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A *POSTERIORI* ERROR ESTIMATES FOR THE LARGE EDDY SIMULATION APPLIED TO INCOMPRESSIBLE FLUIDS

GHINA NASSREDDINE^{1,2}, PASCAL OMNES^{2,3}, AND TONI SAYAH¹

ABSTRACT. We study the two dimensional time dependent Large Eddy Simulation method applied to the incompressible Navier-Stokes system with Smagorinsky's eddy viscosity model and a filter width that depends on the local mesh size. The discrete model is based on the implicit Euler scheme and a conforming finite element method for the time and space discretizations, respectively. We establish a reliable and efficient *a posteriori* error estimation between the numerical LES solution and the exact solution of the original Navier-Stokes system, which involves three types of error indicators respectively related to the filter and to the discretizations in time and space. Numerical results show the effectiveness of adaptive simulations.

Keywords : Large Eddy Simulation, finite element method, *a posteriori* estimation.

1. INTRODUCTION.

Let Ω be a bounded connected open domain in \mathbb{R}^2 , with a Lipschitz-continuous connected boundary $\Gamma = \partial\Omega$. Let $[0, T]$ denote an interval in \mathbb{R} where T is a positive constant. For a positive constant viscosity ν , we consider the following time-dependent Navier-Stokes system, in which we choose a vanishing initial condition for simplification only:

$$(P) \left\{ \begin{array}{ll} \frac{\partial \mathbf{u}}{\partial t}(t, \mathbf{x}) - \nu \Delta \mathbf{u}(t, \mathbf{x}) + \mathbf{u}(t, \mathbf{x}) \cdot \nabla \mathbf{u}(t, \mathbf{x}) + \nabla p(t, \mathbf{x}) & = \mathbf{f}(t, \mathbf{x}) & \text{in }]0, T[\times \Omega, \\ \operatorname{div} \mathbf{u}(t, \mathbf{x}) & = 0 & \text{in }]0, T[\times \Omega, \\ \mathbf{u}(t, \mathbf{x}) & = \mathbf{0} & \text{on } [0, T] \times \Gamma, \\ \mathbf{u}(0, \mathbf{x}) & = \mathbf{0} & \text{on } \Omega, \end{array} \right.$$

where \mathbf{f} represents a density of body forces and supposed to be in $[L^2((0, T) \times \Omega)]^2$ (although a more general setting would be to consider $\mathbf{f} \in L^2(0, T; X') = L^2(0, T; H^{-1}(\Omega)^2)$, where X' is the dual of the Sobolev space $X = H_0^1(\Omega)^2$). The unknowns are the velocity \mathbf{u} and the pressure p of the fluid. Note that the hypothesis of vanishing initial condition is only here for the sake of simplicity and may easily be removed.

Since an exact solution of System (P) is in general out of reach, one may resort to a numerical approximation of it. A first method of simulation in fluid mechanics is the direct numerical simulation (DNS) of the flow at all significant length scales. Since it is necessary to capture all fluctuations of the velocity and pressure fields having an impact on the flow, DNS is very expensive and, for high Reynolds numbers, is even not achievable. Under these circumstances, one often prefers the large eddy simulation (LES) method, which consists in solving the large scales and in modelling the influence of small scales by adding a supplementary non-linear diffusion term in the momentum equation. In LES, large scales are defined by a spatial average of velocity, pressure and external forcing terms. A common method is to define this

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spatial average by convolution of these quantities with an appropriate filter function of width denoted by δ . The velocity field will be decomposed as:

$$\mathbf{u}(t, \mathbf{x}) = \bar{\mathbf{u}}(t, \mathbf{x}) + \mathbf{u}'(t, \mathbf{x}),$$

with $\bar{\mathbf{u}}(t, \mathbf{x})$ the filtered part of the velocity field and $\mathbf{u}'(t, \mathbf{x}) = \mathbf{u}(t, \mathbf{x}) - \bar{\mathbf{u}}(t, \mathbf{x})$, the residue whose effect on the large scale motions is the main issue in LES. Indeed, the convolution of the non-linear term $\bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}}$ is not equal to the non-linear term applied to the convolution $\bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}}$ and the difference between these two terms needs to be modeled. There is vast literature on this issue and we refer to monographs [27, 10, 21] for a mathematically oriented review. The Smagorinsky model is the simplest model of LES that uses the assumption of local balance between production and dissipation of turbulent kinetic energy to express the turbulent viscosity according to the large scales (see [21]). The resulting filtered problem is expressed as:

$$(\bar{P}) \left\{ \begin{array}{ll} \frac{\partial \bar{\mathbf{u}}}{\partial t} - \nu \Delta \bar{\mathbf{u}} - \nabla \cdot (\tilde{\nu} \mathbb{D}(\bar{\mathbf{u}})) + \bar{\mathbf{u}} \cdot \nabla (\bar{\mathbf{u}}) + \nabla \bar{p} & = \bar{\mathbf{f}} \quad \text{in }]0, T[\times \Omega, \\ \operatorname{div} \bar{\mathbf{u}} & = \mathbf{0} \quad \text{in } [0, T] \times \Omega, \\ \mathcal{B}(\bar{\mathbf{u}}) & = \mathbf{0} \quad \text{on } [0, T] \times \Gamma, \\ \bar{\mathbf{u}}(0, \cdot) & = \mathbf{0} \quad \text{on } \Omega, \end{array} \right.$$

where $\tilde{\nu} = (c_s \delta)^2 \|\mathbb{D}(\bar{\mathbf{u}})\|_F$, with $\mathbb{D}(\bar{\mathbf{u}}) = \frac{1}{2}(\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)$ and $\|A\|_F = (\sum_{i,j=1}^2 a_{ij}^2)^{1/2}$ for any matrix A. The

value of the constant c_s is an input of the model and a debated question among specialists (see, e.g., [10, p. 76]). Moreover, finding relevant boundary conditions for $\bar{\mathbf{u}}$ in bounded domains is a central issue in LES (see, e.g., [10, Part IV]); in (\bar{P}) , we have denoted by \mathcal{B} a general boundary operator that we do not discuss here. The boundary condition issue is also closely related to that of filtering with non constant width δ , which is commonly used in practice (the value of δ is typically adjusted to the local size of the mesh cells in numerical simulations) but introduces further problems since there is now a commutation error between the operations of filtering and differentiating.

In view of all these questions, our approach here is not to further refine LES models, but rather to consider a discretization of the Smagorinsky model with turbulent viscosity adjusted to the local size of the mesh cells and to derive *a posteriori* error estimates between its solution and the exact solution of (P) , which will allow us to locally refine or coarsen the mesh and/or the time steps according to local error indicators. Adaptive LES is an active field of research in the quest for accurate simulations with an affordable computational cost. Among works using mesh adaptation in the context of the Smagorinsky model for incompressible flows, we may cite for example [1, 3, 19, 26]. However, these works are either based on local error metrics driven by the Hessian of the numerical solution, a process that is independent of the model and numerical method, or only mention *a posteriori* estimates, without indicating their actual expression, nor giving a proof of their derivation, reliability and efficiency.

The present work provides a rigorous theory based on *a posteriori* error estimates for adaptivity in LES simulations of incompressible flows. For a general introduction to *a posteriori* error estimation, we refer for example to the books of Verfürth [30] or Ainsworth and Oden [2]. As far as time-dependent models are concerned, a large number of contributions may be found. To cite only a few of them, we can refer, for example, to Ladevèze [22] for constitutive relation error estimators for time-dependent non-linear FE analysis, Verfürth [31] for the heat equation, Bernardi and Verfürth [9] for the time dependent Stokes equations, Bernardi and Süli [8] for the time and space adaptivity for the second-order wave equation, Bergam, Bernardi and Mghazli [4] for some parabolic equations, Ern and Vohralik [16] for estimation based on potential and flux reconstruction for the heat equation, and Bernardi and Sayah [7] for the time dependent Stokes equations with mixed boundary conditions. In [25], Nassreddine and Sayah treated the time dependent Navier-Stokes equations in two dimensions. They use [6, 7] for the linear (Stokes) part of the problem and apply the continuous and discrete Gronwall lemma to treat the non-linear term. In this contribution, we extend the method of [25] to the LES problem, for which we shall obtain three types of computable error indicators; the first one is linked to the time discretization, the second one to the filter and the last one to the space discretization. This leads to an adaptive strategy in which we sometimes change the time step and sometimes adapt the mesh.

The outline of the paper is as follows: Section 2 is devoted to some functional analysis tools and to the variational formulation of the continuous Navier-Stokes system. In section 3, we introduce the discrete filtered problem and we recall its main properties. In sections 4 and 5 we study the *a posteriori* error indicators and derive quasi-optimal estimates, in the sense that we need to require higher regularity of the exact solution. In section 6, we present numerical results in which uniform and adaptive refinement strategies are applied and comparisons with a reference DNS simulation show the superiority of the LES adaptive strategy.

2. PRELIMINARIES

In this section, we begin by some notations and definitions which will be used later on and we recall the continuous and discrete Gronwall Lemmas.

We denote by $[L^p(\Omega)]^2$ the space of measurable functions \mathbf{v} such that $|\mathbf{v}|^p$ is integrable. For $\mathbf{v} \in [L^p(\Omega)]^2$, the norm is defined by

$$\|\mathbf{v}\|_{[L^p(\Omega)]^2} = \left(\int_{\Omega} |\mathbf{v}(\mathbf{x})|^p d\mathbf{x} \right)^{\frac{1}{p}}.$$

Let $\alpha = (\alpha_1, \alpha_2)$ be a couple of non negative integers and $|\alpha| = \alpha_1 + \alpha_2$. We define the partial derivative ∂^α by

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}.$$

Then, for any positive integer m and number $p \geq 1$, we recall the classical Sobolev space

$$[W^{m,p}(\Omega)]^2 = \{\mathbf{v} \in [L^p(\Omega)]^2, \partial^\alpha \mathbf{v} \in [L^p(\Omega)]^2, \quad \forall |\alpha| \leq m\}$$

equipped with the following semi-norm and norm :

$$|\mathbf{v}|_{m,p,\Omega} = \left\{ \sum_{|\alpha|=m} \int_{\Omega} |\partial^\alpha \mathbf{v}(\mathbf{x})|^p d\mathbf{x} \right\}^{1/p} \quad \text{and} \quad \|\mathbf{v}\|_{m,p,\Omega} = \left\{ \sum_{k \leq m} |\mathbf{v}|_{k,p,\Omega}^p \right\}^{1/p}.$$

When $p = 2$, this space is the Hilbert space $[H^m(\Omega)]^2$. In view of the boundary conditions in system (P), we thus consider the space

$$X := H_0^1(\Omega)^2 = \{\mathbf{v} \in H^1(\Omega)^2, \mathbf{v} = 0 \text{ on } \Gamma\}.$$

We denote by M the space of functions in $L^2(\Omega)$ with a zero mean-value on Ω .

Lemma 2.1. *For any $1 \leq p < +\infty$, there exists a positive constant S_p only depending on Ω such that*

$$\forall \mathbf{v} \in X, \quad \|\mathbf{v}\|_{L^p(\Omega)^d} \leq S_p \|\mathbf{v}\|_X. \quad (1)$$

Lemma 2.2. *(See [29, page 291]) We have the following inequality for every $\mathbf{v} \in X$*

$$\|\mathbf{v}\|_{L^4(\Omega)^2} \leq 2^{1/4} \|\mathbf{v}\|_{L^2(\Omega)^2}^{1/2} \|\mathbf{v}\|_X^{1/2}. \quad (2)$$

Remark 2.3. *Restriction of the analysis to dimension 2 is mainly due to Lemma 2.2 which is valid in dimension 2 only. A related result exists in dimension 3 (see, e.g. [29, Lemma 3.5, page 295]) but it states that $\|\mathbf{v}\|_{L^4(\Omega)^3} \leq 2^{1/2} \|\mathbf{v}\|_{L^2(\Omega)^3}^{1/4} \|\nabla \mathbf{v}\|_{L^2(\Omega)^{3 \times 3}}^{3/4}$, which then prevents from pursuing like in dimension 2.*

We introduce the kernel of the divergence operator

$$V = \{\mathbf{v} \in X; \forall q \in M, \int_{\Omega} q(\mathbf{x}) \operatorname{div} \mathbf{v}(\mathbf{x}) d\mathbf{x} = 0\},$$

which is a closed subspace of X and coincides with

$$V = \{\mathbf{v} \in X; \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\}.$$

Remark 2.4. *The spaces M and X satisfy a uniform inf-sup condition (see [18]): There exists a constant $\beta_* > 0$ such that*

$$\forall q \in M, \quad \sup_{\mathbf{v} \in X} \frac{\int_{\Omega} q(\mathbf{x}) \operatorname{div} \mathbf{v}(\mathbf{x}) d\mathbf{x}}{\|\mathbf{v}\|_X} \geq \beta_* \|q\|_{L^2(\Omega)}.$$

As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval $]a, b[$ with values in a separable functional space, say Y . More precisely, let $\|\cdot\|_Y$ denote the norm of Y ; then for any r , $1 \leq r \leq \infty$, we define

$$L^r(a, b; Y) = \left\{ \mathbf{f} \text{ measurable in }]a, b[; \int_a^b |\mathbf{f}(t)|_Y^r dt < \infty \right\},$$

equipped with the norm

$$\|\mathbf{f}\|_{L^r(a, b; Y)} = \left(\int_a^b |\mathbf{f}(t)|_Y^r dt \right)^{1/r}.$$

If $r = \infty$, then

$$L^\infty(a, b; Y) = \left\{ \mathbf{f} \text{ measurable in }]a, b[; \sup_{t \in [a, b]} \|\mathbf{f}(t)\|_Y < \infty \right\}$$

equipped with the norm

$$\|\mathbf{f}\|_{L^\infty(a, b; Y)} = \sup_{t \in [a, b]} \|\mathbf{f}(t)\|_Y.$$

Definition 2.5. We introduce the trilinear form c defined by :

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{v}) \cdot \mathbf{w} dx.$$

Lemma 2.6. For every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in X$ we have

$$|c(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \|\mathbf{u}\|_{L^4(\Omega)^2} \|\mathbf{v}\|_X \|\mathbf{w}\|_{L^4(\Omega)^2} \leq S_4^2 \|\mathbf{u}\|_X \|\mathbf{v}\|_X \|\mathbf{w}\|_X.$$

Lemma 2.7. We assume that $\mathbf{u}, \mathbf{v} \in X$ and $\operatorname{div} \mathbf{u} = 0$, then $c(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0$.

As far as Problem (P) is concerned, we recall classical existence and uniqueness results that may be found, for example in [11, Th. V.1.4] and [29, Th. 3.1 and Th. 3.2] which require the definition $H := \{\mathbf{v} \in [L^2(\Omega)]^d, \operatorname{div} \mathbf{v} = 0, \gamma_n \mathbf{v} = 0\}$, where $\gamma_n \mathbf{v}$ is the normal trace operator on Γ .

Theorem 2.8. In dimension 2, with $\mathbf{f} \in [L^2((0, T) \times \Omega)]^2$, Problem (P) has a unique solution (\mathbf{u}, p) belonging to $L^\infty(0, T; H) \cap L^2(0, T; V) \times W^{-1, \infty}(0, T; M)$. Moreover, $\frac{\partial \mathbf{u}}{\partial t} \in L^2(0, T; V')$ and $\mathbf{u} \in C^0(0, T; H)$. Moreover, it holds that

$$\|\mathbf{u}\|_{L^\infty(0, T; L^2(\Omega)^2)}^2 + \nu \|\mathbf{u}\|_{L^2(0, T; X)}^2 \leq C \|\mathbf{f}\|_{[L^2((0, T) \times \Omega)]^2}^2, \quad (3)$$

with C a constant depending on Ω and ν . Moreover, the spaces to which (\mathbf{u}, p) belong allow to write that

$$(FV) \begin{cases} \forall \mathbf{v} \in X, \quad \left\langle \frac{\partial \mathbf{u}}{\partial t}(t), \mathbf{v} \right\rangle + \nu (\nabla \mathbf{u}(t), \nabla \mathbf{v}) + c(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) - (\operatorname{div} \mathbf{v}, p(t)) = (\mathbf{f}(t), \mathbf{v}) \\ \forall q \in M, \quad (\operatorname{div} \mathbf{u}(t), q) = 0 \\ \mathbf{u}(0) = \mathbf{0}. \end{cases}$$

In the next lemmas, we introduce the generalized and discrete Gronwall lemmas.

Lemma 2.9. (Generalized Gronwall Lemma) [32, p. 292] and [15, p. 252]. Let:

- (1) f, g and k be an integrable functions defined from $\mathbb{R}^+ \mapsto \mathbb{R}$,
- (2) $g \geq 0, k \geq 0$,
- (3) $g \in \mathcal{L}_\infty$,
- (4) gk is an integrable function on \mathbb{R}^+ .

If $z : \mathbb{R}^+ \mapsto \mathbb{R}$ satisfies

$$z(t) \leq f(t) + g(t) \int_0^t k(\tau) z(\tau) d\tau, \quad \forall t \in \mathbb{R}^+ \quad (4)$$

then:

$$z(t) \leq f(t) + g(t) \int_0^t k(\tau) f(\tau) \exp\left(\int_\tau^t k(s) g(s) ds\right) d\tau. \quad (5)$$

Lemma 2.10. (*Discrete Gronwall Lemma*) [32, p. 294] let $(y_n)_n$, $(f_n)_n$ and $(g_n)_n$ be three positive sequences that verify:

$$\forall n \geq 0, \quad y_n \leq f_n + \sum_{k=0}^{n-1} g_k y_k.$$

Then, we have:

$$\forall n \geq 0, \quad y_n \leq f_n + \sum_{k=0}^{n-1} f_k g_k \exp\left(\sum_{j=k}^{n-1} g_j\right). \quad (6)$$

Finally, we shall make an extensive use of the following inequalities, the first of which is Young's inequality with ε and is used three times to obtain the second:

Lemma 2.11. For any $(a, b, c, d) \in \mathbb{R}_+^4$ and any $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathbb{R}_+^3$, it holds that

$$ab \leq \frac{1}{2}\left(\varepsilon_1 a^2 + \frac{1}{\varepsilon_1} b^2\right) \quad \text{and} \quad abcd \leq \frac{1}{4}\left(\varepsilon_1 \varepsilon_2 a^4 + \frac{\varepsilon_1}{\varepsilon_2} b^4 + \frac{\varepsilon_3}{\varepsilon_1} c^4 + \frac{1}{\varepsilon_1 \varepsilon_3} d^4\right).$$

In particular, we shall use these inequalities as follows: For any $(a, b, c, d) \in \mathbb{R}_+^4$, and any $(\beta, \gamma, \delta) \in \mathbb{R}_+^3$, there exist $(\alpha_1, \alpha_2) \in \mathbb{R}_+^{*2}$ such that

$$ab \leq \alpha_1 a^2 + \beta b^2 \quad \text{and} \quad abcd \leq \alpha_2 a^4 + \beta b^4 + \gamma c^4 + \delta d^4.$$

3. THE DISCRETE PROBLEM

From now on, we assume that Ω is a polyhedron. In order to describe the time discretization with an adaptive choice of local time steps, we introduce a partition of the interval $[0, T]$ into subintervals $[t_{n-1}, t_n]$, $1 \leq n \leq N$, such that $0 = t_0 < t_1 < \dots < t_N = T$. We denote by τ_n the length of $[t_{n-1}, t_n]$, by τ the N -tuple (τ_1, \dots, τ_N) , by $|\tau|$ the maximum of the τ_n , $1 \leq n \leq N$, and finally by σ_τ the regularity parameter

$$\sigma_\tau = \max_{2 \leq n \leq N} \frac{\tau_n}{\tau_{n-1}}. \quad (7)$$

In what follows, we work with a regular family of partitions, i.e. we assume that σ_τ is bounded independently of τ . This has the practical implication that the time step should not be modified too abruptly in the adaptive algorithm described in Section 6.3.

We introduce the operator π_τ (resp. $\pi_{l,\tau}$): For any Banach space Y and any function g continuous from $]0, T[$ (resp. $[0, T[$) into Y , $\pi_\tau g$ (resp. $\pi_{l,\tau} g$) denotes the step function which is constant and equal to $g(t_n)$ (resp. $g(t_{n-1})$) on each interval $]t_{n-1}, t_n]$ (resp. $[t_{n-1}, t_n[$), $1 \leq n \leq N$. Similarly, with any sequence $(\phi_n)_{0 \leq n \leq N}$ in Y , we associate the step function $\pi_\tau \phi_\tau$ (resp. $\pi_{l,\tau} \phi_\tau$) which is constant and equal to ϕ_n (resp. ϕ_{n-1}) on each interval $]t_{n-1}, t_n]$ (resp. $[t_{n-1}, t_n[$), $1 \leq n \leq N$.

