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Continuous 2-colorings and topological dynamics

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Abstract. We first consider the class $\mathfrak{K}$ of graphs on a zero-dimensional metrizable compact space with continuous chromatic number at least three. We provide a concrete basis of size continuum for $\mathfrak{K}$ made up of countable graphs, comparing them with the quasi-order $\preceq_c^i$ associated with injective continuous homomorphisms. We prove that the size of such a basis is sharp, using odometers. However, using odometers again, we prove that there is no antichain basis in $\mathfrak{K}$, and provide infinite descending chains in $\mathfrak{K}$. Our method implies that the equivalence relation of flip conjugacy of minimal homeomorphisms of 2^ω is Borel reducible to the equivalence relation associated with $\preceq_c^i$. We also prove that there is no antichain basis in the class of graphs on a zero-dimensional Polish space with continuous chromatic number at least three. We study the graphs induced by a continuous function, and show that any $\preceq_c^i$-basis for the class of graphs induced by a homeomorphism of a zero-dimensional metrizable compact space with continuous chromatic number at least three must have size continuum, using odometers or subshifts.

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1 Introduction

The present article is the continuation of the study of definable colorings initiated in [K-S-T], and continued in [L-Z1] and [C-M-Sc-V1]. All our relations will be binary. The motivation for this work goes back to the following so called $\mathcal{G}_0$-dichotomy, essentially proved in [K-S-T].

Theorem 1.1 *(Kechris, Solecki, Todorčević) There is a Borel relation $\mathcal{G}_0$ on 2^ω such that, for any Polish space X, and for any analytic relation R on X, exactly one of the following holds:*

(1) there is $c\colon X\to\omega$ Borel such that $c(x)\neq c(y)$ if $(x,y)\in R$ (a countable Borel coloring *of R),*

(2) there is $\varphi\colon 2^\omega\to X$ continuous such that $\mathcal{G}_0\subseteq(\varphi\times\varphi)^{-1}(R)$.

If (1) holds, then we say that R has countable Borel chromatic number (a relation R on a set X is a *digraph* if it does not meet the *diagonal* $\Delta(X):=\{(x,x)\mid x\in X\}$ of X; the *Borel chromatic number* $\chi_B(X,R)$ of a digraph (X,R) is the minimum cardinal $\kappa\leq\aleph_0$ for which there is a Borel coloring of R taking values in κ (equipped with the discrete topology) if it exists, $2^{\aleph_0}$ otherwise). If (2) holds, then we say that φ is a *continuous homomorphism* from $(2^\omega,\mathcal{G}_0)$ into (X,R), and denote this by $(2^\omega,\mathcal{G}_0)\preceq_c(X,R)$. This result had a lot of developments since. We refer to [K-Ma] for a survey, and to [B0], [B1] and [G-J-Kr-Se] for recent work in continuous combinatorics, which is the topic of the present work. It is natural to ask for a level by level version of Theorem 1.1, with respect to the Borel hierarchy (see the introduction in [K]). This work was initiated in [L-Z1], where the authors prove the following.

Theorem 1.2 *(Lecomte, Zelený) Let $\xi\in\{1,2,3\}$. Then we can find a zero-dimensional Polish space $\mathbb{X}_\xi$, and a Borel relation $\mathbb{R}_\xi$ on $\mathbb{X}_\xi$ such that for any (zero-dimensional if $\xi=1$) Polish space X, and for any analytic relation R on X, exactly one of the following holds:*

(1) there is a countable $\mathbf{\Delta}^0_\xi$-measurable coloring of R,

(2) there is $\varphi\colon\mathbb{X}_\xi\to X$ continuous such that $\mathbb{R}_\xi\subseteq(\varphi\times\varphi)^{-1}(R)$.

[C-M-So, Theorem 4.4] gives a version of this for analytic spaces when $\xi=2$, and this is also possible when $\xi=1$. More recently, the existence of versions of Theorem 1.1 for finite Borel colorings was decided. In [T-V], the authors rule out the most straightforward analogs of the $\mathcal{G}_0$-dichotomy for graphs of Borel chromatic number at least κ, where $4\leq\kappa\leq\omega$ (recall that a *graph* is a symmetric digraph). The difficult remaining case has been solved in [C-M-Sc-V1], where the authors introduce a Borel graph $\mathbb{L}_0$ on a zero-dimensional Polish space $\mathbb{X}_0$ satisfying the following.

Theorem 1.3 *(Carroy, Miller, Schrittesser, Vidnyánszky) Let X be a Polish space, and G be an analytic graph on X. Exactly one of the following holds:*

(1) G has Borel chromatic number at most two,

(2) there is $\varphi\colon\mathbb{X}_0\to X$ continuous such that $\mathbb{L}_0\subseteq(\varphi\times\varphi)^{-1}(G)$.

All this leads to the following question.

Question 1. Fix a countable ordinal $\xi\geq 1$. Is there a Borel graph $\mathbb{G}_\xi$ on a zero-dimensional analytic space $\mathbb{A}_\xi$ which is $\preceq_c$-minimum among analytic graphs on a (zero-dimensional if $\xi=1$) analytic space with $\mathbf{\Delta}^0_\xi$ chromatic number at least three? We will also consider metrizable separable, Polish, compact, and finite spaces.

Since the very beginning of the study of definable chromatic numbers in [K-S-T], injective definable homomorphisms were considered (see also [K-Ma, Sections 4 and 8], [L, Theorem 10], [L-Za, Theorem 1.13], and [L-Z2]). So it is natural to ask the same question with injective continuous homomorphisms instead of continuous homomorphisms (with the notation $\preceq_c^i$ instead of $\preceq_c$). In [C-M-Sc-V2], the authors announce the existence of a continuum sized family of closed graphs on a Polish space with Borel chromatic number at least three which are pairwise $\preceq_B^i$-incompatible in the class of analytic graphs on a Hausdorff space with Borel chromatic number at least three.

We consider the quasi-orders $\preceq_c^i$ and $\preceq_c$ on various classes (a *quasi-order* is a reflexive transitive relation). Let Γ be a class, and $\leq$ be a quasi-order on Γ. A subclass $\mathfrak{B}$ of Γ is a *basis* for Γ if any element Γ is $\leq$-above an element of $\mathfrak{B}$. We are looking for basis as small as possible, for the inclusion. In other words, we want the elements of $\mathfrak{B}$ to be pairwise $\leq$-incomparable. A subclass $\mathfrak{B}$ satisfying this property is called an *antichain*. So we are looking for antichain basis, when they exist. In the best case, the antichain basis is a singleton $\{b\}$, and we say that b is *minimum* among elements of Γ. This is the case in Theorems 1.1, 1.2 and 1.3, but it is not always possible. We are interested in the following questions, very natural when we study a quasi-order.

(1) Is there an antichain basis?

(2) If there is no antichain basis, is there a reasonably simple basis?

(3) What is the minimal size of a basis?

(4) Are there big antichains?

(5) Are there infinite descending chains?

(6) Can we find minimal elements?

(7) Can we embed a complex quasi-order?

In this article, our spaces will be metrizable separable, except in Theorem 1.7 and its two lemmas. As above, the *continuous chromatic number* (*CCN* for short) $\chi_c(X,R)$ of a digraph (X,R) is the minimum cardinal $\kappa\leq\aleph_0$ for which there is a continuous coloring of R taking values in κ (equipped with the discrete topology) if it exists, $2^{\aleph_0}$ otherwise. We mainly focus on continuous 2-colorings, even if some other cardinalities will be considered. The case of continuous 2-colorings is much more complex than in Theorem 1.2 for $\xi=1$ and ω colors, the latter case corresponding directly to the definition of the product topology. In Section 2, we will see that the odd cycles $(2p+3,C_{2p+3})$ are witnesses for the fact that any $\preceq_c^i$-basis for the class of graphs on a zero-dimensional metrizable separable (or Polish, or metrizable compact, or finite) space (*0DMS*, *0DP*, *0DMC* for short) with CCN at least three must be infinite. In the compact case, our main results are as follows. Let $\mathfrak{K}$ be the class of graphs on a 0DMC space with CCN at least three.

Theorem 1.4 *We can find a concrete family $\big((\mathbb{K}_\alpha,\mathbb{G}_\alpha)\big)_{\alpha\in 2^\omega}$, where $\mathbb{K}_\alpha$ is a compact subset of 2^ω and $\mathbb{G}_\alpha$ is a countable graph on $\mathbb{K}_\alpha$, such that, for any 0DMC space X and any graph G on X, exactly one of the following holds:*

(1) G has CCN at most two,

(2) we can find $\alpha\in 2^\omega$ and $\varphi:\mathbb{K}_\alpha\to X$ injective continuous such that $\mathbb{G}_\alpha\subseteq(\varphi\times\varphi)^{-1}(G)$.

In other words, $\big((\mathbb{K}_\alpha,\mathbb{G}_\alpha)\big)_{\alpha\in 2^\omega}$ is a $\preceq_c^i$-basis (and thus a $\preceq_c$-basis) for $\mathfrak{K}$.

Recall that an *oriented graph* is an antisymmetric digraph. Theorem 1.4 and most of our results admit versions for digraphs and oriented graphs. We will come back to this in the last section. It is simpler to work with graphs in Theorem 1.4. Note also that in [C-M-Sc-V1], the authors prove that there is no version of Theorem 1.3 for oriented graphs.

Recall that $D_2(\mathbf{\Pi}^0_1)$ is the class of differences of two closed sets, while

$$\mathbf{\Sigma}^0_1 \oplus \mathbf{\Pi}^0_1 := \{(O \cap C) \cup (F \setminus C) \mid C \in \mathbf{\Delta}^0_1 \wedge O \in \mathbf{\Sigma}^0_1 \wedge F \in \mathbf{\Pi}^0_1\}$$

is the self dual class just after $\mathbf{\Delta}^0_1$ in the Wadge hierarchy of Borel sets (see [K, 22.B and 22.E]). We provide another concrete $\preceq_c$-basis for $\mathfrak{K}$, which is not a $\preceq^i_c$-basis, but is made up of countable $D_2(\mathbf{\Pi}^0_1)$ graphs with $\mathbf{\Sigma}^0_1 \oplus \mathbf{\Pi}^0_1$ (and thus $\mathbf{\Delta}^0_2$ and Borel) chromatic number two, and whose vertices have degree at most one. We will see in Section 4 that our basis are not $\preceq_c$-basis for the class of countable graphs on a 0DP space with CCN at least three.

We next prove that the size of such $\preceq^i_c$-basis is sharp. In order to prove this, we use minimal Cantor dynamical systems. These systems have been widely studied (see, for example, [I-Me], [Ka], [Ku], [Lo], [Me], [P], [Sa-Tö]). A *dynamical system* (X, f) is given by a homeomorphism f of a metrizable compact space X. If X is homeomorphic to 2^ω, then we say that (X, f) is a *Cantor dynamical system*. A dynamical system (or f) is *minimal* if $\mathrm{Orb}_f(x) := \{f^i(x) \mid i \in \mathbb{Z}\}$ is dense in X for each $x \in X$. If (Y, g) is another dynamical system, we say that these systems (or f, g) are *orbit-equivalent* if there is a homeomorphism $\varphi : X \to Y$ such that $\varphi[\mathrm{Orb}_f(x)] = \mathrm{Orb}_g\big(\varphi(x)\big)$ for any $x \in X$. It was known that there is a family of size continuum made up of minimal Cantor dynamical systems which are pairwise not orbit equivalent (see [I-Me]). We consider a property stronger than orbit equivalence, namely flip-conjugacy. We say that two dynamical systems $(X, f), (Y, g)$ (or f, g) are *conjugate* (resp., *flip-conjugate*) if there is a homeomorphism $\varphi : X \to Y$ such that $\varphi \circ f = g \circ \varphi$ (resp., $\varphi \circ f = g \circ \varphi$ or $\varphi \circ f = g^{-1} \circ \varphi$). We provide a family of size continuum made up of minimal Cantor dynamical systems (in fact odometers) which are pairwise not flip conjugate, and associate to each homeomorphism of this family a graph on a 0DMC space, ensuring the following properties.

Theorem 1.5 *There is a $\preceq_c$-antichain (and thus $\preceq^i_c$-antichain) $\big((\mathbb{K}_\alpha, \mathbb{G}_\alpha)\big)_{\alpha \in 2^\omega}$, where*

(a) $\mathbb{K}_\alpha$ is a 0DMC space,

(b) $\mathbb{G}_\alpha$ is a countable $D_2(\mathbf{\Pi}^0_1)$ graph on $\mathbb{K}_\alpha$ with CCN three and $\mathbf{\Sigma}^0_1 \oplus \mathbf{\Pi}^0_1$ chromatic number two, and whose vertices have degree at most one,

(c) $(\mathbb{K}_\alpha, \mathbb{G}_\alpha)$ is $\preceq^i_c$-minimal in $\mathfrak{K}$.

In particular, any $\preceq^i_c$-basis for $\mathfrak{K}$ must have size at least continuum.

The minimal examples are particularily important, since they have to be part of any basis, up to equivalence. Note that Theorem 1.5 shows that if $(\mathbb{A}_1, \mathbb{G}_1)$ exists, then we must have $\chi_c(\mathbb{A}_1, \mathbb{G}_1) = 3$, $\chi_B(\mathbb{A}_1, \mathbb{G}_1) = 2$, and $(\mathbb{A}_1, \mathbb{G}_1)$ must be strictly $\preceq_c$-below $(\mathbb{X}_0, \mathbb{L}_0)$. We will also see in Section 4 that $\mathbb{A}_1$ cannot be compact. Theorem 1.5 shows that our quasi-orders have large antichains. Moreover, they are not well-founded.

Theorem 1.6 *There is a $\preceq_c$ and $\preceq^i_c$-descending chain $\big((\mathbb{K}, \mathbb{G}_p)\big)_{p \in \omega}$, where*

(a) $\mathbb{K}$ is a 0DMC space,

(b) $\mathbb{G}_p$ is a countable $D_2(\mathbf{\Pi}^0_1)$ graph on $\mathbb{K}$ with CCN three and $\mathbf{\Sigma}^0_1 \oplus \mathbf{\Pi}^0_1$ chromatic number two, and whose vertices have degree at most one.

Theorems 1.5 and 1.6 contrast with [C] where it is proved that the closed subsets of a zero-dimensional Polish space are well-quasi-ordered by bi-continuous embeddability (so this quasi-order has finite antichains and descending chains).

We now give a countable $D_2(\mathbf{\Pi}^0_1)$ graph on a 0DP space which is not compact.

Notation. Recall that if R is a relation on a set X and $l\in\omega$, then

$$R^l:=\{(x,y)\in X^2 \mid \exists (x_i)_{i\leq l}\in X^{l+1}\ \ \forall i<l\ \ (x_i,x_{i+1})\in R\wedge (x,y)=(x_0,x_l)\},$$

$R^{-1}:=\{(x,y)\in X^2 \mid (y,x)\in R\}$, and $s(R):=R\cup R^{-1}$ is the *symmetrization* of R. We now define our graph, on the copy $\mathcal{N}:=(\{c,a,\overline{a}\}\cup\omega)^\omega$ of the Baire space ω^ω. We set

$$\mathbb{O}_m:=\{(c^{k+1}a^{j+1}\overline{a}^\infty,k0^{j+1}\overline{a}^\infty)\mid j,k\in\omega\}\cup\{(ki^{j+1}a^\infty,k(i+1)^{j+1}\overline{a}^\infty)\mid j,k\in\omega\wedge i\leq 2k\}\cup \{(k(2k+1)^{j+1}a^\infty,c^{k+1}\overline{a}^{j+1}a^\infty)\mid j,k\in\omega\}.$$

and $\mathbb{G}_m:=s(\mathbb{O}_m)$. The idea of this example is to decompose the graph in levels indexed by k, and that the level k is an approximation of the odd cycle on $2k+3$ points, the approximation being improved when k increases. Note that the vertices of $\mathbb{G}_m$ have degree at most one.

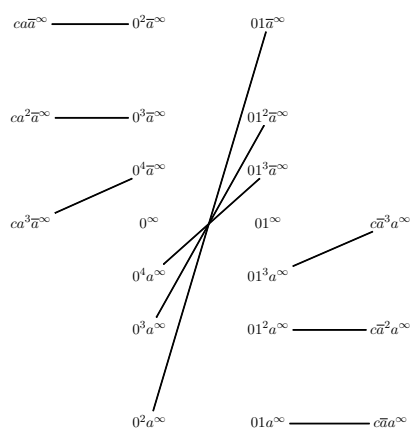
Theorem 1.7 *Let X be a first countable topological space, and G be a graph on X. The following are equivalent:*

(1) $\Delta(X)\cap\overline{\bigcup_{p\in\omega}\overline{G}^{2p+1}}$ is not empty,

(2) $(\mathcal{N},\mathbb{G}_m)\preceq_c(X,G)$.

Moreover, these conditions imply that G has no continuous 2-coloring.

The countable G_δ subset $\mathbb{P}:=\overline{\mathrm{proj}[\mathbb{G}_m]}\setminus\{c^{k+1}\varepsilon^\infty\mid k\in\omega\wedge\varepsilon\in\{a,\overline{a}\}\}$ of $\mathcal{N}$ has the properties that $(\mathbb{P},\mathbb{G}_m)$ has CCN three and $(\mathbb{P},\mathbb{G}_m)\prec_c(\mathcal{N},\mathbb{G}_m)$. The next picture describes the level of $\mathbb{G}_m$ corresponding to $k=0$, seen in $\mathbb{P}$ (so that the sequences $(c^{k+1}\varepsilon^{j+1}\overline{\varepsilon}^\infty)_{j\in\omega}$ become discrete).



Using subgraphs of $(\mathbb{P},\mathbb{G}_m)$, we prove the following.

Theorem 1.8 *Let G be a graph on a 0DMS space Z, satisfying $(Z,G)\preceq^i_c(\mathbb{P},\mathbb{G}_m)$ and having CCN at least three. Then there is a family $\big((P_\alpha,G_\alpha)\big)_{\alpha\in 2^\omega}$ of graphs on a 0DP space with CCN three, $\preceq^i_c$-below (Z,G), and pairwise $\preceq^i_c$-incompatible in the class of graphs on a 0DMS space with CCN at least three.*

In particular, there is no $\preceq^i_c$-antichain basis in the class of graphs on a 0DMS (or 0DP) space with CCN at least three, and any $\preceq^i_c$-basis for one of these classes must have size at least continuum.

Theorem 1.8 shows that $(\mathbb{P}, \mathbb{G}_m)$ is not $\preceq_c^i$-minimal among graphs on a 0DP (or 0DMS) space with CCN at least three. One can prove that this still holds for $\preceq_c$, but we will not do it here. Theorem 1.8 also gives a negative answer to the version of Question 1 for $\preceq_c^i$. One can prove that no subgraph of one of the examples of Theorem 1.5 is $\preceq_c^i$-minimal among graphs on a 0DP space with CCN at least three, but we will not do it here. We also prove a version of Theorem 1.8 in the compact case.

Theorem 1.9 *We can find a countable graph $(3^\omega, \mathbb{G})$ in $\mathfrak{K}$ such that, for each (K, G) in $\mathfrak{K}$ satisfying $(K, G) \preceq_c^i (3^\omega, \mathbb{G})$, there is a $\preceq_c^i$-antichain $\big((3^\omega, G_\alpha)\big)_{\alpha \in 2^\omega}$ of graphs with CCN three and $\preceq_c^i$-below (K, G). In particular, there is no $\preceq_c^i$-antichain basis in $\mathfrak{K}$.*

We now stated our main results concerning general graphs. The case of graphs induced by a function has been particularily considered in [K-S-T], and also in [Co-M], [Pe] and [TV] for instance. Also, we give at the end of this article a table summarizing the properties of our two quasi-orders for graphs. It is remarkable that the same properties hold for graphs induced by a partial homeomorphism with countable domain, up to, possibly, the existence of the $\preceq_c^i$-concrete basis. If

$$f : \mathrm{Domain}(f) \subseteq X \to \mathrm{Range}(f) \subseteq X$$

is a partial function, then the graph *induced* by f is $G_f := s\big(\mathrm{Graph}(f)\big) \setminus \Delta(X)$. In [K-S-T], it is proved that if X is a standard Borel space and f is a Borel function on X (i.e., has a Borel graph), then the Borel chromatic number of G_f is in $\{0, 1, 2, 3, \aleph_0\}$. So it is natural to ask the following.

Question 2. Let X be a 0DMS space, and $f : X \to X$ be a partial continuous function with analytic domain. What are the possible values for $\chi_c(X, G_f)$?

We prove the following.

Theorem 1.10 *Let X be a 0DMC space, and $f : X \to X$ be a partial continuous injection.*

(a) If the domain of f is open, then $\chi_c(X, G_f) \in \omega \cup \{2^{\aleph_0}\}$, and all these values are possible with fixed point free partial homeomorphisms on a countable space.

(b) If the domain of f is closed, then $\chi_c(X, G_f) \in \{0, 1, 2, 3, 2^{\aleph_0}\}$, and all these values are possible with (total) homeomorphisms of a countable space.

Moreover, we can find a countable Polish space X and a fixed point free partial homeomorphism $f : X \to X$ with open domain and $\chi_c(X, G_f) = \aleph_0$.

Our method in the proof of Theorem 1.8 shows that there is no $\preceq_c^i$-antichain basis for the class of graphs induced by a partial homeomorphism on a 0DP space with CCN at least three. Also, the method used to prove Theorem 1.5 shows that any $\preceq_c^i$-basis for the class of graphs induced by a partial fixed point free continuous involution with countable open domain on a 0DMC space with CCN at least three must have size continuum.

Theorem 1.10(b) leads to consider, for $\kappa \leq 3$, the class $\mathfrak{G}_\kappa$ of graphs induced by a (total) homeomorphism of a 0DMC space with CCN strictly bigger than κ. We will see that in Theorem 1.2 we can take $\mathbb{X}_1 := \{0^\infty\} \cup \{0^n 1^\infty \mid n \in \omega\}$ and $\mathbb{R}_1 := \{(0^{2n}1^\infty, 0^{2n+1}1^\infty) \mid n \in \omega\}$. Note that the graph $s(\mathbb{R}_1)$ is G_{f_1}, where $f_1 : \mathbb{X}_1 \to \mathbb{X}_1$ is the total homeomorphism defined by $f_1(0^\infty) := 0^\infty$ and $f_1(0^{2n+\varepsilon}1^\infty) := 0^{2n+1-\varepsilon}1^\infty$. We prove the following.

Theorem 1.11 *(a)* $(1,\emptyset)$ *is* $\preceq_c^i$*-minimum in* $\mathfrak{G}_0$.
(b) $(2, G_{\varepsilon\mapsto 1-\varepsilon})$ *is* $\preceq_c^i$*-minimum in* $\mathfrak{G}_1$.
(c) Any $\preceq_c^i$*-basis for* $\mathfrak{G}_2$ *must have size continuum.*
(d) $(\mathbb{X}_1, G_{f_1})$ *is* $\preceq_c^i$*-minimum in* $\mathfrak{G}_3$.

Moreover, the $(\mathfrak{G}_\kappa, \preceq_c^i)$*'s and the* $(\mathfrak{G}_\kappa, \preceq_c)$*'s are not well-founded. They also contain antichains of size continuum in the case of* $\preceq_c^i$ *or when* $\kappa\neq 3$.

We can also evaluate the descriptive complexity of the $\mathfrak{G}_\kappa$'s. In order to do that, we code the class $\mathfrak{G}_\kappa$. By [K, 7.8], any 0DMC space is homeomorphic to a subspace of 2^ω, so we can restrict our attention to compact subspaces of 2^ω. The Ryll-Nardzewski theorem (see [Kn-R, Corollary 2 and Remark 3]) shows that any homeomorphism on such a subspace can be extended to a homeomorphism of 2^ω. The extension map is injective and, conversely, the restriction map is not. But the chromatic number of the graph on the subspace does not depend on the extension, so the fact that the restriction map is not injective creates no problem. The space $\mathcal{K}(X)$ of compact subsets of a metrizable compact space X, equipped with the Vietoris topology, is a metrizable compact space. The set $\mathcal{H}(2^\omega)$ of homeomorphisms of 2^ω can be equipped with a topology in such a way that it is a Polish group. We set $\mathcal{P}:=\{(X,f)\in\mathcal{K}(2^\omega)\times\mathcal{H}(2^\omega) \mid f[X]=X\}$ and code $\mathfrak{G}_\kappa$ with $\mathcal{O}_\kappa:=\{(X,f)\in\mathcal{P} \mid \chi_c(X, G_{f_{|X}})>\kappa\}$.

Theorem 1.12 *$\mathcal{P}$ is a Polish space. $\mathcal{O}_0$ is a $\mathbf{\Delta}^0_1$ subset of $\mathcal{P}$, $\mathcal{O}_1$ is $\mathbf{\Sigma}^0_1$-complete, and $\mathcal{O}_2, \mathcal{O}_3$ are $\mathbf{\Pi}^0_2$-complete. Moreover, the set $\mathcal{O}_2^{\aleph_0}:=\{(X,f)\in\mathcal{O}_2 \mid X$ is countable$\}$ is $\mathbf{\Pi}^1_1$-complete.*

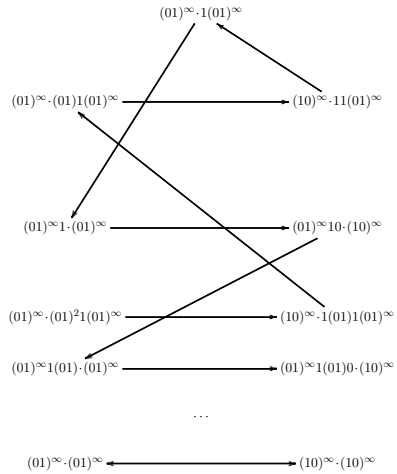
Another motivation for studying graphs induced by a function is related to Cantor dynamical systems. We will see that if f,g are minimal homeomorphisms of a Cantor space X,Y respectively, then f,g are flip-conjugate exactly when $(X,G_f)\preceq_c^i (Y,G_g)$. Similar considerations also motivate our study of oriented graphs: in this case, $\mathrm{Graph}(f), \mathrm{Graph}(g)$ are oriented graphs, and f,g are conjugate exactly when $\big(X,\mathrm{Graph}(f)\big)\preceq_c^i \big(Y,\mathrm{Graph}(g)\big)$. This also leads to study graphs induced by a total homeomorphism. The next result is a version of Theorem 1.5 for graphs induced by a total homeomorphism.

Theorem 1.13 *There is a* $\preceq_c$*-antichain (and thus* $\preceq_c^i$*-antichain)* $\big((\mathcal{C}_\alpha, G_{f_\alpha})\big)_{\alpha\in 2^\omega}$*, where*
(a) $\mathcal{C}_\alpha$ *is homeomorphic to* 2^ω*,*
(b) f_α *is a minimal homeomorphism of* $\mathcal{C}_\alpha$ *(in fact an odometer), and* $(\mathcal{C}_\alpha, G_{f_\alpha})$ *has CCN three,*
(c) $(\mathcal{C}_\alpha, G_{f_\alpha})$ *is* $\preceq_c^i$*-minimal in* $\mathfrak{G}_2$ *and in the class of closed graphs on a 0DMC space with CCN at least three.*

In particular, any $\preceq_c^i$*-basis for one of these classes must have size continuum.*

We also provide a concrete $\preceq_c^i$-basis, made up of graphs induced by an odometer, for the class of elements of $\mathfrak{G}_2$ induced by a minimal equicontinuous Cantor dynamical system, and, under the axiom of choice, a $\preceq_c^i$-antichain basis for this class. However, we will see that such a basis is far from being a basis for $\mathfrak{G}_2$, because of the subshifts associated with irrational rotations, proving a version of Theorem 1.13 for them and $\preceq_c^i$. Thanks to subshifts, we also prove a version of Theorem 1.6 for graphs induced by a total homeomorphism.

The next result shows that the situation in the compact case is different from that in the case of spaces which are not compact. The next picture describes a countable compact subset K_0 of $2^{\mathbb{Z}}$, a two-sided subshift, as well as a homeomorphism $h_0:=\sigma_{|K_0}:K_0\to K_0$ which is not minimal.



The sequence $(01)^\infty\cdot(01)^\infty$ is the element α of $2^{\mathbb{Z}}$ satisfying $\alpha(2n+\varepsilon) := \varepsilon$ if $n\in\omega$ and $\varepsilon\in 2$ on the positive side, and $\alpha(-2n-1-\varepsilon) := 1-\varepsilon$ on the negative side. Here, our space K_0 is $\mathrm{Orb}_\sigma\big((01)^\infty\cdot(01)^\infty\big)\bigcup\mathrm{Orb}_\sigma\big((01)^\infty\cdot 1(01)^\infty\big)$, where $\sigma:2^{\mathbb{Z}}\to 2^{\mathbb{Z}}$ is the shift map.

Theorem 1.14 *We can find a countable (0D)MC space K_0 and a homeomorphism h_0 of K_0 such that (a) (K_0, G_{h_0}) has CCN three, and is $\preceq_c^i$-minimal in $\mathfrak{G}_2$ and in the class of closed graphs on a 0DMC space with CCN at least three,*
(b) if S is a 0DMS (resp., 0DP) space, f is a homeomorphism of S with the properties that (S, G_f) has CCN at least three and $(S, G_f)\preceq_c^i (K_0, G_{h_0})$, then there is a finer 0DMS (resp., 0DP) topology τ on K_0 with the properties that h_0 is a homeomorphism of (K_0,τ), $\big((K_0,\tau), G_{h_0}\big)$ has CCN three, and $\big((K_0,\tau), G_{h_0}\big)$ is strictly $\preceq_c^i$-below (S, G_f).

In particular, there is no $\preceq_c^i$-antichain basis for the class of graphs induced by a (total) homeomorphism of a 0DMS (or 0DP) space with CCN at least three. Also, any $\preceq_c^i$-basis for one of these classes must have size continuum.

We saw a number of results describing classes with complex structures. In order to get simpler structures, we can try to study smaller classes, even if a big class may have a minimum element and not a subclass. Using graphs in the style of (K_0, G_{h_0}), one can try to study the class of graphs induced by a homeomorphism of a countable MC space with CCN at least three, in this direction. We provide examples of arbitrarily high Cantor-Bendixson rank.

Theorem 1.15 *(a) Let $\xi\geq 1$ be a countable ordinal, finite or of the form $\eta+3$. Then there is a countable two-sided subshift Σ with Cantor-Bendixson rank ξ, such that $(\Sigma, G_{\sigma_{|\Sigma}})$ has CCN three, and is $\preceq_c^i$-minimal in $\mathfrak{G}_2$ and in the class of closed graphs on a 0DMC space with CCN at least three.*
(b) There is a family $(\Sigma_\alpha)_{\alpha\in 2^\omega}$ of countable two-sided subshifts with Cantor-Bendixson rank three sharing these properties, and such that the family $\big((\Sigma_\alpha, G_{\sigma_{|\Sigma_\alpha}})\big)_{\alpha\in 2^\omega}$ is a $\preceq_c^i$-antichain. In particular, any $\preceq_c^i$-basis for $\mathfrak{G}_2$ or the class of graphs induced by a homeomorphism of a countable (0D)MC space with CCN at least three must have size $2^{\aleph_0}$.

At this moment, it is still possible to have a $\preceq_c^i$-antichain basis for the class of graphs induced by a homeomorphism of a countable MC space with CCN at least three. Note that the graphs given by Theorem 1.15(a) form a $\preceq_c^i$-antichain. The situation for the other values of ξ is not clear.

Indeed, recall that if (X,f) is a dynamical system where a compatible metric d on X is fixed, then (X,f) is *expansive* if $\exists\varepsilon>0\ \ \forall x\neq y\in X\ \ \exists n\in\mathbb{Z}\ \ d\big(f^n(x),f^n(y)\big)\geq\varepsilon$. The σ_Σ's are expansive, and there is no expansive homeomorphism of a countable MC space with Cantor-Bendixson rank $\lambda+1$ if λ is a limit ordinal (see [Ki-Kat-Pa, Theorem 3.2]). We leave this open for future work.

For Question (7), we prove the following.

Theorem 1.16 *We can embed the quasi-order of inclusion on the power set of ω into*

(a) the quasi-order $\preceq_c^i$ on the class of graphs induced by a (total) homeomorphism of a countable 0DMC (and thus 0DP, 0DMS) space with CCN three,

(b) the quasi-order $\preceq_c$ on the class of countable graphs on a 0DMC (and thus 0DP, 0DMS) space with CCN three.

We can say more about the association between homeomorphisms and graphs mentioned above. The space $\mathbb{M}$ of minimal homeomorphisms of 2^ω is a Polish space. The map associating $(2^\omega,G_f)$ to $f\in\mathbb{M}$ is continuous. Moreover, the graph $(2^\omega,G_f)$ has CCN two or three. The equivalence relations of flip-conjugacy and conjugacy on $\mathbb{M}$ are denoted by FCO and CO respectively. The equivalence relation $\preceq_c^i\cap(\preceq_c^i)^{-1}$ associated with the quasi-order $\preceq_c^i$ on the space

$$\mathcal{S}_m:=\{(2^\omega,K)\in\{2^\omega\}\times\mathcal{K}(2^\omega\times 2^\omega)\mid K\cap\Delta(2^\omega)=\emptyset\wedge 2\leq\chi_c(2^\omega,K)\leq 3\}$$

is denoted by $\equiv_c^i$ (we will check that $\mathcal{S}_m$ is a Polish space). The standard way to compare analytic equivalence relations on standard Borel spaces is the Borel reducibility $\leq_B$ (see, for instance, [G]). Recall that if X,Y are standard Borel spaces and E,F are analytic equivalence relations on X,Y respectively, then $(X,E)\leq_B(Y,F)\Leftrightarrow\exists\varphi:X\to Y$ Borel with $E=(\varphi\times\varphi)^{-1}(F)$.

Theorem 1.17 *The relations FCO, CO and $\equiv_c^i$ are analytic, and FCO is Borel reducible to $\equiv_c^i$.*

We can also use our countable graphs to prove a version of Theorem 1.17 for graphs of CCN at least three. Using oriented graphs instead of graphs, one can prove that CO is Borel reducible to $\equiv_c^i$. Note that the relation $=^+$ on $\mathbb{R}^\omega$ defined by $x=^+y\Leftrightarrow\{x_i\mid i\in\omega\}=\{y_i\mid i\in\omega\}$ is Borel reducible to CO (this is proved in [Ka]).

The present work is organized as follows. In Section 2, we briefly discuss the case of graphs on a finite set. We then study general graphs. Section 3 is about our positive basis results: we prove Theorems 1.4 and 1.7, give our second basis, and start to prepare the proof of Theorem 1.8. In Section 4, we work in 0DMS spaces and prove Theorem 1.8. In Section 5, we study the relation between graphs and dynamical systems, and prove the main part of the version of Theorem 1.17 for graphs of CCN at least three. In Section 6, we start to use odometers and prove Theorems 1.5 and 1.6. In Section 7, we begin our study with the graphs induced by a function and prove general facts. We then study in Section 8 the graphs induced by a partial function and prove Theorem 1.10. In Section 9, we keep on using odometers and prove Theorems 1.13, 1.16(b) and 1.9. Section 10 is devoted to the study of graphs induced by a subshift. In particular, we study the homeomorphisms of a countable compact space and prove Theorem 1.15. In Section 11, we work in 0DMS spaces and prove Theorems 1.14 and 1.16(a). In Section 12, we prove Theorems 1.11 and 1.12. In Section 13, we study equivalence relations and prove two versions of Theorem 1.17. Section 14 is devoted to the versions of our results for digraphs and oriented graphs. Finally, we summarize our work about general graphs in a table in Section 15, which leaves some other open questions for the future.

2 General graphs on a finite set

We briefly discuss the finite case, already showing that the quasi-orders $\preceq_c$ and $\preceq_c^i$ are quite different. In this finite case, we put the discrete topology on the space, so that continuity is automatic. It is known that a graph has chromatic number at most two exactly when it is bipartite, and when it has no odd cycle (see [A-D-H, 2.1]). Thus a graph G on a set X has chromatic number at least three exactly when $\Delta(X) \cap (\bigcup_{p\in\omega} G^{2p+3})$ is not empty. Recall that a *walk* in a relation (X, R) is a sequence $(x_i)_{i\leq n} \in X^{n+1}$ such that $(x_i, x_{i+1}) \in R$ for each $i < n$. A walk $(x_i)_{i\leq n}$ is *odd* if n is odd, *closed* if $x_0 = x_n$, and a *cycle* if it is closed, $n \geq 3$ and $(x_i)_{i<n}$ is injective. We denote, for any natural number p, the symmetric cycle on $2p+3$ by C_{2p+3}.

Theorem 2.1 *Let X be a finite set, and G be a graph on X. The following are equivalent:*

(1) (X, G) has chromatic number at least three,

(2) $\Delta(X) \cap (\bigcup_{p\in\omega} G^{2p+3}) \neq \emptyset$,

(3) there is $p \in \omega$ with $(2p+3, C_{2p+3}) \preceq^i (X, G)$.

Corollary 2.2 *Let $\mathfrak{F} := \big((2p+3, C_{2p+3})\big)_{p\in\omega}$.*

(a) $\mathfrak{F}$ is a $\preceq^i$-antichain basis for the class of graphs on a finite set with chromatic number at least three. In particular, the elements of $\mathfrak{F}$ are $\preceq_c^i$-minimal among graphs on a 0DMS (or 0DP, or 0DMC) space with CCN at least three, and any $\preceq_c^i$-basis for these classes must be infinite.

(b) $\mathfrak{F}$ is a $\preceq$-basis for the class of graphs on a finite set with chromatic number at least three, and is strictly $\preceq$-decreasing. In particular, there is neither $\preceq$-antichain basis, nor finite basis for this class. Also, no graph is $\preceq$-minimal in this class.

Proof. By [He-N, Corollary 1.4], $(2p+3, C_{2p+3}) \not\preceq (2q+3, C_{2q+3})$ if $p < q$.

(a) Theorem 2.1 gives the basis. This is an antichain by the argument just above and by injectivity.

(b) We apply (a), the argument above again, and [He-N, Corollary 1.4]. □

Remarks. (1) Let X be a finite set, G be a graph on X, Y be a set, H be a graph on Y with the property that $(Y, H) \preceq^i (X, G)$, with witness φ. We set $V := \varphi[Y]$ and $E := (\varphi \times \varphi)[H]$. Note that V is a subset of X, E is a graph on V contained in G with $(V, E) \preceq^i (Y, H)$ with witness φ^{-1}, and also $(Y, H) \preceq^i (V, E)$. This and the finiteness of X implies that there is no infinite $\preceq^i$-descending chain in the class of graphs on a finite set.

(2) There are infinite $\preceq$-antichains in the class of graphs on a finite set with chromatic number at least three. Indeed, this comes from [He-N, Theorem 2.23 and Proposition 3.4]. Following their notation, $S(i, i)$ and $S(j, j)$ are $\preceq$-incomparable if $i \neq j \geq 3$ are odd, and have chromatic number at least three.

3 General graphs on a 0DMC space: the basis

3.1 A first basis

We now define the concrete family announced in Theorem 1.4.

Notation. We denote the set of increasing unbounded sequences of natural numbers by

$$\mathcal{S}:=\{\delta\in\omega^\omega \mid \forall k\in\omega\ \ \delta(k)\leq\delta(k+1)\wedge\forall N\in\omega\ \ \exists k\in\omega\ \ \delta(k)\geq N\}.$$

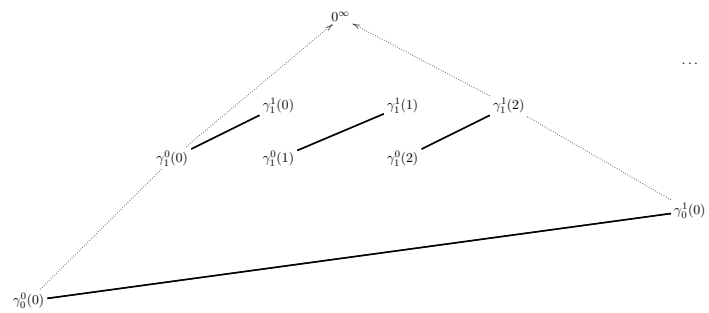
- In the proof of Theorem 1.4, it will be convenient to replace the index set 2^ω with the set $\mathcal{I}$ that we now define. We denote a typical element of $\prod_{k\in\omega}\ (2^\omega)^{2k+1}\times(2^\omega)^{2k+1}$ by

$$\gamma:=\left(\left(\left(\gamma^0_k(i)\right)_{i\leq 2k},\left(\gamma^1_k(i)\right)_{i\leq 2k}\right)\right)_{k\in\omega}.$$

We also set

$$\mathcal{I}:=\big\{\gamma\in\textstyle\prod_{k\in\omega}\ (2^\omega)^{2k+1}\times(2^\omega)^{2k+1}\mid \gamma^0_k(0),\gamma^1_k(2k)\to 0^\infty\wedge\forall k\in\omega\ \ \forall i\leq 2k\ \ \gamma^0_k(i)\neq\gamma^1_k(i)\wedge \\ \exists\delta\in\mathcal{S}\ \ \forall k\in\omega\ \ \forall i<2k\ \ \gamma^1_k(i)|\delta(k)=\gamma^0_k(i+1)|\delta(k)\big\}.$$

- We then define, for $\gamma\in\mathcal{I}$, a countable graph $\mathbb{G}_\gamma$ on 2^ω by $\mathbb{G}_\gamma:=s\big(\big\{\big(\gamma^0_k(i),\gamma^1_k(i)\big)\mid k\in\omega\wedge i\leq 2k\big\}\big)$ and set $\mathbb{K}_\gamma:=\overline{\mathrm{proj}[\mathbb{G}_\gamma]}^{2^\omega}$, so that $\mathbb{G}_\gamma$ is a graph on the compact set $\mathbb{K}_\gamma$. The next picture represents $\mathbb{G}_\gamma$.



We first prove the exactly part of Theorem 1.4.

Proposition 3.1.1 *Let $\gamma\in\mathcal{I}$. Then $(\mathbb{K}_\gamma,\mathbb{G}_\gamma)$ has CCN at least three.*

Proof. Note that $0^\infty\in\mathbb{K}_\gamma$. If $(C,\neg C)$ is a coloring of $\mathbb{G}_\gamma$ into clopen subsets of $\mathbb{K}_\gamma$ with $0^\infty\in C$, then the compactness of $\mathbb{K}_\gamma$ gives $l\in\omega$ with $\alpha,\beta\in C$ or $\alpha,\beta\notin C$ if $\alpha,\beta\in\mathbb{K}_\gamma$ and $\alpha|l=\beta|l$. Note that $\gamma^0_k(0)\in C$ if $k\geq k_0$, where $k_0\in\omega$ is also large enough with $\delta(k_0)\geq l$. Assume that $k\geq k_0$. An induction on $i<2k$ shows that $\gamma^1_k(i),\gamma^0_k(i+1)\notin C$ if i is even, $\gamma^1_k(i),\gamma^0_k(i+1)\in C$ if i is odd, and $\gamma^1_k(2k)\notin C$. This implies that $0^\infty\notin C$, which is the desired contradiction. Thus $\chi_c(\mathbb{K}_\gamma,\mathbb{G}_\gamma)\geq 3$. $\square$

Notation. Let $\mathfrak{C}:=(\omega\setminus 2)^\omega$. Fix $\mathbf{d}=(d_j)_{j\in\omega}\in\mathfrak{C}$.

- In the sequel, we denote, for $S\subseteq\omega$ finite, by $\pi_{j\in S}\ d_j$ the natural number, and by $\prod_{j\in S}\ d_j$ the set of finite sequences of natural numbers. In particular, we set, for $l\in\omega$, $\prod_l:=\prod_{j<l}\ d_j$.

- We set $\mathcal{C}:=\mathcal{C}_{\mathbf{d}}:=\prod_{j\in\omega}\ d_j$, so that $\mathcal{C}$ is homeomorphic to 2^ω. As usual, $N_s:=\{\alpha\in\mathcal{C}\mid s\subseteq\alpha\}$ is a basic clopen set if $s\in\bigcup_{l\in\omega}\ \prod_l$. We extend this notation to other sequential spaces of this kind.

- If R is a relation on $\mathcal{C}$, and $n\in\omega$, then we set $R_n:=\{(s,t)\in\prod^2_n\mid(N_s\times N_t)\cap R\neq\emptyset\}$ and

$${}^nR:=\{(\alpha,\beta)\in\mathcal{C}^2\mid\exists(\alpha',\beta')\in R\ \ \alpha|n=\alpha'|n\wedge\beta|n=\beta'|n\}.$$

Theorem 3.1.2 *Let* $\mathbf{d}\in\mathfrak{C}$*, and* G *be a graph on* $\mathcal{C}$*. The following are equivalent:*
(1) $(\mathcal{C},G)$ *has CCN at least three,*
(2) $\Delta(\mathcal{C})\cap\bigcap_{n\in\omega}\ (\bigcup_{p\in\omega}\ ({}^nG)^{2p+1})$ *is not empty,*
(3) the relations $(\prod_n, G_n)$ *have an odd closed walk,*
(4) there is $\gamma\in\mathcal{I}$ *with* $(\mathbb{K}_\gamma,\mathbb{G}_\gamma)\preceq^i_c(\mathcal{C},G)$.

Proof. (4) $\Rightarrow$ (1) This comes from Proposition 3.1.1.

(1) $\Rightarrow$ (2) We first prove that $\bigcap_{n\in\omega}\big(\overline{{}^n\big(\Delta(\mathcal{C})\big)\cup\bigcup_{p\in\omega}\ ({}^nG)^{2p+2}}\cap\overline{\bigcup_{p\in\omega}\ ({}^nG)^{2p+1}}\big)$ is not empty. We argue by contradiction, which by compactness of $\mathcal{C}$ gives $n\in\omega$ and a clopen relation O on $\mathcal{C}$ separating $\bigcup_{p\in\omega}\ ({}^nG)^{2p+1}$ from ${}^n\big(\Delta(\mathcal{C})\big)\cup\bigcup_{p\in\omega}\ ({}^nG)^{2p+2}$. The compactness of $\mathcal{C}$ gives sequences $(s_j)_{j\leq m}$ and $(t_j)_{j\leq m}$ of finite sequences with the property that $O=\bigcup_{j\leq m}\ N_{s_j}\times N_{t_j}$, and we may assume that all these finite sequences have the same length l, and that $l\geq n$.

We define a subset of $\prod_l$ by $V:=\{s_j\mid j\leq m\}\cup\{t_j\mid j\leq m\}$. Note that G_l is a graph on $\prod_l$ since, for each $s\in\prod_l$, $(N_s\times N_s)\cap G\subseteq{}^n\big(\Delta(\mathcal{C})\big)\cap O$. Let $(\mathcal{C}_i)_{i\in I}$ be the family of the connected components of G_l restricted to V. Fix $i\in I$. As $(\mathcal{C}_i, G_l\cap\mathcal{C}_i^2)$ is a connected graph, we can find an acyclic connected graph $\mathcal{G}_i$ on $\mathcal{C}_i$ with $\mathcal{G}_i\subseteq G_l\cap\mathcal{C}_i^2$. This gives a coloring $c_i:\mathcal{C}_i\to 2$ of $\mathcal{G}_i$.

We set $C:=\bigcup_{i\in I}\ \{N_s\mid c_i(s)=0\}$, so that C is a clopen subset of $\mathcal{C}$. It remains to prove that $G\cap(C^2\cup(\neg C)^2)=\emptyset$, since this contradicts (a). Towards a contradiction, suppose that there is (α,β) in $G\cap C^2$, for example (the other case is similar). As $G\subseteq O$, we can find $j\leq m$ with $(\alpha,\beta)\in N_{s_j}\times N_{t_j}$. In particular, (s_j,t_j) is in $G_l\cap V^2$, which gives $i\in I$ with $s_j,t_j\in\mathcal{C}_i$. As $\alpha,\beta\in C$, $c_i(s_j)=c_i(t_j)$. Let $L\in\omega$, $(u_k)_{k\leq L}$ be the $\mathcal{G}_i$-path from s_j to t_j, and, for $k<L$, $(\alpha_k,\beta_k)\in(N_{u_k}\times N_{u_{k+1}})\cap G$. Note that L is even since $c_i(s_j)=c_i(t_j)$. Also, $(\alpha,\beta_0),(\beta_0,\beta_1),(\beta_1,\beta_2),\dots,(\beta_{L-3},\beta_{L-2}),(\beta_{L-2},\beta)\in{}^nG$, so that (α,β) is in ${}^n\big(\Delta(\mathcal{C})\big)\cup\bigcup_{p\in\omega}\ ({}^nG)^{2p+2}\subseteq\neg O$, which is absurd.

Pick $(\alpha,\beta)\in\bigcap_{n\in\omega}\big(\overline{{}^n\big(\Delta(\mathcal{C})\big)\cup\bigcup_{p\in\omega}\ ({}^nG)^{2p+2}}\cap\overline{\bigcup_{p\in\omega}\ ({}^nG)^{2p+1}}\big)$. If $\alpha=\beta$, then for each n there is p such that $N^2_{\alpha|n}$ meets $({}^nG)^{2p+1}$. Let $(\gamma_i)_{i\leq 2p+1}\in\mathcal{C}^{2p+2}$ such that $\gamma_0,\gamma_{2p+1}\in N_{\alpha|n}$, and, for each $i\leq 2p$, $(\gamma_i,\gamma_{i+1})\in{}^nG$. Note that $(\alpha,\gamma_1),(\gamma_{2p},\alpha)\in{}^nG$, so that we may assume that $\gamma_0=\gamma_{2p+1}=\alpha$ and $(\alpha,\alpha)\in({}^nG)^{2p+1}$. Thus $(\alpha,\alpha)\in\Delta(\mathcal{C})\cap\bigcap_{n\in\omega}\ (\bigcup_{p\in\omega}\ ({}^nG)^{2p+1})$. So we may assume that $\alpha\neq\beta$. Note that $(\alpha,\beta)\notin\overline{{}^n\big(\Delta(\mathcal{C})\big)}$ if n is large enough, and that the intersection above is decreasing with respect to n. This implies that $(\alpha,\beta)\in\bigcap_{n\in\omega}\ \Big(\overline{\bigcup_{p\in\omega}\ ({}^nG)^{2p+2}}\cap\overline{\bigcup_{p\in\omega}\ ({}^nG)^{2p+1}}\Big)$. So we can pick, for each $n\in\omega$ and each $\varepsilon\in 2$, a natural number p^ε_n, and

$$(\alpha_{2n+1+\varepsilon},\beta_{2n+1+\varepsilon})\in(N_{\alpha|n}\times N_{\beta|n})\cap({}^nG)^{2p^\varepsilon_n+1+\varepsilon}.$$

Let $(\gamma^{2n+1+\varepsilon}_i)_{i\leq 2p^\varepsilon_n+1+\varepsilon}\in\mathcal{C}^{2p^\varepsilon_n+2+\varepsilon}$ such that $\gamma^{2n+1+\varepsilon}_0=\alpha_{2n+1+\varepsilon}$, $\gamma^{2n+1+\varepsilon}_{2p^\varepsilon_n+1+\varepsilon}=\beta_{2n+1+\varepsilon}$, and, for each $i\leq 2p^\varepsilon_n+\varepsilon$, $(\gamma^{2n+1+\varepsilon}_i,\gamma^{2n+1+\varepsilon}_{i+1})\in{}^nG$. Fix $n\in\omega$. Note that $(\gamma^{2n+1}_0,\gamma^{2n+2}_1),(\gamma^{2n+2}_{2p^1_n+1},\gamma^{2n+1}_{2p^0_n+1})$ are in ${}^nG$. This implies that $(\alpha,\alpha)\in({}^nG)^{2p^0_n+1+2p^1_n}\subseteq\bigcup_{p\in\omega}\ ({}^nG)^{2p+1}$.

(2) $\Rightarrow$ (4) We choose, for each $j\in\omega$, $l_j\geq 1$ with $d_j\leq 2^{l_j}$. This defines an injection $i_j:d_j\to 2^{l_j}$. We define, for $x\in\mathcal{C}$, $\psi(x)\in 2^\omega$ by $\psi(x):=\frown_{j\in\omega}\ i_j\big(x(j)\big)$. Note that ψ is a continuous injection, and thus a homeomorphism onto its range R. We set $H:=(\psi\times\psi)[G]$, so that H is a graph on 2^ω. Moreover, if $n\in\omega$ and $(x,y)\in{}^nG$, then $\big(\psi(x),\psi(y)\big)\in{}^nH$. Pick $(x,x)\in\bigcap_{n\in\omega}(\bigcup_{p\in\omega}({}^nG)^{2p+1})$, and set $\alpha:=\psi(x)$.

Then $(\alpha,\alpha)\in\bigcap_{n\in\omega}(\bigcup_{p\in\omega}({}^nH)^{2p+1})$. This gives, for each $n\in\omega$, a natural number p_n with $(\alpha,\alpha)\in({}^nH)^{2p_n+1}$. Note that, extracting a subsequence if necessary, we may assume that $(p_n)_{n\in\omega}$ is constant or strictly increasing. Let $(\gamma^n_j)_{j\leq 2p_n+1}$ in $(2^\omega)^{2p_n+2}$ such that $\gamma^n_0=\alpha$, $\gamma^n_{2p_n+1}=\alpha$, and, for each $i\leq 2p_n$, $(\gamma^n_i,\gamma^n_{i+1})$ in ${}^nH$. This gives, when $i\leq 2p_n$, $(\alpha^n_i,\beta^n_i)\in(N_{\gamma^n_i|n}\times N_{\gamma^n_{i+1}|n})\cap H$.

We now define $\upsilon\in\prod_{k\in\omega}\ (2^\omega)^{2k+1}\times(2^\omega)^{2k+1}$ as follows. If $k\leq p_0$ and $i\leq 2k$, then we set $\upsilon^0_k(i):=\alpha^0_i$ and $\upsilon^1_k(i):=\beta^0_i$.

Assume first that $(p_n)_{n\in\omega}$ is constant. If $k>p_0$ and $i\leq 2p_0$, then we set $\upsilon^0_k(i):=\alpha^k_i$ and $\upsilon^1_k(i):=\beta^k_i$. If $k>p_0$ and $2p_0<i\leq 2k$, then we set

$$\upsilon^0_k(i):=\begin{cases}\alpha^k_{2p_0} \text{ if } i \text{ is even,}\\ \beta^k_{2p_0} \text{ if } i \text{ is odd,}\end{cases}\qquad \upsilon^1_k(i):=\begin{cases}\beta^k_{2p_0} \text{ if } i \text{ is even,}\\ \alpha^k_{2p_0} \text{ if } i \text{ is odd.}\end{cases}$$

Assume now that $(p_n)_{n\in\omega}$ is strictly increasing. If $p_j<k\leq p_{j+1}$ and $i\leq 2p_j$, then we set $\upsilon^0_k(i):=\alpha^{p_j}_i$ and $\upsilon^1_k(i):=\beta^{p_j}_i$. If $p_j<k\leq p_{j+1}$ and $2p_j<i\leq 2k$, then we set

$$\upsilon^0_k(i):=\begin{cases}\alpha^{p_j}_{2p_j} \text{ if } i \text{ is even,}\\ \beta^{p_j}_{2p_j} \text{ if } i \text{ is odd,}\end{cases}\qquad \upsilon^1_k(i):=\begin{cases}\beta^{p_j}_{2p_j} \text{ if } i \text{ is even,}\\ \alpha^{p_j}_{2p_j} \text{ if } i \text{ is odd.}\end{cases}$$

Note that $\upsilon^0_k(0)=\alpha^{m_k}_0$, with $\lim_{k\to\infty} m_k=\infty$. As $\alpha^{m_k}_0\in N_{\gamma^{m_k}_0|m_k}=N_{\alpha|m_k}$, $\big(\upsilon^0_k(0)\big)_{k\in\omega}$ converges to α. Similarly, if $(p_n)_{n\in\omega}$ is constant, then $\upsilon^1_k(2k)=\beta^k_{2p_0}$ if k is large enough. As $\beta^k_{2p_0}$ is in $N_{\gamma^k_{2p_0+1}|k}=N_{\alpha|k}$, $\upsilon^1_k(2k)|k=\alpha|k$. If the sequence $(p_n)_{n\in\omega}$ is strictly increasing, then $\upsilon^1_k(2k)=\beta^{p_j}_{2p_j}$ if k is large enough. As $\beta^{p_j}_{2p_j}\in N_{\gamma^{p_j}_{2p_j+1}|p_j}=N_{\alpha|p_j}$, $\upsilon^1_k(2k)|p_j=\alpha|p_j$. Thus $\big(\upsilon^1_k(2k)\big)_{k\in\omega}$ converges to α.

Note that we chose υ in such a way that $\big(\upsilon^0_k(i),\upsilon^1_k(i)\big)$ is in the graph H, so that $\upsilon^0_k(i)\neq\upsilon^1_k(i)$. If $i<2k$, then we also ensured that

$$\begin{cases}\upsilon^1_k(i)|0=\upsilon^0_k(i+1)|0 \text{ if } k\leq p_0,\\ \upsilon^1_k(i)|k=\upsilon^0_k(i+1)|k \text{ if } k>p_0\wedge i<2p_0,\\ \upsilon^1_k(i)=\upsilon^0_k(i+1) \text{ if } k>p_0\wedge i\geq 2p_0,\\ \upsilon^1_k(i)|p_j=\upsilon^0_k(i+1)|p_j \text{ if } p_j<k\leq p_{j+1}\wedge i<2p_j,\\ \upsilon^1_k(i)=\upsilon^0_k(i+1) \text{ if } p_j<k\leq p_{j+1}\wedge i\geq 2p_j.\end{cases}$$

This defines an element ζ of $\mathcal{S}$ as desired. Note that the map $h:2^\omega\to 2^\omega$ defined by

$$h(\beta)(n):=\begin{cases}\beta(n) \text{ if } \alpha(n)=0,\\ 1-\beta(n) \text{ if } \alpha(n)=1,\end{cases}$$

is a homeomorphism sending α to 0^∞. We set $\gamma^\varepsilon_k(i):=h\big(\upsilon^\varepsilon_k(i)\big)$, which defines γ in the set $\prod_{k\in\omega}\ (2^\omega)^{k+1}\times(2^\omega)^{k+1}$. The sequence $\delta:=\zeta$ is in $\mathcal{S}$ and is a witness for the fact that $\gamma\in\mathcal{I}$. Moreover, the map h^{-1} is a witness for the fact that $(2^\omega,\mathbb{G}_\gamma)\preceq^i_c(2^\omega,H)$. We set $K:=h[R]$. As R is compact, K is too. Note that $H\subseteq R^2$, so that $\mathbb{G}_\gamma\subseteq K^2$. Thus $\mathrm{proj}[\mathbb{G}_\gamma]\subseteq K$ and $\mathbb{K}_\gamma=\overline{\mathrm{proj}[\mathbb{G}_\gamma]}\subseteq K$. We are done since $(\mathbb{K}_\gamma,\mathbb{G}_\gamma)\preceq^i_c(K,\mathbb{G}_\gamma)\preceq^i_c(R,H)\preceq^i_c(\mathcal{C},G)$.

(2) $\Rightarrow$ (3) Assume that $(\alpha,\alpha)\in\Delta(\mathcal{C})\cap\bigcap_{n\in\omega}\ (\bigcup_{p\in\omega}\ ({}^nG)^{2p+1})$. Fix $n\in\omega$. We can find p and $(\gamma_i)_{i\leq 2p+1}\in\mathcal{C}^{2p+2}$ with $\gamma_0=\gamma_{2p+1}=\alpha$ and $(\gamma_i,\gamma_{i+1})\in{}^nG$ if $i\leq 2p$. This gives, for each $i\leq 2p$, $(\alpha_i,\beta_i)\in G$ with $\alpha_i|n=\gamma_i|n$ and $\beta_i|n=\gamma_{i+1}|n$. We set, for each $i\leq 2p+1$, $s_i:=\gamma_i|n$. Then $(s_i)_{i\leq 2p+1}$ is an odd closed walk in $(\prod_n,G_n)$.

