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# Moment inequalities for sums of weakly dependent random fields

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We derive both Azuma-Hoeffding and Burkholder-type inequalities for partial sums over a rectangular grid of dimension  $d$  of a random field satisfying a weak dependency assumption of projective type: the difference between the expectation of an element of the random field and its conditional expectation given the rest of the field at a distance more than  $\delta$  is bounded, in  $L^p$  distance, by a known decreasing function of  $\delta$ . The analysis is based on the combination of a multi-scale approximation of random sums by martingale difference sequences, and of a careful decomposition of the domain. The obtained results extend previously known bounds under comparable hypotheses, and do not use the assumption of commuting filtrations.

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*Keywords:* Concentration inequalities; Burkholder-type inequalities; multidimensional martingales; multiparameter processes; weakly dependent processes

## 1. Introduction

Let  $(X_t)_{t \in \mathbb{Z}^d}$  be a real-valued integrable random field over the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $\mathcal{R}$  be a hyperrectangular domain of  $\mathbb{Z}^d$  and define  $S_{\mathcal{R}} := \sum_{t \in \mathcal{R}} (X_t - \mathbb{E}[X_t])$  the deviation of the process sum over  $\mathcal{R}$  from its mean. In this work we are interested in upper bounding the moments of  $S_{\mathcal{R}}$ , under a weak dependency assumption of projective type, expressed as follows, for some  $p \in [2, \infty]$ :

**Assumption 1 (WD( $p$ )).** For  $t = (t_1, \dots, t_d) \in \mathbb{Z}^d$  and  $k \in \mathbb{N}_{>0}$ , define the  $\sigma$ -algebra

$$\mathcal{M}_{t,k} := \mathfrak{G}\{X_u : u = (u_1, \dots, u_d) \in \mathbb{Z}^d, \sup_i (t_i - u_i) \geq k\},$$

it holds

$$\forall t \in \mathbb{Z}^d, \forall r > 0 : \|\mathbb{E}[X_t | \mathcal{M}_{t,r}] - \mathbb{E}[X_t]\|_p \leq M_p \varphi_p(r), \quad (1)$$

where  $\varphi_p(\cdot)$  is a non-increasing function such that  $\varphi_p(0) = 1$ , and  $M_p$  is a real constant.

The sigma-field  $\mathcal{M}_{t,k}$  corresponds to the sigma-field  $\mathbf{F}_{t-k}$  introduced by [Basu and Dorea \(1979\)](#) for martingale random fields.

**Remark 1.** The LHS of Equation (1) can be also seen as a type of generalized "variational" dependence measure between  $X_t$  and  $\mathcal{M}_{t,k}$ , and can be related to families of recently introduced covariance-based

dependence coefficients (see [Dedecker and Prieur, 2005](#), [Maume-Deschamps, 2006](#)). Namely, let  $Y$  be a real random variable over  $(\Omega, \mathcal{F}, \mathbb{P})$  and let  $\mathcal{A} \subset \mathcal{F}$  be some  $\sigma$ -field. Then for  $p \geq 1$  it holds:

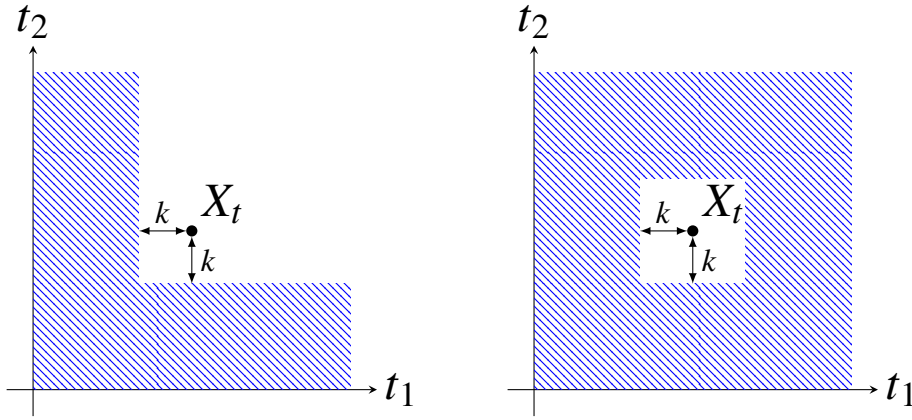
$$\|\mathbb{E}[Y|\mathcal{A}] - \mathbb{E}[Y]\|_p = \sup\left\{\text{Cov}[X, Y] : \|X\|_q = 1, X \text{ is } \mathcal{A}\text{-measurable}\right\},$$

where  $p^{-1} + q^{-1} = 1$ . To see this, notice that  $\text{Cov}[X, Y] = \mathbb{E}[XU]$  where  $U := \mathbb{E}[Y|\mathcal{A}] - \mathbb{E}[Y]$  and apply Hölder's inequality with extremal equality characterization (see also Lemma 1.1.2 of [Maume-Deschamps, 2006](#)).

**Remark 2.** A sufficient condition for assumption **WD**( $p$ ) is to have (1) but with  $\mathcal{M}_{t,k}$  replaced by

$$\mathcal{M}'_{t,k} := \mathfrak{S}\{X_u : u \in \mathbb{Z}^d, \|u - t\|_\infty \geq k\}, \quad (2)$$

where  $\|\cdot\|_\infty$  is the standard  $\ell_\infty$  norm on  $\mathbb{R}^d$ . The  $\sigma$ -algebra  $\mathcal{M}'_{t,k}$  may actually be more intuitive or natural (see Figure 1 for an illustration), except in dimension  $d = 1$  where  $\mathcal{M}_{t,k}$  coincides with the “past at distance  $k$ ” which is a natural notion for sequential processes.



**Figure 1.** Illustration of the  $\sigma$ -algebras  $\mathcal{M}_{t,k}$  (left picture) and  $\mathcal{M}'_{t,k}$  (right picture). The blue hashed surface represents the entries generating the  $\sigma$ -algebra.

We will say that the random field is polynomially weakly dependent if Equation (1) holds with  $\varphi_p(r) = \min(1, cr^{-\alpha})$ , and that it is exponentially weakly dependent if Equation (1) holds with  $\varphi_p(r) = \min(1, c \exp(-\gamma r))$ , where  $c \geq 1$ ,  $\alpha > 0$  (resp.  $\gamma > 0$ ) are some constants. In general, we are interested in the situation where  $\varphi_p(k) \rightarrow 0$  as  $k \rightarrow \infty$ , but where this decrease could be slow (a form of “long-range dependency”).

Under **(WD**( $p$ )), for  $p \in [2, \infty)$  we obtain a non-asymptotic upper bound for  $S_{\mathcal{R}}$  in  $L^p$ -norm (Burkholder-type inequality), and for  $p = \infty$  we obtain that  $S_{\mathcal{R}}$  is sub-Gaussian with a bound on its sub-Gaussian norm (Azuma-Hoeffding type inequality).

We use the terminology of *weakly dependent* random fields obeying Assumption 1 by following the notions defined in the seminal works of [Doukhan \(1994\)](#) and [Bickel and Bühlmann \(1999\)](#) and in the monograph of [Dedecker et al. \(2007\)](#) which in a certain sense extend the concept of mixing to multi-parameter processes.

The condition (1) can be seen as a generalization in dimension  $d$  of similar assumptions made in dimension 1 in previous literature, and allowing the approximation of a sequence of random variables by a martingale difference, also related to the notion of mixingale (see [McLeish, 1976](#)). Various results based on assumptions of a related nature are available in the literature, in particular for stationary sequences ([Peligrad and Utev, 2005](#), [Peligrad, Utev and Wu, 2007](#)) in dimension 1, for Banach-valued processes also in dimension 1 ([Dedecker and Merlevede, 2015](#)), and for random processes in general dimension ([Dedecker, 1998, 2001](#)).

Specialized in dimension  $d = 1$ , the condition (1) recovers similar conditions involving the  $L_p$  norm of the projection of the past of random process, considered by [Peligrad and Utev \(2005\)](#) and [Peligrad, Utev and Wu \(2007\)](#). In the work of [Dedecker \(1998, 2001\)](#), in general dimension  $d$  the  $\sigma$ -algebra  $\mathcal{M}_{t,r}$  is intersected with the past  $\sigma$ -algebra of  $t$  in the sense of lexicographical ordering; this makes the condition of [Dedecker \(1998, 2001\)](#) slightly weaker than  $\mathbf{WD}(p)$ ; however the lexicographical order seems to be chosen as an arbitrary total order over  $\mathbb{Z}^d$ . Furthermore, the condition  $\mathbf{WD}(p)$  implies bounds that improve over those of [Dedecker \(2001\)](#) in the long-range dependency regime, as will be discussed later.

In fact, the absence of a natural total order on  $\mathbb{Z}^d$  for  $d > 1$  is precisely what makes the generalization to higher dimension challenging and constitutes the interest of our contribution; in a nutshell, we are able to generalize the dyadic martingale decomposition used by [Peligrad and Utev \(2005\)](#) and [Peligrad, Utev and Wu \(2007\)](#) to multi-scale decomposition in higher dimension by using a specific total order on  $\mathbb{Z}^d$ , and additional arguments. For a comparison to the theory of strong multidimensional martingales, see Section 3.

Since under our definition of polynomially/exponentially weakly-dependent processes it holds  $\varphi_p(0) = 1$ , and since for any fixed  $t$  the sequence  $\|\mathbb{E}[X_t | \mathcal{M}_{t,k}] - \mathbb{E}[X_t]\|_p$  is nonincreasing in  $k \geq 0$  (by Jensen's inequality and the fact that the  $\sigma$ -algebras  $\mathcal{M}_{t,k}$  are nested for fixed  $t$ ), it is natural to think of  $M_p$  as equal to  $\sup_{t \in \mathbb{Z}^d} \|X_t - \mathbb{E}[X_t]\|_p$ , though we do not explicitly require it. Although we do not assume stationarity of the field, the fact that  $M_p$  is independent of  $t$  indicates that our results are more geared towards fields that are in some sense close to being stationary. However, motivated by applications in statistics, it is usual that instead of a full random field over  $\mathbb{Z}^d$ , one wants to consider only a process  $(X_t)_{t \in \mathcal{D}}$  defined over a subdomain  $\mathcal{D}$ , typically a finite hyperrectangular window. Suppose that a version of Assumption  $\mathbf{WD}(p)$  is satisfied for this process but replacing  $\mathbb{Z}^d$  by  $\mathcal{D}$ . Then if we extend  $X_t$  to  $\mathbb{Z}^d$  by "padding" it with random variables that are constant (say equal to zero) on  $\mathbb{Z}^d \setminus \mathcal{D}$ , we obtain a random field satisfying Assumption  $\mathbf{WD}(p)$  and can apply our results (observe that the extended field is not stationary).

