r}$ . If  $d(\zeta, \xi_2) \leq R_{\zeta, r}$ , then  $d_\alpha(\zeta, \xi_2) < r$ , which is a contradiction. Therefore,  $d(\zeta, \xi_2) > R_{\zeta, r}$ . We see that  $d_\alpha(\zeta, \xi_1) \leq d_\alpha(\zeta, \xi_2)$  implies  $d(\zeta, \xi_1) \leq CR_{\zeta, r} <$

$Cd(\zeta, \xi_2)$ . It follows that the identity map from  $(\mathcal{J}_G, d_\alpha)$  to  $(\mathcal{J}_G, d)$  is weakly quasisymmetric.

Consequently (as  $\mathcal{J}_G$  is connected and doubling in the self-replicating case), by [22, Theorem 10.19], the identity map is quasisymmetric.  $\square$

**Proposition 3.6.** *Let  $d$  be an Ahlfors regular quasi-conformal metric on  $\mathcal{J}_G$ . Then*

$$\lim_{n \rightarrow \infty} \sup_{\xi \in \mathcal{J}_G} \text{diam}(T(\xi, n)) \rightarrow 0$$

as  $n \rightarrow \infty$ . If  $d$  is quasisymmetric to a visual metric, then there exist  $C > 0$  and  $\lambda \in (0, 1)$  such that

$$\sup_{\xi \in \mathcal{J}_G} \text{diam}(T(\xi, n)) \leq C\lambda^n$$

for all  $n$ .

*Proof.* The first statement follows from the fact that it is true for a visual metric and that compact spaces have a unique uniformity, so uniform convergence to 0 does not depend on the choice of a metric.

Suppose now that  $d$  is quasisymmetric to a visual metric  $d_\alpha$ . Let  $\eta : [0, \infty) \rightarrow [0, \infty)$  be the homeomorphism such that  $\frac{d(x, y)}{d(x, z)} \leq \eta\left(\frac{d_\alpha(x, y)}{d_\alpha(x, z)}\right)$  for all  $x, y, z \in \mathcal{J}_G$ , where  $d_\alpha$  is a visual metric on  $\mathcal{J}_G$ . Let  $u_1, u_2 \in \mathbf{X}^*$ . Then for every  $x \in \mathcal{T}_{u_2 u_1} \subset \mathcal{T}_{u_1}$  there exists  $y \in \mathcal{T}_{u_2 u_1}$  such that  $2d(x, y) \geq \text{diam}_d(\mathcal{T}_{u_2 u_1})$ . Similarly, there exists  $z \in \mathcal{T}_{u_1}$  such that  $d_\alpha(x, z) \geq \text{diam}_{d_\alpha}(\mathcal{T}_{u_1})/2$ . Then

$$\frac{\text{diam}(\mathcal{T}_{u_2 u_1})}{\text{diam}(\mathcal{T}_{u_1})} \leq \frac{2d(x, y)}{d(x, z)} \leq 2\eta\left(\frac{d_\alpha(x, y)}{d_\alpha(x, z)}\right) \leq 2\eta\left(\frac{\text{diam}_{d_\alpha}(\mathcal{T}_{u_2 u_1})}{\text{diam}_{d_\alpha}(\mathcal{T}_{u_1})/2}\right).$$

It follows from the definition of the visual metric that  $\text{diam}_{d_\alpha}(\mathcal{T}_v) \asymp e^{-|v|\alpha}$ . It follows that there exists  $C > 1$  such that  $\frac{\text{diam}_{d_\alpha}(\mathcal{T}_{u_2 u_1})}{\text{diam}_{d_\alpha}(\mathcal{T}_{u_1})} \leq C e^{-|u_2|\alpha}$  for all  $u_1, u_2 \in \mathbf{X}^*$ .

Since  $\eta : [0, \infty) \rightarrow [0, \infty)$  is a homeomorphism, this implies that there exists  $m$  and  $\lambda_0 \in (0, 1)$  such that  $\frac{\text{diam}(\mathcal{T}_{u_2 u_1})}{\text{diam}(\mathcal{T}_{u_1})} \leq \lambda_0$  for all  $u_2 \in \mathbf{X}^m$ . This implies the statement of the proposition.  $\square$

#### 4. CONTRACTION COEFFICIENTS

**4.1.  $\ell_p$ -contraction.** Let  $(G, \mathfrak{B})$  be a contracting finitely generated group. Fix some finite generating set of  $G$ , and let  $l(g)$  be the corresponding word length.

The  $\ell^p$ -contraction coefficient of the group is

$$\eta_p = \lim_{n \rightarrow \infty} \sqrt[n]{\limsup_{l(g) \rightarrow \infty} \frac{(\sum_{v \in \mathbf{X}^n} l(g|_v)^p)^{1/p}}{l(g)}}.$$

The  $\ell_\infty$ -contraction coefficient  $\eta_\infty$  is defined in the similar way, replacing the  $\ell^p$  norm of the vector  $(l(g|_v))_{v \in \mathbf{X}^n}$  by its  $\ell_\infty$  norm  $\max_{v \in \mathbf{X}^n} l(g|_v)$ . The existence of the first limit in the formula defining  $\eta_p$  follows from the lemma below.

**Lemma 4.1.** *Denote*

$$\eta_{p, n} = \limsup_{l(g) \rightarrow \infty} \frac{(\sum_{v \in \mathbf{X}^n} l(g|_v)^p)^{1/p}}{l(g)}.$$

Then  $\eta_{p, n_1 + n_2} \leq \eta_{p, n_1} \eta_{p, n_2}$ .

*Proof.* For every  $\lambda_i > \eta_{p, n_i}$  and every  $g$  long enough, we have

$$\sum_{v \in X^{n_i}} l(g|_v)^p \leq \lambda_i^p l(g)^p.$$

It follows that there exists a constant  $C$  not depending on  $g$  such that

$$\sum_{v \in X^{n_i}} l(g|_v)^p \leq \lambda_i^p (l(g) + C)^p$$

for all  $g \in G$ .

Suppose at first that  $p \geq 1$ . We have then, using the triangle inequality for the  $p$ -norm:

$$\begin{aligned} \sum_{v \in X^{n_1+n_2}} l(g|_v)^p &= \sum_{v_1 \in X^{n_1}} \sum_{v_2 \in X^{n_2}} l(g|_{v_1 v_2})^p \leq \sum_{v_1 \in X^{n_1}} \lambda_2^p (l(g|_{v_1}) + C)^p \leq \\ &\left( \lambda_2 C |X^{n_1}| + \lambda_2 \left( \sum_{v_1 \in X^{n_1}} l(g|_{v_1})^p \right)^{1/p} \right)^p \leq \\ &(\lambda_2 C |X^{n_1}| + \lambda_2 \lambda_1 (l(g) + C))^p \end{aligned}$$

which implies that

$$\eta_{p, n_1+n_2} \leq \lambda_1 \lambda_2,$$

for all  $\lambda_1 > \eta_{p, n_1}$  and  $\lambda_2 > \eta_{p, n_2}$ .

The case  $p \in (0, 1)$  is analogous, but using the triangle inequality for the norm  $(x_i) \mapsto \sum |x_i|^p$ .  $\square$

It is shown in [37, Proposition 4.3.12] that  $\eta_p$  does not depend on the choice of the finite generating set or the choice of the basis  $X \subset \mathfrak{B}$ . (This is true for all  $p \in (0, \infty]$ .)

It is shown in [37, Proposition 4.3.15] that for all  $0 < p < q \leq \infty$  we have

$$(5) \quad \eta_p \geq \eta_q \geq |X|^{-1}, \quad |X|^{-1/p} \eta_p \leq |X|^{-1/q} \eta_q.$$

In particular, the function  $p \rightarrow \eta_p$  is non-increasing and continuous.

**Definition 4.2.** The *critical exponent*  $p_c(G, \mathfrak{B})$  of a contracting group is the infimum of the set of values  $p$  such that  $\eta_p < 1$ .

**Proposition 4.3.** *Suppose that  $(G, \mathfrak{B})$  is a contracting group. Fix a basis  $X \subset \mathfrak{B}$  and a finite generating set  $S$  containing the nucleus. Let  $p \geq 1$  be such that  $\eta_p < 1$ . Then for every  $l \geq 2$  there exists a constant  $C$  such that for every  $g \in G$  we have*

$$\sum_{v \in X^*, l(g|_v) \geq l} l(g|_v)^p \leq C l(g)^p.$$

*Proof.* Note that since  $G$  is contracting and the generating set contains the nucleus, for every  $l$  there exists  $m$  such that if  $l(g) \leq l$ , then  $l(g|_v) \leq 1$  for all  $v \in X^*$  such that  $|v| \geq m$ . Therefore if we fix  $l_1 < l_2$ , every vertex in the set  $\{v \in X^* : l_1 \leq l(g|_v) < l_2\}$  is at a uniformly bounded distance either below the root, or below a  $w$  such that  $l(g|_w) \geq l_2$ . It follows that there exists some constant  $C_1$  such that

$$\sum_{v \in X^*, l(g|_v) \geq l_1} l(g|_v)^p \leq C_1 \left( \sum_{v \in X^*, l(g|_v) \geq l_2} l(g|_v)^p \right) + C_1.$$

Consequently, it is enough to prove the proposition for any large  $l$ . It is also easy to see that we can replace  $X$  by any power  $X^{n_0}$ , i.e., consider in the sum only words  $v$  of length divisible by  $n_0$ .

If  $\eta_p < 1$ , then there exist  $n_0, l_0$ , and  $\eta < 1$  such that

$$\sum_{v \in X^{n_0}} l(g|_v)^p \leq \eta l(g)^p$$

for all  $g \in G$  such that  $l(g) \geq l_0$ . If we redefine the length by setting

$$\bar{l}(g) = \begin{cases} l(g) & \text{if } l(g) \geq l_0 \\ 0 & \text{otherwise} \end{cases},$$

then we get

$$\sum_{v \in X^{n_0}} \bar{l}(g|_v)^p \leq \eta \bar{l}(g)^p$$

for all  $g \in G$ . Consequently,

$$\sum_{v \in X^{kn_0}} \bar{l}(g|_v)^p \leq \eta^k \bar{l}(g)^p$$

for every  $k \geq 0$  and every  $g \in G$ , hence

$$\sum_{v \in (X^{n_0})^*, l(g|_v) \geq l_0} l(g|_v)^p \leq \sum_{k=0}^{\infty} \sum_{v \in X^{kn_0}} \bar{l}(g|_v)^p \leq \frac{1}{1-\eta} l(g)^p,$$

which finishes the proof.  $\square$

**Proposition 4.4.** *Let  $I : \mathcal{X} \otimes \mathfrak{B} \rightarrow \mathcal{X}$  be a contracting model of the group  $(G, \mathfrak{B})$ . For every  $p > p_c(G, \mathfrak{B})$  and every  $\delta > 0$  there exists  $C$  such that*

$$\sum_{v \in X^*, d(I^{|v|}(\xi_1 \otimes v), I^{|v|}(\xi_2 \otimes v)) \geq \delta} d(I^{|v|}(\xi_1 \otimes v), I^{|v|}(\xi_2 \otimes v))^p \leq C d(\xi_1, \xi_2)^p$$

for all  $\xi_1, \xi_2 \in \mathcal{X}_G$ .

*Proof.* Denote the sum in the proposition by  $\Sigma(\delta, \xi_1, \xi_2)$ . If  $\delta_1 > \delta$ , then the set of summands in  $\Sigma(\delta_1, \xi_1, \xi_2)$  is a subset of the set of summands of  $\Sigma(\delta, \xi_1, \xi_2)$ .

