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# Conservation of mass-momentum-energy versus conservation of acceleration

Jean-Paul Caltagirone

calta@ipb.fr

Bordeaux INP, University of Bordeaux, CNRS UMR-5295,  
Arts et Métiers Institute of Technology, INRAE, I2M Bordeaux,  
351 Cours de la Libération - Bât. A11, 33405, Talence – France

## Abstract

The conservation laws of mass, momentum and energy of classical mechanics are replaced by the conservation of acceleration alone, one energy per unit of mass and per unit of length. The concept of conservation of momentum is discussed when the motion is accelerated. This notion, often presented as a principle when phenomena are stationary, ignores the wave-like character of wave propagation for time-dependent motions. As this is a wave equation, it is naturally relativistic. The discrete formulation enables accelerated and non-accelerated phenomena to be approached with continuity, whatever the celerity of the media, as a physical homology. A simple example compares the concept of momentum in classical mechanics with that of discrete mechanics.

## Keywords

Discrete Mechanics; Conservation of acceleration; Conservation of momentum; Special Relativity

## 1. Introduction

The notion of momentum  $\mathbf{p} = m \mathbf{v}$  in mechanics is often associated with its conservation for an isolated system, where  $m$  is the mass and  $\mathbf{v}$  the velocity measured in an inertial reference frame. In the absence of external forces, or if their resultant is zero, the momentum of a material system is a constant of motion  $d\mathbf{p}/dt = 0$ . In general, this notion is presented as a principle applicable to all types of stationary or accelerated motion [1]. This is the case in many textbooks, for example Landau & Lifchitz [7], p44, [7]. In the case of relativistic motion, the quantity  $m$  is the moving mass defined last century by A. Einstein [4] by the relation  $m = m_0/\gamma$  where  $m_0$  is the rest mass and  $\gamma = 1/\sqrt{1 - |\mathbf{v}|^2/c_0^2}$ , the Lorentz factor [9] with  $c_0$  the celerity of light. At the time, this form of moving mass allowed us to return to the concept used in classical mechanics when velocity is much lower than the celerity of light. The physics described by these relations is therefore assumed to be of the same nature as that where the velocity of any medium is compared to the celerity of light.

Discrete mechanics develops another point of view where velocity  $\mathbf{v}$  is associated with a local frame of reference satisfying Galileo's principle of relativity. The principle of weak equivalence is reformulated in terms of acceleration, i.e. the intrinsic acceleration of a particle or material medium is equal to the sum of the accelerations applied to it. The notion of mass is replaced by that of compressional energy, which is associated with longitudinal wave propagations. These waves can be very different in nature, such as swell waves, acoustic waves or the propagation of light,  $c_0$ . This formal physical homology amounts to comparing the velocity of a mobile moving through a medium to the celerity of the medium itself, and not to the celerity of light. P.H. Hugoniot [6] had already understood this in 1889, when he stated his theorem that the velocity of a gas in a pipe of constant cross-section cannot exceed the celerity of gas. More than 15 years later, A. Einstein reached the same conclusion for light in a vacuum. Since then, the "velocity of sound" has been largely exceeded in the presence of appropriate curvatures such as those found in nozzles.

The solutions of this new formalism are also those of the equations of classical mechanics, the Navier-Stokes equation for fluids [3], the Navier-Lamé equation for solids, those of the wave equation, etc. in their domain of validity. It also makes it possible to recover emblematic results

from special and general relativity. The equation of motion of discrete mechanics is relativistic, and naturally so, since it can be transformed into a wave equation. Indeed, the fundamental derivation principles of this formalism are those established by J.C. Maxwell [10].

The very simple case of the collision of two finite-dimensional particles allows us to compare the concepts very directly, and to deduce an in-depth analysis of the observed divergences.

## 2. Rebuttal of the use of momentum

The concept of momentum, sometimes presented as a principle, expresses that the material derivative of the product of mass and velocity  $\mathbf{q} = m \mathbf{v}$  is equal to the sum of the forces. In continuum mechanics, however, it is expressed as the product of mass and velocity  $\mathbf{q} = \rho \mathbf{v}$  and the right hand side becomes the sum of forces per unit volume  $\mathbf{f}$ :

$$\frac{d(m \mathbf{v})}{dt} = \sum \mathbf{F} \iff \rho \frac{d\mathbf{v}}{dt} = \sum \mathbf{f}. \quad (1)$$

What poses a problem of consistency is the equality between the term of the first member associated with one medium or a single set of different media and the second, which expresses the sum of the forces applied to each of the media. If we divide the second relation of (1) by the density, it reads:

$$\gamma \equiv \frac{d\mathbf{v}}{dt} = \frac{1}{\rho} \sum \mathbf{f}, \quad (2)$$

but what is the meaning of this density  $\rho$ ? In continuum mechanics,  $\gamma$  is the local acceleration and the force per unit volume is also local  $\mathbf{f}_i = \rho_i \mathbf{g}_i$  where  $\mathbf{g}$  is a force per unit mass, such as gravity. The density  $\rho$  is then presented as an average in some sense of the densities of the medium. Whatever the definition, the solution to a variable-density problem depends on the choice made.

