Asymptotic behavior of large eigenvalues of the two-photon Rabi model

Anne Boutet de Monvel, Lech Zielinski

ASYMPTOTIC BEHAVIOR OF LARGE EIGENVALUES OF THE TWO-PHOTON RABI MODEL

ANNE BOUTET DE MONVEL[1] AND LECH ZIELINSKI[2]

Abstract. We investigate the asymptotic behavior of large eigenvalues for the two-photon Rabi Hamiltonian, i.e., for the two-photon Jaynes–Cummings model without the rotating wave approximation. We prove that the spectrum of this Hamiltonian consists of two eigenvalues sequences $(E_n^+)_{n=0}^\infty$, $(E_n^-)_{n=0}^\infty$, satisfying the same two-term asymptotic formula with remainder $\mathrm{O}(n^{-1/3})$ when n tends to infinity. We also propose a conjecture on a three-term asymptotics formula modeled on the GRWA for the one-photon Rabi model.

1. Introduction

The simplest interaction between a two-level atom and a classical light field is described by the Rabi model [20, 21]. The quantum Rabi model couples a quantized single-mode radiation and a two-level quantum system according to the idea that each photon creation accompanies an atomic de-excitation, and each photon annihilation accompanies atomic excitation (see [23] for the microscopic derivation of the quantum Rabi model in Cavity Quantum Electrodynamics). In the pioneer work [17], E. T. Jaynes and F. W. Cummings introduced the rotating wave approximation (RWA) of the quantum Rabi model. The model of [17] (with the RWA) is explicitly diagonalizable and explains a range of experimental phenomena, but further developments in engineering of quantum systems, shows the necessity of understanding the full Rabi Hamiltonian given by the formula (1.3). We refer to [27] for an exhaustive overview of theoretical and experimental works in relation with the quantum Rabi model and its various generalizations (see also [6]).

In this paper we investigate the two-photon quantum Rabi model. The corresponding Hamiltonian is given by the formula (1.2) which differs from (1.3) by the fact that the atomic excitation/de-excitation appears via annihilation/creation of *two* photons. This model was used in [13] to describe a two-level atom interacting with squeezed light and we refer to [8] for the overview of works about the two-photon quantum Rabi model (see also [11]).

The purpose of this paper is to investigate the asymptotic behavior of large eigenvalues of the two-photon quantum Rabi Hamiltonian. It appears that this problem can be reduced to the analog problem for some infinite Jacobi matrices. The asymptotic behavior of large eigenvalues of classes of infinite Jacobi matrices with discrete spectrum was initiated by J. Janas and S. Naboko [16] and continued in [1, 15, 18]. However, in the case of a non-degenerated two-level system, the corresponding Jacobi matrices do not belong to the classes of operators considered in [1, 15, 16, 18] and the asymptotic behavior of large eigenvalues is quite different. It appears that one deals with Jacobi matrices that have diagonal entries perturbed by periodic oscillations, but the contribution of these oscillations to the behavior of the n-th eigenvalue tends to 0 when $n \to \infty$. This phenomenon for the quantum Rabi Hamiltonian (1.3), was first investigated by E. A. Tur [24, 25] (see also [26] and [2–5]). In this paper we prove that a similar phenomenon still holds for the two-photon Rabi Hamiltonian (1.2). The main result of this paper is the two-term asymptotic formula (1.4) given in Theorem 1.1. However, our approach allows us to conjecture the three-term

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asymptotic formula (1.6) which is the two-photon version of the three-term asymptotic formula proved in [4].

1.1. **The two-photon Rabi model.** The two-photon Rabi model is given by a Hamiltonian $\hat{H}_{\rm Rabi}$ acting in a complex Hilbert space $\mathcal{H}_{\rm Rabi} = \mathcal{H}_{\rm atom} \otimes \mathcal{H}_{\rm field}$. This Hamiltonian is of the form

$$\hat{H}_{\rm Rabi} = \hat{H}_{\rm atom} \otimes I_{\mathcal{H}_{\rm field}} + I_{\mathcal{H}_{\rm atom}} \otimes \hat{H}_{\rm field} + \hat{H}_{\rm int}, \tag{1.1}$$

where $\hat{H}_{\rm atom}$ is the Hamiltonian of the two-level atomic system, $\hat{H}_{\rm field}$ is the Hamiltonian of the light field, and $\hat{H}_{\rm int}$ is the interaction term.