(3) $\Rightarrow$ (2) Let $(s_i^k)_{i\leq 2p_k+1}$ be an odd closed walk in $(\prod_k,G_k)$. As $(s_i^k,s_{i+1}^k)\in G_k$, we can find $(\alpha_i^k,\beta_i^k)\in(N_{s_i^k}\times N_{s_{i+1}^k})\cap G$ if $i\leq 2p_k$. The compactness of $\mathcal{C}$ provides $\alpha\in\mathcal{C}$ and $(k_n)_{n\in\omega}$ strictly increasing such that $\alpha_0^{k_n}|n=\alpha|n$ for each n. Note that

- $(\alpha_0^{k_n},\beta_0^{k_n})\in G$, $\alpha|n=\alpha_0^{k_n}|n$ and $\alpha_1^{k_n}|n=s_1^{k_n}|n=\beta_0^{k_n}|n$,
- $(\alpha_i^{k_n},\beta_i^{k_n})\in G$ and $\alpha_{i+1}^{k_n}|n=s_{i+1}^{k_n}|n=\beta_i^{k_n}|n$ if $1\leq i<2p_{k_n}$,
- $(\alpha_{2p_{k_n}}^{k_n},\beta_{2p_{k_n}}^{k_n})\in G$ and $\alpha|n=\alpha_0^{k_n}|n=s_0^{k_n}|n=s_{2p_{k_n}+1}^{k_n}|n=\beta_{2p_{k_n}}^{k_n}|n$,

This implies that $(\alpha,\alpha_1^{k_n},\dots,\alpha_{2p_{k_n}}^{k_n},\alpha)$ is in $({}^nG)^{2p_{k_n}+1}$. □

Remark. The cycle C_3 on 3 (pairwise different) points p,q,r is a graph with (continuous) chromatic number 3, and we may assume that $p=0^\infty$, $q=1^\infty$, $r=(01)^\infty$. In this case, the element γ of $\mathcal{I}$ given by Theorem 3.1.2(3) highly lacks of "injectivity". For instance, we can take $\gamma\in\mathcal{I}$ given by the equalities $\gamma_k^0(0):=0^\infty=:\gamma_k^1(2k)$, $\gamma_k^0(2l+1):=1^\infty=:\gamma_k^1(2l)$, and $\gamma_k^0(2l+2):=(01)^\infty=:\gamma_k^1(2l+1)$.

Proof of Theorem 1.4. By Proposition 3.1.1, (1) and (2) cannot hold simultaneously. By [K, 4.2], X is Polish, and by [K, 7.8], X is homeomorphic to a subspace of 2^ω, which has to be compact and therefore closed. So we may assume that X is a closed subset of 2^ω. Assume that the problem is solved for $X=2^\omega$, and that (1) does not hold. Note that (1) does not hold in 2^ω since it does not hold in X. This gives $\gamma\in\mathcal{I}$ and $\varphi:\mathbb{K}_\gamma\to 2^\omega$ injective continuous with $\mathbb{G}_\gamma\subseteq(\varphi\times\varphi)^{-1}(G)$. In particular, $\mathbb{G}_\gamma$ is contained in the closed set $\left(\varphi^{-1}(X)\right)^2$. Thus $\mathrm{proj}[\mathbb{G}_\gamma]\subseteq\varphi^{-1}(X)$ and $\mathbb{K}_\gamma=\overline{\mathrm{proj}[\mathbb{G}_\gamma]}\subseteq\varphi^{-1}(X)$, so that φ is a witness for the fact that $(\mathbb{K}_\gamma,\mathbb{G}_\gamma)\preceq_c^i(X,G)$. So we may assume that $X=2^\omega$. It remains to apply Theorem 3.1.2. □

3.2 A second basis

We now provide another $\preceq_c$-basis, closer to the examples used later.

Notation. It will be convenient to use the index set $\mathcal{J}$ that we now define. Fix $\mathbf{d}=(d_j)_{j\in\omega}\in\mathfrak{C}$. We denote a typical element of $\left((\bigcup_{m\in\omega}\ \prod_{m+1})^{<\omega}\right)^\omega$ by $\beta:=\Big(\big(s_l(i)\big)_{i<\lambda_l}\Big)_{l\in\omega}$. We then set

$$\mathcal{J}:=\Big\{\beta\in\big((\bigcup_{m\in\omega}\ \prod_{m+1})^{<\omega}\big)^\omega\mid(\lambda_l)_{l\in\omega}\in\mathcal{S}\wedge\forall l\in\omega\ \ \lambda_l>0\text{ is even }\wedge\forall i<\lambda_l\ \ |s_l(i)|=l+1\Big\},$$

$$\mathcal{J}^c:=\Big\{\beta\in\mathcal{J}\mid\forall i\in\omega\ \ \big(s_l(i)0^\infty\big)_{l\in\omega,\lambda_l>i}\text{ converges to some }\gamma_i\in\mathcal{C}\Big\}.$$

Let $c,a,\overline{a}$ be pairwise different not in ω, and $\overline{\overline{a}}:=a$. We define, for $\beta\in\mathcal{J}$, a countable digraph $\mathbb{O}_\beta$ on $\mathcal{K}_{\mathbf{d}}:=\prod_{j\in\omega}\ (d_j\cup\{c,a,\overline{a}\})$ by

$$\mathbb{O}_\beta:=\{(c^{l+1}a\overline{a}^\infty,s_l(0)\overline{a}a^\infty)\mid l\in\omega\}\cup\{(s_l(i)a^{i+1}\overline{a}^\infty,s_l(i+1)\overline{a}^{i+2}a^\infty)\mid l\in\omega\wedge i\leq\lambda_l-2\}\cup$$
$$\{(s_l(\lambda_l-1)a^{\lambda_l}\overline{a}^\infty,c^{l+1}\overline{a}a^\infty)\mid l\in\omega\}.$$

This allows us to define the graph $\mathbb{G}_\beta := s(\mathbb{O}_\beta)$. We then set $\mathbb{K}_\beta := \overline{\mathrm{proj}[\mathbb{G}_\beta]}^{\mathcal{K}_{\mathbf{d}}}$, so that $\mathbb{K}_\beta$ is a 0DMC space and $\mathbb{G}_\beta$ is a graph on $\mathbb{K}_\beta$ whose vertices have degree at most one.

Lemma 3.2.1 *Let $\mathbf{d} \in \mathfrak{C}$, and $\beta \in \mathcal{J}$. Then $(\mathbb{K}_\beta, \mathbb{G}_\beta)$ has CCN at least three and $\mathbf{\Sigma}^0_1 \oplus \mathbf{\Pi}^0_1$ chromatic number two. If moreover $s_l(2i+\varepsilon)(0) = \varepsilon$ for each $l \in \omega$, each $i \leq \frac{\lambda_l - 2}{2}$ and each $\varepsilon \in 2$, then $(\mathbb{K}_\beta, \mathbb{G}_\beta)$ has CCN three.*

Proof. If $(C, \neg C)$ is a coloring of $\mathbb{G}_\beta$ into clopen subsets of $\mathbb{K}_\beta$ which are not empty and satisfy $c^\infty \in C$, then $N_{c^{j_0}} \subseteq C$. The compactness of $\mathbb{K}_\beta$ gives $l_0 \geq j_0$ with $x, y \in C$ or $x, y \notin C$ if $x, y \in \mathbb{K}_\beta$ and $x|l_0 = y|l_0$. Assume that $l \geq l_0$. An induction on $i < \lambda_l$ shows that $s_l(i)\overline{a}^{i+1}a^\infty, s_l(i)a^{i+1}\overline{a}^\infty \notin C$ if i is even, $s_l(i)\overline{a}^{i+1}a^\infty, s_l(i)a^{i+1}\overline{a}^\infty \in C$ if i is odd, and $c^{l+1}\overline{a}a^\infty \notin C$, which is the desired contradiction. Thus $\chi_c(\mathbb{K}_\beta, \mathbb{G}_\beta) \geq 3$.

We define an open subset of $\mathbb{K}_\beta$ by $O := \big\{x \in \mathbb{K}_\beta \mid \exists n \in \omega \;\; x(n) = a \wedge x|n \in \prod_{j<n} \; (d_j \cup \{c\})\big\}$. The $\mathbf{\Sigma}^0_1 \oplus \mathbf{\Pi}^0_1$ partition $(O, \neg O)$ of $\mathbb{K}_\beta$ is a witness for the fact that $2 \leq \chi_{\mathbf{\Sigma}^0_1 \oplus \mathbf{\Pi}^0_1}(\mathbb{K}_\beta, \mathbb{G}_\beta) \leq 2$.

If moreover $s_l(2i+\varepsilon)(0) = \varepsilon$ for each $l \in \omega$, each $i \leq \frac{\lambda_l - 2}{2}$ and each $\varepsilon \in 2$, then the clopen partition (N_c, N_0, N_1) of $\mathbb{K}_\beta$ is a witness for the fact that $\chi_c(\mathbb{K}_\beta, \mathbb{G}_\beta) \leq 3$. □

Lemma 3.2.2 *Let $\mathbf{d} \in \mathfrak{C}$, and $\beta \in \mathcal{J}$. Then the graph $\mathbb{G}_\beta$ is $D_2(\mathbf{\Pi}^0_1)$.*

Proof. We check that $\overline{\mathbb{G}_\beta} = \mathbb{G}_\beta \cup \big(\overline{\mathbb{G}_\beta} \cap (\{c^\infty\} \cup \mathcal{C})^2\big)$. For the left to right inclusion, assume that $(x, y) \in \overline{\mathbb{G}_\beta} \setminus \mathbb{G}_\beta$. We may assume that (x, y) is the limit of a sequence $\big((x_m, y_m)\big)_{m \in \omega}$ such that $d\big((x_m, y_m), (\{c^\infty\} \cup \mathcal{C})^2\big) \leq 2^{-m}$. We are done since $\{c^\infty\} \cup \mathcal{C}$ is closed in $\mathbb{K}_\beta$. As the first union is the disjoint union of $\mathbb{G}_\beta$ and a closed relation on $\mathbb{K}_\beta$, $\mathbb{G}_\beta$ is $D_2(\mathbf{\Pi}^0_1)$. □

The point c^∞ will often be crucial to ensure a big CCN.

Lemma 3.2.3 *Let $\mathbf{d} \in \mathfrak{C}$, $\beta \in \mathcal{J}$ with $s_l(2i+\varepsilon)(0) = \varepsilon$ for each $l \in \omega$, each $i \leq \frac{\lambda_l - 2}{2}$ and each $\varepsilon \in 2$, X be a topological space, and G be a digraph on X having CCN at least three such that $(X, G) \preceq_c (\mathbb{K}_\beta, \mathbb{G}_\beta)$, with witness φ. Then $c^\infty \in \varphi[X]$.*

Proof. We argue by contradiction. Let $C := \varphi^{-1}(N_0 \cup \bigcup_{l \in \omega} \; N_{c^{l+1}\overline{a}})$. Then $(C, \neg C)$ is a coloring of G into clopen sets since $C = \varphi^{-1}(N_0 \cup \bigcup_{l \in \omega} \; N_{c^{l+1}\overline{a}} \cup \{c^\infty\})$, which is absurd. □

We now prove that, for $\mathbf{d} = 2^\infty$, $\big((\mathbb{K}_\beta, \mathbb{G}_\beta)\big)_{\beta \in \mathcal{J}^c}$ is a $\preceq_c$-basis for $\mathfrak{K}$.

Theorem 3.2.4 *Let X be a 0DMC space, and G be a graph on X. Then exactly one of the following holds:*

(1) (X, G) has CCN at most two,

(2) there is $\beta \in \mathcal{J}^c$ (for $\mathbf{d} = 2^\infty$) such that $(\mathbb{K}_\beta, \mathbb{G}_\beta) \preceq_c (X, G)$.

Proof. By Theorem 1.4 and Lemma 3.2.1, it is enough to prove that if $\gamma \in \mathcal{I}$, then we can find $\beta \in \mathcal{J}^c$ (for $\mathbf{d} = 2^\infty$) such that $(\mathbb{K}_\beta, \mathbb{G}_\beta) \preceq_c (\mathbb{K}_\gamma, \mathbb{G}_\gamma)$. As $\lim_{k \to \infty} \gamma^\varepsilon_k\big(\varepsilon \cdot (2k)\big) = 0^\infty$, we can find $\delta_\varepsilon \in \mathcal{S}$ such that $0^{\delta_\varepsilon(k)} \subseteq \gamma^\varepsilon_k\big(\varepsilon \cdot (2k)\big)$ for each $k \in \omega$. We define $\Delta \in \mathcal{S}$ by setting $\Delta(k) := \min\big(\delta(k), \delta_0(k), \delta_1(k)\big)$. Let $(k_q)_{q \in \omega} \in \mathcal{S}$ such that $\Delta(k_q) > q$ and $k_q > 0$. We set $\lambda'_q := 2k_q$, so that $(\lambda'_q)_{q \in \omega} \in \mathcal{S}$. We then set, for $i < 2k_q$, $s'_q(i) := \gamma^1_{k_q}(i)|(q+1)$. This allows us to set $\beta' := \big(\big(s'_q(i)\big)_{i < 2k_q}\big)_{q \in \omega}$. Note that $\beta' \in \mathcal{J}$. This defines $(\mathbb{K}_{\beta'}, \mathbb{G}_{\beta'})$, and we will define β later.

We now check that $(\mathbb{K}_{\beta'}, \mathbb{G}_{\beta'}) \preceq_c (\mathbb{K}_\gamma, \mathbb{G}_\gamma)$. We have to define $\varphi : \mathbb{K}_{\beta'} \to \mathbb{K}_\gamma$.

We first define a function $\varphi_0:\mathrm{proj}[\mathbb{G}_{\beta'}]\to\mathrm{proj}[\mathbb{G}_\gamma]\subseteq\mathbb{K}_\gamma$, by setting

$$\begin{cases}\varphi_0(c^{q+1}a\overline{a}^\infty):=\gamma^0_{k_q}(0),\\ \varphi_0(c^{q+1}\overline{a}a^\infty):=\gamma^1_{k_q}(2k_q),\\ \varphi_0(s'_q(i)a^{i+1}\overline{a}^\infty):=\gamma^0_{k_q}(i+1)\text{ if }i<2k_q,\\ \varphi_0(s'_q(i)\overline{a}^{i+1}a^\infty):=\gamma^1_{k_q}(i)\text{ if }i<2k_q.\end{cases}$$

Note that $\mathbb{G}_{\beta'}\subseteq(\varphi_0\times\varphi_0)^{-1}(\mathbb{G}_\gamma)$. Let us prove that φ_0 is uniformly continuous on the projection $\mathrm{proj}[\mathbb{G}_{\beta'}]$. We set, for $x\in\mathrm{proj}[\mathbb{G}_{\beta'}]$, $\delta(x):=\min\{n\in\omega\mid x(n)\in\{a,\overline{a}\}\}$. Note that if $\varepsilon\in\{a,\overline{a}\}$, then $\varphi_0(c^{q+1}\varepsilon\overline{\varepsilon}^\infty)|(q+1)=0^{q+1}$ and $\varphi_0(s'_q(i)\varepsilon^{i+1}\overline{\varepsilon}^\infty)|(q+1)=s'_q(i)$, by the choice of $(k_q)_{q\in\omega}$. Let $q_0\in\omega$. We want to find $n\in\omega$ such that $\varphi_0(x)|(q_0+1)=\varphi_0(y)|(q_0+1)$ if $x|(n+1)=y|(n+1)$ and $x,y\in\mathrm{proj}[\mathbb{G}_{\beta'}]$. If $\delta(x),n>q_0$, then $\delta(y)>q_0$ and $\varphi_0(x)|(q_0+1)=\varphi_0(y)|(q_0+1)$ by the previous facts. Note that there are finitely many $z\in\mathrm{proj}[\mathbb{G}_{\beta'}]$ with $\delta(z)\leq q_0$. We choose $n>q_0$ large enough so that $z=t$ if $z,t\in\mathrm{proj}[\mathbb{G}_{\beta'}]$, $\delta(z),\delta(t)\leq q_0$ and $z|n=t|n$, so that n is as desired. The theorem of extension of uniformly continuous maps (see [Bo, chapter II, §3, Section 6, Theorem 2]) provides $\varphi:\overline{\mathrm{proj}[\mathbb{G}_{\beta'}]}^{\mathcal{K}_{2^\infty}}\to\mathbb{K}_\gamma$ continuous extending φ_0. As $\overline{\mathrm{proj}[\mathbb{G}_{\beta'}]}^{\mathcal{K}_{2^\infty}}=\mathbb{K}_{\beta'}$, the map φ is as desired.

It remains to find $\beta\in\mathcal{J}^c$ with $(\mathbb{K}_\beta,\mathbb{G}_\beta)\preceq_c(\mathbb{K}_{\beta'},\mathbb{G}_{\beta'})$. By compactness of $\{0,1,c,a,\overline{a}\}^\omega$, we can find $(\gamma_i)_{i<\lambda'_0}\in(2^\omega)^{\lambda'_0}$ and $(q^0_j)_{j\in\omega}$ strictly increasing such that $q^0_0=0$ and, for each $i<\lambda'_{q^0_0}$, $\big(s'_{q^0_j}(i)0^\infty\big)_{j\in\omega}$ converges to γ_i. Extracting a further subsequence if necessary, we may assume that $s'_{q^0_j}(i)|j=\gamma_i|j$ if $i<\lambda'_{q^0_0}$. We can find $(\gamma'_i)_{i<\lambda'_{q^0_1}}$ and $(q^1_j)_{j\in\omega}$ strictly increasing such that $q^0_0=q^0_1$, $\{q^1_j\mid j\in\omega\}\subseteq\{q^0_m\mid m>0\}$, and, for each $i<\lambda'_{q^1_0}$, $(s'_{q^1_j}(i)0^\infty)_{j\in\omega}$ converges to γ'_i. As $(q^1_j)_{j\in\omega}$ is a subsequence of $(q^0_m)_{m\in\omega}$, $\gamma'_i=\gamma_i$ if $i<\lambda'_{q^0_0}$. For this reason, we may set, for $i<\lambda'_{q^1_0}$, $\gamma_i:=\gamma'_i$. Note that, extracting a further subsequence if necessary, we may assume that $s'_{q^1_j}(i)|j=\gamma_i|j$ if $i<\lambda'_{q^1_0}$. Then, inductively, we can find $(\gamma_i)_{i<2\lambda'_{q^k_1}}$ and $(q^{k+1}_j)_{j\in\omega}$ strictly increasing with the properties that $q^{k+1}_0=q^k_1$, $\{q^{k+1}_j\mid j\in\omega\}\subseteq\{q^k_m\mid m>0\}$, and, for each $i<\lambda'_{q^{k+1}_0}$, $\big(s'_{q^{k+1}_j}(i)0^\infty\big)_{j\in\omega}$ converges to γ_i. We can also ensure that $s'_{q^{k+1}_j}(i)|j=\gamma_i|j$ if $i<\lambda'_{q^{k+1}_0}$. Note that $q^{k+1}_0=q^k_1>q^k_0$. We then set $\lambda_l:=\lambda'_{q^{2l+2}_0}$ and, for $i<\lambda_l$, $s_l(i):=s'_{q^{2l+2}_0}(i)|(l+1)$. Fix $i\in\omega$. Let l minimal such that $i<\lambda'_{q^{l+1}_0}$. Note that $q^{2l+2}_0=q^{2l+1}_1>q^{2l+1}_0=q^{2l}_1>q^{2l}_0=q^{2l-1}_1>\ldots=q^{l+1}_1>q^{l+1}_0$, which shows that $q^{2l+2}_0=q^{l+1}_{j_{l+1}}$ for some $j_{l+1}\geq l+1$. This gives $s'_{q^{l+1}_{j_{l+1}}}(i)|j_{l+1}=\gamma_i|j_{l+1}$, and $\gamma_i|(l+1)=s'_{q^{2l+2}_0}(i)|(l+1)=s_l(i)$. In other words, $\big(s_l(i)0^\infty\big)_{\lambda_l>i}$ converges to γ_i for each $i\in\omega$, so that $\beta:=\big(\big(s_l(i)\big)_{i<\lambda_l}\big)_{l\in\omega}\in\mathcal{J}^c$.

Let us prove that $(\mathbb{K}_\beta,\mathbb{G}_\beta)\preceq^i_c(\mathbb{K}_{\beta'},\mathbb{G}_{\beta'})$. We define $\psi:\mathbb{K}_\beta\to\mathbb{K}_{\beta'}$, which will be the identity on $\{c^\infty\}\cup(\mathbb{K}_\beta\cap 2^\omega)$. Let $\varepsilon\in\{a,\overline{a}\}$, $l\in\omega$, and $i\in\omega$ with either $i=0$ if $s=c^{l+1}$, or $s=s_l(i)$. We define $\psi(s\varepsilon^{i+1}\overline{\varepsilon}^\infty)$ in such a way that $s\subseteq\psi(s\varepsilon^{i+1}\overline{\varepsilon}^\infty)$ and $\psi(s\varepsilon^{i+1}\overline{\varepsilon}^\infty)$ ends with $\varepsilon^{i+1}\overline{\varepsilon}^\infty$. We set

$$\begin{cases}\psi(c^{l+1}a\overline{a}^\infty):=c^{q^{2l+2}_0+1}a\overline{a}^\infty,\\ \psi(c^{l+1}\overline{a}a^\infty):=c^{q^{2l+2}_0+1}\overline{a}a^\infty,\\ \psi(s_l(i)a^{i+1}\overline{a}^\infty):=s'_{q^{2l+2}_0}(i)a^{i+1}\overline{a}^\infty\text{ if }i<\lambda_l=\lambda'_{q^{2l+2}_0},\\ \psi(s_l(i)\overline{a}^{i+1}a^\infty):=s'_{q^{2l+2}_0}(i)\overline{a}^{i+1}a^\infty\text{ if }i<\lambda'_{q^{2l+2}_0}.\end{cases}$$

The map ψ is injective continuous as desired. □

Remark. The $\mathbb{K}_\beta$'s are infinite, so that $\big((\mathbb{K}_\beta,\mathbb{G}_\beta)\big)_{\beta\in\mathcal{J}^c}$ is not a $\preceq^i_c$-basis, because of the (finite) odd cycles. On the other hand, the $(\mathbb{K}_\beta,\mathbb{G}_\beta)$'s have $\mathbf{\Sigma}^0_1\oplus\mathbf{\Pi}^0_1$ chromatic number two, by Lemma 3.2.1. In particular, they have Borel chromatic number two.

The next two results will help us to compare the subgraphs of the $\mathbb{G}_\beta$'s.

Lemma 3.2.5 *Let $\mathbf{d}\in\mathfrak{C}$, $\beta\in\mathcal{J}$, $V_\mathbf{d}\subseteq\mathbb{K}_\beta$, $E_\mathbf{d}\subseteq\mathbb{G}_\beta\cap V_\mathbf{d}^2$, $\mathbf{d}'$, β', $V_{\mathbf{d}'}$, $E_{\mathbf{d}'}$ having the corresponding properties, satisfying $(V_\mathbf{d},E_\mathbf{d})\preceq_c(V_{\mathbf{d}'},E_{\mathbf{d}'})$ with witness φ, and $x\in\overline{proj[s(E_\mathbf{d})]}^{\mathbb{K}_\beta}\cap V_\mathbf{d}\cap\mathcal{C}_\mathbf{d}$ with $\varphi(x)\notin\mathcal{C}_{\mathbf{d}'}$. Then we can find $t\in\bigcup_{l\in\omega}\ \prod_l$ such that*

- either $\varphi[\overline{proj[s(E_\mathbf{d})]}^{\mathbb{K}_\beta}\cap V_\mathbf{d}\cap\mathcal{C}_\mathbf{d}\cap N_t]\subseteq\{c^\infty\}$,

- or we can find $\varepsilon\in\{a,\overline{a}\}$, $l,m\in\omega$ and $s\in\{c^{l+1}\}\cup\{s_l(i)\mid i<\lambda_l\}$ with the property that $\varphi[\overline{proj[s(E_\mathbf{d})]}^{\mathbb{K}_\beta}\cap V_\mathbf{d}\cap\mathcal{C}_\mathbf{d}\cap N_t]\subseteq\{s\varepsilon^{m+1}\overline{\varepsilon}^\infty\}$.

Proof. If $\varphi(x)$ is of the form $s\varepsilon^{m+1}\overline{\varepsilon}^\infty$, then there is $m_0\in\omega$ such that $\varphi(z)\supseteq s\varepsilon^{m+1}\overline{\varepsilon}$ if z is in $V_\mathbf{d}\cap N_{x|m_0}$, by continuity of φ. Assume that $z\in\overline{\mathrm{proj}[s(E_\mathbf{d})]}^{\mathbb{K}_\beta}\cap V_\mathbf{d}\cap\mathcal{C}_\mathbf{d}\cap N_{x|m_0}$. Then $z=\lim_{j\to\infty}z_j$, where z_j is in $\mathrm{proj}[s(E_\mathbf{d})]\cap N_{x|m_0}$, which gives $u_j\in V_\mathbf{d}$ with $(z_j,u_j)\in s(E_\mathbf{d})\subseteq V_\mathbf{d}^2$. This implies that $\big(\varphi(z_j),\varphi(u_j)\big)\in s(E_{\mathbf{d}'})$, $\varphi(z_j)=s\varepsilon^{m+1}\overline{\varepsilon}^\infty$ and $\varphi(z)=s\varepsilon^{m+1}\overline{\varepsilon}^\infty$.

If now $\varphi(x)=c^\infty$, then there is $m_0\in\omega$ such that $\varphi(z)(0)=c$ if $z\in V_\mathbf{d}\cap N_{x|m_0}$, by continuity of φ. As in the previous case, we get (z_j,u_j). This time, $\varphi(z_j)(0)=c$, so that $\varphi(z_j)$ is of the form $c^{k_j+1}\varepsilon\overline{\varepsilon}^\infty$ and $\varphi(z)\in\{c^\infty\}\cup\big\{c^{k+1}\varepsilon\overline{\varepsilon}^\infty\mid k\in\omega\wedge\varepsilon\in\{a,\overline{a}\}\big\}$. By the previous point, we may assume that $\varphi(z)=c^\infty$. □

We get a condition sufficient to send $\mathcal{C}_\mathbf{d}$ into $\mathcal{C}_{\mathbf{d}'}$.

Lemma 3.2.6 *Let $\mathbf{d}\in\mathfrak{C}$, $\beta\in\mathcal{J}$, $V_\mathbf{d}\subseteq\mathbb{K}_\beta$, $E_\mathbf{d}\subseteq\mathbb{G}_\beta\cap V_\mathbf{d}^2$, $\mathbf{d}'$, β', $V_{\mathbf{d}'}$, $E_{\mathbf{d}'}$ having the corresponding properties, satisfying $\mathcal{C}_\mathbf{d}\subseteq\overline{V_\mathbf{d}\cap\mathcal{C}_\mathbf{d}}^{\mathcal{C}_\mathbf{d}}\cap\overline{proj[s(E_\mathbf{d})]}^{\mathbb{K}_\beta}$, and also $(V_\mathbf{d},E_\mathbf{d})\preceq^i_c(V_{\mathbf{d}'},E_{\mathbf{d}'})$ with witness φ. Then $\varphi[V_\mathbf{d}\cap\mathcal{C}_\mathbf{d}]\subseteq\mathcal{C}_{\mathbf{d}'}$.*

Proof. Towards a contradiction, suppose that there is $x\in V_\mathbf{d}\cap\mathcal{C}_\mathbf{d}$ with $\varphi(x)\notin\mathcal{C}_{\mathbf{d}'}$. Note that x is in $\overline{\mathrm{proj}[s(E_\mathbf{d})]}^{\mathbb{K}_\beta}$. Lemma 3.2.5 provides $t\in\bigcup_{l\in\omega}\ \prod_l$ such that $\varphi[V_\mathbf{d}\cap\mathcal{C}_\mathbf{d}\cap N_t]$ has at most one element, which contradicts the injectivity of φ since $\mathcal{C}_\mathbf{d}\subseteq\overline{V_\mathbf{d}\cap\mathcal{C}_\mathbf{d}}^{\mathcal{C}_\mathbf{d}}$. □

We give a condition sufficient to send c^∞ to itself.

Lemma 3.2.7 *Let $\mathbf{d}\in\mathfrak{C}$, $\beta\in\mathcal{J}$ with $s_l(2i+\varepsilon)(0)=\varepsilon$ for each $l\in\omega$, each $i\leq\frac{\lambda_l-2}{2}$ and each $\varepsilon\in 2$, $V_\mathbf{d}\subseteq\mathbb{K}_\beta$, $E_\mathbf{d}\subseteq\mathbb{G}_\beta\cap V_\mathbf{d}^2$ satisfying $\chi_c(V_\mathbf{d},E_\mathbf{d})\geq 3$, $\mathbf{d}'$, β', $V_{\mathbf{d}'}$, $E_{\mathbf{d}'}$ having the corresponding properties, satisfying $V_\mathbf{d}\setminus(\{c^\infty\}\cup\mathcal{C}_\mathbf{d})\subseteq proj[s(E_\mathbf{d})]$, and $(V_\mathbf{d},E_\mathbf{d})\preceq_c(V_{\mathbf{d}'},E_{\mathbf{d}'})$ with witness φ satisfying $\varphi[V_\mathbf{d}\cap\mathcal{C}_\mathbf{d}]\subseteq\mathcal{C}_{\mathbf{d}'}$. Then $\varphi[V_\mathbf{d}\setminus(\{c^\infty\}\cup\mathcal{C}_\mathbf{d})]\subseteq V_{\mathbf{d}'}\setminus(\{c^\infty\}\cup\mathcal{C}_{\mathbf{d}'})$ and $\varphi(c^\infty)=c^\infty$.*

Proof. As $V_\mathbf{d}\setminus(\{c^\infty\}\cup\mathcal{C}_\mathbf{d})\subseteq\mathrm{proj}[s(E_\mathbf{d})]$, $\varphi[V_\mathbf{d}\setminus(\{c^\infty\}\cup\mathcal{C}_\mathbf{d})]\subseteq\mathrm{proj}[s(E_{\mathbf{d}'})]\subseteq V_{\mathbf{d}'}\setminus(\{c^\infty\}\cup\mathcal{C}_{\mathbf{d}'})$. It remains to apply Lemma 3.2.3. □

3.3 Lower bounds

In this subsection, we prove Theorem 1.7, among other things. We first recover an implication in the style of (2) ⇒ (1) in Theorem 3.1.2.

Lemma 3.3.1 *Let X be a first countable topological space, and G be a graph on X with the property that $\Delta(X) \cap \overline{\bigcup_{p\in\omega} \overline{G}^{2p+1}}$ is not empty. Then (X, G) has no continuous 2-coloring.*

Proof. Let $x \in X$ with $(x, x) \in \overline{\bigcup_{p\in\omega} \overline{G}^{2p+1}}$. As X is first countable, we can find $\big((x_n, y_n)\big)_{n\in\omega}$ in $(X^2)^\omega$ converging to (x, x) with $(x_n, y_n) \in \bigcup_{p\in\omega} \overline{G}^{2p+1}$. Let $(p_n)_{n\in\omega} \in \omega^\omega$ with $(x_n, y_n) \in \overline{G}^{2p_n+1}$. Let $(z_i^n)_{i\leq 2p_n+1} \in X^{2p_n+2}$ with $z_0^n = x_n$, $z_{2p_n+1}^n = y_n$, and $(z_i^n, z_{i+1}^n) \in \overline{G}$ if $i \leq 2p_n$. This gives $\big((x_j^{n,i}, y_j^{n,i})\big)_{j\in\omega} \in G^\omega$ converging to (z_i^n, z_{i+1}^n). If $(C, X\setminus C)$ is a coloring of G into clopen subsets of X which are not empty and satisfy $x \in C$, then $x_n, y_n \in C$ for some n large enough. In particular, $x_j^{n,0} \in C$ if j is large enough. An induction on $i \leq 2p_n+1$ shows that $y_j^{n,i} \notin C$ if j is large enough, $z_{i+1}^n \notin C$ and $x_j^{n,i+1} \notin C$ if j is large enough when if i is even, and $y_j^{n,i} \in C$ if j is large enough, $z_{i+1}^n \in C$, $x_j^{n,i+1} \in C$ if j is large enough when i is odd, which is the desired contradiction. Thus $\chi_c(X, G) \geq 3$. □

Lemma 3.3.2 *$(\mathcal{N}, \mathbb{G}_m)$ has CCN three and $\mathbf{\Sigma}^0_1 \oplus \mathbf{\Pi}^0_1$ chromatic number two.*

Proof. Note that $(c^\infty, c^\infty) \in \Delta(\mathcal{N}) \cap \overline{\bigcup_{p\in\omega} \overline{\mathbb{O}_m}^{2p+1}}$. Indeed, if $k \in \omega$, then $(c^{k+1}a^\infty, c^{k+1}\overline{a}^\infty)$ is in $\overline{\mathbb{O}_m}^{2k+3}$, with witness $\big(c^{k+1}a^\infty, k0^\infty, k1^\infty, \ldots, k(2k+1)^\infty, c^{k+1}\overline{a}^\infty\big)$. Lemma 3.3.1 implies that $\chi_c(\mathcal{N}, \mathbb{G}_m) \geq 3$. The clopen partition $(N_a \cup N_{\overline{a}} \cup N_c, \bigcup_{k,i\in\omega} N_{k(2i)}, \bigcup_{k,i\in\omega} N_{k(2i+1)})$ shows that $\chi_c(\mathcal{N}, \mathbb{G}_m) = 3$.

We define an open subset of $\mathcal{N}$ by $O := \big\{x \in \mathcal{N} \mid \exists n \in \omega \;\; x(n) = a \wedge x|n \in (\omega \cup \{c\})^n\big\}$. The $\mathbf{\Sigma}^0_1 \oplus \mathbf{\Pi}^0_1$ partition $(O, \neg O)$ of $\mathcal{N}$ is a witness for the fact that $\chi_{\mathbf{\Sigma}^0_1 \oplus \mathbf{\Pi}^0_1}(\mathcal{N}, \mathbb{G}_m) = 2$. □

Proposition 3.3.3 *The graph $\mathbb{G}_m$ is $D_2(\mathbf{\Pi}^0_1)$.*

Proof. Note that

$$\overline{\mathbb{O}_m} = \mathbb{O}_m \cup \{(c^{k+1}a^\infty, k0^\infty) \mid k \in \omega\} \cup \{(ki^\infty, k(i+1)^\infty) \mid k \in \omega \wedge i \leq 2k\} \cup \{(k(2k+1)^\infty, c^{k+1}\overline{a}^\infty) \mid k \in \omega\}.$$

Thus $\overline{\mathbb{O}_m}$ is the disjoint union of $\mathbb{O}_m$ and a closed relation on $\mathcal{N}$, so that $\mathbb{O}_m$ is $D_2(\mathbf{\Pi}^0_1)$. The proof for $\mathbb{G}_m$ is similar. □

Lemma 3.3.4 *Let X be a first countable topological space, and G be a graph on X. The following are equivalent:*

(1) $\Delta(X) \cap \overline{\bigcup_{p\in\omega} \overline{G}^{2p+1}} \neq \emptyset$,
(2) $(\mathcal{N}, \mathbb{G}_m) \preceq_c (X, G)$.

Proof. (2) $\Rightarrow$ (1) Let φ be a witness for the fact that (a) holds. We set $x := \varphi(c^\infty)$. Let U be an open neighborhood of x, so that $\varphi^{-1}(U) \times \varphi^{-1}(U)$ is an open neighborhood of (c^∞, c^∞) and contains $N_{c^k} \times N_{c^k}$ for some $k \in \omega$. As $(c^{k+1}a^\infty, c^{k+1}\overline{a}^\infty) \in \overline{\mathbb{O}_m}^{2k+3}$, $\big(\varphi(c^{k+1}a^\infty), \varphi(c^{k+1}\overline{a}^\infty)\big)$ is in $\overline{G}^{2k+3} \cap (U \times U)$. This shows that $(x, x) \in \overline{\bigcup_{p\in\omega} \overline{G}^{2p+1}}$.

(1) $\Rightarrow$ (2) Let $(x, x) \in \Delta(X) \cap \overline{\bigcup_{p\in\omega} \overline{G}^{2p+1}}$, and (U_n) be a decreasing countable basis of open neighborhoods of x. There is, for each $n \in \omega$, $p_n \geq 1$ such that $\overline{G}^{2p_n+1} \cap U_n^2 \neq \emptyset$. Note that we may assume that the sequence $(p_n)_{n\in\omega}$ is constant or strictly increasing.

Let $(x_n, y_n) \in \overline{G}^{2p_n+1} \cap U_n^2$, with witness $(z_i^n)_{i \leq 2p_n+1}$, and $\big((x_j^{n,i}, y_j^{n,i})\big)_{j\in\omega} \in G^\omega$ converging to (z_i^n, z_{i+1}^n).

The map $\varphi : \mathcal{N} \to X$ sends c^∞ to x. If $k < p_0$, then φ sends

- $c^{k+1}a^\infty$ to $z_0^0 = x_0$, $\bigcup_{q\neq a} N_{c^{k+1}a^{j+1}q}$ to $x_j^{0,0}$,
- ki^∞ to z_{i+1}^0 if $i \leq 2k+1$, $N_{ki^{j+1}\overline{a}}$ to $y_j^{0,i}$ if $i \leq 2k+1$, $\bigcup_{m\neq i,\overline{a}} N_{ki^{j+1}m}$ to $x_j^{0,i+1}$ if $i \leq 2k+1$,
- $N_{ki} \cup N_\varepsilon$ to x if $i > 2k+1$ or $i \notin \omega$, and $\varepsilon \in \{a, \overline{a}\}$,
- $\bigcup_{m\neq c,a} \{c^{k+1}m^\infty\}$ to z_{2k+3}^0, $\bigcup_{m\neq c,a, \text{ and } q\neq m} N_{c^{k+1}m^{j+1}q}$ to $y_j^{0,2k+2}$.

If $(p_n)_{n\in\omega}$ is constant and $k \geq p_0$, then φ sends

- $c^{k+1}a^\infty$ to $z_0^k = x_k$, $\bigcup_{q\neq a} N_{c^{k+1}a^{j+1}q}$ to $x_j^{k,0}$,
- ki^∞ to z_{i+1}^k if $i \leq 2p_0-2$, $N_{ki^{j+1}\overline{a}}$ to $y_j^{k,i}$ if $i \leq 2p_0-2$, $\bigcup_{m\neq i,\overline{a}} N_{ki^{j+1}m}$ to $x_j^{k,i+1}$ if $i \leq 2p_0-2$,
- $k(2l+1)^\infty$ to $z_{2p_0}^k$ if $p_0-1 \leq l < k$, $\bigcup_{m\neq 2l+1} N_{k(2l+1)^{j+1}m}$ to $y_j^{k,2p_0-1}$ if $p_0-1 \leq l < k$,
- $k(2l)^\infty$ to $z_{2p_0-1}^k$ if $p_0-1 < l \leq k$, $\bigcup_{m\neq 2l} N_{k(2l)^{j+1}m}$ to $x_j^{k,2p_0-1}$ if $p_0-1 < l \leq k$,
- $k(2k+1)^\infty$ to $z_{2p_0}^k$, $N_{k(2k+1)^{j+1}\overline{a}}$ to $y_j^{k,2p_0-1}$, $\bigcup_{m\neq 2k+1,\overline{a}} N_{k(2k+1)^{j+1}m}$ to $x_j^{k,2p_0}$.
- $N_{ki} \cup N_\varepsilon$ to x if $i > 2k+1$ or $i \notin \omega$, and $\varepsilon \in \{a, \overline{a}\}$,
- $\bigcup_{m\neq c,a} \{c^{k+1}m^\infty\}$ to $z_{2p_0+1}^k$, $\bigcup_{m\neq c,a, \text{ and } q\neq m} N_{c^{k+1}m^{j+1}q}$ to $y_j^{k,2p_0}$.

If $(p_n)_{n\in\omega}$ is strictly increasing and $p_n \leq k < p_{n+1}$, then φ sends

- $c^{k+1}a^\infty$ to $z_0^n = x_n$, $\bigcup_{q\neq a} N_{c^{k+1}a^{j+1}q}$ to $x_j^{n,0}$,
- ki^∞ to z_{i+1}^n if $i \leq 2p_n-2$, $N_{ki^{j+1}\overline{a}}$ to $y_j^{n,i}$ if $i \leq 2p_n-2$, $\bigcup_{m\neq i,\overline{a}} N_{ki^{j+1}m}$ to $x_j^{n,i+1}$ if $i \leq 2p_n-2$,
- $k(2l+1)^\infty$ to $z_{2p_n}^n$ if $p_n-1 \leq l < k$, $\bigcup_{m\neq 2l+1} N_{k(2l+1)^{j+1}m}$ to $y_j^{n,2p_n-1}$ if $p_n-1 \leq l < k$.
- $k(2l)^\infty$ to $z_{2p_n-1}^n$ if $p_n-1 < l \leq k$, $\bigcup_{m\neq 2l} N_{k(2l)^{j+1}m}$ to $x_j^{n,2p_n-1}$ if $p_n-1 < l \leq k$,
- $k(2k+1)^\infty$ to $z_{2p_n}^n$, $N_{k(2k+1)^{j+1}\overline{a}}$ to $y_j^{n,2p_n-1}$, $\bigcup_{m\neq 2k+1,\overline{a}} N_{k(2k+1)^{j+1}m}$ to $x_j^{n,2p_n}$.
- $N_{ki} \cup N_\varepsilon$ to x if $i > 2k+1$ or $i \notin \omega$, and $\varepsilon \in \{a, \overline{a}\}$,
- $\bigcup_{m\neq c,a} \{c^{k+1}m^\infty\}$ to $z_{2p_n+1}^n$, $\bigcup_{m\neq c,a, \text{ and } q\neq m} N_{c^{k+1}m^{j+1}q}$ to $y_j^{n,2p_n}$.

Note that φ is a witness for (2). □

Theorem 1.7 is now a consequence of Lemmas 3.3.4 and 3.3.1.

Remark. We saw in the proof of Proposition 3.3.3 that

$$\overline{\mathbb{O}_m} = \mathbb{O}_m \cup \{(c^{k+1}a^\infty, k0^\infty) \mid k \in \omega\} \cup \{(ki^\infty, k(i+1)^\infty) \mid k \in \omega \wedge i \leq 2k\} \cup \{(k(2k+1)^\infty, c^{k+1}\overline{a}^\infty) \mid k \in \omega\}.$$

Moreover,

$$\overline{\text{proj}[\mathbb{G}_m]} = \text{proj}[\mathbb{G}_m] \cup \big\{c^{k+1}\varepsilon^\infty \mid k \in \omega \wedge \varepsilon \in \{a, \overline{a}\}\big\} \cup \{c^\infty\} \cup \{ki^\infty \mid k \in \omega \wedge i \leq 2k+1\}$$

is a closed countable subset of $\mathcal{N}$. As $\Delta(\mathbb{P}) \cap \overline{\bigcup_{p\in\omega} \overline{\mathbb{G}_m}^{2p+1}}$ is empty (its only possible element could be (c^∞, c^∞), this is not the case since the $c^{k+1}\varepsilon^\infty$'s are not in $\mathbb{P}$), $(\mathbb{P}, \mathbb{G}_m) \prec_c (\mathcal{N}, \mathbb{G}_m)$ by Lemma 3.3.4, as announced in the introduction.

We now characterize the subgraphs of $\mathbb{G}_m$ having a big CCN. In fact, we will need some generalizations of $\mathbb{G}_m$ in the sequel, that we now describe.

Notation. We define a copy of $(\mathbb{P},\mathbb{G}_m)$ and subgraphs of it. We set, for $\delta\in 2^\omega$, $\mathbb{G}_\delta := s(\mathbb{O}_\delta)$, where

$$\mathbb{O}_\delta := \{(c^{k+1}0ja^\infty, k0^{j+2}\overline{a}^\infty) \mid \delta(k)=1 \wedge j\in\omega\} \cup \{(ki0^{j+1}a^\infty, k(i+1)0^{j+1}\overline{a}^\infty) \mid \delta(k)=1 \wedge i\leq 2k \wedge j\in\omega\} \cup \{(k(2k+1)0^{j+1}a^\infty, c^{k+1}1j\overline{a}^\infty) \mid \delta(k)=1 \wedge j\in\omega\},$$

and $\mathbb{P}_\delta = \mathrm{proj}[\mathbb{G}_\delta] \cup \{c^\infty\} \cup \{ki0^\infty \mid \delta(k)=1 \wedge i\leq 2k+1\}$. Note that the vertices of $\mathbb{G}_\delta$ have degree at most one. We also set $\mathbb{P}_\infty := \{\alpha\in 2^\omega \mid \exists^\infty n\in\omega \;\; \alpha(n)=1\}$.

Lemma 3.3.5 $(\mathbb{P}_{1\infty},\mathbb{G}_{1\infty}) \equiv^i_c (\mathbb{P},\mathbb{G}_m)$.

Proof. We define $\varphi:\mathbb{P}_{1\infty}\to\mathbb{P}$ by $\varphi(c^\infty) := c^\infty$, $\varphi(ki0^\infty) := ki^\infty$, $\varphi(c^{k+1}0ja^\infty) := c^{k+1}a^{j+1}\overline{a}^\infty$, $\varphi(ki0^{j+1}\varepsilon^\infty) := ki^{j+1}\varepsilon^\infty$, and $\varphi(c^{k+1}1j\overline{a}^\infty) := c^{k+1}\overline{a}^{j+1}a^\infty$. The map φ is a witness for the fact that $(\mathbb{P}_{1\infty},\mathbb{G}_{1\infty}) \preceq^i_c (\mathbb{P},\mathbb{G}_m)$, and φ^{-1} is a witness for the fact that $(\mathbb{P},\mathbb{G}_m) \preceq^i_c (\mathbb{P}_{1\infty},\mathbb{G}_{1\infty})$. $\square$

Lemma 3.3.6 *Let $\delta\in\mathbb{P}_\infty$. Then $(\mathbb{P}_\delta,\mathbb{G}_\delta)$ has CCN three and $\mathbf{\Sigma}^0_1\oplus\mathbf{\Pi}^0_1$ chromatic number two.*

Proof. For the upper bounds, we prove that $\chi_c(\mathbb{P}_{1\infty},\mathbb{G}_{1\infty})\leq 3$ and $\chi_{\mathbf{\Sigma}^0_1\oplus\mathbf{\Pi}^0_1}(\mathbb{P}_{1\infty},\mathbb{G}_{1\infty}) \leq 2$ since $\mathbb{P}_\delta\subseteq\mathbb{P}_{1\infty}$ and $\mathbb{G}_\delta\subseteq\mathbb{G}_{1\infty}$. This comes from Lemmas 3.3.2 and 3.3.5. For the first lower bound, towards a contradiction, suppose that there is a clopen subset C of $\mathbb{P}_\delta$ with $\mathbb{G}_\delta\cap(C^2\cup(\mathbb{P}_\delta\setminus C)^2)=\emptyset$. We may assume that $c^\infty\in C$, which gives $k_0\geq 1$ such that $N_{c^{k_0+1}}\cap\mathbb{P}_\delta\subseteq C$. Assume that $k\geq k_0$ and $\delta(k)=1$. As $(c^{k+1}0ja^\infty, k0^{j+2}\overline{a}^\infty)\in\mathbb{G}_\delta$ and $c^{k+1}0ja^\infty\in C$, $k0^{j+2}\overline{a}^\infty\in\mathbb{P}_\delta\setminus C$. Thus $k0^\infty\in\mathbb{P}_\delta\setminus C$, which gives $j_0\in\omega$ such that $k0^{j+2}a^\infty\in\mathbb{P}_\delta\setminus C$ if $j\geq j_0$. As $(k0^{j+2}a^\infty, k10^{j+1}\overline{a}^\infty)\in\mathbb{G}_\delta$, $k10^{j+1}\overline{a}^\infty\in C$ if $j\geq j_0$. Thus $k10^\infty\in C$. An induction on $i\leq 2k+1$ shows that $ki0^\infty\in C$ if i is odd, and $ki0^\infty\in\mathbb{P}_\delta\setminus C$ if i is even. This implies that $c^{k+1}1j\overline{a}^\infty\in\mathbb{P}_\delta\setminus C$ if j is large enough, which is absurd. $\square$

Proposition 3.3.7 *Let $\delta\in 2^\omega$. Then $\mathbb{G}_\delta$ is a $D_2(\mathbf{\Pi}^0_1)$ graph on the 0DP space $\mathbb{P}_\delta$.*

Proof. As $\mathbb{P}_\delta$ is a closed subset of $\mathcal{N}$, it is a 0DP space. Note that

$$\overline{\mathbb{O}_\delta} = \mathbb{O}_\delta \cup \big\{\big(ki0^\infty, k(i+1)0^\infty\big) \mid \delta(k)=1 \wedge i\leq 2k\big\}.$$

Thus $\overline{\mathbb{O}_\delta}$ is the disjoint union of $\mathbb{O}_\delta$ and a closed relation on $\mathcal{N}$, so that $\mathbb{O}_\delta$ is $D_2(\mathbf{\Pi}^0_1)$. The proof for $\mathbb{G}_\delta$ is similar. $\square$

We now characterize the subgraphs of $\mathbb{G}_\delta$ having a big CCN.

Lemma 3.3.8 *Let $V\subseteq\mathbb{P}_\delta$, and $E\subseteq\mathbb{G}_\delta\cap V^2$. The following are equivalent:*

(1) the digraph (V,E) has CCN at least three,

(2) $c^\infty\in V$ and there is $I\subseteq\{k\in\omega \mid \delta(k)=1\}$ infinite such that, for each $k\in I$,

- *(a) $\forall i\leq 2k+1 \;\; ki0^\infty\in V$,*
- *(b) $\exists^\infty j\in\omega \;\; (c^{k+1}0ja^\infty, k0^{j+2}\overline{a}^\infty)\in s(E)$,*
- *(c) $\forall i\leq 2k \;\; \exists^\infty j\in\omega \;\; (ki0^{j+1}a^\infty, k(i+1)0^{j+1}\overline{a}^\infty)\in s(E)$,*
- *(d) $\exists^\infty j\in\omega \;\; (k(2k+1)0^{j+1}a^\infty, c^{k+1}1j\overline{a}^\infty)\in s(E)$.*

Proof. Note first that $\chi_c(V,E) \leq \chi_c\big(V, s(E)\big) \leq \chi_c(\mathbb{P}_\delta, \mathbb{G}_\delta) = 3$, by Lemma 3.3.6. We may and will assume that $E = s(E)$ is a graph.

(2) $\Rightarrow$ (1) Towards a contradiction, suppose that there is a clopen subset C of V with the property that $E \cap (C^2 \cup (V \setminus C)^2) = \emptyset$. We may assume that $c^\infty \in C$, which gives $k_0 \in \omega$ such that $N_{c^{k_0+1}} \cap V \subseteq C$. Assume that $k \geq k_0$ is in I. By (b), there are infinitely many j's with $(c^{k+1}0ja^\infty, k0^{j+2}\overline{a}^\infty) \in E \subseteq V^2$, which implies that $c^{k+1}0ja^\infty \in C$ and $k0^{j+2}\overline{a}^\infty \notin C$. By (a), $k0^\infty \in V$, so that $k0^\infty \in V \setminus C$. By (c), there are infinitely many j's with the property that $(k0^{j+2}a^\infty, k10^{j+1}\overline{a}^\infty) \in E \subseteq V^2$, so that we may assume that $k0^{j+2}a^\infty \in V \setminus C$ and thus $k10^{j+1}\overline{a}^\infty \in C$. By (a) again, $k10^\infty \in V$, so that $k10^\infty \in C$. An induction on $i \leq 2k+1$ shows that $ki0^\infty \in V \setminus C$ if i is even, and $ki0^\infty \in C$ if i is odd. By (d), this gives infinitely many j's such that $c^{k+1}1j\overline{a}^\infty \in V \setminus C$, which is the desired contradiction.

(1) $\Rightarrow$ (2) If $c^\infty \notin V$, then we set $C := \big(\bigcup_{k\in\omega} \big(\bigcup_{i\leq 2k+1 \text{ even}} N_{ki} \cup N_{c^{k+1}1}\big)\big) \cap V$. Then $(C, V\setminus C)$ is a coloring of E into clopen sets, which is absurd. If (2) does not hold, then there is $k_0 \in \omega$ such that one of the properties (a)-(d) does not hold if $k \geq k_0$. We will use the following notation.

- If (a) does not hold, then $i_k \leq 2k+1$ will be minimal with $ki_k0^\infty \notin V$,
- If (b) does not hold, then $j_k \in \omega$ will be minimal such that $(c^{k+1}0ja^\infty, k0^{j+2}\overline{a}^\infty) \notin E$ if $j \geq j_k$,
- If (c) does not hold, then $i_k \leq 2k$ and $j_k \in \omega$ will be minimal such that $(ki_k0^{j+1}a^\infty, k(i_k{+}1)0^{j+1}\overline{a}^\infty)$ is not in E if $j \geq j_k$,
- If (d) does not hold, then $j_k \in \omega$ will be minimal such that $(k(2k+1)0^{j+1}a^\infty, c^{k+1}1j\overline{a}^\infty) \notin E$ if $j \geq j_k$.

We then set

$$\begin{aligned}
C' := \big(\bigcup_{k<k_0} \big(\bigcup_{i\leq 2k+1 \text{ even}} N_{ki} \cup N_{c^{k+1}1}\big) \cup \\
\bigcup_{k\geq k_0,\neg(a),i_k \text{ even}} \big(\bigcup_{i<i_k \text{ even}} N_{ki} \cup \bigcup_{j\in\omega} N_{ki_k0^{j+1}\overline{a}} \cup \bigcup_{i_k<i\leq 2k+1 \text{ odd}} N_{ki}\big) \cup \\
\bigcup_{k\geq k_0,\neg(a),i_k \text{ odd}} \big(\bigcup_{i<i_k \text{ even}} N_{ki} \cup \bigcup_{j\in\omega} N_{ki_k0^{j+1}a} \cup \bigcup_{i_k<i\leq 2k+1 \text{ odd}} N_{ki}\big) \cup \\
\bigcup_{k\geq k_0,(a),\neg(b)} \big(\bigcup_{j<j_k} N_{k0^{j+2}\overline{a}} \cup \bigcup_{i\leq 2k+1 \text{ odd}} N_{ki}\big) \cup \\
\bigcup_{k\geq k_0,(a),(b),\neg(c),i_k \text{ even}} \big(\bigcup_{i<i_k \text{ even}} N_{ki} \cup \big(N_{ki_k} \setminus (\bigcup_{j<j_k} N_{ki_k0^{j+1}a})\big) \cup \bigcup_{i_k<i\leq 2k+1 \text{ odd}} N_{ki}\big) \cup \\
\bigcup_{k\geq k_0,(a),(b),\neg(c),i_k \text{ odd}} \big(\bigcup_{i<i_k \text{ even}} N_{ki} \cup \bigcup_{j<j_k} N_{ki_k0^{j+1}a} \cup \bigcup_{i_k<i\leq 2k+1 \text{ odd}} N_{ki}\big) \cup \\
\bigcup_{k\geq k_0,(a),(b),(c),\neg(d)} \big(\bigcup_{i\leq 2k+1 \text{ even}} N_{ki} \cup \bigcup_{j<j_k} N_{k(2k+1)0^{j+1}a}\big)\big) \cap V.
\end{aligned}$$

Then $(C', V\setminus C')$ is a coloring of E into clopen sets, which is absurd. $\square$

4 General graphs on a 0DMS space

Remark. We study the limits of Theorem 3.1.2. In its proof, we used the compactness of $\mathcal{C}_{\mathbf{d}}$. This is essential. Indeed, if we replace $\mathcal{C}_{\mathbf{d}}$ with ω^ω or $\mathcal{N}$, then the notation ${}^nG$ still makes sense and the following hold. The implications (4) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) still hold, with the same proof. Also, the implication (4) $\Rightarrow$ (2) still holds, using uniform continuity.

(a) The implication (1) $\Rightarrow$ (4) does not hold. Indeed, if X is a 0DMC space and G is a graph on X with CCN at least three, then (X,G) is not $\preceq_c$-below $(\mathcal{N}, \mathbb{G}_m)$. Indeed, we argue by contradiction to see that, which gives a continuous map $\varphi : X \to \mathcal{N}$. We set $V := \varphi[X]$ and $E := (\varphi\times\varphi)[G]$, so that the graph (V,E) has CCN three.

The compactness of X implies that the first coordinate of the elements of $V\setminus(N_c\cup N_a\cup N_{\overline{a}})$ is bounded by some natural number k_0. We set $C:=\big(\bigcup_{k\leq k_0}\ (\bigcup_{i\leq 2k+1\text{ even}}\ N_{ki}\cup N_{c^{k+1}\overline{a}})\big)\cap V$, so that C is a clopen subset of V and $E\cap(C^2\cup(V\setminus C)^2)=\emptyset$, which is the desired contradiction. In particular, if $(\mathbb{A}_1,\mathbb{G}_1)$ exists, then $\mathbb{A}_1$ cannot be compact, and $\big((\mathbb{K}_\alpha,\mathbb{G}_\alpha)\big)_{\alpha\in 2^\omega}$ given by Theorem 1.4 is not a $\preceq_c$-basis for the class of countable graphs on a 0DP space with CCN at least three.

(b) The implication (2) $\Rightarrow$ (4) does not hold. Indeed, note that $(c^\infty,c^\infty)\in\bigcap_{n\in\omega}\ (\bigcup_{p\in\omega}\ ({}^n\mathbb{G}_m)^{2p+1})$.

(c) The implication (2) $\Rightarrow$ (1) does not hold. Indeed, consider the following countable graph on ω^ω:

$$\mathbb{T}:=s\big(\{(0^{2k+1}1^\infty,(2k+2)0^\infty)\mid k\in\omega\}\cup\big\{\big((2k+2)i0^k1^\infty,(2k+2)(i+1)0^\infty\big)\mid k\in\omega\wedge i\leq 2k\big\}\cup \big\{\big((2k+2)(2k+1)0^k1^\infty,0^{2k+2}1^\infty\big)\mid k\in\omega\big\}\big).$$

Then $(0^\infty,0^\infty)\in\Delta(\omega^\omega)\cap\bigcap_{n\in\omega}\ ({}^n\mathbb{T})^{2n+1}$. We set $C:=N_0\cup\bigcup_{k\in\omega,j\leq 2k}\ N_{(2k+2)(j+1)0^{k+1}}$. Then $\mathbb{T}\cap(C^2\cup(\neg C)^2)=\emptyset$ and C is a clopen subset of ω^ω, so that $\chi_c(\omega^\omega,\mathbb{T})=2$.

We now turn to the proof of Theorem 1.8.

Lemma 4.1 *Let $(Q,\leq)$ be quasi-order for which there is $q_0\in Q$ such that, for any $q\leq q_0$, there are $\leq$-incomparable $p_0,p_1\leq q$. Then $(Q,\leq)$ has no antichain basis.*

Proof. Towards a contradiction, suppose that there is an antichain basis B for $(Q,\leq)$. As B is a basis, there is $q\in B$ with $q\leq q_0$. Our assumption gives $p_0,p_1\leq q$ with $p_0\perp p_1$. As B is a basis, there is, for each $\varepsilon\in 2$, $q_\varepsilon\in B$ with $q_\varepsilon\leq p_\varepsilon$. As $q_\varepsilon\leq q$ are in the antichain B, $q_0=q=q_1$. Thus $p_0\leq q=q_1\leq p_1$ and $p_0\leq p_1$, which contradicts the $\leq$-incomparability of p_0,p_1. □

Recall the graph $(\mathbb{P}_\delta,\mathbb{G}_\delta)$ defined before Lemma 3.3.5.