We now describe the space discretization. For each n , $0 \leq n \leq N$, we consider $(\mathcal{T}_{nh})_h$, a partition of Ω by triangles that belongs to a regular family of triangulations in the usual sense:

- $\bar{\Omega}$ is the union of all elements of \mathcal{T}_{nh} ;
- the intersection of two different elements of \mathcal{T}_{nh} , if not empty, is a vertex or a whole edge of both of them; this, in particular, implies that meshes remain conforming (no hanging nodes) during the adaptive refinement process.
- the ratio of the diameter h_κ of an element κ in \mathcal{T}_{nh} to the diameter of its inscribed circle is bounded by a constant independent of n and h .

As usual, h denotes the maximal diameter of the elements of all \mathcal{T}_{nh} , $0 \leq n \leq N$, while for each n , h_n denotes the maximal diameter of the elements of \mathcal{T}_{nh} . For each κ in \mathcal{T}_{nh} and each non-negative integer k , we denote by $P_k(\kappa)$ the space of restrictions to κ of polynomials of 2 variables and total degree at most k .

In what follows, $c, c', C \dots$ stand for generic constants which may vary from line to line but are always independent of h and n . From now on, we call finite element space associated to \mathcal{T}_{nh} a space of functions such that their restrictions to any element κ of \mathcal{T}_{nh} belong to a space of polynomials of fixed degree.

For each n and h , we associate with \mathcal{T}_{nh} two finite element spaces X_{nh} and M_{nh} which are contained in X and M , respectively, and such that the following inf-sup condition holds for a constant $\beta > 0$, which is independent of n and h ,

$$\forall q_h \in M_{nh}, \quad \sup_{\mathbf{v}_h \in X_{nh}} \frac{\int_{\Omega} q_h(\mathbf{x}) \operatorname{div} \mathbf{v}_h(\mathbf{x}) \, d\mathbf{x}}{\|\mathbf{v}_h\|_X} \geq \beta \|q_h\|_{L^2(\Omega)}. \quad (8)$$

There exist many examples of finite element spaces satisfying this condition. We give an example involving continuous discrete pressures. Velocities are discretized with the "Mini-Element"

$$X_{nh} = \{\mathbf{v}_h \in X; \forall \kappa \in \mathcal{T}_{nh}, \mathbf{v}_h|_{\kappa} \in P_b(\kappa)^2\},$$

where the space $P_b(\kappa)$ is spanned by functions in $P_1(\kappa)$ and the bubble function on κ (for each element κ , the bubble function is equal to the product of the barycentric coordinates associated with the vertices of κ). The pressure is discretized with classical continuous finite elements of order one

$$M_{nh} = \{q_h \in M \cap H^1(\Omega); \forall \kappa \in \mathcal{T}_{nh}, q_h|_{\kappa} \in P_1(\kappa)\}.$$

As usual, we denote by V_{nh} the kernel of the divergence

$$V_{nh} = \{\mathbf{v}_h \in X_{nh}; \forall q_h \in M_{nh}, \int_{\Omega} q_h(\mathbf{x}) \operatorname{div} \mathbf{v}_h(\mathbf{x}) \, d\mathbf{x} = 0\}.$$

As discussed in the Introduction, the fully discrete problem which we consider is a variational version of (\bar{P}) with a local choice of the filter size, in which we use homogeneous Dirichlet boundary conditions for the velocity:

$$(\overline{FV})_{n,h} \left\{ \begin{array}{l} \text{Having } \bar{\mathbf{u}}_h^{n-1} \in X_{(n-1)h}, \text{ find } (\bar{\mathbf{u}}_h^n, \bar{p}_h^n) \in X_{nh} \times M_{nh} \text{ solution of :} \\ \forall \mathbf{v}_h \in X_{nh}, \quad \frac{1}{\tau_n} (\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}_h^{n-1}, \mathbf{v}_h) + \nu (\nabla \bar{\mathbf{u}}_h^n, \nabla \mathbf{v}_h) + (\nu_t(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n), \mathbb{D}(\mathbf{v}_h)) \\ \quad + d(\bar{\mathbf{u}}_h^{n-1}, \bar{\mathbf{u}}_h^n, \mathbf{v}_h) - (\bar{p}_h^n, \operatorname{div} \mathbf{v}_h) = (\bar{\mathbf{f}}^n, \mathbf{v}_h), \\ \forall q_h \in M_{nh}, \quad (\operatorname{div} \bar{\mathbf{u}}_h^n, q_h) = 0, \end{array} \right.$$

where the turbulent viscosity is locally defined on each mesh cell κ by

$$\nu_t(\bar{\mathbf{v}})|_{\kappa} = (c_s h_{\kappa})^2 \|\mathbb{D}(\bar{\mathbf{v}})\|_F$$

and where the trilinear form d on X^3 is defined by

$$d(\mathbf{u}, \mathbf{v}, \mathbf{w}) = c(\mathbf{u}, \mathbf{v}, \mathbf{w}) + \frac{1}{2} \int_{\Omega} (\operatorname{div} \mathbf{u}) \mathbf{v} \cdot \mathbf{w} \, d\mathbf{x}.$$

We initialize the process by choosing $\bar{\mathbf{u}}_h^0 = 0$. We also choose $\bar{\mathbf{f}}^n(\mathbf{x}) = \frac{1}{\tau_n} \int_{t^{n-1}}^{t^n} \bar{\mathbf{f}}(t, \mathbf{x}) \, dt$ for almost all $\mathbf{x} \in \Omega$, where the source term $\bar{\mathbf{f}}$ results from a local average of the original source term \mathbf{f} ; moreover we suppose that $\bar{\mathbf{f}} \in [L^2((0, T) \times \Omega)]^2$.

Note that this problem is linear, since we discretize all non-linear terms semi-explicitly.

We begin by showing a bound for the solution $\bar{\mathbf{u}}_h^n$ of Problem $(\overline{FV})_{n,h}$.

Remark 3.1. We have: $d(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \forall (\mathbf{u}, \mathbf{v}) \in X^2$.

Theorem 3.2. At each time step, knowing $\bar{\mathbf{u}}_h^{n-1} \in X_{(n-1)h}$, Problem $(\overline{FV})_{n,h}$ admits a unique solution $(\bar{\mathbf{u}}_h^n, \bar{p}_h^n)$ with values in $X_{nh} \times M_{nh}$. This solution satisfies, for $m = 1, \dots, N$,

$$\|\bar{\mathbf{u}}_h^m\|_{L^2(\Omega)}^2 + \nu \sum_{n=1}^m \tau_n \|\bar{\mathbf{u}}_h^n\|_X^2 + 2 \sum_{n=1}^m \tau_n (\nu_t(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n), \mathbb{D}(\bar{\mathbf{u}}_h^n)) \leq C \|\bar{\mathbf{f}}\|_{[L^2((0, T) \times \Omega)]^2}^2. \quad (9)$$

Proof. For $\bar{\mathbf{u}}_h^{n-1} \in X_{(n-1)h}$, it is clear that Problem $(\overline{FV})_{n,h}$ has a unique solution $(\bar{\mathbf{u}}_h^n, \bar{p}_h^n)$ as a consequence of the coercivity of the corresponding bilinear form on $X_{nh} \times X_{nh}$ and the inf-sup condition (8). The proof of (9) follows by testing $(\overline{FV})_{n,h}$ with $\bar{\mathbf{u}}_h^n$ and summing the resulting inequalities. \square

4. A POSTERIORI ERROR ANALYSIS

We now intend to prove *a posteriori* error estimates between the exact solution (\mathbf{u}, p) of Problem (FV) and the numerical solution of Problem $(\overline{FV}_{n,h})$. The main result in this respect is given by Corollary 4.16. Several steps are needed to obtain this result. We begin by constructing the *a posteriori* indicators in Section 4.1 and then bound the error by the estimates in Section 4.2.

4.1. Construction of the error indicators. We first introduce the space

$$Z_{nh} = \{\mathbf{g}_h \in L^2(\Omega)^2; \forall \kappa \in \mathcal{T}_{nh}, \mathbf{g}_h|_{\kappa} \in P_1(\kappa)^2\},$$

and, for $1 \leq n \leq N$, we fix an approximation $\bar{\mathbf{f}}_h^n$ of the data $\bar{\mathbf{f}}^n$ in Z_{nh} . This is a technicality which is needed in Section 5 in which we shall make use of Properties 4.1 and 4.2 below, which are valid only for polynomial functions.

Next, for every element κ in \mathcal{T}_{nh} , we denote by

- ε_{κ} the set of edges of κ that are not contained in $\partial\Omega$,
- Δ_{κ} the union of elements of \mathcal{T}_{nh} that share at least one common vertex with κ ,
- h_e the diameter of the edge $e \in \varepsilon_{\kappa}$,
- $[\cdot]_e$ the jump through e for each edge e in ε_{κ} (specifying its sign is not necessary).
- \mathbf{n}_{κ} the unit outward normal vector to κ on $\partial\kappa$.

For the proofs of the next theorems, we introduce for an element κ of \mathcal{T}_{nh} , the bubble function ψ_{κ} (resp. ψ_e for the edge e) which is equal to the product of the 3 barycentric coordinates associated with the vertices of κ (resp. of the 2 barycentric coordinates associated with the vertices of e). We also consider a lifting operator \mathcal{L}_e defined on polynomials on e that vanish on ∂e into polynomials on the at most two elements (κ, κ') containing e and vanishing on $\partial\kappa \setminus e$ and $\partial\kappa' \setminus e$. This lifting operator is constructed by affine transformation from a fixed operator on the reference element. We recall the next results from [30, Lemma 3.3].

Property 4.1. Denoting by $P_r(\kappa)$ the space of polynomials of degree smaller than r on κ , we have

$$\forall v \in P_r(\kappa), \quad \begin{cases} c\|v\|_{L^2(\kappa)} \leq \|v\psi_{\kappa}^{1/2}\|_{L^2(\kappa)} \leq c'\|v\|_{L^2(\kappa)}, \\ |v|_{H^1(\kappa)} \leq ch_{\kappa}^{-1}\|v\|_{L^2(\kappa)}. \end{cases}$$

Property 4.2. Denoting by $P_r(e)$ the space of polynomials of degree smaller than r on e , we have

$$\forall v \in P_r(e), \quad c\|v\|_{L^2(e)} \leq \|v\psi_e^{1/2}\|_{L^2(e)} \leq c'\|v\|_{L^2(e)},$$

and, for all polynomials v in $P_r(e)$ vanishing on ∂e , if κ is an element which contains e ,

$$\|\mathcal{L}_e v\|_{L^2(\kappa)} + h_e |\mathcal{L}_e v|_{H^1(\kappa)} \leq ch_e^{1/2}\|v\|_{L^2(e)}.$$

We also introduce a Clément type regularization operator \mathcal{C}_{nh} [13] which has the following properties, see [5, section IX.3]: For any function \mathbf{w} in $H^1(\Omega)^2$, $\mathcal{C}_{nh}\mathbf{w}$ belongs to the continuous affine finite element space and satisfies for any κ in \mathcal{T}_{nh} and e in ε_{κ} ,

$$\|\mathbf{w} - \mathcal{C}_{nh}\mathbf{w}\|_{L^2(\kappa)^2} \leq ch_{\kappa}\|\mathbf{w}\|_{H^1(\Delta_{\kappa})^2} \quad \text{and} \quad \|\mathbf{w} - \mathcal{C}_{nh}\mathbf{w}\|_{L^2(e)^2} \leq ch_e^{1/2}\|\mathbf{w}\|_{H^1(\Delta_{\kappa})^2}. \quad (10)$$

Note that we use the variant of \mathcal{C}_{nh} which ensures that $\mathcal{C}_{nh}\mathbf{w}$ belongs to $H_0^1(\Omega)^d$ when \mathbf{w} belongs to $H_0^1(\Omega)^d$ (see [13]). Furthermore, we introduce the Scott-Zhang operator \mathcal{F}_{nh} [12] which has the following properties : For any function $\mathbf{v} \in H^s(\Omega)^2$, we have

$$\forall h_n \leq 1, \forall \mathbf{v} \in H^s(\Omega)^2, \|\mathbf{v} - \mathcal{F}_{nh}\mathbf{v}\|_{H^t(\Omega)^2} \leq Ch_n^{s-t}|\mathbf{v}|_{H^s(\Omega)^2}, \quad (11)$$

where $s \in]\frac{1}{2}, 1[$ and $t \in [0, s]$ (see [12]). For the *a posteriori* error studies, we consider the piecewise affine function $\bar{\mathbf{u}}_h$ which takes in the interval $[t_{n-1}, t_n]$ the values

$$\bar{\mathbf{u}}_h(t) = \frac{t - t_{n-1}}{\tau_n}(\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}_h^{n-1}) + \bar{\mathbf{u}}_h^{n-1}, \quad (12)$$

and the piecewise constant function \bar{p}_h equal to \bar{p}_h^n on the interval $[t_{n-1}, t_n]$.

Property 4.3. For any dimension d and for any non negative integer r , there exists a constant \tilde{c} such that for any polynomial function \mathbf{v}_h of degree r on κ

$$\|\mathbf{v}_h\|_{L^\infty(\kappa)} \leq \tilde{c}h_\kappa^{-d/2} \|\mathbf{v}_h\|_{L^2(\kappa)}. \quad (13)$$

We prove optimal *a posteriori* error estimates by using the norm:

$$\begin{aligned} [[\mathbf{u} - \bar{\mathbf{u}}_h]](t_n) &= \left(\|\mathbf{u}(t_n) - \bar{\mathbf{u}}_h(t_n)\|_{L^2(\Omega)}^2 \right. \\ &\quad \left. + \nu \max \left(\int_0^{t_n} \|\mathbf{u}(t) - \bar{\mathbf{u}}_h(t)\|_X^2 dt, \sum_{m=1}^n \int_{t_{m-1}}^{t_m} \|\mathbf{u}(t) - \pi_\tau \bar{\mathbf{u}}_h(t)\|_X^2 dt \right) \right)^{1/2}. \end{aligned} \quad (14)$$

To prove the upper bound, we follow the idea used by Bernardi and Verfurth [9] or Bernardi and Sayah [7] for the Stokes problem in order to uncouple time and space errors. But in this work, the non linear term coming from the Navier-Stokes system requires more sophisticated calculations.

We introduce an auxiliary problem corresponding to the time discretization and calculate upper bounds for the errors between the corresponding solution and the exact solution firstly and the discrete filtered solution of $(FV)_{n,h}$ secondly. Finally, we combine the obtained errors to derive the desired upper bound for the *a posteriori* error estimation.

Setting $\mathbf{u}^0 = \mathbf{0}$ and setting $\mathbf{f}^n(\mathbf{x}) = \frac{1}{\tau_n} \int_{t^{n-1}}^{t^n} \mathbf{f}(t, \mathbf{x}) dt$ for almost all $\mathbf{x} \in \Omega$, we introduce the following time semi-discrete problem:

$$(P_{aux}) \begin{cases} \text{Knowing } \mathbf{u}^{n-1} \in X, \text{ find } (\mathbf{u}^n, p^n) \in X \times M \text{ solution of} \\ \forall \mathbf{v} \in X, \quad \frac{1}{\tau_n} (\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{v}) + \nu (\nabla \mathbf{u}^n, \nabla \mathbf{v}) + (\mathbf{u}^{n-1} \nabla \mathbf{u}^n, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p^n) = (\mathbf{f}^n, \mathbf{v}), \\ \forall q \in M, \quad (\operatorname{div} \mathbf{u}^n, q) = 0. \end{cases}$$

Lemma 4.4. Problem (P_{aux}) has a unique solution because the bilinear form is coercive (owing to the fact that $\operatorname{div} \mathbf{u}^{n-1} = 0$) and because of the inf-sup condition. Furthermore, we have:

$$\|\mathbf{u}^m\|_{L^2(\Omega)}^2 + \nu \sum_{n=1}^m \tau_n \|\mathbf{u}^n\|_X^2 \leq C \|\mathbf{f}\|_{[L^2((0,T) \times \Omega)]}^2. \quad (15)$$

We define the piecewise affine function \mathbf{u}_τ by its value on the interval $[t_{n-1}, t_n]$:

$$\mathbf{u}_\tau(t) = \frac{t - t_{n-1}}{\tau_n} (\mathbf{u}^n - \mathbf{u}^{n-1}) + \mathbf{u}^{n-1} = -\frac{t_n - t}{\tau_n} (\mathbf{u}^n - \mathbf{u}^{n-1}) + \mathbf{u}^n, \quad (16)$$

and we define p_τ as the piecewise constant function equal to p^n on the interval $]t_{n-1}, t_n]$. An easy calculation leads to the following lemma.

Lemma 4.5. By combining Problems (FV) and (P_{aux}) , we observe that the pair $(\mathbf{u} - \mathbf{u}_\tau, p - p_\tau)$ satisfies $(\mathbf{u} - \mathbf{u}_\tau)(0) = \mathbf{0}$, and, for $t \in]t_{n-1}, t_n]$, $1 \leq n \leq N$ and for $(\mathbf{v}(t), q) \in X \times M$,

$$\begin{cases} \left\langle \frac{\partial}{\partial t} (\mathbf{u} - \mathbf{u}_\tau)(t), \mathbf{v}(t) \right\rangle + \nu (\nabla (\mathbf{u}(t) - \mathbf{u}_\tau(t)), \nabla \mathbf{v}(t)) + \int_\Omega [\mathbf{u}(t) \cdot \nabla \mathbf{u}(t) - \mathbf{u}_\tau(t) \cdot \nabla \mathbf{u}_\tau(t)] \cdot \mathbf{v}(t) dx \\ \quad - (\operatorname{div} \mathbf{v}(t), p(t) - p_\tau(t)) = (\mathbf{f}(t) - \mathbf{f}^n(t), \mathbf{v}(t)) + \langle R^\tau(\mathbf{u}_\tau)(t), \mathbf{v} \rangle. \\ \int_\Omega q(t, \mathbf{x}) \operatorname{div}(\mathbf{u}(t, \mathbf{x}) - \mathbf{u}_\tau(t, \mathbf{x})) dx = 0. \end{cases} \quad (17)$$

where

$$\langle R^\tau(\mathbf{u}_\tau)(t), \mathbf{v} \rangle = -\nu (\nabla (\mathbf{u}_\tau - \pi_\tau \mathbf{u}_\tau)(t), \nabla \mathbf{v}) - \int_\Omega [\mathbf{u}_\tau(t) \cdot \nabla \mathbf{u}_\tau(t) - \pi_{l,\tau} \mathbf{u}_\tau(t) \cdot \nabla \pi_\tau \mathbf{u}_\tau(t)] \cdot \mathbf{v} dx. \quad (18)$$