The two-level atomic system is defined by a self-adjoint operator $\hat{H}_{\rm atom}$ acting in the two-dimensional complex Hilbert space $\mathcal{H}_{\rm atom} = \mathbb{C}^2$. We denote by $E_{\rm g} \leq E_{\rm e}$ its eigenvalues, that is, the energies of the ground state and of the excited state. To simplify the model we assume the eigenvalues are $\pm\frac{1}{2}\Delta$ where $\Delta \coloneqq E_{\rm e} - E_{\rm g}$ is the level separation energy. Identifying a basis of eigenvectors $\{\mathrm{e}_{\rm e}, \mathrm{e}_{\rm g}\}$ with the canonical basis of $\mathcal{H}_{\rm atom} = \mathbb{C}^2$ we have

$$\hat{H}_{\rm atom} = \frac{\Delta}{2}\,\sigma_3,$$

where $\sigma_3 = \sigma_z \coloneqq \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the usual Pauli matrix. Henceforth we denote $\mathbf{e}_1 \coloneqq \mathrm{e}_{\rm e}$ and $\mathbf{e}_{-1} \coloneqq \mathrm{e}_{\rm g}$.

$\mathcal{H}_{\rm field}$ is a complex Hilbert space equipped with an orthonormal basis $\{\mathrm{e}_n\}_0^\infty$. The Hamiltonian $\hat{H}_{\rm field}$ is the self-adjoint operator in $\mathcal{H}_{\rm field}$ characterized by

$$\hat{H}_{\rm field}\,\mathrm{e}_n = n\mathrm{e}_n, \quad n = 0,\, 1,\, 2, \ldots$$

The photon annihilation and creation operators are defined in $\mathcal{H}_{\rm field}$ by

$$\hat{a}\,\mathrm{e}_n = \sqrt{n}\,\mathrm{e}_{n-1}, \quad \hat{a}^\dagger \mathrm{e}_n = \sqrt{n+1}\,\mathrm{e}_{n+1}, \quad n = 0,\, 1,\, 2,\, \ldots$$

(with $\hat{a}\,\mathrm{e}_0 = 0$). In particular, $\hat{a}^\dagger\hat{a} = \hat{N}$, where $\hat{N}$ is the photon number operator characterized by $\hat{N}\mathrm{e}_n = n\,\mathrm{e}_n$. Therefore $\hat{H}_{\rm field}$ can be written as

$$\hat{H}_{\rm field} = \hat{N} = \hat{a}^\dagger\hat{a}.$$

Finally, the interaction term $\hat{H}_{\rm int}$ is given by

$$\hat{H}_{\rm int} = g\sigma_1 \otimes \left(\hat{a}^2 + (\hat{a}^\dagger)^2\right),$$

where $g > 0$ is the coupling constant and $\sigma_1 = \sigma_x \coloneqq \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the Pauli matrix.

To summarize, (1.1) takes the explicit form

$$\hat{H}_{\rm Rabi} = \frac{\Delta}{2}\,\sigma_3 \otimes I_{\mathcal{H}_{\rm field}} + I_{\mathcal{H}_{\rm atom}} \otimes \hat{N} + g\sigma_1 \otimes \left(\hat{a}^2 + (\hat{a}^\dagger)^2\right). \tag{1.2}$$

Remark. The usual (one-photon) quantum Rabi Hamiltonian is given by

$$\frac{\Delta}{2}\,\sigma_3 \otimes I_{\mathcal{H}_{\rm field}} + I_{\mathcal{H}_{\rm atom}} \otimes \hat{N} + g\sigma_1 \otimes (\hat{a} + \hat{a}^\dagger). \tag{1.3}$$

1.2. **Main result.** Our aim is to prove

Theorem 1.1 (large eigenvalues of $\hat{H}_{\rm Rabi}$)**.** *Let $\hat{H}_{\rm Rabi}$ be the two-photon Rabi Hamiltonian given by (1.2), with parameters $\Delta \geq 0$ and $g > 0$. We make the additional assumption $g < \frac{1}{2}$. Then $\hat{H}_{\rm Rabi}$ is self-adjoint, has discrete spectrum, its eigenvalues can be enumerated in two nondecreasing sequences $\{E_n^+\}_0^\infty$ and $\{E_n^-\}_0^\infty$:*

$$\sigma(\hat{H}_{\rm Rabi}) = \{E_n^+\}_0^\infty \cup \{E_n^-\}_0^\infty,$$

and the large n behavior of these eigenvalues is given by

$$E_n^\pm = \left(n + \frac{1}{2}\right)\sqrt{1 - 4g^2} - \frac{1}{2} + \mathrm{O}(n^{-1/3}). \tag{1.4}$$

Remark. In the degenerated case $\Delta = 0$ we have an exact formula for $E_n^+ = E_n^-$:

$$E_n^{\pm} = \left(n + \frac{1}{2}\right)\sqrt{1 - 4g^2} - \frac{1}{2}. \tag{1.5}$$

The proof of (1.5), based on a paper by Emary and Bishop [10], is given in Section 6.2.

By analogy with the Generalized Rotating-Wave Approximation (GRWA) proposed by Irish [14] for the one-photon Rabi model (see also Feranchuk, Komarov, and Ulyanenkov [12]), and proved in [4], we propose the following conjecture for the two-photon Rabi model.