It follows from the definition of a contracting model that we have  $d(I^{|v|}(\xi_1 \otimes v), I^{|v|}(\xi_2 \otimes v)) \leq C \lambda^{|v|} d(\xi_1, \xi_2)$  for some constants  $C > 1$  and  $\lambda \in (0, 1)$ . Consequently, for every summand  $d(I^{|v|}(\xi_1 \otimes v), I^{|v|}(\xi_2 \otimes v))^p$  of  $\Sigma(\delta_1, \xi_1, \xi_2)$  there is at most some fixed number of summands of  $\Sigma(\delta, \xi_1, \xi_2)$  of the form  $d(I^{|vu|}(\xi_1 \otimes vu), I^{|vu|}(\xi_2 \otimes vu))^p$  and all the values of these summands are bounded by  $C^p d(I^{|v|}(\xi_1 \otimes v), I^{|v|}(\xi_2 \otimes v))^p$ . It follows that there exists a constant  $K$  not depending on  $\xi_1, \xi_2$  such that  $\Sigma(\delta, \xi_1, \xi_2) \leq K \Sigma(\delta_1, \xi_1, \xi_2)$ .

Consequently, it is enough to prove the proposition for an arbitrary  $\delta$ . Let  $T \subset \mathcal{X}$  be a compact set intersecting every  $G$ -orbit and such that  $I(T \otimes x) \subset T$  for every  $x \in X$ . We can find such a set since the maps  $\xi \mapsto I(\xi \otimes x)$  are contractions. There exist  $g_1, g_2 \in G$  such that  $\xi_i \in T \cdot g_i$ . Since the action of  $G$  on  $\mathcal{X}$  is proper, isometric, and co-compact, there exists a constant  $L > 1$  such that

$$L^{-1} l(g_1 g_2^{-1}) - L \leq d(\xi_1, \xi_2) \leq L l(g_1 g_2^{-1}) + L.$$

In particular, if  $d(\xi_1, \xi_2) > 2L$ , then

$$(2L)^{-1} l(g_1 g_2^{-1}) \leq d(\xi_1, \xi_2) \leq 2L l(g_1 g_2^{-1}).$$

For every  $v \in X^*$ , we have  $I^{|v|}(\xi_i \otimes v) = I^{|v|}(\xi'_i \cdot g_i \otimes v) = I^{|v|}(\xi'_i \otimes g_i(v)) \cdot g_i|_v$  for some  $\xi'_i \in T$ . By the choice of  $T$ , we have  $I^{|v|}(\xi'_i \otimes g_i(v)) \in T$ . Note that  $(g_1|_v)(g_2|_v)^{-1} = (g_1 g_2^{-1})|_{g_2(v)}$ . Therefore, if  $d(I^{|v|}(\xi_1 \otimes v), I^{|v|}(\xi_2 \otimes v)) > 2L$ , then it is equal, up to multiplicative constants to the word length of  $(g_1 g_2^{-1})|_{g_2(v)}$ . We also can guarantee that the word length is greater than any fixed number by bounding from below the distance

$d(I^{|v|}(\xi_1 \otimes v), I^{|v|}(\xi_2 \otimes v))$ . It follows that the sum  $\Sigma(\delta, \xi_1, \xi_2)$ , if  $\delta$  is large enough, is bounded from above by a constant multiple of the sum  $\sum_{v \in X^*, l(g|_v) \geq l} l(g|_v)^p$ , where  $g = g_1 g_2^{-1}$  and  $l$  is an arbitrary constant. Proposition 4.3 then finishes the proof.  $\square$

**Definition 4.5.** The *portrait* of an element  $g \in G$  is the set  $P(g)$  of finite words  $v \in X^*$  such that  $g|_u$  does not belong to the nucleus  $\mathcal{N}$  of  $G$  for any proper prefix  $u$  of  $v$ . If  $g \in \mathcal{N}$ , then we set  $P(g) = \{\emptyset\}$ .

The portrait is prefix-closed, i.e., a rooted subtree of  $X^*$ . Let us denote by  $L(g)$  the set of leaves of this subtree, i.e., the set of words  $v \in X^*$  such that  $g|_v \in \mathcal{N}$  but  $g|_u \notin \mathcal{N}$  for any proper prefix  $u$  of  $v$ .

The following bounds on the size of the portrait is proved in the same way as Proposition 4.3.

**Proposition 4.6.** Let  $(G, \mathfrak{B})$  be a contracting group, choose a basis  $X \subset \mathfrak{B}$ . Let  $p \geq 1$  be such that  $\eta_p < 1$ . Then there exists a constant  $C > 1$  such that

$$|P(g) \cap X^n| \leq C \eta_p^n l(g)^p$$

and

$$|P(g)| \leq C l(g)^p$$

for every  $g \in G$  and  $n \geq 0$ .

The portrait  $P(g)$  of an element  $g$  together with the labeling of the elements  $v \in P(g)$  by the permutations the sections  $g|_v$  induce on the first level  $X \subset X^*$  and the labeling of the elements of  $L(g)$  by the sections  $g|_v \in \mathcal{N}$  uniquely determine  $g$ . This, together with Proposition 4.6 implies that the growth of the group  $G$  is bounded from above by  $e^{Cn^\alpha}$  for every  $\alpha > p_c(G, \mathfrak{B})$ .

#### 4.2. Critical exponent and conformal dimension.

**Theorem 4.7.** Let  $(G, \mathfrak{B})$  be a finitely generated contracting group. Its critical exponent  $p_c$  is not greater than  $\text{ARdim } \mathcal{J}_G$ .

*Proof.* Let  $d$  be an Ahlfors-regular quasiconformal metric on  $\mathcal{J}_G$ . Let  $\nu$  be the corresponding Hausdorff measure, and let  $\alpha$  be its Hausdorff dimension. Take  $\beta > \alpha$ . It is enough to prove that  $p_c \leq \beta$ , i.e., that  $\eta_\beta < 1$ .

Let  $S$  be a finite symmetric generating set of  $G$ . Consider a product  $g = s_m \dots s_2 s_1$  of elements of  $S$ . Let  $n_0$  be such that all sections of elements of  $S$  in words of length  $n_0$  belong to the nucleus. Then for every  $u \in X^n$  and  $v_0 \in X^{n_0}$  consider the sequence

$$v_0 u, \quad s_1(v_0 u), \quad s_2 s_1(v_0 u), \dots, \quad s_m \dots s_2 s_1(v_0 u).$$

Denote by  $u_0, u_1, \dots, u_m$  the corresponding suffixes of length  $n$ . Then  $u_i = s_i|_{s_{i-1} \dots s_1(v_0)}(u_{i-1})$ , so the tiles  $\mathcal{T}_{u_i}$  and  $\mathcal{T}_{u_{i-1}}$  intersect in  $\mathcal{J}_G$ , and the tiles  $\mathcal{T} \otimes u_i$  and  $\mathcal{T} \otimes u_{i-1}$  intersect in  $\mathcal{X}_G$ .

Choose  $\xi_{i, v_0 u} \in \mathcal{T}_{u_i} \cap \mathcal{T}_{u_{i-1}}$  for every  $i = 1, 2, \dots, m$ . Let  $R_{i, v_0 u}$  and  $r_{i, v_0 u}$  be the smallest and the largest radii such that

$$B(\xi_{i, v_0 u}, r_{i, v_0 u}) \subseteq T(\xi_{i, v_0 u}, n) \subseteq B(\xi_{i, v_0 u}, R_{i, v_0 u}).$$

We know that  $R_{i, v_0 u}/r_{i, v_0 u} \in [1, C]$  and that  $\mu(T(\xi_{i, v_0 u}, n)) \geq C^{-1} r_{i, v_0 u}^\alpha \geq C^{-1-\alpha} R_{i, v_0 u}^\alpha$ , for some constant  $C > 1$ .

Denote by  $M_n$  the maximum over  $\xi \in \mathcal{J}_G$  of the minimal radius  $R$  such that  $T(\xi, n) \subset B(\xi, R)$ . It follows from Proposition 3.6 that  $M_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Consequently (using  $\beta > 1$ ),

$$\begin{aligned} (R_{1,v_0u} + R_{2,v_0u} + \cdots + R_{m,v_0u})^\beta &\leq m^{\beta-1} (R_{1,v_0u}^\beta + R_{2,v_0u}^\beta + \cdots + R_{m,v_0u}^\beta) \leq \\ & m^{\beta-1} M_n^{\beta-\alpha} (R_{1,v_0u}^\alpha + R_{2,v_0u}^\alpha + \cdots + R_{m,v_0u}^\alpha) \leq \\ & m^{\beta-1} M_n^{\beta-\alpha} C^{1+\alpha} (\nu(T(\xi_{1,v_0u}, n)) + \nu(T(\xi_{2,v_0u}, n)) + \cdots + \nu(T(\xi_{m,v_0u}, n))). \end{aligned}$$

We have  $\sum_{v_0 \in \mathcal{X}^{n_0}, u \in \mathcal{X}^n} \nu(T(\xi_{i,v_0u}, n)) \leq |\mathcal{X}|^{n_0} \cdot |\mathcal{N}|^2 \cdot \nu(\mathcal{J}_G)$ , hence adding all the inequalities over all  $v_0 \in \mathcal{X}^{n_0}, u \in \mathcal{X}^n$ , we will get

$$\sum_{v_0 \in \mathcal{X}^{n_0}, u \in \mathcal{X}^n} (R_{1,v_0u} + R_{2,v_0u} + \cdots + R_{m,v_0u})^\beta \leq |\mathcal{X}|^{n_0} \cdot |\mathcal{N}|^2 \cdot \nu(\mathcal{J}_G) \cdot C^{1+\alpha} \cdot M_n^{\beta-\alpha} \cdot m^\beta.$$

**Lemma 4.8.** *There exists a constant  $C_1 > 0$ , depending only on  $G, \mathcal{X}$ , and  $d$ , such that  $l(g|_{v_0u}) \leq C_1(R_{1,v_0u} + R_{2,v_0u} + \cdots + R_{m,v_0u}) + C_1$ .*

*Proof.* Let us define a metric on  $\mathcal{X}_G$ . Let  $\zeta_1, \zeta_2 \in \mathcal{X}_G$ . Take all  $v_1, v_2 \in \mathfrak{B}^{\otimes n}$  for all  $n \geq 0$  such that  $\zeta_i \in (\mathcal{T}S) \otimes v_i$  and  $(\mathcal{T}S) \otimes v_1 \cap (\mathcal{T}S) \otimes v_2 \neq \emptyset$ , and then take the infimum  $d_0(\zeta_1, \zeta_2)$  of  $\text{diam}((\mathcal{T}S) \otimes v_1/G) + \text{diam}((\mathcal{T}S) \otimes v_2/G)$ . (Here  $\mathfrak{B}^{\otimes 0} = G$ .) Denote the infimum by  $d_0(\zeta_1, \zeta_2)$  (it may be infinite).

We obviously have that  $d_0(\zeta_1, \zeta_2)$  is not smaller than the distance between the images of  $\zeta_1$  and  $\zeta_2$  in  $\mathcal{J}_G$ .

Define then  $d_1(\zeta_1, \zeta_2)$  as the infimum of  $d_0(\xi_0, \xi_1) + d_0(\xi_1, \xi_2) + \cdots + d_0(\xi_{n-1}, \xi_n)$  over all sequences  $\xi_i$  such that  $\xi_0 = \zeta_1$  and  $\xi_n = \zeta_2$ . Then  $d_1$  is a finite metric on  $\mathcal{X}_G$ . It is obviously  $G$ -invariant. For every  $\xi \in \mathcal{X}_G$ , the diameter of the union of the tiles of the  $n$ th level of  $\mathcal{X}_G$  containing  $\xi$  is not larger than the diameter of the union of the tiles of the  $n$ th level of  $\mathcal{J}_G$  containing the image of  $\xi$ . Consequently, the metric  $d_1$  is compatible with the topology on  $\mathcal{X}_G$ .