Lemma 4.2 *Let $\delta\in 2^\omega$, and G be a graph on a 0DMS space Z, with CCN at least three and satisfying $(Z,G)\ \preceq_c^i\ (\mathbb{P}_\delta,\mathbb{G}_\delta)$. Then there is $\delta'\in\mathbb{P}_\infty$ such that $\{k\in\omega\mid\delta'(k)=1\}\subseteq\{k\in\omega\mid\delta(k)=1\}$ and $(\mathbb{P}_{\delta'},\mathbb{G}_{\delta'})\ \preceq_c^i\ (Z,G)$.*

Proof. Assume that $(Z,G)\ \preceq_c^i\ (\mathbb{P}_\delta,\mathbb{G}_\delta)$, with witness φ. We set $V:=\varphi[Z]$ and $E:=(\varphi\times\varphi)[G]$, so that, by Lemma 3.3.8, $c^\infty\in V$ and the set $I\subseteq\{k\in\omega\mid\delta(k)=1\}$ of k's satisfying (a)-(d) is infinite. We set $(\eta_a,\eta_{\overline{a}}):=(0,1)$, and define

- a singleton $\{\nu\}:=\varphi^{-1}(\{c^\infty\})$,
- singletons $N^{k,i}:=\{n^{k,i}\}:=\varphi^{-1}(\{ki0^\infty\})$ (for $k\in I$ and $i\leq 2k+1$),
- infinite sets $J_k^a:=J_{k,0}^{\overline{a}}:=\{j\in\omega\mid(c^{k+1}0ja^\infty,k0^{j+2}\overline{a}^\infty)\in E\}$ and

$$J_k^{\overline{a}}:=J_{k,2k+1}^a:=\{j\in\omega\mid(k(2k+1)0^{j+1}a^\infty,c^{k+1}1j\overline{a}^\infty)\in E\}$$

(for $k\in I$),

- infinite sets $J_{k,i}^a:=J_{k,i+1}^{\overline{a}}:=\{j\in\omega\mid(ki0^{j+1}a^\infty,k(i+1)0^{j+1}\overline{a}^\infty)\in E\}$ (for $k\in I$ and $i\leq 2k$),
- singletons $Z^{k,\varepsilon,j}:=\{z^{k,\varepsilon,j}\}:=\varphi^{-1}(\{c^{k+1}\eta_\varepsilon j\varepsilon^\infty\})$ (for $k\in I$, $\varepsilon\in\{a,\overline{a}\}$ and $j\in J_k^\varepsilon$),
- singletons $Z^{k,i,j,\varepsilon}:=\{z^{k,i,j,\varepsilon}\}:=\varphi^{-1}(\{ki0^{j+1}\varepsilon^\infty\})$ (for $k\in I$, $i\leq 2k+1$, $j\in J_{k,i}^\varepsilon$, and $\varepsilon\in\{a,\overline{a}\}$).

By [K, 7.8], we may assume that $Z\subseteq\omega^\omega$. We set

$$I':=\big\{k\in I\mid\forall i\leq 2k\ \ (n^{k,i},n^{k,i+1})\in\overline{\{(z^{k,i,j,a},z^{k,i+1,j,\overline{a}})\mid j\in J_{k,i}^a\}}\big\}.$$

If $k\in I\setminus I'$, then $i_k\leq 2k$ will be minimal with $(n^{k,i_k},n^{k,i_k+1})\notin\overline{\{(z^{k,i_k,j,a},z^{k,i_k+1,j,\overline{a}})\mid j\in J^a_{k,i_k}\}}$. This gives $l_k\in\omega$ with $(N_{n^{k,i_k}|l_k}\times N_{n^{k,i_k+1}|l_k})\cap\{(z^{k,i_k,j,a},z^{k,i_k+1,j,\overline{a}})\mid j\in J^a_{k,i_k}\}=\emptyset$. If we set $J^{a,-}_{k,i_k}:=\{j\in J^a_{k,i_k}\mid z^{k,i_k,j,a}\notin N_{n^{k,i_k}|l_k}\}$ and $J^{a,+}_{k,i_k}:=\{j\in J^a_{k,i_k}\mid z^{k,i_k+1,j,\overline{a}}\notin N_{n^{k,i_k+1}|l_k}\}$, then $J^a_{k,i_k}=J^{a,-}_{k,i_k}\cup J^{a,+}_{k,i_k}$.

Let us prove that

$$\forall l\in\omega\ \ \exists^\infty k\in I'\ \ \forall p\in\omega\ \ \exists j\in J^a_k\ \ Z^{k,a,j}\cap N_{\nu|l},Z^{k,0,j,\overline{a}}\cap N_{n^{k,0}|p}\neq\emptyset\wedge$$
$$\forall p\in\omega\ \ \exists j\in J^{\overline{a}}_k\ \ Z^{k,2k+1,j,a}\cap N_{n^{k,2k+1}|p},Z^{k,\overline{a},j}\cap N_{\nu|l}\neq\emptyset.$$

Towards a contradiction, suppose that we can find $l_0,k_0\in\omega$ such that, for each $k\geq k_0$ in I', either there is $p^a_k\in\omega$ such that, for $j\in J^a_k$, $Z^{k,a,j}\cap N_{\nu|l_0}=\emptyset$ or $Z^{k,0,j,\overline{a}}\cap N_{n^{k,0}|p^a_k}=\emptyset$, or there is $p^{\overline{a}}_k\in\omega$ such that, for $j\in J^{\overline{a}}_k$, $Z^{k,2k+1,j,a}\cap N_{n^{k,2k+1}|p^{\overline{a}}_k}=\emptyset$ or $Z^{k,\overline{a},j}\cap N_{\nu|l_0}=\emptyset$. If $k\geq k_0$ and (b) from Lemma 3.3.8 does not hold, then j_k will be minimal such that $(c^{k+1}0ja^\infty,k0^{j+2}\overline{a}^\infty)\notin s(E)$ if $j\geq j_k$.

We set, for $\varepsilon\in\{a,\overline{a}\}$, $S^\varepsilon:=\{k\in I'\mid k\geq k_0\wedge p^\varepsilon_k\text{ exists}\}$. We also set, for $k\in S^a$,

$$S^{k,a}:=\{j\in J^a_k\mid Z^{k,0,j,\overline{a}}\cap N_{n^{k,0}|p^a_k}=\emptyset\}$$

and, for $k\in S^{\overline{a}}$, $S^{k,\overline{a}}:=\{j\in J^{\overline{a}}_k\mid Z^{k,2k+1,j,a}\cap N_{n^{k,2k+1}|p^{\overline{a}}_k}=\emptyset\}$. We then set

$$\begin{aligned}C:=&\bigcup_{k<k_0}\ \varphi^{-1}(\bigcup_{i\leq 2k+1\text{ even}}\ N_{ki}\cup N_{c^{k+1}1})\cup\\
&\bigcup_{k\geq k_0,\neg(a),i_k\text{ even}}\ \varphi^{-1}(\bigcup_{i<i_k\text{ even}}\ N_{ki}\cup\bigcup_{j\in\omega}\ N_{ki_k0^{j+1}\overline{a}}\cup\bigcup_{i_k<i\leq 2k+1\text{ odd}}\ N_{ki})\cup\\
&\bigcup_{k\geq k_0,\neg(a),i_k\text{ odd}}\ \varphi^{-1}(\bigcup_{i<i_k\text{ even}}\ N_{ki}\cup\bigcup_{j\in\omega}\ N_{ki_k0^{j+1}a}\cup\bigcup_{i_k<i\leq 2k+1\text{ odd}}\ N_{ki})\cup\\
&\bigcup_{k\geq k_0,(a),\neg(b)}\ \varphi^{-1}(\bigcup_{j<j_k}\ N_{k0^{j+2}\overline{a}}\cup\bigcup_{i\leq 2k+1\text{ odd}}\ N_{ki})\cup\\
&\bigcup_{k\geq k_0,(a),(b),\neg(c),i_k\text{ even}}\ \varphi^{-1}\big(\bigcup_{i<i_k\text{ even}}\ N_{ki}\cup\big(N_{ki_k}\setminus(\bigcup_{j<j_k}\ N_{ki_k0^{j+1}a})\big)\cup\bigcup_{i_k<i\leq 2k+1\text{ odd}}\ N_{ki}\big)\cup\\
&\bigcup_{k\geq k_0,(a),(b),\neg(c),i_k\text{ odd}}\ \varphi^{-1}\big(\bigcup_{i<i_k\text{ even}}\ N_{ki}\cup\bigcup_{j<j_k}\ N_{ki_k0^{j+1}a}\cup\bigcup_{i_k<i\leq 2k+1\text{ odd}}\ N_{ki}\big)\cup\\
&\bigcup_{k\geq k_0,(a),(b),(c),\neg(d)}\ \varphi^{-1}(\bigcup_{i\leq 2k+1\text{ even}}\ N_{ki}\cup\bigcup_{j<j_k}\ N_{k(2k+1)0^{j+1}a})\cup\\
&\bigcup_{k\geq k_0,k\in I\setminus I',i_k\text{ even}}\ \big(\varphi^{-1}(\bigcup_{i<i_k\text{ even}}\ N_{ki}\cup\bigcup_{j\in\omega}\ N_{ki_k0^{j+1}\overline{a}}\cup\bigcup_{j\notin J^{a,-}_{k,i_k}}\ N_{ki_k0^{j+1}a}\cup\\
&\bigcup_{j\in\omega}\ N_{k(i_k+1)0^{j+1}a}\cup\bigcup_{j\notin J^{a,+}_{k,i_k}}\ N_{k(i_k+1)0^{j+1}\overline{a}}\cup\bigcup_{i_k+1<i\leq 2k+1\text{ odd}}\ N_{ki})\cup N_{n^{k,i_k}|l_k}\cup N_{n^{k,i_k+1}|l_k}\big)\cup\\
&\bigcup_{k\geq k_0,k\in I\setminus I',i_k\text{ odd}}\ \big(\varphi^{-1}(\bigcup_{i<i_k\text{ even}}\ N_{ki}\cup\bigcup_{j\notin J^{a,-}_{k,i_k}}\ N_{ki_k0^{j+1}a}\cup\\
&\bigcup_{j\notin J^{a,+}_{k,i_k}}\ N_{k(i_k+1)0^{j+1}\overline{a}}\cup\bigcup_{i_k+1<i\leq 2k+1\text{ odd}}\ N_{ki})\cup N_{n^{k,i_k}|l_k}\cup N_{n^{k,i_k+1}|l_k}\big)\cup\\
&\bigcup_{k\in S^{\overline{a}}}\ \varphi^{-1}(\bigcup_{i\leq 2k+1\text{ even}}\ N_{ki}\cup\bigcup_{j\in S^{k,\overline{a}}}\ N_{k(2k+1)0^{j+1}a}\cup\bigcup_{j\in J^{\overline{a}}_k\setminus S^{k,\overline{a}}}\ N_{c^{k+1}1j\overline{a}})\cup\\
&\bigcup_{k\in S^a\setminus S^{\overline{a}}}\ \varphi^{-1}(\bigcup_{j\in J^a_k\setminus S^{k,a}}\ N_{c^{k+1}0ja}\cup\bigcup_{j\in S^{k,a}}\ N_{k0^{j+2}\overline{a}}\cup\bigcup_{i\leq 2k+1\text{ odd}}\ N_{ki}).\end{aligned}$$

As $\varphi^{-1}(N_k\cap\mathbb{P}_\delta)$ is a clopen subset of Z, the only possible limit point of $(z^{k,i,j,\varepsilon})_{j\in\omega}$ is $n^{k,i}$. Also, the only possible limit point of $(z^{k,\varepsilon,j})_{j,k\in\omega}$ is ν. This implies that $(C,Z\setminus C)$ is a coloring of G into clopen sets, which is absurd.

We then set, for $l\in\omega$,

$$S_l:=\{k\in I'\mid\forall p\in\omega\ \ \exists j\in J^a_k\ \ Z^{k,a,j}\cap N_{\nu|l},Z^{k,0,j,\overline{a}}\cap N_{n^{k,0}|p}\neq\emptyset\wedge$$
$$\forall p\in\omega\ \ \exists j\in J^{\overline{a}}_k\ \ Z^{k,2k+1,j,a}\cap N_{n^{k,2k+1}|p},Z^{k,\overline{a},j}\cap N_{\nu|l}\neq\emptyset\},$$

so that $(S_l)_{l\in\omega}$ is decreasing.

We inductively define $k_0 := \min S_0$, and $k_{l+1} := \min S_{l+1} \cap (k_l, \infty)$, so that $(k_l)_{l\in\omega}$ is stricly increasing. We define $\delta' \in 2^\omega$ by $\delta'(k) = 1 \Leftrightarrow \exists l \in \omega \;\; k = k_l$, so that $\delta' \in \mathbb{P}_\infty$. We pick, for $l, p \in \omega$ and $i \leq 2k_l$,

- $j_{l,p,0} \in J^a_{k_l}$, $z_{k_l,a,j_{l,p,0}} \in N_{\nu|l}$ and $z_{k_l,0,j_{l,p,0},\overline{a}} \in N_{n^{k_l,0}|p}$, ensuring the injectivity of $(j_{l,p,0})_{p\in\omega}$,
- $j_{l,p,i+1} \in J^a_{k_l,i}$, $z_{k_l,i,j_{l,p,i+1},a} \in N_{n^{k_l,i}|p}$, $z_{k_l,i+1,j_{l,p,i+1},\overline{a}} \in N_{n^{k_l,i+1}|p}$, ensuring the injectivity of $(j_{l,p,i+1})_{p\in\omega}$,
- $j_{l,p,2k_l+2} \in J^{\overline{a}}_{k_l}$, $z_{k_l,2k_l+1,j_{l,p,2k_l+2},a} \in N_{n^{k_l,2k_l+1}|p}$ and $z_{k_l,\overline{a},j_{l,p,2k_l+2}} \in N_{\nu|l}$, ensuring the injectivity of $(j_{l,p,2k_l+2})_{p\in\omega}$.

We are now ready to construct $\psi : \mathbb{P}_{\delta'} \to Z$. Note that

$$\mathbb{P}_{\delta'} = \big\{c^{k_l+1}\eta_\varepsilon j\varepsilon^\infty \mid j, l \in \omega \wedge \varepsilon \in \{a, \overline{a}\}\big\} \cup \big\{k_l i 0^{j+1}\varepsilon^\infty \mid j, l \in \omega \wedge i \leq 2k_l+1 \wedge \varepsilon \in \{a, \overline{a}\}\big\} \cup \{c^\infty\} \cup \{k_l i 0^\infty \mid l \in \omega \wedge i \leq 2k_l+1\}.$$

We first set $\psi(c^\infty) := \nu$. If $l \in \omega$, then ψ sends $c^{k_l+1}0pa^\infty$ to $z_{k_l,a,j_{l,p,0}}$, $k_l i 0^\infty$ to $n^{k_l,i}$ if $i \leq 2k_l+1$, $k_l i 0^{p+1}a^\infty$ to $z_{k_l,i,j_{l,p,i+1},a}$ and $k_l i 0^{p+1}\overline{a}^\infty$ to $z_{k_l,i,j_{l,p,i},\overline{a}}$ if $i \leq 2k_l+1$, and $c^{k_l+1}1p\overline{a}^\infty$ to $z_{k_l,\overline{a},j_{l,p,2k_l+2}}$. Note that ψ is as desired. $\square$

Lemma 4.3 *Let $\delta, \delta' \in \mathbb{P}_\infty$ with $(\mathbb{P}_\delta, \mathbb{G}_\delta) \preceq^i_c (\mathbb{P}_{\delta'}, \mathbb{G}_{\delta'})$. Then there is $k_0 \in \omega$ with the property that $\{k \geq k_0 \mid \delta(k) = 1\} \subseteq \{k \in \omega \mid \delta'(k) = 1\}$.*

Proof. Let $\varphi : \mathbb{P}_\delta \to \mathbb{P}_{\delta'}$ be a witness for the fact that $(\mathbb{P}_\delta, \mathbb{G}_\delta) \preceq^i_c (\mathbb{P}_{\delta'}, \mathbb{G}_{\delta'})$. We set $V := \varphi[\mathbb{P}_\delta]$ and $E := (\varphi\times\varphi)[\mathbb{G}_\delta]$, so that $\chi_c(V, E) = 3$. By Lemma 3.3.8, $c^\infty \in V$. Moreover, $\varphi\big[\mathrm{proj}[\mathbb{G}_\delta]\big] \subseteq \mathrm{proj}[\mathbb{G}_{\delta'}]$.

Let us prove that $\varphi[\mathbb{P}_\delta \setminus (\mathrm{proj}[\mathbb{G}_\delta] \cup \{c^\infty\})] \subseteq \mathbb{P}_{\delta'} \setminus (\mathrm{proj}[\mathbb{G}_{\delta'}] \cup \{c^\infty\})$. Towards a contradiction, suppose that we can find k_1 such that $\delta(k_1) = 1$ and $i \leq 2k_1+1$ with $\varphi(k_1 i 0^\infty) \in \mathrm{proj}[\mathbb{G}_{\delta'}] \cup \{c^\infty\}$. We set $(\eta_a, \eta_{\overline{a}}) := (0, 1)$. If $\varphi(k_1 i 0^\infty) \in \mathrm{proj}[\mathbb{G}_{\delta'}]$, then there are i', j', k', ε with the property that either $\varphi(k_1 i 0^\infty) = c^{k'+1}\eta_\varepsilon j'\varepsilon^\infty$, or $\varphi(k_1 i 0^\infty) = k' i' 0^{j'+1}\varepsilon^\infty$. The continuity of φ provides a natural number j_0 with $\varphi[N_{k_1 i 0^{j_0+1}}] \subseteq N_{c^{k'+1}\eta_\varepsilon j'\varepsilon}$ or $\varphi[N_{k_1 i 0^{j_0+1}}] \subseteq N_{k' i' 0^{j'+1}\varepsilon}$. This implies that the sequence $\big(\varphi(k_1 i 0^{j+1}a^\infty)\big)_{j\geq j_0}$ is constant, which contradicts the injectivity of φ. If $\varphi(k_1 i 0^\infty) = c^\infty$, then $\delta'' \in 2^\omega$ defined by $\delta''(k) = 1 \Leftrightarrow \delta(k) = 1 \wedge k \neq k_1$ is in $\mathbb{P}_\infty$ and $\varphi_{|\mathbb{P}_{\delta''}}$ is a witness for the fact that $(\mathbb{P}_{\delta''}, \mathbb{G}_{\delta''}) \preceq^i_c (\mathbb{P}_{\delta'}, \mathbb{G}_{\delta'})$. But the injectivity of φ implies that $c^\infty \notin \varphi[\mathbb{P}_{\delta''}]$, which implies that $\chi_c(\mathbb{P}_{\delta''}, \mathbb{G}_{\delta''}) \leq 2$ by Lemma 3.3.8, and contradicts Lemma 3.3.6.

This implies that $\varphi(c^\infty) = c^\infty$, and gives k_0 with $\varphi[N_{c^{k_0+1}}] \subseteq N_c$. Pick $k \geq k_0$ with $\delta(k) = 1$. This gives, for each $p \in \omega$, j_p, k_p, ε_p with $\delta'(k_p) = 1$ and $\varphi(c^{k+1}0pa^\infty) = c^{k_p+1}\eta_{\varepsilon_p} j_p \varepsilon_p^\infty$. Extracting a subsequence if necessary, we may assume that the ε_p's are equal to ε. Thus

$$\varphi(k0^{p+2}\overline{a}^\infty) = \begin{cases} k_p 0^{j_p+2}\overline{a}^\infty \text{ if } \varepsilon = a, \\ k_p(2k_p+1)0^{j_p+1}a^\infty \text{ if } \varepsilon = \overline{a}. \end{cases}$$

The continuity of φ implies that $\big(\varphi(k0^{p+2}\overline{a}^\infty)\big)_{p\in\omega}$ converges to $\varphi(k0^\infty)$. Extracting a subsequence if necessary, we may assume that the k_p's are equal to k' and $(j_p)_{p\in\omega}$ is injective. Thus $\varphi(k0^\infty)$ is $k'0^\infty$ or $k'(2k'+1)0^\infty$. Now note that the sequence $\big(\varphi(k0^{p+2}a^\infty)\big)_{p\in\omega}$ also converges to $\varphi(k0^\infty)$. This gives, for each $p \in \omega$, j'_p, ε'_p with $(j'_p)_{p\in\omega}$ is injective (up to an extraction) and

$$\varphi(k0^{p+2}a^\infty) = \begin{cases} k' 0^{j'_p+2}{\varepsilon'_p}^\infty \text{ if } \varepsilon = a, \\ k'(2k'+1)0^{j'_p+1}{\varepsilon'_p}^\infty \text{ if } \varepsilon = \overline{a}. \end{cases}$$

Extracting a subsequence if necessary, we may assume that the ε'_p's are equal to ε'. Thus

$$\varphi(k10^{p+1}\overline{a}^\infty)=\begin{cases}\begin{cases} k'10^{j'_p+1}\overline{a}^\infty \text{ if } \varepsilon'=a,\\ c^{k'+1}0(j'_p+1)a^\infty \text{ if } \varepsilon'=\overline{a},\end{cases} & \text{if } \varepsilon=a,\\ \\ \begin{cases} c^{k'+1}1(j'_p+1)\overline{a}^\infty \text{ if } \varepsilon'=a,\\ k'(2k')0^{j'_p+1}a^\infty \text{ if } \varepsilon'=\overline{a},\end{cases} & \text{if } \varepsilon=\overline{a}.\end{cases}$$

As $\big(\varphi(k10^{p+1}\overline{a}^\infty)\big)_{p\in\omega}$ converges to $\varphi(k10^\infty)\in\mathbb{P}_{\delta'}\setminus(\text{proj}[\mathbb{G}_{\delta'}]\cup\{c^\infty\})$, $\varphi(k10^{p+1}\overline{a}^\infty)\notin N_c$, and the second and third cases are not possible if p is large enough. Thus $\varepsilon'=\varepsilon$, and $\varphi(k10^\infty)$ is $k'10^\infty$ or $k'(2k')0^\infty$. Now note that the sequence $\big(\varphi(k10^{p+1}a^\infty)\big)_{p\in\omega}$ also converges to $\varphi(k10^\infty)$. This gives, for each $p\in\omega$, j''_p,ε''_p with $(j''_p)_{p\in\omega}$ is injective (up to an extraction) and

$$\varphi(k10^{p+1}a^\infty)=\begin{cases} k'10^{j''_p+1}\varepsilon''^\infty_p \text{ if } \varepsilon=a,\\ k'(2k')0^{j''_p+1}\varepsilon''^\infty_p \text{ if } \varepsilon=\overline{a}.\end{cases}$$

Extracting a subsequence if necessary, we may assume that the ε''_p's are equal to ε''. If $k>0$, then the continuity of φ implies that $\big(\varphi(k20^{p+1}\overline{a}^\infty)\big)_{p\in\omega}$ converges to $\varphi(k20^\infty)\in\mathbb{P}_{\delta'}\setminus(\text{proj}[\mathbb{G}_{\delta'}]\cup\{c^\infty\})$. This implies that $\varphi(k20^\infty)\notin N_c$, and $\varphi(k20^{p+1}\overline{a}^\infty)\notin N_c$ if p is large enough. So we may assume that

$$\varphi(k20^{p+1}\overline{a}^\infty)=\begin{cases}\begin{cases} k'20^{j''_p+1}\overline{a}^\infty \text{ if } \varepsilon''=a,\\ k'0^{j''_p+2}a^\infty \text{ if } \varepsilon''=\overline{a},\end{cases} & \text{if } \varepsilon=a,\\ \\ \begin{cases} k'(2k'+1)0^{j''_p+1}\overline{a}^\infty \text{ if } \varepsilon''=a,\\ k'(2k'-1)0^{j''_p+1}a^\infty \text{ if } \varepsilon''=\overline{a},\end{cases} & \text{if } \varepsilon=\overline{a}.\end{cases}$$

The injectivity of φ and the value of $\varphi(k0^\infty)$ imply that second and third cases are not possible if p is large enough. Thus $\varepsilon''=\varepsilon$, and $\varphi(k20^\infty)$ is $k'20^\infty$ or $k'(2k'-1)0^\infty$. This implies that $k'>0$. If now $k=0$, then

$$\varphi(c^{k+1}1p\overline{a}^\infty)=\begin{cases} c^{k'+1}1j''_p\overline{a}^\infty \text{ if } \varepsilon=a,\\ c^{k'+1}0(j''_p+1)a^\infty \text{ if } \varepsilon=\overline{a}.\end{cases}$$

since $\varphi(c^{k+1}1p\overline{a}^\infty)\in N_c$. This implies that $k'=0$. Iterating this argument, we see that $k'=k$ and $\delta'(k)=1$. Thus $\delta'(k)=1$ if $\delta(k)=1$ and $k\geq k_0$. □

Proof of Theorem 1.8. Lemmas 3.3.5 and 4.2 provide $\delta\in\mathbb{P}_\infty$ such that $(\mathbb{P}_\delta,\mathbb{G}_\delta)\preceq^i_c\big(Z,G\big)$. We enumerate injectively $\{n\in\omega\mid\delta(n)=1\}=:\{n_p\mid p\in\omega\}$. Let $(p_n)_{n\in\omega}$ be the sequence of prime numbers. We define, for each $\alpha\in2^\omega$, $S_\alpha\subseteq\omega$ by $S_\alpha:=\{p_0^{\alpha(0)+1}\dots p_n^{\alpha(n)+1}\mid n\in\omega\}$. Note that S_α is infinite, and $S_\alpha\cap S_\beta$ is finite if $\alpha\neq\beta$. We define $\delta_\alpha\in2^\omega$ by $\delta_\alpha(n)=1\Leftrightarrow\exists p\in S_\alpha\ \ n=n_p$. Note that $\delta_\alpha\in\mathbb{P}_\infty$ and $\delta_\alpha(n)\leq\delta(n)$ for each n. We set $(P_\alpha,G_\alpha):=(\mathbb{P}_{\delta_\alpha},\mathbb{G}_{\delta_\alpha})$, so that $P_\alpha\subseteq\mathbb{P}_\delta$, $G_\alpha\subseteq\mathbb{G}_\delta$, $(P_\alpha,G_\alpha)\preceq^i_c(Z,G)$, and $\chi_c(P_\alpha,G_\alpha)=3$ by Lemma 3.3.6.

Let us prove that $(P_\alpha,G_\alpha),(P_{\alpha'},G_{\alpha'})$ are $\preceq^i_c$-incompatible among graphs on a 0DMS space with CCN at least three if $\alpha\neq\alpha'$. We argue by contradiction, which provides a 0DMS space Z and a graph G on Z with CCN at least three and $(Z,G)\preceq^i_c(P_\alpha,G_\alpha),(P_{\alpha'},G_{\alpha'})$. Lemma 4.2 gives $\delta'\in\mathbb{P}_\infty$ such that $\{k\in\omega\mid\delta'(k)=1\}\subseteq\{k\in\omega\mid\delta_\alpha(k)=1\}$ and $(\mathbb{P}_{\delta'},\mathbb{G}_{\delta'})\preceq^i_c(Z,G)$. Lemma 4.3 gives $k_0\in\omega$ such that $\{k\geq k_0\mid\delta'(k)=1\}\subseteq\{k\in\omega\mid\delta_{\alpha'}(k)=1\}$. This implies that $\{k\in\omega\mid\delta'(k)=1\}$ is finite, contradicting $\delta'\in\mathbb{P}_\infty$.

Lemma 4.1 then implies that there is no $\preceq_c^i$-antichain basis in the class of graphs on a 0DMS (or 0DP) space with CCN at least three. If B is a basis for the class of graphs on a 0DP space with CCN at least three and $\alpha \in 2^\omega$, then there is $b_\alpha \in B$ with $b_\alpha \preceq_c^i (P_\alpha, G_\alpha)$. The previous point shows that the sequence $(b_\alpha)_{\alpha \in 2^\omega}$ is injective, so that B has size at least continuum. □

5 General graphs and dynamical systems

We now prepare the proof of Theorem 1.17. We establish preliminary results holding not only for our examples, and clarify the relation between Cantor dynamical systems and our graphs. We first introduce examples in the style of the $\mathbb{G}_\gamma$'s defined at the beginning of Section 3.

Notation. Fix $\mathbf{d} \in \mathfrak{C}$ (defined before Theorem 3.1.2). We associate to $\mathbf{d}$ the following objects:

- an increasing unbounded sequence $(n_l)_{l \in \omega}$ of natural numbers, sometimes denoted by $(n_l^{\mathbf{d}})_{l \in \omega}$,
- sequences $(L_l)_{l \in \omega}, (R_l)_{l \in \omega}$ of integers with $R_l - L_l = 2n_l + 1$.

Let $f : \mathcal{C} \to \mathcal{C}$ be a homeomorphism, so that $(\mathcal{C}, f)$ is a Cantor dynamical system. We will associate a graph to $(\mathcal{C}, f)$, as follows. Recall the definition of $\mathcal{J}$ at the beginning of Subsection 3.2. We define, for $l \in \omega$ and $L_l \leq i \leq R_l$, $f_{l,i} := f_{l,i}^{\mathbf{d}} := f^i(0^\infty)|(l+1)$ in $\prod_{l+1}$. This defines $\beta \in \mathcal{J}$ by setting $\lambda_l := 2n_l + 2$, and $s_l(i) := f_{l, L_l + i}$. We set $\mathcal{C}^+ := \mathbb{K}_\beta$, $\mathbb{O}_f := \mathbb{O}_\beta$, and $\mathbb{G}_f := s(\mathbb{O}_f)$, so that $\mathbb{G}_f = \mathbb{G}_\beta$.

Lemma 5.1 *$(\mathcal{C}^+, \mathbb{G}_f)$ has CCN at least three and $\mathbf{\Sigma}_1^0 \oplus \mathbf{\Pi}_1^0$ chromatic number two. If moreover $d_0 = 2$ and $f(x)(0) \neq x(0)$ for each $x \in \mathcal{C}$, then $(\mathcal{C}^+, \mathbb{G}_f)$ has CCN three.*

Proof. We first apply Lemma 3.2.1. For the end, we argue as in the proof of Lemma 3.2.1. □

We now want to compare the subgraphs of the $\mathbb{G}_f$'s.

Lemma 5.2 *Let X be a topological space, $f : X \to X$ be a homeomorphism, Y, g having the corresponding properties, $x \in X$, and $\varphi : Orb_f(x) \to Y$ such that $\varphi\big(f(z)\big) = g\big(\varphi(z)\big)$ for each $z \in Orb_f(x)$. Then $\varphi\big(f^i(x)\big) = g^i\big(\varphi(x)\big)$ for each $i \in \mathbb{Z}$. Similarly, if $\varphi\big(f(z)\big) = g^{-1}\big(\varphi(z)\big)$ for each $z \in Orb_f(x)$, then $\varphi\big(f^i(x)\big) = g^{-i}\big(\varphi(x)\big)$ for each $i \in \mathbb{Z}$. In particular, $\varphi[Orb_f(x)] = Orb_g\big(\varphi(x)\big)$ in both cases.*

Proof. Inductively, we see that $\varphi\big(f^i(z)\big) = g^i\big(\varphi(z)\big)$ for each $i \in \omega$ and each $z \in \mathrm{Orb}_f(x)$. In particular, $\varphi(x) = \varphi\Big(f^i\big(f^{-i}(x)\big)\Big) = g^i\Big(\varphi\big(f^{-i}(x)\big)\Big)$, so that $\varphi\big(f^{-i}(x)\big) = g^{-i}\big(\varphi(x)\big)$. This implies that $\varphi\big(f^i(x)\big) = g^i\big(\varphi(x)\big)$ if $i \in \mathbb{Z}$. The other case is similar. □

Definition 5.3 *We say that the tuple $(\mathbf{d}, \mathbf{d}', f_{\mathbf{d}}, f_{\mathbf{d}'}, V_{\mathbf{d}}, V_{\mathbf{d}'}, E_{\mathbf{d}}, E_{\mathbf{d}'}, \varphi)$ is a* continuous tuple *if $\mathbf{d} \in \mathfrak{C}$, $f_{\mathbf{d}} : \mathcal{C}_{\mathbf{d}} \to \mathcal{C}_{\mathbf{d}}$ is a homeomorphism, $V_{\mathbf{d}} \subseteq \mathcal{C}_{\mathbf{d}}^+$, $E_{\mathbf{d}} \subseteq \mathbb{G}_{f_{\mathbf{d}}} \cap V_{\mathbf{d}}^2$ is a graph, $\chi_c(V_{\mathbf{d}}, E_{\mathbf{d}}) \geq 3$, $\mathbf{d}', f_{\mathbf{d}'}, V_{\mathbf{d}'}, E_{\mathbf{d}'}$ have the corresponding properties, and $(V_{\mathbf{d}}, E_{\mathbf{d}}) \preceq_c (V_{\mathbf{d}'}, E_{\mathbf{d}'})$ with witness φ.*

The next results are steps towards flip-conjugacy.

Lemma 5.4 *Let $(\mathbf{d}, \mathbf{d}', f_{\mathbf{d}}, f_{\mathbf{d}'}, V_{\mathbf{d}}, V_{\mathbf{d}'}, E_{\mathbf{d}}, E_{\mathbf{d}'}, \varphi)$ be a continuous tuple, and*

$$\big(x, f_{\mathbf{d}}(x)\big) \in \overline{E_{\mathbf{d}}}^{(\mathcal{C}_{\mathbf{d}}^+)^2} \cap V_{\mathbf{d}}^2$$

with $\varphi(x), \varphi\big(f_{\mathbf{d}}(x)\big) \in \mathcal{C}_{\mathbf{d}'}$. Then $\varphi\big(f_{\mathbf{d}}(x)\big) = f_{\mathbf{d}'}\big(\varphi(x)\big)$ or $\varphi\big(f_{\mathbf{d}}(x)\big) = f_{\mathbf{d}'}^{-1}\big(\varphi(x)\big)$.

Proof. Note that $\big(x, f_{\mathbf{d}}(x)\big) = \lim_{l\to\infty} (x_l, u_l)$, where $(x_l, u_l) \in E_{\mathbf{d}}$. This successively implies that $\big(\varphi(x_l), \varphi(u_l)\big) \in E_{\mathbf{d}'} \subseteq \mathbb{G}_{f_{\mathbf{d}'}}$ and $\Big(\varphi(x), \varphi\big(f_{\mathbf{d}}(x)\big)\Big) = \lim_{l\to\infty} \big(\varphi(x_l), \varphi(u_l)\big)$. As $\varphi(x), \varphi\big(f_{\mathbf{d}}(x)\big)$ are in $\mathcal{C}_{\mathbf{d}'}$, the first coordinate of $\varphi(x_l), \varphi(u_l)$ is in d_0 if l is large enough. The definition of $\mathbb{G}_{f_{\mathbf{d}'}}$ provides $k_l \in \omega$, $i_l \in \mathbb{Z}$, and $(\rho_l, \varepsilon_l) \neq (\theta_l, \eta_l)$ in $\{(0,a), (1,\overline{a})\}$ with the properties that $i_l \leq 2n_{k_l}$,

$$\big(\varphi(x_l), \varphi(u_l)\big) = (f_{\mathbf{d}'}^{L_{k_l}+i_l+\rho_l}(0^\infty)|(k_l+1)\varepsilon_l^{i_l+\rho_l+1}\overline{\varepsilon_l}^\infty, f_{\mathbf{d}'}^{L_{k_l}+i_l+\theta_l}(0^\infty)|(k_l+1)\eta_l^{i_l+\theta_l+1}\overline{\eta_l}^\infty)$$

for such a l. Extracting a subsequence if necessary, we may assume that the sequence (ρ_l) is constant. Moreover, the fact that $\varphi(x) \in \mathcal{C}_{\mathbf{d}'}$ implies that we may also assume that the sequence (k_l) is strictly increasing. This implies that $\Big(\varphi(x), \varphi\big(f_{\mathbf{d}}(x)\big)\Big)$ is at distance zero from the closed set $\mathrm{Graph}(f_{\mathbf{d}'})^{1-2\rho_0} \cap V_{\mathbf{d}'}^2$. Thus $\varphi\big(f_{\mathbf{d}}(x)\big) = f_{\mathbf{d}'}\big(\varphi(x)\big)$ or $\varphi\big(f_{\mathbf{d}}(x)\big) = f_{\mathbf{d}'}^{-1}\big(\varphi(x)\big)$. □

Corollary 5.5 *Let $(\mathbf{d}, \mathbf{d}', f_{\mathbf{d}}, f_{\mathbf{d}'}, V_{\mathbf{d}}, V_{\mathbf{d}'}, E_{\mathbf{d}}, E_{\mathbf{d}'}, \varphi)$ be a continuous tuple such that φ is injective, $\mathcal{C}_{\mathbf{d}} \subseteq \overline{V_{\mathbf{d}} \cap \mathcal{C}_{\mathbf{d}}}^{\mathcal{C}_{\mathbf{d}}}$ and $\mathrm{Graph}(f_{\mathbf{d}}) \subseteq \overline{E_{\mathbf{d}}}^{(\mathcal{C}_{\mathbf{d}}^+)^2}$, and assume that $x, f_{\mathbf{d}}(x) \in V_{\mathbf{d}} \cap \mathcal{C}_{\mathbf{d}}$. Then either $\varphi\big(f_{\mathbf{d}}(x)\big) = f_{\mathbf{d}'}\big(\varphi(x)\big)$, or $\varphi\big(f_{\mathbf{d}}(x)\big) = f_{\mathbf{d}'}^{-1}\big(\varphi(x)\big)$.*

Proof. By Lemma 3.2.6, $\varphi(x), \varphi\big(f_{\mathbf{d}}(x)\big) \in \mathcal{C}_{\mathbf{d}'}$. It remains to apply Lemma 5.4. □

Lemma 5.6 *Let X be a 0DMS space, $f: X \to X$ be a homeomorphism, $V \subseteq X$, I be a subset of $\{x \in V \mid f(x) \in V\}$, Y, g, W having the corresponding properties, and $\varphi: V \to W$ be a continuous injection. We assume that $\varphi\big(f(x)\big) = g\big(\varphi(x)\big)$ or $\varphi\big(f(x)\big) = g^{-1}\big(\varphi(x)\big)$ if $x \in I$.*
(a) Assume that g^2 is fixed point free, $x, f(x) \in I$, and $f^2(x) \neq x$. Then $\varphi\big(f(x)\big) = g\big(\varphi(x)\big)$ and $\varphi\big(f^2(x)\big) = g^2\big(\varphi(x)\big)$, or $\varphi\big(f(x)\big) = g^{-1}\big(\varphi(x)\big)$ and $\varphi\big(f^2(x)\big) = g^{-2}\big(\varphi(x)\big)$.
(b) Assume that g^2 is fixed point free, $Orb_f(x)$ is a dense subset of I, and $f^2(x) \neq x$. Then either $\varphi\big(f(z)\big) = g\big(\varphi(z)\big)$ for each $z \in I$, or $\varphi\big(f(z)\big) = g^{-1}\big(\varphi(z)\big)$ for each $z \in I$.
(c) Assume that g^2 is fixed point free, $Orb_f(x) \subseteq I$, and $f^2(x) \neq x$. Then $\varphi[Orb_f(x)] = Orb_g\big(\varphi(x)\big)$.

Proof. (a) We set $P := \{z \in I \mid \varphi\big(f(z)\big) = g\big(\varphi(z)\big)\}$ and $M := \{z \in I \mid \varphi\big(f(z)\big) = g^{-1}\big(\varphi(z)\big)\}$, so that (P, M) is a covering of I into closed sets. As g^2 is fixed point free, P, M are disjoint. Thus P is clopen in I. If $x \in P$ and $f(x) \in M$, then $\varphi\big(f^2(x)\big) = g^{-1}\Big(\varphi\big(f(x)\big)\Big) = \varphi(x)$, which contradicts the fact that $f^2(x) \neq x$ since φ is injective. The argument is similar if we exchange P and M.

(b) By (a), either $\mathrm{Orb}_f(x) \subseteq P$, or $\mathrm{Orb}_f(x) \subseteq M$. By density, $P \in \{I, \emptyset\}$.

(c) If $\mathrm{Orb}_f(x) \subseteq P$, then, by Lemma 5.2, $\varphi\big(f^i(x)\big) = g^i\big(\varphi(x)\big)$ if $i \in \mathbb{Z}$ and we are done. Otherwise, by (a), we are in the similar case $\mathrm{Orb}_f(x) \subseteq M$, so that $\varphi[\mathrm{Orb}_f(x)] = \mathrm{Orb}_g\big(\varphi(x)\big)$ in both cases. □

Lemma 5.7 *Let $(\mathbf{d}, \mathbf{d}', f_{\mathbf{d}}, f_{\mathbf{d}'}, V_{\mathbf{d}}, V_{\mathbf{d}'}, E_{\mathbf{d}}, E_{\mathbf{d}'}, \varphi)$ be a continuous tuple such that $f_{\mathbf{d}}^2, f_{\mathbf{d}'}^2$ are fixed point free, φ is injective, $\mathcal{C}_{\mathbf{d}} \subseteq \overline{V_{\mathbf{d}} \cap \mathcal{C}_{\mathbf{d}}}^{\mathcal{C}_{\mathbf{d}}}$ and $\mathrm{Graph}(f_{\mathbf{d}}) \subseteq \overline{E_{\mathbf{d}}}^{(\mathcal{C}_{\mathbf{d}}^+)^2}$.*
(a) Assume that $x, f_{\mathbf{d}}(x), f_{\mathbf{d}}^2(x) \in V_{\mathbf{d}} \cap \mathcal{C}_{\mathbf{d}}$. Then $\varphi\big(f_{\mathbf{d}}(x)\big) = f_{\mathbf{d}'}\big(\varphi(x)\big)$ and $\varphi\big(f_{\mathbf{d}}^2(x)\big) = f_{\mathbf{d}'}^2\big(\varphi(x)\big)$, or $\varphi\big(f_{\mathbf{d}}(x)\big) = f_{\mathbf{d}'}^{-1}\big(\varphi(x)\big)$ and $\varphi\big(f_{\mathbf{d}}^2(x)\big) = f_{\mathbf{d}'}^{-2}\big(\varphi(x)\big)$.
(b) If $Orb_{f_{\mathbf{d}}}(x)$ is a dense subset of $I := \{z \in V_{\mathbf{d}} \cap \mathcal{C}_{\mathbf{d}} \mid f_{\mathbf{d}}(z) \in V_{\mathbf{d}}\}$ for some x, then either $\varphi\big(f_{\mathbf{d}}(z)\big) = f_{\mathbf{d}'}\big(\varphi(z)\big)$ for each $z \in I$, or $\varphi\big(f_{\mathbf{d}}(z)\big) = f_{\mathbf{d}'}^{-1}\big(\varphi(z)\big)$ for each $z \in I$.
(c) Assume that $Orb_{f_{\mathbf{d}}}(x) \subseteq V_{\mathbf{d}} \cap \mathcal{C}_{\mathbf{d}}$. Then $\varphi[Orb_{f_{\mathbf{d}}}(x)] = Orb_{f_{\mathbf{d}'}}\big(\varphi(x)\big)$.

Proof. We apply Lemma 5.6 to $X := \mathcal{C}_{\mathbf{d}}$, $f := f_{\mathbf{d}}$, $V := V_{\mathbf{d}} \cap \mathcal{C}_{\mathbf{d}}$, I defined in (b), $Y := \mathcal{C}_{\mathbf{d}'}$, $g := f_{\mathbf{d}'}$, $W := V_{\mathbf{d}'} \cap \mathcal{C}_{\mathbf{d}'}$, and $\varphi := \varphi_{|V}$, which is possible by Lemma 3.2.6 and Corollary 5.5. $\square$

Lemma 5.8 *Let $\mathbf{d}, \mathbf{d}' \in \mathfrak{C}$, and $f_{\mathbf{d}} : \mathcal{C}_{\mathbf{d}} \to \mathcal{C}_{\mathbf{d}}$, $f_{\mathbf{d}'} : \mathcal{C}_{\mathbf{d}'} \to \mathcal{C}_{\mathbf{d}'}$ be minimal homeomorphisms such that* $\mathrm{Graph}(f_{\mathbf{d}}) \subseteq \overline{\mathbb{G}_{f_{\mathbf{d}}}}^{(\mathcal{C}_{\mathbf{d}}^+)^2}$. *If $(\mathcal{C}_{\mathbf{d}}^+, \mathbb{G}_{f_{\mathbf{d}}}) \preceq_c^i (\mathcal{C}_{\mathbf{d}'}^+, \mathbb{G}_{f_{\mathbf{d}'}})$, then $f_{\mathbf{d}}, f_{\mathbf{d}'}$ are flip-conjugate.*

Proof. Let $\varphi : \mathcal{C}_{\mathbf{d}}^+ \to \mathcal{C}_{\mathbf{d}'}^+$ be a witness for the fact that $(\mathcal{C}_{\mathbf{d}}^+, \mathbb{G}_{f_{\mathbf{d}}}) \preceq_c^i (\mathcal{C}_{\mathbf{d}'}^+, \mathbb{G}_{f_{\mathbf{d}'}})$. By Lemma 5.1, $(\mathbf{d}, \mathbf{d}', f_{\mathbf{d}}, f_{\mathbf{d}'}, \mathcal{C}_{\mathbf{d}}^+, \mathcal{C}_{\mathbf{d}'}^+, \mathbb{G}_{f_{\mathbf{d}}}, \mathbb{G}_{f_{\mathbf{d}'}}, \varphi)$ is a continuous tuple satisfying the assumptions of Lemmas 3.2.6 and 5.7. In particular, $\varphi[\mathcal{C}_{\mathbf{d}}] \subseteq \mathcal{C}_{\mathbf{d}'}$ and the map $\psi := \varphi_{|\mathcal{C}_{\mathbf{d}}} : \mathcal{C}_{\mathbf{d}} \to \mathcal{C}_{\mathbf{d}'}$ is a witness for the fact that $f_{\mathbf{d}}, f_{\mathbf{d}'}$ are flip-conjugate. Indeed, Lemma 5.7 implies that $\varphi[\mathrm{Orb}_{f_{\mathbf{d}}}(x)] = \mathrm{Orb}_{f_{\mathbf{d}'}}\big(\varphi(x)\big)$ if $x \in \mathcal{C}_{\mathbf{d}}$. As $f_{\mathbf{d}'}$ is minimal, the compact set $\psi[\mathcal{C}_{\mathbf{d}}]$ is dense in $\mathcal{C}_{\mathbf{d}'}$, showing that ψ is onto, and thus a homeomorphism by compactness of $\mathcal{C}_{\mathbf{d}}$. $\square$

Notation. For the converse, we give a definition of the sequences $(n_l)_{l\in\omega}$, $(L_l)_{l\in\omega}$ and $(R_l)_{l\in\omega}$. We define $(n_l)_{l\in\omega}$ by $n_l := l$, so that $(n_l)_{l\in\omega}$ is increasing unbounded.

- We define a map $\zeta : \omega \to \mathbb{Z}$ having the property that each integer appears infinitely many times in the range of ζ.

- We define sequences $(L_l)_{l\in\omega}, (R_l)_{l\in\omega}$ of integers by $L_{2m} := R_{2m+1} := \zeta(m)$ and $R_l - L_l := 2l+1$, so that the sequences $\big(f^{L_l}(0^\infty)\big)_{l\in\omega}$, $\big(f^{R_l}(0^\infty)\big)_{l\in\omega}$ are dense if f is minimal.

Note also that $\{f^i(0^\infty) \mid \exists^\infty l \in \omega \;\; L_l \leq i < R_l\}$ is dense in $\mathcal{C}$ if f is minimal, which implies that $\mathrm{Graph}(f_{\mathbf{d}}) \subseteq \overline{\mathbb{G}_{f_{\mathbf{d}}}}^{(\mathcal{C}_{\mathbf{d}}^+)^2}$.

Lemma 5.9 *Let $\mathbf{d}, \mathbf{d}' \in \mathfrak{C}$, $f_{\mathbf{d}} : \mathcal{C}_{\mathbf{d}} \to \mathcal{C}_{\mathbf{d}}$, $f_{\mathbf{d}'} : \mathcal{C}_{\mathbf{d}'} \to \mathcal{C}_{\mathbf{d}'}$ be minimal homeomorphisms, and $(n_l)_{l\in\omega}, (L_l)_{l\in\omega}, (R_l)_{l\in\omega}$ just defined. If $f_{\mathbf{d}}, f_{\mathbf{d}'}$ are flip-conjugate, then $(\mathcal{C}_{\mathbf{d}}^+, \mathbb{G}_{f_{\mathbf{d}}}) \preceq_c^i (\mathcal{C}_{\mathbf{d}'}^+, \mathbb{G}_{f_{\mathbf{d}'}})$.*

Proof. As $f_{\mathbf{d}}, f_{\mathbf{d}'}$ are flip-conjugate, we get a homeomorphism $\psi : \mathcal{C}_{\mathbf{d}} \to \mathcal{C}_{\mathbf{d}'}$. We have to define a function $\varphi : \mathcal{C}_{\mathbf{d}}^+ \to \mathcal{C}_{\mathbf{d}'}^+$. We first set $\varphi(c^\infty) := c^\infty$, and $\varphi(x) := \psi(x)$ if $x \in \mathcal{C}_{\mathbf{d}}$. As $f_{\mathbf{d}'}$ is uniformly continuous, for any $l \in \omega$, there is $U \geq l$ such that, for any $y, z \in \mathcal{C}_{\mathbf{d}'}$, $f_{\mathbf{d}'}(y)|(l+1) = f_{\mathbf{d}'}(z)|(l+1)$ if $y|(U+1) = z|(U+1)$, which defines $U : \omega \to \omega$.

Assume first that $\psi \circ f_{\mathbf{d}} = f_{\mathbf{d}'} \circ \psi$. By Lemma 5.2, $\psi\big(f_{\mathbf{d}}^i(0^\infty)\big) = f_{\mathbf{d}'}^i\big(\psi(0^\infty)\big)$ for each $i \in \mathbb{Z}$. We define $\big(\varphi(f_{l,L_l+i}^{\mathbf{d}} \varepsilon^{i+1} \overline{\varepsilon}^\infty)\big)_{i \leq 2l+1}$ by induction on l, ensuring that $\varphi(f_{l,L_l+i}^{\mathbf{d}} \varepsilon^{i+1} \overline{\varepsilon}^\infty)$ is of the form $f_{l'(l,i,\varepsilon), L_{l'(l,i,\varepsilon)} + i'(l,i,\varepsilon)}^{\mathbf{d}'} \varepsilon^{i'(l,i,\varepsilon)+1} \overline{\varepsilon}^\infty$ with $l'(l, i+1, \overline{a}) = l'(l, i, a)$ and $i'(l, i+1, \overline{a}) = i'(l, i, a)$ if $i \leq 2l$. Fix $l \in \omega$. Let $M := \sup\big\{l'(k, j, \eta) \mid k < l \wedge j \leq 2k+1 \wedge \eta \in \{a, \overline{a}\}\big\}$.

- We choose $m := l'(l, 0, \overline{a}) > \max(l, M)$ such that $f_{\mathbf{d}'}^{L_m}(0^\infty)|(l+1) = f_{\mathbf{d}'}^{L_l}\big(\psi(0^\infty)\big)|(l+1)$, which is possible since the sequence $\big(f_{\mathbf{d}'}^{L_m}(0^\infty)\big)_{m\in\omega}$ is dense. We set $\varphi(c^{l+1} a \overline{a}^\infty) := c^{m+1} a \overline{a}^\infty$ and $\varphi(f_{l,L_l}^{\mathbf{d}} \overline{a} a^\infty) := f_{m,L_m}^{\mathbf{d}'} \overline{a} a^\infty$. Note that $\big(\varphi(c^{l+1} a \overline{a}^\infty), \varphi(f_{l,L_l}^{\mathbf{d}} \overline{a} a^\infty)\big) \in \mathbb{G}_{f_{\mathbf{d}'}}$, as desired.

- Now fix $i \leq 2n_l$. As $f_{\mathbf{d}'}$ is minimal, we can find $k_i \in \mathbb{Z}$ with

$$f_{\mathbf{d}'}^{k_i}(0^\infty)|\big(U(l)+1\big) = f_{\mathbf{d}'}^{L_l+i}\big(\psi(0^\infty)\big)|\big(U(l)+1\big).$$

Note that $f_{\mathbf{d}'}^{k_i+1}(0^\infty)|(l+1) = f_{\mathbf{d}'}^{L_l+i+1}\big(\psi(0^\infty)\big)|(l+1)$. We choose $m' := l'(l, i, a) > l'(l, i, \overline{a})$ with the property that $L_{m'} \leq k_i = L_{m'} + i_i' < R_{m'}$. We set $\varphi(f_{l,L_l+i}^{\mathbf{d}} a^{i+1} \overline{a}^\infty) := f_{m',k_i}^{\mathbf{d}'} a^{i_i'+1} \overline{a}^\infty$ and $\varphi(f_{l,L_l+i+1}^{\mathbf{d}} \overline{a}^{i+2} a^\infty) := f_{m',k_i+1}^{\mathbf{d}'} \overline{a}^{i_i'+2} a^\infty$. Note that $\big(\varphi(f_{l,L_l+i}^{\mathbf{d}} a^{i+1} \overline{a}^\infty), \varphi(f_{l,L_l+i+1}^{\mathbf{d}} \overline{a}^{i+2} a^\infty)\big)$ is in $\mathbb{G}_{f_{\mathbf{d}'}}$, as desired.

- We then choose $m'' := l'(l, 2l+1, a) > l'(l, 2l+1, \overline{a})$ with $f_{\mathbf{d}'}^{R_{m''}}(0^\infty)|(l+1) = f_{\mathbf{d}'}^{R_l}\big(\psi(0^\infty)\big)|(l+1)$, which is possible since $\big(f_{\mathbf{d}'}^{R_m}(0^\infty)\big)_{m\in\omega}$ is dense. We set $\varphi(f_{l,R_l}^{\mathbf{d}} a^{2l+2}\overline{a}^\infty) := f_{m'',R_{m''}}^{\mathbf{d}'} a^{2m''+2}\overline{a}^\infty$ and $\varphi(c^{l+1}\overline{a}a^\infty) := c^{m''+1}\overline{a}a^\infty$. Note that $\big(\varphi(f_{l,R_l}^{\mathbf{d}} a^{2l+2}\overline{a}^\infty), \varphi(c^{l+1}\overline{a}a^\infty)\big) \in \mathbb{G}_{f_{\mathbf{d}'}}$, as desired.

This completes the definition of φ. Our construction implies that φ is a homomorphism from $(\mathcal{C}_{\mathbf{d}}^+, \mathbb{G}_{f_{\mathbf{d}}})$ into $(\mathcal{C}_{\mathbf{d}'}^+, \mathbb{G}_{f_{\mathbf{d}'}})$. Our choice of the $l'(l,i,\varepsilon)$'s implies the injectivity of φ. For the continuity, note first that the sequence $\big(\varphi(c^{l+1}\varepsilon\overline{\varepsilon}^\infty)\big)_{l\in\omega}$ converges to c^∞ since $m, m'' \geq l$. If now $(f_{l_k,L_{l_k}+i_k}^{\mathbf{d}} \varepsilon_k^{i_k+1}\overline{\varepsilon_k}^\infty)_{k\in\omega}$ converges to $x \in \mathcal{C}_{\mathbf{d}}^+$, then we may assume that $x \in \mathcal{C}_{\mathbf{d}}$. Note that the sequence $\big(f_{\mathbf{d}}^{L_{l_k}+i_k}(0^\infty)\big)_{k\in\omega}$ converges to x, and $\Big(f_{\mathbf{d}'}^{L_{l_k}+i_k}\big(\psi(0^\infty)\big)\Big)_{k\in\omega}$ converges to $\psi(x) = \varphi(x)$. Our construction ensures that $f_{\mathbf{d}'}^{L_{l'(l,i,\varepsilon)}+i'(l,i,\varepsilon)}(0^\infty)|(l+1) = f_{\mathbf{d}'}^{L_l+i}\big(\psi(0^\infty)\big)|(l+1)$, and

$$\varphi(f_{l_k,L_{l_k}+i_k}^{\mathbf{d}} \varepsilon_k^{i_k+1}\overline{\varepsilon_k}^\infty) = f_{l'(l_k,i_k,\varepsilon_k),L_{l'(l_k,i_k,\varepsilon_k)}+i'(l_k,i_k,\varepsilon_k)}^{\mathbf{d}'} \varepsilon_k^{i'(l_k,i_k,\varepsilon_k)+1}\overline{\varepsilon_k}^\infty$$

$$= f_{\mathbf{d}'}^{L_{l'(l_k,i_k,\varepsilon_k)}+i'(l_k,i_k,\varepsilon_k)}(0^\infty)|\big(l'(l_k,i_k,\varepsilon_k)+1\big)\varepsilon_k^{i'(l_k,i_k,\varepsilon_k)+1}\overline{\varepsilon_k}^\infty.$$

As $l'(l,i,\varepsilon) \geq l$, we get

$$\varphi(f_{l_k,L_{l_k}+i_k}^{\mathbf{d}} \varepsilon_k^{i_k+1}\overline{\varepsilon_k}^\infty)|(l_k+1) = f_{\mathbf{d}'}^{L_{l'(l_k,i_k,\varepsilon_k)}+i'(l_k,i_k,\varepsilon_k)}(0^\infty)|(l_k+1) = f_{\mathbf{d}'}^{L_{l_k}+i_k}\big(\psi(0^\infty)\big)|(l_k+1).$$

Thus $\big(\varphi(f_{l_k,L_{l_k}+i_k}^{\mathbf{d}} \varepsilon_k^{i_k+1}\overline{\varepsilon_k}^\infty)\big)_{k\in\omega}$ converges to $\varphi(x)$, proving the continuity of φ.

The case where $\psi \circ f_{\mathbf{d}} = f_{\mathbf{d}'}^{-1} \circ \psi$ is similar. □

Notation. We set $(X,R) \equiv_c^i (Y,S) \Leftrightarrow (X,R) \preceq_c^i (Y,S) \wedge (Y,S) \preceq_c^i (X,R)$, so that $\equiv_c^i$ is the equivalence relation associated with the quasi-order $\preceq_c^i$.

Corollary 5.10 *Let $\mathbf{d}, \mathbf{d}' \in \mathfrak{C}$, $f_{\mathbf{d}} : \mathcal{C}_{\mathbf{d}} \to \mathcal{C}_{\mathbf{d}}$, $f_{\mathbf{d}'} : \mathcal{C}_{\mathbf{d}'} \to \mathcal{C}_{\mathbf{d}'}$ be minimal homeomorphisms, and $(n_l)_{l\in\omega}, (L_l)_{l\in\omega}, (R_l)_{l\in\omega}$ just defined. Then $(\mathcal{C}_{\mathbf{d}}^+, \mathbb{G}_{f_{\mathbf{d}}}) \equiv_c^i (\mathcal{C}_{\mathbf{d}'}^+, \mathbb{G}_{f_{\mathbf{d}'}})$ if and only if $f_{\mathbf{d}}, f_{\mathbf{d}'}$ are flip-conjugate.*

6 General graphs and odometers

We now provide a countable graph on a(n infinite) 0DMC space with CCN three which is strictly $\preceq_c$-below the odd cycles. Our example is based on odometers. We give some notation useful for the sequel.

Notation. Fix $\mathbf{d} \in \mathfrak{C}$ (defined before Theorem 3.1.2). The *odometer* $o := o_{\mathbf{d}} : \mathcal{C} \to \mathcal{C}$ is defined by

$$o(\alpha) := \begin{cases} 0^\infty \text{ if } \forall j \in \omega \;\; \alpha(j) = d_j - 1, \\ 0^n\big(\alpha(n)+1\big)\alpha(n+1)\alpha(n+2)\ldots \text{ if } \alpha(n) < d_n - 1 \wedge \forall j < n \;\; \alpha(j) = d_j - 1. \end{cases}$$

As $\mathrm{Orb}_o^+(\alpha) := \{o^i(\alpha) \mid i > 0\}$ sees all the words of length n in the first n coordinates for any x, o is a minimal homeomorphism. We sometimes extend the definition of o to finite sequences.