Lemma 4.6. *By combining Problems (P_{aux}) and ($\overline{FV}_{n,h}$), we observe for $\mathbf{v} \in X$ and $\mathbf{v}_h \in X_{nh}$:*

$$\begin{aligned} & \frac{1}{\tau_n} ((\mathbf{u}^n - \mathbf{u}^{n-1}) - (\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}_h^{n-1}), \mathbf{v}) + \nu (\nabla(\mathbf{u}^n - \bar{\mathbf{u}}_h^n), \nabla \mathbf{v}) + \int_{\Omega} [\mathbf{u}^{n-1} \cdot \nabla \mathbf{u}^n - \bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n] \cdot \mathbf{v} \, d\mathbf{x} \\ & - \frac{1}{2} (\operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p^n - \bar{p}_h^n) \\ & = \langle \bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n + R^{h,1}(\bar{\mathbf{u}}_h)(t), \mathbf{v} - \mathbf{v}_h \rangle + \langle R^{h,2}(\bar{\mathbf{u}}_h)(t), \mathbf{v} \rangle + (\mathbf{f}^n - \bar{\mathbf{f}}^n, \mathbf{v}) \end{aligned} \quad (19)$$

and

$$\int_{\Omega} q(t, \mathbf{x}) \operatorname{div}(\mathbf{u}^n - \bar{\mathbf{u}}_h^n)(\mathbf{x}) \, d\mathbf{x} = - \int_{\Omega} q(t, \mathbf{x}) \operatorname{div}(\bar{\mathbf{u}}_h^n)(\mathbf{x}) \, d\mathbf{x}, \quad (20)$$

where the residuals $R^{h,1}(\bar{\mathbf{u}}_h)$ and $R^{h,2}(\bar{\mathbf{u}}_h)$ are defined by:

$$\begin{aligned} \langle R^{h,1}(\bar{\mathbf{u}}_h)(t), \mathbf{w} \rangle &= (\bar{\mathbf{f}}_h^n - \frac{1}{\tau_n}(\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}_h^{n-1}), \mathbf{w}) + (\operatorname{div} \mathbf{w}, \bar{p}_h^n) - \int_{\Omega} (\bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n) \cdot \mathbf{w} \, d\mathbf{x} \\ &- \frac{1}{2} (\operatorname{div} \bar{\mathbf{u}}_h^{n-1}(t) \bar{\mathbf{u}}_h^n(t), \mathbf{w}) - \nu (\nabla \bar{\mathbf{u}}_h^n, \nabla \mathbf{w}) - (\nu_{th}(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n), \mathbb{D}(\mathbf{w})) \\ &- ((\nu_t(\bar{\mathbf{u}}_h^{n-1}) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1})) \mathbb{D}(\bar{\mathbf{u}}_h^n), \mathbb{D}(\mathbf{w})), \end{aligned} \quad (21)$$

where ν_{th} is a piecewise constant approximation of ν_t (a technicality needed in Section 5 in which we shall use Properties 4.1 and 4.2 valid for polynomial functions) defined on each κ and for every $\bar{\mathbf{v}} \in X$ by:

$$\nu_{th}(\bar{\mathbf{v}})|_{\kappa} = \frac{1}{|\kappa|} \int_{\kappa} \nu_t(\bar{\mathbf{v}}(\mathbf{x})) \, d\mathbf{x} \quad (22)$$

and where

$$\langle R^{h,2}(\bar{\mathbf{u}}_h)(t), \mathbf{v} \rangle = (\nu_t(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n), \mathbb{D}(\mathbf{v})). \quad (23)$$

In order to derive the upper bounds, we use the integration by parts formula to rewrite the residual operators $R^r(\bar{\mathbf{u}}_h)(t)$, $R^{h,1}(\bar{\mathbf{u}}_h)(t)$ and $R^{h,2}(\bar{\mathbf{u}}_h)(t)$ in the following forms:

$$\begin{aligned} \langle R^r(\bar{\mathbf{u}}_h)(t), \mathbf{v} \rangle &= \frac{t_n - t}{\tau_n} \sum_{\kappa \in \mathcal{T}_{nh}} \left\{ \nu \int_{\kappa} \nabla(\mathbf{u}^n - \mathbf{u}^{n-1})(\mathbf{x}) : \nabla \mathbf{v}(t, \mathbf{x}) \, d\mathbf{x} \right. \\ &\quad \left. + \int_{\kappa} [(\mathbf{u}^{n-1} \cdot \nabla(\mathbf{u}^n - \mathbf{u}^{n-1})) \cdot \mathbf{v}(t, \mathbf{x}) \, d\mathbf{x} \right\} \\ &- \frac{t - t_{n-1}}{\tau_n} \sum_{\kappa \in \mathcal{T}_{nh}} \left\{ \int_{\kappa} [(\mathbf{u}^n - \mathbf{u}^{n-1}) \cdot \nabla \mathbf{u}_\tau(t, \mathbf{x})] \cdot \mathbf{v}(t, \mathbf{x}) \, d\mathbf{x} \right\}, \end{aligned} \quad (24)$$

$$\begin{aligned} \langle R^{h,1}(\bar{\mathbf{u}}_h)(t), \mathbf{v} - \mathbf{v}_h \rangle &= \\ &\sum_{\kappa \in \mathcal{T}_{nh}} \left\{ \int_{\kappa} [\bar{\mathbf{f}}_h^n - \frac{1}{\tau_n}(\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}_h^{n-1}) + \nu \Delta \bar{\mathbf{u}}_h^n + \nabla \cdot (\nu_{th}(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n)) - \bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n - \frac{1}{2} \operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n \right. \\ &- \nabla \bar{p}_h^n] \cdot (\mathbf{v} - \mathbf{v}_h)(\mathbf{x}) \, d\mathbf{x} - \frac{1}{2} \sum_{e \in \mathcal{E}_{\kappa}} \int_e [(\nu \nabla \bar{\mathbf{u}}_h^n + \nu_{th}(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n) - \bar{p}_h^n \mathbb{I}) \cdot \mathbf{n}(\sigma)] \cdot (\mathbf{v} - \mathbf{v}_h)(\sigma) \, d\sigma \\ &\left. - \int_{\kappa} [(\nu_t(\bar{\mathbf{u}}_h^{n-1}) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1})) \mathbb{D}(\bar{\mathbf{u}}_h^n)] : \mathbb{D}(\mathbf{v} - \mathbf{v}_h)(\mathbf{x}) \, d\mathbf{x} \right\}, \end{aligned} \quad (25)$$

$$\langle R^{h,2}(\bar{\mathbf{u}}_h)(t), \mathbf{v} \rangle = \sum_{\kappa \in \mathcal{T}_{nh}} \int_{\kappa} \nu_t(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n) : \mathbb{D}(\mathbf{v})(\mathbf{x}) \, d\mathbf{x}. \quad (26)$$

All this leads to the following definition of the error indicators:

Definition 4.7. *For each κ in \mathcal{T}_{nh} ,*

$$(\eta_{n,\kappa}^\tau)^2 = \tau_n |\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}_h^{n-1}|_{H^1(\kappa)}^2, \quad (27)$$

$$\begin{aligned} (\eta_{n,\kappa}^{h,1})^2 &= h_\kappa^2 \left\| \bar{\mathbf{f}}_h^n - \frac{1}{\tau_n} (\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}_h^{n-1}) + \nu \Delta \bar{\mathbf{u}}_h^n + \nabla \cdot (\nu_{th} (\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n)) - \bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n \right. \\ &\quad \left. - \frac{1}{2} (\operatorname{div} \bar{\mathbf{u}}_h^{n-1}) \bar{\mathbf{u}}_h^n - \nabla \bar{p}_h^n \right\|_{L^2(\kappa)}^2 + \frac{1}{2} \sum_{e \in \varepsilon_\kappa} h_e \left\| (\nu \nabla \bar{\mathbf{u}}_h^n + \nu_{th} (\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n) - \bar{p}_h^n \mathbb{I}) \cdot \mathbf{n} \right\|_{L^2(e)}^2 \\ &\quad + \|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\kappa)}^2 + \|(\nu_t (\bar{\mathbf{u}}_h^{n-1}) - \nu_{th} (\bar{\mathbf{u}}_h^{n-1})) \mathbb{D}(\bar{\mathbf{u}}_h^n)\|_{L^2(\kappa)}^2, \end{aligned} \quad (28)$$

$$(\eta_{n,\kappa}^{h,2})^2 = \|\nu_t (\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n)\|_{L^2(\kappa)}^2. \quad (29)$$

Remark 4.8. *Even if these indicators are a little complex, each term in them is easy to compute since it only depends on the discrete solution and involves (usually low degree) polynomials.*

Lemma 4.9. *The following estimates hold for $1 \leq n \leq N$,*

(1) *For all \mathbf{v} in X and $\mathbf{v}_h = \mathcal{C}_{nh} \mathbf{v}$:*

$$|\langle R^{h,1}(\bar{\mathbf{u}}_h), \mathbf{v} - \mathbf{v}_h \rangle| \leq C \left(\sum_{\kappa \in \mathcal{T}_{nh}} (\eta_{n,\kappa}^{h,1})^2 \right)^{1/2} \|\mathbf{v}\|_X. \quad (30)$$

(2) *For all $\mathbf{v} \in X$:*

$$|\langle R^{h,2}(\bar{\mathbf{u}}_h), \mathbf{v} \rangle| \leq C \left(\sum_{\kappa \in \mathcal{T}_{nh}} (\eta_{n,\kappa}^{h,2})^2 \right)^{1/2} \|\mathbf{v}\|_X. \quad (31)$$

Proof. We proceed in two steps, one for each estimate.

1) We derive the result from formula (25) with $\mathbf{v}_h = \mathcal{C}_{nh} \mathbf{v}$, by using the continuous Cauchy-Schwarz inequality, the properties of \mathcal{C}_{nh} given by (10) and the discrete Cauchy-Schwarz inequality.

2) From (26), using the continuous and discrete Cauchy-Schwarz inequalities, we derive (31). \square

4.2. Upper bounds of the error. In this section, we establish in Corollary 4.16 the upper bound corresponding to the difference in the solutions of Problems (FV) and $(\bar{F}V_{n,h})$ with a condition on τ_n and h_n . The main idea is to decompose the error through the introduction of the solution \mathbf{u}_τ of Problem (P_{aux}) , which leads to the intermediary Theorems 4.10 and 4.12, in which we apply the continuous and discrete Gronwall lemmas.

Theorem 4.10. *Let $\mathbf{u} \in L^\infty(0, T, L^3(\Omega)^2)$. The following a posteriori error estimate holds between the velocity \mathbf{u} of Problem (FV) and the velocity \mathbf{u}_τ associated with the solutions $(\mathbf{u}^n)_{0 \leq n \leq N}$ of Problem (P_{aux}) : For $1 \leq m \leq N$,*

$$\begin{aligned} \|\mathbf{u}(t_m) - \mathbf{u}_\tau(t_m)\|_{L^2(\Omega)^2}^2 + \nu \int_0^{t_m} \|\mathbf{u}(s) - \mathbf{u}_\tau(s)\|_X^2 ds \\ \leq C \left(\|\mathbf{f} - \pi_\tau \mathbf{f}\|_{L^2((0,t_m) \times \Omega)}^2 + \sum_{n=1}^m \tau_n \|\mathbf{u}^n - \bar{\mathbf{u}}_h^n\|_X^2 + \sum_{n=1}^m \sum_{\kappa \in \mathcal{T}_{nh}} (\eta_{n,\kappa}^\tau)^2 \right). \end{aligned} \quad (32)$$

Proof. By taking $\mathbf{v} = \mathbf{u} - \mathbf{u}_\tau$ and $q = p - p_\tau$ in (17), we obtain:

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{v}(t)\|_{L^2(\Omega)^2}^2 + \nu \|\mathbf{v}(t)\|_X^2 = (\mathbf{f}(t) - \mathbf{f}^n(t), \mathbf{v}(t)) + \langle R^\tau(\mathbf{u}_\tau)(t), \mathbf{v}(t) \rangle - \int_\Omega (\mathbf{v}(t) \cdot \nabla \mathbf{u}(t)) \cdot \mathbf{v}(t). \quad (33)$$

Let us bound the right-hand side of equation (33). The first and last terms can be bounded by using (1) and (2), respectively, as well as Lemma 2.11, as follows:

$$|(\mathbf{f}(t) - \mathbf{f}^n, \mathbf{v}(t))| \leq C \|\mathbf{f}(t) - \pi_\tau \mathbf{f}\|_{L^2(\Omega)^2}^2 + \frac{\nu}{6} \|\mathbf{v}(t)\|_X^2, \quad (34)$$

$$\begin{aligned} \left| \int_\Omega (\mathbf{v}(t) \cdot \nabla \mathbf{u}(t)) \cdot \mathbf{v}(t) \right| &\leq \|\mathbf{u}(t)\|_X \|\mathbf{v}(t)\|_{L^4(\Omega)^2}^2 \leq \sqrt{2} \|\mathbf{u}(t)\|_X \|\mathbf{v}(t)\|_{L^2(\Omega)^2} \|\mathbf{v}(t)\|_X \\ &\leq C \|\mathbf{u}(t)\|_X^2 \|\mathbf{v}(t)\|_{L^2(\Omega)^2}^2 + \frac{\nu}{6} \|\mathbf{v}(t)\|_X^2. \end{aligned} \quad (35)$$

Furthermore, the residual term in the right hand side of Equation (33) can be decomposed as:

$$\langle R^\tau(\mathbf{u}_\tau)(t), \mathbf{v} \rangle = T_1 + T_2 + T_3, \quad (36)$$

$$\text{where } T_1|_{[t_{n-1}, t_n]} = \frac{t_n - t}{\tau_n} \sum_{\kappa \in \mathcal{T}_{nh}} \nu \int_{\kappa} \nabla(\mathbf{u}^n - \mathbf{u}^{n-1})(\mathbf{x}) : \nabla \mathbf{v}(t, \mathbf{x}) \, d\mathbf{x},$$

$$T_2|_{[t_{n-1}, t_n]} = \frac{t_n - t}{\tau_n} \sum_{\kappa \in \mathcal{T}_{nh}} \int_{\kappa} (\mathbf{u}^{n-1} \cdot \nabla \mathbf{v}(t, \mathbf{x})) \cdot (\mathbf{u}^n - \mathbf{u}^{n-1}) \, d\mathbf{x}$$

$$\text{and } T_3|_{[t_{n-1}, t_n]} = -\frac{t - t_{n-1}}{\tau_n} \sum_{\kappa \in \mathcal{T}_{nh}} \int_{\kappa} [(\mathbf{u}^n - \mathbf{u}^{n-1}) \cdot \nabla \mathbf{v}(t, \mathbf{x})] \cdot \mathbf{u}_\tau(t, \mathbf{x}) \, d\mathbf{x}.$$

Using the continuous and then discrete Cauchy-Schwarz inequalities, the fact that $\left| \frac{t - t_n}{\tau_n} \right| \leq 1$ for all $t \in [t_{n-1}, t_n]$ and using Lemma 2.11, we bound T_1 in the following way:

$$|T_1| \leq \nu \sum_{\kappa \in \mathcal{T}_{nh}} |\mathbf{u}^n - \mathbf{u}^{n-1}|_{H^1(\kappa)^2} |\mathbf{v}|_{H^1(\kappa)^2} \leq C \sum_{\kappa \in \mathcal{T}_{nh}} |\mathbf{u}^n - \mathbf{u}^{n-1}|_{H^1(\kappa)^2}^2 + \frac{\nu}{36} \|\mathbf{v}\|_X^2. \quad (37)$$

In order to bound T_2 , we insert $\mathbf{u}(t, \mathbf{x})$ and \mathbf{u}_τ :

$$\begin{aligned} |T_2| &\leq \left| \sum_{\kappa \in \mathcal{T}_{nh}} \int_{\kappa} [(\mathbf{u}^{n-1} - \mathbf{u}_\tau(t, \mathbf{x})) \cdot \nabla \mathbf{v}(t, \mathbf{x})] \cdot (\mathbf{u}^n - \mathbf{u}^{n-1}) \, d\mathbf{x} \right| \\ &\quad + \left| \sum_{\kappa \in \mathcal{T}_{nh}} \int_{\kappa} [(\mathbf{u}_\tau(t, \mathbf{x}) - \mathbf{u}(t, \mathbf{x})) \cdot \nabla \mathbf{v}(t, \mathbf{x})] \cdot (\mathbf{u}^n - \mathbf{u}^{n-1}) \, d\mathbf{x} \right| \\ &\quad + \left| \sum_{\kappa \in \mathcal{T}_{nh}} \int_{\kappa} (\mathbf{u}(t, \mathbf{x}) \cdot \nabla \mathbf{v}(t, \mathbf{x})) \cdot (\mathbf{u}^n - \mathbf{u}^{n-1}) \, d\mathbf{x} \right| \\ &= T_{2,1} + T_{2,2} + T_{2,3}. \end{aligned} \quad (38)$$

Using that $\mathbf{u}_\tau = \frac{t - t_{n-1}}{\tau_n} (\mathbf{u}^n - \mathbf{u}^{n-1}) + \mathbf{u}^{n-1}$, the fact that $\left| \frac{t - t_{n-1}}{\tau_n} \right| \leq 1$ and Lemma 2.11, we obtain:

$$\begin{aligned} T_{2,1} &\leq \sum_{\kappa \in \mathcal{T}_{nh}} \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{L^4(\kappa)^2} |\mathbf{v}|_{H^1(\kappa)^2} \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{L^4(\kappa)^2} \\ &\leq C \sum_{\kappa \in \mathcal{T}_{nh}} \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{L^4(\kappa)^2}^4 + \frac{\nu}{36} \|\mathbf{v}\|_X^2. \end{aligned}$$

Using (2) as well as the fact that (15) and the triangular inequality imply that $\|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{L^2(\Omega)^2}^2$ is bounded by a constant, we obtain

$$T_{2,1} \leq C \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{L^2(\Omega)^2}^2 \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_X^2 + \frac{\nu}{36} \|\mathbf{v}\|_X^2 \leq C \sum_{\kappa \in \mathcal{T}_{nh}} |\mathbf{u}^n - \mathbf{u}^{n-1}|_{H^1(\kappa)^2}^2 + \frac{\nu}{36} \|\mathbf{v}\|_X^2. \quad (39)$$

On the other hand, using Lemma 2.11, it holds that

$$\begin{aligned} T_{2,2} &\leq \sum_{\kappa \in \mathcal{T}_{nh}} \|\mathbf{u}_\tau(t) - \mathbf{u}(t)\|_{L^4(\kappa)^2} |\mathbf{v}|_{H^1(\kappa)^2} \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{L^4(\kappa)^2} \\ &\leq C \sum_{\kappa \in \mathcal{T}_{nh}} \|\mathbf{u}_\tau(t) - \mathbf{u}(t)\|_{L^4(\kappa)^2}^2 \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{L^4(\kappa)^2}^2 + \frac{\nu}{36} \|\mathbf{v}\|_X^2. \end{aligned}$$

Using the discrete Cauchy-Schwarz inequality, then (2), (3) and Lemma 4.4, we obtain:

$$\begin{aligned} T_{2,2} &\leq C \|\mathbf{u}_\tau(t) - \mathbf{u}(t)\|_{L^4(\Omega)^2}^2 \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{L^4(\Omega)^2}^2 + \frac{\nu}{36} \|\mathbf{v}\|_X^2 \\ &\leq C \|\mathbf{u}_\tau(t) - \mathbf{u}(t)\|_X \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_X + \frac{\nu}{36} \|\mathbf{v}\|_X^2. \end{aligned}$$

Since $\mathbf{v} = \mathbf{u} - \mathbf{u}_\tau$ and using Lemma 2.11 once again, we obtain

$$T_{2,2} \leq C \sum_{\kappa \in \mathcal{T}_{nh}} |\mathbf{u}^n - \mathbf{u}^{n-1}|_{H^1(\kappa)^2}^2 + \frac{2\nu}{36} \|\mathbf{v}\|_X^2. \quad (40)$$

Now, we bound $T_{2,3}$ by taking into consideration that \mathbf{u} is supposed to be in $L^\infty(0, T, L^3(\Omega))$; then we use the discrete Cauchy-Schwarz inequality, Lemmas 2.11 and 2.1 and we obtain

$$\begin{aligned} T_{2,3} &\leq \sum_{\kappa \in \mathcal{T}_{nh}} \|\mathbf{u}(t)\|_{L^3(\kappa)^2} \|\mathbf{v}\|_{H^1(\kappa)^2} \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{L^6(\kappa)} \\ &\leq C \sum_{\kappa \in \mathcal{T}_{nh}} |\mathbf{u}^n - \mathbf{u}^{n-1}|_{H^1(\kappa)^2}^2 + \frac{\nu}{36} \|\mathbf{v}\|_X^2. \end{aligned} \quad (41)$$

In order to bound $|T_3|$, we use the fact that $\left| \frac{t - t_{n-1}}{\tau_n} \right| \leq 1$ for all $t \in [t_{n-1}, t_n]$ and we insert \mathbf{u} :

$$|T_3| \leq \left| \sum_{\kappa \in \mathcal{T}_{nh}} \int_{\kappa} [(\mathbf{u}^n - \mathbf{u}^{n-1}) \cdot \nabla \mathbf{v}(t, \mathbf{x})] \cdot (\mathbf{u}_\tau - \mathbf{u})(t, \mathbf{x}) \, d\mathbf{x} \right| + \left| \sum_{\kappa \in \mathcal{T}_{nh}} \int_{\kappa} [(\mathbf{u}^n - \mathbf{u}^{n-1}) \cdot \nabla \mathbf{v}(t, \mathbf{x})] \cdot \mathbf{u}(t, \mathbf{x}) \, d\mathbf{x} \right|.$$

By using that $\mathbf{v} = \mathbf{u}_\tau - \mathbf{u}$ and that $c(\mathbf{u}, \mathbf{v}, \mathbf{v}) = \mathbf{0}$ the first term in the right-hand side vanishes. The second term is treated exactly like $T_{2,3}$ and we obtain

$$T_3 \leq C \sum_{\kappa \in \mathcal{T}_{nh}} |\mathbf{u}^n - \mathbf{u}^{n-1}|_{H^1(\kappa)^2}^2 + \frac{\nu}{36} \|\mathbf{v}\|_X^2. \quad (42)$$