Let's consider the case of a particle whose parts are made up of media of density  $\rho_1$  and  $\rho_2$  respectively, in a constant gravity field  $\mathbf{g}$ . Let's assume that the average density is equal to  $\rho = (\rho_1 + \rho_2)/2$ . When the particle is subject to the acceleration of gravity, the law of classical mechanics (2) applies:

$$\frac{\rho_1 + \rho_2}{2} \frac{d\mathbf{v}}{dt} = \rho_1 \mathbf{g} + \rho_2 \mathbf{g}, \quad (3)$$

or  $\gamma = 2\mathbf{g}$ , an absurd result, the weak equivalence principle is violated. Any other average would lead to an absurd result. The problem is that the first member of (3) is associated with the momentum of the particle as a whole, while the second member is linked to each of its parts. If we formulate the problem in terms of mass, we obtain  $d(m_1 + m_2) \mathbf{v}/dt = (m_1 + m_2) \mathbf{g}$  and another difficulty arises because the quantity  $m = (m_1 + m_2)$  is the moving mass of the first member, which depends on velocity in special relativity. If  $m$  is extracted from the derivative of the particle, we find  $\gamma = \mathbf{g}$  but in this case it's a question of equality between accelerations and not conservation of momentum. The case of media immersed in a fluid is different, as Archimedes' acceleration must be applied.

This trivial example shows that the formulation (2) is intrinsically flawed. The momentum  $\mathbf{q} = m\mathbf{v}$  is present in many areas of physics, from particle mechanics to the theory of special and general relativity. In fluid mechanics, particularly for two-phase flows, it poses serious problems for defining the density associated with each term of the Navier-Stokes equation.

We have to go back to the end of the 16th century, to the time of Galileo, to understand the profound meaning of the principle of weak equivalence. Under the effect of terrestrial gravitation, two bodies of different masses fall with the same velocity and acceleration. The meaning given

at the time, and which remains true today, is intimately linked to mass, i.e. gravitational mass is equal to inertial mass. Observation of the fall of two bodies shows above all that they fall with the same velocity and acceleration at each instant. This is a purely kinematic problem, and mass has no effect on the outcome. Newton's fundamental principle of dynamics, or second principle, can therefore be revisited in terms of the conservation of acceleration:

$$\boldsymbol{\gamma} = \boldsymbol{h}. \quad (4)$$

The intrinsic acceleration of a particle or material medium  $\boldsymbol{\gamma}$  is equal to the sum of the accelerations  $\boldsymbol{h}$  imposed on it.

The theory of relativity takes up this notion of momentum by considering that mass depends on velocity. The principle of equivalence of mass and energy does not alter this point of view, and retains both notions. The fundamental law of dynamics revisited (4) definitively discards the notion of mass, since acceleration already represents energy per unit of mass and per unit of length.

### 3. Exchange of energies during wave propagation

Let's consider a one-dimensional elastic medium whose longitudinal celerity is equal to  $c_l$ , the wavefront velocity, the celerity of sound in a solid, or  $c_0$  the celerity of light in a vacuum. At the initial instant  $t^o$ , the medium is compressed by subjecting its boundary located at  $\boldsymbol{x} = 0$  to an imposed normal velocity  $\boldsymbol{v}$ . What is the energy required to maintain this velocity  $\boldsymbol{v}$  for a time  $dt$ ? This value is easy to determine if the phenomenon under consideration is hyperbolic; indeed, no disturbance can be felt at a distance  $dh$  greater than that defined by the celerity, i.e.  $dh = c_l dt$ , where  $dh$  is the discrete horizon. Calculation of the energy per unit mass  $\phi^o$  corresponding to propagation can be obtained by modeling the compression of a one-dimensional medium. The energy of compression per unit mass is the integral of  $v dp$  where  $v = 1/\rho$  is the specific volume, but  $dp/\rho$  is also expressed from the compressibility coefficient  $\chi_T$ . The elemental energy becomes  $dp/\rho = -dt/(\rho \chi_T) \nabla \cdot \boldsymbol{v}$  where  $\sqrt{1/(\rho \chi_T)} = c_l$  is the longitudinal celerity. For isentropic motions, celerity is equal to  $c_l = \sqrt{\gamma/(\rho \chi_T)}$  where  $\gamma$  is the ratio of heats of mass at constant pressure and volume. In discrete mechanics, the energy per unit mass, the retarded potential  $\phi^o$  is therefore of the form:

$$\phi^o = - \int_{t^o}^{t^o+dt} c_l^2 \nabla \cdot \boldsymbol{v} d\tau, \quad (5)$$

between the initial time  $t^o$  and its current value at  $t^o + dt$ .