The paper is organized as follows. In Section 2 we prove the main results. In Section 3 we discuss the notion of multidimensional martingales and compare the results to the known bounds under different dependence measures. The multidimensional hierarchical martingale construction and supplementary technical lemmata are stated and proved in Section 4.

## 2. Main results

For any index set  $\mathbb{T}$  and  $\sigma$ -fields  $(\mathcal{F}_\alpha)_{\alpha \in \mathbb{T}}$  we write  $\vee_{\alpha \in \mathbb{T}} \mathcal{F}_\alpha$  to denote the smallest  $\sigma$ -field which contains all  $\mathcal{F}_\alpha$  for  $\alpha \in \mathbb{T}$  and  $\wedge_{\alpha \in \mathbb{T}} \mathcal{F}_\alpha$  for the  $\sigma$ -field which is the intersection of all  $\sigma$ -fields  $\mathcal{F}_\alpha$ .

For a finite subset  $\mathcal{D} \subset \mathbb{Z}^d$ , we denote  $S_{\mathcal{D}} := \sum_{t \in \mathcal{D}} (X_t - \mathbb{E}[X_t])$  for the centered partial sum of  $(X_t)_{t \in \mathcal{D}}$ . We denote  $a \wedge b$ ,  $a \vee b$  the minimum and maximum of two real numbers  $a, b$ . We use the standard notation  $\lceil a \rceil$ ,  $[a]$ ,  $\lfloor a \rfloor$  for the ceil, integer and floor part of a real number  $a$ . For any  $k \in \mathbb{N}_{>0}$  we denote  $\llbracket k \rrbracket := \{1, \dots, k\}$ . We denote the cardinality of a finite set  $A$  as  $|A|$ . For a centered real random

variable  $Z$  we denote  $\|Z\|_{\text{SG}} := \inf \{c \geq 0 : \mathbb{E}[\exp(\lambda Z)] \leq \exp(\frac{\lambda^2 c^2}{2}), \text{ for all } \lambda \in \mathbb{R}\}$  its subgaussian norm<sup>1</sup>.

The following theorem constitutes our main contribution.

**Theorem 1.** *Let  $\mathcal{R} = \prod_{i=1}^d \llbracket N_i \rrbracket$  be a  $d$ -dimensional rectangle of sidelengths  $N_i \geq 1$ ,  $i \in \llbracket d \rrbracket$ , and  $m(\mathcal{R}) := \max_{i=1, \dots, d} \lfloor \log_2 N_i \rfloor$ . Let  $\delta = (\delta_k)_{k \geq 1}$  be a fixed nondecreasing sequence of nonnegative integers such that*

$$\sum_{k=1}^{m(\mathcal{R})} \delta_k 2^{-k} \leq \frac{1}{4d^2}. \quad (3)$$

Let  $p \in [2, \infty]$  be fixed, and let  $(X_t)_{t \in \mathbb{Z}^d}$  be a random field such that Assumption **WD**( $p$ ) is satisfied.

- if  $p \in [2, \infty)$ , then

$$\|S_{\mathcal{R}}\|_p \leq 4\sqrt{p}\Psi_p(\delta, \mathcal{R}),$$

- if  $p = \infty$ , then

$$\|S_{\mathcal{R}}\|_{\text{SG}} \leq 10\Psi_{\infty}(\delta, \mathcal{R}),$$

where (putting  $\delta_0 = 0$ )

$$\Psi_p(\delta, \mathcal{R}) := 2M_p \sqrt{|\mathcal{R}|} \left( 2 + \sum_{k=1}^{m(\mathcal{R})+1} \varphi_p(\delta_{k-1} + 1) \sqrt{|C_k \cap \mathcal{R}|} \right), \quad (4)$$

with  $C_k := \llbracket 2^k \rrbracket^d$ .

Before proving the theorem, we give in the next corollary the rates obtained via the above bound for the partial sums of a weakly-dependent random field over cubes in  $\mathbb{Z}^d$  under the assumption that the random field  $(X_t)_{t \in \mathbb{Z}^d}$  is either polynomially or exponentially weakly-dependent. In a nutshell, these are obtained by choosing the sequence  $\delta = (\delta_k)_{k \geq 1}$  such that the constraint (3) is fulfilled and the value of the function  $\Psi_p(\delta, \mathcal{R})$  is close to its minimum.

**Corollary 2.** *Let  $\mathcal{D}_n = \llbracket n \rrbracket^d$  be a  $d$ -dimensional cube with side-length  $n \geq 2$ , and put  $N = |\mathcal{D}_n| = n^d$ . Consider a random field  $(X_t)_{t \in \mathbb{Z}^d}$  which satisfies the weak dependency Assumption **WD**( $p$ ) for a given  $p \in [2, +\infty]$  and the function  $\varphi_p(\cdot)$ . Then*

- If  $p \in [2, +\infty)$  and  $\varphi_p(r) \leq cr^{-\alpha}$ ,  $\alpha > 0$ ,  $c \geq 1$ ,

$$\left\| N^{-1} S_{\mathcal{D}_n} \right\|_p \leq c C_{d, \alpha} \sqrt{p} M_p N^{-\left(\frac{1}{2} \wedge \frac{\alpha}{d}\right)}, \quad \text{if } \alpha \neq d/2; \quad (5)$$

$$\left\| N^{-1} S_{\mathcal{D}_n} \right\|_p \leq c \tilde{C}_d \sqrt{p} M_p N^{-\frac{1}{2}} (\log_2 N)^{\frac{d}{2}+1}, \quad \text{if } \alpha = d/2, \quad (6)$$

where  $C_{d, \alpha} := 24 \cdot 2^{d+\alpha} d^{2\alpha} \left| 1 - 2^{\frac{d-\alpha}{2+\alpha}} \right|^{-(1+\alpha)}$ , and  $\tilde{C}_d := 15 \cdot (8d)^{d/2}$ .

<sup>1</sup>It is well-known that the subgaussian norm is equivalent to the Orlicz  $\psi_2$ -norm for centered variables. We use the subgaussian norm here since it is the one appearing naturally in the Hoeffding-Azuma inequality.

• If  $p = \infty$  and  $\varphi_\infty(r) \leq ct^{-\alpha}$ , then

$$\left\| N^{-1} S_{\mathcal{D}_n} \right\|_{\text{SG}} \leq 2.5cC_{d,\alpha} M_\infty N^{-\left(\frac{1}{2} \wedge \frac{\alpha}{d}\right)}, \quad \text{if } \alpha \neq d/2; \quad (7)$$

$$\left\| N^{-1} S_{\mathcal{D}_n} \right\|_{\text{SG}} \leq 2.5c\tilde{C}_d M_\infty N^{-\frac{1}{2}} (\log_2 N)^{\frac{d}{2}+1}, \quad \text{if } \alpha = d/2. \quad (8)$$

Furthermore, if  $\varphi_p(t) \leq c \exp(-\gamma t^\eta)$ , then it holds  $\|N^{-1} S_{\mathcal{D}}\|_p \leq cM_p C_{\gamma,\eta,d} N^{-\frac{1}{2}}$  ( $\|N^{-1} S_{\mathcal{D}}\|_{\text{SG}} \leq cM_\infty C_{\gamma,\eta,d} N^{-\frac{1}{2}}$  when  $p = \infty$  correspondingly) for some factor  $C_{\gamma,\eta,d} > 0$  only depending on  $\gamma, \eta, d$ .

**Proof** Let  $\varphi_p(t) \leq ct^{-\alpha}$ . Applying Theorem 1 to the cube  $\mathcal{D} = \llbracket n \rrbracket^d$ , putting  $m := \lfloor \log_2 n \rfloor$  we obtain that  $\|S_{\mathcal{D}}\|_p$  resp.  $\|S_{\mathcal{D}}\|_{\text{SG}}$  is upper bounded proportionally to

$$\Psi_p(\delta, \mathcal{D}) \leq 2cM_p \sqrt{N} \left( 2 + \sum_{k=1}^{m+1} (\delta_{k-1} + 1)^{-\alpha} \sqrt{2^{kd}} \right),$$

provided  $\delta$  satisfies the constraint (3).

Using Lagrange multiplier method to minimize  $\Psi_p(\delta, \mathcal{D})$  under the constraint over  $\delta$  yields

$$\tilde{\delta}_k = \frac{1}{4d^2} Z_{\alpha,m}^{-1} (2\rho)^k \quad \text{with } Z_{\alpha,m} := \sum_{k=1}^m \rho^k, \quad \rho := 2^{\frac{d-\alpha}{1+\alpha}}.$$

It is easy to check that  $(2\rho) \geq 1$  so that  $(\tilde{\delta}_k)$  is a nondecreasing sequence. Take  $\delta_k := \lfloor \tilde{\delta}_k \rfloor$ , we have  $\delta_k \leq \tilde{\delta}_k$  so that constraint (3) is satisfied, also  $(\delta_k)$  is nondecreasing as required. Moreover,  $\delta_k + 1 \geq \tilde{\delta}_k$  so that (recall  $\delta_0 = 0$ ):

$$\Psi_p(\delta, \mathcal{D}) \leq 2cM_p \sqrt{N} \left( 2 + 2^{\frac{d}{2}} \left( 1 + \sum_{k=1}^m 2^{\frac{kd}{2}} \tilde{\delta}_k^{-\alpha} \right) \right) \leq 2cM_p \sqrt{N} \left( 2 + 2^{\frac{d}{2}} \left( 1 + (4d^2)^\alpha Z_{\alpha,m}^{1+\alpha} \right) \right).$$

If  $\alpha > \frac{d}{2}$  then  $Z_{\alpha,m} \leq Z_{\alpha,\infty} = \rho(1-\rho)^{-1}$  and we get

$$\Psi_p(\delta, \mathcal{D}) \leq \frac{1}{4} cM_p C_{\alpha,d} \sqrt{N}, \quad C_{\alpha,d} := 24 \cdot 2^{d+\alpha} d^{2\alpha} |1-\rho|^{-(1+\alpha)}.$$

If  $\alpha < \frac{d}{2}$  then  $Z_{\alpha,m} \leq \rho^{m+1} (\rho-1)^{-1} \leq N^{\frac{1-\alpha}{1+\alpha}} \rho (\rho-1)^{-1}$  and

$$\Psi_p(\delta, \mathcal{D}) \leq \frac{1}{4} cM_p C_{\alpha,d} N^{1-\frac{\alpha}{d}}.$$

Lastly, in the case  $\alpha = \frac{d}{2}$  it holds  $Z_{\alpha,m} = m \leq d^{-1} \log_2 N$  and we get

$$\Psi_p(\delta, \mathcal{D}) \leq 0.25 \cdot cM_p \tilde{C}_d \sqrt{N} (\log N)^{\frac{d}{2}+1}, \quad \tilde{C}_d := 15 \cdot (8d)^{\frac{d}{2}}.$$