We set $\mathfrak{D}:=\{\mathbf{d}\in\mathfrak{C}\mid d_0=2\wedge\forall j\geq 1\ \ d_j \text{ is odd}\}$.

- We define, for $\mathbf{d}\in\mathfrak{D}$, $(n_l)_{l\in\omega}\in\mathcal{S}$ by $n_0:=0$ and $n_{l+1}:=\frac{(\pi_{1\leq j\leq l+1}\ d_j)-1}{2}$. Note that

$$o^{2n_l+1}(0^\infty)=1^\frown{}_{1\leq j\leq l}\ \frac{d_j-1}{2}0^\infty$$

converges to $\mu:=1^\frown{}_{j\geq 1}\ \frac{d_j-1}{2}$ as l goes to infinity, and $o^{4n_l+1}(0^\infty)=\frown{}_{j\leq l}\ (d_j-1)0^\infty$.

- We define, for $l\in\omega$ and $i\leq 2n_l+1$, $o_{l,i}:=o^{\mathbf{d}}_{l,i}:=o^i(0^\infty)|(l+1)\in\prod_{l+1}$. This defines $\beta\in\mathcal{J}^c$, by setting $\lambda_l:=2n_l+2$ and $s_l(i):=o_{l,i}$. We set $\mathcal{C}^+:=\mathbb{K}_\beta$, $\mathbb{O}_o:=\mathbb{O}_\beta$, and $\mathbb{G}_o:=s(\mathbb{O}_o)$, so that $\mathbb{G}_o=\mathbb{G}_\beta$.

Proposition 6.1 *Let $\mathbf{d}\in\mathfrak{D}$. Then $(\mathcal{C}^+,\mathbb{G}_o)$ has CCN three, $\mathbf{\Sigma}^0_1\oplus\mathbf{\Pi}^0_1$ chromatic number two, and is strictly $\preceq_c$-below the examples of Corollary 2.2.*

Proof. Lemma 3.2.1 proves the assertions about chromatic numbers. Let $p\in\omega$. We define a function $\varphi:\mathcal{C}^+\to 2p+3$ as follows. Fix $l_0\in\omega$ minimal such that $n_{l_0}\geq p$. We set $\varphi(x):=0$ if $x(0)=c$. If $i\leq 4n_{l_0}+1$ and $o^i(0^{l_0+1})\subseteq x$ (recall that o can be extended to finite sequences), then we set

$$\varphi(x):=\begin{cases} i+1 \text{ if } i\leq 2p+1,\\ 2p+2 \text{ if } 2p+1<i\leq 2n_{l_0}+1\wedge i \text{ is odd},\\ 2p+1 \text{ if } 2p+1<i\leq 2n_{l_0}+1\wedge i \text{ is even},\\ 2p+2n_{l_0}+3-i \text{ if } 2n_{l_0}+1<i\leq 2p+2n_{l_0}+1,\\ 2 \text{ if } 2p+2n_{l_0}+1<i\leq 4n_{l_0}+1\wedge i \text{ is odd},\\ 1 \text{ if } 2p+2n_{l_0}+1<i\leq 4n_{l_0}+1\wedge i \text{ is even}.\end{cases}$$

It remains to define $\varphi(o_{l,i}\varepsilon^{i+1}\overline{\varepsilon}^\infty)$ if $\varepsilon\in\{a,\overline{a}\}$, $l<l_0$ and $i\leq 2n_l+1$. We set $\varphi(o_{l,i}\varepsilon^{i+1}\overline{\varepsilon}^\infty):=i+1$. This defines a continuous homomorphism φ from $(\mathcal{C}^+,\mathbb{G}_o)$ into $(2p+3,C_{2p+3})$. The inequality $(\mathcal{C}^+,\mathbb{G}_o)\prec_c(2p+3,C_{2p+3})$ is strict because of Corollary 2.2(b). $\square$

Remark. We clarify the limits of Theorem 2.1. In its proof, we used the finiteness of X. This is essential. Indeed, if we replace χ with χ_c, $\preceq^i$ with $\preceq^i_c$ and X with $\mathcal{C}^+$, then the following hold. The implications (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1) and (3) $\Leftrightarrow$ (2) still hold. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) do not hold, because of Proposition 6.1.

We now characterize the subgraphs of $\mathbb{G}_o$ having a big CCN.

Lemma 6.2 *Let $\mathbf{d}\in\mathfrak{D}$, $V\subseteq\mathcal{C}^+$, and $E\subseteq\mathbb{G}_o\cap V^2$. The following are equivalent:*

(1) the digraph (V,E) has CCN at least three,

(2) the following hold:

(a) $0^\infty,\mu,c^\infty\in V$ and $\mathcal{C}\subseteq\overline{V\cap\mathcal{C}}^{\mathcal{C}}$,

(b) $\{(c^\infty,0^\infty),(\mu,c^\infty)\}\cup\mathrm{Graph}(o)\subseteq\overline{s(E)}^{(\mathcal{C}^+)^2}$.

Proof. Note first that $\chi_c(V,E)\leq\chi_c\big(V,s(E)\big)\leq\chi_c(\mathcal{C}^+,\mathbb{G}_o)=3$, by Lemma 3.2.1. We may and will assume that $E=s(E)$ is a graph.

(1) $\Rightarrow$ (2).(a) For 0^∞, we argue by contradiction. Let $C:=(N_1\cup\bigcup_{l\in\omega}\ N_{0^{l+1}\overline{a}})\cap V$. Then $(C,V\backslash C)$ is a coloring of E into clopen sets since $C=(N_1\cup\bigcup_{l\in\omega}\ N_{0^{l+1}\overline{a}}\cup\{0^\infty\})\cap V$, which is absurd. For μ, we argue similarly, with $(N_0\cup\bigcup_{l\in\omega}\ N_{\mu|(l+1)a})\cap V$. For c^∞, we apply Lemma 3.2.3.

Claim. Let $l\in\omega$. Then either for each $0<i<2n_l+1$ there is $x\in V\cap\mathcal{C}$ with $o_{l,i}\subseteq x$, or for each $2n_l+1<j<4n_l+2$ there is $y\in V\cap\mathcal{C}$ with $o_{l,j}\subseteq y$.

Indeed, towards a contradiction, suppose that l_0, i, j exist. Assume, for example, that i, j are even, the other cases being similar. We extend the notation $o_{l,i}$ for $i \leq 4n_l+1$ (not only $i \leq 2n_l+1$), and set

$$C' := \big(\bigcup_{1\leq l\leq l_0} \ (N_{c^l\overline{a}} \cup \bigcup_{s\in\prod_l} \ N_{s\overline{a}}) \cup \bigcup_{k<i,k \text{ even}} \ N_{o_{l_0,k}} \cup \bigcup_{l\geq l_0, s\in\prod_{l_0<j\leq l} \ d_j} \ N_{o_{l_0,i}s\overline{a}} \cup$$
$$\bigcup_{i<k<j,k \text{ odd}} \ N_{o_{l_0,k}} \cup \bigcup_{l\geq l_0, t\in\prod_{l_0<j\leq l} \ d_j} \ N_{o_{l_0,j}ta} \cup \bigcup_{j<k<4n_{l_0}+2,k \text{ even}} \ N_{o_{l_0,k}}\big) \cap V.$$

Then $(C', V\setminus C')$ is a coloring of E into clopen sets, which is absurd. $\diamond$

Now let $l\in\omega$, $t\in\prod_l$, $i<2n_l+1$ with $t0=o_{l,i}$, and $2n_l+1\leq j<4n_l+2$ with $t(d_l-1)=o_{l,j}$. We may assume that $i\neq 0$ and $j\neq 2n_l+1$ since $0^\infty, \mu\in V$. Then there is $x\in V\cap\mathcal{C}$ with $t\subseteq x$ by the claim, so that $x\in V\cap\mathcal{C}\cap N_t$.

(b) For $(c^\infty, 0^\infty)$, we argue by contradiction. If $l\geq l_0$ is large enough, then $(c^{l+1}a\overline{a}^\infty, 0^{l+1}\overline{a}a^\infty)$ is not in E. Let $C'' := (\bigcup_{1\leq l\leq l_0} \ (N_{c^l\overline{a}} \cup \bigcup_{s\in\prod_l} \ N_{s\overline{a}}) \cup \bigcup_{1s\in\prod_{j\leq l_0} \ d_j} \ N_{1s}) \cap V$. Then $(C'', V\setminus C'')$ is a coloring of E into clopen sets, which is absurd. For (μ, c^∞), we argue similarly, with the clopen set $(\bigcup_{1\leq l\leq l_0} \ (N_{c^l\overline{a}} \cup \bigcup_{s\in\prod_l} \ N_{s\overline{a}}) \cup \bigcup_{0s\in\prod_{j\leq l_0} \ d_j} \ N_{0s}) \cap V$.

In order to prove that $\mathrm{Graph}(o) \subseteq \overline{E}^{(\mathcal{C}^+)^2}$, towards a contradiction, suppose that there is $i_0\in\omega$ such that $\big(o^{i_0}(0^\infty), o^{i_0+1}(0^\infty)\big) \notin \overline{E}^{(\mathcal{C}^+)^2}$. If $l\geq l_0$ is large enough, then $i_0\leq 2n_l$ and

$$(o_{l,i_0}a^{i_0+1}\overline{a}^\infty, o_{l,i_0+1}\overline{a}^{i_0+2}a^\infty) \notin E.$$

We set $P_- := \{o^i(0^{l_0+1}) \mid i<i_0\}$, $P_+ := \{o^i(0^{l_0+1}) \mid i_0<i<\pi_{j\leq l_0} \ d_j\}$, which defines a partition $(P_-, \{o_{l_0,i_0}\}, P_+)$ of $\prod_{l_0+1}$. Assume first that i_0 is even. Let

$$C''' := (\bigcup_{1\leq l\leq l_0} (N_{c^l\overline{a}} \cup \bigcup_{s\in\prod_l} N_{s\overline{a}}) \cup \bigcup_{s\in P_-\cup\{o_{l_0,i_0}\}, s(0)=0} N_s \cup \bigcup_{s\in P_+, s(0)=1} N_s) \cap V.$$

Then $(C''', V\setminus C''')$ is a coloring of E into clopen sets, which is absurd. If i_0 is odd, then we consider

$$C''' := (\bigcup_{1\leq l\leq l_0} (N_{c^l\overline{a}} \cup \bigcup_{s\in\prod_l} N_{s\overline{a}}) \cup N_c \cup \bigcup_{s\in P_-\cup\{o_{l_0,i_0}\}, s(0)=1} N_s \cup \bigcup_{s\in P_+, s(0)=0} N_s) \cap V$$

(for instance $c^{l+1}a\overline{a}^\infty \in C'''$ and $o_{l,0}\overline{a}a^\infty \notin C'''$).

(2) $\Rightarrow$ (1) Towards a contradiction, suppose that there is a clopen subset C of V with the property that $E\cap\big(C^2\cup(V\setminus C)^2\big)=\emptyset$, and by (a) we may assume that $c^\infty\in C$. (b) gives infinitely many l's such that $(c^{l+1}a\overline{a}^\infty, 0^{l+1}\overline{a}a^\infty)$ is in E, and infinitely many l's with $(\mu|(l+1)a^{2n_l+2}\overline{a}^\infty, c^{l+1}\overline{a}a^\infty)\in E$. For these large enough l's, $0^{l+1}\overline{a}a^\infty\notin C$ and $\mu|(l+1)a^{2n_l+2}\overline{a}^\infty\notin C$. By (a), $0^\infty, \mu\in V\setminus C$. This gives $l_0\in\omega$ such that $V\cap(N_{0^{l_0+1}}\cup N_{\mu|(l_0+1)})\subseteq V\setminus C$.

(a) provides $x_1\in V\cap\mathcal{C}$ with $o_{l_0,1}\subseteq x_1$, which gives $l_1>l_0$ such that $V\cap N_{x_1|l_1}\subseteq C$ or $V\cap N_{x_1|l_1}\subseteq V\setminus C$. Let $i_1\leq 2n_{l_1}$ with $o_{l_1,i_1}=(x_1|l_1)0$. (a) provides $x_2\in V\cap\mathcal{C}$ with $o_{l_1,i_1+1}\subseteq x_2$, which gives $l_2>l_1$ such that $V\cap N_{x_2|l_2}\subseteq C$ or $V\cap N_{x_2|l_2}\subseteq V\setminus C$. Continuing like this we get, for each $j\leq 2n_{l_0}$, $x_{j+1}\in V\cap\mathcal{C}$ with $o_{l_j,i_j+1}\subseteq x_{j+1}$, and $l_{j+1}>l_j$ such that $V\cap N_{x_{j+1}|l_{j+1}}\subseteq C$ or $V\cap N_{x_{j+1}|l_{j+1}}\subseteq V\setminus C$, with $i_0:=0$.

By (b), $\big(o^{i_1-1}(0^\infty), o^{i_1}(0^\infty)\big)\in\overline{E}^{(\mathcal{C}^+)^2}$, so that E meets $N_{0^{l_0+1}}\times N_{o_{l_1,i_1}}$. As $V\cap N_{0^{l_0+1}}$ is contained in $V\setminus C$, this implies that $V\cap N_{o_{l_1,i_1}}\subseteq C$. By (b), $\big(o^{i_2-1}(0^\infty), o^{i_2}(0^\infty)\big)\in\overline{E}^{(\mathcal{C}^+)^2}$, so that E meets $N_{o_{l_1,i_1}}\times N_{o_{l_2,i_2}}$. As $V\cap N_{o_{l_1,i_1}}\subseteq C$, this implies that $V\cap N_{o_{l_2,i_2}}\subseteq V\setminus C$.

More generally, if $1 \leq j \leq 2n_{l_0}+1$, then $V \cap N_{o_{l_j,i_j}} \subseteq C$ when j is odd, and $V \cap N_{o_{l_j,i_j}} \subseteq V \setminus C$ when j is even. As $\mu|(l_0+1)$ is an initial segment of $o_{l_{2n_{l_0}+1},i_{2n_{l_0}+1}}$,

$$V \cap N_{o_{l_{2n_{l_0}+1}},i_{2n_{l_0}+1}} \subseteq C \cap N_{\mu|(l_0+1)} \subseteq C \setminus C,$$

which is the desired contradiction. □

The compactness ensures some surjectivity.

Lemma 6.3 *Let $\mathbf{d} \in \mathfrak{D}$, $V \subseteq \mathcal{C}^+$, X be a 0DMC space, and G be a digraph on X having CCN at least three such that $(X,G) \preceq_c (V, \mathbb{G}_o)$, with witness φ. Then φ is onto $\mathcal{C} \subseteq V$.*

Proof. Note that $(\varphi[X], (\varphi \times \varphi)[G])$ has CCN three. By Lemma 6.2, $\mathcal{C}$ is contained in $\overline{\varphi[X] \cap \mathcal{C}}^{\mathcal{C}}$. As $\varphi[X]$ is compact, $\overline{\varphi[X] \cap \mathcal{C}}^{\mathcal{C}} = \varphi[X] \cap \mathcal{C}$, and thus $\mathcal{C} \subseteq \varphi[X]$. □

We now prove some minimality of the $\mathbb{G}_o$'s.

Theorem 6.4 *Let $\mathbf{d} \in \mathfrak{D}$, V be a compact subspace of $\mathcal{C}^+$, and $E \subseteq \mathbb{G}_o \cap V^2$ be a graph with CCN at least three. Then $(\mathcal{C}^+, \mathbb{G}_o) \preceq_c^i (V, E)$.*

Proof. Note that $\mathcal{C}^+ = \mathrm{proj}[\mathbb{G}_o] \cup \{c^\infty\} \cup \mathcal{C}$. By Lemma 6.2 and compactness of V, $\{c^\infty\} \cup \mathcal{C} \subseteq V$. We have to define $\varphi : \mathcal{C}^+ \to V$. The map φ will be the identity on $\{c^\infty\} \cup \mathcal{C}$. Let $\varepsilon \in \{a, \overline{a}\}$, $l \in \omega$, and either $s = c^{l+1}$ and $i = 0$, or $s = o_{l,i}$. We define $\varphi(s\varepsilon^{i+1}\overline{\varepsilon}^\infty)$ by induction on l, in such a way that $s \subseteq \varphi(s\varepsilon^{i+1}\overline{\varepsilon}^\infty)$ and $\varphi(s\varepsilon^{i+1}\overline{\varepsilon}^\infty)$ finishes with some $\varepsilon^{j+1}\overline{\varepsilon}^\infty$. As $(c^\infty, 0^\infty) \in \overline{E} := \overline{E}^{(\mathcal{C}^+)^2}$, we can find $\big(\varphi(ca\overline{a}^\infty), \varphi(0\overline{a}a^\infty)\big)$ in E as desired. As $(0^\infty, 10^\infty) \in \overline{E}$, we can find $\big(\varphi(0a\overline{a}^\infty), \varphi(1\overline{a}^2a^\infty)\big)$ in E as desired. As $(\mu, c^\infty) \in \overline{E}$, we can find $\big(\varphi(1a^2\overline{a}^\infty), \varphi(c\overline{a}a^\infty)\big) \in E$ as desired. Note that we are done for $l = 0$. The general case is similar, we ensure the injectivity of φ by avoiding the finitely many previously chosen sequences. □

The next lemma will provide several $\preceq_c$-antichains.

Lemma 6.5 *Let $\mathbf{d}, \mathbf{d}' \in \mathfrak{C}$ such that the d_j, d'_l's are prime, d'_l is not in $\{d_j \mid j \in \omega\}$ if $d'_l \neq 3$ and l is large enough, $\mathbf{d}'$ is unbounded, and $(\mathcal{C}_{\mathbf{d}}, G_{o_{\mathbf{d}}}) \preceq_c (\mathcal{C}_{\mathbf{d}'}, G_{o_{\mathbf{d}'}})$ with witness φ. Then φ is not onto.*

Proof. We argue by contradiction. As φ is uniformly continuous, there is, for each $l \in \omega$, $L := L(l) \geq l$ with the property that $\varphi(x)|(l+1) = \varphi(y)|(l+1)$ if $x, y \in \mathcal{C}_{\mathbf{d}}$ and $x|(L+1) = y|(L+1)$. We choose $l \geq 2$ with the property that $d'_{l+1} \notin \{d_j \mid j \in \omega\}$, which is possible for infinitely many l's. As φ is onto, we can find a surjection $\Gamma_l : \prod_{L+1} \to \prod_{j \leq l} d'_j$ such that $\varphi[\mathcal{C}_{\mathbf{d}} \cap N_s] \subseteq N_{\Gamma_l(s)}$. We enumerate $\prod_{L+1} := \{s_i^L \mid i < \pi_{j \leq L}\, d_j\}$ in the order defined by $o_{\mathbf{d}}$, starting with 0^{L+1}. Note that, respecting this order, $\{s_i^{L(l+1)} \mid i < \pi_{j \leq L(l+1)}\, d_j\}$ is

$$\{s_i^L 0^{L(l+1)-L} \mid i < \pi_{j \leq L}\, d_j\} \cup \ldots \cup \{s_i^L {}^\frown_{L<j\leq L(l+1)}\, (d_j - 1) \mid i < \pi_{j \leq L}\, d_j\}.$$

This implies that $\{\Gamma_{l+1}(s_i^{L(l+1)}) \mid i < \pi_{j \leq L(l+1)}\, d_j\}$ is

$$\{\Gamma_l(s_i^L)\varepsilon_i^0 \mid i < \pi_{j \leq L}\, d_j\} \cup \ldots \cup \{\Gamma_l(s_i^L)\varepsilon_i^{(\pi_{L<j\leq L(l+1)}\, d_j)-1} \mid i < \pi_{j \leq L}\, d_j\}.$$

As $o_{\mathbf{d}'}(x)|2 \neq o_{\mathbf{d}'}^{-1}(x)|2$ for each $x \in \mathcal{C}_{\mathbf{d}'}$, $o_{\mathbf{d}'}\big(\Gamma_l(s_i^L)\big) \neq o_{\mathbf{d}'}^{-1}\big(\Gamma_l(s_i^L)\big)$ if $i < \pi_{j\leq L}\ d_j$. As φ is a homomorphism, $\Gamma_l(s_{i+1}^L)$ can only be the image or the inverse image of $\Gamma_l(s_i^L)$ by the map $o_{\mathbf{d}'}$ if $i+1<\pi_{j\leq L}\ d_j$. Similarly, $\Gamma_{l+1}(s_{k+1}^{L(l+1)})$ can only be the image or the inverse image of $\Gamma_{l+1}(s_k^{L(l+1)})$ by the map $o_{\mathbf{d}'}$ if $k+1 < \pi_{j\leq L(l+1)}\ d_j$. If $s_k^{L(l+1)}$ extends s_i^L, then $\Gamma_{l+1}(s_k^{L(l+1)})$ extends $\Gamma_l(s_i^L)$, and $\Gamma_l(s_{i+1}^L)$ is the image of $\Gamma_l(s_i^L)$ if and only if $\Gamma_{l+1}(s_{k+1}^{L(l+1)})$ is the image of $\Gamma_{l+1}(s_k^{L(l+1)})$ since $o_{\mathbf{d}'}\big(\Gamma_l(s_i^L)\big) \neq o_{\mathbf{d}'}^{-1}\big(\Gamma_l(s_i^L)\big)$ if $i < \pi_{j\leq L}\ d_j$. This implies that $m := \varepsilon^k_{\pi_{j\leq L}\ d_j-1} - \varepsilon_0^k \pmod{d'_{l+1}}$ does not depend on k. Similarly, either $\varepsilon_0^{k+1} = \varepsilon^k_{\pi_{j\leq L}\ d_j-1}$ for each k, or $\varepsilon_0^{k+1} \equiv \varepsilon^k_{\pi_{j\leq L}\ d_j-1}+1 \pmod{d'_{l+1}}$ for each k, or $\varepsilon_0^{k+1} \equiv \varepsilon^k_{\pi_{j\leq L}\ d_j-1}-1 \pmod{d'_{l+1}}$ for each k.

Assume first that $\varepsilon_0^{k+1} = \varepsilon^k_{\pi_{j\leq L}\ d_j-1}$ for each k. An induction shows that

$$\varepsilon^k_{\pi_{j\leq L}\ d_j-1} \equiv \varepsilon_0^0 + (k+1)m \pmod{d'_{l+1}}.$$

Thus $\varepsilon_0^0 \equiv \varepsilon_0^0 + (\pi_{L<j\leq L(l+1)}\ d_j)m \pmod{d'_{l+1}}$ and d'_{l+1} divides m since d'_{l+1} is not in $\{d_j \mid j \in \omega\}$.

Assume now that $\varepsilon_0^{k+1} \equiv \varepsilon^k_{\pi_{j\leq L}\ d_j-1}+1 \pmod{d'_{l+1}}$ for each k. An induction shows that

$$\varepsilon^k_{\pi_{j\leq L}\ d_j-1} \equiv \varepsilon_0^0 + (k+1)(m+1) - 1 \pmod{d'_{l+1}}.$$

Thus $\varepsilon_0^0 \equiv \varepsilon_0^0 + (\pi_{L<j\leq L(l+1)}\ d_j)(m+1) \pmod{d'_{l+1}}$ and d'_{l+1} divides $m+1$.

Assume now that $\varepsilon_0^{k+1} \equiv \varepsilon^k_{\pi_{j\leq L}\ d_j-1}-1 \pmod{d'_{l+1}}$ for each k. An induction shows that

$$\varepsilon^k_{\pi_{j\leq L}\ d_j-1} \equiv \varepsilon_0^0 + (k+1)(m-1) + 1 \pmod{d'_{l+1}}.$$

Thus $\varepsilon_0^0 \equiv \varepsilon_0^0 + (\pi_{L<j\leq L(l+1)}\ d_j)(m-1) \pmod{d'_{l+1}}$ and d'_{l+1} divides $m-1$.

In all cases, this shows that ε_i^k does not depend on k. This argument can be extended to any length strictly greater than l. This cannot always hold since the sequence $\mathbf{d}'$ is unbounded. □

In our applications of Lemma 6.5, d_l, d'_l can be 3 for infinitely many l's.

Theorem 6.6 *There is a map $\Phi : 2^\omega \to \mathfrak{D}$ such that $(\mathcal{C}^+_{\Phi(\alpha)}, \mathbb{G}_{o_{\Phi(\alpha)}}) \not\preceq_c (\mathcal{C}^+_{\Phi(\beta)}, \mathbb{G}_{o_{\Phi(\beta)}})$ if $\alpha \neq \beta$.*

Proof. Let $(p_n)_{n\in\omega}$ be the sequence of prime numbers. We define, for each $\alpha \in 2^\omega$, $S_\alpha \subseteq \omega$ by $S_\alpha := \{0\} \cup \{p_0^{\alpha(0)+1} \dots p_n^{\alpha(n)+1} \mid n \in \omega\}$. Note that S_α is infinite, contains 0, and $S_\alpha \cap S_\beta$ is finite if $\alpha \neq \beta$. In this proof, we consider $(d_\alpha)_0 = 2$, $(d_\alpha)_j = 3$ if $j \notin S_\alpha$, $(d_\alpha)_j = p_{j+1}$ if $0 < j \in S_\alpha$, so that $\Phi(\alpha) := \mathbf{d}_\alpha \in \mathfrak{D}$ is unbounded, the $(d_\alpha)_j$'s are prime, $(d_\beta)_l$ is not in $\{(d_\alpha)_j \mid j \in \omega\}$ if $\alpha \neq \beta$, $(d_\beta)_l \neq 3$ and l is large enough.

So assume that $\mathbf{d}, \mathbf{d}' \in \mathfrak{D}$, the d_j, d'_l's are prime, d'_l is not in $\{d_j \mid j \in \omega\}$ if $d'_l \neq 3$ and l is large enough, $\mathbf{d}'$ is unbounded, and $(\mathcal{C}^+_{\mathbf{d}}, \mathbb{G}_{o_{\mathbf{d}}}) \preceq_c (\mathcal{C}^+_{\mathbf{d}'}, \mathbb{G}_{o_{\mathbf{d}'}})$ with witness φ. By Lemmas 3.2.3, 6.3 and Proposition 6.1, φ is onto $\{c^\infty\} \cup \mathcal{C}_{\mathbf{d}'}$. As φ is uniformly continuous, there is, for each $l \in \omega$, $L := L(l) \geq l$ with the property that $\varphi(x)|(l+1) = \varphi(y)|(l+1)$ if $x, y \in \mathcal{C}^+_{\mathbf{d}}$ and $x|(L+1) = y|(L+1)$.

Claim 1. If $l \in \omega$, then there is $L'_0 \geq L(l)$ such that, for each $L' \geq L'_0$, each $i \leq 2n_{L'}+1$ and each $\varepsilon \in \{a, \overline{a}\}$, $\varphi(o^{\mathbf{d}}_{L',i}\varepsilon^{i+1}\overline{\varepsilon}^\infty)|(l+1) \in \prod_{j\leq l}\ (d'_j \cup \{c\})$.

Indeed, towards a contradiction, suppose that there is l_0 such that, for each $k \geq L(l_0)$, we can find $L'_k \geq k$, i_k, ε_k and $m_k \leq l_0$ such that $\varphi(o^{\mathbf{d}}_{L'_k,i_k}\varepsilon_k^{i_k+1}\overline{\varepsilon_k}^\infty)(m_k) \in \{a, \overline{a}\}$. As φ is a homomorphism, the uniform continuity of φ and an induction on $i-i_k$ show that $\varphi(o^{\mathbf{d}}_{L'_k,i}\varepsilon^{i+1}\overline{\varepsilon}^\infty)(m_k), \varphi(c^{L'_k+1}\varepsilon\overline{\varepsilon}^\infty)(m_k)$ are in $\{a, \overline{a}\}$ if $i \leq 2n_{L'_k}+1$ and $\varepsilon \in \{a, \overline{a}\}$. If now $i \in \omega$, then $\big(o^i_{\mathbf{d}}(0^\infty), o^{i+1}_{\mathbf{d}}(0^\infty)\big)$ is the limit of couples of the form $(o^{\mathbf{d}}_{L'_k,i}a^{i+1}\overline{a}^\infty, o^{\mathbf{d}}_{L'_k,i+1}\overline{a}^{i+2}a^\infty)$, where we may assume that $(m_k)_k$ is constant. The continuity of φ implies that $\varphi\big(o^i_{\mathbf{d}}(0^\infty)\big)(m_0)$ is in $\{a, \overline{a}\}$. This shows that $\mathcal{C}^+_{\mathbf{d}} \subseteq \varphi^{-1}(\mathrm{proj}[\mathbb{G}_{o_{\mathbf{d}'}}])$ and $c^\infty \notin \varphi[\mathcal{C}^+_{\mathbf{d}}]$, which is absurd. $\diamond$

Claim 2. $\varphi[\mathcal{C}_{\mathbf{d}}] = \mathcal{C}_{\mathbf{d}'}$ and $\varphi(c^\infty) = c^\infty$.

Indeed, by Lemma 3.2.7 and Proposition 6.1, it is enough to see that $\varphi[\mathcal{C}_{\mathbf{d}}] \subseteq \mathcal{C}_{\mathbf{d}'}$. By Lemma 3.2.5 and Proposition 6.1, it is enough to see that $\varphi[\mathcal{C}_{\mathbf{d}} \cap N_t] \not\subseteq \{c^\infty\}$ and $\varphi[\mathcal{C}_{\mathbf{d}} \cap N_t] \not\subseteq \{s\varepsilon^{m+1}\overline{\varepsilon}^\infty\}$ if $t \in \bigcup_{l\in\omega} \prod_l$. We argue by contradiction.

If the singleton is of the form $\{s\varepsilon^{m+1}\overline{\varepsilon}^\infty\}$ with $s = c^{k+1}$ or $s \in \prod_{j\leq k} d'_j$, then we choose $i \in \omega$ with $t \subseteq o^i(0^\infty)$, so that $\varphi\big(o^i(0^\infty)\big) = s\varepsilon^{m+1}\overline{\varepsilon}^\infty$. We may assume that $s\varepsilon^{m+1}\overline{\varepsilon} \subseteq \varphi(z)$ if $z \in \mathcal{C}^+_{\mathbf{d}} \cap N_t$. We apply Claim 1 to $l := |s|$, which gives L'_0. We choose $L' \geq \max(|t|, L'_0)$ with the property that $i \leq 2n_{L'}+1$. Then $\varphi(o^{\mathbf{d}}_{L',i}a^{i+1}\overline{a}^\infty)|(l+1) = s\varepsilon \in \prod_{j\leq l} (d'_j \cup \{c\})$, which is the desired contradiction. If the singleton is $\{c^\infty\}$, then we may assume that $\varphi(z)(0) = c$ if $z \in \mathcal{C}^+_{\mathbf{d}} \cap N_t$. We fix $l > |t|$ with $n^{\mathbf{d}'}_l > 2n^{\mathbf{d}}_{|t|+1}+2$. Let $w \in \mathcal{C}_{\mathbf{d}'} \setminus \{\varphi(c^\infty)\}$, and $u \in \mathcal{C}_{\mathbf{d}}$ with $\varphi(u) = w$. Note that there is $i \leq 2n^{\mathbf{d}}_{L+1}+1$ with $o^{\mathbf{d}}_{L+1,i} = u|(L+1)0$, and $\varphi(u|(L+1)0a^{i+1}\overline{a}^\infty)|(l+1) = w|(l+1)$. Also, there is $k \leq 2n^{\mathbf{d}}_{L+1}+1$ such that $t \subseteq o^{\mathbf{d}}_{L+1,k}$ and $|k-i| \leq 2n^{\mathbf{d}}_{|t|+1}+1$. Note that $\varphi(o^{\mathbf{d}}_{L+1,k}a^{k+1}\overline{a}^\infty)(0) = c$, and $\varphi(o^{\mathbf{d}}_{L+1,k}a^{k+1}\overline{a}^\infty)|(l+1) = c^{l+1}$ since φ is uniformly continuous. Also, there is $i' < \pi_{j\leq l}\, d'_j$ with $w|(l+1) = o^{\mathbf{d}'}_{l,i'}$. Note that $i' \leq 2n^{\mathbf{d}}_{|t|+1}+1$ or $|2n^{\mathbf{d}'}_l+1-i'| \leq 2n^{\mathbf{d}}_{|t|+1}+1$ since φ is a homomorphism. Thus $\{y|(l+1) \mid y \in \mathcal{C}_{\mathbf{d}'}\}$ cannot be $\prod_{j\leq l} d'_j$, which contradicts the fact that φ is onto $\mathcal{C}_{\mathbf{d}'}$. $\diamond$

We set $\psi := \varphi_{|\mathcal{C}_{\mathbf{d}}}$. By Claim 2, ψ takes values in $\mathcal{C}_{\mathbf{d}'}$ and is onto $\mathcal{C}_{\mathbf{d}'}$. By Lemma 6.5, it remains to see that $\varphi\times\varphi$ sends $G_{o_{\mathbf{d}}}$ into $G_{o_{\mathbf{d}'}}$. As $o_{\mathbf{d}}$ is a minimal homeomorphism, it is fixed point free and thus $G_{o_{\mathbf{d}}} = s\big(\mathrm{Graph}(o_{\mathbf{d}})\big)$, and similarly with $\mathbf{d}'$. By Proposition 6.1 and Lemma 6.2, $\mathrm{Graph}(o_{\mathbf{d}})$ is contained in $\overline{\mathbb{G}_{o_{\mathbf{d}}}}^{(\mathcal{C}^+_{\mathbf{d}})^2}$, so that $\varphi\times\varphi$ sends $G_{o_{\mathbf{d}}}$ into $\overline{\mathbb{G}_{o_{\mathbf{d}'}}}^{(\mathcal{C}^+_{\mathbf{d}'})^2} \cap \mathcal{C}^2_{\mathbf{d}'}$ by Claim 2. It remains to note that $\overline{\mathbb{G}_{o_{\mathbf{d}'}}}^{(\mathcal{C}^+_{\mathbf{d}'})^2}$ is contained in $\mathbb{G}_{o_{\mathbf{d}'}} \cup s\big(\{(c^\infty, 0^\infty), (\mu, c^\infty)\} \cup \mathrm{Graph}(o_{\mathbf{d}'})\big)$ to conclude. $\square$

In the compact case, the $\preceq^i_c$-minimality can be seen on subgraphs.

Lemma 6.7 *Let (X, G) in $\mathfrak{K}$. The following are equivalent:*

(1) (X, G) is $\preceq^i_c$-minimal in $\mathfrak{K}$,

(2) $(X, G) \preceq^i_c (V, E)$ if V is a compact subset of X and $E \subseteq G$ is a graph on V with $\chi_c(V, E) \geq 3$.

Proof. It is enough to see that (2) implies (1). So let X' be a 0DMC space, and G' be a graph on X' with $\chi_c(X', G') \geq 3$ and $(X', G') \preceq^i_c (X, G)$ with witness φ. We set $V := \varphi[X']$ and also $E := (\varphi\times\varphi)[G']$. As X' is compact, so is V, and $E \subseteq G$ is a graph on V. Note that φ is a witness for the fact that $(X', G') \preceq^i_c (V, E)$, so that $\chi_c(V, E) \geq 3$. By (2), $(X, G) \preceq^i_c (V, E)$. By compactness of X' again, φ^{-1} is a witness for the fact that $(V, E) \preceq^i_c (X', G')$, which as desired implies that $(X, G) \preceq^i_c (X', G')$. $\square$

Proof of Theorem 1.5. Fix $\mathbf{d}\in\mathfrak{D}$. Note that $(\mathcal{C}^+,\mathbb{G}_o)$ satisfies the properties (a) and (b), by Lemma 3.2.2 and Proposition 6.1. For (c), i.e., the minimality of $(\mathcal{C}^+,\mathbb{G}_o)$, we apply Proposition 6.1, Theorem 6.4 and Lemma 6.7. It remains to apply Theorem 6.6. $\square$

Remark. [P, Theorem 11.38] shows that the $o_{\Phi(\alpha)}$'s involved in Theorem 1.5 are pairwise not flip-conjugate, as announced in the introduction just before Theorem 1.5.

Proof of Theorem 1.6. Fix $\mathbf{d}\in\mathfrak{D}$. The idea is to modify $\mathbb{G}_o$. Let d be a letter not in $\omega\cup\{c,a,\overline{a}\}$. We set, for $l\in\omega$,

$$\begin{aligned}\mathbb{H}_l:=\{(c^{l+1}d^{j+1}a\overline{a}^\infty, o_{l,0}d^{j+1}\overline{a}a^\infty)\mid j\in\omega\}\cup\\ \{(o_{l,i}d^{j+1}a^{i+1}\overline{a}^\infty, o_{l,i+1}d^{j+1}\overline{a}^{i+2}a^\infty)\mid j\in\omega\wedge i\leq 2n_l\}\cup\\ \{(o_{l,2n_l+1}d^{j+1}a^{2n_l+2}\overline{a}^\infty, c^{l+1}d^{j+1}\overline{a}a^\infty)\mid j\in\omega\}.\end{aligned}$$

We then set $\mathbb{O}_p:=\bigcup_{l\geq p}\mathbb{H}_l$ and $\mathbb{G}_p:=s(\mathbb{O}_p)$, so that $\mathbb{G}_p$ is a countable graph on the compact space $\mathcal{K}:=\prod_{j\in\omega}(d_j\cup\{c,a,\overline{a},d\})$. We set $\mathbb{K}:=\overline{\mathrm{proj}[\mathbb{G}_0]}^{\mathcal{K}}$, so that $\mathbb{K}$ is a 0DMC space and $\mathbb{G}_p$ is a graph on $\mathbb{K}$, $(\mathbb{G}_p)_{p\in\omega}$ is $\subseteq$-decreasing and thus $\big((\mathbb{K},\mathbb{G}_p)\big)_{p\in\omega}$ is $\preceq_c$ and $\preceq_c^i$-decreasing. As in the proofs of Lemmas 3.2.1 and 3.2.2, we see that $\mathbb{G}_p$ has CCN three, $\mathbf{\Sigma}^0_1\oplus\mathbf{\Pi}^0_1$ chromatic number two, and is $D_2(\mathbf{\Pi}^0_1)$.

It remains to see that $(\mathbb{K},\mathbb{G}_p)$ is not $\preceq_c$-below $(\mathbb{K},\mathbb{G}_{p+1})$. Towards a contradiction, suppose that there is $\varphi:\mathbb{K}\to\mathbb{K}$. The continuity of φ implies that $\overline{\mathbb{G}_p}\subseteq(\varphi\times\varphi)^{-1}(\overline{\mathbb{G}_{p+1}})$. Note that

$$(c^{p+1}d^\infty, o_{p,0}d^\infty,\dots,o_{p,2n_p+1}d^\infty)$$

is a $\overline{\mathbb{G}_p}$-cycle of length $2n_p+3$, and therefore has to be sent in a $\overline{\mathbb{G}_{p+1}}$-cycle of length at most $2n_p+3$. But such a $\overline{\mathbb{G}_{p+1}}$-cycle does not exist since

$$\begin{aligned}\overline{\mathbb{G}_p}:=\mathbb{G}_p\cup s\big(\{(c^{l+1}d^\infty, o_{l,0}d^\infty)\mid l\geq p\}\cup\{(o_{l,i}d^\infty, o_{l,i+1}d^\infty)\mid l\geq p\wedge i\leq 2n_l\}\cup\\ \{(o_{l,2n_l+1}d^\infty, c^{l+1}d^\infty)\mid l\geq p\}\cup\{(c^\infty,0^\infty),(\mu,c^\infty)\}\cup\mathrm{Graph}(o)\big).\end{aligned}$$

This finishes the proof. $\square$

7 Graphs induced by a function: general facts

Remarks. (1) In the case of finite spaces, the quasi-order $\preceq$ on the class of graphs induced by a partial bijection with chromatic number at least three is linear. Indeed, such a space can be decomposed in pairwise f-unrelated injective walks of the form $\{x,f(x),\dots,f^l(x)\}$. As the chromatic number is at least three, one of these walks gives an odd cycle. The graph induced by the bijection is $\preceq$-equivalent to its odd cycle of minimal length. As the odd cycles are $\preceq$-comparable, so too are all these graphs.

(2) Note that the map $A\mapsto\oplus_{p\in A}(2p+3,C_{2p+3})$ is an embedding of the quasi-order of inclusion on the set of finite subsets of ω into the quasi-order $\preceq^i$ on the class of graphs induced by a bijection on a finite set with chromatic number at least three.

Under some relatively weak assumptions, we can characterize when the CCN of G_f is big. Note that we extend Theorem 1.7 under these assumptions.

Theorem 7.1 *Let X be a 0DMS space of cardinality at least two, $f:X\to X$ be a homeomorphism, and $x\in X$ with $\overline{Orb_f(x)}=X$. The following are equivalent:*

(1) (X,G_f) has CCN at least three,

(2) $\overline{\{f^{2n}(x)\mid n\in\mathbb{Z}\}}\cap\overline{\{f^{2p+1}(x)\mid p\in\mathbb{Z}\}}$ is not empty,

(3) $\Delta(X)\cap\overline{\bigcup_{p\in\omega}\ G_f^{2p+1}}$ is not empty,

(4) $\Delta(X)\cap\overline{\bigcup_{p\in\omega}\ \overline{G_f}^{2p+1}}$ is not empty,

(5) $(\mathcal{N},\mathbb{G}_m)\preceq_c(X,G_f)$.

Proof. (1) $\Rightarrow$ (2) Note that f sends $\{f^{2n}(x)\mid n\in\mathbb{Z}\}$ onto $\{f^{2n+1}(x)\mid n\in\mathbb{Z}\}$. As f is a homeomorphism, f sends $C_e:=\overline{\{f^{2n}(x)\mid n\in\mathbb{Z}\}}$ onto $C_o:=\overline{\{f^{2p+1}(x)\mid p\in\mathbb{Z}\}}$. Note that

$$X=\overline{\mathrm{Orb}_f(x)}=C_e\cup C_o.$$

If C_e is disjoint from C_o, then (C_e,C_o) defines a continuous coloring of (X,G_f), contradicting (1).

(2) $\Rightarrow$ (3) By (2), there is y in the intersection. Let O be an open neighborhood of y, and m,n be integers with $f^{2n}(x),f^{2m+1}(x)\in O$. As X has cardinality at least two, $f_{|\mathrm{Orb}_f(x)}$ is fixed point free. We put $p:=|m|+|n|$, so that $f^{2n}(x),\dots,x,\dots,f^{2m+1}(x)$ is a witness for the fact that O^2 meets G_f^{2p+1}.

(3) $\Rightarrow$ (4) Note that $G_f\subseteq\overline{G_f}$.

(4) $\Rightarrow$ (1) We apply Lemma 3.3.1.

(4) $\Leftrightarrow$ (5) We apply Lemma 3.3.4. □

Notation. The set of fixed points of a map f is very much related to the CCN of G_f. Let X be a set, and $f:X\to X$ be a partial map. The set $F_1:=\{x\in\mathrm{Domain}(f)\mid f(x)=x\}$ of fixed points of f is sometimes also denoted by F_1^f.

Proposition 7.2 *Let X be a 0DMS space, and $f:X\to X$ be a partial continuous function. If F_1 is not an open subset of Domain(f), then $\chi_c(X,G_f)=2^{\aleph_0}$.*

Proof. Let $(C_i)_{i\in\omega}$ be a partition of X into clopen sets. As F_1 is not open in Domain(f), we can find $x\in F_1$ and $(x_n)_{n\in\omega}\in(\mathrm{Domain}(f)\setminus F_1)^\omega$ converging to x. Note that $f(x_n)$ is different from x_n, and $\big(f(x_n)\big)_{n\in\omega}$ converges to $f(x)=x$. Let i with $x\in C_i$. Then we may assume that $x_n,f(x_n)\in C_i$. This implies that $\big(x_n,f(x_n)\big)\in G_f\cap C_i^2$. □

Corollary 7.3 *Let X be a 0DMS space, and $f:X\to X$ be a partial continuous function with closed domain.*

(a) Exactly one of the following holds:

(1) F_1 is an open subset of Domain(f),

(2) $\chi_c(X,G_f)=2^{\aleph_0}$.

(b) If F_1 is an open subset of Domain$(f)\in\mathbf{\Delta}^0_1(X)$ and f is injective, then $\chi_c(X,G_f)=0$ if $X=\emptyset$, 1 if $F_1=$Domain(f) and $X\neq\emptyset$, $\chi_c(X\setminus F_1,G_f\cap(X\setminus F_1)^2)$ if $F_1\neq$Domain(f).

Proof. (a) Assume that (1) holds. Note that $s\big(\mathrm{Graph}(f_{|\mathrm{Domain}(f)\setminus F_1})\big)$ and $\Delta(X)$ are disjoint closed relations on the metrizable space X. By [K, 22.16], there is a clopen relation C on X separating $\Delta(X)$ from $s\big(\mathrm{Graph}(f_{|\mathrm{Domain}(f)\setminus F_1})\big)$. This relation gives a countable continuous coloring of (X, G_f) since X is zero-dimensional and second countable. So (2) does not hold.

If F_1 is not an open subset of $\mathrm{Domain}(f)$, then we apply Proposition 7.2.

(b) If $F_1 \neq \mathrm{Domain}(f)$, then we can find $2 \leq n \leq \omega$ and a continuous coloring $c : X \setminus F_1 \to n$ of $(X\setminus F_1, G_f \cap (X\setminus F_1)^2)$, by (a). As f is injective, $f[\mathrm{Domain}(f)\setminus F_1] \cap F_1 = \emptyset$, so that F_1 and $X\setminus F_1$ are f-invariant. The extension of c by 0 on F_1 is a continuous coloring of (X, G_f). Conversely, any continuous coloring of (X, G_f) gives a coloring of $(X\setminus F_1, G_f \cap (X\setminus F_1)^2)$, by restriction. $\square$

In the introduction, we announced a version of Theorem 1.2 for analytic spaces when $\xi = 1$. Here are the argument and some precisions. Recall that $\mathbb{X}_1 := \{0^\infty\} \cup \{0^n 1^\infty \mid n \in \omega\}$, $f_1 : \mathbb{X}_1 \to \mathbb{X}_1$ is defined by $f_1(0^\infty) := 0^\infty$ and $f_1(0^{2n+\varepsilon}1^\infty) := 0^{2n+1-\varepsilon}1^\infty$, and $\mathbb{R}_1 := \{(0^{2n}1^\infty, 0^{2n+1}1^\infty) \mid n \in \omega\}$. We also define $f_0 : \mathbb{X}_1 \to \mathbb{X}_1$ by $f_0(\alpha) := 0^\infty$.

Proposition 7.4 *(a) (Lecomte-Zelený) Let X be a zero-dimensional Lindelöf first countable space, and R be a relation on X. Then exactly one of the following holds:*

(1) there is a countable continuous coloring of R,

(2) there is $f : \mathbb{X}_1 \to X$ continuous such that $\mathbb{R}_1 \subseteq (f \times f)^{-1}(R)$.

In particular, $(\mathbb{X}_1, G_{f_1})$ is $\preceq_c$-minimum in the class of graphs on a 0DMS space with uncountable CCN.

(b) $\{(\mathbb{X}_1, G_{f_0}), (\mathbb{X}_1, G_{f_1})\}$ is a $\preceq_c^i$-antichain basis for the class of graphs on a 0DMS space with uncountable CCN.

Proof. (a) If $\Delta(X) \cap \overline{R} = \emptyset$, then for each $x \in X$ there is a clopen neighborhood C_x of x with $R \cap C_x^2 = \emptyset$. As X is Lindelöf, the covering $(C_x)_{x\in X}$ of X can be replaced with a covering $(C_n)_{n\in\omega}$. Replacing C_n with $C_n \setminus (\bigcup_{m<n} C_m)$ if necessary, we may assume that the C_n's are pairwise disjoint, which gives a countable continuous coloring of R. If there is $(x, x) \in \overline{R}$, then the fact that X is first countable provides a sequence (x_n) converging to x with $(x_{2n}, x_{2n+1}) \in R$. It remains to set $\varphi(0^\infty) := x$ and $\varphi(0^n 1^\infty) := x_n$ to see that (2) holds. If C is a clopen subset of $\mathbb{X}_1$ containing 0^∞, then we can find n with $0^{2n}1^\infty, 0^{2n+1}1^\infty \in C$, so that $(0^{2n}1^\infty, 0^{2n+1}1^\infty) \in G_{f_1} \cap C^2$ and $(0^{2n}1^\infty, 0^\infty) \in G_{f_0} \cap C^2$. This implies that the $(\mathbb{X}_1, G_{f_\varepsilon})$'s have uncountable CCN. It remains to note that $s(\mathbb{R}_1) = G_{f_1}$ to see that (1) and (2) cannot hold simultaneously.

(b) Let X be a 0DMS space, and R be a graph on X with uncountable CCN. We use the proof of (a), which gives (x_n). As $x_{2n} \neq x_{2n+1}$, we may assume, extracting a subsequence if necessary, that the sequence $\big((x_{2n}, x_{2n+1})\big)_{n\in\omega}$ is injective, and that $(x_{2n})_{n\in\omega}$ or $(x_{2n+1})_{n\in\omega}$ is injective too. By symmetry, we may assume that $(x_{2n})_{n\in\omega}$ is injective and does not take the value x. We may also assume that $(x_{2n+1})_{n\in\omega}$ is either injective and does not take the value x, or is constant with value x. If $(x_{2n+1})_{n\in\omega}$ is injective, then we may assume that $\{x_{2n} \mid n \in \omega\}$ and $\{x_{2n+1} \mid n \in \omega\}$ are disjoint. In this case, we define $\varphi : \mathbb{X}_1 \to X$ by $\varphi(0^\infty) := x$, $\varphi(0^{2n}1^\infty) := x_{2n}$, and $\varphi(0^{2n+1}1^\infty) := x_{2n+1}$, so that φ is an injective continuous homomorphism from $(\mathbb{X}_1, G_{f_1})$ into (X, R). If $(x_{2n+1})_{n\in\omega}$ is constant with value x, then we define a function $\psi : \mathbb{X}_1 \to X$ by $\psi(0^\infty) := x$, and $\psi(0^n 1^\infty) := x_{2n}$, so that ψ is an injective continuous homomorphism from $(\mathbb{X}_1, G_{f_0})$ into (X, R). This shows that $\{(\mathbb{X}_1, G_{f_0}), (\mathbb{X}_1, G_{f_1})\}$ is a $\preceq_c^i$-basis.

If $f:\mathbb{X}_1\to\mathbb{X}_1$ is a witness for $(\mathbb{X}_1,G_{f_1})\preceq^i_c(\mathbb{X}_1,G_{f_0})$, then f sends $\{0^{2n}1^\infty,0^{2n+1}1^\infty\}$ onto some $\{0^{p_n}1^\infty,0^\infty\}$, which contradicts the injectivity of f. If $g:\mathbb{X}_1\to\mathbb{X}_1$ is a witness for the inequality $(\mathbb{X}_1,G_{f_0})\preceq^i_c(\mathbb{X}_1,G_{f_1})$, then g sends $(0^p1^\infty,0^\infty)$ to some $(0^{2n_p+\varepsilon_p}1^\infty,0^{2n_p+1-\varepsilon_p}1^\infty)$, and $(n_p)_{p\in\omega}$, $(\varepsilon_p)_{p\in\omega}$ have to be constant, which contradicts the injectivity of g. This implies that $\{(\mathbb{X}_1,G_{f_0}),(\mathbb{X}_1,G_{f_1})\}$ is a $\preceq^i_c$-antichain. Note then that $(\mathbb{X}_1,G_{f_0}),(\mathbb{X}_1,G_{f_1})$ are in our class. $\square$

Example. Recall $(\mathbb{X}_1,f_1)$ defined before Proposition 7.4. Then $\mathbb{X}_1\neq\emptyset$ is a 0DMC space, and f_1 is a homeomorphism whose only fixed point is 0^∞. By Proposition 7.2, $\chi_c(\mathbb{X}_1,G_{f_1})=2^{\aleph_0}$. If we restrict f_1 to the open set $\mathbb{X}_1\setminus\{0^\infty\}$, then F_1 becomes open in the domain of the restriction, and $\chi_c(\mathbb{X}_1,G_{f_{1|\mathbb{X}_1\setminus\{0^\infty\}}})=2^{\aleph_0}$. Indeed, if $(C_i)_{i\in\omega}$ be a partition of $\mathbb{X}_1$ into clopen sets, then there is i with $0^\infty\in C_i$. We may assume that $0^{2n}1^\infty,0^{2n+1}1^\infty\in C_i$. This implies that $(0^{2n}1^\infty,0^{2n+1}1^\infty)$ is in $G_{f_{1|\mathbb{X}_1\setminus\{0^\infty\}}}\cap C_i^2$. This shows that we cannot extend Corollary 7.3(a) when the domain of f is open.

We now turn to the study of involutions.

Proposition 7.5 *Let $X\neq\emptyset$ be a 0DMS space, and $f\colon X\to X$ be a fixed point free continuous involution. Then $\chi_c(X,G_f)=2$.*

Proof. Note that f is a continuous bijection with inverse f, so that it is a homeomorphism. If $x\in X$, then $f(x)\neq x$ since f is fixed point free, which gives a clopen neighborhood N of x with $f(x)\notin N$. As f is a homeomorphism, $C^0:=N\cap f^{-1}(X\setminus N)$ and $C^1:=f[C^0]=f[N]\setminus N$ are disjoint clopen subsets of X. In particular, $C^0=f[C^1]$ and $C:=C^0\cup C^1$ is a f-invariant clopen neighborhood of x. As X has the Lindelöf property, we can cover X with countably many such f-invariant clopen sets, say $(C_n)_{n\in\omega}$. In particular, $\cup_{p<n}C_n$, $X\setminus(\cup_{p<n}C_n)$ and $O_n:=C_n\setminus(\cup_{p<n}C_n)$ are f-invariant clopen sets with union X. If $U\subseteq C$ is a f-invariant clopen set, then we set $U^\varepsilon:=U\cap C^\varepsilon$, so that U is the disjoint union of U^0 and U^1, and $U^{1-\varepsilon}=f[U^\varepsilon]$ for each $\varepsilon\in2$. We can apply this to $O_n\subseteq C_n$, so that X is the disjoint union of the family of clopen sets $(O_n^\varepsilon)_{n\in\omega,\varepsilon\in2}$, and $O_n^{1-\varepsilon}=f[O_n^\varepsilon]$ for each $(n,\varepsilon)\in\omega\times2$. We then define $c:X\to2$ by $c(x):=\varepsilon$ if $x\in O_n^\varepsilon$ for some n, and c is a continuous coloring of (X,G_f). $\square$

Proposition 7.5 implies some minimality of G_{f_1}.

Proposition 7.6 *$(\mathbb{X}_1,G_{f_1})$ is $\preceq^i_c$-minimal, but not $\preceq_c$-minimal, in $\mathfrak{K}$.*

Proof. Let V be compact subset of $\mathbb{X}_1$, and $E\subseteq G_{f_1}$ be a graph on V with $\chi_c(V,E)\geq3$. By Lemma 6.7, it is enough to see that $(\mathbb{X}_1,G_{f_1})\preceq^i_c(V,E)$. As $P:=\mathrm{proj}[E]\subseteq V\cap\mathbb{X}_1\setminus\{0^\infty\}$, we can find, for each $\varepsilon\in2$, $S_\varepsilon\subseteq\omega$ with $P=\{0^{2n+\varepsilon}1^\infty\mid\varepsilon\in2\wedge n\in S_\varepsilon\}$. As E is a graph,

$$n\in S_0\Leftrightarrow0^{2n}1^\infty\in P\Leftrightarrow0^{2n+1}1^\infty\in P\Leftrightarrow n\in S_1,$$

so that $S_0=S_1$. Thus $E=G_{f_{1|P}}$ is the graph induced by the fixed point free involution $f_{1|P}:P\to P$, which is continuous since P is discrete. As $P\subseteq\mathbb{X}_1$ is not empty, $\chi_c(P,E)=2$ by Proposition 7.5. This gives $C\in\mathbf{\Delta}^0_1(P)$ with $E\cap\big(C^2\cup(P\setminus C)^2\big)=\emptyset$. If $0^\infty\notin V$, then C is a clopen subset of the discrete space V, $E\cap\big(C^2\cup(V\setminus C)^2\big)=\emptyset$ and $\chi_c(V,E)\leq2$, which cannot be. Thus $0^\infty\in V$. Note also that there are infinitely many p's with $(0^{2p}1^\infty,0^{2p+1}1^\infty)\in E$, otherwise $(0^{2p}1^\infty,0^{2p+1}1^\infty)\notin E$ if $p\geq p_0$. We then set $C':=(\bigcup_{p<p_0}N_{0^{2p}1})\cap V$, so that C' is a clopen subset of V, $E\cap\big((C')^2\cup(V\setminus C')^2\big)=\emptyset$, and $\chi_c(V,E)\leq2$. This implies that $(0^\infty,0^\infty)\in\overline{E}^{V^2}$, and gives an injective sequence $(p_n)_{n\in\omega}$ with $(0^{2p_n}1^\infty,0^{2p_n+1}1^\infty)\in E$. We then define $g:\mathbb{X}_1\to V$ by $g(0^\infty):=0^\infty$ and $g(0^{2n+\varepsilon}1^\infty):=0^{2p_n+\varepsilon}1^\infty$, so that g is a witness for $(\mathbb{X}_1,G_{f_1})\preceq^i_c(V,E)$, as desired.

We now set $(0^+,1^+,2^+)=(1,2,0)$, $X:=\{\varepsilon^\infty \mid \varepsilon\in 3\}\cup\{\varepsilon^{n+1}(\varepsilon^+)^\infty \mid \varepsilon\in 3\wedge n\in\omega\}$, and $G:=s(\{(\varepsilon^{2p+1}(\varepsilon^+)^\infty,(\varepsilon^+)^{2p+2}\big((\varepsilon^+)^+\big)^\infty) \mid \varepsilon\in 3\wedge p\in\omega\})$. Note that X is a 0DMC space, and G is a graph on X. As $(\varepsilon^\infty,(\varepsilon^+)^\infty)\in\overline{G}$, $(0^\infty,0^\infty)\in\Delta(X)\cap\overline{G}^3$, and thus $\chi_c(X,G)\geq 3$ by Theorem 1.7. In particular, $(X,G)\in\mathfrak{K}$. The map $\alpha\mapsto\alpha(0)$ is a 3-continuous coloring of (X,G), so that $\chi_c(X,G)=3$. As $(\mathbb{X}_1,G_{f_1})$ has uncountable CCN, $(\mathbb{X}_1,G_{f_1})\not\preceq_c(X,G)$. However, $(X,G)\preceq_c(\mathbb{X}_1,G_{f_1})$, with witness $\varepsilon^\infty\mapsto 0^\infty$ and $\varepsilon^{n+1}(\varepsilon^+)^\infty\mapsto 0^{n+1}1^\infty$. □

The proof of of Theorem 1.8 implies, as announced in the introduction, the following.

Theorem 7.7 *There is no $\preceq_c^i$-antichain basis for the class of graphs induced by a partial homeomorphism on a 0DMS (or 0DP) space with CCN at least three. In fact, we can even restrict this class to the case where the spaces are countable Polish and the functions are fixed point free involutions with open domain.*

Proof. By Proposition 3.3.7, $\mathbb{P}_\delta$ is countable Polish. Note that

$$\text{proj}[\mathbb{G}_\delta]=\mathbb{P}_\delta\setminus(\{c^\infty\}\cup\{ki0^\infty \mid \delta(k)=1\wedge i\leq 2k+1\})$$

is an open subset of $\mathbb{P}_\delta$. We define $f_\delta:\text{proj}[\mathbb{G}_\delta]\to\text{proj}[\mathbb{G}_\delta]$ by $f_\delta(\alpha):=$ the unique $\beta\in\text{proj}[\mathbb{G}_\delta]$ with $(\alpha,\beta)\in\mathbb{G}_\delta$, so that f_δ is a fixed point free involution, and $\mathbb{G}_\delta=\text{Graph}(f_\delta)=G_{f_\delta}$. As $\text{proj}[\mathbb{G}_\delta]$ is discrete, f_δ is continuous, and thus a homeomorphism. By Lemma 3.3.6, $\chi_c(\mathbb{P}_\delta,G_{f_\delta})=3$ if δ has infinitely many ones. Let (X,G_f) in our class with $(X,G_f)\preceq_c^i(\mathbb{P}_{1^\infty},G_{f_{1^\infty}})$. Theorem 1.8 provides a family $(\delta_\gamma)_{\gamma\in 2^\omega}$ in $\mathbb{P}_\infty$ such that $(\mathbb{P}_{\delta_\gamma},G_{f_{\delta_\gamma}})\preceq_c^i(X,G_f)$ and the $(\mathbb{P}_{\delta_\gamma},G_{f_{\delta_\gamma}})$'s are pairwise $\preceq_c^i$-incompatible in our class. We then apply Lemma 4.1. □

The next results will help us to prove a condition sufficient to get the minimality of some G_f's.