It is possible to assign orders of magnitude to each term of the expression (5), (i) the velocity is equivalent to  $\boldsymbol{v}$  the velocity imposed in  $\boldsymbol{x} = 0$ , (ii) the time  $dt$  is of the order of magnitude of  $dh/c_l$  and (iii) the characteristic length equal to  $dh$ . A simple analysis leads to the expression of the energy per unit mass required to propagate a wave of celerity  $c_l$  at a velocity level equal to  $\boldsymbol{v}$ ; we find in absolute value:

$$\phi^o = c_l \boldsymbol{v}. \quad (6)$$

In one dimension of space, the velocity of a particle or material medium cannot exceed the celerity of the medium; for example, the velocity of a massless photon cannot exceed the celerity of light  $c_0$  on a straight trajectory. The equivalent phenomenon is observed for a gas of acoustic celerity  $c_l$ , which cannot exceed this value in a pipe of constant cross-section. So, if  $\boldsymbol{v} = c_l$ , the energy value is equal to  $\phi = c_l^2$ . In the case of light propagating in a vacuum, this expression becomes  $\phi = E/m = c_0^2$  where  $E$  is total energy and  $m$  is mass. This expression is none other than the famous special relativity formula  $E = m c_0^2$ . It should be noted, however, that this is

a simple proportionality of mass and energy, contrary to what is generally accepted. There is indeed a principle of equivalence that expresses a homology between mass and energy, but its demonstration relies on the conservation laws of both. What's more, we're not talking about total energy, but only expansion energy. Discrete mechanics states that the intrinsic acceleration of a particle or material medium is equal to the sum of the expansion and rotation accelerations. The former is linked to the divergence of the velocity  $\nabla \cdot \mathbf{v}$  and the latter to its rotational motion  $\nabla \otimes \mathbf{v}$ . The acceleration extended to a segment  $\Gamma$  of length  $dh$  is the energy:

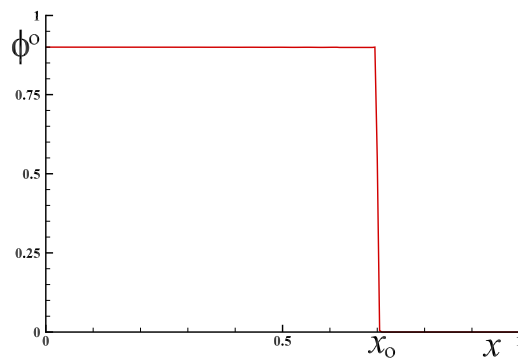
$$\phi = \int_{\Gamma} \boldsymbol{\gamma} \cdot \mathbf{t} \, dl, \quad (7)$$

where  $\mathbf{t}$  is the unit vector associated with the oriented segment. The law of discrete motion is written as  $\boldsymbol{\gamma} = -\nabla\phi + \nabla \otimes \boldsymbol{\psi}$  where the symbol  $\nabla \otimes$  corresponds to the dual curl; it replaces the fundamental law of mechanics expressed in terms of momentum. Thus, the law of acceleration expresses the conservation of total energy, the sum of the effects of compression and rotation (conservation of angular momentum in classical mechanics).

There is some confusion over the notion of energy conservation, where the quantity  $|\mathbf{v}|^2$  is supposed to express energy conservation, but this is a misinterpretation. The term  $|\mathbf{v}|^2/2$  that appears in Bernoulli's law is part of a stationary formulation of motion and not in the case of wave propagation. To clarify the notion of energy required for the propagation of a compressional wave, the law of discrete mechanics [2],

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} = -\nabla \left( \phi^o + \frac{1}{2} |\mathbf{v}|^2 - c_l^2 dt \nabla \cdot \mathbf{v} \right), \\ \phi^o - c_l^2 dt \nabla \cdot \mathbf{v} \mapsto \phi^o, \end{cases} \quad (8)$$

is repeated, removing any rotational effects that are not relevant to this presentation. The symbol  $\mapsto$  reflects the update of the retarded potential  $\phi^o$ . This potential corresponds to the potential energy per unit mass and  $|\mathbf{v}|^2/2$  is the kinetic energy per unit mass; the last term reflects the potential exchange between compression energy and kinetic energy. These three terms form an oscillator that allows energy to change nature without dissipation. The vector equation of (8) is, in the absence of the rotation terms, a law of conservation of total energy.