For the case  $\varphi_p(t) \leq c \exp(-\gamma t^\eta)$ , it suffices to apply Theorem 1 directly with  $\delta_k = \lfloor \tilde{\delta}_k \rfloor$ ,  $\tilde{\delta}_k = 2^{k/2} (\sqrt{2}-1)/4d^2$ . Note that such choice of  $\delta_k$  ensures (3). Thus we obtain that:

$$\Psi_p(\delta, \mathcal{D}) \leq 2cM_p \sqrt{N} \left( 2 + \sum_{k=1}^{m+1} \exp\left(-C_{\gamma,\eta,d} \cdot 2^{\frac{k\eta}{2}}\right) 2^{\frac{kd}{2}} \right) \leq cM_p C_{\gamma,\eta,d}^1 \sqrt{N},$$

where  $C_{\gamma,\eta,d} > 0$ ,  $C_{\gamma,\eta,d}^1 > 0$  some finite factors and the last inequality is due to

$$\sum_{k=1}^{\infty} \exp\left(-C_{\gamma,\eta,d} \cdot 2^{\frac{k\eta}{2}}\right) 2^{\frac{kd}{2}} < \infty.$$

Finally the calculations for the case  $p = \infty$  are identical and the claim follows.  $\square$

**Remark 3 (On the optimality of the bound from Corollary 2).** In the case  $\varphi_p(r) = \min(1, cr^{-\alpha})$  with  $\alpha > \frac{d}{2}$  or  $\varphi_p(r) = \min(1, c \exp(-\gamma r))$  ( $\gamma > 0$ ) the bounds (5) and (7) yield asymptotic behaviour (as  $N$  grows) of order  $\|N^{-1}S_{\mathcal{D}}\|_p = O(N^{-\frac{1}{2}})$  and  $\|N^{-1}S_{\mathcal{D}}\|_{SG} = O(N^{-1/2})$  respectively, which is known to be optimal if the field is i.i.d. Let us consider the case of polynomially weakly dependent processes with rate  $\varphi_p(r) = \min(1, cr^{-\alpha})$  and  $\alpha < \frac{d}{2}$  (we will not comment here on the optimality of the additional  $\log(N)$  factor if  $\alpha = \frac{d}{2}$ ). Consider the field  $X_t(\omega) := m_0 \xi \mathbb{1}_{t \in \llbracket n \rrbracket^d}$ , where  $m_0 = n^{-\alpha}$  and  $\xi$  is a Rademacher variable. In other words  $X_t$  is constant equal to  $\pm m_0$  with a randomly flipped sign over the cube of side-length  $n$  and 0 elsewhere. One can readily check that  $(X_t)_{t \in \mathbb{Z}^d}$  satisfies **WD**( $p$ ) with  $M_p = 1$  and furthermore  $|N^{-1}S_{\mathcal{D}}| = m_0 = N^{-\frac{\alpha}{d}}$ . This example shows that inequalities (5),(7) are tight under the given assumptions, up to multiplicative factors only depending on  $p$  and  $d$ . In other words the bounds of Corollary 2 are saturated by either of the extremal cases of an i.i.d. field or a constant field on the cube. Concerning the latter, note that the exhibited distribution saturating the bound depends on the side-length  $n$ . It would be more satisfactory to exhibit an example of a *fixed* field distribution satisfying the polynomial weak mixing condition with  $\alpha < \frac{d}{2}$  ("long range" or "heavy tail" dependence) and such that the rates (5),(7) are tight asymptotically as  $N$  grows.

### 3. Discussion and comparison to known results

In this section we compare our approach and bounds to some existing results for dependent real random fields. In many of such results (see for instance [Dedecker, 2001](#), [Doukhan, Léon and Portal, 1984](#), [Rio, 2000](#)) an exponential inequality for partial sums of bounded random fields is derived from  $L_p(\mathbb{P})$  bound by optimizing over the value  $p \geq 2$ . The analysis of the properties of dependent random fields are commonly based on approximations by multidimensional martingales. For this reason we first review briefly the different notions of multidimensional martingales, associated Burkholder-type inequalities for this reference case, and the relation to approximations of dependent random fields. In the following section we also discuss the case of Bernoulli random fields, that is, stationary transforms of an i.i.d. random field.

Notice that the weak dependency condition (1) when  $d = 1$  is stronger than the so-called "mixingale type" condition (see for example [McLeish, 1976](#)), for which the process is not necessarily adapted to the conditioning filtration. The former is mentioned in the works of [Dedecker et al. \(2007\)](#), [Dehling and Philipp \(1982\)](#) to characterize fading correlation between the past and the future of a discrete stochastic process.

#### 3.1. Multiparameter martingale difference fields

Although our interest concerns chiefly dependent random fields, we first review briefly several notions of multiparameter martingale difference fields as reference situations. For  $k, \ell \in \mathbb{Z}^d$  we denote  $k \leq_{c_w} \ell$  if the inequality holds coordinate-wise, i.e.  $k_i \leq \ell_i$  for all  $i \in \llbracket d \rrbracket$ ,  $k <_{c_w} \ell$  if the inequality is strict for at least one coordinate, and  $k \ll_{c_w} \ell$  if the inequality is strict for *all* coordinates. Simplifying the general point of view to focus on canonical filtrations in our setting, define the  $d$  marginal filtrations  $(\mathcal{F}_i^{(k)}) := \mathfrak{G}\{X_{(t_1, \dots, t_d)} : t_k \leq i\}_{i \in \mathbb{Z}}$ ,  $k \in \llbracket d \rrbracket$ ; and for  $t = (t_1, \dots, t_d) \in \mathbb{Z}^d$ , let  $\mathcal{F}_t^\wedge := \bigwedge_{k \in \llbracket d \rrbracket} \mathcal{F}_{t_k}^{(k)}$ , and  $\mathcal{F}_t^\vee := \bigvee_{k \in \llbracket d \rrbracket} \mathcal{F}_{t_k}^{(k)}$ . Note that  $\mathcal{F}_t^\vee = \mathcal{M}_{t,0}$ . The field  $(X_t)_{t \in \mathbb{Z}^d}$  is then called a

- (i) *weak* martingale difference field, if  $\mathbb{E}[X_t | \mathcal{F}_{t'}^\wedge] = 0$  for all  $t' <_{c_w} t$ ;
- (ii) *ortho-martingale* difference field, if  $\mathbb{E}[X_t | \mathcal{F}_{t'_k}^{(k)}] = 0$  for all  $t = (t_1, \dots, t_d) \in \mathbb{Z}^d$ ,  $k \in \llbracket d \rrbracket$  and  $t'_k < t_k$ ;
- (iii) *strong* martingale difference field, if  $\mathbb{E}[X_t | \mathcal{F}_{t'}^\vee] = 0$  for all  $t' \ll_{c_w} t$ .

The above terminology is in line with the general definitions introduced in the seminal paper of [Cairoli and Walsh \(1975\)](#) (see also [Basu and Dorea, 1979](#)). Since  $\mathcal{F}_{t'}^\wedge \subseteq \mathcal{F}_{t'_k}^{(k)} \subseteq \mathcal{F}_{t'}^\vee$ , we have the straightforward implication  $(iii) \Rightarrow (ii) \Rightarrow (i)$ . It is also well-known that  $(i)$  implies  $(ii)$  under the additional assumption of *commuting* marginal  $\sigma$ -fields also known as condition (F4) of [Cairoli and Walsh \(1975\)](#), see [Khosnevisan \(2002\)](#) for a modern account and equivalent formulations of the assumption. However, we stress that  $(iii)$  does not imply commuting marginal  $\sigma$ -fields in general.

The projective-type dependency assumption  $\mathbf{WD}(p)$ , in the edge case  $\varphi_p(0) = 1$  and  $\varphi_p(t) = 0$  for  $t > 0$ , reduces to the strong martingale difference field assumption  $(iii)$ . On the other hand, under assumption  $(iii)$  a Burkholder-type inequality for random fields can be derived as a straightforward consequence of Burkholder's inequality in dimension 1. This holds since  $(iii)$  implies a (one-dimensional) martingale difference condition under the filtration  $\mathcal{F}_t^{<\text{lex}} := \mathfrak{S}\{X_{t'} : t' <_{\text{lex}} t\}$  and the one-dimensional Burkholder's inequality can be applied to the elements of the partial sum ordered in lexicographic order, i.e, we obtain

$$\|S_{\llbracket n \rrbracket^d}\|_p \leq C_p n^{\frac{d}{2}} \max_{t \in \llbracket n \rrbracket^d} \|X_t\|_p \quad (9)$$

by this direct argument. Therefore, in the particular case  $(iii)$  our results do not bring something new (note also that in that case the optimal multiplicative constant in the bound does not depend on the dimension  $d$ , while ours does). Still, from this point of view the assumption  $\mathbf{WD}(p)$  in the general case can be interpreted as a form of limited departure of the random field from a strong martingale difference field.

Concerning the other types of multi-parameter martingales, in the bivariate case a Burkholder-type inequality was established by [Gundy and Varopoulos \(1976\)](#) (Lemma 1 there) under assumptions  $(i)+(F4)$ , and under assumption  $(ii)$  by [Méttraux \(1978\)](#). For general dimension  $d$ , [Fazekas \(2005\)](#) established Burkholder-type inequalities, again under the assumptions  $(i)+(F4)$ . Both [Gundy and Varopoulos \(1976\)](#) and [Fazekas \(2005\)](#) use an iterative approach over the dimension. These results are not recovered by our approach since the assumptions are different.

Recent developments have used notions of approximation of a random field by ortho-martingales to obtain various results of a related nature ([El Machkouri and Giraud, 2016](#), [Giraud, 2018, 2019](#), [Volný, 2015](#)). However, we note that these works use the terminology “ortho-martingale” to mean in fact  $(i)+(F4)$ . We insist that we do not use the assumption (F4) of commuting marginal filtrations in the present work, so that we see our results as going in a different direction in nature.