Finally, gathering (33), (34), (35), (36), (37), (38), (39), (40), (41) and (42), we obtain on each $[t_{n-1}, t_n]$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{v}(t)\|_{L^2(\Omega)^2}^2 + \nu \|\mathbf{v}(t)\|_X^2 &\leq \frac{\nu}{2} \|\mathbf{v}(t)\|_X^2 + C \|\mathbf{f}(t) - \pi_\tau \mathbf{f}\|_{L^2(\Omega)^2}^2 + C \sum_{\kappa \in \mathcal{T}_{nh}} |\mathbf{u}^n - \mathbf{u}^{n-1}|_{H^1(\kappa)^2}^2 \\ &\quad + C \|\mathbf{u}(t)\|_X^2 \|\mathbf{v}(t)\|_{L^2(\Omega)^2}^2. \end{aligned} \quad (43)$$

Then, we simplify (43), take into account the definitions of $\pi_\tau \mathbf{u}$ and $\pi_{l,\tau} \mathbf{u}$, and integrate equation between 0 and t ; we obtain

$$\begin{aligned} \|\mathbf{v}(t)\|_{L^2(\Omega)^2}^2 + \nu \int_0^t \|\mathbf{v}(s)\|_X^2 \, ds &\leq C \left(\|\mathbf{f} - \pi_\tau \mathbf{f}\|_{[L^2((0,t) \times \Omega)]^2}^2 + \|\pi_\tau \mathbf{u} - \pi_{l,\tau} \mathbf{u}\|_{L^2(0,t,X)}^2 \right) \\ &\quad + C \int_0^t \|\mathbf{u}(\tau)\|_X^2 \|\mathbf{v}(\tau)\|_{L^2(\Omega)^2}^2 \, d\tau. \end{aligned}$$

We apply the Gronwall Lemma 2.9 with functions given by the following form :

$$\begin{aligned} z(t) &= \|\mathbf{v}(t)\|_{L^2(\Omega)^2}^2 + \nu \int_0^t \|\mathbf{v}(s)\|_X^2 \, ds, \quad f(t) = C \left(\|\mathbf{f} - \pi_\tau \mathbf{f}\|_{[L^2((0,t) \times \Omega)]^2}^2 + \|\pi_\tau \mathbf{u} - \pi_{l,\tau} \mathbf{u}\|_{L^2(0,t,X)}^2 \right), \\ g(t) &= C \text{ and } k(\tau) = \|\mathbf{u}(\tau)\|_X^2. \end{aligned}$$

We obtain the following bound:

$$\begin{aligned} \|\mathbf{u}(t) - \mathbf{u}_\tau(t)\|_{L^2(\Omega)^2}^2 + \nu \int_0^t \|\mathbf{u}(\tau) - \mathbf{u}_\tau(\tau)\|_X^2 \, d\tau &\leq C \left(\|\mathbf{f} - \pi_\tau \mathbf{f}\|_{[L^2((0,t) \times \Omega)]^2}^2 + \|\pi_\tau \mathbf{u} - \pi_{l,\tau} \mathbf{u}\|_{L^2(0,t,X)}^2 \right) \\ &\quad + C \int_0^t f(\tau) \|\mathbf{u}(\tau)\|_X^2 \exp \left(C \int_\tau^t \|\mathbf{u}(s)\|_X^2 \, ds \right) \, d\tau. \end{aligned}$$

Since f is an increasing function of time, we bound it by $f(t_m)$; we use (3) to bound the integrals of $\|\mathbf{u}\|_X^2$. Applying the resulting inequality at $t = t_m$, we get:

$$\|\mathbf{u}(t_m) - \mathbf{u}_\tau(t_m)\|_{L^2(\Omega)^2}^2 + \nu \int_0^{t_m} \|\mathbf{u}(\tau) - \mathbf{u}_\tau(\tau)\|_X^2 \, d\tau \leq C \left(\|\mathbf{f} - \pi_\tau \mathbf{f}\|_{[L^2((0,t_m) \times \Omega)]^2}^2 + \sum_{n=1}^m \tau_n \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_X^2 \right). \quad (44)$$

To obtain (32), we use the triangle inequality below in the last term in the right-hand side of (44), then use hypothesis (7) and definition (27)

$$\|\mathbf{u}^n - \mathbf{u}^{n-1}\|_X \leq \|\mathbf{u}^n - \bar{\mathbf{u}}_h^n\|_X + \|\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}_h^{n-1}\|_X + \|\mathbf{u}^{n-1} - \bar{\mathbf{u}}_h^{n-1}\|_X.$$

□

To derive an *a posteriori* estimate between the solution \mathbf{u} of problem (FV) and the solution $\bar{\mathbf{u}}_h$ corresponding to the solutions $\bar{\mathbf{u}}_h^n$ of $(\overline{FV}_{n,h})$, it suffices to get an *a posteriori* estimate between the solution \mathbf{u}_τ of Problem (P_{aux}) and $\bar{\mathbf{u}}_h$ and to apply the triangle inequality using the previous theorem. In order to get an *a posteriori error* estimate between the solutions \mathbf{u}_τ and $\bar{\mathbf{u}}_h$, we introduce the operator Π (see [9]) defined from X into itself as follows: For each \mathbf{v} in X , $\Pi\mathbf{v}$ denotes the velocity \mathbf{w} of the unique weak solution (\mathbf{w}, r) in $X \times M$ of the Stokes problem:

$$\begin{cases} \forall \mathbf{t} \in X, & (\nabla \mathbf{w}, \nabla \mathbf{t}) - (\operatorname{div} \mathbf{t}, r) = 0, \\ \forall q \in M, & (\operatorname{div} \mathbf{w}, q) = (\operatorname{div} \mathbf{v}, q). \end{cases} \quad (45)$$

The next lemma states some properties of the operator Π . We refer to [9, Lemma 4.2] or [24, Lemma 4.2.4]

Lemma 4.11. *The operator Π has the following properties:*

- (1) For all \mathbf{v} in V , $\Pi\mathbf{v}$ is zero,
- (2) The following estimates hold for all \mathbf{v} in X ,

$$\|\mathbf{v} - \Pi\mathbf{v}\|_X \leq \|\mathbf{v}\|_X \quad \text{and} \quad \|\Pi\mathbf{v}\|_X \leq \frac{1}{\beta_*} \|\operatorname{div} \mathbf{v}\|_{L^2(\Omega)}.$$

- (3) $\forall \mathbf{v}_h \in V_{nh}$ and $1 \leq n \leq N$:

$$\|\Pi\mathbf{v}_h\|_{L^2(\Omega)^2} \leq ch_n^{1/2} \|\operatorname{div} \mathbf{v}_h\|_{L^2(\Omega)}.$$

We are now in a position to prove *a posteriori* estimates between the solution \mathbf{u}_τ of Problem (P_{aux}) and the solution $\bar{\mathbf{u}}_h$ of $(\overline{FV}_{n,h})$.

Theorem 4.12. *Suppose there exists a positive constant C_s such that for all $1 \leq n \leq N$ we have $h_n \leq C_s \tau_n$. The following *a posteriori error* estimate holds between the solutions \mathbf{u}^m and $\bar{\mathbf{u}}_h^m$ of Problems (P_{aux}) and $(\overline{FV}_{n,h})$.*

$$\begin{aligned} \|\mathbf{u}^m - \bar{\mathbf{u}}_h^m\|_{L^2(\Omega)^2}^2 + \nu \sum_{n=1}^m \tau_n \|\mathbf{u}^n - \bar{\mathbf{u}}_h^n\|_X^2 \\ \leq C \sum_{n=1}^m \tau_n \left(\sum_{\kappa \in \mathcal{T}_{nh}} (h_\kappa^2 \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\kappa)}^2 + \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\kappa)}^2 + (\eta_{n,\kappa}^{h,1})^2 + (\eta_{n,\kappa}^{h,2})^2) \right). \end{aligned} \quad (46)$$

Proof. For abbreviation we set $\mathbf{e}^n = \mathbf{u}^n - \bar{\mathbf{u}}_h^n$ and $\varepsilon^n = p^n - \bar{p}_h^n$, $0 \leq n \leq N$. For any $1 \leq n \leq N$, we have

$$\frac{1}{2} \|\mathbf{e}^n\|_{L^2(\Omega)^2}^2 - \frac{1}{2} \|\mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^2 + \frac{1}{2} \|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^2 + \nu \tau_n \|\mathbf{e}^n\|_X^2 = (\mathbf{e}^n - \mathbf{e}^{n-1}, \mathbf{e}^n) + \nu \tau_n (\nabla \mathbf{e}^n, \nabla \mathbf{e}^n). \quad (47)$$

Adding and subtracting $\Pi\mathbf{e}^n$ in both terms in the right-hand side of (47) and since $\operatorname{div}(\mathbf{e}^n - \Pi\mathbf{e}^n) = 0$, we obtain:

$$\begin{aligned} (\mathbf{e}^n - \mathbf{e}^{n-1}, \mathbf{e}^n) + \nu \tau_n (\nabla \mathbf{e}^n, \nabla \mathbf{e}^n) &= (\mathbf{e}^n - \mathbf{e}^{n-1}, \Pi\mathbf{e}^n) + \nu \tau_n (\nabla \mathbf{e}^n, \nabla \Pi\mathbf{e}^n) \\ &\quad + (\mathbf{e}^n - \mathbf{e}^{n-1}, \mathbf{e}^n - \Pi\mathbf{e}^n) + \nu \tau_n (\nabla \mathbf{e}^n, \nabla (\mathbf{e}^n - \Pi\mathbf{e}^n)) \\ &\quad - \tau_n (\operatorname{div}(\mathbf{e}^n - \Pi\mathbf{e}^n), \varepsilon^n). \end{aligned} \quad (48)$$

By taking $\mathbf{v} = \mathbf{e}^n - \Pi\mathbf{e}^n$ in (19), we have for every $\mathbf{v}_h \in X_{nh}$

$$\begin{aligned} (\mathbf{e}^n - \mathbf{e}^{n-1}, \mathbf{e}^n) + \nu \tau_n (\nabla \mathbf{e}^n, \nabla \mathbf{e}^n) &= (\mathbf{e}^n - \mathbf{e}^{n-1}, \Pi\mathbf{e}^n) + \nu \tau_n (\nabla \mathbf{e}^n, \nabla \Pi\mathbf{e}^n) + \tau_n (\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n, \mathbf{v} - \mathbf{v}_h) \\ &\quad + \tau_n (\mathbf{f}^n - \bar{\mathbf{f}}^n, \mathbf{v}) + \tau_n \langle R^{h,1}(\bar{\mathbf{u}}_h), \mathbf{v} - \mathbf{v}_h \rangle + \tau_n \langle R^{h,2}(\bar{\mathbf{u}}_h), \mathbf{v} \rangle \\ &\quad - \tau_n (\mathbf{u}^{n-1} \cdot \nabla \mathbf{u}^n - \bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n, \mathbf{v}) + \frac{1}{2} \tau_n (\operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n, \mathbf{v}). \end{aligned} \quad (49)$$

Next, we evaluate all terms in the right-hand side of (49) separately by repeatedly using both inequalities in Lemma 2.11 for various values of (β, γ, δ) . Taking into account that $\Pi \mathbf{e}^n = -\Pi \bar{\mathbf{u}}_h^n$ and using Lemma 4.11, the first and second terms can be bounded as:

$$(\mathbf{e}^n - \mathbf{e}^{n-1}, \Pi \mathbf{e}^n) \leq \frac{1}{4} \|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^2 + C\tau_n \|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\Omega)}^2 \quad (50)$$

and

$$\nu \tau_n (\nabla \mathbf{e}^n, \nabla \Pi \mathbf{e}^n) \leq \frac{\nu \tau_n}{28} \|\mathbf{e}^n\|_X^2 + C\tau_n \|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\Omega)}^2. \quad (51)$$

To estimate the third and fourth terms of (49), we take $\mathbf{v}_h = \mathcal{C}_{nh} \mathbf{v}$, use estimation (10) and Lemma 4.11 to derive :

$$\begin{aligned} & \tau_n (\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n, \mathbf{v} - \mathcal{C}_{nh} \mathbf{v}) + \tau_n (\mathbf{f}^n - \bar{\mathbf{f}}^n, \mathbf{v}) \\ & \leq \tau_n \left(\sum_{\kappa \in \mathcal{T}_{nh}} ch_\kappa \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\kappa)^2} \|\mathbf{v}\|_{H^1(\Delta_\kappa)^2} + S_2 \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\Omega)^2} \|\mathbf{v}\|_X \right) \\ & \leq C\tau_n \left(\sum_{\kappa \in \mathcal{T}_{nh}} (h_\kappa^2 \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\kappa)^2}^2 + \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\kappa)^2}^2) \right)^{1/2} \|\mathbf{v}\|_{H^1(\Omega)^2} \\ & \leq C\tau_n \left(\sum_{\kappa \in \mathcal{T}_{nh}} (h_\kappa^2 \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\kappa)^2}^2 + \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\kappa)^2}^2) \right) + \frac{\nu \tau_n}{28} \|\mathbf{e}^n\|_X^2. \end{aligned} \quad (52)$$

The fifth and sixth terms of (49) can be bounded as (cf Lemma 4.9)

$$\tau_n \langle R^{h,1}(\bar{\mathbf{u}}_h)(t), \mathbf{v} - \mathbf{v}_h \rangle \leq C\tau_n \left(\sum_{\kappa \in \mathcal{T}_{nh}} (\eta_{n,\kappa}^{h,1})^2 \right)^{1/2} \|\mathbf{v}\|_X \leq C \sum_{\kappa \in \mathcal{T}_{nh}} \tau_n (\eta_{n,\kappa}^{h,1})^2 + \frac{\nu \tau_n}{28} \|\mathbf{e}^n\|_X^2. \quad (53)$$

$$\tau_n \langle R^{h,2}(\bar{\mathbf{u}}_h)(t), \mathbf{v} \rangle \leq C\tau_n \left(\sum_{\kappa \in \mathcal{T}_{nh}} (\eta_{n,\kappa}^{h,2})^2 \right)^{1/2} \|\mathbf{v}\|_X \leq C \sum_{\kappa \in \mathcal{T}_{nh}} \tau_n (\eta_{n,\kappa}^{h,2})^2 + \frac{\nu \tau_n}{28} \|\mathbf{e}^n\|_X^2. \quad (54)$$

Finally, we bound the last two terms of (49). We have the relation:

$$\begin{aligned} & \tau_n (\mathbf{u}^{n-1} \cdot \nabla \mathbf{u}^n - \bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n, \mathbf{v}) + \frac{\tau_n}{2} (\operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n, \mathbf{v}) \\ & = -\tau_n (\mathbf{e}^{n-1} \cdot \nabla \mathbf{u}^n, \mathbf{v}) - \frac{\tau_n}{2} (\operatorname{div} \mathbf{e}^{n-1} \mathbf{u}^n, \mathbf{v}) - \tau_n (\bar{\mathbf{u}}_h^{n-1} \cdot \nabla \mathbf{e}^n, \mathbf{v}) - \frac{\tau_n}{2} (\operatorname{div} \bar{\mathbf{u}}_h^{n-1} \mathbf{e}^n, \mathbf{v}). \end{aligned}$$

We set $A = A_1 + A_2$ where $A_1 = -\tau_n (\mathbf{e}^{n-1} \cdot \nabla \mathbf{u}^n, \mathbf{v})$ and $A_2 = -\frac{\tau_n}{2} (\operatorname{div} \mathbf{e}^{n-1} \mathbf{u}^n, \mathbf{v})$, and $B = -\tau_n (\bar{\mathbf{u}}_h^{n-1} \cdot \nabla \mathbf{e}^n, \mathbf{v}) - \frac{\tau_n}{2} (\operatorname{div} \bar{\mathbf{u}}_h^{n-1} \mathbf{e}^n, \mathbf{v})$. We first bound A_1 by using the $L^4 - L^2 - L^4$ inequality, the fact that $\mathbf{v} = \mathbf{e}^n - \Pi \mathbf{e}^n$ and $\Pi \mathbf{e}^n = -\Pi \bar{\mathbf{u}}_h^n$, then (2) and Lemma 4.11; we obtain

$$\begin{aligned} |A_1| & \leq C\tau_n \|\mathbf{e}^{n-1}\|_{L^4(\Omega)^2} \|\mathbf{u}^n\|_X (\|\mathbf{e}^n\|_{L^4(\Omega)^2} + \|\Pi \mathbf{e}^n\|_{L^4(\Omega)^2}) \\ & \leq C\tau_n \|\mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^{1/2} \|\mathbf{e}^{n-1}\|_X^{1/2} \|\mathbf{u}^n\|_X (\|\mathbf{e}^n\|_{L^2(\Omega)^2}^{1/2} \|\mathbf{e}^n\|_X^{1/2} + (h_n^{1/2} \|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\Omega)})^{1/2} \|\mathbf{e}^n\|_X^{1/2}). \end{aligned} \quad (55)$$

We bound separately the two terms that result from the expansion of the product in the right-hand side of (55), that we denote by $A_{1,1}$ and $A_{1,2}$. First, using Lemma 2.11 we have

$$A_{1,1} \leq \tau_n (C \|\mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^2 \|\mathbf{u}^n\|_X^2 + \frac{\nu}{28\sigma_\tau} \|\mathbf{e}^{n-1}\|_X^2 + \frac{\nu}{28} \|\mathbf{e}^n\|_X^2) + \delta \tau_n \|\mathbf{e}^n\|_{L^2(\Omega)^2}^2 \|\mathbf{u}^n\|_X^2$$

Inserting \mathbf{e}^{n-1} in the last term and using that $\tau_n \|\mathbf{u}^n\|_X^2$ is bounded by (15), we can choose δ such that

$$A_{1,1} \leq \tau_n (C \|\mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^2 \|\mathbf{u}^n\|_X^2 + \frac{\nu}{28\sigma_\tau} \|\mathbf{e}^{n-1}\|_X^2 + \frac{\nu}{28} \|\mathbf{e}^n\|_X^2) + \frac{1}{16} \|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^2. \quad (56)$$

Secondly, using Lemma 2.11, we have:

$$A_{1,2} \leq C\tau_n (\|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\Omega)} \|\mathbf{e}^n\|_X + h_n^{1/2} \|\mathbf{u}^n\|_X \|\mathbf{e}^{n-1}\|_{L^2(\Omega)^2} \|\mathbf{e}^{n-1}\|_X)$$

Using the relation $h_n \leq C_s \tau_n$ and the fact that (15) allows to bound $\tau_n^{1/2} \|\mathbf{u}^n\|_X$ by a constant, we have, using Lemma 2.11 twice

$$\begin{aligned} A_{1,2} &\leq C\tau_n \|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\Omega)} \|\mathbf{e}^n\|_X + C\tau_n \|\mathbf{u}^n\|_X \|\mathbf{e}^{n-1}\|_{L^2(\Omega)^2} \|\mathbf{e}^{n-1}\|_X \\ &\leq \tau_n (C \|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\Omega)}^2 + \frac{\nu}{28} \|\mathbf{e}^n\|_X^2) + \tau_n (C \|\mathbf{u}^n\|_X^2 \|\mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^2 + \frac{\nu}{28\sigma_\tau} \|\mathbf{e}^{n-1}\|_X^2). \end{aligned} \quad (57)$$

Finally, summing (56) and (57), we get

$$|A_1| \leq \tau_n (C \|\mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^2 \|\mathbf{u}^n\|_X^2 + \frac{2\nu}{28\sigma_\tau} \|\mathbf{e}^{n-1}\|_X^2 + \frac{2\nu}{28} \|\mathbf{e}^n\|_X^2 + C \|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\Omega)}^2) + \frac{1}{16} \|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^2. \quad (58)$$

As far as A_2 is concerned, we first use that $\operatorname{div} \mathbf{e}^{n-1} = -\operatorname{div} \bar{\mathbf{u}}_h^{n-1}$, then use (2), Lemma 2.11 and a triangular inequality. We get, for any $\beta > 0$

$$\begin{aligned} |A_2| &\leq C\tau_n \|\operatorname{div} \mathbf{u}_h^{n-1}\|_{L^2(\Omega)} \|\mathbf{u}^n\|_{L^4(\Omega)^2} \|\mathbf{v}\|_{L^4(\Omega)^2} \\ &\leq C\tau_n \|\operatorname{div} \mathbf{u}_h^{n-1}\|_{L^2(\Omega)}^2 + \beta\tau_n \|\mathbf{u}^n\|_{L^2(\Omega)^2} \|\mathbf{u}^n\|_X (\|\mathbf{e}^n\|_{L^2(\Omega)^2} \|\mathbf{e}^n\|_X + \|\Pi\mathbf{e}^n\|_{L^2(\Omega)^2} \|\Pi\mathbf{e}^n\|_X) \\ &\leq C\tau_n \|\operatorname{div} \mathbf{u}_h^{n-1}\|_{L^2(\Omega)}^2 + \beta(A_{2,1} + A_{2,2} + A_{2,3}) \end{aligned} \quad (59)$$