Lemma 7.8 *Let X be a 0DMC space, $f:X\to X$ be a minimal homeomorphism, and $C\neq\emptyset$ be a clopen subset of X. Then there is $L>0$ such that, for each $x\in X$, we can find $0<l<L$ with $f^l(x)\in C$, and $0<l'<L$ with $f^{-l'}(x)\in C$.*

Proof. This is standard. By minimality, $X\subseteq\bigcup_{m\in\mathbb{Z}}\{x\in X \mid f^m(x)\in C\}$. The compactness of X gives $M>0$ with $X\subseteq\bigcup_{-M<m<M}\{x\in X \mid f^m(x)\in C\}$. We put $L:=2M$. If $x\in X$, then there is $-M<m<M$ with $f^{m+M}(x)\in C$, and $0<l:=m+M<L$. This is similar for l'. □

Lemma 7.8 allows us to define $r_C:X\to L$ by $r_C(x):=\min\{l<L \mid f^l(x)\in C\}$, and r_C is continuous. Similarly, we can define $r'_C:X\to L$ continuous by $r'_C(x):=\min\{l'<L \mid f^{-l'}(x)\in C\}$.

Lemma 7.9 *Let X be a 0DMC space, $f:X\to X$ be a minimal homeomorphism with the property that $\chi_c(X,G_f)\geq 3$, and $E\subseteq G_f$ be a graph. The following are equivalent:*

(1) $\chi_c(X,E)\geq 3$,

(2) E is dense in G_f.

Proof. (1) $\Rightarrow$ (2) Towards a contradiction, suppose that we can find open subsets U,V of X with $G_f\cap(U\times V)\neq\emptyset$ and $E\cap(U\times V)=\emptyset$. Pick $(x,y)\in G_f\cap(U\times V)$. If $y=f(x)$, then $x\in U\cap f^{-1}(V)$, and there is a clopen neighborhood $C\subseteq U\cap f^{-1}(V)$ of x with $C\cap f[C]=\emptyset$ since f is fixed point free. By symmetry of E, $(C\times f[C])\cap E=(f[C]\times C)\cap E=\emptyset$. We first define $s:X\to\omega$ by $s(x):=r_C(x)+r'_{f[C]}(x)$, so that s is continuous.

We then define $c : X \to 2$ by $c(x) := \text{parity}\big(r_C(x)\big)$ if $x \notin f[C]$, and $c(x) := \text{parity}\big(s(x)\big)$ if $x \in f[C]$. Note that c is continuous. It is enough to see that c is a coloring of (X, E). It is enough to see that $c\big(f(x)\big) \neq c(x)$ if $\big(x, f(x)\big) \in E$. Note that $x \notin C$. If $x \notin f[C]$, then $f(x) \notin f[C]$ and $r_C(x) = r_C\big(f(x)\big) + 1$. If $x \in f[C]$, then $f(x) \notin f[C]$ and $s(x) = r_C(x)$, and we conclude similarly.

(2) $\Rightarrow$ (1) Towards a contradiction, suppose that there is a clopen subset C of X with the property that $E \cap \big(C^2 \cup (X \setminus C)^2\big) = \emptyset$. As C is clopen and E is dense in G_f,

$$G_f \cap \big(C^2 \cup (X \setminus C)^2\big) \subseteq \overline{E} \cap \big(C^2 \cup (X \setminus C)^2\big) = \emptyset,$$

which is the desired contradiction. $\square$

Lemma 7.9 essentially implies that G_f is minimal if f is.

Lemma 7.10 *Let X be a 0DMC space, $f : X \to X$ be a minimal homeomorphism such that (X, G_f) has CCN at least three, K be a 0DMC space, G be a closed graph on K such that $(K, G) \preceq_c^i (X, G_f)$. Then exactly one of the following holds:*

(1) (K, G) has CCN at most two,

(2) $(X, G_f) \preceq_c^i (K, G)$.

In other words, (X, G_f) is $\preceq_c^i$-minimal in $\mathfrak{G}_2$ and in the class of closed graphs on a 0DMC space with CCN at least three.

Proof. Let V be compact subset of X, and $E \subseteq G_f$ be a compact graph on V with $\chi_c(V, E) \geq 3$ (which implies that $\chi_c(X, E) \geq 3$). As in the proof of Lemma 6.7, it is enough to see that $(X, G_f) \preceq_c^i (V, E)$ to see that (1) or (2) holds. Let $P := \text{proj}[E]$, which is compact like E. The next two claims give our result.

Claim 1. $P = V = X$.

Indeed, by compactness it is enough to see that P is dense in X. Towards a contradiction, suppose that there is a clopen subset $C \neq \emptyset$ of X disjoint from P, so that r_C is defined. We define $c : P \to 2$ by $c(x) := \text{parity}\big(r_C(x)\big)$. Note that c is continuous. It is enough to see that c is a coloring of (P, E), since this implies that $\chi_c(V, E) \leq 2$ by [E, Theorem 2.1(1)]. It is enough to see that $c\big(f(x)\big) \neq c(x)$ if $\big(x, f(x)\big) \in E$ since f is fixed point free and by symmetry. The equality $r_C(x) = r_C\big(f(x)\big) + 1$ gives the result. $\diamond$

Claim 2. $E = G_f$.

Indeed, Lemma 7.9 implies that E is dense in G_f. It remains to note that E is compact. $\diamond$

Claims 1 and 2 imply that $(V, E) = (X, G_f)$.

For $\mathfrak{G}_2$, assume that K is a 0DMC space, h is a homeomorphism of K, $\chi_c(K, G_h) \geq 3$, and $(K, G_h) \preceq_c^i (X, G_f)$. Then $\chi_c(K, G_h) \leq \chi_c(X, G_f) < 2^{\aleph_0}$ by Corollary 7.3. Corollary 7.3 then implies that F_1^h is an open subset of K, $F_1^h \neq K$ and $\chi_c(K, G_h) = \chi_c(K \setminus F_1^h, G_h \cap (K \setminus F_1^h)^2)$. Note that $K \setminus F_1^h$ is a 0DMC space, $h_{|K \setminus F_1^h}$ is a homeomorphism of $K \setminus F_1^h$, $G_{h_{|K \setminus F_1^h}} = G_h \cap (K \setminus F_1^h)^2$, $\chi_c(K \setminus F_1^h, G_{h_{|K \setminus F_1^h}}) \geq 3$ and $(K \setminus F_1^h, G_{h_{|K \setminus F_1^h}}) \preceq_c^i (X, G_f)$. Moreover, $h_{|K \setminus F_1^h}$ is fixed point free, so that $G_{h_{|K \setminus F_1^h}}$ is closed. Note that $(X, G_f) \preceq_c^i (K \setminus F_1^h, G_{h_{|K \setminus F_1^h}}) \preceq_c^i (K, G_h)$, by the first part of the present theorem. In other words, (X, G_f) is $\preceq_c^i$-minimal in $\mathfrak{G}_2$. $\square$

Remark. Proposition 7.6 shows that the converse of Lemma 7.10 does not hold, since 0^∞ is a fixed point of f_1.

The next result is in the style of Corollary 5.10.

Lemma 7.11 *Let X, Y be 0DMC spaces of cardinality at least three, and $f : X \to X$, $g : Y \to Y$ be minimal homeomorphisms. Then f, g are flip-conjugate if and only if $(X, G_f) \preceq_c^i (Y, G_g)$, with the same witness. In particular, $(X, G_f) \preceq_c^i (Y, G_g)$ implies that $(Y, G_g) \preceq_c^i (X, G_f)$.*

Proof. Let φ be a witness for the fact that $(X, G_f) \preceq_c^i (Y, G_g)$. If $x \in X$, then $\big(x, f(x)\big) \in G_f$ since f is fixed point free, so that $\Big(\varphi(x), \varphi\big(f(x)\big)\Big) \in G_g$. So $\varphi\big(f(x)\big) = g\big(\varphi(x)\big)$ or $\varphi(x) = g\Big(\varphi\big(f(x)\big)\Big)$. We apply Lemma 5.6 to f, $V := I := X$, g, $W := Y$, and φ. In particular, $\varphi[\mathrm{Orb}_f(x)] = \mathrm{Orb}_g\big(\varphi(x)\big)$. As g is minimal, the compact set $\varphi[X]$ is dense in Y, showing that φ is onto, and thus a homeomorphism by compactness of X.

Conversely, assume that f, g are flip-conjugate, which gives a homeomorphism $\varphi : X \to Y$ with $\varphi \circ f = g \circ \varphi$ or $\varphi \circ f = g^{-1} \circ \varphi$. If $x \in X$, then $g\big(\varphi(x)\big) = \varphi\big(f(x)\big)$ or $\varphi(x) = g\Big(\varphi\big(f(x)\big)\Big)$, so that $\Big(\varphi(x), \varphi\big(f(x)\big)\Big) \in G_g$. Thus φ is a witness for the fact that $(X, G_f) \preceq_c^i (Y, G_g)$. □

8 Possible chromatic numbers

The main goal of this section is to prove Theorem 1.10. The next result is essentially [Kra-St, Corollary 2.3].

Theorem 8.1 *(Krawczyk-Steprans) Let $X \neq \emptyset$ be a 0DMC space, and $f : X \to X$ be a fixed point free continuous map. Then $\chi_c(X, G_f) \in \{2, 3\}$.*

Proof. [Kra-St, Corollary 2.3] shows that $\chi_c(X, G_f) \leq 3$. As X is not empty, $\chi_c(X, G_f) \neq 0$. As f is fixed point free, $\chi_c(X, G_f) \neq 1$. □

Corollary 8.2 *Let X be a 0DMC space, and $f : X \to X$ be a continuous injection. Then $\chi_c(X, G_f)$ is in $\{0, 1, 2, 3, 2^{\aleph_0}\}$, and all these values are possible with homeomorphisms of a countable metrizable compact space.*

Proof. If F_1 is not an open subset of X, then $\chi_c(X, G_f) = 2^{\aleph_0}$ by Proposition 7.2. If F_1 is an open subset of X, then we may assume that $\chi_c(X, G_f) = \chi_c(X \setminus F_1, G_f \cap (X \setminus F_1)^2)$ by Corollary 7.3. In other words, we may assume that X is not empty and f is fixed point free. It remains to apply Theorem 8.1 for the possible values.

If $X = \emptyset$, then $\chi_c(X, G_f) = 0$. If $X = 1$ and $f = \mathrm{Id}$, then $\chi_c(X, G_f) = 1$. If $X = 2$ and $f(\varepsilon) := 1 - \varepsilon$, then $c : 2 \to 2$ defined by $c(\varepsilon) := \varepsilon$ is a continuous coloring of (X, G_f), so that $\chi_c(X, G_f) = 2$. If $X = 3$ and $f(\varepsilon) := \varepsilon^+$ where $(0^+, 1^+, 2^+) = (1, 2, 0)$, then $c : 3 \to 3$ defined by $c(\varepsilon) := \varepsilon$ is a continuous coloring of (X, G_f), so that $\chi_c(X, G_f) = 3$ since $(0, 1)$, $(1, 2)$ and $(2, 0)$ are in G_f. We conclude with the example just after Corollary 7.3. □

We will now extend Theorem 8.1 to some partial injections. In order to do that, we prove a fixed point free version of the Ryll-Nardzewski theorem (see [Kn-R]). We need to emphasize one point in the [Kn-R] proof, and give the full proof for completeness.

Theorem 8.3 *Let X be a Cantor space, P, Q be closed nowhere dense subsets of X, and $h : P \to Q$ be a fixed point free homeomorphism. Then there is a fixed point free homeomorphism $h^* : X \to X$ extending h.*

Proof. We may assume that $X = 2^\omega$ and, considering $\varepsilon\alpha \mapsto (1-\varepsilon)\alpha$, that P, Q are not empty. Let $S := \{s \in 2^{<\omega} \mid N_s \cap P = \emptyset\}$, so that $2^\omega \setminus P = \bigcup_{s\in S} N_s$ since P is closed. As P is not empty and nowhere dense, P is not clopen and S is infinite. We enumerate S in the increasing order of the lengths of the finite binary sequences, which gives $\{s_i \mid i \in \omega\}$. Note that we may assume that the N_{s_i}'s are pairwise disjoint, so that the length of s_i goes to infinity.

Claim 1. (a) The sequence $\big(d(N_{s_i}, P)\big)_{i\in\omega}$ converges to zero.

(b) Let $p \in P$ and $l \in \omega$. Then there is one (and thus infinitely many) $i \in \omega$ with $2^{-l} \geq d(N_{s_i}, p)$.

Indeed, we argue by contradiction for (a), which gives $l \in \omega$ such that, for each $j \in \omega$, there is $i \geq j$ such that, for each $\alpha \in 2^\omega$ and each $\gamma \in P$, $\gamma|l \not\subseteq s_i\alpha$. This provides a strictly increasing sequence $(i_k)_{k\in\omega}$ such that, for each $k \in \omega$ and each $\gamma \in P$, $\gamma|l \not\subseteq s_{i_k}0^\infty$. Extracting a further subsequence if necessary, we may assume that $(s_{i_k}0^\infty)_{k\in\omega}$ converges to some $\delta \in 2^\omega$, by compactness. As the length of s_i goes to infinity, $\delta \notin P$. This gives j with $\delta \in N_{s_j}$. Thus N_{s_j} meets infinitely many $N_{s_{i_k}}$'s, which is the desired contradiction.

For (b), towards a contradiction, suppose that we can find $p \in P$ and $l \in \omega$. As $2^\omega \setminus P = \bigcup_{i\in\omega} N_{s_i}$ and P is nowhere dense, we can find i with $s_i 0^\infty \in N_{p|l}$. ◇

Similarly, we can find $\{t_j \mid j \in \omega\} \subseteq 2^{<\omega}$ such that $2^\omega \setminus Q$ is the disjoint union of the N_{t_j}'s, the length of t_j goes to infinity, the sequence $\big(d(N_{t_j}, Q)\big)_{j\in\omega}$ converges to zero, and, for each $q \in Q$ and each $l \in \omega$, there is $j \in \omega$ with $2^{-l} \geq d(N_{t_j}, q)$. We fix, for each i, $p_i \in P$ with $d(N_{s_i}, p_i) = d(N_{s_i}, P)$, as well as, for each j, $q_j \in Q$ with $d(N_{t_j}, q_j) = d(N_{t_j}, Q)$. Note that, by Claim 1,

(I) there is $f : \omega \to \omega$ injective such that $d(N_{s_i}, p_i) \geq d\big(N_{t_{f(i)}}, h(p_i)\big)$,

(II) there is $g : \omega \to \omega$ injective such that $d(N_{t_j}, q_j) \geq d\big(N_{s_{g(j)}}, h^{-1}(q_j)\big)$.

[Ba, Theorem 1] provides partitions (I', I''), (J', J'') of ω with $f[I'] = J'$ and $g[J''] = I''$.

Claim 2. Let $s, t \in 2^{<\omega}$. Then there is a fixed point free homeomorphism $\phi : N_s \to N_t$.

Indeed, if s, t are incompatible, we just set $\phi(s\alpha) := t\alpha$. If $s \subseteq t$, then we set

$$\phi(s\alpha) := t\big(1-\alpha(|t|-|s|)\big)\alpha(0)\ldots\alpha(|t|-|s|-1)\alpha(|t|-|s|+1)\alpha(|t|-|s|+2)\ldots$$

If $t \subsetneq s$, then we set $\phi(s\alpha) := t\alpha(1)\ldots\alpha(|s|-|t|)\big(1-\alpha(0)\big)\alpha(|s|-|t|+1)\alpha(|s|-|t|+2)\ldots$ ◇

If $i' \in I'$, Claim 2 provides a fixed point free homeomorphism $\varphi_{i'} : N_{s_{i'}} \to N_{t_{f(i')}}$. The sum of these maps provides a fixed point free homeomorphism $\varphi : U' := \bigcup_{i'\in I'} N_{s_{i'}} \to V' := \bigcup_{j'\in J'} N_{t_{j'}}$, since f is injective. Similarly, if $j'' \in J''$, Claim 2 provides a fixed point free homeomorphism $\psi_{j''} : N_{t_{j''}} \to N_{s_{g(j'')}}$. This gives a fixed point free homeomorphism

$$\psi : V'' := \bigcup_{j''\in J''} N_{t_{j''}} \to U'' := \bigcup_{i''\in I''} N_{s_{i''}},$$

by injectivity of g.

As $2^\omega \setminus P$ is the sum of U', U'' and $2^\omega \setminus Q$ is the sum of V', V'', the function $k : 2^\omega \setminus P \to 2^\omega \setminus Q$ defined by

$$k(\alpha) := \begin{cases} \varphi(\alpha) \text{ if } \alpha \in U' \\ \psi^{-1}(\alpha) \text{ if } \alpha \in U'' \end{cases}$$

is a fixed point free homeomorphism. It remains to prove that the bijection $h^* : 2^\omega \to 2^\omega$ defined by

$$h^*(\alpha) := \begin{cases} h(\alpha) \text{ if } \alpha \in P \\ k(\alpha) \text{ if } \alpha \notin P \end{cases}$$

is continuous at each point of P. So let $p \in P$, and $(p_n)_{n\in\omega} \in (2^\omega \setminus P)^\omega$ converging to p. There is i_n with $p_n \in N_{s_{i_n}}$, and the set $\{i_n \mid n \in \omega\}$ is infinite. Note that $d(N_{s_{i_n}}, p_{i_n})$ tends to 0. As the length of s_i goes to infinity, $\big(d(p_n, p_{i_n})\big)_{n\in\omega}$ converges to zero, as well as $\big(d(p, p_{i_n})\big)_{n\in\omega}$. Thus $(p_{i_n})_{n\in\omega}$ converges to p, and $\big(h(p_{i_n})\big)_{n\in\omega}$ converges to $h(p)$. Let (i'_n) be the sequence of i_n's in I'. As $d(N_{s_{i'_n}}, p_{i'_n}) \geq d\big(N_{t_{f(i'_n)}}, h(p_{i'_n})\big)$, $\Big(d\big(N_{t_{f(i'_n)}}, h(p_{i'_n})\big)\Big)_n$ tends to zero. Call p'_n the fixed element of $N_{s_{i'_n}}$, as in the notation $p_n \in N_{s_{i_n}}$. Note that the point $\varphi(p'_n) = \varphi_{i'_n}(p'_n)$ is in $N_{t_{f(i'_n)}}$. As the length of t_j goes to infinity, $\Big(d\big(\varphi(p'_n), h(p_{i'_n})\big)\Big)_n$ converges to zero, as well as $\Big(d\big(\varphi(p'_n), h(p)\big)\Big)_n$. So we proved that $\Big(d\big(\varphi(p'_n), h(p)\big)\Big)_n$ converges to zero if $p'_n \in N_{s_{i'_n}}$ and $\big(d(p'_n, p)\big)_{n\in\omega}$ converges to zero, i.e., $\Big(d\big(h^*(p'_n), h^*(p)\big)\Big)_n$ converges to zero if $\big(d(p'_n, p)\big)_{n\in\omega}$ does. Similarly, $\Big(d\big(\psi(q''_n), h^{-1}(q)\big)\Big)_n$ converges to zero if $q \in Q$, $q''_n \in N_{t_{j''_n}}$ and $\big(d(q''_n, q)\big)_{n\in\omega}$ converges to zero. Consider now the sequence (i''_n) be the sequence of i_n's in I''. Let $j''_n \in J''$ with $g(j''_n) = i''_n$. Note that there is $q''_n \in N_{t_{j''_n}}$ with $p''_n = \psi(q''_n) = \psi_{j''_n}(q''_n)$. Let $F := \{q_{j''_n} \mid n\}$. We will check that $\overline{F} \setminus F \subseteq \{h(p)\}$. Let $q \in \overline{F} \setminus F$. Assume, for the simplicity of the notation, that $(q_{j''_n})_n$ converges to q. As $\big(d(N_{t_j}, Q)\big)_{j\in\omega}$ converges to zero, $\big(d(N_{t_{j''_n}}, q_{j''_n})\big)_n$ converges to zero. As the length of t_j goes to infinity, $\big(d(q''_n, q_{j''_n})\big)_n$ converges to zero, as well as $\big(d(q''_n, q)\big)_n$, $\Big(d\big(\psi(q''_n), h^{-1}(q)\big)\Big)_n$ and $\Big(d\big(p''_n, h^{-1}(q)\big)\Big)_n$. Thus $p = h^{-1}(q)$, as desired. Thus $\big(d(q''_n, h(p)\big)_n$ converges to zero. So $\Big(d\big(\psi^{-1}(p''_n), h(p)\big)\Big)_n$ converges to zero if $\big(d(p''_n, p)\big)_n$ does, and $\Big(d\big(h^*(p''_n), h^*(p)\big)\Big)_n$ converges to zero if $\big(d(p''_n, p)\big)_n$ does. □

Corollary 8.4 *Let X be a 0DMC space, and $f : X \to X$ be a fixed point free partial continuous injection whose domain is not empty and closed. Then $\chi_c(X, G_f) \in \{2, 3\}$.*

Proof. By [K, 7.8], we may assume that $X \subseteq 2^\omega$. Note that the map $I : 2^\omega \to 2^\omega$ defined by $I(\alpha) := (0, \alpha(0), 0, \alpha(1), \ldots)$ is a homeomorphism onto its nowhere dense range. If moreover we define $g : I[\text{Domain}(f)] \to I[\text{Range}(f)]$ by $g(y) := I\Big(f\big(I^{-1}(y)\big)\Big)$, then g is a fixed point free partial continuous injection whose domain is not empty and closed, $(X, G_f) \preceq_c \big(I[X], G_g\big)$ with witness I, and thus $2 \leq \chi_c(X, G_f) \leq \chi_c\big(I[X], G_g\big)$. So we may assume that X is closed nowhere dense in 2^ω, as well as Domain(f) and Range(f). By compactness, f is a fixed point free homeomorphism from Domain(f) onto Range(f). Theorem 8.3 provides a fixed point free homeomorphism $f^* : 2^\omega \to 2^\omega$ extending f. By Theorem 8.1, $\chi_c(2^\omega, G_{f^*}) \in \{2, 3\}$. Thus $\chi_c(X, G_f) \in \{2, 3\}$. □

Remark. The conclusion of this corollary does not hold if the domain of f is open, by the example after Corollary 7.3.

Corollary 8.5 *Let X be a 0DMC space, and $f:X\to X$ be a partial continuous injection with closed domain. Then $\chi_c(X,G_f)\in\{0,1,2,3,2^{\aleph_0}\}$, and all these values are possible with homeomorphisms of a countable metrizable compact space.*

Proof. If F_1 is not an open subset of Domain(f), then $\chi_c(X,G_f)=2^{\aleph_0}$ by Proposition 7.2. If F_1 is an open subset of Domain(f), then it is a clopen subset of Domain(f). This gives an open subset O of X with $F_1=O\cap\text{Domain}(f)$. Note that F_1 and $X\setminus O$ are disjoint closed subsets of the zero-dimensional metrizable space X, which gives a clopen subset C' of X with $F_1\subseteq C'\subseteq O$, so that $F_1=C'\cap\text{Domain}(f)$. Note then that $f[F_1]=F_1$ and $f[\text{Domain}(f)\setminus F_1]$ are disjoint compact subsets of X, which gives a clopen subset C'' of X with $F_1\subseteq C''\subseteq X\setminus f[\text{Domain}(f)\setminus F_1]$. We set $C:=C'\cap C''$, so that C is also a clopen subset of X with $F_1=C\cap\text{Domain}(f)$. Note that $F:=(X\setminus C)\cap f^{-1}(X\setminus C)$ is a clopen subset of Domain(f), and thus a closed subset of X. Moreover, $G_{f_{|F}}=G_f\cap(X\setminus C)^2$. Note that $X\setminus C$ is a 0DMC space, and $f_{|F}:X\setminus C\to X\setminus C$ is a fixed point free partial continuous injection with closed domain. Corollary 8.4 provides a continuous coloring $c':X\setminus C\to 3$ of $(X\setminus C,G_f\cap(X\setminus C)^2)$. We extend c' by 0 on C, which defines $c:X\to 3$ continuous. If $f(x)\neq x$ is defined, then $x\in\text{Domain}(f)\setminus F_1\subseteq X\setminus C$. If $f(x)\in F_1$, then $f^2(x)=f(x)$, and $f(x)=x$ by injectivity of f, which is absurd, proving that $f(x)\notin F_1$. If $f(x)\in\text{Domain}(f)$, then $f(x)\in X\setminus C$ and $c(x)\neq c\big(f(x)\big)$ since $c':X\setminus C\to 3$ is a coloring of $(X\setminus C,G_f\cap(X\setminus C)^2)$. If $f(x)\notin\text{Domain}(f)$, then either $f(x)\in X\setminus C$ and $c(x)\neq c\big(f(x)\big)$ again, or $f(x)\in C\subseteq C''\subseteq X\setminus f[\text{Domain}(f)\setminus F_1]$, which is absurd. So c is a coloring of (X,G_f).

We then apply Corollary 8.2. □

Proof of Theorem 1.10. The case (b) of a closed domain comes from Corollary 8.5. In fact, if G is an arbitrary graph on X, two cases can happen. Either $\Delta(X)$ meets $\overline{G}$ in (x,x), in which case, for any countable partition $(C_i)_{i\in\omega}$ of X into clopen sets, there is i with $x\in C_i$, and G meets C_i^2, so that $\chi_c(X,G)=2^{\aleph_0}$. Or $\Delta(X)$ does not meet $\overline{G}$, in which case the compactness of X provides a finite continuous coloring of G. So $\chi_c(X,G)$ cannot be $\aleph_0$. For the values $0,1,2^{\aleph_0}$ in the open case (a), we use the proof of Corollary 8.2.

So let $1\leq n<\omega$. We set $K_n:=\{p^\infty\mid p\leq n\}\cup\{p^{j+1}m^\infty\mid p\neq m\leq n\wedge j\in\omega\}$ and Domain$(f_n):=K_n\setminus\{p^\infty\mid p\leq n\}$, so that Domain$(f_n)$ is an open subset of the countable metrizable compact space K_n. We set $(0^+,1^+,\dots,n^+):=(1,2,\dots,n,0)$ and, for $p\leq n$ and $r<n$, $p^{+^{r+1}}:=p^+$ if $r=0$, $p^{+^{r+1}}:=(p^{+^r})^+$ if $r>0$. If $\alpha\in$ Domain(f_n), then we can find $p\neq m\leq n$, $q\in\omega$, and $r<n$ with $\alpha=p^{nq+r+1}m^\infty$. We then set $f_n(\alpha):=(p^{+^{r+1}})^{nq+r+1}(m^{+^{r+1}})^\infty$. Note that f_n takes values in Domain(f_n), is continuous, and $f_n^{n+1}=\text{Id}$. In particular, f_n is a bijection and $f_n^{-1}=f_n^n$ is continuous, so that f_n is a homeomorphism. The map $\alpha\mapsto\alpha(0)$ is a continuous $(n+1)$-coloring of (K_n,G_{f_n}) (in fact, f_n is fixed point free). If $c:K_n\to n$ is continuous, then we can find $p<l\leq n$ with $c(p^\infty)=c(l^\infty)=:i$. This gives $q\in\omega$ with $N_{p^q}\cup N_{l^q}\subseteq c^{-1}(\{i\})$. Let $r<n$ with $p^{+^{r+1}}=l$. Then $\big(p^{nq+r+1}(p+1)^\infty,(p^{+^{r+1}})^{nq+r+1}\big((p+1)^{+^{r+1}}\big)^\infty\big)\in G_{f_n}\cap\big(c^{-1}(\{i\})\big)^2$, so that the function c is not a coloring of (K_n,G_{f_n}). Thus $\chi_c(K_n,G_{f_n})=n+1$.

We then set $X:=\oplus_{n\geq 1}K_n$, Domain$(f):=\oplus_{n\geq 1}$ Domain(f_n), and $f(n,\alpha):=\big(n,f_n(\alpha)\big)$. Then X is a countable Polish space X, Domain(f) is an open subset of X, and f is a fixed point free partial homeomorphism from Domain(f) onto it. As $(K_n,G_{f_n})\preceq_c(X,G_f)$, $\chi_c(X,G_f)\geq\aleph_0$. The map $(n,\alpha)\mapsto\alpha(0)$ is an $\aleph_0$-coloring of (X,G_f), so that $\chi_c(X,G_f)=\aleph_0$. □

In the case of spaces which are not compact, the first space to look at is ω.

Corollary 8.6 *Let $f:\omega\to\omega$ be a partial (continuous) injection. Then $\chi_c(\omega,G_f)\in\{1,2,3\}$, and all these values are possible.*

Proof. Let $n\in\omega$. If there is $p\in\omega$ with $f^p(n)=n$, we take it minimal, in which case the orbit of n is $\{f^i(n)\mid 0\le i<p\}$. If p is odd, then we set $c\big(f^i(n)\big):=0$ if $i<p-1$ is even, $c\big(f^i(n)\big):=1$ if $i<p$ is odd, $c\big(f^{p-1}(n)\big):=2$, so that c is a coloring of G_f on the orbit. If p is even, then we set $c\big(f^i(n)\big):=0$ if $i<p$ is even, $c\big(f^i(n)\big):=1$ if $i<p$ is odd, so that c is a coloring of G_f on the orbit. If there is no $p\in\omega$ with $f^p(n)=n$, then we can find ordinals $\xi,\eta\le\omega$ such that the orbit of n is $\{f^i(n)\mid -\xi<i<\eta\}$. We set $c\big(f^i(n)\big):=0$ if i is even, $c\big(f^i(n)\big):=1$ if i is odd, so that c is a coloring of G_f on the orbit. We defined a continuous coloring $c:\omega\to 3$ of (ω,G_f) since ω is discrete. So $\chi_c(X,G_f)\le 3$, and $\chi_c(X,G_f)\ne 0$ since ω is not empty.

If $f=\mathrm{Id}$, then $\chi_c(X,G_f)=1$. If $f(2n+\varepsilon):=2n+(1-\varepsilon)$, then $\chi_c(X,G_f)=2$ by Proposition 7.5. If $f(3n+\varepsilon):=3n+\varepsilon^+$, where $(0^+,1^+,2^+)=(1,2,0)$, then $c:\omega\to 3$ defined by $c(3n+\varepsilon):=\varepsilon$ is a continuous coloring of (X,G_f), so that $\chi_c(X,G_f)=3$ since $(0,1),(1,2),(2,0)\in G_f$. □

The next natural space to look at is the Baire space ω^ω.

9 Graphs induced by a function and odometers

We now study graphs induced by a homeomorphism of an uncountable 0DMC space.

Remark. We set, for $\mathbf{d}\in\mathfrak{D}$, $X_o:=\overline{\mathrm{proj}[\mathbb{G}_o]}^{\mathcal{K}_{\mathbf{d}}}$, so that X_o is a 0DMC space. This space is $X_o=\mathrm{proj}[\mathbb{G}_o]\cup\{c^\infty\}\cup\mathcal{C}$, so that $\mathrm{proj}[\mathbb{G}_o]=X_o\setminus(\{c^\infty\}\cup\mathcal{C})$ is a countable open subset of X_o. We define $f_o:\mathrm{proj}[\mathbb{G}_o]\to\mathrm{proj}[\mathbb{G}_o]$ by $f_o(\alpha):=$ the unique $\beta\in\mathrm{proj}[\mathbb{G}_o]$ with $(\alpha,\beta)\in\mathbb{G}_o$, so that f_o is a fixed point free involution, and $\mathbb{G}_o=\mathrm{Graph}(f_o)=G_{f_o}$. As $\mathrm{proj}[\mathbb{G}_o]$ is discrete, f_o is continuous, and thus a homeomorphism. By Proposition 6.1, $\chi_c(X_o,G_{f_o})=3$.

The proof of Theorem 1.5 provides a $\preceq_c$-antichain made up of $\preceq_c^i$-minimal graphs in the class of graphs induced by a partial fixed point free continuous involution with countable open domain on a 0DMC space with CCN at least three. In particular, any $\preceq_c^i$-basis for this class must have size continuum, as announced in the introduction.

We now turn to the proof of Theorem 1.13, i.e., we study the G_o's instead of the $\mathbb{G}_o$'s.

Notation. We set $\mathcal{O}:=\{\mathbf{d}\in\mathfrak{C}\mid\forall j\in\omega\ \ d_j\text{ is odd}\}$.

Proposition 9.1 *Let $\mathbf{d}=(d_j)_{j\in\omega}\in\mathfrak{C}$. Then $\chi_c(\mathcal{C},G_o)=3$ if $\mathbf{d}\in\mathcal{O}$, $\chi_c(\mathcal{C},G_o)=2$ otherwise.*

Proof. The key remark is that $o^{\pi_{j<l}\ d_j}(0^\infty)\in N_{0^l}$ for each $l\in\omega$. Assume that there is $j_0\in\omega$ such that d_{j_0} is even. We define $c:\mathcal{C}\to 2$ by $c(\alpha):=\mathrm{parity}(i)$ if $i<\pi_{j\le j_0}\ d_j$ and $o^i(0^{j_0+1})\subseteq\alpha$. Then c is continuous, and a coloring of $(\mathcal{C},G_o)$ by the key remark and the fact that $\pi_{j\le j_0}\ d_j$ is even. Conversely, assume that there is a coloring $c':\mathcal{C}\to 2$ of $(\mathcal{C},G_o)$. Let $\varepsilon:=c'(0^\infty)$, and also $C:=(c')^{-1}(\{\varepsilon\})$. As C is a clopen subset of $\mathcal{C}$, there is $l_0\in\omega$ with $N_{0^{l_0}}\subseteq C$. An induction shows that $o^i(0^\infty)\in C$ if i is even, $o^i(0^\infty)\notin C$ if i is odd since c' is a coloring of $(\mathcal{C},G_o)$. The key remark shows that $\pi_{j<l}\ d_j$ is even if $l\ge l_0$. This gives $j\in\omega$ such that d_j is even. This shows that $\chi_c(\mathcal{C},G_o)\ge 3$ if $\mathbf{d}\in\mathcal{O}$. Let $\mathbf{d}\in\mathcal{O}$. We define $c'':\mathcal{C}\to 3$ by $c''(\alpha):=\mathrm{parity}\big(\alpha(0)\big)$ if $\alpha(0)<d_0-1$ and $c''(\alpha):=2$ if $\alpha(0)=d_0-1$. Then c'' is continuous, and a coloring of $(\mathcal{C},G_o)$. □

Proposition 9.1 implies that we can apply Lemma 7.10 to $(X, G_f) = (\mathcal{C}, G_o)$ if $\mathbf{d} = (d_j)_{j\in\omega} \in \mathcal{O}$. It is also important to assume that (X, G_f) has CCN at least three in Lemma 7.10. Indeed, if $\mathbf{d} \in \mathfrak{C}\setminus\mathcal{O}$, then $\mathcal{C}_\mathbf{d}$ is a 0DMC space, $o_\mathbf{d}$ is a minimal homeomorphism with $\chi_c(\mathcal{C}_\mathbf{d}, G_{o_\mathbf{d}}) \geq 2$ by Proposition 9.1, and the strict inequality $(2, G_{\varepsilon\mapsto 1-\varepsilon}) \prec_c^i (\mathcal{C}_\mathbf{d}, G_{o_\mathbf{d}})$ holds.

A consequence of Lemma 7.11 is the existence, announced in the introduction, of $\preceq_c^i$-basis in the case of equicontinuity.

Definition 9.2 *We say that a dynamical system (X, f), where a compatible metric d on X is fixed, is* equicontinuous *if* $\forall x \in X \;\; \forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall y \in B_d(x, \delta) \;\; \forall n \in \omega \;\; d\big(f^n(x), f^n(y)\big) < \varepsilon$.

This means that the family $(f^n)_{n\in\omega}$ is equicontinuous. For instance, if $\mathbf{d} \in (\omega\setminus 2)^\omega$, then $(\mathcal{C}, o)$ is equicontinuous (see [Ku, 4.1.2]). We set

$$\mathfrak{G}_2^e := \{(X, G_f) \in \mathfrak{G}_2 \mid (X, f) \text{ is equicontinuous} \wedge \exists x \in X \;\; \mathrm{Orb}_f^+(x) \text{ is dense infinite}\}.$$

Proposition 9.3 *(a) $\{(\mathcal{C}, G_o) \mid \mathbf{d} \in (\omega\setminus 2)^\omega \cap \mathcal{O}\}$ is a $\preceq_c^i$-basis for $\mathfrak{G}_2^e$.*
(b) Under the axiom of choice, there is a $\preceq_c^i$-antichain basis for $\mathfrak{G}_2^e$.

Proof. (a) Let $(X, G_f) \in \mathfrak{G}_2^e$. [Ku, Theorem 2.9, Corollary 2.34, and Section 4.1 (in particular Theorem 4.4)] provide $\mathbf{d} \in \mathfrak{C}$ such that $(\mathcal{C}, o)$ is conjugate to (X, f). This gives a homeomorphism $\varphi : \mathcal{C} \to X$ with $\varphi \circ o = f \circ \varphi$. If $\beta = o(\alpha)$, then $\varphi(\beta) = f\big(\varphi(\alpha)\big)$, so that $(\mathcal{C}, G_o) \preceq_c^i (X, G_f)$. Similarly, $(X, G_f) \preceq_c^i (\mathcal{C}, G_o)$, so that $\chi_c(\mathcal{C}, G_o) \geq \chi_c(X, G_f) \geq 3$ since $(X, G_f) \in \mathfrak{G}_2$. Proposition 9.1 then implies that $\mathbf{d} \in \mathcal{O}$.

(b) Lemma 7.11 implies that $\preceq_c^i$ is an equivalence relation on $\{(\mathcal{C}, G_o) \mid \mathbf{d} \in (\omega\setminus 2)^\omega \cap \mathcal{O}\}$. Using the axiom of choice, we can pick an element in each equivalence class, which provides the desired $\preceq_c^i$-antichain basis. □

We now get a $\preceq_c$-antichain in the style of Theorem 6.6.

Theorem 9.4 *There is a map $\Phi : 2^\omega \to \mathcal{O}$ such that $(\mathcal{C}_{\Phi(\alpha)}, G_{o_{\Phi(\alpha)}}) \not\preceq_c (\mathcal{C}_{\Phi(\beta)}, G_{o_{\Phi(\beta)}})$ if $\alpha \neq \beta$.*

Proof. Let $(p_n)_{n\in\omega}$ be the sequence of prime numbers. We define, for each $\alpha \in 2^\omega$, $S_\alpha \subseteq \omega$ by $S_\alpha := \{p_0^{\alpha(0)+1} \dots p_n^{\alpha(n)+1} \mid n \in \omega\}$. Note that S_α is infinite, and $S_\alpha \cap S_\beta$ is finite if $\alpha \neq \beta$. In this proof, we consider $(d_\alpha)_j = 3$ if $j \notin S_\alpha$, $(d_\alpha)_j = p_{j+1}$ if $j \in S_\alpha$, so that $\Phi(\alpha) := \mathbf{d}_\alpha \in \mathcal{O}$ is unbounded, the $(d_\alpha)_j$'s are prime, and $(d_\beta)_l$ is not in $\{(d_\alpha)_j \mid j \in \omega\}$ if $\alpha \neq \beta$, $(d_\beta)_l \neq 3$ and l is large enough.

If $(\mathcal{C}_{\Phi(\alpha)}, G_{o_{\Phi(\alpha)}}) \preceq_c (\mathcal{C}_{\Phi(\beta)}, G_{o_{\Phi(\beta)}})$ with witness φ, then we set $V := \varphi[\mathcal{C}_{\Phi(\alpha)}]$ and

$$E := (\varphi \times \varphi)[G_{o_{\Phi(\alpha)}}],$$

so that V is a compact subset of $\mathcal{C}_{\Phi(\beta)}$ and $E \subseteq G_{o_{\Phi(\beta)}}$ is a compact graph on V with $\chi_c(V, E) \geq 3$, by Proposition 9.1. Claim 1 in the proof of Lemma 7.10 shows that $V = \mathcal{C}_{\Phi(\beta)}$, so that φ is onto, which contradicts Lemma 6.5. □

Remark. By Lemma 7.11, the $o_{\Phi(\alpha)}$'s involved in Theorem 9.4 are pairwise not flip-conjugate.

Proof of Theorem 1.13. Theorem 9.4 provides a map $\Phi : 2^\omega \to \mathcal{O}$. We now can set $\mathcal{C}_\alpha := \mathcal{C}_{\Phi(\alpha)}$ and $f_\alpha := o_{\Phi(\alpha)}$, and we are done by Proposition 9.1 and Lemma 7.10. □

We now turn to the version of Theorem 1.16(b). In fact, we prove something stronger since it is possible to consider always the same space, with restrictions of the graph induced by a fixed odometer to different countable dense subsets.

Notation. Fix $\mathbf{d} := 3^\infty$, so that $\mathbf{d} \in \mathcal{O}$ and $\chi_c(3^\omega, G_o) = 3$, by Proposition 9.1. We set, for $l \in \omega$, $i_l := 3^{l+2}$, so that $0^{l+2} \subseteq o^{i_l}(0^\infty)$. We set, for $A \subseteq \omega$ infinite, $S_A := \{0\} \cup \{i_l + i \mid l \in A \wedge i < 3^l\}$, so that $S_A \subseteq \omega$ is infinite and contains arbitrarily large intervals appearing in the definition of $\mathbb{G}_o$.

Lemma 9.5 *The map $A \mapsto S_A$ is injective. Moreover, $A \subseteq B$ is equivalent to $S_A \subseteq S_B$.*

Proof. This comes from the fact that $i_l + 3^l - 1 < i_{l+1}$ for each $l \in \omega$. □

We also set, for $S \subseteq \omega$, $D_S := \{o^i(0^\infty) \mid i \in S\}$.

Lemma 9.6 *The graphs $(3^\omega, G_{o_{|D_{S_A}}})$ are countable and have CCN three, for each $A \subseteq \omega$ infinite.*

Proof. Fix $A \subseteq \omega$ infinite. Let us prove that D_{S_A} is dense in 3^ω. Let $t \in 3^{<\omega}$, and $i < 3^{|t|}$ with $t \subseteq o^i(0^\infty)$. We choose $l' > |t|$ with $l' \in A$. Then $i_{l'} + i \in S_A$ and $t \subseteq o^{i_{l'}+i}(0^\infty) \in D_{S_A}$. It remains to prove that $\chi_c(3^\omega, G_{o_{|D}}) \geq 3$ if D is dense in 3^ω. Towards a contradiction, suppose that there is a clopen subset C of 3^ω with $G_{o_{|D}} \cap \big(C^2 \cup (3^\omega \setminus C)^2\big) = \emptyset$. As $\chi_c(3^\omega, G_o) = 3$, we may assume that $G_o \cap C^2 \neq \emptyset$, which implies that $C \cap o^{-1}(C)$ is not empty. The density of D gives $\alpha \in D \cap C \cap o^{-1}(C)$. Then $\big(\alpha, o(\alpha)\big) \in G_{o_{|D}} \cap C^2$ since o is fixed point free, which is the desired contradiction. □

One can prove that if $A, B \subseteq \omega$ are infinite, then $(3^\omega, G_{o_{|D_{S_A}}}) \preceq_c^i (3^\omega, G_{o_{|D_{S_B}}})$ is equivalent to $A \subseteq B$. But we will prove a better result.

Lemma 9.7 *We can find a sequence $(S_q)_{q \in \omega}$ of pairwise disjoint infinite subsets of ω such that, for any $l \in \omega$, $p \neq q$, $3l < r \in S_p$ and $s \in S_q$, $|r - s| > l$.*

Proof. Fix a bijection $b : \omega^2 \to \omega$, for instance $b(q, j) := \frac{(q+j)(q+j+1)}{2} + j$. We set $r_j^q := 2^{b(q,j)}$ and $S_q := \{r_j^q \mid j \in \omega\}$. Then $(S_q)_{q \in \omega}$ is a sequence of pairwise disjoint infinite subsets of ω. Fix $l \in \omega$, $p \neq q$, and assume that $r = r_i^p \in S_p$ and $s = r_j^q \in S_q$. Note that

$$r - s = 2^{b(p,i)} - 2^{b(q,j)} = \begin{cases} 2^{b(p,i)}(1 - 2^{b(q,j)-b(p,i)}) \text{ if } b(p,i) > b(q,j), \\ \\ 2^{b(q,j)}(2^{b(p,i)-b(q,j)} - 1) \text{ if } b(p,i) < b(q,j). \end{cases}$$

The first term is at least $2^{b(p,i)-1}$, and is bigger than l if $r > 2l$. If $2^{b(q,j)-1} > l$, then the second term is smaller than $-l$ and we are done. If $2^{b(q,j)-1} \leq l$, then the second term is at least $2^{b(p,i)} - 2l$, and is bigger than l if $r > 3l$. □

Proof of Theorem 1.16(b). Let $\psi : \omega \to 3^{<\omega}$ be the bijection defined by the length and o: $\emptyset$ for the length 0, 0, 1, 2 for the length 1, 0^2, 10, 20, 01, 1^2, 21, 02, 12, 2^2 for the length 2, ... Note that $|\psi(k)| \leq k$. Recall the sets $S_q := \{r_j^q \mid j \in \omega\}$ given by Lemma 9.7. If $s = \psi(k) \in 3^{<\omega}$, then there is $i'_k < 3^k$ with $s \subseteq o^{i'_k}(0^\infty)$, and thus $s \subseteq o^{i_{r_k^0} + i'_k}(0^\infty)$. Recall that $D_S := \{o^i(0^\infty) \mid i \in S\}$ if $S \subseteq \omega$, so that D_S is countable, and $G_{o_{|D_S}}$ is a countable graph on 3^ω. We proved that D_S is dense in 3^ω if S contains $\{i_{r_k^0} + i'_k \mid k \in \omega\}$. In this case, $G_{o_{|D_S}}$ is dense in G_o, and thus $\chi_c(3^\omega, G_{o_{|D_S}}) = 3$ by Proposition 9.1 and Lemma 7.9. We set, for $A \subseteq \omega$, $S_A := \{i_{r_k^0} + i'_k \mid k \in \omega\} \cup \{i_{r_k^{n+1}} \mid n \in A \wedge k \in \omega\}$, so that $\chi_c(3^\omega, G_{o_{|D_{S_A}}}) = 3$ and $(3^\omega, G_{o_{|D_{S_A}}}) \preceq_c^i (3^\omega, G_{o_{|D_{S_B}}})$ if $A \subseteq B$.

Assume now that $n \in A \setminus B$ and, towards a contradiction, that $(3^\omega, G_{o_{|D_{S_A}}}) \preceq_c (3^\omega, G_{o_{|D_{S_B}}})$ with witness φ. If $\alpha \in D_{S_A}$, then $\big(\alpha, o(\alpha)\big) \in G_{o_{|D_{S_A}}}$, so that $\Big(\varphi(\alpha), \varphi\big(o(\alpha)\big)\Big) \in G_{o_{|D_{S_B}}}$. Thus $\varphi\big(o(\alpha)\big) = o^{\pm 1}\big(\varphi(\alpha)\big)$. As the set $\{\alpha \in 3^\omega \mid \varphi\big(o(\alpha)\big) = o^{\pm 1}\big(\varphi(\alpha)\big)\}$ is closed and contains the dense set D_{S_A}, $\varphi\big(o(\alpha)\big) = o^{\pm 1}\big(\varphi(\alpha)\big)$ holds for each $\alpha \in 3^\omega$. In particular, $\varphi[\{o^i(0^\infty) \mid i \in \omega\}] \subseteq \mathrm{Orb}_o(0^\infty)$.

As φ is uniformly continuous, there is $l \in \omega$ such that $\varphi(\alpha)|1 = \varphi(\beta)|1$ if $\alpha|l = \beta|l$. As we work with the odometer on 3^ω, $o\big(\varphi(\alpha)\big)|1 \neq o^{-1}\big(\varphi(\alpha)\big)|1$, and thus $\varphi\big(o(\alpha)\big) = o\big(\varphi(\alpha)\big)$ is equivalent to $\varphi\big(o(\alpha)\big)|1 = o\big(\varphi(\alpha)\big)|1$. The previous discussion allows us to define, for each natural number $r \leq 3^l$, $f(r) \in \mathbb{Z}$ with $\varphi\big(o^r(0^\infty)\big) = o^{f(r)}(0^\infty)$. Note that $f(r+1) = f(r) \pm 1$ if $r < 3^l$, so that

$$f(0) - 3^l < f(r) < f(0) + 3^l.$$

We set $d := \mathrm{Card}(\{r < 3^l \mid f(r+1) = f(r)+1\}) - \mathrm{Card}(\{r < 3^l \mid f(r+1) = f(r)-1\})$, so that $d \in \mathbb{Z} \setminus \{0\}$ and $-3^l \leq d \leq 3^l$. Note that any natural number i has a unique decomposition $3^l q + r$, where $q \in \omega$ and $r < 3^l$. The previous discussion shows that $\varphi\big(o^i(0^\infty)\big) = o^{dq + f(r)}(0^\infty)$. We apply this to $i_{r_k^{n+1}} \in S_A$, where k is large enough to ensure that $r_k^{n+1} + 2 \geq l$, so that $\varphi\big(o^{i_{r_k^{n+1}}}(0^\infty)\big) = o^{d3^{r_k^{n+1}+2-l} + f(0)}(0^\infty)$. The previous discussion shows the existence of $i \in S_B$, say $i_{r_{j_k}^{n_k}} + i'_{j_k}$, and $\varepsilon \in 2$ with the property that $d3^{r_k^{n+1}+2-l} + f(0) = i + \varepsilon$. In particular, taking k large enough, we see that $d \geq 1$. Moreover,

$$\frac{d}{3^l} + \frac{f(0)}{3^{r_k^{n+1}+2}} = 3^{r_{j_k}^{n_k} - r_k^{n+1}} + \frac{i'_{j_k} + \varepsilon}{3^{r_k^{n+1}+2}},$$

showing that $r_{j_k}^{n_k} \leq r_k^{n+1}$ if k is large enough. Similarly, $d3^{r_k^{n+1} - r_{j_k}^{n_k} - l} + \frac{f(0)}{3^{r_{j_k}^{n_k}+2}} = 1 + \frac{i'_{j_k} + \varepsilon}{3^{r_{j_k}^{n_k}+2}}$, showing that $r_k^{n+1} \leq r_{j_k}^{n_k} + l$ if k is large enough. This shows that $0 \leq r_k^{n+1} - r_{j_k}^{n_k} \leq l$. As $i \in S_B$, $n_k = 0$ or $n_k - 1 \in B$, showing that $n+1 \neq n_k$ since $n \notin B$. It remains to apply Lemma 9.7 to $p := n+1$, $q := n_k$, $r := r_k^{n+1}$ with k large enough so that $r_k^{n+1} > 3l$, and $s := r_{j_k}^{n_k}$ to get the desired contradiction. So we proved that $(3^\omega, G_{o_{|D_{S_A}}}) \not\preceq_c (3^\omega, G_{o_{|D_{S_B}}})$ if $A \not\subseteq B$.

It remains to check that the map $A \mapsto G_{o_{|D_{S_A}}}$ is injective. First, the map $A \mapsto S_A$ is injective by Lemma 9.5. Then the map $S \mapsto D_S$ from $\mathcal{P}(\omega) \equiv 2^\omega$ into $2^{\mathrm{Orb}_o(0^\infty)}$ is injective. Finally, the map $D \mapsto G_{o_{|D}}$ from $2^{\mathrm{Orb}_o(0^\infty)}$ into $2^{\mathrm{Orb}_o(0^\infty)^2}$ is injective by minimality of o. $\square$

Proof of Theorem 1.9. Let o be the odometer on 3^ω, and recall S_ω defined before Lemma 9.5. We set $\mathbb{G} := G_{o_{|D_{S_\omega}}}$. By Lemma 9.6, the graph $(3^\omega, \mathbb{G})$ is countable and has CCN three, and is therefore in $\mathfrak{K}$. Let (K, G) in $\mathfrak{K}$ satisfying $(K, G) \preceq_c^i (3^\omega, \mathbb{G})$, with witness φ. We set $V := \varphi[K]$ and $E := (\varphi \times \varphi)[G]$, so that V is a compact subset of 3^ω and $E \subseteq \mathbb{G}$ is a graph on V. Also $(K, G) \preceq_c^i (V, E)$ with witness φ, so that $\chi_c(V, E) = 3$. Claim 1 in the proof of Lemma 7.10 shows that $V = 3^\omega$. Note that $(3^\omega, E) \preceq_c^i (K, G)$ with witness φ^{-1}. So it is enough to find a $\preceq_c^i$-antichain $\big((3^\omega, G_\alpha)\big)_{\alpha \in 2^\omega}$ of graphs with CCN three and $\preceq_c^i$-below $(3^\omega, E)$, by Lemma 4.1. We first inductively construct a sequence $(\alpha_n)_{n \in \omega}$ of points of D_{S_ω} satisfying the following:

(1) $\forall \varepsilon \in 2$, $E_\varepsilon := \bigsqcup_{p \in \omega} \big\{\big(\alpha_{2p+\varepsilon}, o(\alpha_{2p+\varepsilon})\big), \big(o(\alpha_{2p+\varepsilon}), \alpha_{2p+\varepsilon}\big)\big\}$ is dense in E,
(2) $E_0 \cap E_1 = \emptyset$,
(3) $\alpha_n = o^{i_{l_n} + i'_n}(0^\infty)$,
(4) $(l_n)_{n \in \omega}$ is injective.

Lemma 7.9 implies that E is dense in G_o. As o is a homeomorphism of the perfect (= without isolated point) space $3^\omega \neq \emptyset$, $G_o, E \neq \emptyset$ are also perfect. Let $(B_q)_{q\in\omega}$ be a basis for the topology of E made up of sets which are not empty. We first choose $(\beta_0, \gamma_0) \in B_0$. As $E \subseteq G_o$, either $\gamma_0 = o(\beta_0)$, or $\beta_0 = o(\gamma_0)$. The point α_0 is β_0 in the first case, γ_0 in the second one, so that $\big\{\big(\alpha_0, o(\alpha_0)\big), \big(o(\alpha_0), \alpha_0\big)\big\} \subseteq E$ since E is symmetric, and $\alpha_0 \in D_{S_\omega}$ since o is minimal. As $B_0 \neq \emptyset$ is perfect, we can then choose $(\beta_1, \gamma_1) \in B_0 \setminus \big\{\big(\alpha_0, o(\alpha_0)\big), \big(o(\alpha_0), \alpha_0\big)\big\}$. Here again, either $\gamma_1 = o(\beta_1)$, or $\beta_1 = o(\gamma_1)$. The point α_1 is β_1 in the first case, γ_1 in the second one, so that

$$\big\{\big(\alpha_1, o(\alpha_1)\big), \big(o(\alpha_1), \alpha_1\big)\big\} \subseteq E \setminus \big\{\big(\alpha_0, o(\alpha_0)\big), \big(o(\alpha_0), \alpha_0\big)\big\}$$

and $\alpha_1 \in D_{S_\omega}$. It remains to iterate this construction in the other B_q's. At this point, we only ensured (1) and (2). As $i < 3^l$ in the definition of S_ω, we can also ensure (3) and (4).

We ensured that the graph E_0 is dense in G_o, as well as any $E_0 \cup D$ if $D \subseteq E_1$ is a graph. Lemma 7.9 implies that $\chi_c(3^\omega, E_0 \cup D) = 3$. Let $(p_n)_{n\in\omega}$ be the sequence of prime numbers. We define, for each $\alpha \in 2^\omega$, $S_\alpha \subseteq \omega$ by $S_\alpha := \{p_0^{\alpha(0)+1} \dots p_n^{\alpha(n)+1} \mid n \in \omega\}$. Note that S_α is infinite, and $S_\alpha \cap S_\beta$ is finite if $\alpha \neq \beta$. We then set $D_\alpha := \bigcup_{p\in S_\alpha} \big\{\big(\alpha_{2p+1}, o(\alpha_{2p+1})\big), \big(o(\alpha_{2p+1}), \alpha_{2p+1}\big)\big\}$, so that $D_\alpha \subseteq E_1$ is a graph. We put $G_\alpha := E_0 \cup D_\alpha$, so that G_α is a graph with CCN three and $\preceq_c^i$-below $(3^\omega, E)$.

It remains to see that $\big((3^\omega, G_\alpha)\big)_{\alpha\in 2^\omega}$ is a $\preceq_c^i$-antichain. So let $\alpha \neq \beta \in 2^\omega$, and assume, towards a contradiction, that $(3^\omega, G_\alpha) \preceq_c^i (3^\omega, G_\beta)$ with witness ψ. We set

$$D_0 := \{\alpha_{2k} \mid k \in \omega\} \cup \{\alpha_{2k+1} \mid k \in S_\alpha\}.$$

Claim. D_0 is dense in 3^ω.

Indeed, let $\emptyset \neq s \in 3^{<\omega}$. Note that G_o meets $N_s \times N_{o(s)}$. As E_0 is dense in G_o,

$$G_\alpha = G_{o_{|D_0}} = \text{Graph}(o_{|D_0}) \cup \text{Graph}(o_{|D_0})^{-1}$$

also meets this clopen set. As o is the odometer on 3^ω, $\text{Graph}(o_{|D_0})$ meets this clopen set, and D_0 meets N_s. $\diamond$

If $x \in D_0$, then $\big(x, o(x)\big) \in G_\alpha$, so that $\Big(\psi(x), \psi\big(o(x)\big)\Big) \in G_\beta$, and either $\psi\big(o(x)\big) = o\big(\psi(x)\big)$, or $\psi\big(o(x)\big) = o^{-1}\big(\psi(x)\big)$. This leads to define $P := \{x \in 3^\omega \mid \psi\big(o(x)\big) = o\big(\psi(x)\big)\}$ and

$$M := \{x \in 3^\omega \mid \psi\big(o(x)\big) = o^{-1}\big(\psi(x)\big)\}.$$

Note that P, M are closed, and disjoint by minimality of o. Moreover, $D_0 \subseteq P \cup M$, so that $3^\omega = P \cup M$ by the claim. If $y \in P$, then $o(y) \in P$ by injectivity of ψ and minimality of o, and similarly with M. In particular, the dense set $\{o^i(0^\infty) \mid i \in \omega\}$ is contained in either P, or M. Thus $P = 3^\omega$ or $M = 3^\omega$, which means that $\psi \circ o = o \circ \psi$ or $\psi \circ o = o^{-1} \circ \psi$.

Fix $p \in S_\alpha \setminus S_\beta$. If $\psi \circ o = o^{-1} \circ \psi$, then there is $n \in \omega$ with $\psi(\alpha_{2p+1}) = o(\alpha_n)$ by minimality of o. In particular, $\psi\big(o^{i_{l_{2p+1}} + i'_{2p+1}}(0^\infty)\big) = o^{i_{l_n} + i'_n + 1}(0^\infty)$. If $i \in \omega$, then

$$\psi\big(o^{i_{l_{2p+1}} + i'_{2p+1} + i}(0^\infty)\big) = o^{i_{l_n} + i'_n + 1 - i}(0^\infty).$$

If we choose i large enough so that $o^{i_{l_{2p+1}} + i'_{2p+1} + i}(0^\infty) \in \{\alpha_{2k} \mid k \in \omega\} \cup \{\alpha_{2k+1} \mid k \in S_\alpha\}$ and $i_{l_n} + i'_n + 1 - i < 0$, then we get a contradiction with the minimality of o. This shows that $\psi \circ o = o \circ \psi$. The minimality of o provides $n \in \omega$ with $\psi(\alpha_{2p+1}) = \alpha_n$, and $n \neq 2p+1$ since $p \notin S_\beta$.