Choose a point  $\xi_0 \in \mathcal{T}$ , and consider the sequence

$$\xi_0 \otimes v_0u, \quad \xi_0 \cdot s_1 \otimes v_0u, \quad \xi_0 \cdot s_2 s_1 \otimes v_0u, \dots, \quad \xi_0 \cdot s_m \cdots s_2 s_1 \otimes v_0u.$$

Then  $R_{1,v_0u} + R_{2,v_0u} + \cdots + R_{m,v_0u}$  is an upper bound on  $d_1(\xi_0 \otimes v_0u, \xi_0 \cdot g \otimes v_0u)$ . The lemma follows then from Lemma 2.11.  $\square$

Note that each  $R_{i,v_0u}$  is bounded from below by some number  $r_n > 0$ . Consequently,

$$\begin{aligned} \left( \sum_{v \in \mathcal{X}^{n+n_0}} l(g|_v)^\beta \right)^{1/\beta} &\leq \\ C_1 \left( \sum_{v_0 \in \mathcal{X}^{n_0}, u \in \mathcal{X}^n} (R_{1,v_0u} + R_{2,v_0u} + \cdots + R_{m,v_0u} + 1)^\beta \right)^{1/\beta} &\leq \\ C_1 \left( \sum_{v_0 \in \mathcal{X}^{n_0}, u \in \mathcal{X}^n} (R_{1,v_0u} + R_{2,v_0u} + \cdots + R_{m,v_0u})^\beta \right)^{1/\beta} + |\mathcal{X}^n|^{1/\beta} &\leq \\ & KM_n^{1-\frac{\alpha}{\beta}} \cdot l(g) + |\mathcal{X}^n|^{1/\beta}, \end{aligned}$$

which implies that  $\limsup_{l(g) \rightarrow \infty} \frac{(\sum_{v \in \mathcal{X}^{n+n_0}} l(g|_v)^\beta)^{1/\beta}}{l(g)} \leq KM_n^{1-\frac{\alpha}{\beta}}$ . Since  $M_n \rightarrow 0$  as  $n \rightarrow \infty$ , Lemma 4.1 implies that  $\eta_\beta < 1$ .  $\square$

**Remark 4.9.** The inequality in Theorem 4.7 is not sharp. For example, it is known that  $\eta_1$  for the first Grigorchuk group  $G$  is strictly less than 1 (this is the classical sum-contraction

of Grigorchuk [17]), which implies by the inequality (5) that its critical exponent is strictly less than 1. On the other hand, the limit space of the Grigorchuk group is a segment, hence we have  $\text{ARdim } \mathcal{J}_G = 1$ . In fact, it follows from the norm contracting inequality in [5] and the explicit sequence of group elements which satisfies the reverse inequality (see [6][Prop 4.7]), we have that its critical exponent is equal to  $\alpha_0 = \log 2 / \log \lambda_0 \simeq 0.7674$ , where  $\lambda_0$  is the positive real root of the polynomial  $X^3 - X^2 - 2X - 4$ . Recall the definition of critical constant for Liouville property in the Introduction. From the random walk with nontrivial Poisson boundary constructed in [15] for the purpose of volume lower estimate, we have that  $\text{Cr}_{\text{Liouv}}(G)$  is also equal to  $\alpha_0$  and  $\alpha_0$  is the growth exponent of  $G$ .

**4.3. Thurston obstructions.** Let  $(G, \mathfrak{B})$  be a self-similar group. Consider the linear span  $V$  of conjugacy classes of infinite order elements of  $G$ . We denote by  $[g]$  the conjugacy class of an element  $g \in G$  seen as an element of  $V$ . If  $g$  is of finite order, then we define  $[g]$  to be equal to zero. Choose a basis  $X \subset \mathfrak{B}$  and the associated self-similar action on  $X^*$ . Let  $g \in G$ . Consider the action of  $g$  on  $X$ , and for every cycle  $x_1 \mapsto x_2 \mapsto \dots \mapsto x_k \mapsto x_1$  of the action, consider the conjugacy class  $[g|_{x_k} \cdots g|_{x_2} g|_{x_1}] = [(g^k)|_{x_1}]$ . Note that it does not depend on the choice of the initial element  $x_1$  of the cycle. Choose  $p \geq 1$ , and denote by  $T_p([g])$  the sum of the elements  $k^{1-p}[(g^k)|_{x_1}] \in V$  taken over all cycles of the action of  $g$  on  $X$ . Since a change of the basis  $X$  corresponds to conjugation of the wreath recursion by an element of the wreath product, the value of  $T_p([g])$  does not depend on the choice of  $X$ .

For example, if  $a = \sigma(1, a)$  is the binary odometer, then we have  $T_p([a]) = 2^{1-p}[a]$ .

We call the linear operator  $T_p$  the *Thurston's  $p$ -map*.

**Lemma 4.10.** *If  $T_p$  is the Thurston's  $p$ -map for  $(G, \mathfrak{B})$ , then the Thurston's  $p$ -map for  $(G, \mathfrak{B}^{\otimes n})$  is  $T_p^n$ .*

*Proof.* Let  $x_1 \mapsto x_2 \mapsto \dots \mapsto x_k \mapsto x_1$  be a cycle of the action of  $g$  on  $X$ . Denote  $h = (g^k)|_{x_1}$ . For every cycle  $v_1 \mapsto v_2 \mapsto \dots \mapsto v_m \mapsto v_1$  of the action of  $h$  on  $X^n$  we get the cycle

$$\begin{aligned} x_1 v_1 &\mapsto x_2 g|_{x_1}(v_1) \mapsto \dots \mapsto x_k (g^{k-1})_{x_1}(v_1) \mapsto \\ &x_1 v_2 \mapsto x_2 g|_{x_1}(v_2) \mapsto \dots \mapsto x_k (g^{k-1})_{x_1}(v_2) \mapsto \\ &\dots \\ &x_1 v_m \mapsto x_2 g|_{x_1}(v_m) \mapsto \dots \mapsto x_k (g^{k-1})_{x_1}(v_m) \mapsto x_1 v_1 \end{aligned}$$

of length  $km$  of the action of  $g$  on  $X^{n+1}$ . We have  $(g^{km})|_{x_1 v_1} = h^m|_{v_1}$ . A proof of the lemma now follows by induction.  $\square$

**Theorem 4.11.** *Suppose that  $U < V$  is  $T_p$ -invariant subspace spanned by a finite set of conjugacy classes. If the  $\ell^p$ -contraction coefficient  $\eta_p$  of  $(G, \mathfrak{B})$  is less than 1, then the spectral radius of  $T_p|_U$  is also less than 1.*

In particular, we can use this theorem to find lower estimates for  $p_c(G, \mathfrak{B})$ , and hence for  $\text{ARdim}(G, \mathfrak{B})$ .

A particular case of this theorem is related to one direction of the Thurston's theorem (see [13]) characterizing post-critically finite branched self-coverings of the sphere that are realizable as complex rational function. In Thurston's theorem one considers subspaces spanned by conjugacy classes defined by disjoint simple closed curves of the punctured sphere (with some additional conditions). Such collections of curves are called *multicurves*.

*Proof.* Let  $\{[g_1], [g_2], \dots, [g_m]\}$  be a finite spanning set of  $U$ . Let  $N$  be a natural number such that the action of each element  $g_i^N$  on  $X$  is identical. Denote by  $L_{i,kN}$  the length of  $g_i^{iN}$  (with respect to some fixed finite generating set of the group). We have then  $L_{i,k_1 k_2 N} \leq k_1 L_{i,k_2 N}$  for all  $i = 1, \dots, m, k_1, k_2 \geq 1$ . Equivalently, we have

$$L_{i,k_2 N} \geq k_1^{-1} L_{i,k_1 k_2 N}.$$

Note also that  $L_{i,kN} \rightarrow \infty$  as  $kN \rightarrow \infty$ , since we assume that the elements  $g_i$  are of infinite order.

Assume that  $\eta_p < 1$ . Let  $\eta$  be such that  $\eta_p < \eta < 1$ . After replacing  $X$  by  $X^n$  for a large  $n$  (using Lemma 4.10) and assuming that  $N$  is large enough, we will have

$$\sum_{x \in X} l(g_i^{Nk}|_x)^p \leq \eta l(g_i^{Nk})^p$$

for all  $i$  and  $k$ .

For cycle  $x_1 \mapsto x_2 \mapsto \dots \mapsto x_r \mapsto x_1$  of  $g_i$ , the corresponding sections  $g_i^{Nk}|_{x_i}$  are conjugated to  $(g_i^r|_{x_1})^{\frac{Nk}{r}}$ . If  $g_i^r|_{x_1}$  has infinite order, then it is conjugate to one of the elements  $g_j$ . Since the number of corresponding conjugators is finite, we get

$$l(g_i^{Nk}|_{x_i}) \geq L_{j,Nk/r} - C.$$

Choose an arbitrary  $\rho \in (\eta, 1)$ , so that  $\eta^{-1}\rho > 1$ . Assuming that  $N$  is large enough, we will get  $l(g_i^{Nk}|_{x_i}) \geq \rho L_{j,Nk/r}$  for all  $k \geq 1$ . Consequently,

$$\sum_{i=1}^r l(g_i^{Nk}|_{x_i})^p \geq \rho^r L_{j,Nk/r}^p \geq \rho^r (r^{-1} L_{j,Nk})^p = \rho^{r(1-p)} L_{j,Nk}^p.$$

It follows that

$$L_{i,Nk}^p = l(g_i^{Nk})^p \geq \eta^{-1} \sum_{x \in X} l(g_i^{Nk}|_x)^p \geq \eta^{-1} \rho \cdot \phi(T_p([g_i])),$$

where  $\phi$  is the linear functional defined by

$$\phi([g_j]) = L_{j,Nk}^p.$$

Consider the matrix of the dual operator  $T_p^*$  in the basis of  $U^*$  dual to the basis  $([g_1], [g_2], \dots, [g_m])$ . The above inequality means that the  $i$ th coordinate  $T_p^*(\phi)([g_i]) = \phi(T_p([g_i]))$  of  $T_p^*(\phi) \in U^*$  is less than or equal to  $\eta\rho^{-1}$  times the  $i$ th coordinate  $\phi([g_i])$  of  $\phi$ . Since all coordinates of  $\phi$  are positive, Perron-Frobenius theorem implies that the spectral radius of  $T_p^*$  is strictly less than 1. Consequently, the spectral radius of  $T_p$  is also strictly less than 1.  $\square$

As an example of an application of Theorem 4.11 (and Thurston's theorem), consider a mating of two cubic polynomials, found by M. Shishikura and L. Tan, see [43].

The iterated monodromy group is generated by

$$\begin{aligned} a_1 &= (12)(1, 1, a_3), & b_1 &= (1, b_3, 1), \\ a_2 &= (a_1, 1, 1), & b_2 &= (b_1, 1, 1), \\ a_3 &= (02)(a_3^{-1}, 1, a_2 a_3), & b_3 &= (012)(b_3^{-1}, 1, b_2 b_3). \end{aligned}$$

Note that  $a_1 a_2 a_3 = b_1 b_2 b_3 = \tau$ , where  $\tau = (012)(1, 1, \tau)$ . The groups  $\langle a_1, a_2, a_3 \rangle$  and  $\langle b_1, b_2, b_3 \rangle$  are the iterated monodromy groups of the mated cubic polynomials.

It is checked directly that the group is contracting with the nucleus consisting of the elements  $a_3, a_2 a_3, b_3, b_2 b_3, a_1 a_2 a_3 = b_1 b_2 b_3$ , their inverses, and the identity element. Moreover, the wreath recursion is contracting for the free group (of rank 5), hence the limit space has topological dimension 1, see [38, Theorem 4.4].