### 3.2. Bernoulli random fields

A Burkholder-type inequality for so-called Bernoulli random fields is given by [El Machkouri, Volný and Wu \(2013\)](#). In that work a class of random fields of the form  $X_t = g(\varepsilon_{t-s}, s \in \mathbb{Z}^d)$ ,  $t \in \mathbb{Z}^d$  where  $(\varepsilon_i)_{i \in \mathbb{Z}^d}$  are i.i.d. random variables and  $g$  is some bounded measurable function is considered. Typical examples which belongs to this class are linear random fields and Volterra random fields (see examples of [Sang and Xiao, 2018](#) and Section 2 of [Giraud, 2019](#)). Functional central limit theorems ([Biermé and Durieu, 2014](#), [Klicnarová, Volný and Wang, 2016](#)), a large deviation principle ([Sang and Xiao, 2018](#)) and a variant of law of the iterated logarithm for partial sums of this class of random fields ([Giraud,](#)



2022) have been established. For these types of random fields a following dependence measure (originally introduced by Wu, 2005) is introduced. Namely, for a stochastic random field  $(X_t)_{t \in \mathbb{Z}^d}$ , such that  $X_t \in L_p(\mathbb{P})$  define the coefficient  $\delta_{t,p} = \|X_t - X_t^*\|_p$ , where the coupled process  $X_t^*$  is given by  $X_t^* = g(\varepsilon_{t-s}^*, s \in \mathbb{Z}^d)$  and  $\varepsilon_j^* = \varepsilon_j \mathbb{1}_{j \neq 0} + \varepsilon_0' \mathbb{1}_{j=0}$ , with  $\varepsilon_0'$  being an independent copy of  $\varepsilon_0$ . In this framework, Proposition 1 of El Machkouri, Volný and Wu (2013) implies following the Burkholder-type inequality for the partial sums of  $X_t$ :

$$\|S_{\llbracket n \rrbracket^d}\|_p \leq \sqrt{2pn}^{d/2} \Delta_p,$$

where  $\Delta_p = \sum_{i \in \mathbb{Z}^d} \delta_{i,p} < \infty$ .

In the one-dimensional case, a particular situation is when the field  $(X_t)_{t \in \mathbb{Z}}$  is *causal* with respect to the underlying i.i.d. field  $(\varepsilon_t)_{t \in \mathbb{Z}}$ , that is,  $X_t = g(\varepsilon_{t-s}, s \geq 0)$ . In this situation, as pointed out by Peligrad, Utev and Wu (2007), if we denote  $\mathcal{F}_k := \sigma((\varepsilon_t)_{t \leq k})$ , it holds by Jensen's and stationarity

$$\begin{aligned} \|\mathbb{E}[X_t | \mathcal{M}_{t,k}] - \mathbb{E}[X_t]\|_p &= \|\mathbb{E}[X_k | \mathcal{M}_{k,k}] - \mathbb{E}[X_k]\|_p \\ &\leq \|\mathbb{E}[X_k | \mathcal{F}_0] - \mathbb{E}[X_k]\|_p \\ &\leq \|g(\varepsilon_k, \varepsilon_{k-1}, \dots) - g(\varepsilon_k, \varepsilon_{k-1}, \dots, \varepsilon_0, \varepsilon_{-1}', \varepsilon_{-2}', \dots)\|_p, \end{aligned}$$

so that assumption **WD** can be related to quantities that are tractable in many models (see Peligrad, Utev and Wu, 2007 for more discussion). Notice that an analogous notion of a "causal field" in higher dimension would be of the form  $X_t = g(\varepsilon_u, u \succ_{cw} t)$  (with the notation  $\succ_{cw}$  as introduced in Section 3.1), a notion that has been considered in past literature (see e.g. Giraudo, 2021a, Corollary 3.5). Observe that in the Bernoulli random field model, the marginal filtrations for the i.i.d. field  $(\varepsilon_t)_{t \in \mathbb{Z}^d}$  are commuting, so that tools based on ortho-martingale approximation mentioned in the previous section can be applied as well (see e.g. Giraudo, 2021a,b). In such a setting we can, by the same token as in dimension 1, relate assumption **WD(p)** to tractable quantities. To deal with non-causal Bernoulli fields however will require to relax assumption **WD(p)** to the non-adapted case (wherein we condition with respect to the field  $(\varepsilon_t)_{t \in \mathbb{Z}^d}$  rather than  $(X_t)_{t \in \mathbb{Z}^d}$ ); which is out of the scope of the present work.

### 3.3. $L_p(\mathbb{P})$ – projective criterion

In the work of Dedecker (2001), a Burkholder-type inequality for random fields is obtained under an  $L_p$ –projective dependence criterion; it takes the form

$$\|S_{\llbracket n \rrbracket^d}\|_p \leq \sqrt{2p \sum_{t \in \llbracket n \rrbracket^d} b_{t,p/2}(X)}, \quad (10)$$

where  $b_{t,\alpha}(X) = \|X_t^2\|_\alpha + \sum_{k \in V_t^1} \|X_k \mathbb{E}_{|k-t|}[X_t]\|_\alpha$  (we assume the field is centered for simplicity),  $p \geq 2$ ,  $|x - y| = \max_{1 \leq i \leq d} |x_i - y_i|$ ,  $V_t^1$  denotes the set of all elements which precede  $t$  in lexicographic order on  $\mathbb{Z}^d$  and  $\mathbb{E}_\ell[X_t]$  is the conditional expectation with respect to the  $\sigma$ –algebra  $\mathcal{F}_{V_t^\ell}$ , where  $V_t^\ell$  is the subset of elements from  $V_t^1$  which are at distance at least  $\ell$  from  $t$ . In the case where  $(X)_{t \in \mathcal{D}}$  is a strong martingale difference random field then  $b_{t,p/2} = \|X_t^2\|_{p/2} = \|X_t\|_p^2$  and (10) implies (9) with a constant  $C_{p,d} = \mathcal{O}(\sqrt{p})$ .

Let us analyze briefly the estimates obtained via bound (10) under the projective weak dependency assumption **WD'(p)** (that is, **WD(p)** with  $\mathcal{M}'_{t,r}$  in place of  $\mathcal{M}_{t,r}$ , see (2)). By construction of the  $\sigma$ –

algebras, it holds  $\mathcal{F}_{V_t} \subseteq \mathcal{M}'_{t,r}$ , thus under assumption  $\mathbf{WD}'(p)$ , by the generalized Hölder's inequality it holds  $\|X_k \mathbb{E}_{|k-t|}[X_t]\|_{p/2} \leq \|X_k\|_p \|\mathbb{E}[X_t | \mathcal{M}'_{t,|k-t|}]\|_p \leq M_p^2 \varphi_p(|k-t|)$ . Hence, for the right-hand-side of Inequality (10) when  $p \geq 2$  we get

$$\sqrt{2p \sum_{t \in \llbracket n \rrbracket^d} b_{t,p/2}(X)} \leq M_p \sqrt{2p \left( \sum_{t \in \llbracket n \rrbracket^d} \left( 1 + \sum_{r=1}^{\infty} r^{d-1} \varphi_p(r) \right) \right)},$$

Now, if  $\varphi_p(r) \lesssim r^{-\alpha}$ , plugging this into (10) yields a bound of order  $O(n^{-(\frac{d}{2} \wedge \frac{\alpha}{2})})$  for  $n^{-d} \|S_{\llbracket n \rrbracket^d}\|_p$ , while Corollary 2 entails a bound in  $O(n^{-(\frac{d}{2} \wedge \alpha)})$ . Thus, under assumption  $\mathbf{WD}'(p)$  the estimate obtained from Corollary 2 improves over that obtained via the bound (10) (in particular extending the range of  $\alpha$  for which a convergence of order  $O(n^{-d/2})$  is granted). On the other hand, it should be noted (Dedecker, 2020) that under the stronger assumption of control of the  $\alpha$ -mixing coefficient between the  $\sigma$ -algebras  $\mathfrak{S}(X_t)$  and  $\mathcal{M}'_{t,k}$ , using covariance inequalities of Rio (1993), one can infer both assumption  $\mathbf{WD}'(p)$  for a certain function  $\varphi_p(k)$  but also  $\|X_k \mathbb{E}_{|k-t|}[X_t]\|_{p/2} \lesssim \varphi_p(|k-t|)^2$ . In this scenario, if  $\varphi_p(r) \lesssim r^{-\alpha}$ , bound (10) and Corollary 2 yield a bound of the same order as Dedecker (2001).

## 4. Proof of main result

To prove Theorem 1 it turns out to be more convenient to consider rectangles starting at the origin. We therefore introduce the notation  $\llbracket k \rrbracket_0 := \{0, \dots, k-1\}$  if  $k \geq 1$  and  $\llbracket k \rrbracket_0 = \emptyset$  if  $k = 0$ , and below we consider rectangles of the form  $\mathcal{R} = \prod_{i=1}^d \llbracket N_i \rrbracket_0$ .

For sets  $A, B \subseteq \mathbb{Z}^d$ , and  $c \in \mathbb{Z}$ ,  $v \in \mathbb{Z}^d$  we will use the standard notation

$$cA := \{ca : a \in A\} \subseteq \mathbb{Z}^d;$$

$$A + B := \{a + b : a \in A, b \in B\} \subseteq \mathbb{Z}^d.$$

(If  $A = \emptyset$  we denote  $cA := \emptyset$ ,  $A + B := \emptyset$ .)

Before we begin the proof proper, we give a few words of informal overview of its structure. The main principle of the proof is a decomposition of each term  $X_t - \mathbb{E}[X_t]$  as a "telescopic" series with terms  $\mathbb{E}[X_t | \mathcal{F}_k^<(t)] - \mathbb{E}[X_t | \mathcal{F}_{k-1}^<(t)]$ , where  $\mathcal{F}_k^<(t)$  is a suitable filtration generated by blocks (which we will call *cells*) of the random field at a dyadic scale  $2^k$ . Each of these terms can be then summed over the corresponding dyadic cells and gives rise to a martingale difference "at scale  $k$ " (i.e. indexed by dyadic integer vectors); for each  $k$  the corresponding dyadic martingale can be controlled by the Marcinkiewicz-Zygmund inequality. This canvas follows the general line of arguments used by Peligrad and Utev (2005), Peligrad, Utev and Wu (2007) in dimension  $d = 1$ .

Going to higher dimension, we run into the following additional challenges:

- There is no natural total order on  $\mathbb{Z}^d$ . In order to use martingale arguments, we need to carefully define a specific total order on  $\mathbb{Z}^d$  to get the martingale structure while ensuring a form of compatibility of filtrations across scales; this construction will be explained in detail in Section 4.2.
- We need to control  $\|\mathbb{E}[X_t | \mathcal{F}_k^<(t)] - \mathbb{E}[X_t | \mathcal{F}_{k-1}^<(t)]\|_p$  using the  $\mathbf{WD}(p)$  assumption, and this is directly linked to  $\varphi_p(d_k(t))$ , where  $d_k(t)$  is the (supremum) distance of  $t$  to the border of the dyadic cell it belongs to. Directly summing these quantities over cells gives rise to good estimates in dimension 1 (see Peligrad, Utev and Wu, 2007), but it turns out that the estimates obtained this

way for rectangles in dimension  $d \geq 2$  are suboptimal. The reason is that there are “too many” elements in the cell that are close to its boundary, making a too large contribution in the sum over the cell. This is why, to alleviate this issue, we will at first exclude the elements which are close to boundaries of a cell at any scale, which we will call the (multiscale) “frame”, so that the remaining elements are then sufficiently “separated” from the cell boundaries. The sum of elements in the “frame” will then be dealt with by an inductive argument. This framed decomposition and the inductive argument will be explained first, in the coming section.