In particular, $\psi\big(o^{i_{l_{2p+1}}+i'_{2p+1}}(0^\infty)\big) = o^{i_{l_n}+i'_n}(0^\infty)$. If $i\in\omega$, then

$$\psi\big(o^{i_{l_{2p+1}}+i'_{2p+1}+i}(0^\infty)\big) = o^{i_{l_n}+i'_n+i}(0^\infty).$$

Applying this to $i = i_{l_{2q+1}}+i'_{2q+1}-i_{l_{2p+1}}-i'_{2p+1}\geq 0$, we get

$$\psi\big(o^{i_{l_{2q+1}}+i'_{2q+1}}(0^\infty)\big) = o^{i_{l_n}+i'_n+i_{l_{2q+1}}+i'_{2q+1}-i_{l_{2p+1}}-i'_{2p+1}}(0^\infty),$$

which has to be of the form $o^{i_{l_m}+i'_m}(0^\infty)$ if $q\in S_\alpha$. In particular,

$$3^{l_{2q+1}+2}+i'_{2q+1}+i_{l_n}+i'_n-i_{l_{2p+1}}-i'_{2p+1}=3^{l_m+2}+i'_m$$

and $3^{l_{2q+1}-l_m}+\frac{i'_{2q+1}+i_{l_n}+i'_n-i_{l_{2p+1}}-i'_{2p+1}}{3^{l_m+2}}=1+\frac{i'_m}{3^{l_m+2}}<2$, showing that $l_{2q+1}\leq l_m$ if q is large enough. Also, $1+\frac{i'_{2q+1}+i_{l_n}+i'_n-i_{l_{2p+1}}-i'_{2p+1}}{3^{l_{2q+1}+2}}=3^{l_m-l_{2q+1}}+\frac{i'_m}{3^{l_{2q+1}+2}}<2$, showing that $l_m\leq l_{2q+1}$ if q is large enough. Thus $m=2q+1$ if q is large enough, by (4), and $i'_m=i'_{2q+1}$ if q is large enough. This implies that $n=2p+1$, which is the desired contradiction finishing the proof. □

Remark. In fact, $\mathbb{G}$ and the G_α's are induced by a partial homeomorphism with countable domain, so that there is no $\preceq^i_c$-antichain basis for the class of graphs on a 0DMC space induced by a partial homeomorphism with countable domain with CCN at least three.

10 Subshifts

We now prove a version of Theorem 1.6 for graphs induced by a homeomorphism, as announced in the introduction. The proof of Theorem 1.11 will provide descending chains of graphs of uncountable CCN, and here we get CCN three. We consider subshifts, which are widely studied particular dynamical systems. We refer to the book [Ku] for basic notions and definitions.

Definition 10.1 *(a) An* alphabet *is a finite set of cardinality at least two.*
(b) Let A be an alphabet, and $\mathbb{X}\in\{\mathbb{Z},\omega\}$. The shift map $\sigma : A^{\mathbb{X}}\to A^{\mathbb{X}}$ *is defined by the formula* $\sigma(\alpha)(k):=\alpha(k+1)$.

Recall that the sets of the form $[w]_p:=\{\beta\in 2^{\mathbb{Z}} \mid \forall j<|w|\ \ w(j)=\beta(p+j)\}$, where $w\in 2^{<\omega}$ and $p\in\mathbb{Z}$, form a basis made up of clopen subsets of the space $2^{\mathbb{Z}}$, which is therefore homeomorphic to 2^ω. If $\mathbb{X}=\mathbb{Z}$, then the shift map is a homeomorphism, so that $(A^{\mathbb{Z}},\sigma)$ is a Cantor dynamical system. Corollary 7.3 shows that the fixed points of a homeomorphism f are important in the computation of the CCN of G_f.

Notation. If A is an alphabet and $\emptyset\neq w\in A^{<\omega}$, then $w^{\mathbb{Z}}\in A^{\mathbb{Z}}$ is defined by $(w^{\mathbb{Z}})(k|w|+j)=w(j)$ if $k\in\mathbb{Z}$ and $j<|w|$.

Proposition 10.2 *Let A be an alphabet, and $i>0$ be a natural number. Then $\sigma^i(\alpha)=\alpha$ holds exactly when there is $w\in A^i$ with $\alpha=w^{\mathbb{Z}}$ (in this case, we say that α is a* periodic point *of σ). In particular, the fixed points of σ are exactly the constant sequences.*

Proof. If $k\in\mathbb{Z}$, then $\sigma^i(\alpha)(k)=\alpha(k+i)$. For the left to right implication, we consider $w:=\alpha|i$ defined by $w(j):=\alpha(j)$ if $j<i$. □

Definition 10.3 *Let A be an alphabet. A* two-sided subshift *is a closed subset Σ of $A^{\mathbb{Z}}$ with the property that $\sigma[\Sigma]=\Sigma$.*

Note that a two-sided subshift defines a dynamical system, by restriction.

Notation. If A is an alphabet, $\alpha\in A^{\leq\omega}\cup A^{\mathbb{Z}}$ and $w\in A^{\leq\omega}$, then we write $w\sqsubseteq\alpha$ when w appears in α, i.e., when there is $k\in\mathbb{Z}$ such that $w(j)=\alpha(k+j)$ for each $j<|w|$. In particular, if $\alpha\in A^{\mathbb{Z}}$, then $\alpha^+:=\big(\alpha(0),\alpha(1),\dots\big)\sqsubseteq\alpha$.

Example. If A is an alphabet and $F\subseteq A^{<\omega}$, then $\Sigma_F:=\{\alpha\in A^{\mathbb{Z}}\mid\forall w\sqsubseteq\alpha\ \ w\notin F\}$ is the set of biinfinite words without subword in F. This is a two-sided subshift, and any two-sided subshift is of this form (see [Sa-Tö, Section 2]).

The next notion will be crucial in our study of subshifts.

Definition 10.4 *Let A be an alphabet. A* substitution *on A is a map $\tau:A^{<\omega}\to A^{<\omega}$ satisfying $\tau(uv)=\tau(u)\tau(v)$ for all $u,v\in A^{<\omega}$.*

A substitution is determined by the images of the letters of the alphabet. Some authors require that $\tau^{-1}(\{\emptyset\})=\{\emptyset\}$, which will be the case in our examples. We now provide infinite descending chains of graphs induced by a homeomorphism of a 0DMC space with CCN exactly three.

Theorem 10.5 *There is a $\preceq_c$ and $\preceq_c^i$-descending chain $\big((\Sigma_p,G_{\sigma_{|\Sigma_p}})\big)_{p\in\omega}$, where Σ_p is a two-sided subshift, $(\sigma_{|\Sigma_p})^2$ is fixed point free, and $(\Sigma_p,G_{\sigma_{|\Sigma_p}})$ has CCN three.*

Proof. We consider the generalized Fibonacci sequence of natural numbers defined by $f_0:=2$, $f_1:=3$, and $f_{p+2}:=f_p+f_{p+1}$. Note that $f_p>0$, $(f_p)_{p\in\omega}$ is strictly increasing, and f_p is even exactly when 3 divides p, by induction. Also, $f_{p+5}>8f_p$ since

$$f_{p+5}=f_{p+3}+f_{p+4}=f_{p+1}+2f_{p+2}+f_{p+3}=f_{p+1}+2f_p+2f_{p+1}+f_{p+1}+f_{p+2}=5f_{p+1}+3f_p.$$

In particular, $8f_{9p+5}<f_{9p+14}$.

This leads to define a $\subseteq$-increasing sequence $(F_p)_{p\in\omega}$ of subsets of $2^{<\omega}$ by

$$F_p:=\{0^2,1^3\}\cup\{w^8\mid w\in 2^{<\omega}\wedge 0<8|w|<f_{9p+5}\}.$$

This allows us to define the two-sided subshifts $\Sigma_p:=\Sigma_{F_p}$. Note that $(\Sigma_p)_{p\in\omega}$ is $\subseteq$-decreasing, so that $\big((\Sigma_p,G_{\sigma_{|\Sigma_p}})\big)_{p\in\omega}$ is $\preceq_c^i$-decreasing. Also, $w^{\mathbb{Z}}\notin\Sigma_p$ if $0<8|w|<f_{9p+5}$, so that $\sigma^i(\alpha)\neq\alpha$ if $\alpha\in\bigcup_{0<8i<f_{9p+5}}\Sigma_p$. In particular, $(\sigma_{|\Sigma_p})^2$ is fixed point free for each p since $f_5=21$.

We finally define a sequence $(w_p)_{p\in\omega}$ of finite binary sequences by $w_0:=01$, $w_1:=101$ and $w_{p+2}:=w_pw_{p+1}$. Note that $|w_p|=f_p$, inductively, so that $\sigma^{f_p}(w_p^{\mathbb{Z}})=w_p^{\mathbb{Z}}$. Here is the key fact.

Claim 1. Let $p\in\omega$. Then $w_{9p+5}^{\mathbb{Z}}\in\Sigma_p$.

Indeed, we consider the subsitution $\tau:2^{<\omega}\to 2^{<\omega}$ defined by $\tau(0):=1$ and $\tau(1):=01$. Note that $w_p=\tau^{p+1}(1)$. Indeed, $\tau^2(1)=\tau(01)=\tau(0)\tau(1)=101=w_1$ and

$$\begin{aligned}\tau^{p+3}(1)&=\tau^{p+2}\big(\tau(1)\big)=\tau^{p+2}(01)=\tau^{p+2}(0)\tau^{p+2}(1)=\tau^{p+1}\big(\tau(0)\big)\tau^{p+2}(1)=\tau^{p+1}(1)\tau^{p+2}(1)\\&=w_pw_{p+1}=w_{p+2}.\end{aligned}$$

If $w = \big(w(0),\ldots,w(|w|-1)\big) \in 2^{<\omega}$, then we set $w^{-1} = \big(w(|w|-1),\ldots,w(0)\big)$. The sequence $(w_p^{-1})_{p\in\omega}$ of *Fibonacci words* is strictly $\subseteq$-increasing, so that its elements are initial segments of the *infinite Fibonacci word* $\Phi \in 2^\omega$. By [Kar, Section 4], Φ contains no fourth power, i.e., $v^4 \not\sqsubseteq \Phi$ if $\emptyset \neq v \in 2^{<\omega}$. An induction shows that $w_q^{\mathbb{Z}} \in \Sigma_{\{0^2,1^3\}}$. We argue by contradiction to prove our claim, which gives p and w with $0 < 8|w| < f_{9p+5}$ and $w^8 \sqsubseteq w_{9p+5}^{\mathbb{Z}}$. Note that $w^4 \sqsubseteq w_{9p+5}$ since $8|w| < f_{9p+5} = |w_{9p+5}|$. In particular, if $v := w^{-1}$, then $v^4 \sqsubseteq \Phi$, which cannot be. $\diamond$

Claim 2. Let $p \in \omega$. Then we can find $\alpha \in \Sigma_p$ and $k \in \omega$ with $2k+3 \leq f_{9p+5}$, $\sigma^{2k+3}(\alpha) = \alpha$ and $\sigma^i(\alpha) \neq \alpha$ if $0 < i < 2k+3$.

Indeed, we choose $\alpha := w_{9p+5}^{\mathbb{Z}}$. By Claim 1, $\alpha \in \Sigma_p$. As $|w_{9p+5}| = f_{9p+5}$, $\sigma^{f_{9p+5}}(\alpha) = \alpha$, and f_{9p+5} is odd. Let n be odd and minimal with $\sigma^n(\alpha) = \alpha$. As $\sigma_{|\Sigma_p}$ is fixed point free, $n \geq 3$, which gives $k \in \omega$ with $n = 2k+3$, so that $2k+3 \leq f_{9p+5}$ and $\sigma^{2k+3}(\alpha) = \alpha$. If $0 < i < 2k+3$ and $\sigma^i(\alpha) = \alpha$, then i has to be even by minimality of n. Note then that $0 < n-i < n$ is odd and $\sigma^{n-i}(\alpha) = \alpha$, which contradicts the minimality of n. $\diamond$

Claim 2 implies that $\big(\sigma^i(\alpha)\big)_{i\leq 2k+3}$ is a $G_{\sigma_{|\Sigma_p}}$-cycle, so that $(2k+3, C_{2k+3}) \preceq_c^i (\Sigma_p, G_{\sigma_{|\Sigma_p}})$. In particular, $\chi_c(\Sigma_p, G_{\sigma_{|\Sigma_p}}) \geq 3$. By Theorem 8.1, $\chi_c(\Sigma_p, G_{\sigma_{|\Sigma_p}}) = 3$. Assume now, towards a contradiction, that $(\Sigma_p, G_{\sigma_{|\Sigma_p}}) \preceq_c (\Sigma_{p+1}, G_{\sigma_{|\Sigma_{p+1}}})$ with witness φ. Let α be given by Claim 2. As $\big(\sigma^i(\alpha)\big)_{i\leq 2k+3}$ is a $G_{\sigma_{|\Sigma_p}}$-cycle and odd cycles must map to odd cycles of at most equal length, $(\Sigma_{p+1}, G_{\sigma_{|\Sigma_{p+1}}})$ contains a cycle of length $2l+3 \leq 2k+3$. This implies that $\sigma^{2l+3}\big(\varphi(\alpha)\big) = \varphi(\alpha)$, which cannot be since $0 < 8(2l+3) \leq 8(2k+3) \leq 8f_{9p+5} < f_{9p+14}$. $\square$

Lemma 7.10 shows that many odometers induce minimal graphs with CCN three. We will now see that it is also the case with subshifts.

Notation. Let $r \in (0, \frac{1}{2}) \setminus \mathbb{Q}$. We consider the irrational rotation $R_r : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ (well-)defined by $R_r([x]) := [x+r]$. We (well-)define $\phi_r : \mathbb{R}/\mathbb{Z} \to 2$ by $\phi_r([x]) := 0$ if $x \in [0,r)$ mod 1, $\phi_r([x]) := 1$ otherwise, and set $\Sigma_r^2 := \overline{\Big\{\Big(\phi_r\big(R_r^n([x])\big)\Big)_{n\in\mathbb{Z}} \mid [x] \in \mathbb{R}/\mathbb{Z}\Big\}}$.

The following result is mentioned in [MB, Section 4].

Theorem 10.6 *(Hedlund) Let $r \in (0, \frac{1}{2}) \setminus \mathbb{Q}$. Then $(\Sigma_r^2, \sigma_{|\Sigma_r^2})$ is a minimal two-sided subshift.*

Notation. If $\mathbb{X} \in \{\mathbb{Z}, \omega\}$ and $\Sigma \subseteq A^{\mathbb{X}}$, then we denote by $\mathcal{L}(\Sigma) := \{w \in A^{<\omega} \mid \exists \alpha \in \Sigma \;\; w \sqsubseteq \alpha\}$ the set of finite words word occurring in Σ. If moreover $n \in \omega$, then we set $\mathcal{L}_n(\Sigma) := \mathcal{L}(\Sigma) \cap A^n$.

Let $r \in (0, \frac{1}{2}) \setminus \mathbb{Q}$. By Theorem 10.6, $(\Sigma_r^2, \sigma_{|\Sigma_r^2})$ is a two-sided subshift. We set $\mathcal{L}^r := \mathcal{L}(\Sigma_r^2)$. If moreover $n \in \omega$, then we set $\mathcal{L}_n^r := \mathcal{L}^r \cap 2^n$, so that $\mathcal{L}_n^r = \mathcal{L}_n(\Sigma_r^2)$.

Definition 10.7 *A subshift Σ is* uniformly recurrent *if, for each $w \in \mathcal{L}(\Sigma)$, there is a natural number l such that, for each $v \in \bigcup_{n\geq l} \mathcal{L}_n(\Sigma)$, $w \sqsubseteq v$.*

By [Sa-Tö, Section 2], the following result holds.

Theorem 10.8 *Any minimal two-sided subshift is uniformly recurrent.*

Lemma 10.9 *Let $r \in (0, \frac{1}{2}) \setminus \mathbb{Q}$.*
(a) The map $\sigma_{|\Sigma_r^2}$ is fixed point free.
(b) Σ_r^2 is homeomorphic to 2^ω.

Proof. By Theorems 10.6 and 10.8, Σ_r^2 is uniformly recurrent. By [MB, Section 4], $(\Sigma_r^2, \sigma_{|\Sigma_r^2})$ has no periodic point

(a) If $\sigma_{|\Sigma_r^2}$ has a fixed point α, Proposition 10.2 gives $a \in 2$ with $\alpha = a^{\mathbb{Z}}$. In particular, α is a periodic point of Σ_r^2, which cannot be. Thus $\sigma_{|\Sigma_r^2}$ is fixed point free.

(b) Let $\alpha \in \Sigma_r^2$. Assume now that α is isolated in Σ_r^2, which gives $\emptyset \neq w \sqsubseteq \alpha$ and $p \in \mathbb{Z}$ with $\Sigma_r^2 \cap [w]_p = \{\alpha\}$. As Σ_r^2 is uniformly recurrent, there is l such that $w \sqsubseteq \alpha|[p+|w|, p+|w|+l]$. This gives a natural number i with $w \subseteq \alpha|[p+|w|+i, \infty)$, so that $\sigma^{|w|+i}(\alpha) \in [w]_p$. Thus $\sigma^{|w|+i}(\alpha) = \alpha$ and α is periodic. This contradiction shows that Σ_r^2 is perfect, and thus homeomorphic to 2^ω, by [K, 7.4], finishing the proof. $\square$

The following result is also mentioned in [MB, Section 4].

Theorem 10.10 *Let $r \neq r' \in (0, \frac{1}{2}) \setminus \mathbb{Q}$. Then the homeomorphisms $\sigma_{|\Sigma_r^2}$ and $\sigma_{|\Sigma_{r'}^2}$ are not conjugate.*

We are now ready to prove that the Σ_r^2's induce minimal graphs with CCN exactly three.

Theorem 10.11 *Let $r \in (0, \frac{1}{2}) \setminus \mathbb{Q}$.*
(a) The dynamical system $(\Sigma_r^2, \sigma_{|\Sigma_r^2})$ is minimal and the graph $(\Sigma_r^2, G_{\sigma_{|\Sigma_r^2}})$ has CCN three.

(b) The graph $(\Sigma_r^2, G_{\sigma_{|\Sigma_r^2}})$ is $\preceq_c^i$-minimal in $\mathfrak{G}_2$ and in the class of closed graphs on a 0DMC space with CCN at least three.

Proof. (a) Note that Σ_r^2 is not empty. By Theorem 10.6, $(\Sigma_r^2, \sigma_{|\Sigma_r^2})$ is minimal. By Lemma 10.9, Σ_r^2 has cardinality at least two and $\sigma_{|\Sigma_r^2}$ is fixed point free. By Theorem 8.1, $(\Sigma_r^2, G_{\sigma_{|\Sigma_r^2}})$ has CCN two or three. By Theorem 7.1 and minimality, it is enough to find $\alpha \in \Sigma_r^2$ such that the intersection $\overline{\{\sigma^{2p}(\alpha) \mid p \in \mathbb{Z}\}} \cap \overline{\{\sigma^{2p+1}(\alpha) \mid p \in \mathbb{Z}\}}$ is not empty.

If $[x] \in \mathbb{R}/\mathbb{Z}$, then $\sigma\Big(\big(\phi_r\big(R_r^n([x])\big)\big)_{n\in\mathbb{Z}}\Big) = \big(\phi_r\big(R_r^{n+1}([x])\big)\big)_{n\in\mathbb{Z}}$, so that

$$\sigma^{2p+\varepsilon}\Big(\big(\phi_r\big(R_r^n([x])\big)\big)_{n\in\mathbb{Z}}\Big) = \big(\phi_r\big(R_r^{n+2p+\varepsilon}([x])\big)\big)_{n\in\mathbb{Z}}.$$

Note then that $\phi_{r|(\mathbb{R}/\mathbb{Z})\setminus([0]\cup[r])}$ is continuous. So it is enough to find

$$[x] \in (\mathbb{R}/\mathbb{Z}) \setminus \big\{R_r^l([y]) \mid l \in \mathbb{Z} \wedge y \in \{0, r\}\big\}$$

such that $\overline{\big\{\big(R_r^{n+2p+1}([x])\big)_{n\in\mathbb{Z}} \mid p \in \mathbb{Z}\big\}}$ is not empty. Pick $[x]$ in

$$(\mathbb{R}/\mathbb{Z}) \setminus \big\{R_r^l([y]) \mid l \in \mathbb{Z} \wedge y \in \{0, r\}\big\}$$

arbitrary. It is enough to check that $\big(R_r^n([x])\big)_{n\in\mathbb{Z}} \in \overline{\big\{\big(R_r^{n+2p+1}([x])\big)_{n\in\mathbb{Z}} \mid p \in \mathbb{Z}\big\}}$. Note that the restriction b of the canonical map from $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ to $[0, 1)$ is a bijection. The map b is a homeomorphism from $[0, 1)$ equipped with $\tau := \{b^{-1}(O) \mid O \text{ open in } \mathbb{R}/\mathbb{Z}\}$ onto $\mathbb{R}/\mathbb{Z}$. As usual, $[0, 1)$ is equipped with the τ-compatible metric defined by $d(x, y) := \min(|x-y|, 1-|x-y|)$. The previous identification through b defines a compatible metric D on $\mathbb{R}/\mathbb{Z}$ for which the R_r's are isometries. Let q be a natural number, and, for $n \in [-q, q] \cap \mathbb{Z}$, $0 < a_n < r_n := b^{-1}\big(R_r^n([x])\big) < b_n < 1 \bmod 1$. We choose a natural number $m > 0$ such that $\frac{1}{m} < \min_{n\in[-q,q]\cap\mathbb{Z}} \min(r_n - a_n, b_n - r_n)$. The next claim is inspired by the proof of [Ku, Proposition 1.32].

Claim. Let $r \in (0, \frac{1}{2}) \setminus \mathbb{Q}$, $x \in \mathbb{R}$, and $m > 0$ be a natural number. Then there is $p \in \mathbb{Z}$ such that, for each $n \in \mathbb{Z}$, $D\big(R_r^{n+2p+1}([x]), R_r^n([x])\big) < \frac{1}{m}$.

Indeed, as $r \notin \mathbb{Q}$, the m classes $[x]$, $R_r^2([x])$, …, $R_r^{2m-2}([x])$ are pairwise different. As b is defined on $[0, 1)$, we can find $0 \leq i < j < m$ with the property that $D\big(R_r^{2i}([x]), R_r^{2j}([x])\big) < \frac{1}{m}$. As R_r is an isometry, $D\big([x], R_r^{2(j-i)}([x])\big) = D\big(R_r^{2i}([x]), R_r^{2j}([x])\big)$. Now $R_r^{2(j-i)} = R_b$ is also a rotation, and either $0 < b < \frac{1}{m}$, or $1 - \frac{1}{m} < b < 1$. In both cases, for any $y \in \mathbb{R}/\mathbb{Z}$, there is $k > 0$ with $D\big(R_r^{2k(j-i)}([x]), y\big) < \frac{1}{m}$. Applying this to $y := R_r([x])$, and putting $p := k(i-j)$, we get $D\big(R_r^{2p+1}([x]), [x]\big) < \frac{1}{m}$. It remains to apply again the fact that R_r is an isometry. ◇

The claim provides $p \in \mathbb{Z}$ such that, for each $n \in \mathbb{Z}$, $D\big(R_r^{n+2p+1}([x]), b(r_n)\big) < \frac{1}{m}$. In particular, $b^{-1}\big(R_r^{n+2p+1}([x])\big) \in (a_n, b_n)$ mod 1 if $n \in [-q, q] \cap \mathbb{Z}$, as desired.

(b) We apply (a) and Lemma 7.10. □

Theorem 10.11 gives a version of Theorem 1.13 for subshifts.

Corollary 10.12 *There is a $\preceq_c^i$-antichain $(\Sigma_r, G_{\sigma_{|\Sigma_r}})_{r \in \mathbb{R}}$, where*

(a) Σ_r is a two-sided subshift homeomorphic to 2^ω,

(b) $\sigma_{|\Sigma_r}$ is a minimal homeomorphism of Σ_r, and $G_{\sigma_{|\Sigma_r}}$ has CCN three,

(c) $(\Sigma_r, G_{\sigma_{|\Sigma_r}})_{r \in \mathbb{R}}$ is $\preceq_c^i$-minimal in $\mathfrak{G}_2$ and in the class of closed graphs on a 0DMC space with CCN at least three.

Proof. Let $r \in (0, \frac{1}{2}) \setminus \mathbb{Q}$. By Theorem 10.6 and Lemma 10.9(b), $(\Sigma_r^2, G_{\sigma_{|\Sigma_r^2}})$ is a minimal two-sided subshift homeomorphic to 2^ω. By Lemma 10.9(a), $\sigma_{|\Sigma_r^2}$ is fixed point free, so that Σ_r^2 is a closed graph, with CCN three by Theorem 10.11(a). As $(\Sigma_r^2, G_{\sigma_{|\Sigma_r^2}})$ has cardinality at least three and is minimal, the vertices of Σ_r^2 have degree two. By Theorem 10.11(b), $(\Sigma_r^2, G_{\sigma_{|\Sigma_r^2}})$ is $\preceq_c^i$-minimal in $\mathfrak{G}_2$ and in the class of closed graphs on a 0DMC space with CCN at least three.

So it is enough to find a subfamily of $(\Sigma_r^2, G_{\sigma_{|\Sigma_r^2}})_{r \in (0, \frac{1}{2}) \setminus \mathbb{Q}}$ which is a $\preceq_c^i$-antichain. By Lemma 7.11, it is enough to ensure that the homeomorphisms corresponding to the elements of the subfamily are pairwise not flip conjugate. By Theorem 10.10, the $\sigma_{|\Sigma_r^2}$'s are pairwise not conjugate. Thus $\sigma_{|\Sigma_r^2}$ is flip-conjugate to $\sigma_{|\Sigma_{r'}^2}$ exactly when $\sigma_{|\Sigma_r^2}$ is conjugate to $\sigma_{|\Sigma_{r'}^2}^{-1}$. The key remark is that if $\sigma_{|\Sigma_r^2}$ is conjugate to $\sigma_{|\Sigma_{r'}^2}^{-1}$ and $\sigma_{|\Sigma_{r''}^2}^{-1}$, then $\sigma_{|\Sigma_{r'}^2}$ is conjugate to $\sigma_{|\Sigma_{r''}^2}$, which implies that $r' = r''$. We inductively construct a injective family $(r_\xi)_{\xi < 2^{\aleph_0}}$ of elements of $(0, \frac{1}{2}) \setminus \mathbb{Q}$ such that the $\sigma_{|\Sigma_{r_\xi}^2}$'s are pairwise not flip-conjugate. r_0 is an arbitrary element of $(0, \frac{1}{2}) \setminus \mathbb{Q}$. Assume that $1 \leq \eta < 2^{\aleph_0}$ and $(r_\xi)_{\xi < \eta}$ are constructed. The key remark shows that, for each $\xi < \eta$, there is at most one element r'_ξ of $(0, \frac{1}{2}) \setminus \mathbb{Q}$ such that $\sigma_{|\Sigma_{r_\xi}^2}$ is conjugate to $\sigma_{|\Sigma_{r'_\xi}^2}^{-1}$. If r'_ξ does not exist, then we set $r'_\xi := r_\xi$. We choose $r_\eta \in (0, \frac{1}{2}) \setminus (\mathbb{Q} \cup \bigcup_{\xi < \eta} \{r_\xi, r'_\xi\})$, so that $(r_\xi)_{\xi \leq \eta}$ is as desired. □

Corollary 10.12 gives a second proof of the fact, met in Theorem 1.13, that any $\preceq_c^i$-basis for $\mathfrak{G}_2$ must have size continuum. We now prove that the basis given by Proposition 9.3(a) is far from being a basis for $\mathfrak{G}_2$, as announced in the introduction.

Proposition 10.13 *Let* $\mathbf{d} \in (\omega \setminus 2)^\omega \cap \mathcal{O}$ *and* $r \in (0, \frac{1}{2}) \setminus \mathbb{Q}$. *Then* $(\mathcal{C}, G_o)$ *and* $(\Sigma^2_r, G_{\sigma_{|\Sigma^2_r}})$ *are* $\preceq^i_c$*-incompatible in the class of closed graphs on a 0DMC space with CCN at least three.*

Proof. Towards a contradiction, suppose that we can find a 0DMC space K and a closed graph G on K which is $\preceq^i_c$-below our two graphs. Thus $(\Sigma^2_r, G_{\sigma_{|\Sigma^2_r}}) \preceq^i_c (\mathcal{C}, G_o)$, by Corollary 10.12. Theorem 10.6 and Lemma 10.9 allow us to apply Lemma 7.11, so that $\sigma_{|\Sigma^2_r}$ and o are flip-conjugate, with witness say φ. We already saw that $(\mathcal{C}, o)$ is equicontinuous, i.e., $(o^n)_{n\in\omega}$ is equicontinuous. In fact, as o is an isometry, $(o^n)_{n\in\mathbb{Z}}$ is equicontinuous. The uniform continuity of φ and φ^{-1} implies that $(\Sigma^2_r, \sigma_{|\Sigma^2_r})$ is also equicontinuous. Theorem 10.6 and [Ku, Proposition 3.68(2)] imply that $(\Sigma^2_r, \sigma_{|\Sigma^2_r})$ is expansive, which gives $\varepsilon > 0$ such that, for each $\alpha \neq \beta \in \Sigma^2_r$, there is $n \in \mathbb{Z}$ with $d\big(\sigma^n(\alpha), \sigma^n(\beta)\big) \geq \varepsilon$. Lemma 10.9(b) gives $\alpha \in \Sigma^2_r$. The equicontinuity of $(\Sigma^2_r, \sigma_{|\Sigma^2_r})$ gives $\delta > 0$ such that, for each $\beta \in B(\alpha, \delta)$ and each $k \in \mathbb{Z}$, $d\big(\sigma^k(\alpha), \sigma^k(\beta)\big) < \varepsilon$. Lemma 10.9(b) gives $\beta \in B(\alpha, \delta) \setminus \{\alpha\}$, which is the desired contradiction. $\square$

Remark. Theorem 1.13 and Corollary 10.12 provide examples $\preceq^i_c$-minimal in $\mathfrak{G}_2$. None of them is $\preceq_c$-minimal in $\mathfrak{K}$. Indeed, there is a dense orbit. Let (X, G_f) be one of them. Theorem 3.2.4 provides $\beta \in \mathcal{J}^c$ such that $(\mathbb{K}_\beta, \mathbb{G}_\beta) \preceq_c (X, G_f)$ and the vertices of $\mathbb{G}_\beta$ have degree at most one. Assume that $(X, G_f) \preceq_c (\mathbb{K}_\beta, \mathbb{G}_\beta)$, towards a contradiction. Then the dense orbit of f has to be sent to a two-point set because of the degree. So X has to be sent to this closed set, by density. But this contradicts the fact that (X, G_f) has CCN at least three. The examples are not $\preceq^i_c$-minimal in $\mathfrak{K}$. Indeed, consider the dense orbit $D = \mathrm{Orb}_f(x)$. Assume that $(X, G_f) \preceq^i_c (X, G_{f_{|D}})$ with witness φ, towards a contradiction. The proof of Lemma 7.11 shows that $\varphi[D] = \mathrm{Orb}_f\big(\varphi(x)\big) = D$ since $(f_{|D})^2$ is fixed point free. Thus $\varphi^2[G_{f_{|D}}] = G_{f_{|D}}$. By injectivity of φ, there is no more room for $\varphi[X \setminus D]$.

We now turn to the proof of Theorem 1.15. By [K, 6.C], the countable MC spaces can be analyzed through their Cantor-Bendixson rank. Recall that if X is a topological space, then the *Cantor-Bendixson derivative* of X is $X' := \{x \in X \mid x \text{ is a limit point of } X\}$. The *iterated Cantor-Bendixson derivatives* are defined by $X^0 := X$, $X^{\alpha+1} := (X^\alpha)'$, and $X^\lambda := \bigcap_{\alpha<\lambda} X^\alpha$ if λ is a limit ordinal. Note that if f is a homeomorphism of X, then all the derivatives are f-invariant, i.e., $f[X^\alpha] = X^\alpha$ if α is an ordinal. If $X \neq \emptyset$ is a countable metrizable compact space, then the *Cantor-Bendixson rank* of X is the minimal countable ordinal α with $X^\alpha = \emptyset$, which is a successor ordinal by compactness. The odd cycles provide examples of graphs induced by a homeomorphism of a countable (0D)MC space with Cantor-Bendixson rank one whose CCN is three, and which are $\preceq^i_c$-minimal in $\mathfrak{G}_2$. We now provide examples for higher ranks, including the example (K_0, h_0) mentioned in the introduction.

Proof of Theorem 1.15. (a) For $\xi = 1$, we can take $\Sigma := \mathrm{Orb}_\sigma\big((012)^\infty \cdot (012)^\infty\big) \subseteq 3^\mathbb{Z}$, which defines a cycle on three points, and we apply Corollary 2.2. Assume now that $2 \leq \xi = n+2 < \omega$. We set, for $j, m \in \omega$, $w^0_j := 01$, $w^{m+1}_0 := 1^2$, and $w^{m+1}_{j+1} := (01)^{j+1} 1^2 {}^\frown_{k\leq j+1} w^m_k$. We then set $\alpha_0 := (01)^\infty \cdot (01)^\infty$, and, for $m \in \omega$, $\alpha_{m+1} := (01)^\infty \cdot 1^2 {}^\frown_{j\in\omega} w^m_j$ and $\beta_m := (01)^\infty \cdot 1 {}^\frown_{j\in\omega} w^m_j$. Finally, $\Sigma = \bigcup_{m\leq n} \mathrm{Orb}_\sigma(\alpha_m) \cup \mathrm{Orb}_\sigma(\beta_n)$, so that $\Sigma = K_0$ and $\sigma_{|\Sigma} = h_0$ if $\xi = 2$.

Σ is by definition countable, and $\sigma[\Sigma] = \Sigma$. We then set, for $1 \leq i \leq n+1$,

$$\Sigma^{(i)} = \bigcup_{m\leq n+1-i} \mathrm{Orb}_\sigma(\alpha_m).$$

Claim 1. (a) Let $1 \leq i \leq n+1$. *Then* $\Sigma^{(i)} = \overline{Orb_\sigma(\alpha_{n+1-i})}$.

(b) $\Sigma = \overline{Orb_\sigma(\beta_n)}$.

Indeed, for (a), we argue by induction on $n+1-i$. Note first that

$$\mathrm{Orb}_\sigma(\alpha_0) = \{(01)^\infty \cdot (01)^\infty, (10)^\infty \cdot (10)^\infty\}$$

is closed, so that $\Sigma^{(n+1)} = \overline{\mathrm{Orb}_\sigma(\alpha_0)}$.

Let us prove that $\alpha_m \in \overline{\mathrm{Orb}_\sigma(\alpha_{m+1})}$ if $m \in \omega$, which holds for $m = 0$. Note that

$$\alpha_{m+2} = (01)^\infty \cdot 1^2 {}^\frown_{j\in\omega} w_j^{m+1} = (01)^\infty \cdot 1^2 1^2 {}^\frown_{j\in\omega} \left((01)^{j+1} 1^2 {}^\frown_{k\leq j+1} w_k^m\right).$$

If $\alpha \in 2^{\mathbb{Z}}$ and $a \leq b$ are integers, then we define $\alpha_{[a,b]} \in 2^{b-a+1}$ by $\alpha_{[a,b]}(l) = \alpha(a+l)$ if $l \leq b-a$. Note that $(01)^{j+1} 1^2 {}^\frown_{k\leq j+1} w_k^m = \alpha_{m+1[-2j-2, 1+\Sigma_{k\leq j+1} |w_k^m|]}$. An induction shows that $|w_k^m| \geq 2$, so that $1 + \Sigma_{k\leq j+1} |w_k^m| \geq 2j+5$. This implies that $\alpha_{m+1} \in \overline{\mathrm{Orb}_\sigma(\alpha_{m+2})}$, as desired.

From this we deduce, inductively and by continuity of σ and σ^{-1}, that

$$\Sigma^{(i)} = \Sigma^{(i+1)} \cup \mathrm{Orb}_\sigma(\alpha_{n+1-i}) \subseteq \overline{\mathrm{Orb}_\sigma(\alpha_{n-i})} \cup \mathrm{Orb}_\sigma(\alpha_{n+1-i}) \subseteq \overline{\mathrm{Orb}_\sigma(\alpha_{n+1-i})}.$$

This shows that $\Sigma^{(i)} \subseteq \overline{\mathrm{Orb}_\sigma(\alpha_{n+1-i})}$ if $1 \leq i \leq n+1$.

Assume then that $1 \leq i \leq n$, $(k_p)_{p\in\omega} \in \mathbb{Z}^\omega$ and $\left(\sigma^{k_p}(\alpha_{n+1-i})\right)_{p\in\omega}$ converges to $\alpha \in 2^{\mathbb{Z}}$. We want to see that $\alpha \in \Sigma^{(i)}$, and we may assume that $i < n$. If $(k_p)_{p\in\omega}$ has a constant subsequence, then $\alpha \in \mathrm{Orb}_\sigma(\alpha_{n+1-i})$ and we are done. So we may assume that $(k_p)_{p\in\omega}$ tends to $\pm\infty$. If $(k_p)_{p\in\omega}$ tends to $-\infty$, then $\alpha \in \mathrm{Orb}_\sigma(\alpha_0)$ since $\alpha_{n+1-i} = (01)^\infty \cdot 1^2 {}^\frown_{j\in\omega} w_j^{n-i}$, and we are done. So we may write $k_p = 2 + \Sigma_{j<l_p} |w_j^{n-i}| + j_p$, where $j_p < |w_{l_p}^{n-i}|$, and $(l_p)_{p\in\omega}$ tends to ∞. As above, $\alpha_{n+1-i} = (01)^\infty \cdot 1^2 1^2 {}^\frown_{j\in\omega} \left((01)^{j+1} 1^2 {}^\frown_{k\leq j+1} w_k^{n-i-1}\right)$, and

$$(01)^{j+1} 1^2 {}^\frown_{k\leq j+1} w_k^{n-i-1} = \alpha_{n-i[-2j-2, 1+\Sigma_{k\leq j+1} |w_k^{n-i-1}|]}.$$

If $(j_p)_{p\in\omega}$ and $(|w_{l_p}^{n-i}| - j_p)_{p\in\omega}$ also tend to ∞, then $\alpha \in \overline{\mathrm{Orb}_\sigma(\alpha_{n-i})} \subseteq \Sigma^{(i+1)} \subseteq \Sigma^{(i)}$, by induction assumption, as desired. Otherwise, we may assume that $(j_p)_{p\in\omega}$ or $(|w_{l_p}^{n-i}| - j_p)_{p\in\omega}$ is constant, so that $\alpha \in \mathrm{Orb}_\sigma(\alpha_0)$ since w_j^{m+1} starts and ends with $(01)^j$, by induction.

The proof of (b) is similar. $\diamond$

Claim 1 implies that Σ is a two-sided subshift.

Claim 2. Σ *has Cantor-Bendixson rank* ξ.

Indeed, let us show that $\Sigma^{(i)}$ is the ith iterated Cantor-Bendixson derivative of Σ if $1 \leq i \leq n+1$. By Claim 1(b), $\Sigma \setminus \mathrm{Orb}_\sigma(\beta_n) \subseteq \Sigma'$. As Σ is countable, it has an isolated point, which has therefore to be in $\mathrm{Orb}_\sigma(\beta_n)$. As σ is a homeomorphism, $\mathrm{Orb}_\sigma(\beta_n)$ is disjoint from Σ', showing that $\Sigma' = \Sigma \setminus \mathrm{Orb}_\sigma(\beta_n) \subseteq \bigcup_{m\leq n} \mathrm{Orb}_\sigma(\alpha_m) = \Sigma^{(1)}$. It remains to see that $\alpha_m \notin \mathrm{Orb}_\sigma(\beta_n)$ if $m \leq n$ to get $\Sigma^{(1)} = \Sigma'$. As all the odd coordinates of the w_j^m's and the α_m's are 1, and we can find an even coordinate of β_n and an odd coordinate of β_n which are 0, we are done.

Fix now $1 \leq i \leq n$. By Claim 1(a), $\Sigma^{(i)} \setminus \mathrm{Orb}_\sigma(\alpha_{n+1-i}) \subseteq (\Sigma^{(i)})'$. As $\Sigma^{(i)}$ is countable, it has an isolated point, which has therefore to be in $\mathrm{Orb}_\sigma(\alpha_{n+1-i})$. As σ is a homeomorphism, $\mathrm{Orb}_\sigma(\alpha_{n+1-i})$ is disjoint from $(\Sigma^{(i)})'$, showing that $(\Sigma^{(i)})' = \Sigma^{(i)} \setminus \mathrm{Orb}_\sigma(\alpha_{n+1-i}) \subseteq \bigcup_{m \leq n-i} \mathrm{Orb}_\sigma(\alpha_m) = \Sigma^{(i+1)}$. It remains to see that $\alpha_m \notin \mathrm{Orb}_\sigma(\alpha_{m+1+p})$ if $m, p \in \omega$ to get $\Sigma^{(i+1)} = (\Sigma^{(i)})'$ and, inductively, that $\Sigma^{(i+1)}$ is the $(i+1)$th iterated Cantor-Bendixson derivative of Σ. Note that if $m \in \omega$, then $w_1^{m+2} := (01)1^21^2w_1^{m+1}$, so that $(011^21^2)^m011^201 \subseteq w_1^{m+1}$, w_1^{m+1} and w_1^{m+2+p} are incompatible if $p \in \omega$. In particular, $\alpha_{m+2} \neq \alpha_{m+3+p}$, and, because of $(01)^\infty$, $\alpha_{m+2} \notin \mathrm{Orb}_\sigma(\alpha_{m+3+p})$. Because of 1^2, $\alpha_0 \notin \mathrm{Orb}_\sigma(\alpha_{1+p})$, and because of 1^21^2, $\alpha_1 \notin \mathrm{Orb}_\sigma(\alpha_{2+p})$.

As $\Sigma^{(n+1)} = \mathrm{Orb}_\sigma(\alpha_0) \neq \emptyset$ is finite, Σ has Cantor-Bendixson rank ξ. $\diamond$

As no sequence in Σ is constant, $\sigma_{|\Sigma}$ is fixed point free. As moreover Σ is not empty, $(\Sigma, G_{\sigma_{|\Sigma}})$ has CCN two or three by Theorem 8.1. Note then that

$$(\alpha_0, \alpha_0) = \lim_{p \to \infty} \left(\sigma^{-2p}(\beta_n), \sigma^{1+\Sigma_{j<p}\ |w_j^n|}(\beta_n)\right) \in \overline{\bigcup_{q \in \omega} G_{\sigma_{|\Sigma}}^{2q+1}},$$

so that $\chi_c(\Sigma, G_{\sigma_{|\Sigma}}) = 3$ by Theorem 1.7. For the minimality of $(\Sigma, G_{\sigma_{|\Sigma}})$, we first prove the following.

Claim 3. Let $V \subseteq \Sigma$, and $E \subseteq G_{\sigma_{|\Sigma}} \cap V^2$ be a graph on V such that (V, E) has CCN three. Then $\big(\alpha, \sigma(\alpha)\big) \in E$ if $\alpha \in$ Orb$_\sigma(\beta_n)$.

Indeed, we argue by contradiction. Let $k \in \mathbb{Z}$ with $\alpha = \sigma^k(\beta_n)$. Recall that the sets of the form $[w]_q := \{\beta \in 2^{\mathbb{Z}} \mid \forall j < |w| \ \ w(j) = \beta(q+j)\}$, where $w \in 2^{<\omega}$ and $q \in \mathbb{Z}$, form a basis made up of clopen subsets of the space $2^{\mathbb{Z}}$.

Assume first that $n = 0$, so that $\Sigma_n = \mathrm{Orb}_\sigma\big((01)^\infty \cdot (01)^\infty\big) \cup \mathrm{Orb}_\sigma\big((01)^\infty \cdot 1(01)^\infty\big)$, and

$$\sigma^{2p+\varepsilon}(\alpha_n) = \begin{cases} (01)^\infty \cdot (01)^\infty \text{ if } \varepsilon = 0, \\ (10)^\infty \cdot (10)^\infty \text{ if } \varepsilon = 1, \end{cases}$$

$$\sigma^{2p+\varepsilon}(\beta_n) = \begin{cases} (01)^\infty 1(01)^{p-1}0 \cdot (10)^\infty \text{ if } p > 0 \wedge \varepsilon = 0, \\ (01)^\infty 1(01)^p \cdot (01)^\infty \text{ if } p \geq 0 \wedge \varepsilon = 1, \\ (01)^\infty \cdot (01)^p 1(01)^\infty \text{ if } p \leq 0 \wedge \varepsilon = 0, \\ (10)^\infty \cdot (10)^{-p-1}11(01)^\infty \text{ if } p < 0 \wedge \varepsilon = 1 \end{cases}$$

if $p \in \mathbb{Z}$ and $\varepsilon \in 2$. We set $C := ([0]_{-k} \cup [1^20]_{-k-1}) \cap V$, so that C is a clopen subset of V and $E \cap \big(C^2 \cup (V \setminus C)^2\big) = \emptyset$ since

$$\begin{array}{c} \ldots, \sigma^{k-2}(\alpha_n), \sigma^k(\alpha_n), \sigma^{k+2}(\alpha_n), \ldots \in C, \\ \ldots, \sigma^{k-3}(\alpha_n), \sigma^{k-1}(\alpha_n), \sigma^{k+1}(\alpha_n), \ldots \notin C, \end{array}$$

$$\begin{array}{c} \ldots, \sigma^{k-4}(\beta_n), \sigma^{k-2}(\beta_n), \sigma^k(\beta_n), \sigma^{k+1}(\beta_n), \sigma^{k+3}(\beta_n), \sigma^{k+5}(\beta_n), \ldots \in C, \\ \ldots, \sigma^{k-5}(\beta_n), \sigma^{k-3}(\beta_n), \sigma^{k-1}(\beta_n), \sigma^{k+2}(\beta_n), \sigma^{k+4}(\beta_n), \sigma^{k+6}(\beta_n), \ldots \notin C, \end{array}$$

which contradicts the fact that (V, E) has CCN three.

In this argument, the case $k \neq 0$ is similar to the case $k = 0$, we just have to translate the basic clopen sets of the form $[w]_q$. It will also be the case in the general case $n \geq 1$ that we now consider, so that we may and will assume that $k = 0$.

We set $C := ([0]_0 \cup [1^20]_0 \cup [1^50]_{-1} \cup [01^40]_{-2}) \cap V$, so that C is a clopen subset of V. We already noticed that the odd coordinates of the w_j^m's and the α_m's are 1, that the w_j^m's have a stricly positive even length, and that w_j^{m+1} starts and ends with $(01)^j$. The definition of the w_j^m's, the α_m's and the β_n's then imply that 0 can only be an even coordinate of α_m, a negative even coordinate of β_n or a positive odd coordinate of β_n. Moreover, 01^l0 can be of the form $\alpha_{m[a,b]}$ only if $l \in \{1,3,5\}$ and a is even, and of the form $\beta_{n[a,b]}$ only if $l=1$ and $a \leq -4$ is even, $l \in \{1,3,5\}$ and $a \geq 3$ is odd, or $l=4$ and $a=-2$. This implies that $E \cap \big(C^2 \cup (V \setminus C)^2\big) = \emptyset$ since

$$\begin{aligned} &\ldots, \sigma^{-2}(\alpha_m), \alpha_m, \sigma^2(\alpha_m), \ldots \in C, \\ &\ldots, \sigma^{-3}(\alpha_m), \sigma^{-1}(\alpha_m), \sigma(\alpha_m), \ldots \notin C \end{aligned}$$

if $m \leq n$ and

$$\begin{aligned} &\ldots, \sigma^{-4}(\beta_n), \sigma^{-2}(\beta_n), \beta_n, \sigma(\beta_n), \sigma^3(\beta_n), \sigma^5(\beta_n), \ldots \in C, \\ &\ldots, \sigma^{-5}(\beta_n), \sigma^{-3}(\beta_n), \sigma^{-1}(\beta_n), \sigma^2(\beta_n), \sigma^4(\beta_n), \sigma^6(\beta_n), \ldots \notin C, \end{aligned}$$

which contradicts the fact that (V,E) has CCN three again. $\diamond$

Assume now that $(K,G) \in \mathfrak{G}_2$ and $(K,G) \preceq_c^i (\Sigma, G_{\sigma_{|\Sigma}})$ with witness φ, which implies that (K,G) has CCN three. As $(K,G) \in \mathfrak{G}_2$, there is a homeomorphism $f : K \to K$ with $G = G_f$. As $\chi_c(K,G) = 3$, the set F_1 of fixed points of f is a clopen subset of K, and $\chi_c(K \setminus F_1, G_f \cap (K \setminus F_1)^2) = 3$ by Corollary 7.3. This implies that we may assume that f is fixed point free, so that G is compact. We set $V := \varphi[K]$ and $E := (\varphi \times \varphi)[G]$, so that $V \subseteq \Sigma$ is a 0DMC space, $E \subseteq G_{\sigma_{|\Sigma}}$ is a compact graph on V, $(K,G) \preceq_c^i (V,E)$ with witness φ, and $(V,E) \preceq_c^i (K,G)$ with witness φ^{-1} by compactness.

By Claim 3, $\big(\alpha, \sigma(\alpha)\big) \in E$ if $\alpha \in \mathrm{Orb}_\sigma(\beta_n)$. The density of $\mathrm{Orb}_\sigma(\beta_n)$ in Σ given by Claim 1 and the compactness of E then imply that $\mathrm{Graph}(\sigma_{|\Sigma}) \subseteq E$. As E is a graph, we get $E = G_{\sigma_{|\Sigma}}$ and therefore $V = \Sigma$. Thus $(\Sigma, G_{\sigma_{|\Sigma}}) \preceq_c^i (K,G)$ and $(\Sigma, G_{\sigma_{|\Sigma}})$ is $\preceq_c^i$-minimal in $\mathfrak{G}_2$ and in the class of closed graphs on a 0DMC space with CCN at least three.

Assume now that $\xi \geq \omega$ is of the form $\eta + 3$. Using ideas similar to those in [Ce-Da-To-Wy], we now provide a two-sided subshift with Cantor-Bendixson rank of the form ξ having the desired minimality property. The first step of our construction is inspired by [Ce-Da-To-Wy, Theorem 4.6]. Fix an infinite countable ordinal η, and a closed countable subset P of 2^ω with Cantor-Bendixson rank $\eta+1$, which exists by [K, 6.13]. The following fact is known. However, we include a proof for completeness.

Claim 4. Let P be a countable Polish space. Then $P \setminus P'$ is dense in P.

Indeed, we set, for $x \in P'$, $O_x := X \setminus \{x\}$, so that O_x is a dense open subset of X. Moreover, $P \setminus P' = \bigcap_{x \in P'} O_x$ is a G_δ subset of the countable space P. It remains to apply the Baire category theorem (see [K, 8.4]). $\diamond$

We enumerate $P \setminus P' := \{\gamma_j \mid j \in \omega\}$ and set, for $j \in \omega$,

$$w_j := 1^2 {}^\frown_{k<j} \big((01)^{\gamma_{(j)_0}(k) + \Sigma_{i<k} (\gamma_{(j)_0}(i)+1)} 1^2\big),$$

and $\delta_\infty := (01)^\infty \cdot 1 {}^\frown_{j \in \omega} (w_j (01)^{j+1})$. Similarly, we define $\Phi_0 : 2^\omega \to 2^\omega$ by

$$\Phi_0(\gamma) := {}^\frown_{k \in \omega} \big((01)^{\gamma(k) + \Sigma_{i<k} (\gamma(i)+1)} 1^2\big).$$

We also define $\Phi : 2^\omega \to 2^{\mathbb{Z}}$ by $\Phi(\gamma) := (01)^\infty \cdot 1^2 \Phi_0(\gamma)$. We then set

$$Q := \mathrm{Orb}_\sigma\big((01)^\infty \cdot (01)^\infty\big) \cup \mathrm{Orb}_\sigma\big((01)^\infty \cdot 1^2 (01)^\infty\big) \cup \bigcup_{\gamma \in P} \mathrm{Orb}_\sigma\big(\Phi(\gamma)\big)$$

and $\Sigma := Q \cup \mathrm{Orb}_\sigma(\delta_\infty)$. Note that P, Q and Σ are countable, and $\sigma[Q] = Q$, $\sigma[\Sigma] = \Sigma$.

Claim 5. Σ is a countable two-sided subshift with Cantor-Bendixson rank ξ, and $\mathrm{Orb}_\sigma(\delta_\infty)$ is dense in Σ.

Indeed, as in [Ce-Da-To-Wy], we check that Q is closed and has Cantor-Bendixson rank ξ. Note first that $\mathrm{Orb}_\sigma\big((01)^\infty \cdot (01)^\infty\big) = \{(01)^\infty \cdot (01)^\infty, (10)^\infty \cdot (10)^\infty\}$ is closed, as well as

$$\mathrm{Orb}_\sigma\big((01)^\infty \cdot (01)^\infty\big) \cup \mathrm{Orb}_\sigma\big((01)^\infty \cdot 1^2 (01)^\infty\big).$$

Let $(\delta_n)_{n\in\omega}$ be a sequence of elements of Q converging to $\delta \in 2^{\mathbb{Z}}$. By the previous remark, we may assume that the δ_n's are in $\bigcup_{\gamma\in P} \mathrm{Orb}_\sigma\big(\Phi(\gamma)\big)$. This gives $(\beta_n)_{n\in\omega} \in P^\omega$ and $(k_n)_{n\in\omega} \in \mathbb{Z}^\omega$ with $\delta_n = \sigma^{k_n}\big(\Phi(\beta_n)\big)$, and we may assume that $(\beta_n)_{n\in\omega}$ converges to $\beta \in P$. If we may assume that $(k_n)_{n\in\omega}$ is constant, then $\delta = \sigma^{k_0}\big(\Phi(\beta)\big) \in Q$ by continuity of Φ_0 and Φ. Otherwise, we may assume that $(k_n)_{n\in\omega}$ is either strictly increasing, or strictly decreasing. In the latter case, $\delta \in \mathrm{Orb}_\sigma\big((01)^\infty \cdot (01)^\infty\big) \subseteq Q$. So we may assume that $(k_n)_{n\in\omega}$ is strictly increasing and $k_0 \geq 2$. Note that

$$S := \big\{\frown_{k<l} \big((01)^{m_k} 1^2\big)(01)^\infty \mid l \in \omega \wedge \forall k < l-1 \;\; m_k < m_{k+1}\big\} \cup \\ \big\{\frown_{k\in\omega} \big((01)^{m_k} 1^2\big) \mid \forall k \in \omega \;\; m_k < m_{k+1}\big\}$$

and $\sigma[S]$ are closed subsets of 2^ω, as well as $C := S \cup \sigma[S]$. We define, for $\beta \in 2^{\mathbb{Z}}$, $\beta^* \in 2^\omega$ by $\beta^*(i) := \beta(i)$ if $i \in \omega$. As Φ_0 takes values in S, $\sigma^2[S] = S$ and $k_n \geq 2$, $\delta_n^* \in C$ for each n, and $\delta^* \in C$. If δ^* contains at most one 1^3, then $\delta \in \mathrm{Orb}_\sigma\big((01)^\infty \cdot (01)^\infty\big) \cup \mathrm{Orb}_\sigma\big((01)^\infty \cdot 1^2 (01)^\infty\big) \subseteq Q$. So we may assume that there are $n_0, n_1 \geq 1$ with $1^2(01)^{n_0}1^2(01)^{n_1}1^2 \subseteq \delta^*$, $(01)^{n_0}1^2(01)^{n_1}1^2 \subseteq \delta^*$ or $(10)^{n_0-1}11^2(01)^{n_1}1^2 \subseteq \delta^*$. As $(\delta_n^*)_{n\in\omega}$ converges to δ^*, we may assume that is also the case for the δ_n^*'s. Note then that, just after this initial segment, δ_n^* can have at most $n_1 + 2$ blocks 01 before having a block 1^2, by definition of Φ_0. This implies that δ^* is of the form $1^2 \frown_{k\in\omega} \big((01)^{n_k} 1^2\big)$, $\frown_{k\in\omega} \big((01)^{n_k} 1^2\big)$ or $(10)^{n_0-1}11^2 \frown_{k\geq 1} \big((01)^{n_k} 1^2\big)$, with $n_{k+1} + 1 \leq n_{k+2} \leq n_{k+1} + 2$. So we may assume that either $1^2 \frown_{k\leq n} \big((01)^{n_k} 1^2\big) \subseteq \delta_n^*$ for each n, $\frown_{k\leq n} \big((01)^{n_k} 1^2\big) \subseteq \delta_n^*$ for each n, or $(10)^{n_0-1}11^2 \frown_{1\leq k\leq n} \big((01)^{n_k} 1^2\big) \subseteq \delta_n^*$ for each n. Note that $\Phi_0(\beta_n)$ has an initial segment of the form either $\frown_{k\leq l} \big((01)^{m_k} 1^2\big) \frown_{k\leq n} \big((01)^{n_k} 1^2\big)$, or $\frown_{k\leq l} \big((01)^{m_k} 1^2\big) 0 (10)^{n_0-1} 11^2 \frown_{1\leq k\leq n} \big((01)^{n_k} 1^2\big)$. As $m_0 < m_1 < \cdots < m_l < n_0$ in both cases, there are only finitely many possible values for the block $\frown_{k\leq l} \big((01)^{m_k} 1^2\big)$. So we may assume that this block does not depend on n. Note then that $\delta_n^* = \sigma^{k_n - 2}\big(\Phi_0(\beta_n)\big)$. This implies that we may assume that $(k_n)_{n\in\omega}$ is constant, which is not the case. This shows that Q is closed.

In order to prove that Q has Cantor-Bendixson rank ξ, we introduce the notion of the *rank of a point*. If $\mathbb{X} \in \{\omega, \mathbb{Z}\}$, P is a countable compact subset of $2^{\mathbb{X}}$ and $\delta \in P$, then the rank $rk_P(\delta)$ of δ in P is the least ordinal α such that $\delta \notin P^{\alpha+1}$. Under this definition, the Cantor-Bendixson rank of P is $\sup\{rk_P(\delta) + 1 \mid \delta \in P\}$ (see [Ce-Da-To-Wy, Section 2]).

Note that $P_1 := \big\{\Phi(\gamma) \mid \gamma \in P\big\} \subseteq Q$, which implies that $rk_{P_1}(\delta) \leq rk_Q(\delta)$ if $\delta \in P_1$. By [Ce-Da-To-Wy, Lemma 3.3], $rk_Q\big(\sigma^k(\delta)\big) = rk_Q(\delta)$ if $k \in \mathbb{Z}$ and $\delta \in Q$.

Thus $rk_Q\big(\sigma^k(\delta)\big) \geq rk_{P_1}(\delta)$ if $k \in \mathbb{Z}$ and $\delta \in P_1$. It follows that $rk_Q\big((01)^\infty \cdot 1^2(01)^\infty\big) \geq \eta+1$ and hence $rk_Q\big((01)^\infty \cdot (01)^\infty\big) \geq \eta+2$.