Post-conjugating the wreath recursion by  $(1, b_3^{-1}, b_3^{-1})$ , and passing to the generating set  $b_1, b_2, b_3, x = b_1^{-1}a_1, y = b_3a_3^{-1}$  (we have  $b_2^{-1}a_2 = yx^{-1}$  in the faithful quotient), we get an equivalent wreath recursion:

$$\begin{aligned} b_1 &= (1, b_3, 1), & x &= (12)(1, 1, y^{-1}), \\ b_2 &= (b_1, 1, 1), & y &= (12)(x, 1, y^{-1}), \\ b_3 &= (012)(1, 1, b_2). \end{aligned}$$

Let  $U$  be the formal linear span of the conjugacy classes  $[x], [x^{-1}], [y], [y^{-1}]$ . Thurston map  $T_2$  acts by the rule

$$\begin{aligned} [x] &\mapsto \frac{1}{2}[y^{-1}], & [x^{-1}] &\mapsto \frac{1}{2}[y], \\ [y] &\mapsto [x] + \frac{1}{2}[y^{-1}], & [y^{-1}] &\mapsto [x^{-1}] + \frac{1}{2}[y]. \end{aligned}$$

It is easy to see that the spectral radius of  $T_2|_U$  is equal to 1. In particular, this means, by Thurston's theorem, that the mating is *obstructed*, i.e., is not equivalent to a rational function. Theorem 4.11 implies that the conformal dimension of the limit space of the group is at least 2.

## 5. RANDOM WALK AND LIOUVILLE PROPERTY

### 5.1. Random walk preliminaries.

5.1.1. *Notation.* Let  $G$  be a group endowed with a symmetric probability measure  $\mu$ . The measure  $\mu$  is said to be *non-degenerate* if its support  $\text{supp } \mu$  generates  $G$  as a semi-group. If  $G$  is finitely generated, the measure  $\mu$  is said to have finite  $p$ -moment for  $p > 0$  if  $\sum \mu(g)l(g)^p < +\infty$ , where  $l(\cdot)$  is the word metric on  $G$  associated to some finite symmetric generating set. We denote by  $(\mathbf{g}_t)$  the (left) random walk associated to  $\mu$ , namely  $\mathbf{g}_t = \mathbf{s}_t \cdots \mathbf{s}_1$ , where  $(\mathbf{s}_i)_{i \geq 1}$  is a sequence of independent random variables with distribution  $\mu$ .

5.1.2. *Capacities and random walk on Schreier graphs.* Assume that  $G$  acts on a countable set  $\Omega$ . Then for each starting point  $\omega \in \Omega$ , the process  $(\mathbf{g}_t \cdot \omega)$  is a Markov chain on  $\Omega$  with transition probabilities  $p(v, w) = \sum_{g \cdot v = w} \mu(g)$ . The assumption that  $\mu$  is symmetric implies that the induced Markov chain is reversible with respect to the counting measure on  $\Omega$ , which we use as the reference measure in the Dirichlet forms below. For  $p \geq 1$ , the associated  $p$ -Dirichlet form on  $\ell^p(\Omega)$  is

$$\mathcal{E}_{p,\mu}(f) = \frac{1}{2} \sum_{g \in G, v \in \Omega} \mu(g) |f(v) - f(g \cdot v)|^p, \quad f \in \ell^p(\Omega).$$

Let  $A, B$  be two disjoint finite subsets of  $\Omega$ . For  $p \geq 1$ , the (effective)  $p$ -capacity between  $A$  and  $B$  is defined as

$$\text{Cap}_{p,\mu}(A, B) = \inf \{ \mathcal{E}_{p,\mu}(f) : f \in \ell^p(\Omega), f|_A = 1, f|_B = 0 \}.$$

For  $p = 2$  this has a well-known probabilistic interpretation. The following is a reformulation of [30, Exercises 2.13 and 2.47], we provide a proof for completeness.

**Lemma 5.1.** *Let  $(G, \mu)$  be a group endowed with a probability measure, and  $(\mathbf{g}_n)$  the associated random walk. Assume that  $G$  acts on a finite set  $\Omega$ . For  $\omega \in \Omega$  and  $B \subset \Omega$ , consider the stopping time:*

$$\mathbf{T}_{\omega \rightarrow B} = \min\{t \geq 1 : \mathbf{g}_t \cdot \omega \in B\}.$$

Then for  $A, B$  disjoint subsets of  $\Omega$ ,

$$\text{Cap}_{2,\mu}(A, B) = \sum_{\omega \in A} \mathbb{P}(\mathbf{T}_{\omega \rightarrow B} < \mathbf{T}_{\omega \rightarrow A}).$$

*Proof.* We use the same notation and terminology on electric networks as in [30]. Since  $\mu$  is symmetric, we may regard the induced random walk transition operator  $P_\mu$  on  $\Omega$  as coming from an electric network with conductance  $c(x, y) = \sum_{g \in G} \mathbf{1}_{\{y=g \cdot x\}} \mu(g)$ , and reversing measure  $\pi(x) = 1, x, y \in \Omega$ . Denote by  $\tau_{\omega \rightarrow A} = \min\{t \geq 0 : \mathbf{g}_t \omega \in A\}$  the hitting time of  $A$ . Let  $v(x) = \mathbb{P}(\tau_{x \rightarrow A} < \tau_{x \rightarrow B})$ . Then  $v$  is a voltage function which is 1 on the source set  $A$ , and 0 on the sink set  $B$ . Considering the first step of the random walk, we have

$$\begin{aligned} \sum_{\omega \in A} \mathbb{P}(\mathbf{T}_{\omega \rightarrow B} < \mathbf{T}_{\omega \rightarrow A}) &= \sum_{\omega \in A} \sum_{x \in \Omega} P_\mu(\omega, x) \mathbb{P}(\tau_{x \rightarrow B} < \tau_{x \rightarrow A}) \\ &= \sum_{\omega \in A} \sum_{x \in \Omega} c(\omega, x) (v(\omega) - v(x)). \end{aligned}$$

Regard  $i(\omega, x) = c(\omega, x)(v(\omega) - v(x))$  as the current on the edge  $(\omega, x)$ . The capacity  $\text{Cap}_{2,\mu}(A, B)$  is the effective conductance between  $A$  and  $B$ , thus equal to the total amount of current flowing out of  $A$  under the voltage function  $v$ , which is  $\sum_{\omega \in A} \sum_{x \in \Omega} i(\omega, x)$ .  $\square$

**5.1.3. Liouville property and entropy.** Let  $\mu$  be a probability measure on  $G$ . A function  $f : G \rightarrow \mathbb{R}$  is  $\mu$ -harmonic if  $f(x) = \sum_{g \in G} f(xg) \mu(g)$  for all  $x \in G$ . By definition the  $\mu$ -random walk has the *Liouville property* if  $\mu$  has trivial Poisson boundary, namely if every bounded  $\mu$ -harmonic function is constant on the subgroup  $\langle \text{supp } \mu \rangle$ , Furstenberg observed that admitting a non-degenerate measure  $\mu$  with the Liouville property implies amenability of the group  $G$  (see [27, Theorem 4.2]). Conversely every amenable group admits a symmetric non-degenerate measure with trivial Poisson boundary, by a result of Kaimanovich and Vershik [27, Theorem 4.4] and Rosenblatt [42].

Recall that given a probability measure  $\nu$  supported on a discrete set  $\Omega$ , the (Shannon) entropy of  $\nu$  is defined as

$$H(\nu) = - \sum_{\omega \in \Omega} \nu(\omega) \log \nu(\omega).$$

If  $\mathbf{X}$  is a discrete random variable taking values in a set  $\Omega$ , the entropy  $H(\mathbf{X})$  is defined as the entropy of the distribution of  $\mathbf{X}$ .

Let  $\mathbf{X}, \mathbf{Y}$  be discrete random variables, and for any value  $y$  taken by  $\mathbf{Y}$  with positive probability, denote by  $\mathbf{X}^{(y)}$  the variable with the conditional distribution of  $\mathbf{X}$  given  $\{\mathbf{Y} = y\}$ . Then the *conditional entropy* of  $\mathbf{X}$  given  $\mathbf{Y}$  is defined as the expected value of  $H(\mathbf{X}^{(y)})$ , namely

$$H(\mathbf{X}|\mathbf{Y}) = \sum_y H(\mathbf{X}^{(y)}) \mathbb{P}(\mathbf{Y} = y).$$

We resume below some basic properties of Shannon entropy.

- Lemma 5.2.** (1) If  $\mathbf{X}$  is a random variable taking values in a finite set  $\Omega$ , then  $H(\mathbf{X}) \leq \log |\Omega|$ .  
(2) The entropy of the joint distribution of two discrete random variables  $(\mathbf{X}, \mathbf{Y})$  satisfies

$$H(\mathbf{X}, \mathbf{Y}) = H(\mathbf{X}|\mathbf{Y}) + H(\mathbf{Y}).$$

- (3) Assume that a random variable  $\mathbf{Y}$  can be expressed as  $\mathbf{Y} = f(\mathbf{X}_1, \dots, \mathbf{X}_k)$  for some discrete random variables  $\mathbf{X}_1, \dots, \mathbf{X}_k$  and some measurable function  $f$ . Then

$$H(\mathbf{Y}) \leq H(\mathbf{X}_1) + \dots + H(\mathbf{X}_k).$$

Another well-known and elementary fact is that every random variable  $\mathbf{X}$  taking values in the non-negative integers  $\mathbb{N}$  and with finite expectation satisfies

$$H(\mathbf{X}) \leq \log(1 + \mathbb{E}[\mathbf{X}]) + 1.$$

This in turn gives the following simple bound for random variable taking values in a group.

**Lemma 5.3.** *Let  $G$  be a finitely generated group and  $l(\cdot)$  be a word metric on  $G$  associated to a finite symmetric generating set  $S$  containing the identity. Then for every  $G$ -valued random variable  $\mathbf{g}$  with finite entropy we have  $H(\mathbf{g}) \leq (\log |S| + 1)\mathbb{E}[l(\mathbf{g})] + 1$ .*

*Proof.* We may assume that the expectation of  $l(\mathbf{g})$  is finite. We have  $H(\mathbf{g}) \leq H(\mathbf{g}|l(\mathbf{g})) + H(l(\mathbf{g}))$ . Let  $\mathbf{g}^{(n)}$  be the variable  $\mathbf{g}$  conditioned to  $l(\mathbf{g}) = n$ . Since  $\mathbf{g}^{(n)}$  belongs to  $S^n$ , we have  $H(\mathbf{g}^{(n)}) \leq n \log |S|$  and  $H(\mathbf{g}|l(\mathbf{g})) \leq \log |S| \mathbb{E}[l(\mathbf{g})]$ . On the other hand  $H(l(\mathbf{g})) \leq \mathbb{E}[l(\mathbf{g})] + 1$ .  $\square$

Assume that  $\mu$  is a probability measure on a countable group  $G$ . Then the limit  $h_\mu = \lim \frac{1}{n} H(\mu^{*n})$  exists and is called the (Avez) asymptotic entropy of  $\mu$ . The entropy criterion, proven by Kaimanovich and Vershik [27] and Derriennic [12], states that if  $H(\mu) < +\infty$ , then  $\mu$  has trivial Poisson boundary if and only if  $h_\mu = 0$ .

For groups acting on rooted trees, the entropy criterion combined with the previous properties of entropy gives the following sufficient condition for boundary triviality. Some tightly related criteria appeared in [9, 26] and were used in several papers.

**Proposition 5.4.** *Let  $G$  be a finitely generated group of automorphisms of a rooted tree  $X^*$  endowed with a word metric, and  $\mu$  be a probability measure on  $G$  with finite entropy. Then there is a constant  $C > 0$  such that for every  $n, t > 0$  we have*

$$H(\mu^{*t}) \leq C \mathbb{E} \left[ \sum_{v \in X^n} l(\mathbf{g}_t|_v) \right] + C |X|^n.$$

In particular, if there exists a sequence  $(\varepsilon_n)$  tending to 0 such that for every  $n$  we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \left[ \sum_{v \in X^n} l(\mathbf{g}_t|_v) \right] \leq \varepsilon_n,$$

then  $\mu$  has trivial Poisson boundary.