Using that $\tau_n \|\mathbf{u}^n\|_X^2$ and $\|\mathbf{u}^n\|_{L^2(\Omega)^2}^2$ are bounded we get

$$\begin{aligned} A_{2,1} &= \tau_n \|\mathbf{u}^n\|_{L^2(\Omega)^2} \|\mathbf{u}^n\|_X \|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{L^2(\Omega)^2} \|\mathbf{e}^n\|_X \\ &\leq \frac{1}{2} \tau_n (\|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^2 \|\mathbf{u}^n\|_X^2 + \|\mathbf{u}^n\|_{L^2(\Omega)^2}^2 \|\mathbf{e}^n\|_X^2) \\ &\leq C \|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^2 + C\tau_n \|\mathbf{e}^n\|_X^2, \end{aligned}$$

$$A_{2,2} = \tau_n \|\mathbf{u}^n\|_{L^2(\Omega)^2} \|\mathbf{u}^n\|_X \|\mathbf{e}^{n-1}\|_{L^2(\Omega)^2} \|\mathbf{e}^n\|_X \leq \tau_n (\|\mathbf{u}^n\|_X^2 \|\mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^2 + C \|\mathbf{e}^n\|_X^2),$$

$$\begin{aligned} A_{2,3} &= \tau_n \|\mathbf{u}^n\|_{L^2(\Omega)^2} \|\mathbf{u}^n\|_X \|\Pi\mathbf{e}^n\|_{L^2(\Omega)^2} \|\Pi\mathbf{e}^n\|_X \leq C\tau_n \|\mathbf{u}^n\|_{L^2(\Omega)^2} \|\mathbf{u}^n\|_X (h_n^{1/2} \|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\Omega)}) \|\mathbf{e}^n\|_X \\ &\leq C\tau_n (\|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\Omega)}^2 + \|\mathbf{u}^n\|_{L^2(\Omega)^2}^2 \tau_n \|\mathbf{u}^n\|_X^2 \|\mathbf{e}^n\|_X^2) \\ &\leq C\tau_n (\|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\Omega)}^2 + \|\mathbf{e}^n\|_X^2). \end{aligned}$$

Since β in (59) can be freely chosen, this yields the following bound for A_2

$$|A_2| \leq C\tau_n \|\operatorname{div} \mathbf{u}_h^{n-1}\|_{L^2(\Omega)}^2 + \frac{1}{16} \|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^2 + \tau_n \frac{3\nu}{28} \|\mathbf{e}^n\|_X^2 + C\tau_n \|\mathbf{u}^n\|_X^2 \|\mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^2. \quad (60)$$

Let us now bound the term B . It can be written as follows:

$$\begin{aligned} B &= -\tau_n (\bar{\mathbf{u}}_h^{n-1} \cdot \nabla (\mathbf{e}^n - \Pi\mathbf{e}^n + \Pi\mathbf{e}^n), \mathbf{e}^n - \Pi\mathbf{e}^n) - \frac{\tau_n}{2} (\operatorname{div} \bar{\mathbf{u}}_h^{n-1} (\mathbf{e}^n - \Pi\mathbf{e}^n + \Pi\mathbf{e}^n), \mathbf{e}^n - \Pi\mathbf{e}^n) \\ &= -\tau_n (\bar{\mathbf{u}}_h^{n-1} \cdot \nabla \Pi\mathbf{e}^n, \mathbf{e}^n - \Pi\mathbf{e}^n) - \frac{\tau_n}{2} (\operatorname{div} \bar{\mathbf{u}}_h^{n-1} \Pi\mathbf{e}^n, \mathbf{e}^n - \Pi\mathbf{e}^n) \\ &= -\tau_n (\bar{\mathbf{u}}_h^{n-1} \cdot \nabla \Pi\mathbf{e}^n, \mathbf{e}^n) - \frac{\tau_n}{2} (\operatorname{div} \bar{\mathbf{u}}_h^{n-1} \Pi\mathbf{e}^n, \mathbf{e}^n) = B_1 + B_2 \end{aligned}$$

Using the fact that $\Pi\mathbf{e}^n = -\Pi\bar{\mathbf{u}}_h^n$, Lemma 4.11, then (2) and Lemma 2.11, we get, for any $\beta > 0$

$$\begin{aligned} |B_1| &\leq C\tau_n \|\bar{\mathbf{u}}_h^{n-1}\|_{L^4(\Omega)} \|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\Omega)} \|\mathbf{e}^n\|_{L^4(\Omega)} \\ &\leq C\tau_n \|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\Omega)}^2 + \beta\tau_n \|\bar{\mathbf{u}}_h^{n-1}\|_{L^2(\Omega)^2} \|\bar{\mathbf{u}}_h^{n-1}\|_X \|\mathbf{e}^n\|_{L^2(\Omega)} \|\mathbf{e}^n\|_X \end{aligned} \quad (61)$$

A term very similar to the second one in the right-hand side of (61) was already bounded in (59), replacing \mathbf{u}^n by $\bar{\mathbf{u}}_h^{n-1}$; since (9) yields similar bounds as (15), choosing β small enough leads to

$$|B_1| \leq C\tau_n \|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\Omega)}^2 + \frac{1}{16} \|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^2 + \tau_n \frac{2\nu}{28} \|\mathbf{e}^n\|_X^2 + C\tau_n \|\bar{\mathbf{u}}_h^{n-1}\|_X^2 \|\mathbf{e}^{n-1}\|_{L^2(\Omega)^2}^2. \quad (62)$$

Moreover, using once more the fact that $\Pi\mathbf{e}^n = -\Pi\bar{\mathbf{u}}_h^n$, Lemma 2.11, then (2) and Lemma 4.11 we get

$$\begin{aligned} |B_2| &\leq C\tau_n \|\operatorname{div} \bar{\mathbf{u}}_h^{n-1}\|_{L^2(\Omega)}^2 + \beta\tau_n \|\Pi\bar{\mathbf{u}}_h^n\|_{L^4(\Omega)^2}^2 \|\mathbf{e}^n\|_{L^4(\Omega)^2}^2 \\ &\leq C\tau_n \|\operatorname{div} \bar{\mathbf{u}}_h^{n-1}\|_{L^2(\Omega)}^2 + \beta\tau_n \|\Pi\bar{\mathbf{u}}_h^n\|_{L^2(\Omega)^2} \|\Pi\bar{\mathbf{u}}_h^n\|_X \|\mathbf{e}^n\|_{L^2(\Omega)^2} \|\mathbf{e}^n\|_X \\ &\leq C\tau_n \|\operatorname{div} \bar{\mathbf{u}}_h^{n-1}\|_{L^2(\Omega)}^2 + C\beta\tau_n h_n^{1/2} \|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\Omega)^2} \|\bar{\mathbf{u}}_h^n\|_X \|\mathbf{e}^n\|_{L^2(\Omega)^2} \|\mathbf{e}^n\|_X. \end{aligned}$$

Using the fact that $h_n \leq C_s \tau_n$, the fact that $\tau_n^{1/2} \|\bar{\mathbf{u}}_h^n\|_X$ is bounded thanks to (9), Lemma 2.11 and inserting \mathbf{e}^{n-1} , we obtain

$$|B_2| \leq C \tau_n \|\operatorname{div} \bar{\mathbf{u}}_h^{n-1}\|_{L^2(\Omega)}^2 + C \beta \tau_n (\|\mathbf{e}^n\|_X^2 + \|\bar{\mathbf{u}}_h^n\|_X^2 \|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{L^2(\Omega)}^2 + \|\bar{\mathbf{u}}_h^n\|_X^2 \|\mathbf{e}^{n-1}\|_{L^2(\Omega)}^2).$$

Using once more that $\tau_n \|\bar{\mathbf{u}}_h^n\|_X^2$ is bounded, we may choose β such that

$$|B_2| \leq C \tau_n \|\operatorname{div} \bar{\mathbf{u}}_h^{n-1}\|_{L^2(\Omega)}^2 + \tau_n \frac{\nu}{28} \|\mathbf{e}^n\|_X^2 + \frac{1}{16} \|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{L^2(\Omega)}^2 + C \tau_n \|\bar{\mathbf{u}}_h^n\|_X^2 \|\mathbf{e}^{n-1}\|_{L^2(\Omega)}^2. \quad (63)$$

Thus, by summing (50), (51), (52), (53), (54), (58), (60), (62) and (63), using Equations (47), (49), the fact that $\mathbf{e}^0 = 0$, and summing over n from 1 to m , we obtain:

$$\begin{aligned} & \frac{1}{2} \|\mathbf{e}^m\|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_{n=1}^m \|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{L^2(\Omega)}^2 + \nu \sum_{n=1}^m \tau_n \|\mathbf{e}^n\|_X^2 \\ & \leq \frac{12\nu}{28} \sum_{n=1}^m \tau_n \|\mathbf{e}^n\|_X^2 + \frac{2\nu}{28\sigma_\tau} \sum_{n=1}^m \tau_n \|\mathbf{e}^{n-1}\|_X^2 + \frac{1}{2} \sum_{n=1}^m \|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{L^2(\Omega)}^2 \\ & + C \sum_{n=1}^m \tau_n \left(\sum_{\kappa \in \mathcal{T}_{n,h}} (h_\kappa^2 \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\kappa)}^2 + \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\kappa)}^2) \right) + C \sum_{n=1}^m \left(\sum_{\kappa \in \mathcal{T}_{n,h}} (\tau_n (\eta_{n,\kappa}^{h,1})^2 + \tau_n (\eta_{n,\kappa}^{h,2})^2) \right) \\ & + C \sum_{n=1}^m \tau_n (\|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\Omega)}^2 + \|\operatorname{div} \bar{\mathbf{u}}_h^{n-1}\|_{L^2(\Omega)}^2) + C \sum_{n=1}^m \tau_n (\|\mathbf{u}^n\|_X^2 + \|\bar{\mathbf{u}}_h^n\|_X^2 + \|\bar{\mathbf{u}}_h^{n-1}\|_X^2) \|\mathbf{e}^{n-1}\|_{L^2(\Omega)}^2. \end{aligned} \quad (64)$$

Moreover, we perform a change of indices, we use (7) to bound τ_{n+1} by $\sigma_\tau \tau_n$; we also use the fact that $\mathbf{e}^0 = 0$ and $\bar{\mathbf{u}}_h^0 = 0$ and the fact that $\|\operatorname{div} \bar{\mathbf{u}}_h^n\|_{L^2(\kappa)}^2 \leq (\eta_{n,\kappa}^{h,1})^2$. Simplifying, we obtain

$$\begin{aligned} & \frac{1}{2} \|\mathbf{e}^m\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \sum_{n=1}^m \tau_n \|\mathbf{e}^n\|_X^2 \leq C \sum_{n=0}^{m-1} \tau_{n+1} (\|\mathbf{u}^{n+1}\|_X^2 + \|\bar{\mathbf{u}}_h^{n+1}\|_X^2 + \|\bar{\mathbf{u}}_h^n\|_X^2) \|\mathbf{e}^n\|_{L^2(\Omega)}^2 \\ & + C \sum_{n=1}^m \tau_n \left(\sum_{\kappa \in \mathcal{T}_{n,h}} (h_\kappa^2 \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\kappa)}^2 + \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\kappa)}^2) \right) + C \sum_{n=1}^m \left(\sum_{\kappa \in \mathcal{T}_{n,h}} (\tau_n (\eta_{n,\kappa}^{h,1})^2 + \tau_n (\eta_{n,\kappa}^{h,2})^2) \right). \end{aligned}$$

We apply the Gronwall Lemma 2.10 with the following functions:

$$\begin{aligned} y_m &= \|\mathbf{e}^m\|_{L^2(\Omega)}^2 + \nu \sum_{n=1}^m \tau_n \|\mathbf{e}^n\|_X^2, \\ f_m &= C \sum_{n=1}^m \tau_n \left(\sum_{\kappa \in \mathcal{T}_{n,h}} (h_\kappa^2 \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\kappa)}^2 + \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\kappa)}^2) + (\eta_{n,\kappa}^{h,1})^2 + (\eta_{n,\kappa}^{h,2})^2 \right) \\ \text{and} \\ g_n &= C \tau_{n+1} (\|\mathbf{u}^{n+1}\|_X^2 + \|\bar{\mathbf{u}}_h^{n+1}\|_X^2 + \|\bar{\mathbf{u}}_h^n\|_X^2). \end{aligned}$$

We obtain:

$$\begin{aligned} & \|\mathbf{e}^m\|_{L^2(\Omega)}^2 + \nu \sum_{n=1}^m \tau_n \|\mathbf{e}^n\|_X^2 \leq f_m \\ & + C \sum_{n=0}^{m-1} f_n \tau_{n+1} (\|\mathbf{u}^{n+1}\|_X^2 + \|\bar{\mathbf{u}}_h^{n+1}\|_X^2 + \|\bar{\mathbf{u}}_h^n\|_X^2) \times \exp \left(C \sum_{j=n}^{m-1} \tau_{j+1} (\|\mathbf{u}^{j+1}\|_X^2 + \|\bar{\mathbf{u}}_h^{j+1}\|_X^2 + \|\bar{\mathbf{u}}_h^j\|_X^2) \right). \end{aligned} \quad (65)$$

By remarking that for every $n \leq m$ it holds that $f_n \leq f_m$, and using the bounds provided by (9) and (15), the last line in (65) can be bounded by $C f_m$, and we finally obtain (46). This concludes the proof of Theorem 4.12 where we proved a *posteriori* error estimate between the velocity \mathbf{u}_τ of Problem (P_{aux}) and the velocity $\bar{\mathbf{u}}_h$ of Problem $(FV_{n,h})$. \square

The following lemma relates the integral of $\|(\mathbf{u}_\tau - \bar{\mathbf{u}}_h)\|_X^2$ with its values at the different time steps

Lemma 4.13. *It holds that*

$$\frac{1}{4} \sum_{n=1}^m \tau_n \|\mathbf{u}^n - \bar{\mathbf{u}}_h^n\|_X^2 \leq \sum_{n=1}^m \int_{t_{n-1}}^{t_n} \|\mathbf{u}_\tau(s) - \bar{\mathbf{u}}_h(s)\|_X^2 ds \leq \frac{1 + \sigma_\tau}{2} \sum_{n=1}^m \tau_n \|\mathbf{u}^n - \bar{\mathbf{u}}_h^n\|_X^2. \quad (66)$$

Proof. For the proof of this lemma, we refer to [6] page 15. \square

Corollary 4.14. *We suppose $h_n \leq C_s \tau_n, \forall n = 1 \dots m$ with $m \in \{1, \dots, N\}$. The following a posteriori error estimate holds between the velocity \mathbf{u} solution of problem (FV) and the solution $\bar{\mathbf{u}}_h$ associated to the solutions $\bar{\mathbf{u}}_h^n$ of problem $(\overline{FV}_{n,h})$:*

$$\begin{aligned} \|\mathbf{u}(t_m) - \bar{\mathbf{u}}_h^m\|_{L^2(\Omega)^2}^2 + \nu \int_0^{t_m} \|\mathbf{u}(s) - \bar{\mathbf{u}}_h(s)\|_X^2 ds &\leq C \left(\sum_{n=1}^m \sum_{\kappa \in \mathcal{T}_{n,h}} (\tau_n (\eta_{n,\kappa}^{h,1})^2 + \tau_n (\eta_{n,\kappa}^{h,2})^2 + (\eta_{n,\kappa}^\tau)^2) \right. \\ &\left. + \sum_{n=1}^m \tau_n \sum_{\kappa \in \mathcal{T}_{n,h}} (h_\kappa^2 \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\kappa)^2}^2 + \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\kappa)^2}^2 + \|\mathbf{f} - \pi_\tau \mathbf{f}\|_{[L^2((0,t_m) \times \Omega)]^2}^2) \right). \end{aligned} \quad (67)$$

Proof. We use the triangle inequality in the left-hand side of (67) by inserting $\mathbf{u}_\tau(t_m) = \mathbf{u}^m$ in its first term and the function \mathbf{u}_τ in its second term. Then, the proof is a direct consequence of Theorems 4.10 on the one hand and of the second inequality in (66) and Theorem 4.12 on the other hand. \square

Next, the following theorem can be found in Theorem 4.10 in [7] as far as the first inequality is concerned and follows the same steps as Theorem 4.10 in [6] as far as the second inequality is concerned.

Theorem 4.15. *The following a posteriori error estimate holds between the solution (\mathbf{u}, p) of Problem (FV) and $(\bar{\mathbf{u}}_h, \pi_\tau p_\tau)$ associated with the solutions of Problem $(\overline{FV}_{n,h})$: For $1 \leq n \leq N$,*

$$\sum_{n=1}^m \int_{t_{n-1}}^{t_n} \|\mathbf{u}(s) - \pi_\tau \bar{\mathbf{u}}_h(s)\|_X^2 ds \leq c \left(\int_0^{t_m} \|\mathbf{u}(s) - \bar{\mathbf{u}}_h(s)\|_X^2 ds + \sum_{n=1}^m \sum_{\kappa \in \mathcal{T}_{n,h}} (\eta_{n,\kappa}^\tau)^2 \right) \quad (68)$$

and

$$\begin{aligned} &\left\| \frac{\partial}{\partial t} (\mathbf{u} - \bar{\mathbf{u}}_h) + \mathbf{u} \cdot \nabla \mathbf{u} - \pi_{l,\tau} \bar{\mathbf{u}}_h \cdot \nabla \pi_\tau \bar{\mathbf{u}}_h - \frac{1}{2} \operatorname{div} (\pi_{l,\tau} \bar{\mathbf{u}}_h) \pi_\tau \bar{\mathbf{u}}_h + \nabla (p - \bar{p}_h) \right\|_{L^2(0,t_m, X')} \\ &\leq C \left(\sum_{n=1}^m \sum_{\kappa \in \mathcal{T}_{n,h}} (\tau_n (\eta_{n,\kappa}^{h,1})^2 + \tau_n (\eta_{n,\kappa}^{h,2})^2 + (\eta_{n,\kappa}^\tau)^2) + \|\mathbf{f} - \pi_\tau \mathbf{f}\|_{[L^2((0,t_m) \times \Omega)]^2}^2 \right. \\ &\left. + \sum_{n=1}^m \sum_{\kappa \in \mathcal{T}_{n,h}} (\tau_n h_\kappa^2 \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\kappa)^2}^2 + \tau_n \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\kappa)^2}^2) \right)^{\frac{1}{2}}. \end{aligned} \quad (69)$$

Finally, Definition (14) of the $[\![\cdot]\!]$ norm, inequalities (68), (67) and (69) lead to the following result:

Corollary 4.16. *We suppose that $h_n \leq C_s \tau_n, \forall n = 1, \dots, m$ with $m \in \{1, \dots, N\}$. The pressure and the velocity verify the following a posteriori error bound:*

$$\begin{aligned} &[\![\mathbf{u} - \bar{\mathbf{u}}_h]\!]^2(t_m) + \left\| \frac{\partial}{\partial t} (\mathbf{u} - \bar{\mathbf{u}}_h) + \mathbf{u} \nabla \mathbf{u} - \pi_{l,\tau} \bar{\mathbf{u}}_h \nabla \pi_\tau \bar{\mathbf{u}}_h - \frac{1}{2} \operatorname{div} \pi_{l,\tau} \bar{\mathbf{u}}_h \pi_\tau \bar{\mathbf{u}}_h + \nabla (p - \bar{p}_h) \right\|_{L^2(0,t_m, X')}^2 \\ &\leq C \left(\sum_{n=1}^m \sum_{\kappa \in \mathcal{T}_{n,h}} (\tau_n (\eta_{n,\kappa}^{h,1})^2 + \tau_n (\eta_{n,\kappa}^{h,2})^2 + (\eta_{n,\kappa}^\tau)^2) + \|\mathbf{f} - \pi_\tau \mathbf{f}\|_{[L^2((0,t_m) \times \Omega)]^2}^2 \right. \\ &\left. + \sum_{n=1}^m \sum_{\kappa \in \mathcal{T}_{n,h}} (\tau_n h_\kappa^2 \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\kappa)^2}^2 + \tau_n \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\kappa)^2}^2) \right). \end{aligned} \quad (70)$$

Remark 4.17. *Due to the use of the Gronwall Lemma and of Lemma 2.11, the dependence of the constant C in (70) with respect to ν involves an exponential of $\frac{1}{\nu}$. This may look very unfavorable as ν is small in turbulent flows. However, from a practical point of view, numerical experiments presented in Section 6.2 show that this dependence is not that dramatic.*

5. UPPER BOUNDS OF THE INDICATORS

In this section, we prove three theorems regarding upper bounds for the error indicators proposed in Section 4. We follow exactly the same steps like in [6, Theorems 4.11 and 4.12] by simply changing the form of the non-linear terms. Taken together, these theorems will establish the following general estimate

$$\tau_n(\eta_{n,\kappa}^{h,1})^2 + \tau_n(\eta_{n,\kappa}^{h,2})^2 + (\eta_{n,\kappa}^\tau)^2 \leq L_{w_\kappa}, \quad (71)$$

where L_{w_κ} is an expression depending on the errors, on the data approximations, and on terms related to the LES viscosity, evaluated on the union of the elements of \mathcal{T}_{nh} sharing a face with κ . The discussion is presented in terms of three theorems, each of which providing an upper bound of one term in the left-hand side of (71). In all, suitable regularity assumptions on the exact solution are performed.