For the other direction, note that the map $(k,\gamma) \mapsto \sigma^k\big(\Phi(\gamma)\big)$ is injective. We now prove by induction on $\rho := rk_{P_1}(\delta)$ that $rk_Q(\delta) = \rho$ if $\delta \in P_1$. If $\rho = 0$, then δ is isolated in P_1. If δ is not isolated in Q, then there is an injective sequence $(\delta_n)_{n\in\omega}$ of elements of Q converging to δ. The discussion above provides $k_0 \in \mathbb{Z}$ and $(\beta_n)_{n\in\omega} \in P^\omega$ converging to $\beta \in P$ with $\delta_n = \sigma^{k_0}\big(\Phi(\beta_n)\big)$. Thus $\delta = \sigma^{k_0}\big(\Phi(\beta)\big)$. The injectivity property shows that $k_0 = 0$, so that $\delta_n \in P_1$, contradicting the fact that δ is isolated in P_1. Suppose now that our claim holds for all ordinals strictly below ρ, and that $rk_Q(\delta) > \rho$. Then we can find an injective sequence $(\delta_n)_{n\in\omega}$ of elements of Q converging to δ. The discussion above provides $k_0 \in \mathbb{Z}$ and $(\beta_n)_{n\in\omega} \in P^\omega$ converging to $\beta \in P$ with $\delta_n = \sigma^{k_0}\big(\Phi(\beta_n)\big)$. The injectivity property shows that $k_0 = 0$, so that $\delta_n \in P_1$. As $rk_{P_1}(\delta) = \rho$, we may assume that $rk_{P_1}(\delta_n) < \rho$, so that $rk_Q(\delta_n) < \rho$ by induction assumption. This contradicts the fact that $\delta_n \in Q$. This implies that $rk_Q(\delta) \leq \eta$ if $\delta \in \bigcup_{\gamma\in P} \mathrm{Orb}_\sigma\big(\Phi(\gamma)\big)$. It follows that $rk_Q(\delta) \leq \eta+1$ if $\delta \in \mathrm{Orb}_\sigma\big((01)^\infty \cdot 1^2(01)^\infty\big)$, and $rk_Q(\delta) \leq \eta+2$ if $\delta \in \mathrm{Orb}_\sigma\big((01)^\infty \cdot (01)^\infty\big)$. This proves that Q has Cantor-Bendixson rank ξ.

Let us prove that Σ is closed. Assume that $(\delta_n)_{n\in\omega}$ is a sequence of elements of Σ converging to $\delta \in 2^{\mathbb{Z}}$. We may assume that the δ_n's are in $\mathrm{Orb}_\sigma(\delta_\infty)$ since Q is closed. This gives $k_n \in \mathbb{Z}$ with the property that $\delta_n = \sigma^{k_n}(\delta_\infty)$. If $(k_n)_{n\in\omega}$ has a constant subsequence, then $\delta \in \mathrm{Orb}_\sigma(\delta_\infty)$ and we are done. So we may assume that $(k_n)_{n\in\omega}$ tends to $\pm\infty$. If $(k_n)_{n\in\omega}$ tends to $-\infty$, then $\delta \in \mathrm{Orb}_\sigma\big((01)^\infty \cdot (01)^\infty\big)$, and we are done. So we may write $k_n = 1 + \Sigma_{j<l_n}\ (|w_j| + 2j + 2) + j_n$, where $j_n < |w_{l_n}| + 2l_n + 2$, and $(l_n)_{n\in\omega}$ tends to ∞. In particular, the distance between δ_n and $C := \{\beta \in 2^{\mathbb{Z}} \mid \exists \varepsilon \in 2\ \ \forall i \in \mathbb{Z}\ \ \beta(2i+\varepsilon) = 1\}$ tends to zero as n tends to infinity, so that δ is in the closed set C. As above we may assume that δ is not in $\mathrm{Orb}_\sigma\big((01)^\infty \cdot (01)^\infty\big) \cup \mathrm{Orb}_\sigma\big((01)^\infty \cdot 1^2(01)^\infty\big)$, which gives m minimal such that $1^2(01)^m1^2$ is a finite subword of δ, and we may also assume that $1^2(01)^m1^2 = \delta_{[0,2m+3]} = \delta_{n[0,2m+3]}$. This implies that we may assume that $j_n = 0$. Note then that

$$\begin{aligned}
\delta_n^* &= \frown_{j\geq l_n}\ (w_j(01)^{j+1}) \\
&= \frown_{j\geq l_n}\ \Big(1^2\Big(\frown_{k<j}\ \big((01)^{\gamma_{(j)_0}(k)+\Sigma_{i<k}\ (\gamma_{(j)_0}(i)+1)}1^2\big)\Big)(01)^{j+1}\Big) \\
&\supseteq 1^2\Big(\frown_{k<l_n}\ \big((01)^{\gamma_{(l_n)_0}(k)+\Sigma_{i<k}\ (\gamma_{(l_n)_0}(i)+1)}1^2\big)\Big)(01)^{l_n+1}
\end{aligned}$$

converges to δ^*. This implies that $(\gamma_{(l_n)_0})_{n\in\omega}$ converges to some $\gamma \in P$, and that $\delta = \Phi(\gamma) \in Q$, showing that Σ is closed.

This discussion above shows that if $\delta \in Q$, then $rk_Q(\delta) = 0$ if and only if there is $\gamma \in P \setminus P'$ with $\delta \in \mathrm{Orb}_\sigma\big(\Phi(\gamma)\big)$. In particular, $rk_\sigma(\delta) \geq 1$ if δ is not of this form. If now $j \in \omega$, then $\Phi(\gamma_j)$ is in $\overline{\mathrm{Orb}(\delta_\infty)}$, showing that Q is a subset of the Cantor-Bendixson derivative Σ' of Σ. As Σ is countable, it has an isolated point which has to be in $\mathrm{Orb}_\sigma(\delta_\infty)$, and $\Sigma' = Q$ since σ is a homeomorphism. This implies that $\Sigma^{k+1} = Q^k$ for each natural number k, and $\Sigma^\theta = Q^\theta$ if θ is infinite, so that Σ has Cantor-Bendixson rank ξ since η is infinite.

The density assertion comes from the previous discussion and Claim 4. $\diamond$

We then partially argue as in the finite case. Note that

$$\big((01)^\infty \cdot (01)^\infty, (01)^\infty \cdot (01)^\infty\big) = \lim_{p\to\infty}\ \big(\sigma^{-2p}(\delta_\infty), \sigma^{1+\Sigma_{j<2p}\ (|w_j|+2j+2)+|w_{2p}|+2p}(\delta_\infty)\big)$$

is in $\overline{\bigcup_{q\in\omega}\ G^{2q+1}_{\sigma_{|\Sigma}}}$, so that $\chi_c(\Sigma, G_{\sigma_{|\Sigma}}) = 3$.

Claim 6. Let $V \subseteq \Sigma$, and $E \subseteq G_{\sigma_{|\Sigma}} \cap V^2$ be a graph on V such that (V,E) has CCN three. Then $\big(\alpha, \sigma(\alpha)\big) \in E$ if $\alpha \in Orb_\sigma(\delta_\infty)$.

Indeed, we argue by contradiction. Let $k \in \mathbb{Z}$ with $\alpha = \sigma^k(\delta_\infty)$. As in the proof of Claim 3, we may assume that $k=0$. We set

$$C := ([0]_0 \cup [1^2 0]_0 \cup [01^4 0]_{-2}) \cap V,$$

so that C is a clopen subset of V. Noted that the odd coordinates of the w_j's are 1, and that the w_j's have a stricly positive even length. The definition of the w_j's and δ_∞ then imply that 0 can only be an even coordinate of the elements appearing in the definition of Q (which is the union of the orbits of these elements), a negative even coordinate of δ_∞ or a positive odd coordinate of δ_∞. Moreover, 01^k0 can be of the form $\alpha_{[a,b]}$ with α appearing in the definition of Q only if $k \in \{1,3\}$ and a is even, and of the form $\delta_{\infty[a,b]}$ only if $k=1$ and $a \leq -4$ is even, $k \in \{1,3\}$ and $a \geq 3$ is odd, or $k=4$ and $a=-2$. In particular,

$$\dots, \sigma^{-2}(\alpha), \alpha, \sigma^2(\alpha), \dots \in C,$$
$$\dots, \sigma^{-3}(\alpha), \sigma^{-1}(\alpha), \sigma(\alpha), \dots \notin C$$

if α appears in the definition of Q, and

$$\dots, \sigma^{-4}(\delta_\infty), \sigma^{-2}(\delta_\infty), \delta_\infty, \sigma(\delta_\infty), \sigma^3(\delta_\infty), \sigma^5(\delta_\infty), \dots \in C,$$
$$\dots, \sigma^{-5}(\delta_\infty), \sigma^{-3}(\delta_\infty), \sigma^{-1}(\delta_\infty), \sigma^2(\delta_\infty), \sigma^4(\delta_\infty), \sigma^6(\delta_\infty), \dots \notin C.$$

This implies that $E \cap \big(C^2 \cup (V \setminus C)^2\big) = \emptyset$, which contradicts the fact that (V,E) has CCN three. $\diamond$

We now conclude as in the finite case.

(b) Let $Q := (q_j)_{j\in\omega} \in \omega^\omega$ converging to infinity. We set $\alpha_0 := (01)^\infty \cdot (01)^\infty$, $\alpha_1 := (01)^\infty \cdot 1^2 (01)^\infty$ and $\beta_Q := (01)^\infty \cdot 1^\frown{}_{j\in\omega} \big((01)^{q_j} 1^2\big)$. This defines $\Sigma_Q = \bigcup_{m\leq 1} \mathrm{Orb}_\sigma(\alpha_m) \cup \mathrm{Orb}_\sigma(\beta_Q)$.

We first essentially argue as in the finite case when $n=1$ to check the individual properties of Σ_Q.

Claim 7. (a) Let $1 \leq i \leq 2$. Then $\Sigma_Q^{(i)} = \overline{Orb_\sigma(\alpha_{2-i})}$.

(b) $\Sigma_Q = \overline{Orb_\sigma(\beta_Q)}$.

Indeed, let us prove that $\alpha_1 \in \overline{\mathrm{Orb}_\sigma(\beta_Q)}$. Fix a natural number N. If j is large enough, then $q_j \geq N$, so that $(01)^N 1^2 (01)^N = \alpha_{1[-2N,2N+1]} = \beta_{Q[1+(\Sigma_{n<j}\ (2q_n+2))+2(q_j-N),(\Sigma_{n\leq j}\ (2q_n+2))+2N]}$. This implies that $\alpha_1 \in \overline{\mathrm{Orb}_\sigma(\beta_Q)}$, as desired.

Assume then that $(k_p)_{p\in\omega} \in \mathbb{Z}^\omega$ and $\big(\sigma^{k_p}(\beta_Q)\big)_{p\in\omega}$ converges to $\alpha \in 2^\mathbb{Z}$. We want to see that $\alpha \in \Sigma_Q$. If $(k_p)_{p\in\omega}$ tends to ∞, then we may write $k_p = 1 + \big(\Sigma_{j<l_p}\ (2q_j+2)\big) + j_p$, where $j_p < 2q_{l_p}+2$, and $(l_p)_{p\in\omega}$ tends to ∞. If $(j_p)_{p\in\omega}$ and $(2q_{l_p}+2-j_p)_{p\in\omega}$ also tend to ∞, then $\alpha \in \overline{\mathrm{Orb}_\sigma(\alpha_0)} \subseteq \Sigma_Q$, as desired. Otherwise, we may assume that $(j_p)_{p\in\omega}$ or $(2q_{l_p}+2-j_p)_{p\in\omega}$ is constant, so that α is in $\mathrm{Orb}_\sigma(\alpha_1) \subseteq \Sigma_Q$. $\diamond$

Claim 8. Σ_Q has Cantor-Bendixson rank 3.

Indeed, as all the odd coordinates of the α_m's are 1, and we can find an even coordinate of β_Q and an odd coordinate of β_Q which are 0, so that $\alpha_m \notin \mathrm{Orb}_\sigma(\beta_Q)$ if $m \leq 1$. It remains to note that $\alpha_0 \notin \mathrm{Orb}_\sigma(\alpha_1)$ to get $\Sigma_Q^{(2)} = (\Sigma_Q^{(1)})'$ and, inductively, that $\Sigma_Q^{(2)}$ is the 2nd iterated Cantor-Bendixson derivative of Σ_Q. $\diamond$

Note then that

$$(\alpha_0,\alpha_0)=\lim_{p\to\infty}\left(\sigma^{-2p}(\beta_Q),\sigma^{1+(\Sigma_{j<p}\ (2q_j+2))+2\lceil\frac{q_p}{2}\rceil}(\beta_Q)\right)\in\overline{\bigcup_{q\in\omega}G^{2q+1}_{\sigma_{|\Sigma_Q}}},$$

so that $\chi_c(\Sigma_Q,G_{\sigma_{|\Sigma_Q}})=3$.

Claim 9. Let $V\subseteq\Sigma_Q$, and $E\subseteq G_{\sigma_{|\Sigma_Q}}\cap V^2$ be a graph on V such that (V,E) has CCN three. Then $\big(\alpha,\sigma(\alpha)\big)\in E$ if $\alpha\in Orb_\sigma(\beta_Q)$.

Indeed, let $k\in\mathbb{Z}$ with $\alpha=\sigma^k(\beta_Q)$. As in the proof of Claim 3, we may assume that $k=0$. Fix $j_0\in\omega$ such that $q_j\geq 1$ if $j\geq j_0$. We set

$$C:=([0]_0\cup\bigcup_{j\leq q_{j_0}}[01^{2j+1}0]_{-2}\cup\bigcup_{j\leq q_{j_0}}[01^{2j+2}0]_{-2})\cap V,$$

so that C is a clopen subset of V. We already noticed that the odd coordinates of the α_m's are 1. The definition of the α_m's and β_Q then imply that 0 can only be an even coordinate of α_m, a negative even coordinate of β_Q or a positive odd coordinate of β_Q. Moreover, 01^k0 can be of the form $\alpha_{m[a,b]}$ only if $k\in\{1,3\}$ and a is even, and of the form $\beta_{Q[a,b]}$ only if $k=1$ and $a\leq -4$ is even, $k\in\{1,3\}$ and $a\geq 1$ is odd, or $2\leq k\leq 2j_0+2$ is even and $a=-2$. This implies that $E\cap\big(C^2\cup(V\setminus C)^2\big)=\emptyset$ since

$$\begin{array}{l}\ldots,\sigma^{-2}(\alpha_m),\alpha_m,\sigma^2(\alpha_m),\ldots\in C,\\ \ldots,\sigma^{-3}(\alpha_m),\sigma^{-1}(\alpha_m),\sigma(\alpha_m),\ldots\notin C\end{array}$$

if $m\leq 1$ and

$$\begin{array}{l}\ldots,\sigma^{-4}(\beta_Q),\sigma^{-2}(\beta_Q),\beta_Q,\sigma(\beta_Q),\sigma^3(\beta_Q),\sigma^5(\beta_Q),\ldots\in C,\\ \ldots,\sigma^{-5}(\beta_Q),\sigma^{-3}(\beta_Q),\sigma^{-1}(\beta_Q),\sigma^2(\beta_Q),\sigma^4(\beta_Q),\sigma^6(\beta_Q),\ldots\notin C,\end{array}$$

which contradicts the fact that (V,E) has CCN three again. $\diamond$

We conclude as in the finite case to get the individual properties of Σ_Q. We now provide a family of size continuum of countable subshifts $(\Sigma_{Q^\nu})_{\nu\in 2^\omega}$. Let $(p_n)_{n\in\omega}$ be the sequence of prime numbers. We set, for $\nu\in 2^\omega$ and $n\in\omega$, $q^\nu_0:=0$ and $q^\nu_{n+1}:=p_0^{\nu(0)+2}\cdots p_n^{\nu(n)+2}-1$, which defines $Q^\nu\in\omega^\omega$ converging to infinity.

Let us check that the family $\big((\Sigma_{Q^\nu},G_{\sigma_{|\Sigma_{Q^\nu}}})\big)_{\nu\in 2^\omega}$ is a $\preceq^i_c$-antichain. Assume, towards a contradiction, that $\nu\neq\nu'$ and $(\Sigma_{Q^\nu},G_{\sigma_{|\Sigma_{Q^\nu}}})\preceq^i_c(\Sigma_{Q^{\nu'}},G_{\sigma_{|\Sigma_{Q^{\nu'}}}})$ with witness φ. Let m_0 be minimal with $\nu(m_0)\neq\nu'(m_0)$. By minimality of $(\Sigma_{Q^{\nu'}},G_{\sigma_{|\Sigma_{Q^{\nu'}}}})$, we may assume that $\nu(m_0)<\nu'(m_0)$. If $x\in\Sigma_{Q^\nu}$, then $\big(x,\sigma(x)\big)\in G_{\sigma_{|\Sigma_{Q^\nu}}}$ since $\sigma_{|\Sigma_{Q^\nu}}$ is fixed point free. Thus $\Big(\varphi(x),\varphi\big(\sigma(x)\big)\Big)\in G_{\sigma_{|\Sigma_{Q^{\nu'}}}}$ and $\varphi\big(\sigma(x)\big)=\sigma^{\pm 1}\big(\varphi(x)\big)$. We choose $x\in\Sigma_{Q^\nu}\setminus\mathrm{Orb}_\sigma(\alpha_0)$, so that $\mathrm{Orb}_\sigma(x)$ is infinite, as well as $\mathrm{Orb}_\sigma\big(\varphi(x)\big)\supseteq\varphi[\mathrm{Orb}_\sigma(x)]$, and $\sigma^2(x)\neq x$. In particular, $\sigma^2_{|\mathrm{Orb}_\sigma(\varphi(x))}$ is fixed point free. We apply Lemma 5.6 and its proof to $V=X=\Sigma_{Q^\nu}$, $f=\sigma_{|\Sigma_{Q^\nu}}$, $I=\mathrm{Orb}_\sigma(x)$, $W=Y=\Sigma_{Q^{\nu'}}$, $g=\sigma_{|\Sigma_{Q^{\nu'}}}$ and φ. The proof of Lemma 5.6 shows that $P\cap M=\emptyset$, either $\varphi\big(\sigma(z)\big)=\sigma\big(\varphi(z)\big)$ for each $z\in\mathrm{Orb}_\sigma(x)$ or $\varphi\big(\sigma(z)\big)=\sigma^{-1}\big(\varphi(z)\big)$ for each $z\in\mathrm{Orb}_\sigma(x)$, and $\varphi[\mathrm{Orb}_\sigma(x)]=\mathrm{Orb}_\sigma\big(\varphi(x)\big)$. In particular, $\varphi[\mathrm{Orb}_\sigma(\alpha_1)],\varphi[\mathrm{Orb}_\sigma(\beta_{Q^\nu})]$ are disjoint infinite orbits in $\Sigma_{Q^{\nu'}}$, so they are $\mathrm{Orb}_\sigma(\alpha_1),\mathrm{Orb}_\sigma(\beta_{Q^{\nu'}})$.

As $\mathrm{Orb}_\sigma(\beta_{Q^{\nu'}})$ is dense in $\Sigma_{Q^{\nu'}}$, the compact set $\varphi[\Sigma_{Q^\nu}]$ is $\Sigma_{Q^{\nu'}}$, so that φ is a homeomorphism from Σ_{Q^ν} onto $\Sigma_{Q^{\nu'}}$. Moreover, φ is a witness for the fact that $\sigma_{|\Sigma_{Q^\nu}}$ and $\sigma_{|\Sigma_{Q^{\nu'}}}$ are flip-conjugate, by density of $\mathrm{Orb}_\sigma(\beta_{Q^\nu})$ in Σ_{Q^ν}. In particular, $\varphi[\Sigma'_{Q^\nu}]=\Sigma'_{Q^{\nu'}}$ and $\varphi[\Sigma''_{Q^\nu}]=\Sigma''_{Q^{\nu'}}$, so that $\varphi[\mathrm{Orb}_\sigma(\alpha_0)]=\mathrm{Orb}_\sigma(\alpha_0)$, $\varphi[\mathrm{Orb}_\sigma(\alpha_1)]=\mathrm{Orb}_\sigma(\alpha_1)$ and $\varphi[\mathrm{Orb}_\sigma(\beta_{Q^\nu})]=\mathrm{Orb}_\sigma(\beta_{Q^{\nu'}})$. This gives $n_0,n_1\in\mathbb{Z}$ with $\varphi(\alpha_1)=\sigma^{n_1}(\alpha_1)$ and $\varphi(\beta_{Q^\nu})=\sigma^{n_0}(\beta_{Q^{\nu'}})$. We then set, for $r\in\omega$,

$$K^\nu_r:=1+(\Sigma_{j<r}\ (2q_j+2))+2q_r.$$

Note that the sequence $\big(\sigma^{K^\nu_r}(\beta_{Q^\nu})\big)_{r\in\omega}$ converges to α_1, so that $\Big(\varphi\big(\sigma^{K^\nu_r}(\beta_{Q^\nu})\big)\Big)_{r\in\omega}$ converges to $\varphi(\alpha_1)=\sigma^{n_1}(\alpha_1)$. As $\varphi\big(\sigma^{K^\nu_r}(\beta_{Q^\nu})\big)=\sigma^{n_0\pm K^\nu_r}(\beta_{Q^{\nu'}})$, this implies that $\big(\sigma^{n_0-n_1\pm K^\nu_r}(\beta_{Q^{\nu'}})\big)_{r\in\omega}$ converges to α_1. As $(K^\nu_r)_{r\in\omega}$ is strictly increasing, this implies that $\sigma_{|\Sigma_{Q^\nu}}$ and $\sigma_{|\Sigma_{Q^{\nu'}}}$ are conjugate and $\big(\sigma^{n_0-n_1+K^\nu_r}(\beta_{Q^{\nu'}})\big)_{r\in\omega}$ converges to α_1. In particular,

$$\sigma^{n_0-n_1+K^\nu_r}(\beta_{Q^{\nu'}})_{[-2,2]}=\big(\beta_{Q^{\nu'}}(n_0-n_1+K^\nu_r-2),\cdots,\beta_{Q^{\nu'}}(n_0-n_1+K^\nu_r+2)\big)=\alpha_{1[-2,2]}=01^30$$

if r is large enough. Using similar notation, this implies that $n_0-n_1+K^\nu_r\in\{K^{\nu'}_m\mid m\in\omega\}$ if r is large enough. In particular, this gives, for r large enough, $m<M\in\omega$ with $n_0-n_1+K^\nu_r=K^{\nu'}_m$ and $n_0-n_1+K^\nu_{r+1}=K^{\nu'}_M$. Thus $K^\nu_{r+1}-K^\nu_r=2q^\nu_{r+1}+2=\Sigma_{m<j\leq M}\ (2q^{\nu'}_j+2)$ and

$$p_0^{\nu(0)+2}\cdots p_r^{\nu(r)+2}=q^\nu_{r+1}+1=\Sigma_{m\leq n<M}\ (q^{\nu'}_{n+1}+1)=\Sigma_{m\leq n<M}\ (p_0^{\nu'(0)+2}\cdots p_n^{\nu'(n)+2}).$$

We may assume that r is large enough to ensure that $r,m\geq m_0$, which implies that $p_{m_0}^{\nu'(m_0)+2}$ divides $p_0^{\nu(0)+2}\cdots p_r^{\nu(r)+2}$, which cannot be since $\nu(m_0)<\nu'(m_0)$. □

Note that Theorem 1.15(b) provides a version of Corollary 10.12 (and thus Theorem 1.13) for countable subshifts (which are not necessary minimal). By minimality and for cardinality reasons, the examples provided by Theorem 1.15(b) are $\preceq^i_c$-incompatible in the class of closed graphs on a 0DMC space with CCN at least three with the examples given by Proposition 10.13.

11 Homeomorphisms of a 0DMS space

In this section, we prove Theorem 1.14, among other things.

Remarks. (a) Let X be a 0DMC space, and $f:X\to X$ be a homeomorphism such that (X,G_f) has CCN at least three. Lemma 7.10 says that if f is minimal, then (X,G_f) is $\preceq^i_c$-minimal in the class of closed graphs on a 0DMC space with CCN at least three. Theorem 1.15 shows that the converse is not true since the finite orbit $\mathrm{Orb}_\sigma(\alpha_0)$ is not dense in the infinite countable space K_0.

(b) Theorem 1.15 also provides $(K_0,G_{h_0})\in\mathfrak{G}_2$ for which it is not possible to find $(K,G_f)\in\mathfrak{G}_2$ with f minimal and $(K,G_f)\preceq^i_c(K_0,G_{h_0})$. In particular, K_0 contains no subshift Σ such that $(\Sigma,\sigma_{|\Sigma})$ is minimal and has CCN at least three, even if it contains $\mathrm{Orb}_\sigma(\alpha_0)$ and $(\mathrm{Orb}_\sigma(\alpha_0),\sigma_{|\mathrm{Orb}_\sigma(\alpha_0)})$ is minimal.

We will see that (K_0,G_{h_0}) is no more minimal in 0DMS (or 0DP) spaces. Let $\mathcal{T}$ be the set of finer 0DMS topologies τ on K_0 such that $\big((K_0,\tau),G_{h_0}\big)$ has CCN at least three and h_0 is a homeomorphism of (K_0,τ).

Lemma 11.1 *Let S be a 0DMS (resp., 0DP) space, f be a homeomorphism of S with the properties that $\chi_c(S,G_f)\geq 3$ and $(S,G_f)\preceq^i_c (K_0,G_{h_0})$. Then there is a finer 0DMS (resp., 0DP) topology τ in $\mathcal{T}$ with the property that $\big((K_0,\tau),G_{h_0}\big)\preceq^i_c (S,G_f)$.*

Proof. By Theorem 1.15, (K_0,G_{h_0}) has CCN three, and thus (S,G_f) has CCN three too. Therefore the set F_1 of fixed points of f is a clopen subset of S, and $\chi_c(S\setminus F_1, G_f\cap (S\setminus F_1)^2)=3$ by Corollary 7.3. So we may assume that f is fixed point free. Let φ be a witness for the fact that $(S,G_f)\preceq^i_c (K_0,G_{h_0})$. We define $V:=\varphi[S]$ and $E:=(\varphi\times\varphi)[G_f]$. The finer topology is

$$\tau:=\{O\subseteq K_0 \mid \varphi^{-1}(O)\in \mathbf{\Sigma}^0_1(S)\}.$$

Note that $\varphi: S\to (V,\tau)$ is a homeomorphism, so that (V,τ) is a 0DMS (resp., 0DP) space. As φ is a witness for the fact that $(S,G_f)\preceq^i_c \big((K_0,\tau),G_{h_0}\big)$, $\chi_c\big((K_0,\tau),G_{h_0}\big)\geq 3$. Also, φ^{-1} is a witness for the fact that $\big((V,\tau),E\big)\preceq^i_c (S,G_f)$.

Let us prove that $V=K_0$ and $E=G_{h_0}$. As φ is a witness for the fact that $(S,G_f)\preceq^i_c (V,E)$, $\chi_c(V,E)\geq 3$. As $\mathrm{Orb}_{h_0}(\beta_0)$ is discrete, there is $\varepsilon\in 2$ with $\sigma^\varepsilon(\alpha_0)\in V$, which gives $x\in S$ with $\varphi(x)=\sigma^\varepsilon(\alpha_0)$. As f is fixed point free, $f(x)\neq x$, which implies that $\big(x,f(x)\big)\in G_f$, $\Big(\varphi(x),\varphi\big(f(x)\big)\Big)\in G_{h_0}$, and $\sigma^{1-\varepsilon}(\alpha_0)=\varphi\big(f(x)\big)\in V$. This implies that $\mathrm{Orb}_\sigma(\alpha_0)\subseteq V$ and $\big\{\big(\alpha_0,\sigma(\alpha_0)\big),\big(\sigma(\alpha_0),\alpha_0\big)\big\}\subseteq E$. As $\chi_c(V,E)\geq 3$ again, there is $\alpha\in\mathrm{Orb}_{h_0}(\beta_0)\cap V$, which gives $y\in S$ with $\varphi(y)=\alpha$.

Let us check that $f^2(y)\neq y$. We argue by contradiction. As just above, $\varphi\big(f(y)\big)=h_0\big(\varphi(y)\big)$ or $\varphi\big(f(y)\big)=h_0^{-1}\big(\varphi(y)\big)$. Assume that $\varphi\big(f(y)\big)=h_0^{-1}\big(\varphi(y)\big)$, the other case being similar. Then $\big(\alpha,h_0(\alpha)\big)\notin E$ since $f(y)=f^{-1}(y)$ is sent to $h_0^{-1}(\alpha)$ by φ and h_0^2 is fixed point free on $\mathrm{Orb}_{h_0}(\beta_0)$. This contradicts Claim 3 in the proof of Theorem 1.15.

As $f^2(y)\neq y$, either $\varphi\big(f(z)\big)=h_0\big(\varphi(z)\big)$ for each $z\in\mathrm{Orb}_f(y)$, or $\varphi\big(f(z)\big)=h_0^{-1}\big(\varphi(z)\big)$ for each $z\in\mathrm{Orb}_f(y)$, by Lemma 5.6. Lemma 5.2 then implies that $\varphi[\mathrm{Orb}_f(y)]=\mathrm{Orb}_{h_0}(\alpha)=\mathrm{Orb}_{h_0}(\beta_0)$. Thus $V=K_0$ and $E=G_{h_0}$. In particular, $\big((K_0,\tau),G_{h_0}\big)\preceq^i_c (S,G_f)$ and τ is a finer 0DMS (resp., 0DP) topology on K_0.

The previous discussion shows that $S=\mathrm{Orb}_f(x)\cup\mathrm{Orb}_f(y)$ by injectivity of φ, and $\varphi\circ f=h_0^{\pm 1}\circ\varphi$. Thus $h_0=\varphi\circ f^{\pm 1}\circ\varphi^{-1}$ is a homeomorphism of (K_0,τ). $\square$

Notation. Lemma 9.7 provides a sequence $(S_q)_{q\in\omega}$ of pairwise disjoint infinite subsets of ω such that, for any $l\in\omega$, $p\neq q$, $3\cdot 2^{l+1}<r\in S_p$ and $s\in S_q$, $|r-s|>2^{l+1}$. We enumerate, here again, S_q in a strictly increasing way by $\{r^q_j \mid j\in\omega\}$. We set, for $l\in\omega$, $j_l:=3\cdot 2^{l+1}+1$ and, for $A\subseteq\omega$,

$$N_A:=\{r^q_{j_l}+r \mid (q>0\Rightarrow q-1\in A)\wedge l\in\omega\wedge -l\leq r\leq l\}.$$

Note that $N_A\subseteq\omega$. Let τ be a finer 0DMS topology on K_0, and $\mathcal{B}^\tau$ be a countable basis, made up of clopen sets and closed under finite intersections, for τ. We set, for $A\subseteq\omega$,

$$\begin{aligned}\mathcal{B}^\tau_A:=\mathcal{B}^\tau\cup\Big\{C\cap(\{(01)^\infty\cdot(01)^\infty\}\cup\textstyle\bigcup_{n\in\bigcap_{-p_0\leq r\leq q_0}(N_A+r)\cap\omega}\{(01)^\infty 1(01)^n\cdot(01)^\infty\}\cup\\ \textstyle\bigcup_{n\in\bigcap_{-p_1\leq r\leq q_1}(N_A+r)\cap\omega}\{(01)^\infty\cdot(01)^{n+1}1(01)^\infty\})\mid C\in\mathcal{B}^\tau\wedge p_0,q_0,p_1,q_1\in\omega\Big\}\cup\\ \Big\{C\cap(\{(10)^\infty\cdot(10)^\infty\}\cup\textstyle\bigcup_{n\in\bigcap_{-p_0\leq r\leq q_0}(N_A+r)\cap\omega}\{(01)^\infty 1(01)^n 0\cdot(10)^\infty\}\cup\\ \textstyle\bigcup_{n\in\bigcap_{-p_1\leq r\leq q_1}(N_A+r)\cap\omega}\{(10)^\infty\cdot 1(01)^n 1(01)^\infty\})\mid C\in\mathcal{B}^\tau\wedge p_0,q_0,p_1,q_1\in\omega\Big\}.\end{aligned}$$

Lemma 11.2 *Let $A\subseteq\omega$ and τ be in $\mathcal{T}$. Then*
(a) $\mathcal{B}^\tau_A$ is the basis for a 0DMS (0DP if τ is) topology t^τ_A in $\mathcal{T}$ finer than τ,
(b) the sequences $\big((01)^\infty 1(01)^{r^0_{j_l}}\cdot(01)^\infty\big)_{l\in\omega}$ and $\big((01)^\infty\cdot(01)^{r^0_{j_l}+1}1(01)^\infty\big)_{l\in\omega}$ are t^τ_A-converging to $(01)^\infty\cdot(01)^\infty$, as well as $\big((01)^\infty 1(01)^{r^{q+1}_{j_l}}\cdot(01)^\infty\big)_{l\in\omega}$ and $\big((01)^\infty\cdot(01)^{r^{q+1}_{j_l}+1}1(01)^\infty\big)_{l\in\omega}$ if $q\in A$.

Proof. Note that $\mathcal{B}^\tau_A$ contains $\mathcal{B}^\tau$, is countable, and closed under finite intersections. Thus $\mathcal{B}^\tau_A$ is the basis for a second countable topology t^τ_A finer than τ. Moreover, $\mathcal{B}^\tau_A$ is made up of closed subsets of K_0, so that t^τ_A is zero-dimensional. This shows that (K_0,t^τ_A) is a 0DP space (see [K, 13.2 and 13.3]) if (K_0,τ) is.

We apply the definitions of N_A and $\mathcal{B}^\tau_A$ to see that (b) holds. Note then that

$$\big((01)^\infty\cdot(01)^{r^0_{j_l}+1}1(01)^\infty,(01)^\infty 1(01)^{r^0_{j_l}}\cdot(01)^\infty\big)=\big(h_0^{-2r^0_{j_l}-2}(\beta^0_0),h_0^{2r^0_{j_l}+1}(\beta^0_0)\big)\in G_{h_0}^{4r^0_{j_l}+3}.$$

Thus $\big((K_0,t^\tau_A),G_{h_0}\big)$ has CCN at least three by (b) and Lemma 3.3.1. It remains to note that $\big((K_0,t^\tau_A),G_{h_0}\big)\preceq_c(K_0,G_{h_0})$ to see that the CCN is three. Note that

$$\begin{aligned}h_0[\{(10)^\infty\cdot(10)^\infty\}\cup\textstyle\bigcup_{n\in\bigcap_{-p_0\le r\le q_0}(N_A+r)\cap\omega}\{(01)^\infty 1(01)^n 0\cdot(10)^\infty\}\cup\\ \textstyle\bigcup_{n\in\bigcap_{-p_1\le r\le q_1}(N_A+r)\cap\omega}\{(10)^\infty\cdot 1(01)^n 1(01)^\infty\}]=\\ \{(01)^\infty\cdot(01)^\infty\}\cup\textstyle\bigcup_{n\in\bigcap_{-p_0\le r\le q_0}(N_A+r)\cap\omega}\{(01)^\infty 1(01)^{n+1}\cdot(01)^\infty\}\cup\\ \textstyle\bigcup_{n\in\bigcap_{-p_1\le r\le q_1}(N_A+r)\cap\omega}\{(01)^\infty\cdot(01)^n 1(01)^\infty\}\end{aligned}$$

is equal, up to a finite open discrete set, to

$$\begin{aligned}\{(01)^\infty\cdot(01)^\infty\}\cup\textstyle\bigcup_{n\in\bigcap_{-p_0+1\le r\le q_0+1}(N_A+r)\cap\omega}\{(01)^\infty 1(01)^n\cdot(01)^\infty\}\cup\\ \textstyle\bigcup_{n\in\bigcap_{-p_1-1\le r\le q_1-1}(N_A+r)\cap\omega}\{(01)^\infty\cdot(01)^{n+1}1(01)^\infty\},\end{aligned}$$

and thus t^τ_A-open. We argue similarly with h_0^{-1} instead of h_0, or with $(01)^\infty\cdot(01)^\infty$ instead of $(10)^\infty\cdot(10)^\infty$, to see that h_0 is a homeomorphism as desired. □

Lemma 11.3 *Let $A,B\subseteq\omega$ with $A\not\subseteq B$, τ be in $\mathcal{T}$, and τ' be in $\mathcal{T}$ finer than τ. Then*

$$\big((K_0,t^{\tau'}_A),G_{h_0}\big)\not\preceq^i_c\big((K_0,t^\tau_B),G_{h_0}\big).$$

Proof. Towards a contradiction, suppose that there is $\varphi:(K_0,t^{\tau'}_A)\to(K_0,t^\tau_B)$. We define $V:=\varphi[K_0]$ and $E:=(\varphi\times\varphi)[G_{h_0}]$. If $z\in\mathrm{Orb}_{h_0}(\beta_0)$, then $\big(z,h_0(z)\big)\in G_{h_0}$ since h_0 is fixed point free, so that $\varphi\big(h_0(z)\big)=h_0^{\pm1}\big(\varphi(z)\big)$. Thus $\varphi[\mathrm{Orb}_{h_0}(\beta_0)]\subseteq\mathrm{Orb}_{h_0}(\beta_0)$ or $\varphi[\mathrm{Orb}_{h_0}(\beta_0)]\subseteq\mathrm{Orb}_{h_0}(\alpha_0)$. By injectivity of φ, the first case holds. As h_0^2 is fixed point free on $\mathrm{Orb}_{h_0}(\beta_0)$, either $\varphi\big(h_0(z)\big)=h_0\big(\varphi(z)\big)$ for each $z\in\mathrm{Orb}_{h_0}(\beta_0)$, or $\varphi\big(h_0(z)\big)=h_0^{-1}\big(\varphi(z)\big)$ for each $z\in\mathrm{Orb}_{h_0}(\beta_0)$, by Lemma 5.6. Lemma 5.2 then implies that $\varphi[\mathrm{Orb}_{h_0}(\beta_0)]=\mathrm{Orb}_{h_0}\big(\varphi(\beta_0)\big)$. This provides $n\in\mathbb{Z}$ with $\varphi(\beta_0)=h_0^n(\beta_0)$. So either $\varphi\big(h_0^i(\beta_0)\big)=h_0^{n+i}(\beta_0)$ for each $i\in\mathbb{Z}$, or $\varphi\big(h_0^i(\beta_0)\big)=h_0^{n-i}(\beta_0)$ for each $i\in\mathbb{Z}$, by Lemma 5.2 again. In particular, if $q\in A\setminus B$ and $l\in\omega$, then

$$\varphi\big((01)^\infty 1(01)^{r^{q+1}_{j_l}}\cdot(01)^\infty\big)=\varphi\big(h_0^{2r^{q+1}_{j_l}+1}(\beta_0)\big)\in\{h_0^{n+2r^{q+1}_{j_l}+1}(\beta_0),h_0^{n-2r^{q+1}_{j_l}-1}(\beta_0)\}$$

is t^τ_B-converging to $\varphi\big((01)^\infty\cdot(01)^\infty\big)$, in $\{(01)^\infty\cdot(01)^\infty,(10)^\infty\cdot(10)^\infty\}$ by injectivity of φ since $\varphi[\mathrm{Orb}_{h_0}(\beta_0)]=\mathrm{Orb}_{h_0}(\beta_0)$.

Assume, for example, that $\varphi\big((01)^\infty\cdot(01)^\infty\big)=(10)^\infty\cdot(10)^\infty$, the other case being similar. Then

$$\{(10)^\infty\cdot(10)^\infty\}\cup\bigcup_{n\in N_B}\ \{(01)^\infty 1(01)^n 0\cdot(10)^\infty\}\cup\bigcup_{n\in N_B}\ \{(10)^\infty\cdot 1(01)^n 1(01)^\infty\}=$$
$$\{(10)^\infty\cdot(10)^\infty\}\cup\bigcup_{n\in N_B}\ \{h_0^{2n+2}(\beta_0^0),h_0^{-2n-1}(\beta_0^0)\}$$

is a t_B^τ-neighborhood of $\varphi\big((01)^\infty\cdot(01)^\infty\big)$, and thus contains $\varphi\big((01)^\infty 1(01)^{r_{j_l}^{q+1}}\cdot(01)^\infty\big)$ if l is large enough.

The injectivity of h_0 provides, for example, $p\in B$, $k\in\omega$ and $-k\le r\le k$ with the property that $n-2r_{j_l}^{q+1}-1=-2(r_{j_k}^{p+1}+r)-1$ if l is large enough (the other case is similar). This implies that $2|r_{j_l}^{q+1}-r_{j_k}^{p+1}|=|n+2r|\le|n|+2k$, k goes to ∞ as l goes to ∞, and $|r_{j_l}^{q+1}-r_{j_k}^{p+1}|\le 2k$ if l is large enough. As $q+1\ne p+1$, this contradicts our choice of j_k since $|r_{j_l}^{q+1}-r_{j_k}^{p+1}|>2^{k+1}>2k$. □

Proof of Theorem 1.14. (a) We apply Theorem 1.15.

(b) Lemma 11.1 provides a finer 0DMS (resp., 0DP) topology τ on K_0 such that h_0 is a homeomorphism of (K_0,τ), $\chi_c\big((K_0,\tau),G_{h_0}\big)\ge 3$ and $\big((K_0,\tau),G_{h_0}\big)\preceq_c^i(S,G_f)$. We apply Lemmas 11.2 and 11.3 to $A:=\{2q\mid q\in\omega\}$, $B:=\{2q+1\mid q\in\omega\}$, and τ. As t_A^τ,t_B^τ are finer than τ, $\big((K_0,t_A^\tau),G_{h_0}\big),\big((K_0,t_B^\tau),G_{h_0}\big)\preceq_c^i\big((K_0,\tau),G_{h_0}\big)$. As $A\not\subseteq B$,

$$\big((K_0,t_A^\tau),G_{h_0}\big)\not\preceq_c^i\big((K_0,t_B^\tau),G_{h_0}\big)$$

and thus $\big((K_0,\tau),G_{h_0}\big)\not\preceq_c^i\big((K_0,t_B^\tau),G_{h_0}\big)$. This proves that $\big((K_0,t_B^\tau),G_{h_0}\big)$ is strictly $\preceq_c^i$-below $\big((K_0,\tau),G_{h_0}\big)$, and also (S,G_f).

Assume now, towards a contradiction, that there is a $\preceq_c^i$-antichain basis $\mathfrak{B}$ for a class in the statement. By Theorem 1.15, (K_0,G_{h_0}) is in this class, which gives $(S,G_f)\in\mathfrak{B}$ with the property that $(S,G_f)\preceq_c^i(K_0,G_{h_0})$. The first part of this theorem provides a finer 0DMS (resp., 0DP) topology τ' on K_0 with the properties that h_0 is a homeomorphism of (K_0,τ'), $\big((K_0,\tau'),G_{h_0}\big)$ has CCN at least three, and $\big((K_0,\tau'),G_{h_0}\big)$ is strictly $\preceq_c^i$-below (S,G_f), which is the desired contradiction.

For the size of the basis, towards a contradiction, suppose we can find $\kappa<2^{\aleph_0}$ and a basis $(B_\gamma)_{\gamma<\kappa}$ for our class. Let $(p_n)_{n\in\omega}$ be the sequence of prime numbers. We define, for each $\alpha\in 2^\omega$, $S_\alpha\subseteq\omega$ by

$$S_\alpha:=\{p_0^{\alpha(0)+1}\dots p_n^{\alpha(n)+1}\mid n\in\omega\}.$$

Note that $S_\alpha\subseteq\omega$ is infinite, and $S_\alpha\cap S_\beta$ is finite if $\alpha\ne\beta$. By Theorem 1.15, we can apply Lemma 11.2 to any S_α and the topology τ_0 on K_0, so that $t_{S_\alpha}^{\tau_0}$ is a 0DP topology in $\mathcal{T}$. As $\kappa<2^{\aleph_0}$, we can find $\gamma<\kappa$ and $\alpha\ne\beta$ with $B_\gamma\preceq_c^i\big((K_0,t_{S_\alpha}^{\tau_0}),G_{h_0}\big)$, $\big((K_0,t_{S_\beta}^{\tau_0}),G_{h_0}\big)\preceq_c^i(K_0,G_{h_0})$. Lemma 11.1 provides a finer 0DMS (resp., 0DP) topology τ in $\mathcal{T}$ with $\big((K_0,\tau),G_{h_0}\big)\preceq_c^i B_\gamma$. We can apply again Lemma 11.2, to $A:=\omega$ and τ, so that t_ω^τ is a 0DMS (resp., 0DP) topology in $\mathcal{T}$ finer than τ, so that $\big((K_0,t_\omega^\tau),G_{h_0}\big)\preceq_c^i\big((K_0,\tau),G_{h_0}\big)\preceq_c^i\big((K_0,t_{S_\alpha}^{\tau_0}),G_{h_0}\big)$. As S_β is infinite, $\omega\not\subseteq S_\alpha$. Lemma 11.3 then implies that $\big((K_0,t_\omega^\tau),G_{h_0}\big)\not\preceq_c^i\big((K_0,t_{S_\alpha}^{\tau_0}),G_{h_0}\big)$, which is the desired contradiction. □

We turn to the proof of Theorem 1.16(a). In fact, we prove something stronger since it is possible to consider always the same graph, with different underlying 0DP spaces.

Proof of Theorem 1.16(a). We define, for $A\subseteq\omega$,

$$K_A:=\{\varepsilon^{n+1}(\varepsilon+1 \bmod 4)1^\infty\mid\varepsilon\in 4\wedge n\in\omega\}\cup\{\varepsilon^{n+1}(\varepsilon-1 \bmod 4)0^\infty\mid\varepsilon\in 4\wedge n\in\omega\}\cup$$
$$\{\varepsilon^\infty\mid\varepsilon\in 5\}\cup\bigcup_{n\in A}\ \{\varepsilon^{n+2}(\varepsilon+1 \bmod 4)s2^\infty\mid\varepsilon\in 4\wedge s\in 2^{n+1}\}.$$

We then enumerate $2^{n+1} := \{s_i^{n+1} \mid i < 2^{n+1}\} \subseteq 2^{<\omega}$ and define a function $h_A : K_A \to K_A$ by $h_A(\varepsilon^\infty) := (\varepsilon+1 \bmod 4)^\infty$ if $\varepsilon \in 4$, $h_A(4^\infty) := 01^\infty$, $h_A(320^\infty) := 4^\infty$,

$$h_A(\varepsilon^{n+1}(\varepsilon+1 \bmod 4)1^\infty) := (\varepsilon+1 \bmod 4)^{n+1}(\varepsilon+2 \bmod 4)1^\infty$$

if $\varepsilon \neq 3$, $h_A(3^{n+1}01^\infty) := 0^{n+2}1^\infty$, $h_A(\varepsilon^{n+1}(\varepsilon-1 \bmod 4)0^\infty) := (\varepsilon+1 \bmod 4)^{n+1}\varepsilon 0^\infty$ if $\varepsilon \neq 3$, as well as $h_A(3^{n+2}20^\infty) := 0^{n+1}30^\infty$ on the one hand,

$$h_A(\varepsilon^{n+2}(\varepsilon+1 \bmod 4)s_i^{n+1}2^\infty) := (\varepsilon+1 \bmod 4)^{n+2}(\varepsilon+2 \bmod 4)s_i^{n+1}2^\infty$$

if $\varepsilon \neq 3$, $h_A(3^{n+2}0s_i^{n+1}2^\infty) := 0^{n+2}1s_{i+1 \bmod 2^{n+1}}^{n+1}2^\infty$ on the other hand. In other words, K_A is the union of the h_A-orbit $\{0^\infty, 1^\infty, 2^\infty, 3^\infty\}$, the orbit $\{4^\infty\} \cup \{\varepsilon^{n+1}(\varepsilon \pm 1 \bmod 4)\eta^\infty \mid \varepsilon \in 2 \wedge n \in \omega \wedge \eta \in 2\}$ in the style of the infinite h_0^0-orbit of K_0^0, and even cycles given by the elements of A. The beginning of the proof of Theorem 1.15 shows that K_A is a countable 0DMC space, h_A is a homeomorphism of K_A, and (K_A, G_{h_A}) has CCN three. If $A \subseteq B$, then $K_A \subseteq K_B$, $G_{h_A} \subseteq G_{h_B}$, which implies that $(K_A, G_{h_A}) \preceq_c^i (K_B, G_{h_B})$. If $A \not\subseteq B$, then let $n \in A \setminus B$. Note that $(K_A, G_{h_A}) \not\preceq_c^i (K_B, G_{h_B})$ because (K_A, G_{h_A}) has a cycle of length $4 \cdot 2^{n+1}$ and (K_B, G_{h_B}) does not. $\square$

As announced in the introduction, one can check that the $\sigma_{|\Sigma}$'s appearing in the statement of Theorem 1.15 are expansive, which leaves the question of infinite Cantor Bendixson ranks uncertain.

Remark. By [K, 33.B], the set $\mathcal{K}_{\aleph_0}(2^\omega)$ of countable compact subsets of 2^ω is $\mathbf{\Pi}^1_1$-complete. By [K, 34.18(3)], the Cantor-Bendixson rank $|\cdot|_{CB}$ is a co-analytic rank on $\mathcal{K}_{\aleph_0}(2^\omega)$. Thus the map $r : (X, f) \mapsto |X|_{CB}$ defines a co-analytic rank on $\mathcal{P} \cap \big(\mathcal{K}_{\aleph_0}(2^\omega) \times \mathcal{H}(2^\omega)\big)$ ($\mathcal{P}$ was defined before Theorem 1.12). By [K, 35.23], $|\cdot|_{CB}$ has to be unbounded. Thus r is unbounded, which implies that the co-analytic set $\mathcal{P} \cap \big(\mathcal{K}_{\aleph_0}(2^\omega) \times \mathcal{H}(2^\omega)\big)$ is not Borel, by [K, 35.23] again. By [K, 34.2], $(X, f) \mapsto |X|_{CB}$ also defines a co-analytic rank on $\mathcal{O}_2^{\aleph_0} := \mathcal{O}_2 \cap \big(\mathcal{K}_{\aleph_0}(2^\omega) \times \mathcal{H}(2^\omega)\big)$. Theorem 1.15 implies that the co-analytic subset $\mathcal{O}_2^{\aleph_0}$ of $\mathcal{P}$ is not Borel. This set is in fact $\mathbf{\Pi}^1_1$-complete. Indeed, define $\{s_i \mid i \in 3\} := \{0^2, 10, 1^2\}$, $f_0 \in \mathcal{H}(2^\omega)$ by $f_0(0\alpha) := 0\alpha$, $f_0(1s\alpha) := 1s\alpha$ if $s \in 2^2 \setminus \{s_i \mid i \in 3\}$, and $f_0(1s_i\alpha) := 1s_{i+1 \bmod 3}\alpha$. The map $X \mapsto (\{0\alpha \mid \alpha \in X\} \cup \{1s_i0^\infty \mid i \in 3\}, f_0)$ is a continuous reduction of $\mathcal{K}_{\aleph_0}(2^\omega)$ to $\mathcal{O}_2^{\aleph_0}$, by [K, 4.29] and since G_{f_0} contains the 3-cycle $\{1s_i0^\infty \mid i \in 3\}$.

12 The classes $\mathfrak{G}_\kappa$

We consider, for $\kappa \leq 3$,

- the class $\mathfrak{G}_\kappa$ of graphs induced by a homeomorphism of a 0DMC space with CCN strictly bigger than κ,

- the class $\mathfrak{H}_\kappa$ of homeomorphisms of 2^ω whose induced graph has CCN strictly bigger than κ.

Proof of Theorem 1.11. (a) The CCN is strictly bigger than 0 if and only if the space is not empty.

(b) The CCN is strictly bigger than 1 if and only if the graph is not empty.

(c) By Theorem 1.13, any $\preceq_c^i$-basis for $\mathfrak{G}_2$ must have size continuum.

(d) We apply Proposition 7.4(b).

For the well-foundedness, fix $n \in \omega$. We enumerate the set of finite binary sequences 2^{n+1} by $\{s_i \mid i < 2^{n+1}\}$, so that $N_{0^n1} = N_{0^n10} \cup \bigcup_{i<2^{n+1}} N_{0^n1^2s_i}$.

We consider the map c_n on N_{0^n1} defined by $c_n(0^n10\alpha):=0^n1^2s_0\alpha$, $c_n(0^n1^2s_i\alpha):=0^n1^2s_{i+1}\alpha$ if $i<2^{n+1}-1$, and $c_n(0^n1^2s_{2^{n+1}-1}\alpha):=0^n10\alpha$. Note that c_n is a homeomorphism, $c_n^{2^{n+1}+1}=\mathrm{Id}_{N_{0^n1}}$, and $c_n^i(\beta)\neq\beta$ if $i\leq 2^{n+1}$. Moreover, the function $h_p:2^\omega\to 2^\omega$, defined by $h_p(\beta):=\beta$ if β is in $\{0^\infty\}\cup\bigcup_{n<p}\ N_{0^n1}$ and $h_p(\beta):=c_n(\beta)$ if $n\geq p$ and $\beta\in N_{0^n1}$, is a homeomorphism whose set $\{0^\infty\}\cup\bigcup_{n<p}\ N_{0^n1}$ of fixed points is not open. By Proposition 7.2, $\chi_c(2^\omega,G_{h_p})=2^{\aleph_0}$, so that $(2^\omega,G_{h_p})$ is in all the $\mathfrak{G}_\kappa$'s. As $G_{h_{p+1}}\subseteq G_{h_p}$, $(2^\omega,G_{h_{p+1}})\preceq_c^i(2^\omega,G_{h_p})$. As G_{h_p} contains a cycle of length $2^{p+1}+1$ and all the cycles in $G_{h_{p+1}}$ have length at least $2^{p+2}+1$, $(2^\omega,G_{h_p})\not\preceq_c(2^\omega,G_{h_{p+1}})$.

Theorem 1.13 provides $\preceq_c$-antichains of size continuum in $\mathfrak{G}_\kappa$ if $\kappa\leq 2$. For $\mathfrak{G}_3$, we use again the c_n's. Let $(S_\alpha)_{\alpha\in 2^\omega}$ be as in the proof of Theorem 9.4. The map $h_\alpha:2^\omega\to 2^\omega$, defined by $h_\alpha(\beta):=\beta$ if $\beta\in\{0^\infty\}\cup\bigcup_{n\notin S_\alpha}\ N_{0^n1}$ and $h_\alpha(\beta):=c_n(\beta)$ if $n\in S_\alpha$ and $\beta\in N_{0^n1}$, is a homeomorphism whose set of fixed points $\{0^\infty\}\cup\bigcup_{n\notin S_\alpha}\ N_{0^n1}$ is not open. By Proposition 7.2, $\chi_c(2^\omega,G_{h_\alpha})=2^{\aleph_0}$, so that $(2^\omega,G_{h_\alpha})$ is in all the $\mathfrak{G}_\kappa$'s. If $\alpha\neq\beta$, then there is $n\in S_\alpha\setminus S_\beta$, so that G_{h_α} contains a cycle of length $2^{n+1}+1$, which is not the case of G_{h_β}. Thus $(2^\omega,G_{h_\alpha})\not\preceq_c^i(2^\omega,G_{h_\beta})$. □

We can also evaluate the descriptive complexity of $\mathfrak{G}_\kappa$ and $\mathfrak{H}_\kappa$. Let $\mathcal{H}(2^\omega)$ be the set of homeomorphisms of 2^ω. We equip $\mathcal{H}(2^\omega)$ with the topology whose basic open sets are of the form

$$O_{U_1,\dots,U_n,V_1,\dots,V_n}:=\{f\in\mathcal{H}(2^\omega)\mid\forall 1\leq i\leq n\ \ f[U_i]=V_i\},$$

where n is a natural number and U_i,V_i are clopen subsets of 2^ω. By [I-Me, Section 2], this defines a structure of Polish group on $\mathcal{H}(2^\omega)$. A compatible complete distance is given by

$$d(f,g):=\sup_{\alpha\in 2^\omega}\ d_{2^\omega}\big(f(\alpha),g(\alpha)\big)+\sup_{\alpha\in 2^\omega}\ d_{2^\omega}\big(f^{-1}(\alpha),g^{-1}(\alpha)\big).$$

Lemma 12.1 *The map $f\mapsto\mathrm{Graph}(f)$ from $\mathcal{H}(2^\omega)$ into $\mathcal{K}(2^\omega\times 2^\omega)$ is continuous.*

Proof. If $O\neq\emptyset$ is an open subset of $2^\omega\times 2^\omega$, and $(s_n)_{n\in\omega},(t_n)_{n\in\omega}$ are sequences of finite binary sequences with $O=\bigcup_{n\in\omega}\ (N_{s_n}\times N_{t_n})$, then

$$\mathrm{Graph}(f)\subseteq O\Leftrightarrow\exists F\subseteq\omega\text{ finite with }\mathrm{Graph}(f)\subseteq U_F:=\bigcup_{n\in F}\ (N_{s_n}\times N_{t_n}).$$

If $\mathrm{Graph}(f)\subseteq U_F$, $l:=\max_{n\in F}\ |t_n|$ and $d(f,g)<2^{-l}$, then $\mathrm{Graph}(g)\subseteq U_F$, which proves that $\{f\in\mathcal{H}(2^\omega)\mid\mathrm{Graph}(f)\subseteq O\}$ is open. Now

$$\mathrm{Graph}(f)\cap O\neq\emptyset\Leftrightarrow\exists n\in\omega\ \ \mathrm{Graph}(f)\cap(N_{s_n}\times N_{t_n})\neq\emptyset\Leftrightarrow\exists n\in\omega\ \ \exists\alpha\in N_{s_n}\ \ f(\alpha)\in N_{t_n},$$

so that $\{f\in\mathcal{H}(2^\omega)\mid\mathrm{Graph}(f)\cap O\neq\emptyset\}$ is open. □

Lemma 12.2 *The map $f\mapsto F_1^f$ from $\mathcal{H}(2^\omega)$ into $\mathcal{K}(2^\omega)$ is Baire class one and not continuous. In fact, $\{f\in\mathcal{H}(2^\omega)\mid F_1^f\subseteq U\}$ is open for each open subset U of 2^ω.*

Proof. If U is an open subset of 2^ω, then $F_1^f\subseteq U\Leftrightarrow\mathrm{Graph}(f)\subseteq\neg\Delta(2^\omega\setminus U)$. This implies that $\{f\in\mathcal{H}(2^\omega)\mid F_1^f\subseteq U\}$ is open by Lemma 12.1. If now $U=\bigcup_{n\in\omega}\ N_{s_n}$ is not empty, then $F_1^f\cap U\neq\emptyset\Leftrightarrow\exists n\in\omega\ \ \mathrm{Graph}(f)\not\subseteq\neg\Delta(N_{s_n})$, so that $\{f\in\mathcal{H}(2^\omega)\mid F_1^f\cap U\neq\emptyset\}$ is $\mathbf{\Sigma}^0_2$. This last set is not open if $U=2^\omega$ since it contains Id, which is the limit of g_n defined by $g_n(\alpha)(p)=\alpha(p)\Leftrightarrow p\leq n$. This finishes the proof. □

Notation. We define a family $(h_s)_{s\in\omega^{<\omega}}$ of functions from 2^ω into itself as follows. If $s\in\omega^{<\omega}$, then we set $s(-1):=0$.

- If $|s|$ is even, then we set $h_s(0^\infty):=0^\infty$,

$$h_s(0^{(\Sigma_{i<j}\ (s(2i)+1))+p}1t\varepsilon\alpha):=0^{(\Sigma_{i<j}\ (s(2i)+1))+p}1t(1-\varepsilon)\alpha$$

if $j<\frac{|s|}{2}$, $p\leq s(2j)$, $t\in 2^{s(2j-1)}$, $\varepsilon\in 2$ and $\alpha\in 2^\omega$, and

$$h_s(0^{(\Sigma_{i<\frac{|s|}{2}}\ (s(2i)+1))+p}1t\varepsilon\alpha):=0^{(\Sigma_{i<\frac{|s|}{2}}\ (s(2i)+1))+p}1t(1-\varepsilon)\alpha$$

if $p\in\omega$, $t\in 2^{s(|s|-1)}$, $\varepsilon\in 2$ and $\alpha\in 2^\omega$.