*Proof.* For every  $n$  and  $t$ , the variable  $\mathbf{g}_t$  is completely determined by the collection of sections  $(\mathbf{g}_t|_v)_{v \in X^n}$  and by the permutation  $\sigma$  describing the action on level  $n$ . Thus

$$H(\mathbf{g}_t) \leq \sum_{v \in X^n} H(\mathbf{g}_t|_v) + H(\sigma).$$

By Lemma 5.3, the first term is bounded by  $C_1 \mathbb{E}[\sum_{v \in X^n} l(\mathbf{g}_t|_v)] + C_1 |X|^n$ . The permutation  $\sigma$  is completely determined by the first-level permutation of all sections  $\mathbf{g}_t|_w$  where  $|w| \leq n-1$ . Thus the total number of possibilities for  $\sigma$  is bounded by  $\exp(C_2 |X|^n)$ , and  $H(\sigma) \leq C_2 |X|^n$  for some constant  $C_2 > 0$ . The proposition follows.  $\square$

**5.2. Critical contraction exponent and  $p$ -capacity.** Let  $(G, \mathfrak{B})$  be a finitely generated contracting group with a basis  $X$ , and  $\mu$  be a symmetric measure on  $G$ . In this subsection we will relate the  $p$ -capacities associated to the action of  $G$  on the levels  $X^n$  with  $p_c(G, \mathfrak{B})$  and  $\text{ARdim } \mathcal{J}_G$ .

**Proposition 5.5.** *Suppose that  $(G, \mathfrak{B})$  is a contracting finitely generated group. Denote by  $l(g)$  the word length with respect to some fixed finite generating set on  $G$ .*

*Let  $d$  be an Ahlfors-regular metric on  $\mathcal{J}_G$  of Hausdorff dimension  $\alpha$ . Let  $\beta > \alpha$  and let  $p > p_c(G, \mathfrak{B})$ . Pick a basis  $X \subset \mathfrak{B}$ .*

*There exists a sequence  $\epsilon_n \rightarrow 0$  such that for every  $g \in G$  and every  $\dots x_2 x_1 \in X^{-\omega}$  we have*

$$\sum_{v \in X^n} d(\dots x_2 x_1 g(v), \dots x_2 x_1 v)^\beta \leq \epsilon_n l(g)^p.$$

*If  $(G, \mathfrak{B})$  is self-replicating, then we can take  $\epsilon_n = C\lambda^n$  for some  $C > 0$  and  $\lambda \in (0, 1)$ .*

*Proof.* Let  $\nu$  be the Hausdorff measure of the metric  $d$ . Take an element  $g \in G$ . For every  $v \in X^n$ , let  $v'$  be the shortest prefix of  $v$  such that  $g|_{v'} \in \mathcal{N}$ . If such a prefix does not exist, then define  $v' = v$ . Let  $v'' \in X^*$  be such that  $v = v'v''$ .

We have  $\dots x_2 x_1 g(v) \in T(\dots x_2 x_1 v, |v''|)$  for every  $v \in X^n$ . It follows that

$$d(\dots x_2 x_1 g(v), \dots x_2 x_1 v)^\alpha \leq C_1 \nu(T(\dots x_2 x_1 v, |v''|))$$

for some constant  $C_1$ . Therefore, there exists  $C_2 > 0$  such that for every sequence  $(k_n)$  we have

$$\begin{aligned} \sum_{v \in X^n} d(\dots x_2 x_1 g(v), \dots x_2 x_1 v)^\beta &\leq C_2 \sum_{v \in X^n} \nu(T(\dots x_2 x_1 v, |v''|))^\frac{\beta}{\alpha} \leq \\ &C_2 \nu(\mathcal{J}_G)^\frac{\beta}{\alpha} \cdot |\{v \in X^n : |v''| \leq k_n\}| + C_2 \sum_{v \in X^n, |v''| > k_n} \nu(T(\dots x_2 x_1 v, |v''|))^\frac{\beta}{\alpha}. \end{aligned}$$

By Proposition 4.6

$$|\{v \in X^n : |v''| \leq k_n\}| \leq C_3 \eta_p^{n-k_n} l(g)^p,$$

for some  $C_3$  not depending on  $k_n$  or on  $g$ . For the second summand, note that if  $T(\dots x_2 x_1 v_1, |v''_1|)$  and  $T(\dots x_2 x_1 v_2, |v''_2|)$  intersect for  $v_1, v_2 \in X^n$ , then either  $v'_1 \neq v'_2$  or  $v'_1 = v'_2$  and  $h(v''_1) = v''_2$  for some  $h \in \mathcal{N}^2$ . It follows that every point of  $\mathcal{J}_G$  is contained in not more than  $|L(g)| \cdot |\mathcal{N}^2|$  sets of the form  $T(\dots x_2 x_1 v, |v''|)$ , recalling that  $L(g)$  is the set of words  $v \in X^*$  such that  $g|_v \in \mathcal{N}$  but  $g|_u \notin \mathcal{N}$  for any proper prefix  $u$  of  $v$ . It follows that

$$\sum_{v \in X^n, |v''| > k_n} \nu(T(\dots x_2 x_1 v, |v''|)) \leq |L(g)| \cdot |\mathcal{N}^2| \cdot \nu(\mathcal{J}_G).$$

By Proposition 4.6,  $|L(g)| \leq C_4 l(g)^p$  for some constant  $C_4$ . Hence

$$\begin{aligned} \sum_{v \in X^n} d(\dots x_2 x_1 g(v), \dots x_2 x_1 v)^\beta &\leq C_2 \sum_{v \in X^n} \nu(T(\dots x_2 x_1 v, |v''|))^\frac{\beta}{\alpha} \leq \\ &C_5 \left( \eta_p^{n-k_n} + \max_{v \in X^n, |v''| > k_n} \nu(T(\dots x_2 x_1 v, |v''|))^\frac{\beta}{\alpha-1} \right) l(g)^p \\ &\leq C_5 \left( \eta_p^{n-k_n} + \sup_{\xi \in \mathcal{J}_G, k > k_n} \nu(T(\xi, k))^\frac{\beta}{\alpha-1} \right) l(g)^p. \end{aligned}$$

Since  $\nu(T(\xi, k)) \asymp \text{diam}(T(\xi, k))^\alpha$ , Proposition 3.6 implies that the coefficient

$$\eta_p^{n-k_n} + \sup_{\xi \in \mathcal{J}_G, k > k_n} \nu(T(\xi, k))^{\frac{\beta}{\alpha}-1}$$

converges to zero if simultaneously  $n - k_n \rightarrow \infty$  and  $k_n \rightarrow \infty$ . Note that this coefficient does not depend on  $g$ . We have proved that there exists a sequence  $\epsilon_n \rightarrow 0$  not depending on  $g$  such that

$$\sum_{v \in X^n} d(\dots x_2 x_1 g(v), \dots x_2 x_1 v)^\beta \leq \epsilon_n \ell(g)^p$$

for all  $g \in G$ .

If the group  $(G, \mathfrak{B})$  is self-replicating, then the second part of Proposition 3.6 implies that (for example, if we choose  $k_n = \lfloor n/2 \rfloor$ ) we can choose  $\epsilon_n$  of the form  $C\lambda^n$  for some  $C > 0$  and  $\lambda \in (0, 1)$ .  $\square$

**Theorem 5.6.** *Let  $(G, \mathfrak{B})$  be a contracting finitely generated self-similar group, and let  $\mu$  be a symmetric probability on  $G$  with finite  $p$ -moment for some  $p > p_c(G, \mathfrak{B})$ .*

*Let  $A$  and  $B$  be disjoint closed subsets of  $\mathcal{J}_G$ . Fix  $\xi_0 = \dots x_2 x_1 \in X^{-\omega}$ , and denote by  $A_n$  and  $B_n$  the sets of words  $v \in X^n$  such that the point of  $\mathcal{J}_G$  represented by the sequence  $\dots x_2 x_1 v$  belongs to  $A$  and  $B$ , respectively. If  $\beta > \text{ARdim } \mathcal{J}_G$  then*

$$\lim_{n \rightarrow \infty} \text{Cap}_{\beta, \mu}(A_n, B_n) = 0.$$

*Moreover if  $(G, \mathfrak{B})$  is self replicating, then there exist  $C > 0$  and  $\lambda \in (0, 1)$  such that*

$$\text{Cap}_{\beta, \mu}(A_n, B_n) \leq C\lambda^n.$$

*Proof.* The algebra of Lipschitz functions  $\mathcal{J}_G \rightarrow \mathbb{R}$  is dense in the algebra of all continuous functions, by Stone-Weierstrass theorem. Therefore, for every  $\epsilon > 0$ , there exists a Lipschitz function  $f_0 : \mathcal{J}_G \rightarrow \mathbb{R}$  such that  $|f_0(a) - 1| < \epsilon$  and  $|f_0(b)| < \epsilon$  for all  $a \in A$  and  $b \in B$ . Let  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  be a Lipschitz (with respect to the usual metric on  $\mathbb{R}$ ) function such that  $\chi(x) = 0$  for all  $x \in (-1/3, 1/3)$  and  $\chi(x) = 1$  for all  $x \in (2/3, 5/3)$ . Then the function  $f = \chi \circ f_0 : \mathcal{J}_G \rightarrow \mathbb{R}$  is a Lipschitz function such that  $f(a) = 0$  and  $f(b) = 1$  for all  $a \in A$  and  $b \in B$ . By Proposition 5.5 there exists  $C > 0$  and a sequence  $(\epsilon_n)$  tending to 0 such that for every  $g \in G$

$$\sum_{v \in X^n} |f(\dots x_2 x_1 g(v)) - f(\dots x_2 x_1 v)|^p \leq C_1 \epsilon_n \ell(g)^p.$$

This finishes the proof.  $\square$

**Remark 5.7.** Given an infinite index subgroup  $H$  of a finitely generated group  $G$ , the (symmetric) critical constant for recurrence  $\text{Cr}(G, H)$  is defined as the infimum of  $p$  such that all symmetric random walks with finite  $p$ -moment induce recurrent random walks on  $G/H$ . This notion was introduced by Erschler [14]. In our setting, Theorem 5.6 implies that if  $\text{ARdim } \mathcal{J}_G < 2$ , and if  $H$  is a stabiliser for the action of  $G$  on the boundary  $X^\omega$  of the tree, then  $\text{Cr}(G, H) \leq p_c(G, \mathfrak{B})$ . This inequality is an equality in some cases (for the Grigorchuk and Basilica group for instance). We do not know whether this inequality is always sharp.

We conclude this subsection by pointing out a consequence of Theorem 5.6 which will be used in the proof of Liouville property next subsection. To state it, we first note that the topological dimension of  $\mathcal{J}_G$  is always bounded above by  $\text{ARdim } \mathcal{J}_G$  (see [31, Theorem

1.4.12]). Hence if  $\text{ARdim } \mathcal{J}_G < 2$ , then Theorem 2.14 implies that  $(G, \mathfrak{B})$  has a contracting model  $\Delta_0$  which is a locally finite connected graph, and we can place ourselves in the setting of Notation 2.15.

**Corollary 5.8.** *Let  $(G, \mathfrak{B})$  be a finitely generated contracting self similar group such that  $\text{ARdim } \mathcal{J}_G < 2$ , and retain Notation 2.15. Let  $\mu$  be a symmetric probability measure on  $G$  with a finite  $p$ -moment, where  $p > p_c(G, \mathfrak{B})$ . Then for any two distinct vertices  $z_1$  and  $z_2$  of  $\Gamma_0$ , we have  $\lim_{n \rightarrow \infty} \text{Cap}_{2, \mu}(U_{z_1, n}, U_{z_2, n}) = 0$ .*

*Proof.* Note first that since  $\Delta_0$  is a contracting model, if we replace the basepoint  $\xi_0$  in Notation 2.15 by a different basepoints  $\xi'_0$ , then the corresponding sets  $U'_{z, n}$  satisfy  $U'_{z, n} = U_{z, n}$  for all  $n$  large enough. Therefore, without loss of generality, we can suppose that  $\xi_0$  is the image of a point  $\hat{\xi}_0 \in \mathcal{X}_G$  under the natural map  $\mathcal{X}_G \rightarrow \Delta_0$  given by Theorem 2.9. The map  $\mathcal{X}_G \rightarrow \Delta_0$  also induces in the quotient a map  $\mathcal{J}_G \rightarrow \Gamma_0$ . We will apply Theorem 5.6 to the sets  $A, B \subset \mathcal{J}_G$  given by the pre-images of  $U_{z_1}, U_{z_2}$ . To this end we choose the sequence  $\dots x_2 x_1 \in X^{-\omega}$  in the statement of Theorem 5.6 so that it represents the same element of  $\mathcal{X}_G$  as  $\hat{\xi}_0$ . With this choice, we have  $U_{z_1, n} = A_n$  and  $U_{z_2, n} = B_n$ .  $\square$

**5.3. Proof of Liouville property under conformal dimension  $< 2$ .** In this subsection we complete the proof of the following statement on Liouville property.