#### 4.1. Framed decomposition and main inductive step

We prove the theorem by induction on the size of the rectangle  $\mathcal{R}$ .

We consider the case  $p < \infty$  only; the arguments for the case  $p = \infty$  are the same for the remainder of the proof. For  $\mathcal{R}$  reduced to a single element  $\mathbf{0}$ , the claim obviously holds. Now, assume the claim is established for any rectangle  $\mathcal{R}' \subsetneq \mathcal{R}$ . We use the following construction: for any integer  $\delta < 2^k$  let

$$\Lambda_{k,\delta} := 2^k \mathbb{N}_{>0} + \llbracket \delta \rrbracket_0. \quad (\text{If } \delta = 0, \text{ then } \Lambda_{k,\delta} := \emptyset.) \quad (11)$$

Let  $\delta = (\delta_k)_{k \geq 1}$  be a fixed sequence of integers with  $\delta_k \leq 2^k$ ,  $k \geq 1$ . Define

$$\Lambda_\delta := \bigcup_{k \geq 1} \Lambda_{k,\delta_k} \quad \mathfrak{F}_\delta := (\mathbb{N} \setminus \Lambda_\delta)^d; \quad \mathfrak{F}_\delta^c := \mathbb{Z}^d \setminus \mathfrak{F}_\delta. \quad (12)$$

We call  $\mathfrak{F}_\delta$  the “framed set” and  $\mathfrak{F}_\delta^c$  the “frame”. These sets are illustrated on Figure 2. We use the decomposition  $\mathcal{R} = (\mathcal{R} \cap \mathfrak{F}_\delta) \uplus (\mathcal{R} \cap \mathfrak{F}_\delta^c)$ . By the triangle inequality

$$\|S_{\mathcal{R}}\|_p \leq \|S_{\mathcal{R} \cap \mathfrak{F}_\delta}\|_p + \|S_{\mathcal{R} \cap \mathfrak{F}_\delta^c}\|_p. \quad (13)$$

The proof proceeds as follows. The first and main term will be controlled by a multiscale martingale decomposition and lead to the crucial estimate  $\|S_{\mathcal{R} \cap \mathfrak{F}_\delta}\|_p \leq \frac{C_p}{2} \Psi_p(\delta, \mathcal{R})$  where  $C_p = 4\sqrt{p}$ . The proof of this estimate is postponed to Proposition 4 below.

The second, remainder term will be dealt with via the induction hypothesis. We first decompose  $\mathfrak{F}_\delta^c$  as a disjoint union of product sets, writing  $\Lambda_\delta^c := \mathbb{N} \setminus \Lambda_\delta$ :

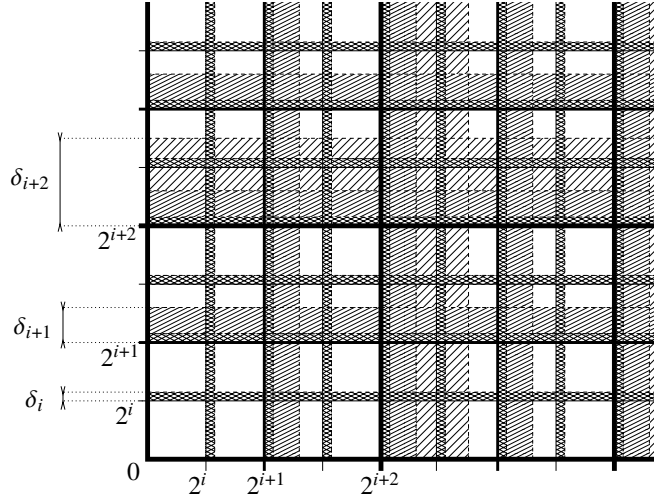
$$\mathfrak{F}_\delta^c = \mathbb{N}^d \setminus (\Lambda_\delta^c)^d = \bigsqcup_{i=1}^d \Delta_i, \quad \Delta_i := \left( \prod_{j=1}^{i-1} \Lambda_\delta^c \times \Lambda_\delta \times \prod_{j=i+1}^d \mathbb{N} \right),$$

therefore, by the triangle inequality,

$$\|S_{\mathcal{R} \cap \mathfrak{F}_\delta^c}\|_p = \left\| \sum_{i=1}^d S_{\mathcal{R} \cap \Delta_i} \right\|_p \leq \sum_{i=1}^d \|S_{\mathcal{R} \cap \Delta_i}\|_p. \quad (14)$$

We introduce the following notation. For a finite set  $B \subset \mathbb{N}$ , and an integer  $j \in \llbracket |B| \rrbracket_0$ , denote  $(j : B)$  the  $(j+1)$ -th element of  $B$  in increasing order. For a finite product set  $\mathbf{A} = \prod_{i=1}^d A_i \subseteq \mathbb{N}^d$ , define  $\mathcal{K}(\mathbf{A}) := \prod_{i=1}^d \llbracket |A_i| \rrbracket_0$ , and for a  $d$ -tuple  $t = (t_1, \dots, t_d) \in \mathcal{K}(\mathbf{A})$ , denote  $(t : \mathbf{A}) = ((t_1 : A_1), \dots, (t_d : A_d))$ , and the “compressed” version of the restriction of the process  $(X_t)_{t \in \mathbb{Z}^d}$  to  $\mathbf{A}$  as

$$\begin{cases} \widetilde{X}_t^{(\mathbf{A})} = X_{t:\mathbf{A}}, & t \in \mathcal{K}(\mathbf{A}), \\ \widetilde{X}_t^{(\mathbf{A})} = 0, & t \notin \mathcal{K}(\mathbf{A}); \end{cases} \quad \text{then it holds} \quad \widetilde{S}_{\mathcal{K}(\mathbf{A})}^{(\mathbf{A})} := \sum_{t \in \mathcal{K}(\mathbf{A})} \widetilde{X}_t^{(\mathbf{A})} = S_{\mathbf{A}}. \quad (15)$$



**Figure 2.** Illustration of “framed set” and the “frame” for the case  $d = 2$  on three consecutive scales. *Hatched regions* belong to the frame  $\mathfrak{F}_\delta^c$ . The *white region* is the framed set  $\mathfrak{F}_\delta$ .

Since  $\Delta_i$  is a product set, so is  $\mathcal{R} \cap \Delta_i$ , and we can apply the above “compression principle”. Using assumption (3), the side-length of the rectangle  $\mathcal{K}(\mathcal{R} \cap \Delta_i)$  along direction  $i$  is bounded by

$$|\Lambda_\delta \cap \llbracket N_i \rrbracket_0| = \left| \bigcup_{k \geq 1} \Lambda_{k, \delta_k} \cap \llbracket N_i \rrbracket_0 \right| \leq \sum_{k=1}^{\lfloor \log_2(N_i) \rfloor} \left\lfloor \frac{N_i}{2^k} \right\rfloor \delta_k \leq N_i \sum_{k=1}^{m(\mathcal{R})} 2^{-k} \delta_k \leq \frac{N_i}{4d^2},$$

while for  $j \neq i$  the side-lengths are bounded by  $N_j$ . Therefore  $\mathcal{K}(\mathcal{R} \cap \Delta_i) \subsetneq \mathcal{R}$  and

$$|\mathcal{R} \cap \Delta_i| \leq \frac{|\mathcal{R}|}{4d^2}. \quad (16)$$

By Lemma 3 below, the “compressed” process  $(\tilde{X}_t^{(\mathcal{R} \cap \Delta_i)})_{t \in \mathbb{Z}^d}$  satisfies the same weak-dependency condition (1) as the original process. Applying the induction hypothesis to the process  $(\tilde{X}_t^{(\mathcal{R} \cap \Delta_i)})_{t \in \mathbb{Z}^d}$  over the rectangle  $\mathcal{K}(\mathcal{R} \cap \Delta_i)$ , we obtain

$$\|S_{\mathcal{R} \cap \Delta_i}\|_p = \|\tilde{S}_{\mathcal{K}(\mathcal{R} \cap \Delta_i)}^{(\mathcal{R} \cap \Delta_i)}\|_p \leq C_p \Psi_p(\delta, \mathcal{K}(\mathcal{R} \cap \Delta_i)). \quad (17)$$

We estimate this upper bound using (16) and straightforward cardinality bounds via:

$$\begin{aligned} \Psi_p(\delta, \mathcal{K}(\mathcal{R} \cap \Delta_i)) &= 2M_p \sqrt{|\mathcal{R} \cap \Delta_i|} \left( 2 + \sum_{k=1}^{m(\mathcal{K}(\mathcal{R} \cap \Delta_i))+1} \varphi_p(\delta_{k-1} + 1) \sqrt{|C_{k,0} \cap \mathcal{K}(\mathcal{R} \cap \Delta_i)|} \right) \\ &\leq \frac{1}{d} M_p \sqrt{|\mathcal{R}|} \left( 2 + \sum_{k=1}^{m(\mathcal{R})+1} \varphi_p(\delta_{k-1} + 1) \sqrt{|C_{k,0} \cap \mathcal{R}|} \right) \\ &= \frac{1}{2d} \Psi_p(\delta, \mathcal{R}). \end{aligned} \quad (18)$$

Finally combining Equation (13), Proposition 4, Equations (14), (17) and (18), we obtain

$$\|S_{\mathcal{R}}\|_p \leq \frac{1}{2}C_p\Psi(\delta, \mathcal{R}) + \sum_{i=1}^d \frac{1}{2d}C_p\Psi_p(\delta, \mathcal{R}) \leq C_p\Psi_p(\delta, \mathcal{R}),$$

and the induction claim is proved.  $\square$

**Lemma 3.** *Let  $p \in [2, \infty]$  be fixed and assume the process  $(X_t)_{t \in \mathbb{N}^d}$  satisfies assumption **WD**( $p$ ) with  $\varphi_p(\cdot)$ . Then for any product set  $\mathbf{A} \subset \mathbb{N}^d$ , the "compressed" version  $(\tilde{X}_t^{(\mathbf{A})})_{t \in \mathbb{N}^d}$  as defined by (15) satisfies assumption **WD**( $p$ ) with the same function  $\varphi_p(\cdot)$ .*

**Proof** For every  $t \notin \mathcal{K}(\mathbf{A})$ , we have by construction that  $\tilde{X}_t^{(\mathbf{A})} = 0$ , it is therefore sufficient to establish property (1) only for elements  $t \in \mathcal{K}(\mathbf{A})$ . Recall  $\mathcal{M}_{t,k} = \mathfrak{S}\{X_u : u \in \mathbb{Z}^d, \sup_i(t_i - u_i) \geq k\}$  and let

$$\begin{aligned} \widetilde{\mathcal{M}}_{t,k}^{(\mathbf{A})} &:= \mathfrak{S}\{\tilde{X}_u^{(\mathbf{A})} : u \in \mathbb{Z}^d, \sup_i(t_i - u_i) \geq k\} \\ &= \mathfrak{S}\{X_{(u:\mathbf{A})} : u \in \mathcal{K}(\mathbf{A}), \sup_i(t_i - u_i) \geq k\}, \end{aligned}$$

wherein we are able to restrict for  $u \in \mathcal{K}(\mathbf{A})$  using again  $\tilde{X}_u^{(\mathbf{A})} = 0$  for  $u \notin \mathcal{K}(\mathbf{A})$ .