Theorem 5.1. *Let $\mathbb{D}(\mathbf{u}) \in L^\infty(0, T, \Omega)$ and suppose that $h_n \leq C_s \tau_n$. The following estimate holds:*

$$\tau_n(\eta_{n,\kappa}^{h,1})^2 \leq CL(w_\kappa), \quad (72)$$

where w_κ denotes the union of the elements of \mathcal{T}_{nh} that share at least a face with κ and

$$\begin{aligned} L(W) &= |\mathbf{u} - \bar{\mathbf{u}}_h^n|_{L^2(t_{n-1}, t_n, H^1(W)^2)}^2 + |\mathbf{u} - \bar{\mathbf{u}}_h^{n-1}|_{L^2(t_{n-1}, t_n, H^1(W)^2)}^2 \\ &+ h_\kappa^2 \|\mathbf{f} - \bar{\mathbf{f}}^n\|_{L^2(t_{n-1}, t_n, L^2(W)^2)}^2 + \tau_n \sum_{\kappa' \subset W} h_{\kappa'}^2 \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\kappa')^2}^2 + \tau_n \sum_{\kappa' \subset W} h_{\kappa'}^2 \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\kappa')^2}^2 \\ &+ \left\| \frac{\partial}{\partial t} (\mathbf{u} - \bar{\mathbf{u}}_h) + \mathbf{u} \cdot \nabla \mathbf{u} - \pi_{t,\tau} \bar{\mathbf{u}}_h \cdot \nabla \pi_\tau \bar{\mathbf{u}}_h - \frac{1}{2} \operatorname{div} \pi_{t,\tau} \bar{\mathbf{u}}_h \pi_\tau \bar{\mathbf{u}}_h + \nabla(p - \bar{p}_h) \right\|_{L^2(t_{n-1}, t_n, H^{-1}(W)^2)}^2 \\ &+ \|\nu_t(\mathbf{u}) - \nu_{th}(\mathbf{u})\|_{L^2(t_{n-1}, t_n, L^2(W))}^2 + \|(\nu_t(\mathbf{u})\mathbb{D}(\mathbf{u}))\|_{L^2(t_{n-1}, t_n, L^2(W))}^2. \end{aligned} \quad (73)$$

Proof. We proceed in 4 steps:

(1) We consider $t \in]t_{n-1}, t_n[$ and we insert $\nu_t(\bar{\mathbf{u}}_h^{n-1})\mathbb{D}(\mathbf{u}(t))$, then $\nu_{th}(\bar{\mathbf{u}}_h^{n-1})\mathbb{D}(\mathbf{u}(t))$, then $\nu_t(\mathbf{u}(t))\mathbb{D}(\mathbf{u}(t))$ and $\nu_{th}(\mathbf{u}(t))\mathbb{D}(\mathbf{u}(t))$ in the fourth term of $(\eta_{n,\kappa}^{h,1})^2$ defined in (28) and we obtain :

$$\begin{aligned} &\|(\nu_t(\bar{\mathbf{u}}_h^{n-1}) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1}))\mathbb{D}(\bar{\mathbf{u}}_h^n)\|_{L^2(\kappa)}^2 \leq C \left(\|(\nu_t(\bar{\mathbf{u}}_h^{n-1}) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1}))\mathbb{D}(\bar{\mathbf{u}}_h^n - \mathbf{u}(t))\|_{L^2(\kappa)}^2 \right. \\ &+ \|(\nu_t(\bar{\mathbf{u}}_h^{n-1}) - \nu_t(\mathbf{u}(t)))\mathbb{D}(\mathbf{u}(t))\|_{L^2(\kappa)}^2 + \|(\nu_{th}(\mathbf{u}(t)) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1}))\mathbb{D}(\mathbf{u}(t))\|_{L^2(\kappa)}^2 \\ &\left. + \|(\nu_t(\mathbf{u}(t)) - \nu_{th}(\mathbf{u}(t)))\mathbb{D}(\mathbf{u}(t))\|_{L^2(\kappa)}^2 \right). \end{aligned} \quad (74)$$

We integrate from t_{n-1} to t_n , we bound the first term in the right-hand side of (74) by:

$$\begin{aligned} &\int_{t_{n-1}}^{t_n} \|(\nu_t(\bar{\mathbf{u}}_h^{n-1}) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1}))\mathbb{D}(\bar{\mathbf{u}}_h^n - \mathbf{u}(t))\|_{L^2(\kappa)}^2 dt \\ &= \int_{t_{n-1}}^{t_n} \left(\int_\kappa \left[(c_s h_\kappa^2 \|\mathbb{D}(\bar{\mathbf{u}}_h^{n-1})\|_F - \frac{1}{|\kappa|} \int_\kappa c_s h_\kappa^2 \|\mathbb{D}(\bar{\mathbf{u}}_h^{n-1})\|_F d\mathbf{x}) \mathbb{D}(\bar{\mathbf{u}}_h^n - \mathbf{u}(t)) \right]^2 d\mathbf{x} \right) dt \\ &\leq c_s^2 h_\kappa^4 \|\mathbb{D}(\bar{\mathbf{u}}_h^{n-1})\|_{L^\infty(\kappa)}^2 \int_{t_{n-1}}^{t_n} \|\mathbb{D}(\bar{\mathbf{u}}_h^n - \mathbf{u})\|_{L^2(\kappa)}^2 dt. \end{aligned}$$

By using the inverse inequality (13), the fact that $h_\kappa \leq c\tau_n$ and equation (9), we have:

$$\begin{aligned} &\int_{t_{n-1}}^{t_n} \|(\nu_t(\bar{\mathbf{u}}_h^{n-1}) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1}))\mathbb{D}(\bar{\mathbf{u}}_h^n - \mathbf{u}(t))\|_{L^2(\kappa)}^2 dt \\ &\leq Ch_\kappa \tau_n \|\mathbb{D}(\bar{\mathbf{u}}_h^{n-1})\|_{L^2(\kappa)}^2 \int_{t_{n-1}}^{t_n} \|\mathbb{D}(\bar{\mathbf{u}}_h^n - \mathbf{u}(t))\|_{L^2(\kappa)}^2 dt \\ &\leq Ch_\kappa |\bar{\mathbf{u}}_h^n - \mathbf{u}|_{L^2(t_{n-1}, t_n, H^1(\kappa)^2)}^2. \end{aligned}$$

By using that $\|\mathbb{D}(\mathbf{u})\|_{L^\infty(0,T,\Omega)}$ is bounded, we bound the second term in the right-hand side of (74) by:

$$\begin{aligned} & \int_{t_{n-1}}^{t_n} \|(\nu_t(\bar{\mathbf{u}}_h^{n-1}) - \nu_t(\mathbf{u}(t)))\mathbb{D}(\mathbf{u})\|_{L^2(\kappa)}^2 dt \\ &= \int_{t_{n-1}}^{t_n} \left(\int_{\kappa} \left[c_s h_\kappa^2 (\|\mathbb{D}(\bar{\mathbf{u}}_h^{n-1})\|_F - \|\mathbb{D}(\mathbf{u}(t))\|_F) \mathbb{D}(\mathbf{u}(t)) \right]^2 d\mathbf{x} \right) dt \\ &\leq c_s^2 h_\kappa^4 \|\mathbb{D}(\mathbf{u})\|_{L^\infty(0,T,\Omega)} \int_{t_{n-1}}^{t_n} \|\mathbb{D}(\mathbf{u} - \bar{\mathbf{u}}_h^{n-1})\|_{L^2(\kappa)}^2 dt \\ &\leq C h_\kappa^4 |\mathbf{u} - \bar{\mathbf{u}}_h^{n-1}|_{L^2(t_{n-1}, t_n, H^1(\kappa)^2)}. \end{aligned}$$

The operator defined in (22) verifies the following property: For every $\bar{\mathbf{v}}_1$ and $\bar{\mathbf{v}}_2 \in Z$, we have:

$$\|\nu_{th}(\bar{\mathbf{v}}_1) - \nu_{th}(\bar{\mathbf{v}}_2)\|_{L^2(\kappa)} \leq c_s h_\kappa^2 \|\mathbb{D}(\bar{\mathbf{v}}_1 - \bar{\mathbf{v}}_2)\|_{L^2(\kappa)}. \quad (75)$$

Indeed we have

$$|\nu_{th}(\bar{\mathbf{v}}_1) - \nu_{th}(\bar{\mathbf{v}}_2)| \leq \frac{1}{|\kappa|} \int_{\kappa} |\nu_t(\bar{\mathbf{v}}_1(\mathbf{x})) - \nu_t(\bar{\mathbf{v}}_2(\mathbf{x}))| d\mathbf{x} \leq \frac{1}{|\kappa|} \int_{\kappa} c_s h_\kappa^2 \|\mathbb{D}(\bar{\mathbf{v}}_1 - \bar{\mathbf{v}}_2)(\mathbf{x})\|_F d\mathbf{x}.$$

By using the Hölder inequality, we obtain (75). This leads to the following bound of the third term in the right-hand side of (74):

$$\begin{aligned} \int_{t_{n-1}}^{t_n} \|(\nu_{th}(\mathbf{u}(t)) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1}))\mathbb{D}(\mathbf{u}(t))\|_{L^2(\kappa)}^2 dt &\leq \|\mathbb{D}(\mathbf{u})\|_{L^\infty(0,T,\Omega)}^2 \int_{t_{n-1}}^{t_n} \|\nu_{th}(\mathbf{u}(t)) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1})\|_{L^2(\kappa)}^2 dt \\ &\leq C h_\kappa^4 |\mathbf{u} - \bar{\mathbf{u}}_h^{n-1}|_{L^2(t_{n-1}, t_n, H^1(\kappa)^2)}. \end{aligned}$$

Finally,

$$\int_{t_{n-1}}^{t_n} \|(\nu_t(\mathbf{u}(t)) - \nu_{th}(\mathbf{u}(t)))\mathbb{D}(\mathbf{u}(t))\|_{L^2(\kappa)}^2 dt \leq C \|\nu_t(\mathbf{u}) - \nu_{th}(\mathbf{u})\|_{L^2(t_{n-1}, t_n, L^2(\kappa))}^2.$$

By regrouping the previous inequalities, we obtain:

$$\begin{aligned} & \int_{t_{n-1}}^{t_n} \|(\nu_t(\bar{\mathbf{u}}_h^{n-1}) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1}))\mathbb{D}(\bar{\mathbf{u}}_h^{n-1})\|_{L^2(\kappa)}^2 dt \leq C \left(h_\kappa |\bar{\mathbf{u}}_h^n - \mathbf{u}|_{L^2(t_{n-1}, t_n, H^1(\kappa)^2)}^2 \right. \\ & \left. + h_\kappa^4 |\mathbf{u} - \bar{\mathbf{u}}_h^{n-1}|_{L^2(t_{n-1}, t_n, H^1(\kappa)^2)}^2 + \|\nu_t(\mathbf{u}) - \nu_{th}(\mathbf{u})\|_{L^2(t_{n-1}, t_n, L^2(\kappa))}^2 \right), \end{aligned} \quad (76)$$

which provides a bound for the fourth term of $(\eta_{n,\kappa}^{h,1})^2$.

Next, the solution (\mathbf{u}, p) of problem (FV) and the solution $(\bar{\mathbf{u}}_h, \bar{p}_h)$ associated with the solution $(\bar{\mathbf{u}}_h^n, \bar{p}_h^n)_{0 \leq n \leq m}$ of Problem $(\overline{FV}_{n,h})$ verify: $\forall \mathbf{v} \in Z$ and $t \in]t_{n-1}, t_n], 1 \leq n \leq m$,

$$\begin{aligned} & (\bar{\mathbf{f}}_h^n - \frac{1}{\tau_n}(\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}_h^{n-1}), \mathbf{v}) + (\operatorname{div} \mathbf{v}, \bar{p}_h^n) - (\bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n, \mathbf{v}) - \frac{1}{2}(\operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n, \mathbf{v}) \\ & - \nu(\nabla \bar{\mathbf{u}}_h^n, \nabla \mathbf{v}) - (\nu_{th}(\bar{\mathbf{u}}_h^{n-1})\mathbb{D}(\bar{\mathbf{u}}_h^n), \mathbb{D}(\mathbf{v})) \\ &= \left\langle \frac{\partial}{\partial t}(\mathbf{u} - \bar{\mathbf{u}}_h)(t), \mathbf{v} \right\rangle + \nu(\nabla(\mathbf{u}(t) - \bar{\mathbf{u}}_h^n), \nabla \mathbf{v}) + \int_{\Omega} [\mathbf{u}(t) \cdot \nabla \mathbf{u}(t) - \bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n] \cdot \mathbf{v}(t) d\mathbf{x} \\ & - \frac{1}{2}(\operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p(t) - \bar{p}_h(t)) - (\mathbf{f}(t) - \bar{\mathbf{f}}^n, \mathbf{v}) - (\bar{\mathbf{f}}^n - \bar{\mathbf{f}}^n, \mathbf{v}) \\ & + \int_{\Omega} (\nu_t(\mathbf{u}(t))\mathbb{D}(\mathbf{u})(t) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1})\mathbb{D}(\bar{\mathbf{u}}_h^n)) : \mathbb{D}(\mathbf{v}) d\mathbf{x} - \int_{\Omega} \nu_t(\mathbf{u}(t))\mathbb{D}(\mathbf{u})(t) : \mathbb{D}(\mathbf{v}) d\mathbf{x}. \end{aligned} \quad (77)$$

This equality will help us bound successively the first and second terms of $(\eta_{n,\kappa}^{h,1})^2$ for a given κ .

(2) Regarding the first term in $(\eta_{n,\kappa}^{h,1})^2$, we set

$$\alpha_h^n = (\bar{\mathbf{f}}_h^n - \frac{1}{\tau_n}(\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}_h^{n-1}) - \nabla \bar{p}_h^n + \nu \Delta \bar{\mathbf{u}}_h^n + \nabla \cdot (\nu_{th}(\bar{\mathbf{u}}_h^{n-1})\mathbb{D}(\bar{\mathbf{u}}_h^n)) - \bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n - \frac{1}{2} \operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n)$$

and choose in (77) :

$$\mathbf{v} = \mathbf{v}_\kappa = \begin{cases} \alpha_h^n \psi_\kappa & \text{on } \kappa, \\ 0 & \text{on } \Omega \setminus \kappa, \end{cases}$$

where ψ_κ is the bubble function which is equal to the product of the barycentric coordinates associated with the vertices of κ . Integrating between t_{n-1} and t_n and using the Cauchy-Schwarz inequality yields

$$\begin{aligned} & \tau_n \|\alpha_h^n \psi_\kappa^{1/2}\|_{L^2(\kappa)}^2 \\ & \leq \int_{t_{n-1}}^{t_n} \left(\nu |\mathbf{u} - \bar{\mathbf{u}}_h^n|_{H^1(\kappa)} |\mathbf{v}_\kappa|_{H^1(\kappa)} + (\|\mathbf{f} - \mathbf{f}^n\|_{L^2(\kappa)} + \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\kappa)} + \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\kappa)}) \|\mathbf{v}_\kappa\|_{L^2(\kappa)} \right) \\ & + \left\| \frac{\partial}{\partial t} (\mathbf{u} - \bar{\mathbf{u}}_h) + \mathbf{u} \cdot \nabla \mathbf{u} - \bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n - \frac{1}{2} \operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n + \nabla(p - \bar{p}_h) \right\|_{X'(\kappa)}(t) \|\mathbf{v}_\kappa\|_{H^1(\kappa)} dt \\ & + \int_{t_{n-1}}^{t_n} \left(\int_\kappa (\nu_t(\mathbf{u})(t) \mathbb{D}(\mathbf{u})(t) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n)) : \mathbb{D}(\mathbf{v}_\kappa) d\mathbf{x} \right) dt - \int_{t_{n-1}}^{t_n} \left(\int_\kappa \nu_t(\mathbf{u})(t) \mathbb{D}(\mathbf{u})(t) : \mathbb{D}(\mathbf{v}_\kappa) d\mathbf{x} \right) dt. \end{aligned}$$

The Cauchy-Schwarz inequality allows to write:

$$\begin{aligned} & \tau_n \|\alpha_h^n \psi_\kappa^{1/2}\|_{L^2(\kappa)}^2 \leq \nu \tau_n^{\frac{1}{2}} \|\mathbf{u} - \bar{\mathbf{u}}_h^n\|_{L^2(t_{n-1}, t_n, H^1(\kappa))} \|\mathbf{v}_\kappa\|_{H^1(\kappa)} \\ & + (\tau_n^{\frac{1}{2}} \|\mathbf{f} - \mathbf{f}^n\|_{L^2(t_{n-1}, t_n, L^2(\kappa))} + \tau_n \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\kappa)} + \tau_n \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\kappa)}) \|\mathbf{v}_\kappa\|_{L^2(\kappa)} \\ & + \tau_n^{\frac{1}{2}} \left\| \frac{\partial}{\partial t} (\mathbf{u} - \bar{\mathbf{u}}_h) + \mathbf{u} \cdot \nabla \mathbf{u} - \bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n - \frac{1}{2} \operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n + \nabla(p - \bar{p}_h) \right\|_{L^2(t_{n-1}, t_n, H^{-1}(\kappa))} \|\mathbf{v}_\kappa\|_{H^1(\kappa)} \\ & + \tau_n^{\frac{1}{2}} (\|\nu_t(\mathbf{u})(t) \mathbb{D}(\mathbf{u})(t) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n)\|_{L^2(t_{n-1}, t_n, L^2(\kappa))} + \|\nu_t(\mathbf{u})(t) \mathbb{D}(\mathbf{u})(t)\|_{L^2(t_{n-1}, t_n, L^2(\kappa))}) \|\mathbf{v}_\kappa\|_{H^1(\kappa)}. \end{aligned}$$

Multiplying the above inequality with h_κ^2 and using Property 4.1, we obtain:

$$\begin{aligned} & \tau_n h_\kappa^2 \|\alpha_h^n \psi_\kappa^{1/2}\|_{L^2(\kappa)}^2 \leq C \tau_n^{\frac{1}{2}} h_\kappa \|\mathbf{v}_\kappa\|_{L^2(\kappa)} (\nu \|\mathbf{u} - \bar{\mathbf{u}}_h^n\|_{L^2(t_{n-1}, t_n, H^1(\kappa))}) \\ & + h_\kappa \|\mathbf{f} - \mathbf{f}^n\|_{L^2(t_{n-1}, t_n, L^2(\kappa))} + \tau_n^{\frac{1}{2}} h_\kappa \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\kappa)} + \tau_n^{\frac{1}{2}} h_\kappa \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\kappa)} \\ & + \left\| \frac{\partial}{\partial t} (\mathbf{u} - \bar{\mathbf{u}}_h)(t) + \mathbf{u} \cdot \nabla \mathbf{u} - \bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n - \frac{1}{2} \operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n + \nabla(p - \bar{p}_h) \right\|_{L^2(t_{n-1}, t_n, H^{-1}(\kappa))} \\ & + \|\nu_t(\mathbf{u})(t) \mathbb{D}(\mathbf{u})(t) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n)\|_{L^2(t_{n-1}, t_n, L^2(\kappa))} + \|\nu_t(\mathbf{u})(t) \mathbb{D}(\mathbf{u})(t)\|_{L^2(t_{n-1}, t_n, L^2(\kappa))}. \end{aligned}$$