**Figure 1.** Evolution of energy according to  $x$  where  $\phi^o = c_l \mathbf{v} = 0.9$  et  $x_0 = dh = c_l dt = 0.7$

Computer resolution of the system (8) for celerity  $c_l = 1$ , time  $dt = 0.7$  and velocity  $\mathbf{v} = 0.9$  gives the simulation result in Figure 1. As expected, the distance traveled by the wavefront is indeed equal to  $x_0 = dh = c_l dt$  and the value of potential energy per unit mass is equal to  $\phi^o = c_l \mathbf{v}$ .

The equation (8) has remarkable properties due to its structure derived from its initial formulation: it is a nonlinear wave equation [3], relativistic by nature. In particular, it verifies the limitation of velocity  $\mathbf{v}$  to the celerity of the medium  $c_l$  in one dimension of space, whatever the medium considered (solid, fluid, vacuum) and the nature of the wave (swell, acoustic waves, light). This essential feature shows that the phenomenon corresponding to the limitation of the velocity of a gas in a duct of constant cross-section fixed by Hugoniot's theorem, is of the same nature as that linked to the velocity of a photon limited to the celerity of light  $c_0$  [5], [12], [11]. These limitations do not depend on the level of acceleration imposed.

#### 4. Collision of two masses

Before analyzing the causes of the potential abandonment of the notion of momentum in the general case of unsteady motion, a very simple case of impact between two masses is explored. Consider two particles of masses  $m_1$  and  $m_2$ , to simplify two cylinders of equal cross-section  $\mathcal{S}$ , densities  $\rho_1$  and  $\rho_2$  and different lengths such that  $m_1 = \rho_1 L_1 \mathcal{S}$  and  $m_2 = \rho_2 L_2 \mathcal{S}$ . The  $\mathcal{S}$  cross-section is large enough to neglect transverse expansion during particle compression. In a Galilean reference frame, the first particle has a velocity  $\mathbf{u}_1^o$  and the second a velocity  $\mathbf{u}_2^o$ . The first particle collides with the second (Newton's pendulum is an example), so what are the quantities  $\mathbf{u}_1$  and  $\mathbf{u}_2$  at the end of their separation? The shock is considered perfectly elastic, with the two media propagating longitudinal waves at velocities  $c_1$  and  $c_2$ .

##### 4.1. In continuum mechanics

Initially, the solution is obtained by considering the principle of conservation of momentum and that of conservation of energy. The conservation of momentum  $\mathbf{q} = (m\mathbf{v})$  is expressed by the equation  $d(m\mathbf{v})/dt = \mathbf{F}$  where  $\mathbf{F}$  is the sum of the forces applied to the physical system under consideration. For an isolated system, for example a set of particles, we have  $d(m\mathbf{v})/dt = 0$ . In the case of two masses, the system of equations is very classical and can be found in many textbooks:

$$\begin{cases} m_1 \mathbf{u}_1^o + m_2 \mathbf{u}_2^o = m_1 \mathbf{u}_1 + m_2 \mathbf{u}_2, \\ m_1 \mathbf{u}_1^{o2} + m_2 \mathbf{u}_2^{o2} = m_1 \mathbf{u}_1^2 + m_2 \mathbf{u}_2^2, \end{cases} \quad (9)$$

whose solution is:

$$\begin{cases} \mathbf{u}_1 = \frac{m_1 - m_2}{m_1 + m_2} \mathbf{u}_1^o + \frac{2 m_2}{m_1 + m_2} \mathbf{u}_2^o, \\ \mathbf{u}_2 = \frac{2 m_1}{m_1 + m_2} \mathbf{u}_1^o + \frac{m_2 - m_1}{m_1 + m_2} \mathbf{u}_2^o. \end{cases} \quad (10)$$

For equal densities, the solution is simplified and the problem is reformulated in a single direction of space. This apparent restriction does not entail any loss of generality, since compressional wave propagation is always associated with a one-dimensional model; extension to other spatial dimensions is achieved by a causal link. The system (10) becomes:

$$\begin{cases} \mathbf{u}_1 = \frac{L_1 - L_2}{L_1 + L_2} \mathbf{u}_1^o + \frac{2 L_2}{L_1 + L_2} \mathbf{u}_2^o, \\ \mathbf{u}_2 = \frac{2 L_1}{L_1 + L_2} \mathbf{u}_1^o + \frac{L_2 - L_1}{L_1 + L_2} \mathbf{u}_2^o. \end{cases} \quad (11)$$

When  $\mathbf{u}_2^o = 0$  we find  $\mathbf{u}_2 = 2L_1/(L_1 + L_2)\mathbf{u}_1^o$ . Two questions then arise, (i) does the velocity of the particle ② depend on the length  $L_1$  of the particle ① whatever its value, does this imply an infinite wave velocity? And (ii) the velocity  $\mathbf{u}_2$  is greater than  $\mathbf{u}_1$ , so what happens to the velocity value of the second particle if the first is relativistic? These questions are discussed further below.