**Theorem 5.9.** *Let  $(G, \mathfrak{B})$  be finitely generated contracting group such that  $\text{ARdim } \mathcal{J}_G < 2$ . Let  $p > p_c(G, \mathfrak{B})$ . Then every symmetric probability measure  $\mu$  on  $G$  with finite  $p$ -moment has trivial Poisson boundary.*

**Corollary 5.10.** *Let  $(G, \mathfrak{B})$  be a finitely generated contracting group. If  $\text{ARdim } \mathcal{J}_G < 2$ , then  $G$  is amenable.*

In the proofs below we take terminology and notation from Subsection 2.5, in particular the set  $U_{z, n}$  and the number of traverses  $\tau_n$  as in Definition 2.16 and Theorem 2.17.

For clarity we first prove the theorem under the simplifying assumption that  $\mu$  is symmetric and finitely supported.

*Proof of Theorem 5.9 (finite support case).* Let  $\mu$  be a symmetric probability measure supported on the finite set  $S$ , and  $\mathbf{g}_t = \mathbf{s}_t \cdots \mathbf{s}_1$  be the associated random walk. We wish to estimate the expected number of traverses  $\mathbb{E}[\tau_n(\mathbf{s}_t \cdots \mathbf{s}_1)]$ . By Corollary 5.8 we have  $\max_{z_1, z_2} \text{Cap}_{2, \mu}(U_{z_1, n}, U_{z_2, n}) \rightarrow 0$  as  $n \rightarrow \infty$ , where the maximum is taken over all distinct vertices  $z_1, z_2$  of  $\Gamma_0$ . Denote this maximum by  $\varepsilon_n$ .

Say that a level  $n$  traverse from  $z_1$  to  $z_2$  starts at time  $i$  at a vertex  $v \in U_{z_1, n}$  if there exists  $w \in U_{z_2, n}$  and  $j \geq i$  so that  $(i, j, v, w)$  is a traverse from  $z_1$  to  $z_2$ . Denote this event by  $A(i, v, z_1, z_2)$ . Let  $N_i$  be the total number of traverses that start at time  $i$ . Then  $N_i = \sum_{z_1, z_2} \sum_{v \in U_{z_1, n}} 1_{A(i, v, z_1, z_2)}$ . It follows from Lemma 5.1 that

$$\mathbb{E}[N_i] \leq \sum_{z_1, z_2} \sum_{v \in U_{z_1, n}} \mathbb{P}(\mathbf{T}_{v \rightarrow U_{z_2, n}} < \mathbf{T}_{v \rightarrow U_{z_1, n}}) \leq |\Gamma_0|^2 \varepsilon_n.$$

Therefore, we have

$$\mathbb{E}[\tau_n(\mathbf{s}_t \cdots \mathbf{s}_1)] \leq \sum_{i=1}^t \mathbb{E}[N_i] \leq |\Gamma_0|^2 \varepsilon_n t.$$

By Theorem 2.17 we deduce that there is  $C > 0$  such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \left[ \sum_{v \in X^n} l(\mathbf{g}_t | v) \right] \leq \liminf_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}[\tau_n(\mathbf{s}_t \cdots \mathbf{s}_1)] \leq C \varepsilon_n.$$

Since  $\varepsilon_n$  tends to 0 as  $n \rightarrow \infty$ , Proposition 5.4 concludes the proof.  $\square$

We now move to the general case.

*Proof of Theorem 5.9.* First note that when  $p_c(G, \mathfrak{B}) < 1$ , the upper volume growth exponent of  $G$  is  $\leq p_c$ . It follows from a general argument that any measure (not necessarily symmetric) of finite  $p$ -moment,  $p > p_c(G, \mathfrak{B})$ , has finite entropy and trivial Poisson boundary, see Corollary 2.3 in [15].

For the rest of the proof assume  $p_c(G, \mathfrak{B}) \geq 1$ . Let  $\mu$  be a measure on  $G$  of finite  $p$ -moment,  $p > p_c(G, \mathfrak{B})$ . In particular,  $\mu$  has finite first moment, and thus expectations of length of sections  $l(\mathbf{g}_t|_v)$  are finite. By Corollary 5.8, we have the capacity estimate

$$(6) \quad \epsilon_n := \max_{z_1, z_2 \in \Gamma_0} \text{Cap}_{2, \mu}(U_{z_1, n}, U_{z_2, n}) \rightarrow 0.$$

The main difference from the finite-support case is that we can no longer apply directly Theorem 2.17 to a word  $\mathbf{s}_t \cdots \mathbf{s}_1$  consisting of samples of  $\mu$  because the support of  $\mu$  is infinite.

Let us follow the proof of Theorem 2.17 and take notations set there. Let  $\mathbf{s}_t \cdots \mathbf{s}_2 \mathbf{s}_1$  be a sample of the random walk, and set  $\mathbf{g}_i = \mathbf{s}_i \cdots \mathbf{s}_2 \mathbf{s}_1$ , with  $\mathbf{g}_0 = 1$ , and  $\xi_i = \xi_0 \cdot \mathbf{g}_i$ . Let  $n$  be large enough and for  $v \in X^n$  fixed choose a geodesic path from  $I^n(\xi_{i-1} \otimes v)$  to  $I^n(\xi_i \otimes v)$  and let  $\gamma = \delta_t \cdots \delta_1$  be their concatenation. By the same arguments as in Theorem 2.17, the length of  $\mathbf{g}|_v$  is bounded by an affine function of the total number of edges crossed by  $\gamma$ . Consider a crossing of an edge  $e$  corresponding to a subcurve  $\gamma|_{[t_1, t_2]}$ , and let  $\delta_i, \delta_j$  be the subcurves containing  $\gamma(t_1)$  and  $\gamma(t_2)$ . If both  $\delta_i$  and  $\delta_j$  have length less than  $1/3$ , then we can associate to the crossing of  $e$  a traverse of the word  $\mathbf{s}_t \cdots \mathbf{s}_2 \mathbf{s}_1$ . The number of edges crossed by  $\gamma$  whose beginning or end is inside a path  $\delta_i$  of length  $\geq 1/3$  is not larger than the length of  $\delta_i$  plus 2, and hence less than a constant times the length of  $\delta_i$ . We obtain the estimate

$$\begin{aligned} \sum_{v \in X^n} \ell(\mathbf{g}_t|_v) &\leq C|X|^n + C\tau_n(\mathbf{s}_t \cdots \mathbf{s}_2 \mathbf{s}_1) \\ &+ C \sum_{i=1}^t \sum_{v \in X^n, d(I^n(\xi_i \otimes v), I^n(\xi_{i-1} \otimes v)) \geq 1/3} d(I^n(\xi_i \otimes v), I^n(\xi_{i-1} \otimes v)), \end{aligned}$$

for some  $C > 0$ . By (6) the expected total number of traverses (summed over all  $v \in X^n$ ), that is,  $\mathbb{E}[\tau_n(\mathbf{s}_t \cdots \mathbf{s}_2 \mathbf{s}_1)]$ , is bounded by  $\epsilon_n \cdot t$  for a sequence  $\epsilon_n$  converging to 0, by the same argument as for finitely supported measures. Next note that each term  $d(I^n(\xi_i \otimes v), I^n(\xi_{i-1} \otimes v))$  in the second sum is bounded above by  $3^{p-1} d(I^n(\xi_i \otimes v), I^n(\xi_{i-1} \otimes v))^p$ . By Proposition 4.4 we have that

$$\begin{aligned} \sum_{v \in X^*, d(I^{|v|}(\xi_i \otimes v), I^{|v|}(\xi_{i-1} \otimes v)) \geq 1/3} d(I^n(\xi_i \otimes v), I^n(\xi_{i-1} \otimes v))^p &\leq \\ C_1 d(\xi_i, \xi_{i-1})^p &\leq (C_2 l(\mathbf{s}_i) + C_2)^p. \end{aligned}$$

The last term has finite expectation, since  $\mu$  has finite  $p$ -moment. Thus the series

$$\sum_{n=0}^{\infty} \mathbb{E} \left( \sum_{v \in X^n, d(I^n(\xi_i \otimes v), I^n(\xi_{i-1} \otimes v)) \geq 1/3} d(I^n(\xi_i \otimes v), I^n(\xi_{i-1} \otimes v))^p \right)$$

is convergent. Therefore its terms converge to 0 as  $n$  tends to  $\infty$ . This finishes the proof by Proposition 5.4.  $\square$

Theorem 1.3 stated in the Introduction follows from Theorem 4.7 and 5.9.

## 6. EXAMPLES

**6.1. Automata of polynomial activity growth.** Let  $G$  be a faithful self-similar group acting on  $X^*$ . Consider, for  $g \in G$  the function

$$\alpha_g(n) = |\{v \in X^n : g|_v \neq 1\}|$$

counting the number of non-trivial sections on the  $n$ th level. Since the total set of sections  $\{g|_v : v \in X^*\}$  is finite,  $\alpha_g(n)$  is the number of paths in a finite directed graph (the *Moore diagram* with the trivial state removed) starting in a vertex. Therefore, the function  $\alpha_g(n)$  is either exponentially growing, or is bounded by a polynomial. It is not hard to show that the subset of elements  $g \in G$  for which  $\alpha_g(n)$  is bounded from above by a polynomial of degree  $d$  is a subgroup of  $G$ . We say that  $G$  is *generated by an automaton of polynomial activity growth* if  $\alpha_g(n)$  is bounded by a polynomial for every  $g \in G$ .

Groups generated by automata of polynomial activity growth were introduced by S. Sidki in [44] (in greater generality than we described here). In particular, he showed that such groups can not have free subgroups.

A special sub-class of such groups are groups generated by *bounded automata*, i.e., consisting of elements  $g$  such that  $\alpha_g(n)$  is bounded. It includes iterated monodromy groups of all post-critically finite polynomials (see [8, 35]).

L. Bartholdi and B. Virág proved amenability of  $\text{IMG}(z^2 - 1)$  in [9] using random walks. Their methods were extended in [7] to prove that all groups generated by bounded automata are amenable. Later, this result was extended to groups for which  $\alpha_g(n)$  is bounded by polynomials of degree 1 (in [2]) and 2 (in [3]).

The following theorem is proved in [39].

**Theorem 6.1.** *If  $(G, \mathfrak{B})$  is a contracting group such that its action on  $X^*$  for some basis  $X \subset \mathfrak{B}$  is a subgroup of the group of automata of polynomial activity growth, then  $\text{ARdim } \mathcal{J}_G \leq 1$ .*

**Corollary 6.2.** *If  $(G, \mathfrak{B})$  satisfies the conditions of the above theorem, and if  $\mu$  is a symmetric probability measure on  $G$  with a finite  $p$ -moment for some  $p > 1$ , then the  $\mu$ -random walk is Liouville. In particular  $G$  is amenable.*

Note that groups generated by bounded automata are always contracting, but this is no longer true for groups generated by automata of polynomial activity of degree at least 1. For example, the group generated by the wreath recursion  $a = \sigma(1, a)$  and  $b = (b, a)$  is generated by automata of linear activity growth and is not contracting. By [4], there are automata groups of polynomial activity that are not Liouville for any finitely supported symmetric non-degenerate measure, so that the contracting assumption is crucial in Corollary 6.2.