Now, we replace \mathbf{v}_κ by $\alpha_h^n \psi_\kappa$ and use Property 4.1 to get rid of $\psi_\kappa^{1/2}$ in the norm in the left-hand side and to get rid of ψ_κ in the norm of \mathbf{v}_κ in the right-hand side. Moreover, we use Lemma 2.11 and we obtain

$$\begin{aligned} & \tau_n h_\kappa^2 \|\alpha_h^n\|_{L^2(\kappa)}^2 \leq \frac{1}{2} \tau_n h_\kappa^2 \|\alpha_h^n\|_{L^2(\kappa)}^2 \\ & + C \left(\|\mathbf{u} - \bar{\mathbf{u}}_h^n\|_{L^2(t_{n-1}, t_n, H^1(\kappa))}^2 + h_\kappa^2 \|\mathbf{f} - \mathbf{f}^n\|_{L^2(t_{n-1}, t_n, L^2(\kappa))}^2 + \tau_n h_\kappa^2 \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\kappa)}^2 + \tau_n h_\kappa^2 \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\kappa)}^2 \right. \\ & + \left\| \frac{\partial}{\partial t} (\mathbf{u} - \bar{\mathbf{u}}_h)(t) + \mathbf{u} \cdot \nabla \mathbf{u} - \bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n - \frac{1}{2} \operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n + \nabla(p - \bar{p}_h) \right\|_{L^2(t_{n-1}, t_n, H^{-1}(\kappa))}^2 \\ & \left. + \|\nu_t(\mathbf{u})(t) \mathbb{D}(\mathbf{u})(t) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n)\|_{L^2(t_{n-1}, t_n, L^2(\kappa))}^2 + \|\nu_t(\mathbf{u})(t) \mathbb{D}(\mathbf{u})(t)\|_{L^2(t_{n-1}, t_n, L^2(\kappa))}^2 \right). \quad (78) \end{aligned}$$

We insert $\nu_t(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n)$ and $\nu_t(\mathbf{u}(t)) \mathbb{D}(\bar{\mathbf{u}}_h^n)$ into $\|\nu_t(\mathbf{u})(t) \mathbb{D}(\mathbf{u})(t) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n)\|_{L^2(\kappa)}^2$:

$$\begin{aligned} & \|\nu_t(\mathbf{u}(t)) \mathbb{D}(\mathbf{u}(t)) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n)\|_{L^2(\kappa)}^2 \\ & \leq C \left(\|\nu_t(\mathbf{u}(t)) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1})\|_{L^2(\kappa)}^2 + \|\nu_t(\mathbf{u}(t)) \mathbb{D}(\mathbf{u}(t)) - \mathbb{D}(\bar{\mathbf{u}}_h^n)\|_{L^2(\kappa)}^2 \right. \\ & \left. + \|\nu_t(\bar{\mathbf{u}}_h^{n-1}) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1})\|_{L^2(\kappa)}^2 \right). \quad (79) \end{aligned}$$

By integrating between t_{n-1} and t_n , and using the inverse inequality (13), the fact $h_\kappa \leq c\tau_n$ and the bound (9), we bound the first term in the right-hand side of (79) by:

$$\begin{aligned} \int_{t_{n-1}}^{t_n} \|(\nu_t(\mathbf{u}(t)) - \nu_t(\bar{\mathbf{u}}_h^{n-1}))\mathbb{D}(\bar{\mathbf{u}}_h^n)\|_{L^2(\kappa)}^2 dt &= \int_{t_{n-1}}^{t_n} \left(\int_\kappa \left[c_s h_\kappa^2 (\|\mathbb{D}(\mathbf{u}(t))\|_F - \|\mathbb{D}(\bar{\mathbf{u}}_h^{n-1})\|_F) \mathbb{D}(\bar{\mathbf{u}}_h^n) \right]^2 d\mathbf{x} \right) dt \\ &\leq c_s^2 h_\kappa^4 \int_{t_{n-1}}^{t_n} \left(\int_\kappa \left[\|\mathbb{D}(\mathbf{u}(t) - \bar{\mathbf{u}}_h^{n-1})\|_F \|\mathbb{D}(\bar{\mathbf{u}}_h^n)\| \right]^2 d\mathbf{x} \right) dt \\ &\leq c_s^2 h_\kappa^4 \|\mathbb{D}(\bar{\mathbf{u}}_h^n)\|_{L^\infty(\kappa)}^2 \int_{t_{n-1}}^{t_n} \|\mathbb{D}(\mathbf{u}(t) - \bar{\mathbf{u}}_h^{n-1})\|_{L^2(\kappa)}^2 dt \\ &\leq Ch_\kappa |\mathbf{u} - \bar{\mathbf{u}}_h^{n-1}|_{L^2(t_{n-1}, t_n, H^1(\kappa)^2)}. \end{aligned} \quad (80)$$

By using that $\mathbb{D}(\mathbf{u}) \in L^\infty(0, T, \Omega)$, the second term in the right-hand side of (79) can be bounded by:

$$\begin{aligned} \int_{t_{n-1}}^{t_n} \|(\nu_t(\mathbf{u}(t))\mathbb{D}(\mathbf{u}(t)) - \mathbb{D}(\bar{\mathbf{u}}_h^n))\|_{L^2(\kappa)}^2 dt &= \int_{t_{n-1}}^{t_n} \left(\int_\kappa \left[c_s h_\kappa^2 \|\mathbb{D}(\mathbf{u}(t))\|_F \mathbb{D}(\mathbf{u}(t) - \bar{\mathbf{u}}_h^n) \right]^2 d\mathbf{x} \right) dt \\ &\leq c_s^2 h_\kappa^4 \|\mathbb{D}(\mathbf{u})\|_{L^\infty(0, T, \Omega)}^2 \int_{t_{n-1}}^{t_n} \|\mathbb{D}(\mathbf{u}(t) - \bar{\mathbf{u}}_h^n)\|_{L^2(\kappa)}^2 dt \\ &\leq Ch_\kappa^4 |\mathbf{u} - \bar{\mathbf{u}}_h^n|_{L^2(t_{n-1}, t_n, H^1(\kappa)^2)}. \end{aligned} \quad (81)$$

Integrating (79) between t_{n-1} and t_n , using the above two inequalities and (76), we obtain:

$$\begin{aligned} &\int_{t_{n-1}}^{t_n} \|\nu_t(\mathbf{u}(t))\mathbb{D}(\mathbf{u}(t)) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1})\mathbb{D}(\bar{\mathbf{u}}_h^n)\|_{L^2(\kappa)}^2 dt \\ &\leq C \left(|\bar{\mathbf{u}}_h^n - \mathbf{u}|_{L^2(t_{n-1}, t_n, H^1(\kappa)^2)}^2 + |\mathbf{u} - \bar{\mathbf{u}}_h^{n-1}|_{L^2(t_{n-1}, t_n, H^1(\kappa)^2)}^2 + \|\nu_t(\mathbf{u}) - \nu_{th}(\mathbf{u})\|_{L^2(t_{n-1}, t_n, L^2(\kappa))}^2 \right). \end{aligned} \quad (82)$$

We replace (82) into (78) and simplify, we obtain:

$$\begin{aligned} \tau_n h_\kappa^2 \|\alpha_h^n\|_{L^2(\kappa)}^2 &\leq C \left(|\mathbf{u} - \bar{\mathbf{u}}_h^n|_{L^2(t_{n-1}, t_n, H^1(\kappa)^2)}^2 + |\mathbf{u} - \bar{\mathbf{u}}_h^{n-1}|_{L^2(t_{n-1}, t_n, H^1(\kappa)^2)}^2 \right. \\ &\quad + h_\kappa^2 \|\mathbf{f} - \mathbf{f}^n\|_{L^2(t_{n-1}, t_n, L^2(\kappa))}^2 + \tau_n h_\kappa^2 \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\kappa)}^2 + \tau_n h_\kappa^2 \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\kappa)}^2 \\ &\quad + \left\| \frac{\partial}{\partial t} (\mathbf{u} - \bar{\mathbf{u}}_h)(t) + \mathbf{u} \cdot \nabla \mathbf{u} - \bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n - \frac{1}{2} \operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n + \nabla(p - \bar{p}_h) \right\|_{L^2(t_{n-1}, t_n, H^{-1}(\kappa)^2)}^2 \\ &\quad \left. + \|\nu_t(\mathbf{u}) - \nu_{th}(\mathbf{u})\|_{L^2(t_{n-1}, t_n, L^2(\kappa))}^2 + \|(\nu_t(\mathbf{u})(t)\mathbb{D}(\mathbf{u})(t))\|_{L^2(t_{n-1}, t_n, L^2(\kappa))}^2 \right). \end{aligned}$$

We replace α_h^n by its value and we obtain:

$$\begin{aligned} \tau_n h_\kappa^2 \left\| \left(\bar{\mathbf{f}}_h^n - \frac{1}{\tau_n} (\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}_h^{n-1}) - \nabla \bar{p}_h^n + \nu \Delta \bar{\mathbf{u}}_h^n + \nabla \cdot (\nu_{th}(\bar{\mathbf{u}}_h^{n-1})\mathbb{D}(\bar{\mathbf{u}}_h^n)) - \bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n \right. \right. \\ \left. \left. - \frac{1}{2} \operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n \right) \right\|_{L^2(\kappa)}^2 \leq CL(\kappa), \end{aligned} \quad (83)$$

where we recall that $L(\cdot)$ is defined in (73).

(3) Regarding the second term in $(\eta_{n,\kappa}^{h,1})^2$, for every $e \in \varepsilon_\kappa$, we note by κ' the other element containing e . We introduce the function:

$$R_{n,e}^{h,1} = [(\nu \nabla \bar{\mathbf{u}}_h^n + \nu_{th}(\bar{\mathbf{u}}_h^{n-1})\mathbb{D}(\bar{\mathbf{u}}_h^n))\mathbf{n} - \bar{p}_h^n \mathbf{n}]_e$$

and, recalling that \mathcal{L}_e is defined at the beginning of Section 4, we choose in (77) $\mathbf{v} = \mathbf{v}_e = \mathcal{L}_e(R_{n,e}^{h,1}\psi_e)$ extended by 0 outside of $\tilde{\kappa} := \kappa \cup \kappa'$. Integrating by parts the diffusion and pressure terms, we notice that the resulting trace terms on e sum up to $\int_e R_{n,e}^{h,1}(R_{n,e}^{h,1}\psi_e)$; moving all other terms in the right-hand side

of the equality, integrating between t_{n-1} and t_n and applying the Cauchy-Schwarz inequality we obtain:

$$\begin{aligned}
\tau_n \|R_{n,e}^{h,1}(\psi_e)\|_{L^2(\epsilon)}^2 &\leq \tau_n^{\frac{1}{2}} \left(\nu |\mathbf{u} - \bar{\mathbf{u}}_h^n|_{L^2(t_{n-1}, t_n, H^1(\bar{\kappa}))} |\mathbf{v}_e|_{H^1(\bar{\kappa})} \right. \\
&+ \left(\|\mathbf{f} - \mathbf{f}^n\|_{L^2(t_{n-1}, t_n, L^2(\bar{\kappa}))} + \tau_n^{\frac{1}{2}} \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\bar{\kappa})} + \tau_n^{\frac{1}{2}} \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\bar{\kappa})} \right) \|\mathbf{v}_e\|_{L^2(\bar{\kappa})} \\
&+ \left\| \frac{\partial}{\partial t} (\mathbf{u} - \bar{\mathbf{u}}_h) + \nabla(p - \bar{p}_h) + \mathbf{u} \cdot \nabla \mathbf{u} - \bar{\mathbf{u}}_h^{n-1} \nabla \cdot \bar{\mathbf{u}}_h^n - \frac{1}{2} \operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n \right\|_{L^2(t_{n-1}, t_n, H^{-1}(\bar{\kappa}))} |\mathbf{v}_e|_{H^1(\bar{\kappa})} \\
&+ \tau_n^{\frac{1}{2}} \|\mathbf{v}_e\|_{L^2(\bar{\kappa})} \|\bar{\mathbf{f}}_h^n - \frac{1}{\tau_n} (\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}_h^{n-1}) + \nu \Delta \bar{\mathbf{u}}_h^n + \nabla \cdot (\nu_{th}(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n)) - \bar{\mathbf{u}}_h^{n-1} \nabla \bar{\mathbf{u}}_h^n \\
&- \frac{1}{2} \operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n - \nabla \bar{p}_h^n \|_{L^2(\bar{\kappa})} \\
&+ \left(\|\nu_t(\mathbf{u})(t) \mathbb{D}(\mathbf{u})(t) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n)\|_{L^2(t_{n-1}, t_n, L^2(\bar{\kappa}))} + \|\nu_t(\mathbf{u})(t) \mathbb{D}(\mathbf{u})(t)\|_{L^2(t_{n-1}, t_n, L^2(\bar{\kappa}))} \right) |\mathbf{v}_e|_{H^1(\bar{\kappa})} \Big).
\end{aligned}$$

Multiplying by h_e , and using Property 4.2 for $\mathbf{v}_e = \mathcal{L}_e(R_{n,e}^{h,1}\psi_e)$, we obtain:

$$\begin{aligned}
\tau_n h_e \|R_{n,e}^{h,1}(\psi_e)\|_{L^2(\epsilon)}^2 &\leq C \tau_n^{\frac{1}{2}} h_e^{\frac{1}{2}} \|R_{n,e}^{h,1}(\psi_e)\|_{L^2(\bar{\kappa})} \left(|\mathbf{u} - \bar{\mathbf{u}}_h^n|_{L^2(t_{n-1}, t_n, H^1(\bar{\kappa}))} \right. \\
&+ h_e \left(\|\mathbf{f} - \mathbf{f}^n\|_{L^2(t_{n-1}, t_n, L^2(\bar{\kappa}))} + \tau_n^{\frac{1}{2}} \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\bar{\kappa})} + \tau_n^{\frac{1}{2}} \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\bar{\kappa})} \right) \\
&+ \left\| \frac{\partial}{\partial t} (\mathbf{u} - \bar{\mathbf{u}}_h) + \nabla(p - \bar{p}_h) + \mathbf{u} \cdot \nabla \mathbf{u} - \bar{\mathbf{u}}_h^{n-1} \nabla \cdot \bar{\mathbf{u}}_h^n - \frac{1}{2} \operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n \right\|_{L^2(t_{n-1}, t_n, H^{-1}(\bar{\kappa}))} \\
&+ h_e \tau_n^{\frac{1}{2}} \|\bar{\mathbf{f}}_h^n - \frac{1}{\tau_n} (\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}_h^{n-1}) + \nu \Delta \bar{\mathbf{u}}_h^n + \nabla \cdot (\nu_{th}(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n)) - \bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n \\
&- \frac{1}{2} \operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n - \nabla \bar{p}_h^n \|_{L^2(\bar{\kappa})} \\
&+ \left. \|\nu_t(\mathbf{u})(t) \mathbb{D}(\mathbf{u})(t) - \nu_{th}(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n)\|_{L^2(t_{n-1}, t_n, L^2(\bar{\kappa}))} + \|\nu_t(\mathbf{u})(t) \mathbb{D}(\mathbf{u})(t)\|_{L^2(t_{n-1}, t_n, L^2(\bar{\kappa}))} \right].
\end{aligned}$$

Using (82), the fact $\psi_e \leq 1$ implies $\psi_e \leq \psi_e^{\frac{1}{2}}$ and using Lemma 2.11, we obtain:

$$\begin{aligned}
\tau_n h_e \|R_{n,e}^{h,1}(\psi_e)\|_{L^2(\epsilon)}^2 &\leq C \left(\nu |\mathbf{u} - \bar{\mathbf{u}}_h^n|_{L^2(t_{n-1}, t_n, H^1(\bar{\kappa}))}^2 + |\mathbf{u} - \bar{\mathbf{u}}_h^{n-1}|_{L^2(t_{n-1}, t_n, H^1(\bar{\kappa}))}^2 \right. \\
&+ h_e^2 \|\mathbf{f} - \mathbf{f}^n\|_{L^2(t_{n-1}, t_n, L^2(\bar{\kappa}))}^2 + \tau_n h_e^2 \|\mathbf{f}^n - \bar{\mathbf{f}}^n\|_{L^2(\bar{\kappa})}^2 + \tau_n h_e^2 \|\bar{\mathbf{f}}^n - \bar{\mathbf{f}}_h^n\|_{L^2(\bar{\kappa})}^2 \\
&+ \left\| \frac{\partial}{\partial t} (\mathbf{u} - \bar{\mathbf{u}}_h) + \nabla(p - \bar{p}_h) + \mathbf{u} \cdot \nabla \mathbf{u} - \bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n - \frac{1}{2} \operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n \right\|_{L^2(t_{n-1}, t_n, H^{-1}(\bar{\kappa}))}^2 \\
&+ \tau_n h_e^2 \|\bar{\mathbf{f}}_h^n - \frac{1}{\tau_n} (\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}_h^{n-1}) + \nu \Delta \bar{\mathbf{u}}_h^n + \nabla \cdot (\nu_{th}(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n)) - \bar{\mathbf{u}}_h^{n-1} \cdot \nabla \bar{\mathbf{u}}_h^n \\
&- \frac{1}{2} \operatorname{div} \bar{\mathbf{u}}_h^{n-1} \bar{\mathbf{u}}_h^n - \nabla \bar{p}_h^n \|_{L^2(\bar{\kappa})}^2 \\
&+ \left. \|\nu_t(\mathbf{u}) - \nu_{th}(\mathbf{u})\|_{L^2(t_{n-1}, t_n, L^2(\bar{\kappa}))}^2 + \|\nu_t(\mathbf{u}) \mathbb{D}(\mathbf{u})\|_{L^2(t_{n-1}, t_n, L^2(\bar{\kappa}))}^2 \right) + \frac{1}{2} \tau_n h_e \|R_{n,e}^{h,1}(\psi_e)\|_{L^2(\epsilon)}^2.
\end{aligned}$$

Then we simplify, use the first property of Proposition 4.2, replace $R_{n,e}^{h,1}$ by its value, sum over $e \subset \partial\kappa$ using that $h_e \leq h_\kappa$ and then use (83); we obtain:

$$\tau_n \left(\sum_{e \in \varepsilon_\kappa} h_e \left\| \left(\nu \nabla \bar{\mathbf{u}}_h^n + \nu_{th}(\bar{\mathbf{u}}_h^{n-1}) \mathbb{D}(\bar{\mathbf{u}}_h^n) \right) \mathbf{n} - \bar{p}_h^n \mathbf{n} \right\|_{L^2(\epsilon)}^2 \right) \leq CL(w_\kappa). \quad (84)$$

(4) Regarding the third term in $(\eta_{n,\kappa}^{h,1})^2$, since $t \mapsto \bar{\mathbf{u}}_h(t, \mathbf{x})$ is an affine function with value $\bar{\mathbf{u}}_h^n(\mathbf{x})$ in $t = t_n$ and using $\operatorname{div} \mathbf{u}(t, \mathbf{x}) = 0$, we have

$$\begin{aligned}
\frac{\tau_n}{4} \|\operatorname{div} \bar{\mathbf{u}}_h^n(\mathbf{x})\|_{L^2(\kappa)}^2 &\leq \|\operatorname{div} \bar{\mathbf{u}}_h(t, \mathbf{x})\|_{L^2(t_{n-1}, t_n, L^2(\kappa))}^2 = \|\operatorname{div}(\mathbf{u}(t, \mathbf{x}) - \bar{\mathbf{u}}_h(t, \mathbf{x}))\|_{L^2(t_{n-1}, t_n, L^2(\kappa))}^2 \\
&\leq \|\mathbf{u}(t, \mathbf{x}) - \bar{\mathbf{u}}_h(t, \mathbf{x})\|_{L^2(t_{n-1}, t_n, H^1(\kappa))}^2. \quad (85)
\end{aligned}$$

Combining (76), (83), (84) and (85), we obtain the desired result. \square

Secondly, we estimate the indicator $\eta_{n,\kappa}^{h,2}$.