#### 4.2. In discrete mechanics

In discrete mechanics, the notions of mass and force are abandoned in favor of a formulation in acceleration. The law of dynamics is replaced by a one-dimensional equation derived from the system:

$$\begin{cases} \frac{d\mathbf{u}}{dt} = -\nabla(\phi^o - dt c_i^2 \nabla \cdot \mathbf{u}), \\ \phi^o - c_i^2 dt \nabla \cdot \mathbf{u} \mapsto \phi^o, \end{cases} \quad (12)$$

where  $\gamma$  is the acceleration of the medium,  $\phi^o$  is the energy accumulated since the initial instant and  $\mathbf{u}$  is the local velocity; the velocities are such that  $c_1 \leq c_2$ . When both particles are in uniform motion, acceleration is zero and the principle of inertia prevails for both media,  $\mathbf{u}^o = cte$ .

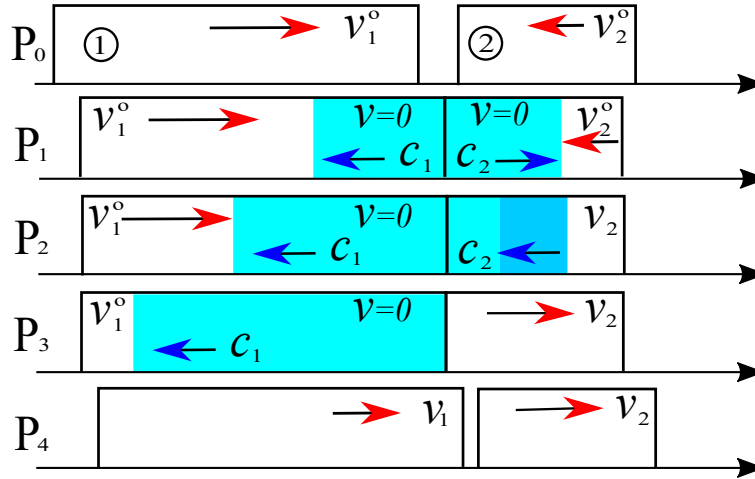
When the two particles come into contact, the  $\Sigma$  interface between the two media itself advances at a velocity  $\mathbf{u}_\Sigma$  which can be calculated by respecting two conditions: (i) the normal velocities are the same  $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}_\Sigma$  and (ii), the scalar potential  $\phi^o = p/\rho = c\mathbf{u}$  is equal on both sides of the interface, giving an interface velocity:

$$\mathbf{u}_\Sigma = \frac{c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2}{c_1 + c_2}. \quad (13)$$

Without loss of generality, we can adopt a global reference frame whose velocity is equal to  $\mathbf{u}_\Sigma$ ; in this frame of reference, the velocity of the particle ① will be equal to  $\mathbf{v}_1 = \mathbf{u}_1 - \mathbf{u}_\Sigma$  and the relative velocity of the particle ② will be equal to  $\mathbf{v}_2 = \mathbf{u}_2 - \mathbf{u}_\Sigma$ . The initial conditions on  $\mathbf{v}$  are therefore  $\mathbf{v}_1^o = \mathbf{u}_1^o - \mathbf{u}_\Sigma$  and  $\mathbf{v}_2^o = \mathbf{u}_2^o - \mathbf{u}_\Sigma$ . Given the principle of relativity, it is of course necessary that  $\mathbf{v}_1^o$  and  $\mathbf{v}_2^o$  satisfy the conditions  $v_i^o \leq c_i$ . Consider the diagrams in figure 2 representing the evolution of the two particles in the new reference frame at different phases:

- the initial phase ( $P_0$ ) presents the two particles ① and ② at velocities of  $\mathbf{v}_1^o$  and  $\mathbf{v}_2^o$  respectively. The uniform kinetic energy per unit mass of the ① medium is equal to  $|\mathbf{v}_1^o|^2/2$  and that of the ② medium is equal to  $|\mathbf{v}_2^o|^2/2$ . Subsequent evolution is marked by local spatial variations in velocity within the two media, due to compressional wave propagation;
- the first phase ( $P_1$ ) of exchange begins when the two particles come into perfect contact, the contact surface remains at rest,  $\mathbf{v}_\Sigma = 0$ . In the very first instants after contact, the left side of ① is always at velocity  $\mathbf{v}_1^o$  and the right side of ② is always at velocity  $\mathbf{v}_2^o$ . Only the right-hand side of ① and the left-hand side of ② undergo accelerations that bring them to a velocity equal to  $\mathbf{v}_\Sigma = 0$ . For a short time, the particle ① undergoes a negative mean acceleration  $\gamma_1$  and the particle ② undergoes a positive acceleration  $\gamma_2$ , but locally, each medium is in motion or at rest. Zones compressed to an energy  $\phi^o = c\mathbf{v}$ , at zero velocities, are represented by the blue color on Figure 2 while other parts are at reference potential or  $\phi^o = 0$ . In the particle ①, the blue zone therefore moves with celerity  $c_1$  and energy  $\phi^o = c_1 \mathbf{v}_1^o$  since the left-hand zone of ① is always moving at velocity  $\mathbf{v}_1^o$ . In the medium ②, the energy is the same since  $c_1 \mathbf{v}_1^o = c_2 \mathbf{v}_2^o$  but the celerity is equal to  $c_2$ . This phase has a duration of  $dt = L_2/c_2$ ;

- the second phase ( $P_2$ ) corresponds to a reflection of the wave on the right wall of the second particle, from time  $t^o + dt$ ; the potential energy is then doubled,  $\phi^1 = 2 \phi^o$  in the zone where the wave progresses to the left at celerity  $c_2$ , and the kinetic energy of the zone close to the wall is correspondingly increased to bring the velocity of ② to its final velocity  $\mathbf{v}_2$ . This wave of celerity  $c_2$  and energy  $\phi^1 = 2 \phi^o$  is directed towards the medium ①;
- the third phase ( $P_3$ ) in figure 2 represents an instant when the  $\phi^1$  energy wave reaches the  $\Sigma$  interface. From a time  $t = t^o + 2 dt$  with  $dt = L_2/c_2$  the two particles dissociate and energy exchange is interrupted. In the particle ② the potential energy  $2 \phi^o$  is transformed into kinetic energy, and it continues its trajectory with a uniform final velocity  $\mathbf{v}_2 = 2 \phi^o / 2 L_2 = c_2 t$ . The medium ①, on the other hand, is not at uniform velocity, its left zone still being at velocity  $\mathbf{v}_1^o$ , while its right zone is at zero velocity;
- the final phase ( $P_4$ ) is characterized by the propagation of waves at celerity  $c_1$  in the medium ①; these waves persist indefinitely if the medium does not dissipate energy. In the general case, the waves dissipate over time and the particle ① takes on a final velocity equal to  $\mathbf{v}_1$ . Depending on the parameter values, velocities  $c_1$  and  $c_2$  and lengths  $L_1$  and  $L_2$ , the final velocities  $\mathbf{v}_1$  and  $\mathbf{v}_2$  can take on different values. However, as each particle is subject to the relativistic constraint in one dimension of space, its velocity cannot exceed the celerity of the medium considered;



**Figure 2.** Shock between a solid of length  $L_1$  initially moving at velocity  $\mathbf{v}_1^o > 0$  and a solid of length  $L_2$  moving at velocity  $\mathbf{v}_2^o$  in the moving reference frame of velocity  $\mathbf{u}_{|\Sigma}$ ; the colored area represents the compression energy transferred.  $\phi^o = c_1 \mathbf{v}_1 = c_2 \mathbf{v}_2$ .

For this example, let's assume that  $c_1 \leq c_2$  and that  $L_1 > L_2$ . The value of the time lapse corresponding to contact is equal to  $dt = 2 L_2/c_2$  because beyond this characteristic time the two particles are separated. This is because the compression waves move back and forth along the length of  $L_2$ , and the entire medium ② is accelerated in the positive direction. At this instant, the energy per unit mass that the medium ① has transferred to the medium ② is equal to  $2 \phi^o = 2 c_1 \mathbf{v}_1^o$  during time  $2 dt$ . As exchanges between the media are no longer possible, the compression energy in the perfectly elastic medium ② will remain there. The value of the mean acceleration of this medium is therefore easily calculated,  $\gamma_2 = 2 \mathbf{v}_1^o / L_2$ . Since velocity is a relative quantity, it can only be evaluated from its velocity at the initial instant,  $\mathbf{v}_2 = \mathbf{v}_2^o + \gamma_2 dt$  or  $\mathbf{v}_2 = c_1/c_2 \mathbf{v}_1^o$ . Propagation at celerity  $c_2$  of the medium ② increases its potential energy by the value  $c_1 \mathbf{v}_1^o$  in the compression phase and decreases its potential energy by the same value in