It follows from the results of [25] that a sufficient condition for a group generated by automata of polynomial activity growth to be amenable is recurrence of the orbital graphs of its action on the boundary of the tree  $X^*$  (see the proof of [25, Theorem 5.1]). This gives another way to conclude amenability of groups satisfying the conditions of Theorem 6.1, see [39]. The statement on the Liouville property in Corollary 6.2 is new. For bounded automata groups, it extends the result of [1] to measures with a moment condition (in that particular case the proof of Theorem 5.9 can actually be modified to include measures with finite first moment, by observing that Proposition 4.6 holds for  $p = 1$ ).

**6.2. Iterated monodromy groups of rational functions.**

**Theorem 6.3.** *Let  $f \in \mathbb{C}(z)$  be a post-critically finite rational function. Then the canonical homeomorphism of the limit space of  $\text{IMG}(f)$  with the Julia set of  $f$  is a quasi-symmetry with respect to the visual metric on the limit space and the spherical metric on the Julia set.*

*Proof.* This is a result of P. Haïssinsky and K. Pilgrim. Namely, it is proved in [20, Theorem 6.15] that the limit dynamical system  $s : \mathcal{J}_G \rightarrow \mathcal{J}_G$  of every self-replicating contracting group is *coarse expanding conformal* with respect to a visual metric (see the definition in [19]).

It is also shown in [20, Theorem 3.3] that for every post-critically finite rational function  $f$  the spherical metric restricted to a neighborhood of the Julia set of  $f$  is also coarse expanding conformal. By [19, Theorem 2.8.2] any topological conjugacy between coarse expanding conformal dynamical systems is a quasi-symmetry.  $\square$

**Corollary 6.4.** *Let  $f \in \mathbb{C}(z)$  be a post-critically finite rational function such that its Julia set is not the whole sphere (equivalently, if  $f$  has a cycle containing a critical point). Then  $\text{IMG}(f)$  is amenable. Moreover, if  $\mu$  is a symmetric measure on  $\text{IMG}(f)$  with finite second moment, then the random walk generated by  $\mu$  is Liouville.*

*Proof.* By Theorems 6.3, 5.9, and 4.7, it is enough to show that the Julia set of  $f$  has Hausdorff dimension strictly less than 2. But this is shown in [32, Corollary 6.2].  $\square$

Let us consider some examples of iterated monodromy groups that are not generated by automata of polynomial activity growth.

Consider  $f(z) = z^2 - \frac{1}{16z^2}$ . This function is representative of the well-studied family of maps of the form  $z^n + \frac{\lambda}{z^m}$ , see the survey [10]. See the Julia set of this function on Figure 1.

Its critical points are  $0, \infty$ , and roots of  $z^4 - 1/16$ . The latter critical points are mapped to  $\pm \frac{i}{2}$ , which are both mapped to  $0$ . Consequently, the post-critical set is  $\{0, \infty, i/2, -i/2\}$ . Let us take the generators  $a, b, c$  given by loops around the points  $0, i/2$ , and  $-i/2$ , respectively, and connected to a basepoint, which we choose to be equal to  $1$ . The preimages of the basepoint  $1$  are  $\pm\sqrt{2} \pm \sqrt{5}/2$  (where the choices of the sign are independent), which are approximately equal to  $\pm 0.24i$  and  $\pm 1.03$ . Let us connect the basepoint to the preimages  $\approx \pm 0.24i$  and  $\approx 1.03$  by straight paths. We can not do it for  $-1.03$ , since such a path will contain  $0$ . Let us connect it to the basepoint by a path passing below  $0$ . We denote the connecting paths from  $1$  to  $\approx 0.24i, \approx -1.03, \approx -0.24i, \approx 1.03$  by  $\ell_1, \ell_2, \ell_3, \ell_4$ , respectively. This will be our chosen basis of the biset associated with the rational function (see 2.4). We also denote  $\ell_1, \ell_2, \ell_3, \ell_4$  just by  $1, 2, 3, 4$  when we write the corresponding wreath recursion.

The generators  $a, b, c$ , and their preimages  $f^{-1}(a), f^{-1}(b), f^{-1}(c)$  together with the connecting paths  $\ell_i$  are shown on Figure 2.

It follows that the iterated monodromy group is generated by the wreath recursion

$$\begin{aligned} a &= (b, 1, c, 1), \\ b &= (23)(14), \\ c &= (12)(34)(a, a^{-1}, 1, 1). \end{aligned}$$

Note that all generators are of order 2. (In particular, we may replace  $a^{-1}$  by  $a$  in the wreath recursion.)

The wreath recursion implies the recurrent rules for the graph of actions on the levels  $X^n$  of the tree, shown on Figure 3.

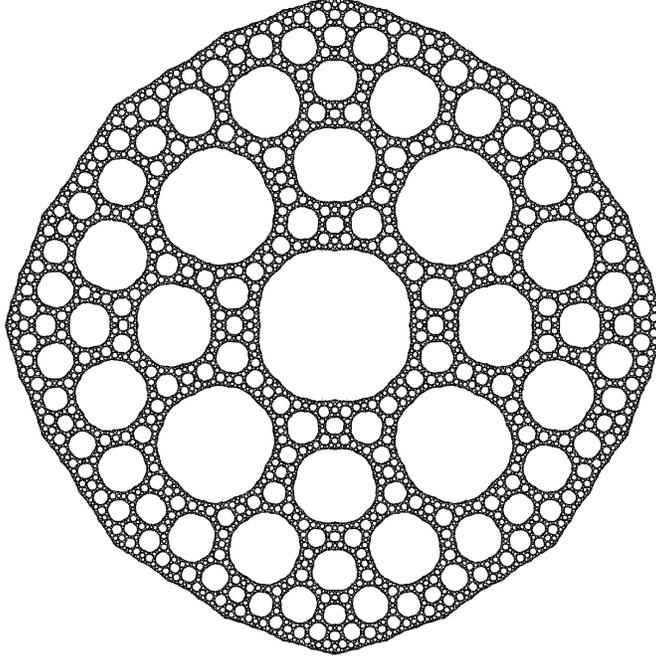


FIGURE 1. The Julia set of  $z^2 - \frac{1}{16z^2}$

The same substitution rules from Figure 3 are also valid for the infinite orbital Schreier graphs of the action on the boundary  $X^\omega$  of the tree  $X^*$ . If we assume that the edges corresponding to  $a$  have length 0, and the edges corresponding to  $b$  and  $c$  have length 1, then we get graphs that are (uniformly) quasi-isometric to the Schreier graphs (since the group  $\langle a \rangle$  is finite). It follows then from the recurrent rules that the shift map  $xv \mapsto v$  contracts the distances twice. Since this map is four-to-one, it follows that the orbital Schreier graphs have quadratic growth.

See Figure 4 where the graph of the action on the 5th level  $X^5$  is shown in a way approximating the Julia set of  $z^2 - \frac{1}{16}z^2$ .

**6.3. Sierpiński carpet group.** Let  $X = \{1, \dots, 8\}$  be an alphabet. Consider the group  $G$  acting on  $X^*$  generated by the set  $S = \{a, b, c, d\}$ , given by the wreath recursion

$$a = (12)(67)(1, 1, a, 1, a, 1, 1, a)$$

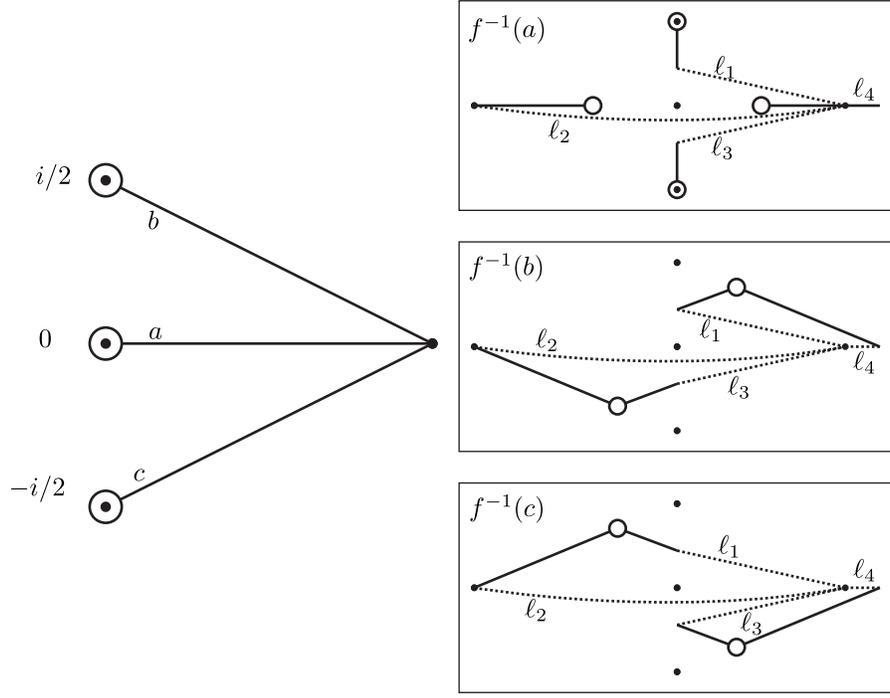
$$b = (46)(58)(b, b, b, 1, 1, 1, 1, 1)$$

$$c = (23)(78)(c, 1, 1, c, 1, c, 1, 1)$$

$$d = (14)(35)(1, 1, 1, 1, 1, d, d, d)$$

Its dual Moore diagram is shown in Figure 5.

The limit space  $\mathcal{J}_G$  can be identified (up to quasimetry) with the classical Sierpiński carpet, obtained by successively subdividing a euclidean square into nine equal squares and removing the middle one. The limit dynamical system  $s: \mathcal{J}_G \rightarrow \mathcal{J}_G$  can be described by


 FIGURE 2. Generators of  $\text{IMG}(z^2 - 1/16z^2)$ 

first dilating the carpet by a factor nine, and then folding it in the natural way, so that each of the eight tiles of the dilated carpet is mapped isometrically onto the original carpet (preserving the orientation for the four corner tiles, and reversing it for the four remaining ones).

The products  $ab, bc, cd$ , and  $ad$  are of order 6. For example,

$$ab = (12)(476)(58)(b, b, ba, 1, a, 1, 1, a),$$

so

$$(ab)^2 = (476)(1, 1, (ba)^2, 1, 1, 1, 1, 1),$$

and

$$(ab)^6 = (1, 1, (ba)^6, 1, 1, 1, 1, 1),$$

which implies that  $(ab)^6 = 1$ .