Theorem 5.2. Let $\mathbb{D}(\mathbf{u}) \in L^\infty(0, T, \Omega)$ and suppose that $h_n \leq C_s \tau_n$. We have the following estimate:

$$\begin{aligned} \tau_n (\eta_{n,\kappa}^{h,2})^2 &\leq C \left(h_\kappa |\bar{\mathbf{u}}_h^{n-1} - \mathbf{u}|_{L^2(t_{n-1}, t_n, H^1(\kappa)^2)}^2 \right. \\ &\quad \left. + h_\kappa^4 |\bar{\mathbf{u}}_h^n - \mathbf{u}|_{L^2(t_{n-1}, t_n, H^1(\kappa)^2)}^2 + \|\nu_t(\mathbf{u})\mathbb{D}(\mathbf{u})\|_{L^2(t_{n-1}, t_n, L^2(\kappa))}^2 \right). \end{aligned} \quad (86)$$

Proof. We write that $\tau_n (\eta_{n,\kappa}^{h,2})^2 = \int_{t_{n-1}}^{t_n} (\eta_{n,\kappa}^{h,2})^2 dt$ and we insert $\nu_t(\mathbf{u})(t)\mathbb{D}(\mathbf{u})(t)$ and $\nu_t(\mathbf{u})(t)\mathbb{D}(\bar{\mathbf{u}}_h^n)$ into $(\eta_{n,\kappa}^{h,2})^2$ and then use triangular inequalities. Using (80) and (81) we obtain the result. \square

Remark 5.3. In Theorems 5.1 and 5.2, the term $\|\nu_t(\mathbf{u})\mathbb{D}(\mathbf{u})\|_{L^2(t_{n-1}, t_n, L^2(\kappa))}^2$ comes from the fact that the LES solves a system that differs from the original Navier-Stokes equations. It is of order h_κ^4 because ν_t scales like h_κ^2 and we remark that it is a higher order term as compared to other error terms.

To conclude, we estimate $\eta_{n,\kappa}^\tau$.

Theorem 5.4. We have the following estimate:

$$(\eta_{n,\kappa}^\tau)^2 \leq C \left(|\mathbf{u} - \bar{\mathbf{u}}_h|_{L^2(t_{n-1}, t_n, H^1(\kappa)^2)}^2 + |\mathbf{u} - \pi_\tau \bar{\mathbf{u}}_h|_{L^2(t_{n-1}, t_n, H^1(\kappa)^2)}^2 \right). \quad (87)$$

Proof. Using the definition of $\bar{\mathbf{u}}_h(t)$, we obtain $\forall t \in [t_{n-1}, t_n]$:

$$\frac{t - t_n}{\tau_n} (\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}_h^{n-1}) = \bar{\mathbf{u}}_h(t) - \pi_\tau \bar{\mathbf{u}}_h(t).$$

Inserting $\mathbf{u}(t)$ in the right-hand side of this equality and taking the gradient of both sides, integrating on κ and using the triangular inequality, we obtain

$$\left| \frac{t - t_n}{\tau_n} \right|^2 \|\nabla(\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}_h^{n-1})(\mathbf{x})\|_{L^2(\kappa)}^2 \leq 2(\|\nabla(\mathbf{u} - \bar{\mathbf{u}}_h)(t, \mathbf{x})\|_{L^2(\kappa)}^2 + \|\nabla(\mathbf{u} - \pi_\tau \bar{\mathbf{u}}_h)(t, \mathbf{x})\|_{L^2(\kappa)}^2).$$

Integrating between t_{n-1} and t_n we obtain the result with $C = 6$. \square

As a final conclusion of Sections 4 and 5, we have obtained the equivalence of the error indicators of Definition 4.7 with the error between the exact solution of the Navier-Stokes equations and the discrete filtered solution, up to terms representing the data oscillation (with respect to time) for \mathbf{f} , the change of source term due to the filtering process ($\bar{\mathbf{f}}^n$ instead of \mathbf{f}^n), the approximation of \mathbf{f}^n by $\bar{\mathbf{f}}_h^n$, that of ν_t by ν_{th} and finally a term related to the additional turbulent diffusion operator.

6. NUMERICAL RESULTS

In this section, we present numerical simulations using the FreeFem++ code (see [20]) and the finite element spaces and time marching scheme discussed in Section 3. They show that LES provides lower error indicators and lower deviations from a reference solution than pure Navier-Stokes simulations; they also show that the efficiency index does not vary much with increasing Reynolds number and, finally, that, compared to uniform refinement, adaptive LES offers a substantial reduction in the number of degrees of freedom needed in the simulations to reach a given error indicator value.

We consider the domain Ω given by Figure 1 with the following properties: $AB = 1.5$, $BC = DE = 1$,

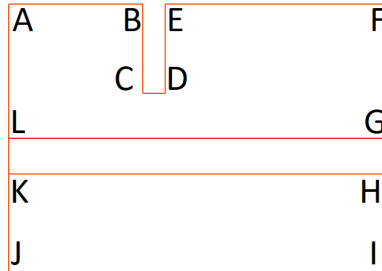


FIGURE 1. The domain

$CD = 0.25$, $EF = 2.45$, $FG = AL = 1.5$, $GH = LK = 0.4$, $HI = KJ = 1.1$, $LG = KH = JI = 4.2$.

Concerning uniform meshes, we divide the edges of $\partial\Omega$ into segments of equal lengths, and we define M to be the number of such mesh segments per unit length. For $M = 8$ this leads to a mesh with around 900 vertices and 1800 triangles, for a total of around 6500 degrees of freedom

We consider $\nu = \frac{1}{Re}$ where Re is the Reynolds number and we choose the density of body forces \mathbf{f} equal to $(-2, 0)$ in the rectangle $LGHK$ and to $(0, 0)$ elsewhere. This implies, as shown on Fig. 3, that the flow will move from the right to the left at the center of the domain; when the flow hits the left boundary, it splits into an upper flow and a lower flow; the $BCDE$ obstacle will cause recirculations in the upper flow. This will in turn generate turbulent interactions with the main flow at the center of the domain. In all the calculations, we chose the Smagorinsky constant to be $c_s = 0.1$.

From the calculation of $(\bar{\mathbf{u}}_h^n, \bar{p}_h^n)$ solution of $(\overline{FV}_{n,h})$ at each time step, we compute the error estimators of Definition 4.7; in the actual calculation we replace ν_{th} by ν_t for the sake of simplicity; this has almost no effect on the results.

6.1. Comparison of error indicators for calculations with or without LES. The indicator is defined by:

$$(\eta_n^h)^2 = \sum_{\kappa \in \mathcal{T}_{nh}} \tau_n (\eta_{n,\kappa}^{h,1})^2 + \tau_n (\eta_{n,\kappa}^{h,2})^2, \quad (\eta_n^\tau)^2 = \sum_{\kappa \in \mathcal{T}_{nh}} (\eta_{n,\kappa}^\tau)^2 \text{ and } \eta_n = ((\eta_n^h)^2 + (\eta_n^\tau)^2)^{\frac{1}{2}}. \quad (88)$$

To study the effect of the addition of the turbulent diffusion term in the LES method, we compare error indicators in time and space for simulations with or without LES for various values of $Re \in \{200, 5000\}$ and $M \in \{8, 16, 32, 64\}$. In all simulations the final time is set to $T = 3$. We choose the time step to be $\Delta t = \frac{1}{4M}$. We define :

$$\eta_{h,1} = \sqrt{\frac{\sum_n \sum_{\kappa \in \mathcal{T}_{nh}} \tau_n (\eta_{n,\kappa}^{h,1})^2}{\sum_n \tau_n |\bar{\mathbf{u}}_h^n|_{H^1(\Omega)}^2}}, \quad \eta_{h,2} = \sqrt{\frac{\sum_n \sum_{\kappa \in \mathcal{T}_{nh}} \tau_n (\eta_{n,\kappa}^{h,2})^2}{\sum_n \tau_n |\bar{\mathbf{u}}_h^n|_{H^1(\Omega)}^2}},$$

$$\eta_\tau = \sqrt{\frac{\sum_n \sum_{\kappa \in \mathcal{T}_{nh}} (\eta_{n,\kappa}^\tau)^2}{\sum_n \tau_n |\bar{\mathbf{u}}_h^n|_{H^1(\Omega)}^2}} \quad \text{and} \quad \eta = \sqrt{\frac{\sum_n (\eta_n)^2}{\sum_n \tau_n |\bar{\mathbf{u}}_h^n|_{H^1(\Omega)}^2}}.$$

The results are in Table 1 for $Re = 200$ and in Table 2 for $Re = 5000$. The total number of degrees of freedom in each simulation is indicated under the column # dof. It is clear that the space and time estimators using the LES method are smaller than those of the Navier-Stokes problem without LES. In addition, improvement is more important for high Reynolds numbers and coarse meshes, which is an indication that the LES method exactly plays the role expected from it. However, these conclusions are based only on observation of indicators; to confirm this trend, we will directly measure the difference between numerical solutions on different meshes and reference solutions obtained on the finest mesh that we could use given the resources at our disposal. This is the subject of the next sub-section.

		NS without LES					NS with LES				
		estimators				time (s)	estimators				time (s)
M	# dof	$\eta_{h,1}$	$\eta_{h,2}$	η_τ	η	CPU	$\eta_{h,1}$	$\eta_{h,2}$	η_τ	η	CPU
8	604K	0.324	0	0.028	0.325	28.5	0.260	7.5×10^{-4}	0.026	0.262	42.2
16	4.75M	0.149	0	0.014	0.150	211	0.139	2.6×10^{-4}	0.014	0.138	312
32	38.2M	0.0630	0	0.008	0.0639	1702	0.0618	8.1×10^{-5}	0.008	0.0623	2333
64	308M	0.0296	0	0.004	0.0299	14858	0.0295	2.1×10^{-5}	0.004	0.0298	19350

TABLE 1. Comparison between estimators with and without LES for $Re = 200$ and $T = 3$.

	NS without LES					NS with LES				
	estimators				time (s)	estimators				time(s)
M	$\eta_{h,1}$	$\eta_{h,2}$	η_τ	η	CPU	$\eta_{h,1}$	$\eta_{h,2}$	η_τ	η	CPU
8	0.552	0	0.060	0.555	26.5	0.396	8.2×10^{-4}	0.039	0.398	39.4
16	0.496	0	0.047	0.498	217	0.357	3.6×10^{-4}	0.044	0.359	307
32	0.444	0	0.036	0.445	1703	0.306	1.6×10^{-4}	0.035	0.308	2410
64	0.292	0	0.024	0.293	14688	0.226	6.4×10^{-5}	0.023	0.228	18880

TABLE 2. Comparison between estimators with and without LES for $Re = 5000$ and $T = 3$.

6.2. Comparisons of errors and estimators. In this section, we compare the relative norms of differences between a reference solution \mathbf{u}_F obtained by a simulation without LES on a fine mesh ($M = 256$ and $\Delta t = 0.25/M$) and solutions \mathbf{u}_C of simulations on coarse meshes without LES on the one hand and with LES on the other hand for different Reynolds numbers. This serves us as a measure of the errors between the numerical solutions obtained on coarse meshes and the exact, unknown solution. We also compare the error estimators and these measures of the errors in order to estimate the unknown constants appearing in the upper bounds of Section 4, see Remark 4.17. The relative difference norm is defined by:

$$e_{FC} = \sqrt{\frac{|\mathbf{u}_C - \mathbf{u}_F|_{H^1(\Omega)}^2}{|\mathbf{u}_F|_{H^1(\Omega)}^2}}.$$

The relative norms of the differences are in Table 3 and Table 4 for $Re \in \{200, 5000\}$ and $T = 3$. We note $M_F = 256$ the number of segments per unit length used to cut the edges of $\partial\Omega$ in the fine mesh and $M_C \in \{8, 16, 32, 64\}$ in the different coarse meshes. In addition, the difference between the solutions with and without LES is also calculated on the fine mesh (this is noted $M_F = 256$ and $M_C = 256$ in the tables below).

The tables show a general trend indicating that the differences between the LES solutions and the reference solution are smaller than those observed without LES. This trend is particularly notable for high Reynolds numbers and coarse meshes, on which solutions without LES are particularly inaccurate. This broadly confirms the observations made from the error estimators in the previous section.

	e_{FC} without LES	e_{FC} with LES
$M_F = 256$ and $M_C = 8$	0.743	0.628
$M_F = 256$ and $M_C = 16$	0.359	0.349
$M_F = 256$ and $M_C = 32$	0.164	0.166
$M_F = 256$ and $M_C = 64$	0.078	0.079
$M_F = 256$ and $M_C = 256$	–	5.19×10^{-4}

TABLE 3. Relative norm of the difference between the solution of the NS problem on a reference fine mesh and solution of the NS problem with and without LES on coarse meshes for $Re = 200$ and $T = 3$.

Figure 2 shows the ratio of e_{FC} in Tables 3 and 4 with estimator η in Tables 1 and 2 (the same vertical scale was chosen on both figures). This figure shows that this ratio, and thus the constant C appearing in (70), seem to increase with the Reynolds number, but not dramatically. On coarse meshes and without LES this constant seems to be larger, but remains moderate with LES.

6.3. Comparison between uniform and adapted LES problems. In the previous sections, we proved that solutions obtained with LES are more accurate than those obtained without LES. Therefore, we only use LES in this sub-section. We are now seeking to obtain more accurate simulations by using an algorithm for adapting the mesh and the time step using the associated estimators, seeking to balance the time estimator and the space estimator. We propose an adaptation algorithm described below. We measure its efficiency by comparing the estimators calculated on the one hand on uniform meshes and with

	e_{FC} without LES	e_{FC} with LES
$M_F = 256$ and $M_C = 8$	3.052	0.983
$M_F = 256$ and $M_C = 16$	2.617	0.963
$M_F = 256$ and $M_C = 32$	1.612	0.840
$M_F = 256$ and $M_C = 64$	0.975	0.694
$M_F = 256$ and $M_C = 256$	—	0.246

TABLE 4. Relative norm of the difference between the solution of the NS problem on a reference fine mesh and solution of the NS problem with and without LES on coarse meshes for $Re = 5000$ and $T = 3$.



FIGURE 2. Estimation of ratio of errors over estimators with (left) and without (right) LES.

constant time steps and on the other hand on meshes and with time steps adapted using our algorithm, which is driven by a tolerance denoted ε set by the user, chosen all the smaller as the desired precision is high. At each time step, knowing $\bar{\mathbf{u}}_h^n$, the following operations are performed

- (1) we calculate $(\bar{\mathbf{u}}_h^{n+1}, \bar{p}_h^{n+1})$,
- (2) we calculate η_n^h and η_n^τ defined in (88),
- (3) If $\eta_n^h + \eta_n^\tau > \varepsilon$, we adapt in time or in space :
 - (a) If $\eta_n^h < \eta_n^\tau$ and if the time step τ_n is greater than a minimum time step $(\Delta t)_{\min}$, we adapt in time so that the next estimator in time is slightly less than the estimator in space
 - (b) If $\eta_n^h < \eta_n^\tau$ and $\tau_n = (\Delta t)_{\min}$, then
 - (i) if $\eta_n^h > \frac{\varepsilon}{2}$, we adapt in space.
 - (ii) Otherwise, adaptation in time is impossible and adaptation in space useless.
 - (c) If $\eta_n^h > \eta_n^\tau$
 - (i) If the maximum number of mesh refinements is not reached, we adapt in space
 - (ii) Otherwise we keep the calculation
- (4) If $\eta_n^h + \eta_n^\tau < 0.9\varepsilon$ we can increase the time step; in doing so we try to balance the space and time estimators
- (5) Otherwise we ensure that the estimators are balanced
 - (a) If $\eta_n^\tau \ll \eta_n^h$, the time step is increased to balance the estimators, even if this means having to refine in space at the following iteration.
 - (b) If the estimators are balanced, nothing is done.

Figure 4 shows the resulting meshes at different values of t , for the particular value $\varepsilon = 0.08$. Since we start from a vanishing velocity at $t = 0$ and since the momentum source term is piecewise constant in the computational domain, large variations of the velocity and its gradient occur near the discontinuity line of the source term; this is well captured by the mesh which concentrates cells around this line at $t = 1$. Then, as complex interactions with boundaries occur, the mesh at $t = 3$ is refined in the vicinity of some parts of the boundary, including, as expected, that of the upper obstacle, since geometric singularities usually affect negatively the regularity of the solution. It is important to note that mesh density is not necessarily driven by velocity modulus, since the estimators rather depend on variations of the velocity gradients. Additionally, many cells are avoided in a wide part of the domain, in particular in the vicinity

of a large part of the boundary. This results in huge savings in terms of computational resources: the smallest cell length is around 2.1×10^{-3} , while the largest is around 5.7×10^{-1} .

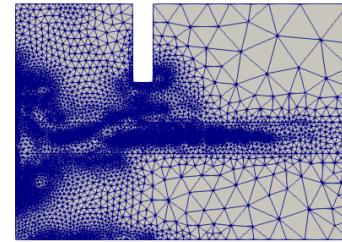
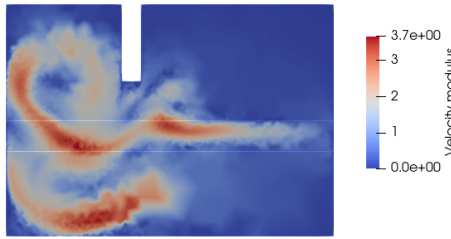
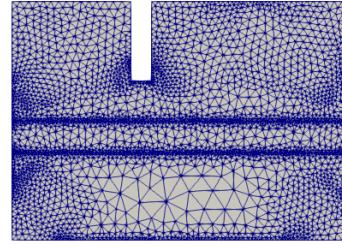
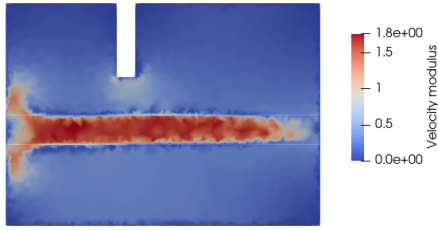


FIGURE 3. Velocity modulus for $Re = 1000$ and $t = 1$ (upper row) and $t = 3$ (lower row, with a different color scale).

FIGURE 4. Adapted meshes for $Re = 1000$ and $t = 1$ (upper row) and $t = 3$ (lower row).

For $Re = 1000$, Figure 5 shows a comparison, in logarithmic scale, of the error indicators for the uniform LES problems and adapted LES problem with respect to the number of degrees of freedom. The uniform curve was obtained by simulations on several uniform meshes with constant time steps. For each of these meshes the number of degrees of freedom is the product of the number of time steps by the number of unknowns discretizing the problem on the considered mesh. The adaptive curve was obtained by choosing several different values of ε (see algorithm above) and for each of these values, we added the number of degrees of freedom at each time step, which may vary during the calculation when the algorithm decides that a mesh adaptation is necessary. This figure shows the effectiveness of the adapted algorithm versus the uniform one.

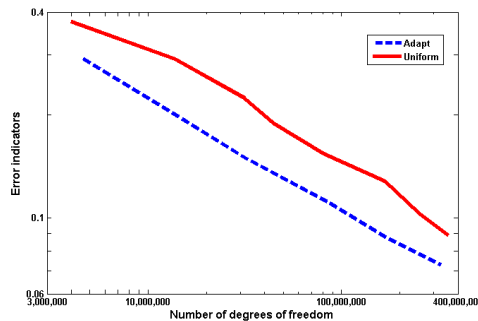


FIGURE 5. Comparison between uniform and adaptive LES estimators with respect to the total numbers of degrees of freedom in logarithmic scale.

7. CONCLUSIONS AND PERSPECTIVES

We have obtained reliable and efficient a posteriori error estimators between the numerical solution of LES equations and the exact solution of the Navier-Stokes problem in two space dimensions. The error indicators are of three types: one related to the time discretization, one to the space discretization and finally one to the additional diffusion term introduced by the Smagorinsky model for large eddies. Numerical simulations have shown the usefulness of such indicators to perform adaptive LES.

A first possible extension of this work is to investigate the case of three space dimensions; even under idealized existence and uniqueness assumptions, we have noticed in Remark 2.3 that one of our main tools in two dimensions does not carry over to three dimensions, so that other techniques are to be found. Furthermore, in practice, using homogeneous Dirichlet boundary conditions for the discrete filtered velocity is not the method of choice because this implies resolving the viscous boundary layers, which may be particularly thin for high Reynolds numbers; practitioners often resort to "laws of the wall", which would then need to be included in the analysis in order to account for the additional error that these boundary conditions generate. Another possible additional research path is to consider other types of finite elements, be they conforming like the Taylor-Hood pairs [28] or non-conforming like the Crouzeix-Raviart [14] or the Matthies-Tobiska [23] families. Finally, other, more sophisticated, LES models are routinely used in Computational Fluid Dynamics, like for example that of Germano et al. [17]; it would be worth extending the present work to these alternative LES models.

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