For every  $t, u \in \mathcal{K}(\mathbf{A})$ , it holds  $\sup_i((t_i : A_i) - (u_i : A_i)) \geq \sup_i(t_i - u_i)$ , so for any  $k > 0$ :

$$\begin{aligned} \widetilde{\mathcal{M}}_{t,k}^{(\mathbf{A})} &= \mathfrak{S}\{X_{(u:\mathbf{A})} : u \in \mathcal{K}(\mathbf{A}), \sup_i(t_i - u_i) \geq k\} \subseteq \mathfrak{S}\{X_{(u:\mathbf{A})} : u \in \mathcal{K}(\mathbf{A}), \sup_i(t_i : A_i - u_i : A_i) \geq k\} \\ &\subseteq \mathfrak{S}\{X_v : \sup_i(t_i : A_i - v_i) \geq k\} \\ &= \mathcal{M}_{(t:\mathbf{A}),k}. \end{aligned}$$

By using Jensen's inequality, for any  $t \in \mathcal{K}(\mathbf{A})$ ,  $k > 0$ , using the assumption **WD**( $p$ ) for process  $(X_t)$  we get:

$$\begin{aligned} \left\| \mathbb{E}\left[\tilde{X}_t^{(\mathbf{A})} \mid \widetilde{\mathcal{M}}_{t,k}^{(\mathbf{A})}\right] - \mathbb{E}\left[\tilde{X}_t^{(\mathbf{A})}\right] \right\|_p &= \left\| \mathbb{E}\left[X_{(t:\mathbf{A})} \mid \widetilde{\mathcal{M}}_{t,k}^{(\mathbf{A})}\right] - \mathbb{E}\left[X_{(t:\mathbf{A})}\right] \right\|_p \\ &\leq \left\| \mathbb{E}\left[X_{(t:\mathbf{A})} \mid \mathcal{M}_{(t:\mathbf{A}),k}\right] - \mathbb{E}\left[X_{(t:\mathbf{A})}\right] \right\|_p \leq M_p \varphi_p(k), \end{aligned}$$

so process  $(X_t^{(\mathbf{A})})$  satisfies **WD**( $p$ ).  $\square$

## 4.2. Multiscale martingale decomposition

As announced previously, the following estimate for  $\|S_{\mathcal{R} \cap \mathfrak{F}_\delta}\|_p$  is crucial. The main argument resides on a multi-scale martingale decomposition, the principle of which was used in dimension  $d = 1$  by [Peligrad, Utev and Wu \(2007\)](#). However, in dimension  $d \geq 2$  the conditioning  $\sigma$ -algebra  $\mathcal{M}_{t,k}$  in Assumption **WD**( $p$ ) cannot be expressed as a "past" for a total order on  $\mathbb{Z}^d$ , and extending the argument requires a more involved construction.

**Proposition 4.** Let  $\mathcal{R} = \prod_{i=1}^d \llbracket N_i \rrbracket_0$  be a  $d$ -dimensional rectangle of side-lengths  $N_i \geq 1$ ,  $i = 1, \dots, d$ , and  $m(\mathcal{R}) := \max_{i=1, \dots, d} \lfloor \log_2 N_i \rfloor$ . Let  $\delta = (\delta_k)_{k \geq 1}$  be a fixed nondecreasing sequence of integers with  $\delta_k \leq 2^k$ ,  $k \geq 1$  (put  $\delta_0 = 0$ ), and let  $\mathfrak{F}_\delta$  be as defined in (12).

Assume the process  $(X_t)_{t \in \mathbb{N}^d}$  satisfies the weak dependency assumption **WD**( $p$ ) for some with  $p \in [2, \infty]$ . If  $p < \infty$ , it holds

$$\|S_{\mathcal{R} \cap \mathfrak{F}_\delta}\|_p \leq \frac{1}{2} C_p \Psi_p(\delta, \mathcal{R}), \quad (19)$$

where  $C_p := 4\sqrt{p}$ ; and if  $p = \infty$ , then it holds

$$\|S_{\mathcal{R} \cap \mathfrak{F}_\delta}\|_{\text{SG}} \leq \frac{1}{2} C_\infty \Psi_p(\delta, \mathcal{R}), \quad (20)$$

where  $C_\infty := 10$  and we recall that  $\Psi_p(\delta, \mathcal{R})$  is given by (4).

The key element of the proof of is a tree-like recursive ordering over  $\mathbb{N}^d$ . To define it we introduce the following notation. For  $t = (t_1, \dots, t_d) \in \mathbb{N}^d$  define the "dyadic projection at scale  $k$ " as  $\pi_k(t) := (\lfloor 2^{-k} t_i \rfloor 2^k)_{1 \leq i \leq d} \in 2^k \mathbb{N}^d$ . Observe that  $\pi_0(t) = t$ , and that  $\pi_k(t) = \mathbf{0}$  for  $k \geq \log_2 \|t\|_\infty$ . Let  $\leq_{\text{lex}}$  denote the lexicographical order on  $\mathbb{N}^d$ . Denote  $<_{\text{lex}}$  to be the associated strict order relation. For two elements  $t, t'$  of  $\mathbb{N}^d$ , define

$$\kappa(t, t') = \min\{k \in \mathbb{N} : \pi_k(t) = \pi_k(t')\} - 1. \quad (21)$$

Note that  $\kappa(t, t')$  is always well-defined, since  $\pi_k(t) = \pi_k(t') = \mathbf{0}$  for  $k \geq \max(\|t\|_\infty, \|t'\|_\infty)$ , hence the minimum in Equation (21) is over a non-empty set. Furthermore  $\kappa(t, t') = -1$  iff  $t = t'$ . We define the following order  $\leq$  on  $\mathbb{N}^d$

$$t \leq t' \quad \text{iff} \quad \text{either } \kappa(t, t') = -1 \text{ or } \pi_{\kappa(t, t')}(t) <_{\text{lex}} \pi_{\kappa(t, t')}(t'). \quad (22)$$

It is straightforward to check that  $\leq$  is a total order over  $\mathbb{N}^d$  since  $\leq_{\text{lex}}$  is a total order over  $\mathbb{N}^d$ . This order can be described as the co-lexicographical order for the (one-to-one) sequence representation  $(\pi_k(t))_{k \geq 0}$  of  $t \in \mathbb{N}^d$ , where the base order for the elements of the sequence is the usual lexicographical order. Equivalently, this is the co-lexicographical order on the (infinite) binary representation  $((\lfloor t_i 2^{-k} \rfloor \bmod 2)_{i=d, \dots, 1})_{k \geq 0}$ , where the vectorization is along (reverse) dimension first, then along scale. See Figure 3 an illustration of the order  $\leq$ .

For  $t \in \mathbb{N}^d$ , let

$$\Pi_k^<(t) := \{t' \in \mathbb{N}^d : \pi_k(t') < \pi_k(t)\}, \quad (23)$$

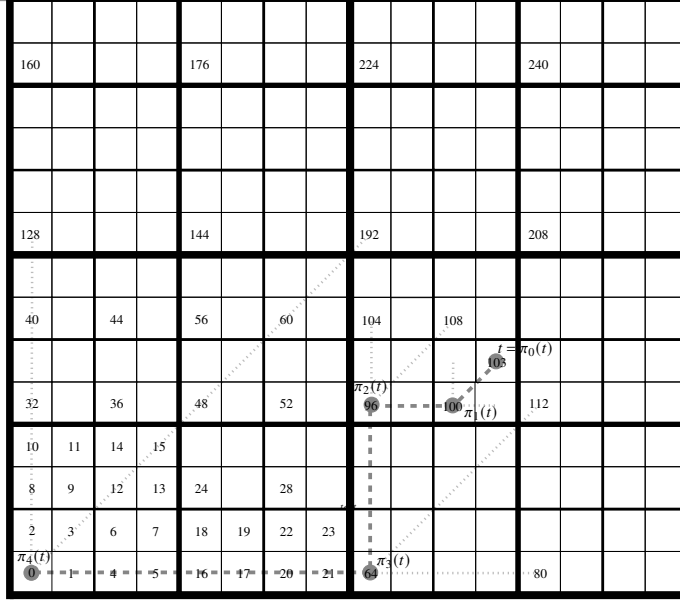
$$\Pi_k(t) := \{t' \in \mathbb{N}^d : \pi_k(t') \leq \pi_k(t)\}, \quad (24)$$

where  $t < t'$  indicates that  $t$  is strictly less than  $t'$  for the order  $\leq$ . Next we need the following Lemma which describes properties of the order  $\leq$ .

**Lemma 5.** Let ordering  $\leq_{\text{cw}}$  be the partial order on  $\mathbb{N}^d$  such that for  $t = (t_1, \dots, t_d) \in \mathbb{N}^d$ ,  $t' = (t'_1, \dots, t'_d) \in \mathbb{N}^d$  we say that  $t \leq_{\text{cw}} t'$  iff  $t_i \leq t'_i$  for all  $i \in \llbracket d \rrbracket$ . The following statements hold true:

- (0) For any  $k, \ell$  such that  $k \leq \ell$  it holds  $\pi_\ell \circ \pi_k = \pi_k \circ \pi_\ell = \pi_\ell$ .
- (i) The partial order  $\leq_{\text{cw}}$  is compatible with both the total orders  $\leq_{\text{lex}}$  and  $\leq$ , meaning that

$$t \leq_{\text{cw}} t' \implies t \leq_{\text{lex}} t' \text{ and } t \leq t'.$$



**Figure 3.** Illustration of the total order  $\leq$  for  $d = 2$ . The numbers indicate the total ordering of the cells of  $\mathbb{N}^2$  according to  $\leq$ . The successive “dyadic projections”  $\pi_k(t)$  for a particular point  $t$  are given (dashed line). For each projection  $\pi_k(t)$ , dotted lines point to the possible positions of  $\pi_{k-1}(t')$  such that  $\pi_k(t) = \pi_k(t')$ .