We have

$$ac = (123)(678)(c, a, 1, c, a, c, a, 1),$$

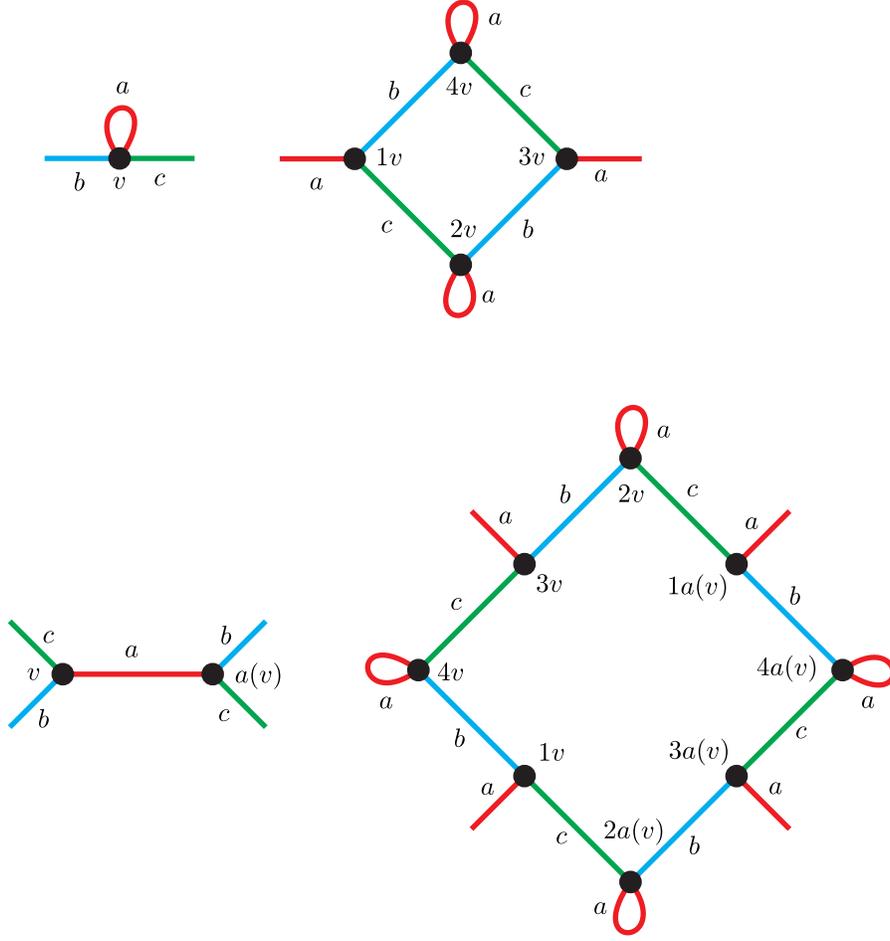
hence  $ac$  has infinite order and

$$(ac)^6 = ((ac)^2, (ca)^2, (ac)^2, 1, 1, (ac)^2, (ca)^2, (ac)^2).$$

It follows that the Thurston's  $p$ -map  $T_p$  satisfies

$$T_p([ac]) = 2 \cdot 3^{1-p}[ac].$$

It is contracting only when  $1 - p < \frac{\log 2}{\log 3}$ , i.e., for  $p > \frac{\log 6}{\log 3}$ . Consequently,  $p_c(G, \mathfrak{B}) \geq \frac{\log 6}{\log 3}$ , by Theorem 4.11. In particular, Theorem 4.7 provides the same lower bound for  $\text{ARdim}(G, \mathfrak{B})$ .

FIGURE 3. Recursion rules for Schreier graphs of  $\text{IMG}(z^2 - 1/16z^2)$ 

On the other hand, the usual metric on Sierpiński carpet implies that  $\text{ARdim}(G, \mathfrak{B}) \leq \frac{\log 8}{\log 3}$ . The exact value of  $\text{ARdim}(G, \mathfrak{B})$  is still unknown, see some estimates in [29].

**6.4. A non-Liouville contracting group.** Not all contracting self-similar groups are Liouville even for finitely supported generating measures. Let us describe one counterexample.

Consider the group  $H$  of affine transformations of the space  $\mathbb{Z}_2^3$  (where  $\mathbb{Z}_2$  is the ring of dyadic integers) of the form  $\vec{x} \mapsto (-1)^k \vec{x} + \vec{v}$ , where  $\vec{v} \in \mathbb{Z}_2^3$  and  $k = 0, 1$ .

We will denote the points of  $\mathbb{Z}_2^3$  as infinite matrices  $\begin{pmatrix} i_{11} & i_{12} & \dots \\ i_{21} & i_{22} & \dots \\ i_{31} & i_{32} & \dots \end{pmatrix}$  in such a way

that  $i_t \in \{0, 1\}$ , and if  $\begin{pmatrix} i_{11} & i_{12} & \dots \\ i_{21} & i_{22} & \dots \\ i_{31} & i_{32} & \dots \end{pmatrix}$  represents  $\vec{x} \in \mathbb{Z}_2^3$ , and the shifted matrix

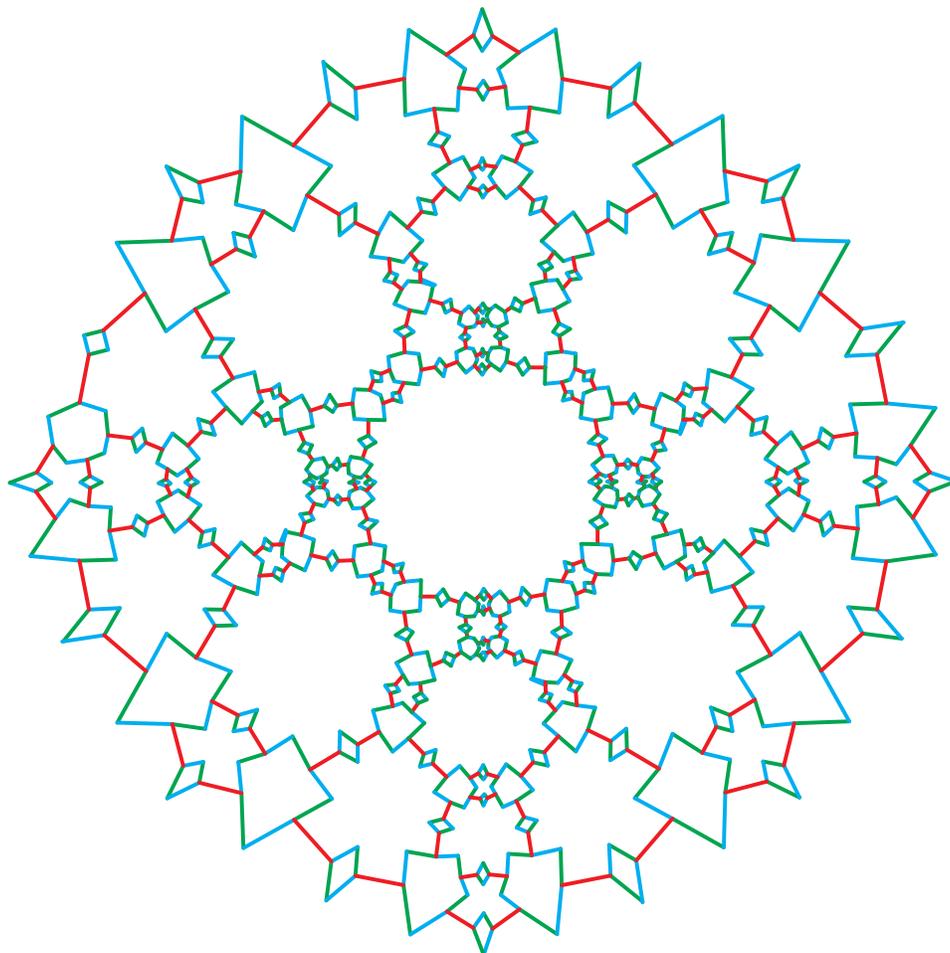


FIGURE 4. The 5th level Schreier graph of  $\text{IMG}(z^2 - 1/16z^2)$

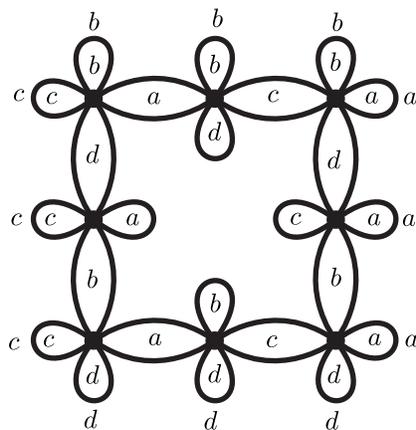


FIGURE 5. The Sierpiński carpet dual automaton

$\begin{pmatrix} i_{12} & i_{13} & \dots \\ i_{22} & i_{23} & \dots \\ i_{32} & i_{33} & \dots \end{pmatrix}$  represents  $\vec{y}$ , then

$$\vec{x} = - \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} + 2\vec{y}.$$

Essentially, we are using the binary numeration system with digits 0 and  $-1$ , but write 1 instead of  $-1$ .

Denote, for  $(i_1, i_2, i_3) \in \{0, 1\}^3$ , by  $a_{i_1 i_2 i_3}$  the transformation  $\vec{x} \mapsto -\vec{x} + \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix}$ . The elements  $a_{i_1 i_2 i_3}$  generate the group  $H$ .

We have

$$a_{i_1 i_2 i_3} \left( - \begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix} + 2\vec{y} \right) = \begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix} - 2\vec{y} + \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = - \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} + 2 \left( -\vec{y} + \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \right)$$

if  $i_t + j_t = 2l_t - k_t$ . Equivalently,  $l_t = \max(i_t, j_t)$  and  $k_t$  is  $i_t + l_t$  modulo 2.

To shorten notation, let us denote the columns  $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$  by 0, 1, 2, 3, 4, 5, 6, 7, respectively. Similarly, we will denote the generators  $a_{i_1 i_2 i_3}$  by  $a_0, a_1, \dots, a_7$ . We have then

$$\begin{aligned} a_0 &= (a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7), \\ a_1 &= (01)(23)(45)(67)(a_1, a_1, a_3, a_3, a_5, a_5, a_7, a_7), \\ a_2 &= (02)(13)(46)(57)(a_2, a_3, a_2, a_3, a_6, a_7, a_6, a_7), \\ a_3 &= (03)(12)(47)(56)(a_3, a_3, a_3, a_3, a_7, a_7, a_7, a_7), \\ a_4 &= (04)(15)(26)(37)(a_4, a_5, a_6, a_7, a_4, a_5, a_6, a_7), \\ a_5 &= (05)(14)(27)(36)(a_5, a_5, a_7, a_7, a_5, a_5, a_7, a_7), \\ a_6 &= (06)(17)(24)(35)(a_6, a_7, a_6, a_7, a_6, a_7, a_6, a_7), \\ a_7 &= (07)(16)(25)(34)(a_7, a_7, a_7, a_7, a_7, a_7, a_7, a_7). \end{aligned}$$

Note that  $a_0$  is not a section of any of the other generators  $a_i$ . Let us “fragment”  $a_0$  into a product of two commuting elements  $b$  and  $c$  given by

$$b = (b, a_1, 1, 1, 1, 1, 1, 1), \quad c = (c, 1, a_2, a_3, a_4, a_5, a_6, a_7).$$

Note that  $b, c \notin G$  and  $bc = a_0$ .

Let  $G$  be the group generated by  $H$  and  $b, c$ .

**Proposition 6.5.** *Let  $\mu$  be the uniform probability measure on the generating set  $\{a_i\}_{i=1, \dots, 7} \cup \{b, c\}$  of  $G$ . Then the  $\mu$ -random walk is not Liouville.*

*Proof.* It follows directly from the recursion that if not all letters of  $v \in X^*$  are 0 and the first non-zero letter of  $v$  is 1, then  $b(v) = a_0(v)$  and  $b|_v = a_0|_v$ . If not all letters of  $v$  are 0 and the first non-zero letter is not 1, then  $b(v) = v$  and  $b|_v = 1$ . Similar statement (but other way around) is true for  $c$ .

Define the map  $\chi : H \rightarrow \{0, 1\}$  by the condition that  $\chi(g) = 0$  if  $g^{-1} \underbrace{00\dots 0}_n \in G$

for all  $n$  large enough, and  $\chi(g) = 1$  otherwise.

Then, according to the first paragraph of the proof, if  $g \in H$  and  $s \in \{a_i\}_{i=1,\dots,7} \cup \{b, c\}$  is a generator, then  $\chi(g) \neq \chi(gs)$  only if  $g(00\dots) = 00\dots$  and  $s \in \{b, c\}$ .

Consider the  $\mu$ -random walk on  $G$ . The induced random walk on the  $G$ -orbit of  $000\dots$  coincides with a random walk on generated by a finitely supported measure on the generating set  $\{a_i\}_{i=0,1,\dots,7} \cup \{1\}$  of  $H$ . Since the Schreier graph is quasi-isometric to  $\mathbb{Z}^3$ , and simple random walk on  $\mathbb{Z}^3$  is transient by Pólya's theorem, it follows from the comparison theorem (see e.g., [30, Theorem 2.17]) that this random walk is transient. Consequently, it will visit  $000\dots$  finitely many times almost surely, and so the value of  $\chi$  along a trajectory of the walk will be eventually constant. The eventual value of  $\chi$  will be a non-trivial event invariant under the tail equivalence relation, hence the Poisson boundary of the random walk is non-trivial.  $\square$

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