the release phase, transforming it into kinetic energy. During this exchange, the particle ① loses energy per unit mass equal to  $\phi = 2 c_1 \mathbf{v}_1^o$  for the duration  $2 dt$  and its acceleration is equal to  $\boldsymbol{\gamma}_1 = -c_1 \mathbf{v}_1/L_1$  and its final velocity after impact is therefore equal to  $\mathbf{v}_1 = \mathbf{v}_1^o(1 - (c_1 L_2/c_2 L_1))$ . The result on the relative velocities  $\mathbf{v}$  is then written:

$$\begin{cases} \mathbf{v}_1 = \left(1 - \frac{c_1 L_2}{c_2 L_1}\right) \mathbf{v}_1^o, & \mathbf{v}_1 \leq c_1, \\ \mathbf{v}_2 = \frac{c_1}{c_2} \mathbf{v}_1^o, & \mathbf{v}_2 \leq c_2. \end{cases} \quad (14)$$

The results corresponding to the velocity  $\mathbf{u}$  from continuum mechanics (CM) and discrete mechanics (DM) are summarized by the relations:

$$C.M. \begin{cases} \mathbf{u}_1 = \frac{L_1 - L_2}{L_1 + L_2} \mathbf{u}_1^o + \frac{2 L_2}{L_1 + L_2} \mathbf{u}_2^o, \\ \mathbf{u}_2 = \frac{2 L_1}{L_1 + L_2} \mathbf{u}_1^o + \frac{L_2 - L_1}{L_1 + L_2} \mathbf{u}_2^o, \end{cases} \neq D.M. \begin{cases} \mathbf{u}_1 = \left(1 - \frac{c_1 L_2}{c_2 L_1}\right) (\mathbf{u}_1^o + \mathbf{u}|_{\Sigma}), \\ \mathbf{u}_2 = \frac{c_1}{c_2} (\mathbf{u}_1^o + \mathbf{u}|_{\Sigma}). \end{cases} \quad (15)$$

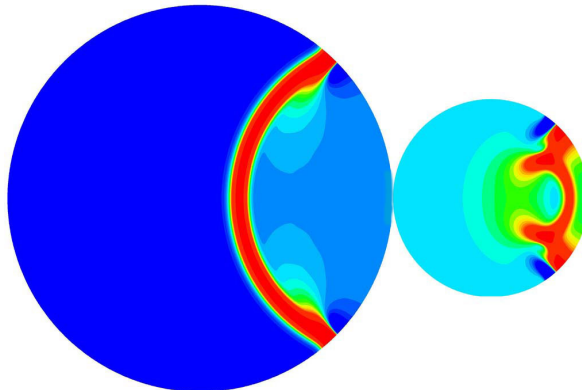
with, in discrete mechanics, the additional constraints  $\mathbf{u}_1 \leq c_1$  and  $\mathbf{u}_2 \leq c_2$ .

In the case of Newton's pendulum, with equal lengths  $L_1 = L_2$ , equal velocities  $c_1 = c_2$  and the particle ② initially at rest ( $\mathbf{u}_2^o = 0$ ), both formalisms lead to the same result,  $\mathbf{u}_1 = 0$  and  $\mathbf{u}_2 = \mathbf{u}_1^o$ . In this case, the velocity of the reference frame  $\mathbf{u}_{\Sigma}$  is globally zero at instant  $2 dt$  and the position of the particle ① is the same as at the initial instant when the two particles collide.

In the general case, the expressions (15) show a divergence. In discrete mechanics, the velocity  $\mathbf{u}_2$  of the particle ② does not depend on the length  $L_1$  of the particle ②; indeed, this particle has no way of "knowing" the upstream conditions beyond a distance  $dh = c_1 dt$ . The reasons for this discrepancy are analysed in depth in the last section.

### 4.3. Collision of two steel spheres

Whatever the dimension of space considered, the discrete equation (8) provides the solution to a problem where rotation is absent. It replaces all Euler equations of momentum  $\mathbf{q} = (\rho \mathbf{v})$ , energy  $E$ , conservation of mass  $\rho$  and state law  $f(p, \rho, T) = 0$  with a single law of total energy per unit mass.



**Figure 3.** Shock between two spheres of radii  $r_1 = 1$  and  $r_2 = 0.5$ ; snapshot of scalar potential  $\phi^o$  during wave propagation in the case of equal celeration  $c_1 = c_2$ .

To illustrate this point, let's consider two spheres of radii  $R$  and  $2R$  that collide, as shown in figure (3) shortly after the initial instant when the spheres come into contact. At this instant, the  $\Sigma$  interface is represented by a disk whose radius depends on the celerity of the two media. When the sphere of radius  $2R$  and velocity  $\mathbf{v}_1^o$  collides with the second, stationary sphere, two compression waves are formed in each medium, propagating at velocities  $c_1$  and  $c_2$ . As soon as the wave in the second medium reaches the radius  $R$ , the potential rises sharply and an expansion wave develops upstream. When this wave reaches the contact zone between spheres of the same radius, they dissociate and the smallest sphere is projected downstream.