(ii) For any  $t \in \mathbb{N}^d$  and  $k, \ell \in \mathbb{N}$  with  $k \geq \ell$  it holds  $\pi_k(t) \leq \pi_\ell(t)$ . In particular, in case  $\ell = 0$  it holds  $\pi_k(t) \leq t$ .

(iii) All applications  $\pi_k$  are monotone nondecreasing with respect to  $\leq$ :

$$\forall k \in \mathbb{N}, \forall t, t' \in \mathbb{N}^d : \quad t \leq t' \Rightarrow \pi_k(t) \leq \pi_k(t').$$

(iv) For an integer  $k$  put  $C_{k,0} := \llbracket 2^k \rrbracket_0^d$ , and, for  $b \in 2^k \mathbb{N}^d$ , put  $C_{k,b} := \{b\} + C_{k,0}$ . For any  $t \in C_{k,b}$ , it holds  $\pi_k(t) = \pi_k(b) = b$ ,  $\Pi_k^<(t) = \Pi_k^<(b)$  and  $\Pi_k(t) = \Pi_k(b)$ .

(v) For any  $t \in \mathbb{N}^d$  and  $k \in \mathbb{N}$ , it holds

$$\Pi_k^<(t) = \{t' \in \mathbb{N}^d : t' < \pi_k(t)\}, \quad (25)$$

$$\Pi_k(t) = \Pi_k^<(t) \cup C_{k, \pi_k(t)}. \quad (26)$$

(vi) For any  $k \in \mathbb{N}_{>0}$  and  $t \in \mathbb{N}^d$ , it holds

$$\Pi_k^<(t) \subseteq \Pi_{k-1}^<(t) \subseteq \Pi_{k-1}(t) \subseteq \Pi_k(t). \quad (27)$$

**Proof (0)** For  $\ell \geq k$ , the equality  $\pi_k \circ \pi_\ell = \pi_\ell$  follows directly from the fact that  $\lfloor 2^{\ell-k} \lfloor 2^{-\ell} t \rfloor \rfloor = 2^{\ell-k} \lfloor 2^{-\ell} t \rfloor$ , while the equality  $\pi_\ell \circ \pi_k = \pi_\ell$  follows from  $\lfloor 2^{k-\ell} \lfloor 2^{-k} t \rfloor \rfloor = \lfloor 2^{-\ell} t \rfloor$  which is easy to check.

(i) The implication for the lexicographical order is obvious; concerning the order  $\leq$ , note that obviously all the mappings  $\pi_k$  for  $k \geq 0$  are non-decreasing for the partial order  $\leq_{\text{ew}}$ , i.e.  $t \leq_{\text{ew}} t'$  implies

$\pi_k(t) \leq_{\text{cw}} \pi_k(t')$ , in turn implying  $\pi_k(t) \leq_{\text{lex}} \pi_k(t')$  for all  $k$ , which finally entails  $t \leq t'$  from the definition.

(ii) The claim follows directly from (i) since it follows from the definition that  $\pi_k(t) \leq_{\text{cw}} \pi_\ell(t)$  if  $k \geq \ell$ .

(iii) Assume  $t < t'$  and let  $\kappa = \kappa(t, t') \geq 0$ . Then by definition of the order  $\leq$ , for  $\ell = \kappa + 1$  it holds  $\pi_\ell(t) = \pi_\ell(t')$ , and therefore also further for any  $\ell > \kappa$ , since  $\pi_\ell = \pi_\ell \circ \pi_{\kappa+1}$  by point (0). Thus, for  $k > \kappa$  it holds  $\pi_k(t) \leq \pi_k(t')$ . On the other hand, it holds  $\pi_\kappa(t) <_{\text{lex}} \pi_\kappa(t')$  and for  $\ell < \kappa$ ,  $\pi_\ell(t) \neq \pi_\ell(t')$ . For  $k \leq \kappa$ , put  $u := \pi_k(t), u' := \pi_k(t')$ , then for any  $\ell$  we have by point (0):  $\pi_\ell(u) = \pi_{\max(\ell, k)}(t)$  and  $\pi_\ell(u') = \pi_{\max(\ell, k)}(t')$ . It follows that  $\kappa(u, u') = \kappa$  and that the conditions for  $u < u'$  are met. In both cases we have  $\pi_k(t) \leq \pi_k(t')$ .

(iv) For  $u \in 2^k \mathbb{N}$ , it holds  $\lfloor 2^{-k}(u+v) \rfloor = u$  iff  $v \in \llbracket 2^k \rrbracket_0$ . It follows that for  $t, t' \in \mathbb{N}^d$ ,  $\pi_k(t) = \pi_k(t')$  iff  $t' \in C_{k, \pi_k(t)}$ . The claims follow from the definitions of  $\pi_k, \Pi_k^<$  and  $\Pi_k$ .

(v) For  $t, t' \in \mathbb{N}^d$ , if  $t' < \pi_k(t)$  then  $\pi_k(t') \leq t' < \pi_k(t)$ , from point (ii). Conversely, if  $t' \geq \pi_k(t)$ , then  $\pi_k(t') \geq \pi_k(\pi_k(t)) = \pi_k(t)$ , by (iii). Hence  $t' < \pi_k(t)$  iff  $\pi_k(t') < \pi_k(t)$ . This establishes (25). Concerning (26), we have seen above (see proof of (iv)) that  $\{t' \in \mathbb{N}^d : \pi_k(t) = \pi_k(t')\} = C_{k, \pi_k(t)}$ , therefore

$$\Pi_k(t) = \Pi_k^<(t) \cup \{t' \in \mathbb{N}^d : \pi_k(t) = \pi_k(t')\} = \Pi_k^<(t) \cup C_{k, \pi_k(t)}.$$

(vi) It holds  $\pi_{k-1}(t) \geq \pi_k(t)$  from (ii). Then from (25), we deduce the inclusion  $\Pi_k^<(t) \subseteq \Pi_{k-1}^<(t)$ . The inclusion  $\Pi_{k-1}^<(t) \subseteq \Pi_k(t)$  is immediate from the definitions (23), (24). Finally, for any  $t' \in \Pi_{k-1}(t)$ , by definition  $\pi_{k-1}(t') \leq \pi_{k-1}(t)$ , so by (iii) and  $\pi_k \circ \pi_{k-1} = \pi_k$ , it holds  $\pi_k(t') \leq \pi_k(t)$ , hence  $t' \in \Pi_k(t)$ , proving the last inclusion.  $\square$

**Remark 4.** The choice of the lexicographical order in the definition (22) is largely arbitrary; any total order on  $\mathbb{N}^d$  that is compatible with the coordinate-wise partial order would work, since it would result in the same properties as above, which are the only ones we will be using in the sequel.

For  $t \in \mathbb{N}^d$  and an integer  $k$ , define

$$\mathcal{F}_k^<(t) := \mathfrak{S}(X_{t'}, t' \in \Pi_k^<(t)), \quad \mathcal{F}_k(t) := \mathfrak{S}(X_{t'}, t' \in \Pi_k(t)), \quad (28)$$

where  $\Pi_k^<, \Pi_k$  are as defined in (23), (24) (and  $\mathfrak{S}(\emptyset)$  is the trivial  $\sigma$ -algebra).

For every element  $t \in \mathbb{N}^d$ , using the fact that  $\Pi_k^<(t) = \emptyset$  for  $k > \log_2 t$ , we write the decomposition

$$X_t - \mathbb{E}[X_t] = \left( X_t - \mathbb{E}[X_t | \mathcal{F}_0^<(t)] \right) + \sum_{k=1}^{\lfloor \log_2 \|t\|_\infty \rfloor + 1} \left( \mathbb{E}[X_t | \mathcal{F}_k^<(t)] - \mathbb{E}[X_t | \mathcal{F}_{k-1}^<(t)] \right).$$

For any finite subset  $A \subset \mathbb{N}^d$ , denoting  $\pi_k(A) = \{\pi_k(t), t \in A\} \subset 2^k \mathbb{N}^d$  and  $\|A\|_\infty = \max_{t \in A} \|t\|_\infty$ , we have  $A = \bigsqcup_{b \in \pi_k(A)} (A \cap C_{k, b})$ , hence:

$$S_A = \sum_{t \in A} (X_t - \mathbb{E}[X_t])$$



$$\begin{aligned}
&= \sum_{t \in A} \left( X_t - \mathbb{E}[X_t | \mathcal{F}_0^{\prec}(t)] \right) + \sum_{k=1}^{\lfloor \log_2 \|A\|_{\infty} \rfloor + 1} \sum_{b \in \pi_k(A)} \sum_{t \in C_{k,b} \cap A} \left( \mathbb{E}[X_t | \mathcal{F}_k^{\prec}(t)] - \mathbb{E}[X_t | \mathcal{F}_{k-1}^{\prec}(t)] \right) \\
&= \sum_{k=0}^{\lfloor \log_2 \|A\|_{\infty} \rfloor + 1} \sum_{b \in \pi_k(A)} Z_{b,k}(A), \tag{29}
\end{aligned}$$

where

$$Z_{t,0}(A) := X_t - \mathbb{E}[X_t | \mathcal{F}_0^{\prec}(t)]; \tag{30}$$

$$\text{and for } k \geq 1: Z_{b,k}(A) := \sum_{t \in C_{k,b} \cap A} \left( \mathbb{E}[X_t | \mathcal{F}_k^{\prec}(t)] - \mathbb{E}[X_t | \mathcal{F}_{k-1}^{\prec}(t)] \right). \tag{31}$$

**Lemma 6.** *Let  $A \subset \mathbb{N}^d$  be a finite set. Let  $k$  be a fixed integer. Then  $(Z_{b,k}(A), \mathcal{F}_k(b))_{b \in \pi_k(A)}$  is a martingale difference, where  $\pi_k(A)$  is ordered by the total order  $\leq$  defined by (22).*

**Proof** We start with the special case  $k = 0$ . In this case, since  $\pi_0(t) = t$ , we have  $\mathcal{F}_0^{\prec}(t) = \mathfrak{S}(X_{t'}, t' < t)$ , and  $\mathcal{F}_0(t) = \mathfrak{S}(X_{t'}, t' \leq t)$ . It is straightforward that  $Z_{0,t}(A)$  is  $\mathcal{F}_0(t)$ -measurable, and that for any  $t' < t$  we have  $\mathcal{F}_0(t') \subseteq \mathcal{F}_0^{\prec}(t)$  thus  $\mathbb{E}[Z_{0,t}(A) | \mathcal{F}_0(t')] = 0$ ; hence the claim. Let  $k \geq 1$  be a fixed integer. The claim for  $(Z_{b,k})$  relies on points (iv) and (vi) of Lemma 5, which straightforwardly imply for any  $t \in C_{k,b}$  that  $\mathcal{F}_k^{\prec}(b) = \mathcal{F}_k^{\prec}(t) \subseteq \mathcal{F}_{k-1}^{\prec}(t) \subseteq \mathcal{F}_k(t) = \mathcal{F}_k(b)$ . Thus,  $Z_{b,k}(A)$  is  $\mathcal{F}_k(b)$ -measurable, and for any  $b' < b$ , since  $\mathcal{F}_k(b') \subseteq \mathcal{F}_k^{\prec}(b)$ , it holds  $\mathbb{E}[Z_{b,k}(A) | \mathcal{F}_k(b')] = 0$ , implying the claim.  $\square$

As announced at the beginning of the proof, the role of excluding elements from the "frame" as constructed in Section 4.1 is to ensure that the remaining elements are sufficiently "separated" from the cell boundaries. We recall that  $\Lambda_{k,\delta} := 2^k \mathbb{N}_{>0} + \llbracket \delta \rrbracket_0$ . We need the following supporting result which estimates the distance from any element of the set  $\mathfrak{F}_{\delta}$  to the boundaries of the cells containing it.