- If $|s|$ is odd, then we set

$$h_s(0^{(\Sigma_{i<j}\ (s(2i)+1))+p}1t\varepsilon\alpha):=0^{(\Sigma_{i<j}\ (s(2i)+1))+p}1t(1-\varepsilon)\alpha$$

if $j\leq\frac{|s|-1}{2}$, $p\leq s(2j)$, $t\in 2^{s(2j-1)}$, $\varepsilon\in 2$ and $\alpha\in 2^\omega$, and

$$h_s(0^{\Sigma_{i\leq\frac{|s|-1}{2}}\ (s(2i)+1)}\alpha):=0^{\Sigma_{i\leq\frac{|s|-1}{2}}\ (s(2i)+1)}\alpha$$

if $\alpha\in 2^\omega$.

Lemma 12.3 *The h_s's are continuous involutions, $\chi_c(2^\omega,G_{h_s})=2^{\aleph_0}$ if $|s|$ is even, $\chi_c(2^\omega,G_{h_s})=2$ if $|s|$ is odd, and $(h_{sn})_{n\in\omega}$ converges to h_s in $\mathcal{H}(2^\omega)$.*

Proof. Note that h_s is a continuous involution, and thus a homeomorphism. If $|s|$ is even, then 0^∞ is the only fixed point of the map h_s, so that $\chi_c(2^\omega,G_{h_s})=2^{\aleph_0}$ by Proposition 7.2. If $|s|$ is odd, then $F_1^{h_s}=N_{0^{\Sigma_{i\leq\frac{|s|-1}{2}}\ (s(2i)+1)}}$ is a clopen subset of 2^ω. By Corollaries 7.3(a) and 8.2, $\chi_c(2^\omega,G_{h_s})\in\{2,3\}$. By Corollary 7.3(b) and Proposition 7.5, $\chi_c(2^\omega,G_{h_s})=2$. Note that the inequality $\sup_{\alpha\in 2^\omega} d_{2^\omega}\big(h_{sn}(\alpha),h_s(\alpha)\big)<2^{-n}$ holds. We are done since the h_s's are involutions. $\square$

Lemma 12.4 *The map $(K,f)\mapsto f[K]$ from $\mathcal{K}(2^\omega)\times\mathcal{C}(2^\omega,2^\omega)$ into $\mathcal{K}(2^\omega)$ is continuous. This is also the case if we replace $\mathcal{C}(2^\omega,2^\omega)$ with $\mathcal{H}(2^\omega)$, 2^ω with $\mathcal{K}_{2^\infty}$. The map $(K,f)\mapsto(f\times f)[K]$ from $\mathcal{K}(2^\omega\times 2^\omega)\times\mathcal{C}(2^\omega,2^\omega)$ into $\mathcal{K}(2^\omega\times 2^\omega)$ is also continuous.*

Proof. Let $O\neq\emptyset$ be an open subset of 2^ω, and $(s_n)_{n\in\omega}$ be a sequence of finite binary sequences with $O=\bigcup_{n\in\omega}\ N_{s_n}$. Note that

$$\begin{aligned} f[K]\subseteq O &\Leftrightarrow \exists F\subseteq\omega \text{ finite with } f[K]\subseteq\textstyle\bigcup_{n\in F}\ N_{s_n}\\ &\Leftrightarrow \exists F\subseteq\omega \text{ finite } \exists C\in\mathbf{\Delta}^0_1(2^\omega)\ K\subseteq C\wedge C\subseteq f^{-1}(\textstyle\bigcup_{n\in F}\ N_{s_n}).\end{aligned}$$

Let $l_F:=\max_{n\in F}|s_n|$. If $\sup_{\alpha\in 2^\omega} d_{2^\omega}\big(f(\alpha),g(\alpha)\big)<2^{-l_F}$ and $C\subseteq f^{-1}(\bigcup_{n\in F}\ N_{s_n})$, then

$$C\subseteq g^{-1}(\bigcup_{n\in F}\ N_{s_n}),$$

so that $\{(K,f)\in\mathcal{K}(2^\omega)\times\mathcal{C}(2^\omega,2^\omega)\mid K\subseteq C\wedge C\subseteq f^{-1}(\bigcup_{n\in F}\ N_{s_n})\}$ is open. This shows that $\{(K,f)\in\mathcal{K}(2^\omega)\times\mathcal{C}(2^\omega,2^\omega)\mid f[K]\subseteq O\}$ is open (even if O is empty).

Now

$$\begin{aligned} f[K]\cap O\neq\emptyset &\Leftrightarrow \exists n\in\omega\ f[K]\cap N_{s_n}\neq\emptyset \\ &\Leftrightarrow \exists n\in\omega\ \exists C\in\mathbf{\Delta}^0_1(2^\omega)\ K\cap C\neq\emptyset\wedge C\subseteq f^{-1}(N_{s_n}), \end{aligned}$$

so that $\{(K,f)\in\mathcal{K}(2^\omega)\times\mathcal{C}(2^\omega,2^\omega)\mid f[K]\cap O\neq\emptyset\}$ is open. □

Theorem 12.5 *$\mathfrak{H}_0=\mathcal{H}(2^\omega)$, $\mathfrak{H}_1$ is a $\mathbf{\Sigma}^0_1\setminus\mathbf{\Pi}^0_1$ subset of $\mathcal{H}(2^\omega)$, while $\mathfrak{H}_2,\mathfrak{H}_3$ are $\mathbf{\Pi}^0_2\setminus\mathbf{\Sigma}^0_2$ subsets of $\mathcal{H}(2^\omega)$.*

Proof. We may restrict our attention to $\mathfrak{H}_2,\mathfrak{H}_3$ since $\mathfrak{H}_1=\mathcal{H}(2^\omega)\setminus\{\mathrm{Id}\}$. Fix now $\kappa<\omega$. Note that $\chi_c(2^\omega,G_f)\leq\kappa$ holds if and only if

$$\exists(C_i)_{i<\kappa}\in\mathbf{\Delta}^0_1(2^\omega)^\kappa\ \ (\forall i\neq j<\kappa\ \ C_i\cap C_j=\emptyset)\wedge(2^\omega\subseteq\bigcup_{i<\kappa}C_i)\wedge(\forall i<\kappa\ \ G_f\cap C_i^2=\emptyset).$$

As κ is finite and $\mathbf{\Delta}^0_1(2^\omega)$ is countable, $\mathbf{\Delta}^0_1(2^\omega)^\kappa$ is countable. So we can restrict our attention to

$$\begin{aligned} G_f\cap C_i^2=\emptyset &\Leftrightarrow 2^\omega\subseteq(2^\omega\setminus C_i)\cup f^{-1}(2^\omega\setminus C_i)\cup F_1^f \\ &\Leftrightarrow \neg(\exists C\in\mathbf{\Delta}^0_1(2^\omega)\setminus\{\emptyset\}\ \ (2^\omega\setminus C_i)\cup f^{-1}(2^\omega\setminus C_i)\cup F_1^f\subseteq 2^\omega\setminus C). \end{aligned}$$

By Lemmas 12.4 and 12.2, $\mathcal{H}(2^\omega)\setminus\mathfrak{H}_\kappa$ is a $\mathbf{\Sigma}^0_2$ subset of $\mathcal{H}(2^\omega)$. In particular, $\mathfrak{H}_2,\mathfrak{H}_3$ are $\mathbf{\Pi}^0_2$. It remains to see that $\mathfrak{H}_2,\mathfrak{H}_3$ are not $\mathbf{\Sigma}^0_2$. We will use the family $(h_s)_{s\in\omega^{<\omega}}$ defined before Lemma 12.3. We set $P:=\overline{\{h_s\mid s\in\omega^{<\omega}\}}$, so that P is a Polish space. Note that $\mathfrak{H}_3\cap P\subseteq\mathfrak{H}_2\cap P$ are dense and co-dense $\mathbf{\Pi}^0_2$ subsets of P, by Lemma 12.3. By Baire's theorem, $\mathfrak{H}_2,\mathfrak{H}_3$ are not $\mathbf{\Sigma}^0_2$. □

We next turn to the $\mathfrak{G}_\kappa$'s.

Proof of Theorem 1.12. Note that $f[X]=X\Leftrightarrow f[X]\subseteq X\wedge f^{-1}[X]\subseteq X$. By Lemma 12.4 and [K, 4.29], $\mathcal{P}$ is a closed subset of the Polish space $\mathcal{K}(2^\omega)\times\mathcal{H}(2^\omega)$, and thus a Polish space. Note also that $\mathcal{O}_0=\mathcal{P}\setminus\big(\{\emptyset\}\times\mathcal{H}(2^\omega)\big)$ is a clopen subset of $\mathcal{P}$ since $\emptyset$ is an isolated point in $\mathcal{K}(2^\omega)$.

Note that $\chi_c(X,G_{f_{|X}})\leq 1\Leftrightarrow X\subseteq F_1^f$, i.e., $f_{|X}=\mathrm{Id}_{|X}$. Let $\big((X_n,h_n)\big)_{n\in\omega}$ be a sequence of elements of $\mathcal{P}\setminus\mathcal{O}_1$ converging to a point (X,h) of $\mathcal{P}$. As $\chi_c(X_n,G_{(h_n)_{|X_n}})\leq 1$, $(h_n)_{|X_n}=\mathrm{Id}_{|X_n}$. Assume, towards a contradiction, that $h_{|X}\neq\mathrm{Id}_{|X}$. This gives $\alpha\in X$ with $h(\alpha)\neq\alpha$, $l\in\omega$ such that $h(\alpha)|l\neq\alpha|l$, and $L\geq l$ such that $g(\beta)|l=h(\alpha)|l\neq\alpha|l=\beta|l$ if $\beta\in N_{\alpha|L}$ and $d(g,h)<2^{-L}$. As $\alpha\in X\cap N_{\alpha|L}$, $X_n\cap N_{\alpha|L}\neq\emptyset$ and $d(h_n,h)<2^{-L}$ if n is large enough. We pick, for such a n, $\beta\in X_n\cap N_{\alpha|L}$, so that $h_n(\beta)|l\neq\beta|l$, contradicting $(h_n)_{|X_n}=\mathrm{Id}_{|X_n}$. This shows that $\mathcal{O}_1$ is open. We define, for $n\in\omega$, a map $g_n:2^\omega\to 2^\omega$ by $g_n(0^\infty):=0^\infty$, $g_n(0^p1\alpha):=0^p1\alpha$ if $p<2n$, and $g_n(0^{2p+\varepsilon}1\alpha):=0^{2p+(1-\varepsilon)}1\alpha$ if $p\geq n$ and $\varepsilon\in 2$, so that g_n is a continuous involution whose set of fixed points $\{0^\infty\}\cup\bigcup_{p<2n}N_{0^p1}$ is not open. By Proposition 7.2, $\chi_c(2^\omega,G_{g_n})=2^{\aleph_0}$. As the sequence (g_n) converges in $\mathcal{H}(2^\omega)$ to Id, $\mathcal{O}_1$ is not closed. Thus $\mathcal{O}_1$ is $\mathbf{\Sigma}^0_1$-complete, by [K, 22.11].

Fix now $\kappa<\omega$. Note that, by [E, Theorem 2.1(1)],

$$\begin{aligned} \chi_c(X,G_{f_{|X}})\leq\kappa &\Leftrightarrow \exists(C_i)_{i<\kappa}\in\mathbf{\Delta}^0_1(X)^\kappa\ \ (\forall i\neq j<\kappa\ \ C_i\cap C_j=\emptyset)\wedge(X\subseteq\textstyle\bigcup_{i<\kappa}C_i)\wedge \\ &\qquad\qquad (\forall i<\kappa\ \ G_f\cap C_i^2=\emptyset) \\ &\Leftrightarrow \exists(C_i)_{i<\kappa}\in\mathbf{\Delta}^0_1(2^\omega)^\kappa\ \ (\forall i\neq j<\kappa\ \ C_i\cap C_j=\emptyset)\wedge(2^\omega\subseteq\textstyle\bigcup_{i<\kappa}C_i)\wedge \\ &\qquad\qquad (\forall i<\kappa\ \ G_f\cap(X\cap C_i)^2=\emptyset). \end{aligned}$$

As κ is finite and $\mathbf{\Delta}^0_1(2^\omega)$ is countable, $\mathbf{\Delta}^0_1(2^\omega)^\kappa$ is countable. So we can restrict our attention to

$$\begin{aligned} G_f \cap (X \cap C_i)^2 = \emptyset &\Leftrightarrow X \subseteq (2^\omega \setminus C_i) \cup f^{-1}(2^\omega \setminus C_i) \cup F_1^f \\ &\Leftrightarrow \neg(\exists C \in \mathbf{\Delta}^0_1(2^\omega)\ \ X \cap C \neq \emptyset \wedge (2^\omega \setminus C_i) \cup f^{-1}(2^\omega \setminus C_i) \cup F_1^f \subseteq 2^\omega \setminus C). \end{aligned}$$

By Lemmas 12.4 and 12.2, $\mathcal{P} \setminus \mathcal{O}_\kappa$ is a $\mathbf{\Sigma}^0_2$ subset of $\mathcal{P}$. In particular, the sets $\mathcal{O}_2, \mathcal{O}_3$ are $\mathbf{\Pi}^0_2$. As $(\mathcal{O}_\kappa)_{2^\omega} = \mathfrak{H}_\kappa$, $\mathcal{O}_2, \mathcal{O}_3$ are not $\mathbf{\Sigma}^0_2$ by Theorem 12.5. Thus $\mathcal{O}_2, \mathcal{O}_3$ are $\mathbf{\Pi}^0_2$-complete, by [K, 22.11]. The $\mathbf{\Pi}^1_1$-completeness of $\mathcal{O}_2^{\aleph_0}$ was proved at the very end of Section 11. $\square$

13 Equivalence relations

Lemma 7.11 and Corollary 5.10 imply that FCO is Borel reducible to different versions of $\equiv^i_c$.

Notation. We set $\mathbb{M} := \{f \in \mathcal{H}(2^\omega) \mid f \text{ is minimal}\}$. By [Me, Lemma 4.1], the set $\mathbb{M}$ is a G_δ subset of $\mathcal{H}(2^\omega)$, and thus a Polish space. If $f, g \in \mathbb{M}$, then f, g are flip-conjugate if and only if there is $\varphi \in \mathcal{H}(2^\omega)$ with $\varphi \circ f = g \circ \varphi$ or $\varphi \circ f = g^{-1} \circ \varphi$, proving that FCO is analytic. Similarly, CO is analytic.

We first consider the case of graphs induced by a function. As in the introduction, we consider the equivalence relation $\equiv^i_c := \preceq^i_c \cap (\preceq^i_c)^{-1}$ on $\mathcal{S}_m$ associated with

$$(2^\omega, K) \preceq^i_c (2^\omega, K') \Leftrightarrow \exists \varphi : 2^\omega \to 2^\omega \text{ injective continuous with } K \subseteq (\varphi \times \varphi)^{-1}(K').$$

We define a map $g : \mathbb{M} \to \mathcal{S}_m$ by $g(f) := (2^\omega, G_f)$ (see Theorem 8.1).

Theorem 13.1 *The equivalence relation $\equiv^i_c$ on the Polish space $\mathcal{S}_m$ is analytic, and g reduces continuously FCO to $\equiv^i_c$. Moreover, the vertices of the graph $g(f)$ have degree two, for each $f \in \mathbb{M}$.*

Proof. By Lemma 7.11, g reduces FCO to $\equiv^i_c$. Let O be an open subset of $2^\omega \times 2^\omega$, and $(C^0_n)_{n\in\omega}$, $(C^1_n)_{n\in\omega}$ be sequences of clopen subsets of 2^ω with $O = \bigcup_{n\in\omega} (C^0_n \times C^1_n)$. If $f \in \mathbb{M}$ and $G_f \subseteq O$, then there is a finite subset F of ω with $G_f = s\big(\text{Graph}(f)\big) \subseteq \bigcup_{n\in F} (C^0_n \times C^1_n)$. Note then that $\bigcup_{n\in F} (C^0_n \times C^1_n) = \bigcup_{S\subseteq F} \big((\bigcap_{n\in S} C^0_n \cap \bigcap_{n\in F\setminus S} 2^\omega \setminus C^0_n) \times (\bigcup_{n\in S} C^1_n)\big)$. Thus

$$\begin{aligned} \text{Graph}(f) \subseteq \bigcup_{n\in F} (C^0_n \times C^1_n) &\Leftrightarrow \forall S \subseteq F\ f[\bigcap_{n\in S} C^0_n \cap \bigcap_{n\in F\setminus S} 2^\omega \setminus C^0_n] \subseteq \bigcup_{n\in S} C^1_n \\ &\Leftrightarrow \forall S \subseteq F\ \exists R_n \in \mathbf{\Delta}^0_1(2^\omega) \\ &\qquad f[\bigcap_{n\in S} C^0_n \cap \bigcap_{n\in F\setminus S} 2^\omega \setminus C^0_n] = R_n \subseteq \bigcup_{n\in S} C^1_n. \end{aligned}$$

This implies that $\{f \in \mathbb{M} \mid G_f \subseteq O\}$ is an open subset of $\mathbb{M}$ since

$$G_f \subseteq O \Leftrightarrow \exists F \subseteq \omega \text{ finite with } \text{Graph}(f) \subseteq \bigcap_{\varepsilon\in 2} \big(\bigcup_{n\in F} (C^\varepsilon_n \times C^{1-\varepsilon}_n)\big).$$

Now $G_f \cap O \neq \emptyset \Leftrightarrow \exists n \in \omega\ \ \exists \varepsilon \in 2\ \ C^\varepsilon_n \cap f^{-1}(C^{1-\varepsilon}_n) \neq \emptyset \Leftrightarrow \exists n \in \omega\ \ \exists \varepsilon \in 2\ \ \exists \alpha \in C^\varepsilon_n\ \ f(\alpha) \in C^{1-\varepsilon}_n$, so that $\{f \in \mathbb{M} \mid G_f \cap O \neq \emptyset\}$ is an open subset of $\mathbb{M}$. Thus g is continuous.

Note that $(2^\omega, K) \in \mathcal{S}_m$ if and only if

$$\begin{aligned} K \cap \Delta(2^\omega) = \emptyset \wedge K \neq \emptyset \wedge \exists (C_i)_{i<3} \in \big(\mathbf{\Delta}^0_1(2^\omega)\big)^3\ \ 2^\omega \subseteq \bigcup_{i<3} C_i \wedge \forall i \neq j < 3\ \ C_i \cap C_j = \emptyset \wedge \\ K \cap (\bigcup_{i<3} C^2_i) = \emptyset, \end{aligned}$$

so that $\mathcal{S}_m$ is an open subset of $\{2^\omega\} \times \mathcal{K}(2^\omega \times 2^\omega)$ and thus a Polish space.

Note then that $\varphi:2^\omega\to 2^\omega$ is injective if and only if $\varphi[O\cap U]=\varphi[O]\cap\varphi[U]$ whenever O,U are clopen subsets of 2^ω. By Lemma 12.4, and [K, 4.19, 4.29, 27.7], $\preceq^i_c$ and thus $\equiv^i_c$ are analytic. $\square$

Notation. We now consider the case of general graphs, and we can ensure that the reduction map associates graphs of continuous chromatic number at least three instead of two or three. Recall that $\mathcal{K}_{2^\infty}:=(2\cup\{c,a,\overline{a}\})^\omega$. As $\mathcal{K}_{2^\infty}\neq\emptyset$ is a perfect 0DMC space, it is homeomorphic to 2^ω via a map i, by [K, 7.4]. We equip $\mathcal{K}(\mathcal{K}_{2^\infty})$ with the Vietoris topology, so that $\mathcal{K}(\mathcal{K}_{2^\infty})$ is a metrizable compact space, by [K, 4.26]. By [K, 4.29], the map $K\mapsto i[K]$ defines a homeomorphism from $\mathcal{K}(\mathcal{K}_{2^\infty})$ onto $\mathcal{K}(2^\omega)$. We set $\mathbb{Q}:=\big\{x\in\mathcal{K}_{2^\infty}\mid\exists l\in\omega\ \ \exists\varepsilon\in\{a,\overline{a}\}\ \ \forall k\geq l\ \ x(k)=\varepsilon\big\}$. Note that $\mathbb{Q}$ is countable, as well as $Q:=i[\mathbb{Q}]$. We set $\mathcal{S}_g:=\{(K,R)\in\mathcal{K}(2^\omega)\times 2^{Q^2}\mid R\subseteq K^2\setminus\Delta(K)\wedge\chi_c(K,R)\geq 3\}$ and equip 2^{Q^2} with the product topology of the discrete topology on 2, so that

$$\{(K,R)\in\mathcal{K}(2^\omega)\times 2^{Q^2}\mid R\subseteq K^2\setminus\Delta(K)\}$$

is a metrizable compact space.

We consider the equivalence relation $\equiv^i_c$ on $\mathcal{S}_g$ associated with

$$(K,R)\preceq^i_c(K',R')\Leftrightarrow\exists\varphi:K\to K'\text{ injective continuous with }R\subseteq(\varphi\times\varphi)^{-1}(R').$$

We equip $2^{\mathbb{Q}^2}$ with the product topology of the discrete topology on 2, so that $2^{\mathbb{Q}^2}$ is homeomorphic to 2^ω. The map $R\mapsto(i\times i)[R]$ defines a homeomorphism from $2^{\mathbb{Q}^2}$ onto 2^{Q^2}, and the equality $i[K]=\overline{\mathrm{proj}\big[(i\times i)[s(R)]\big]}$ holds if $K=\overline{\mathrm{proj}[s(R)]}$. Moreover, $(K,R)\equiv^i_c(i[K],(i\times i)[R])$. We define a map $\mathcal{G}:\mathbb{M}\to\mathcal{S}_g$ by $\mathcal{G}(f):=\big(\overline{\mathrm{proj}\big[(i\times i)[\mathbb{G}_f]\big]},(i\times i)[\mathbb{G}_f]\big)$ (see Lemma 5.1).

Theorem 13.2 *The equivalence relation $\equiv^i_c$ on the Polish space $\mathcal{S}_g$ is analytic, and $\mathcal{G}$ Borel reduces FCO to $\equiv^i_c$. Moreover, the vertices of the graph $\mathcal{G}(f)$ have degree at most one, for each $f\in\mathbb{M}$.*

Proof. As $(K,R)\equiv^i_c(i[K],(i\times i)[R])$, we may replace 2^ω and Q with $\mathcal{K}_{2^\infty}$ and $\mathbb{Q}$ respectively. By Corollary 5.10, $\mathcal{G}$ reduces FCO to $\equiv^i_c$ since $\mathcal{C}^+=\overline{\mathrm{proj}[\mathbb{G}_f]}$. Note that, for each $i\in\mathbb{Z}$, the map $f\mapsto f^i$ defined on $\mathcal{H}(2^\omega)$ is continuous since $\mathcal{H}(2^\omega)$ is a topological group. Note also that the evaluation map $(f,\alpha)\mapsto f(\alpha)$ is continuous since $d_{2^\omega}\big(f(\alpha),f_0(\alpha_0)\big)\leq d(f,f_0)+d_{2^\omega}\big(f_0(\alpha),f_0(\alpha_0)\big)$. This implies that the map from $\mathbb{M}$ into $(2^\omega)^{\mathbb{Z}}$ defined by $f\mapsto\big(f^i(0^\infty)\big)_{i\in\mathbb{Z}}$ is continuous. Here we only consider $\mathbf{d}:=2^\infty$. Recall from the notation before Lemma 5.9 that $L_{2m}:=R_{2m+1}:=\zeta(m)$. The map from $(2^\omega)^{\mathbb{Z}}$ into $2^{\mathbb{Q}^2}$ associating

$$\begin{aligned}s\big(\{(c^{l+1}a\overline{a}^\infty,\gamma_{L_l}|(l+1)\overline{a}a^\infty)\mid l\in\omega\}\cup&\\ \{(\gamma_{L_l+i}|(l+1)a^{i+1}\overline{a}^\infty,\gamma_{L_l+i+1}|(l+1)\overline{a}^{i+2}a^\infty)\mid l\in\omega\wedge i\leq 2l\}\cup&\\ \{(\gamma_{R_l}|(l+1)a^{2l+2}\overline{a}^\infty,c^{l+1}\overline{a}a^\infty)\mid l\in\omega\}\big)&\end{aligned}$$

to $(\gamma_i)_{i\in\mathbb{Z}}$ is continuous, as well as $f\mapsto\mathbb{G}_f$. The map from $2^{\mathbb{Q}^2}$ into $2^{\mathbb{Q}}$ defined by $R\mapsto\mathrm{proj}[s(R)]$ is Baire class one. The map from $2^{\mathbb{Q}}$ into $\mathcal{K}(\mathcal{K}_{2^\infty})$ defined by $S\mapsto\overline{S}$ is Borel, by [K, 12.11]. Thus $\mathcal{G}$ is Borel.

Note that, by [E, Theorem 2.1(1)],

$$\begin{aligned}\chi_c(K,R)\leq 2&\Leftrightarrow\exists C\in\mathbf{\Delta}^0_1(K)\ \ R\cap C^2=R\cap(K\setminus C)^2=\emptyset\\ &\Leftrightarrow\exists C\in\mathbf{\Delta}^0_1(2^\omega)\ \ R\cap C^2=R\cap(2^\omega\setminus C)^2=\emptyset\end{aligned}$$

if $R\subseteq K^2$, so that $\mathcal{S}_g$ is a G_δ subset of $\{(K,R)\in\mathcal{K}(2^\omega)\times 2^{Q^2}\mid R\subseteq K^2\setminus\Delta(K)\}$ and thus a Polish space.

If $(K,G),(L,H)\in\mathcal{S}_g$, then $(K,G)\preceq^i_c(L,H)$ holds if and only if there is $\varphi:K\to L$ injective continuous such that $\big(\varphi(x),\varphi(y)\big)\in H$ if $(x,y)\in G$. By [K, 2.8], this holds if and only if there is $\psi:\mathcal{K}_{2^\infty}\to\mathcal{K}_{2^\infty}$ continuous such that $\psi[K]\subseteq L$, $\psi_{|K}$ is injective, and $\big(\psi(x),\psi(y)\big)\in H$ if $(x,y)\in G$. Note that $\psi_{|K}$ is injective if and only if $\psi[O\cap U\cap K]=\psi[O\cap K]\cap\psi[U\cap K]$ whenever O,U are clopen subsets of $\mathcal{K}_{2^\infty}$. We conclude as in the proof of Theorem 13.1 to see that $\equiv^i_c$ is analytic. $\square$

Remark. As mentioned in the introduction, using oriented graphs instead of graphs, one can prove that CO is Borel reducible to $\equiv^i_c$. In that case, the proof also works in the case of dynamical systems involving continuous maps instead of homeomorphisms, considering forward orbits instead of orbits.

14 Digraphs and oriented graphs

14.1 General digraphs

We start with a version of Theorem 1.4 for digraphs.

Theorem 14.1.1 *We can find a concrete family $\big((\mathbb{K}_\alpha,\mathbb{D}_\alpha)\big)_{\alpha\in 2^\omega}$, where $\mathbb{K}_\alpha$ is a compact subset of 2^ω and $\mathbb{D}_\alpha$ is a countable digraph on $\mathbb{K}_\alpha$, such that, for any 0DMC space X and any digraph D on X, exactly one of the following holds:*

(1) D has CCN at most two,

(2) we can find $\alpha\in 2^\omega$ and $\varphi:\mathbb{K}_\alpha\to X$ injective continuous such that $\mathbb{D}_\alpha\subseteq(\varphi\times\varphi)^{-1}(D)$.

In other words, $\big((\mathbb{K}_\alpha,\mathbb{D}_\alpha)\big)_{\alpha\in 2^\omega}$ is a $\preceq^i_c$-basis (and thus a $\preceq_c$-basis) for the class of digraphs on a 0DMC space with CCN at least three.

Proof. We define, for $(\gamma,\delta)\in\mathcal{I}\times 2^{\{(k,i)\in\omega^2|i\leq 2k\}}$, a countable relation $\mathbb{D}_{\gamma,\delta}$ on 2^ω by

$$\mathbb{D}_{\gamma,\delta}:=\big\{\big(\gamma_k^{\delta(k,i)}(i),\gamma_k^{1-\delta(k,i)}(i)\big)\mid k\in\omega\wedge i\leq 2k\big\},$$

so that $s(\mathbb{D}_{\gamma,\delta})=\mathbb{G}_\gamma$ and $\mathbb{D}_{\gamma,\delta}$ is a digraph on $\mathbb{K}_{\gamma,\delta}:=\mathbb{K}_\gamma$. By Proposition 3.1.1, $\chi_c(\mathbb{K}_\gamma,\mathbb{G}_\gamma)\geq 3$. As $s(\mathbb{D}_{\gamma,\delta})=\mathbb{G}_\gamma$, $\chi_c(\mathbb{K}_\gamma,\mathbb{D}_{\gamma,\delta})\geq 3$ as well. It will be convenient to replace 2^ω with $\mathcal{I}\times 2^{\{(k,i)\in\omega^2|i\leq 2k\}}$. We just proved that (1) and (2) cannot hold simultaneously.

Assume that (1) does not hold. Then $\chi_c\big(X,s(D)\big)\geq 3$. Theorem 1.4 provides $\gamma\in\mathcal{I}$ with $(\mathbb{K}_\gamma,\mathbb{G}_\gamma)\preceq^i_c\big(X,s(D)\big)$, with witness say φ. Let $\mathbb{D}:=\mathbb{G}_\gamma\cap(\varphi\times\varphi)^{-1}(D)$. Note that $s(\mathbb{D})=\mathbb{G}_\gamma$, which gives $\delta\in 2^{\{(k,i)\in\omega^2|i\leq 2k\}}$ with $\mathbb{D}_{\gamma,\delta}\subseteq\mathbb{D}$. $\square$

Considering the $\mathbb{D}_\alpha$'s which are oriented graphs, and using the fact that a digraph $\preceq^i_c$-below an oriented graph is also an oriented graph, we get a (less concrete) basis for oriented graphs.

Corollary 14.1.2 *We can find a $\preceq^i_c$-basis (and thus a $\preceq_c$-basis) of size at most continuum, made up of countable oriented graphs, for the class of oriented graphs on a 0DMC space with CCN at least three.*

By Theorem 1.5, any $\preceq^i_c$-basis for the class of digraphs on a 0DMC space with CCN at least three must have size at least continuum. We will see that the $\preceq^i_c$-basis given by Corollary 14.1.2 must also have size exactly the continuum later. The second basis given by Theorem 3.2.4 also provides a second basis for digraphs, which is also a basis for oriented graphs, more concrete than the first one we just met.

Theorem 14.1.3 *We can find a concrete $\preceq_c$-basis of size continuum, made up of countable oriented graphs, for the class of digraphs on a 0DMC space with CCN at least three.*

Proof. We adapt the proof of Theorem 3.2.4. We set

$$\mathcal{J}_o := \{(\beta,\alpha) \in \mathcal{J} \times (2^{<\omega})^\omega \mid \forall l \in \omega \ \ |\alpha(l)| = \lambda_l + 1\}.$$

and $\mathcal{J}_o^c := \mathcal{J}_o \cap \left(\mathcal{J}^c \times (2^{<\omega})^\omega\right)$. We also set, for $x, y \in \mathcal{K}_{\mathbf{d}}$ and $\varepsilon \in 2$,

$$(x,y)_\varepsilon := \begin{cases} (x,y) \text{ if } \varepsilon = 0, \\ (y,x) \text{ if } \varepsilon = 1. \end{cases}$$

We then define, for $(\beta,\alpha) \in \mathcal{J}_o$, a countable digraph $\mathbb{D}_{\beta,\alpha}$ on $\mathcal{K}_{\mathbf{d}}$ by

$$\begin{aligned} \mathbb{D}_{\beta,\alpha} := \{(c^{l+1}a\overline{a}^\infty, s_l(0)\overline{a}a^\infty)_{\alpha(l)(0)} \mid l \in \omega\} \cup \\ \{(s_l(i)a^{i+1}\overline{a}^\infty, s_l(i+1)\overline{a}^{i+2}a^\infty)_{\alpha(l)(i+1)} \mid l \in \omega \wedge i \leq \lambda_l - 2\} \cup \\ \{(s_l(\lambda_l - 1)a^{\lambda_l}\overline{a}^\infty, c^{l+1}\overline{a}a^\infty)_{\alpha(l)(\lambda_l)} \mid l \in \omega\}, \end{aligned}$$

so that $s(\mathbb{D}_{\beta,\alpha}) = \mathbb{G}_\beta$ and $\mathbb{D}_{\beta,\alpha}$ is an oriented graph on $\mathbb{K}_\beta$. By Lemma 3.2.1, $\chi_c(\mathbb{K}_\beta, \mathbb{G}_\beta) \geq 3$. As $s(\mathbb{D}_{\beta,\alpha}) = \mathbb{G}_\beta$, $\chi_c(\mathbb{K}_\beta, \mathbb{D}_{\beta,\alpha}) \geq 3$ as well. By Theorem 14.1.1, it is enough to prove that if (γ,δ) is in $\mathcal{I} \times 2^{\{(k,i) \in \omega^2 \mid i \leq 2k\}}$, then we can find $(\beta,\alpha) \in \mathcal{J}_o^c$ (for $\mathbf{d} = 2^\infty$) such that $(\mathbb{K}_\beta, \mathbb{D}_{\beta,\alpha}) \preceq_c (\mathbb{K}_\gamma, \mathbb{D}_{\gamma,\delta})$. We first define β' as in the proof of Theorem 3.2.4, and define $\alpha' \in (2^{<\omega})^\omega$ by $\alpha'(q)(i) := \delta(k_q, i)$ if $q \in \omega$ and $i \leq \lambda'_q$. Then $(\beta', \alpha') \in \mathcal{J}_o$, and $(\mathbb{K}_{\beta'}, \mathbb{D}_{\beta',\alpha'}) \preceq_c (\mathbb{K}_\gamma, \mathbb{D}_{\gamma,\delta})$, by the proof of Theorem 3.2.4. We then define β as in the proof of Theorem 3.2.4, and define $\alpha \in (2^{<\omega})^\omega$ by the formula $\alpha(l)(i) := \alpha'(q_0^{2l+2})(i)$ if $l \in \omega$ and $i \leq \lambda_l$. Then $(\beta,\alpha) \in \mathcal{J}_o^c$ and the proof of Theorem 3.2.4 shows that $(\mathbb{K}_\beta, \mathbb{D}_{\beta,\alpha}) \preceq_c^i (\mathbb{K}'_\beta, \mathbb{D}_{\beta',\alpha'})$. □

In order to prove that the basis given by Corollary 14.1.2 has size exactly continuum, we prove the following oriented version of Theorem 6.4.

Theorem 14.1.4 *Let $\mathbf{d} \in \mathfrak{D}$, V be a compact subspace of $\mathcal{C}^+$, and $E \subseteq \mathbb{O}_o \cap V^2$ having CCN three. Then $(\mathcal{C}^+, \mathbb{O}_o) \preceq_c^i (V, E)$.*

Proof. We essentially copy the proof of Theorem 6.4, replacing E with $s(E)$ under the closure symbols, and using the fact that $E \subseteq \mathbb{O}_o$. □

We are now ready to prove a version of Theorem 1.5 for oriented graphs.

Theorem 14.1.5 *There is a $\preceq_c$-antichain (and thus $\preceq_c^i$-antichain) $\left((\mathbb{K}_\alpha, \mathbb{O}_\alpha)\right)_{\alpha \in 2^\omega}$, where*

(a) $\mathbb{K}_\alpha$ is a 0DMC space,

(b) $\mathbb{O}_\alpha$ is a countable $D_2(\mathbf{\Pi}^0_1)$ oriented graph on $\mathbb{K}_\alpha$ with CCN three and $\mathbf{\Sigma}^0_1 \oplus \mathbf{\Pi}^0_1$ chromatic number two, and whose vertices have degree at most one,

(c) $(\mathbb{K}_\alpha, \mathbb{O}_\alpha)$ is $\preceq_c^i$-minimal in the class of digraphs on a 0DMC space with CCN at least three.

In particular, any $\preceq_c^i$-basis for the class of digraphs (or oriented graphs) on a 0DMC space with CCN at least three. must have size at least continuum.

Proof. Fix $\mathbf{d} \in \mathfrak{D}$. By Proposition 6.1, $(\mathcal{C}^+, \mathbb{G}_o)$ has CCN three and $\mathbf{\Sigma}^0_1 \oplus \mathbf{\Pi}^0_1$ chromatic number two. As $s(\mathbb{O}_o) = \mathbb{G}_o$, this is also the case of $(\mathcal{C}^+, \mathbb{O}_o)$. We check that $\mathbb{O}_\alpha$ is $D_2(\mathbf{\Pi}^0_1)$ as in the proof of Lemma 3.2.2, so that (a) and (b) hold. For (c), i.e., the minimality of $(\mathcal{C}^+, \mathbb{O}_o)$, we first note that the proof of Lemma 6.7 works for digraphs instead of graphs. We then apply Theorem 14.1.4. $(\mathcal{C}^+_{\Phi(\alpha)}, \mathbb{G}_{o_{\Phi(\alpha)}}) \not\preceq_c (\mathcal{C}^+_{\Phi(\beta)}, \mathbb{G}_{o_{\Phi(\beta)}})$ if $\alpha \neq \beta$, by Theorem 6.6. As $s(\mathbb{O}_o) = \mathbb{G}_o$ again, $(\mathcal{C}^+_{\Phi(\alpha)}, \mathbb{O}_{o_{\Phi(\alpha)}}) \not\preceq_c (\mathcal{C}^+_{\Phi(\beta)}, \mathbb{O}_{o_{\Phi(\beta)}})$. □

Replacing $\mathbb{G}_p = s(\mathbb{O}_p)$ with $\mathbb{O}_p$, we get a version of Theorem 1.6 for oriented graphs in a straightforward way. This kind of argument will be used several times in the sequel, and we will not always repeat it.

Our version of Theorem 1.8 for digraphs and oriented graphs is as follows.

Theorem 14.1.6 *Let D be a digraph on a 0DMS space Z, with CCN at least three and satisfying $(Z, D) \preceq^i_c (\mathbb{P}, \mathbb{O}_m)$. Then there is a family $\big((P_\alpha, O_\alpha)\big)_{\alpha \in 2^\omega}$ of oriented graphs on a 0DP space with CCN three, $\preceq^i_c$-below (Z, D), and pairwise $\preceq^i_c$-incompatible in the class of digraphs on a 0DMS space with CCN at least three.*

In particular, there is no $\preceq^i_c$-antichain basis in the class of digraphs (or oriented graphs) on a 0DMS (or 0DP) space with CCN at least three, and any $\preceq^i_c$–basis for one of these classes must have size at least continuum.

Proof. Note first that we can modify Lemma 4.2 as follows. Let $\delta \in 2^\omega$, and D be a digraph on a 0DMS space Z, with CCN at least three and satisfying $(Z, D) \preceq^i_c (\mathbb{P}_\delta, \mathbb{O}_\delta)$. Then there is $\delta' \in \mathbb{P}_\infty$ such that $\{k \in \omega \mid \delta'(k) = 1\} \subseteq \{k \in \omega \mid \delta(k) = 1\}$ and $(\mathbb{P}_{\delta'}, \mathbb{O}_{\delta'}) \preceq^i_c (Z, D)$. We complete the proof of Lemma 4.2 as follows. We set $G := s(D)$, so that G is a graph on Z with CCN at least three and $(Z, G) \preceq^i_c (\mathbb{P}_\delta, \mathbb{G}_\delta)$. We then set $R := (\varphi \times \varphi)[D]$, so that $s(R) = E$. We can then follow the proof of Lemma 4.2. For the conditions (b)-(d), the couples are not only in $s(R)$, but also in R since $R \subseteq \mathbb{O}_\delta$ and thus $R^{-1} \subseteq \mathbb{O}_\delta^{-1}$. So we can replace E with R after the first line of the proof, which implies that $(\mathbb{P}_{\delta'}, \mathbb{O}_{\delta'}) \preceq^i_c (Z, D)$.

The version of Lemma 4.3 for oriented graphs is straightforward, and we then follow the proof of Theorem 1.8 to conclude. □

Our version of Theorem 1.9 for digraphs and oriented graphs is as follows.

Theorem 14.1.7 *There is a countable oriented graph $(3^\omega, \mathbb{O})$ in the class of digraphs on a 0DMC space with CCN at least three such that, for each (K, G) in this class satisfying $(K, D) \preceq^i_c (3^\omega, \mathbb{O})$, there is a $\preceq^i_c$-antichain $\big((3^\omega, O_\alpha)\big)_{\alpha \in 2^\omega}$ of oriented graphs with CCN three and $\preceq^i_c$-below (K, D). In particular, there is no $\preceq^i_c$-antichain basis in this class (or the corresponding one for oriented graphs).*

Proof. We just have to follow the proof of Theorem 1.9. The oriented graphs $\mathbb{O} := \mathrm{Graph}(o_{|D_{S_\omega}})$ and $O_\alpha := G_\alpha \cap \mathrm{Graph}(o)$ are convenient since $s(\mathbb{O}) = \mathbb{G}$ and $s(O_\alpha) = G_\alpha$. □

We can prove a version of Theorem 1.16 for oriented graphs in a straightforward way. A straightforward modification of Section 5 gives the following version of Corollary 5.10 for oriented graphs. In order to do that, we replace $\mathbb{G}_{f_\mathbf{d}}$ with $\mathbb{O}_{f_\mathbf{d}}$ in the definition of continuous tuples.

Theorem 14.1.8 *Let $\mathbf{d}, \mathbf{d}' \in \mathfrak{C}$, $f_\mathbf{d} : \mathcal{C}_\mathbf{d} \to \mathcal{C}_\mathbf{d}$, $f_{\mathbf{d}'} : \mathcal{C}_{\mathbf{d}'} \to \mathcal{C}_{\mathbf{d}'}$ be minimal homeomorphisms, and $(n_l)_{l \in \omega}, (L_l)_{l \in \omega}, (R_l)_{l \in \omega}$ defined before Lemma 5.9. Then $(\mathcal{C}^+_\mathbf{d}, \mathbb{O}_{f_\mathbf{d}}) \equiv^i_c (\mathcal{C}^+_{\mathbf{d}'}, \mathbb{O}_{f_{\mathbf{d}'}})$ if and only if $f_\mathbf{d}, f_{\mathbf{d}'}$ are conjugate.*

A straightforward modification of the proof of Proposition 7.4(b) gives the following result. We set $\mathbb{O}_1 := \text{Graph}(f_0) \setminus \{(0^\infty, 0^\infty)\}$.

Proposition 14.1.9 $\{(\mathbb{X}_1, \mathbb{O}_1), (\mathbb{X}_1, \mathbb{O}_1^{-1}), (\mathbb{X}_1, \mathbb{R}_1)\}$ *is a $\preceq_c^i$-antichain basis for the class of digraphs on a 0DMS space with uncountable CCN.*

Our version of Proposition 7.6 for digraphs and oriented graphs is as follows.

Proposition 14.1.10 $(\mathbb{X}_1, \mathbb{R}_1)$ *is $\preceq_c^i$-minimal, but not $\preceq_c$-minimal, in the class of digraphs (or oriented graphs) on a 0DMC space with CCN at least three.*

Proof. We follow the proof of Proposition 7.6 since $s(\mathbb{R}_1) = G_{f_1}$. For $\preceq_c$, we work with the oriented graph $\{(\varepsilon^{2p+1}(\varepsilon^+)^\infty, (\varepsilon^+)^{2p+2}\big((\varepsilon^+)^+\big)^\infty) \mid \varepsilon \in 3 \wedge p \in \omega\}$. □

14.2 Digraphs induced by a partial function

Notation. If $f : \text{Domain}(f) \subseteq X \to \text{Range}(f) \subseteq X$ is a partial function, then the digraph *induced* by f is $D_f := \text{Graph}(f) \setminus \Delta(X)$. Note that $G_f = s(D_f)$, which gives versions of Proposition 7.2 and Corollary 7.3 with D_f instead of G_f in a straightforward way.

We also have the following versions of Theorem 7.7 for digraphs and oriented graphs.

Theorem 14.2.1 *There is no $\preceq_c^i$-antichain basis for the class of digraphs (or oriented graphs) induced by a partial homeomorphism on a 0DMS (or 0DP) space with CCN at least three. In fact, we can even restrict this class to the case where the spaces are countable Polish and the functions are fixed point free with open domain.*

Proof. We follow the proof of Theorem 7.7. We restrict f_δ to

$$D_\delta := \big\{x \in \text{proj}[\mathbb{G}_\delta] \mid \exists n \in \omega \;\; x(n) = a \wedge x|n \in (\omega \cup \{c\})^n\big\},$$

so that $\mathbb{O}_\delta = \text{Graph}(f_{\delta|D_\delta})$ is an oriented graph. We then work with the $(\mathbb{P}_\delta, \mathbb{O}_\delta)$'s since $\mathbb{G}_\delta := s(\mathbb{O}_\delta)$, applying Theorem 14.1.6. □

14.3 Digraphs induced by a total function

Similarly, the versions of Proposition 7.5 and Lemma 7.9 for the D_f's are straightforward. For Lemma 7.9, we just assume that $E \subseteq D_f$. We now give a motivating result.

Lemma 14.3.1 *Let X, Y be 0DMC spaces, and $f : X \to X$, $g : Y \to Y$ be homeomorphisms, g being minimal. Then*

(a) $\text{Graph}(g)$ *is an oriented graph on Y if Y has cardinality at least three,*

(b) f, g are conjugate with witness φ if and only if $\big(X, \text{Graph}(f)\big) \preceq_c^i \big(Y, \text{Graph}(g)\big)$ with witness φ.

Proof. (a) As Y has cardinality at least three and g is minimal, g and g^2 are fixed point free.

(b) The proof is similar to and simpler than the proof of Lemma 7.11. □

The version of Lemma 7.10 for the D_f's is straightforward.

Lemma 14.3.2 *Let X be a 0DMC space, and $f : X \to X$ be a minimal homeomorphism with $\chi_c\big(X, \mathrm{Graph}(f)\big) \geq 3$. Then $\big(X, \mathrm{Graph}(f)\big)$ is $\preceq_c^i$-minimal in the class of closed digraphs (or oriented graphs) on a 0DMC space with CCN at least three. This is the case of $\big(\mathcal{C}, \mathrm{Graph}(o)\big)$ if $\mathbf{d} = (d_j)_{j\in\omega} \in \mathcal{O}$.*

This gives a version of Theorem 1.13 for digraphs and oriented graphs.

Theorem 14.3.3 *There is a $\preceq_c$-antichain (and thus $\preceq_c^i$-antichain) $\Big(\big(\mathcal{C}_\alpha, \mathrm{Graph}(f_\alpha)\big)\Big)_{\alpha\in 2^\omega}$, where*

(a) $\mathcal{C}_\alpha$ is homeomorphic to 2^ω,

(b) f_α is a minimal homeomorphism of $\mathcal{C}_\alpha$, and $\mathrm{Graph}(f_\alpha)$ has CCN three,

(c) $\big(\mathcal{C}_\alpha, \mathrm{Graph}(f_\alpha)\big)$ is $\preceq_c^i$-minimal in the class of closed digraphs (or oriented graphs) on a 0DMC space with CCN at least three.

In particular, any $\preceq_c^i$-basis for the class of digraphs (or oriented graphs) induced by a homeomorphism of a 0DMC space with CCN at least three must have size continuum.

We consider, for $\kappa \leq 3$, the class $\mathfrak{G}_\kappa^o$ of digraphs D_f induced by a homeomorphism f of a 0DMC space with CCN strictly bigger than κ. Replacing G_o with $D_o = \mathrm{Graph}(o)$, we get the version of Proposition 9.3 for the D_f's in a straightforward way, using Lemma 14.3.1. It is worth noting that the version of Theorem 1.11 for the D_f's is different from the one for the graphs.

Theorem 14.3.4 *(a) $(1, \emptyset)$ is $\preceq_c^i$-minimum in $\mathfrak{G}_0^o$.*

(b) Any $\preceq_c^i$-basis for $\mathfrak{G}_1^o$ must have size continuum.

(c) Any $\preceq_c^i$-basis for $\mathfrak{G}_2^o$ must have size continuum.

Moreover, the $(\mathfrak{G}_\kappa^o, \preceq_c^i)$'s and the $(\mathfrak{G}_\kappa^o, \preceq_c)$'s are not well-founded. They also contain antichains of size continuum (except maybe for $\preceq_c$ when $\kappa = 3$).

Proof. (a) See the proof of Theorem 1.11(a).

(b) The situation here is very different from the one for $\mathfrak{G}_1$. Let K be a 0DMC space of cardinality at least two, and f be a minimal homeomorphism of K. Note that f is fixed point free, so that $D_f = \mathrm{Graph}(f)$ is not empty and thus has CCN strictly bigger than 1. Assume that L is a 0DMC space, g is a homeomorphism of L such that D_g has CCN strictly bigger than 1, and $(L, D_g) \preceq_c^i (K, D_f)$. We will see that $(K, D_f) \preceq_c^i (L, D_g)$, which will prove the $\preceq_c^i$-minimality of (K, D_f) in $\mathfrak{G}_1^o$. If the set of fixed points of g is not open, then (L, D_g) has uncountable CCN by the version of Proposition 7.2 for the D_f's, and thus (K, D_f) and (K, G_f) too, which contradicts Theorem 8.1. Thus the set F_1 of fixed points of g is open, which by the version of Corollary 7.3 for the D_f's implies that $\chi_c(L \setminus F_1, D_g \cap (L \setminus F_1)^2) = \chi_c(L, D_f) > 1$. This implies that we may assume that g is fixed point free and thus $D_g = \mathrm{Graph}(g)$. Lemma 14.3.1 then implies that g, f are conjugate and thus $(K, D_f) \preceq_c^i (L, D_g)$. Theorem 14.3.3 now provides a $\preceq_c^i$-antichain of size continuum made up of minimal elements in $\mathfrak{G}_1^o$, which gives the result.

(c) By Theorem 14.3.3, any $\preceq_c^i$-basis for $\mathfrak{G}_2^o$ must have size continuum.

We then argue as in the proof of Theorem 1.11, using the fact that $G_f = s(D_f)$. $\square$

Remark. If (K, D_f) is in $\mathfrak{G}_1^o$ and the set of fixed points of f is open, then (K, D_f) is $\preceq_c^i$-above a similar element (L, D_g) with g fixed point free by the proof of Theorem 14.3.4(b). [W, Theorem 5.2] gives a compact subset M of L such that $g[M] = M \neq \emptyset$ and the dynamical system $(M, g_{|M})$ is minimal. As g is fixed point free, M has cardinality at least two. The proof of Theorem 14.3.4(b) shows that $(M, D_{g_{|M}})$ is $\preceq_c^i$-minimal in $\mathfrak{G}_1^o$, and is $\preceq_c^i$-below (K, D_f). So the elements of $\mathfrak{G}_1^o$ induced by a minimal homeomorphism form a $\preceq_c^i$-basis for the subclass of $\mathfrak{G}_1^o$ whose elements are induced by a homeomorphism with an open set of fixed points. So in order to get an interesting basis, we need to understand the elements (K, D_f) of $\mathfrak{G}_1^o$ whose set of fixed points is not open. In such a case the CCN is $2^{\aleph_0}$, by the version of Proposition 7.2 for the D_f's. Proposition 14.1.9 implies that $(\mathbb{X}_1, \mathbb{R}_1) \preceq_c^i (K, D_f)$. The problem is that $\mathbb{R}_1$ is not of the form D_g.

The versions of Theorems 1.12, 10.5, 10.11 and Proposition 10.13 for the D_f's are direct.

Corollary 14.3.5 *There is a $\preceq_c^i$-antichain $\big(\Sigma_r, \mathrm{Graph}(\sigma_{|\Sigma_r})\big)_{r\in\mathbb{R}}$ of size continuum, where*
(a) Σ_r is a two-sided subshift homeomorphic to 2^ω,
(b) $\sigma_{|\Sigma_r}$ is a minimal homeomorphism of Σ_r, $\mathrm{Graph}(\sigma_{|\Sigma_r})$ is an oriented graph with CCN three,
(c) $\big(\Sigma_r, \mathrm{Graph}(\sigma_{|\Sigma_r})\big)_{r\in\mathbb{R}}$ is $\preceq_c^i$-minimal in the class of closed digraphs (or oriented graphs) on a 0DMC space with CCN at least three.

The version of Theorem 1.14 for digraphs, $\mathfrak{G}_2^o$ and the D_f's is straightforward. For oriented graphs, we modify K_0 and h_0 (note that $\big(\sigma^\varepsilon(\alpha_0), \sigma^{1-\varepsilon}(\alpha_0)\big) \in D_{h_0}$ for each $\varepsilon \in 2$, so that D_{h_0} is not an oriented graph). In order to get a version of Theorem 1.14 for oriented graphs, we set $\alpha_0^o := (0123)^\infty \cdot (0123)^\infty$, $\beta_0^o := (0123)^\infty \cdot 4(0123)^\infty$, $K_0^o := \mathrm{Orb}_\sigma(\alpha_0^o) \cup \mathrm{Orb}_\sigma(\beta_0^o)$ and $h_0^o := \sigma_{|K_0^o}$. The version of Lemma 11.1 is as follows.

Lemma 14.3.6 *Let S be a 0DMS (resp., 0DP) space, f be a homeomorphism of S with the properties that $\chi_c(S, D_f) \geq 3$ and $(S, D_f) \preceq_c^i (K_0^o, D_{h_0^o})$. Then there is a finer 0DMS (resp., 0DP) topology τ in $\mathcal{T}$ with the property that $\big((K_0^o, \tau), D_{h_0^o}\big) \preceq_c^i (S, D_f)$.*

Proof. The argument is a slight variation of that in the proof of Lemma 11.1. For instance, as $\mathrm{Orb}_{h_0^o}(\beta_0^o)$ is discrete, there is $\varepsilon \in 4$ with $\sigma^\varepsilon(\alpha_0^o) \in V$, which gives $x \in S$ with $\varphi(x) = \sigma^\varepsilon(\alpha_0^o)$. As f is fixed point free, $f(x) \neq x$, which implies that $\big(x, f(x)\big) \in D_f$, $\Big(\varphi(x), \varphi\big(f(x)\big)\Big) \in D_{h_0^o}$, and $\sigma^{\varepsilon+1 \bmod 4}(\alpha_0^o) = \varphi\big(f(x)\big) \in V$. Iterating this argument, we see that $\mathrm{Orb}_\sigma(\alpha_0^o) \subseteq V$ and $\big\{\big(\sigma^\varepsilon(\alpha_0^o), \sigma^{\varepsilon+1 \bmod 4}(\alpha_0^o)\big) \mid \varepsilon \in 4\big\} \subseteq E$. $\square$

Notation. We set, for $A \subseteq \omega$,

$$\mathcal{B}_A^\tau := \mathcal{B}^\tau \cup \bigcup_{\varepsilon\in 4} \Big\{C \cap (\{\sigma^\varepsilon(\alpha_0)\} \cup \bigcup_{n\in\bigcap_{-p_0\leq r\leq q_0} (N_A+r)\cap\omega} \{\sigma^{\varepsilon+4n+1}(\beta_0)\} \cup \bigcup_{n\in\bigcap_{-p_1\leq r\leq q_1} (N_A+r)\cap\omega} \{\sigma^{\varepsilon-4n-2}(\beta_0)\}) \mid C \in \mathcal{B}^\tau \wedge p_0, q_0, p_1, q_1 \in \omega\Big\}.$$

The version of Lemma 11.2, that of Lemma 11.3, as well as the rest of the proof of Theorem 1.14 for oriented graphs, are then straightforward. The version of Theorem 1.15 for digraphs and $\mathfrak{G}_2^o$ is straightforward.

14.4 Equivalence relations

We define a map $d:\mathbb{M}\to\mathcal{S}_m$ by $d(f):=\big(2^\omega,\mathrm{Graph}(f)\big)$. Applying Lemma 14.3.1, we get the following result.

Theorem 14.4.1 *The map d reduces continuously CO to $\equiv_c^i$. Moreover, the vertices of the digraph $d(f)$ have degree one, for each $f\in\mathbb{M}$.*

We then define a map $\mathcal{D}:\mathbb{M}\to\mathcal{S}_g$ by $\mathcal{D}(f):=\big(\overline{\mathrm{proj}\big[(i\times i)[\mathbb{G}_f]\big]},(i\times i)[\mathbb{O}_f]\big)$. Applying Theorem 14.1.8, we get the following result.

Theorem 14.4.2 *The map $\mathcal{D}$ Borel reduces CO to $\equiv_c^i$. Moreover, the vertices of the digraph $\mathcal{D}(f)$ have degree at most one, for each $f\in\mathbb{M}$.*

15 A summary for future work

We summarize a number of our results in the following table, which leaves open questions about graphs on a 0DMS space with CCN at least three.

	finite	metrizable compact	Polish or metrizable separable
$\preceq_c^i$	(1) concrete antichain basis of size $\aleph_0$ (2) concrete basis of size $\aleph_0$ (3) any basis is infinite (4) antichain of size $\aleph_0$ made up of minimals (5) no infinite descending chain (6) minimal elements (7) embed $\subseteq$ on $\mathcal{P}_{<\infty}(\omega)$	(1) no antichain basis (2) concrete basis of size $2^{\aleph_0}$ (3) any basis has size at least $2^{\aleph_0}$ (4) antichain of size $2^{\aleph_0}$ made up of minimals (5) infinite descending chain (6) minimal elements (7) embed $\subseteq$ on $\mathcal{P}(\omega)$	(1) no antichain basis (3) any basis has size at least $2^{\aleph_0}$ (4) antichain of size $2^{\aleph_0}$ (5) infinite descending chain (6) minimal elements (7) embed $\subseteq$ on $\mathcal{P}(\omega)$
$\preceq_c$	(1) no antichain basis (2) concrete basis of size $\aleph_0$ (3) any basis is infinite (4) antichain of size $\aleph_0$ (5) infinite descending chain (6) no minimal element	(2) concrete basis of size $2^{\aleph_0}$ (4) antichain of size $2^{\aleph_0}$ (5) infinite descending chain (7) embed $\subseteq$ on $\mathcal{P}(\omega)$	(4) antichain of size $2^{\aleph_0}$ (5) infinite descending chain (7) embed $\subseteq$ on $\mathcal{P}(\omega)$

It is remarkable that the properties in the last two columns are the same for graphs induced by a partial homeomorphism with countable domain, (possibly) up to (2)-$\preceq_c^i$ in the compact case. For graphs induced by a total homeomorphism, (4) and (5) hold, as well as (3), (6) and (7)-$\preceq_c^i$, and (1)-$\preceq_c^i$ in the case of spaces which are not compact. All these results admit versions for digraphs and oriented graphs.

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List of symbols

$\mathcal{G}_0$ 2
$\Delta(X)$ 2
(X,R), $\chi_B(X,R)$ 2
$\preceq_c$ 2
$\mathbb{L}_0$ 2
$(\mathbb{A}_\xi, \mathbb{G}_\xi)$ 2
$\preceq_c^i$, $\preceq_B^i$ 3
CCN, $\chi_c(X,R)$ 3
0DMS, 0DP, 0DMC 3
$\mathfrak{K}$ 3
$D_2(\mathbf{\Pi}_1^0)$, $\mathbf{\Sigma}_1^0 \oplus \mathbf{\Pi}_1^0$ 4
(X,f) 4
$\mathrm{Orb}_f(x)$ 4
R^l, R^{-1}, $s(R)$ 5
$\mathcal{N}$ 5
$\mathbb{O}_m$, $\mathbb{G}_m$, $\mathbb{P}$ 5
G_f 6
$\mathfrak{G}_\kappa$ 6
$\mathbb{X}_1$, $\mathbb{R}_1$, f_1 6
$\mathcal{K}(X)$ 7
$\mathcal{H}(2^\omega)$, $\mathcal{P}$, $\mathcal{O}_\kappa$, $\mathcal{O}_2^{\aleph_0}$ 7
σ, $(01)^\infty \cdot (01)^\infty$, K_0, h_0 7
$\mathbb{M}$, FCO, CO 9
$\mathcal{S}_m$ 9
$\leq_B$ 9
$=^+$ 9
$(2p+3, C_{2p+3})$ 10
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