This example, which has not been quantitatively documented, is only intended to illustrate the phenomena of wave propagation when two media collide. Mass and momentum are not relevant quantities for understanding and simulating physical phenomena, which are by nature complex.

## 5. Discussion

The argument used to challenge the concept of momentum for accelerated motions is intrinsic to continuum mechanics itself. Consider the velocity result  $\mathbf{v}_2$  from the relations (15),  $\mathbf{v}_2 = 2L_1/(L_1 + L_2)\mathbf{v}_1^o$ . If  $L_2 < L_1$  and  $\mathbf{v}_1^o$  is relativistic,  $\mathbf{v}_1^o \approx c_0$  where  $c_0$  is the celerity of light, then the celerity of the second medium is greater than the "speed of light",  $\mathbf{v}_2 > c_0$ . The SRT treatment of this problem consists in introducing a momentum of the form  $\mathbf{p} = m\mathbf{v}/\sqrt{1 - \mathbf{v}^2/c_0^2}$  where  $m$  is the rest mass,  $\gamma = 1/\sqrt{1 - \mathbf{v}^2/c_0^2}$  is the Lorentz factor and set  $\mathbf{p}_1 = -\mathbf{p}_2$  in a reference frame corresponding to the center of mass [8]; the momentum of each colliding medium does not change magnitude after the collision, but reverses the direction of its motion. Under these conditions, the results of special relativity allow us to recover those of classical mechanics, assuming that  $\mathbf{v}_1 \ll c_0$  and  $\mathbf{v}_2 \ll c_0$ . But what does this have to do with the celerity of light  $c_0$  when it comes to the propagation of elastic mechanical waves in a solid material? This profound disagreement is not due to the relativistic or non-relativistic aspect of mechanics, since the theory of relativity itself appeals to the notion of conservation of momentum. What's more, the celerity  $c_0$  bears no relation to the longitudinal celerities  $c_l$  of the two media considered in the previous example. The conservation of acceleration of discrete mechanics provides a fundamentally different and consistent result for this case, where  $\mathbf{v}_2$  is always smaller than the proper celerity of the medium. This constraint is written into the equation of discrete motion (8); indeed, in one dimension of space, the velocity obtained by this equation cannot exceed the local celerity, whatever the acceleration applied to it. In all cases, the result of the collision of two media by conservation of momentum is incompatible with that of discrete mechanics, whether or not the velocities considered are close to the celerity of light  $c_0$ . In discrete mechanics, there is no different treatment depending on the velocity level considered.

The kinetic energy of a material medium of mass  $m$  and velocity  $\mathbf{v}$  is equal to  $E_c = m|\mathbf{v}|^2/2$  but the latter corresponds to pressure, a potential energy per unit volume. This is the pressure recorded, for example, in front of a fixed obstacle in a flow whose upstream velocity is equal to  $\mathbf{v}$  when the motion is not accelerated. In this case:

$$-\nabla \left( \phi^o + \frac{|\mathbf{v}|^2}{2} \right) = 0 \implies \phi^o + \frac{|\mathbf{v}|^2}{2} = cte, \quad (16)$$

Bernoulli's theorem describes this situation very well, where  $p + \rho|\mathbf{v}|^2/2 = cte$  or, with the scalar potential,  $\phi^o + |\mathbf{v}|^2/2 = cte$ .

But the energy per unit mass required to set in motion a medium of length  $dh$  whose celerity is equal to  $c_l$  with a velocity  $\mathbf{v}$  is equal to  $\phi^o = c_l \mathbf{v}$  i.e. a global energy of  $E = m c_l \mathbf{v}$  for a mass

*m.* In this case, the local velocity evolution is not zero, nor is the velocity divergence, and the equation for these phenomena is:

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla \left( \phi^o + \frac{|\mathbf{v}|^2}{2} - c_l^2 dt \nabla \cdot \mathbf{v} \right) = 0. \quad (17)$$

In certain phenomena, such as sonoluminescence or a nuclear explosion, mass or its equivalent, the compression energy  $\phi^o$ , is transformed partly into a radiation emission at celerity  $c_0$  and partly into a shock wave whose velocity cannot exceed the local celerity of the external medium  $c_l$ .

The concept of conservation of momentum, in its various forms, ignores the phenomena of wave propagation (acoustic or light). Its association with the conservation of kinetic energy makes no sense when motion is accelerated, as kinetic energy can be transformed locally into potential energy. The concept of momentum is purely classical.

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