**Lemma 7.** *For any  $k \in \mathbb{N}$ ,  $\delta \in \llbracket 2^k \rrbracket_0$  and  $t \in (\mathbb{N} \setminus \Lambda_{k,\delta})^d$ , it holds*

$$d_{\infty}(t, \Pi_k^{\prec}(t)) \geq \delta + 1.$$

**Proof** Point (i) from Lemma 5 implies that  $\leq$  is compatible with the partial coordinate-wise order  $\leq_{\text{cw}}$ . This implies in particular that any  $t'$  such that  $\pi_k(t) \leq_{\text{cw}} t'$  satisfies  $\pi_k(\pi_k(t)) = \pi_k(t) \leq \pi_k(t')$ , and thus cannot belong to  $\Pi_k^{\prec}(t)$ . Therefore, for any  $t' \in \Pi_k^{\prec}(t)$ , there exists a coordinate  $i$  such that  $t'_i < \pi_k(t_i)$ . In particular,  $\pi_k(t_i) > 0$ , hence  $\pi_k(t_i) \in 2^k \mathbb{N}_{>0}$ . On the other hand, if we assume  $t \in (\mathbb{N} \setminus \Lambda_{k,\delta})^d$  then  $t_i \in \mathbb{N} \setminus (2^k \mathbb{N}_{>0} + \llbracket \delta \rrbracket_0)$ . Since  $t'_i < \pi_k(t_i) \leq t_i$ , it must hold  $t_i - t'_i \geq t_i - \pi_k(t_i) + 1 \geq \delta + 1$ , implying the claim.  $\square$

We now have all ingredients to establish Proposition 4.

**Proof** [of Proposition 4] We use the decomposition (29) with  $A = \mathcal{R} \cap \mathfrak{F}_{\delta}$ , so that by the triangle inequality

$$\|S_{\mathcal{R} \cap \mathfrak{F}_{\delta}}\|_p \leq \sum_{k=0}^{m(\mathcal{R})+1} \left\| \sum_{b \in \pi_k(\mathcal{R} \cap \mathfrak{F}_{\delta})} Z_{b,k}(\mathcal{R} \cap \mathfrak{F}_{\delta}) \right\|_p, \tag{32}$$

where  $Z_{b,k}(\mathcal{R} \cap \mathfrak{F}_{\delta})$  is defined in (30), (31).

We now estimate the norm of the martingale increments  $Z_{b,k}(\mathcal{R} \cap \mathfrak{F}_\delta)$  using Assumption **WD**( $p$ ). We will denote below  $Z_{b,k} = Z_{b,k}(\mathcal{R} \cap \mathfrak{F}_\delta)$  and  $S_k = \pi_k(\mathcal{R} \cap \mathfrak{F}_\delta)$  to lighten notation. As a direct consequence of Lemma 7, for any  $t \in \mathfrak{F}_\delta$  it holds  $\mathcal{F}_k^{\prec}(t) \subset \mathcal{M}_{t, \delta_{k+1}}$  (as defined in Assumption **WD**( $p$ )). Therefore, using this property and Jensen's inequality for  $k = 0$  we get:

$$\begin{aligned} \|Z_{b,0}\|_p &= \|X_b - \mathbb{E}[X_b | \mathcal{F}_0^{\prec}(b)]\|_p \\ &\leq \|\mathbb{E}[X_b] - \mathbb{E}[X_b | \mathcal{F}_0^{\prec}(b)]\|_p + \|X_b - \mathbb{E}[X_b]\|_p \\ &\leq \|X_b - \mathbb{E}[X_b]\|_p + \|\mathbb{E}[X_b | \mathcal{M}_{b,1}] - \mathbb{E}[X_b]\|_p \\ &\leq M_p(\varphi_p(0) + \varphi_p(1)), \end{aligned} \quad (33)$$

while for  $k \geq 1$ :

$$\begin{aligned} \|Z_{b,k}\|_p &= \left\| \sum_{t \in C_{k,b} \cap \mathcal{R} \cap \mathfrak{F}_\delta} (\mathbb{E}[X_t | \mathcal{F}_k^{\prec}(t)] - \mathbb{E}[X_t | \mathcal{F}_{k-1}^{\prec}(t)]) \right\|_p \\ &\leq \sum_{t \in C_{k,b} \cap \mathcal{R} \cap \mathfrak{F}_\delta} \left( \|\mathbb{E}[X_t | \mathcal{F}_k^{\prec}(t)] - \mathbb{E}[X_t]\|_p + \|\mathbb{E}[X_t | \mathcal{F}_{k-1}^{\prec}(t)] - \mathbb{E}[X_t]\|_p \right) \\ &\leq \sum_{t \in C_{k,b} \cap \mathcal{R} \cap \mathfrak{F}_\delta} \left( \|\mathbb{E}[X_t | \mathcal{M}_{t, \delta_{k+1}}] - \mathbb{E}[X_t]\|_p + \|\mathbb{E}[X_t | \mathcal{M}_{t, \delta_{k-1+1}}] - \mathbb{E}[X_t]\|_p \right) \\ &\leq |C_{k,b} \cap \mathcal{R}| M_p(\varphi_p(\delta_k + 1) + \varphi_p(\delta_{k-1} + 1)). \end{aligned} \quad (34)$$

Note that we can subsume (33) into (34) by putting formally  $\delta_{-1} := -1$ . By Lemma 6, the sequence  $(Z_{b,k}, \mathcal{F}_k(b))_{b \in S_k}$  is a martingale difference sequence (over  $b$  for fixed  $k$ ), therefore for  $p \in [2, \infty)$  we apply the Marcinkiewicz-Zygmund inequality for martingales, with optimal constant obtained from Theorem 2.1 of Rio (2009) (see also Theorem 4.3 of Pinelis, 1994 for a related result under the assumption of  $\mathcal{F}_k(b)$ -conditionally symmetric distributed martingale difference sequence) which, together with the nonincreasing character of  $\varphi_p(\cdot)$  and nondecreasing character of  $(\delta_k)_{k \geq 1}$  implies for  $p \geq 2$ :

$$\begin{aligned} \left\| \sum_{b \in S_k} Z_{b,k} \right\|_p &\leq 2\sqrt{p} \left( \sum_{b \in S_k} \|Z_{b,k}\|_p^2 \right)^{\frac{1}{2}} \\ &\leq 2\sqrt{p} M_p(\varphi_p(\delta_k + 1) + \varphi_p(\delta_{k-1} + 1)) \left( \sum_{b \in \pi_k(\mathcal{R})} |C_{k,b} \cap \mathcal{R}|^2 \right)^{\frac{1}{2}} \\ &\leq 4\sqrt{p} M_p \varphi_p(\delta_{k-1} + 1) \left( \sum_{b \in \pi_k(\mathcal{R})} |C_{k,b} \cap \mathcal{R}|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (35)$$

We now concentrate on the estimate for  $\sum_{b \in \pi_k(\mathcal{R})} |C_{k,b} \cap \mathcal{R}|^2$ . Put  $q_i := \lfloor \frac{N_i}{2^k} \rfloor$  and  $r_i := N_i - q_i 2^k$ , for  $i \in \llbracket d \rrbracket$ . Observe that  $\pi_k(\mathcal{R}) = \prod_{i=1}^d (2^k \llbracket q_i + 1 \rrbracket_0)$ ; for  $b = (b_1, \dots, b_d) \in \pi_k(\mathcal{R})$ , the set  $C_{k,b} \cap \mathcal{R}$  is a hyperrectangle with side-lengths:

$$\ell_k(b_i, N_i) := \begin{cases} 2^k & \text{if } b_i < 2^k q_i; \\ r_i & \text{if } b_i = 2^k q_i. \end{cases}$$

Hence, it holds

$$\begin{aligned}
\sum_{b \in \pi_k(\mathcal{R})} |\mathcal{C}_{k,b} \cap \mathcal{R}|^2 &= \sum_{b \in \prod_{i=1}^d (2^k \llbracket q_i+1 \rrbracket_0)} \prod_{i=1}^d \ell_k(b_i, N_i)^2 = \prod_{i=1}^d \sum_{j=0}^{q_i} \ell_k(2^k j, N_i)^2 \\
&= \prod_{i=1}^d (q_i 2^{2k} + r_i^2) \\
&\leq \prod_{i=1}^d ((N_i - r_i) \min(2^k, N_i) + r_i \min(2^k, N_i)) \\
&= \prod_{i=1}^d (N_i \min(2^k, N_i)) \\
&= |R| |R \cap \mathcal{C}_{0,k}|. \tag{36}
\end{aligned}$$

The claimed estimate for  $2 \leq p < \infty$  follows by using (35) and (36) into (32) and straightforward computations. In the case of  $p = \infty$ , we can apply the bounded martingale difference inequality (Azuma, 1967) stating that the sum  $\sum_{b \in S_k} Z_{b,k}$  is sub-Gaussian such that  $\|\sum_{b \in S_k} Z_{b,k}\|_{\text{SG}} \leq (\|\sum_{b \in S_k} Z_{b,k}^2\|_{\infty})^{\frac{1}{2}}$  and using the triangle inequality for the sub-Gaussian norm  $\|\sum_{k=0}^{m(\mathcal{R})+1} \sum_{b \in \pi_k(\mathcal{R} \cap \mathfrak{F}_\delta)} Z_{b,k}(\mathcal{R} \cap \mathfrak{F}_\delta)\|_{\text{SG}}$  over scales  $k \in \llbracket m(\mathcal{R}) + 2 \rrbracket_0$ . All other arguments are as in the case  $p < \infty$ .  $\square$

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