Conjecture. *For $\mu = 0, 1$ and $\frac{1}{2} < \rho < 1$, we have the three-term asymptotics*

$$E_{2k+\mu}^{\pm} + \frac{1}{2} = E_{2k+\mu}^0 \pm \Delta\,\frac{\omega_\mu}{\sqrt{k}}\cos(\alpha_\mu k + \theta_\mu) + \mathrm{O}(k^{-\rho}) \text{ as } k \to \infty, \tag{1.6}$$

where

$$E_{2k+\mu}^0 := \left(2k + \mu + \frac{1}{2}\right)\sqrt{1 - 4g^2} - \frac{1}{2},$$

and ω_μ, α_μ, and θ_μ are real constants, depending only on g (see Section 5.3 for explicit expressions of these constants).

Comment. As we will show in Section 5.3 below, the validity of this three-term asymptotics can be derived from the following improvement of the estimate (5.8) from Lemma 5.6:

$$\lambda_j(L_\gamma^\delta) = \lambda_j(L_\gamma^0) + \delta R_\gamma(j, j) + \mathrm{O}(|j|^{-\rho}) \text{ as } |j| \to \infty. \tag{1.7}$$

1.3. **Plan of the paper.** In Section 2, by splitting the canonical basis $\mathfrak{B}$ of $\mathcal{H}_{\mathrm{Rabi}}$ into four parts $\mathfrak{B}_{\nu,\mu}$, $\nu = \pm 1$, $\mu = 0, 1$ we introduce a natural orthogonal splitting

$$\mathcal{H}_{\mathrm{Rabi}} = \oplus_{\nu,\mu}\mathcal{H}_{\nu,\mu} = \mathcal{H}_{1,0} \oplus \mathcal{H}_{-1,0} \oplus \mathcal{H}_{1,1} \oplus \mathcal{H}_{-1,1} \tag{1.8}$$

which is invariant under the two-photon Rabi Hamiltonian. Moreover, using $\mathfrak{B}_{\nu,\mu}$ we show that the restriction of the two-photon Rabi Hamiltonian to $\mathcal{H}_{\nu,\mu}$ is unitary equivalent to some operator defined in $\ell^2(\mathbb{N})$ by a Jacobi matrix $J_{\nu,\mu}$. Thus the initial problem is reduced to the asymptotic analysis of large eigenvalues of these Jacobi matrices $J_{\nu,\mu}$.

In Section 3 we reduce this analysis to that of a simpler class of Jacobi matrices

$$J_\gamma^\delta := \begin{pmatrix} \delta & g\gamma & 0 & 0 & \dots \\ g\gamma & 1-\delta & g(1+\gamma) & 0 & \dots \\ 0 & g(1+\gamma) & 2+\delta & g(2+\gamma) & \dots \\ 0 & 0 & g(2+\gamma) & 3-\delta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.9}$$

Lemma 3.2 shows that suitable estimates for large eigenvalues of the associated operator $\hat{J}_\gamma^\delta$ imply the assertion of Theorem 1.1.

In Section 4 we replace the operator $\hat{J}_\gamma^\delta$ acting in $\ell^2(\mathbb{N})$ with its natural extension $\tilde{J}_\gamma^\delta$ acting in $\ell^2(\mathbb{Z})$. It appears that the spectrum of $\tilde{J}_\gamma^0$ was explicitly described in [9, 19]. Following [9] we use the Fourier transform $\mathcal{F}_0 \colon \mathrm{L}^2(-\pi, \pi) \to \ell^2(\mathbb{Z})$ defined by (4.1) and consider the operator

$$L_\gamma^\delta := \mathcal{F}_0^{-1}\tilde{J}_\gamma^\delta\mathcal{F}_0 \tag{1.10}$$

acting in $\mathrm{L}^2(-\pi, \pi)$. In the case $\delta = 0$ this operator, L_γ^0, is a first order differential operator in $\mathrm{L}^2(-\pi, \pi)$ with periodic boundary conditions (or a differential operator with smooth coefficients on the circle). Our analysis of L_γ^0 given in Section 4 is a modification of the approach used in [9]. The main difference is that we use a unitary equivalence in order to obtain an explicit expression for an orthonormal basis of eigenvectors of L_γ^0.

In Section 5 we use the form of eigenvectors of L_γ^0 to estimate the difference between the eigenvalues of L_γ^δ and L_γ^0 thanks to an idea coming from [26]. Since the operator L_γ^δ is unitary equivalent to $\tilde{J}_\gamma^\delta$, we obtain the asymptotic behavior of the eigenvalues of $\tilde{J}_\gamma^\delta$.

In Section 6 we complete the proof of Theorem 1.1, using that $\hat{J}_\gamma^\delta$ is the restriction of $\tilde{J}_\gamma^\delta$ to $\ell^2(\mathbb{N})$.

1.4. **Notations and conventions.** $\ell^2(\mathbb{Z})$ is the Hilbert space of square-summable complex sequences $\tilde{x}\colon \mathbb{Z} \to \mathbb{C}$ indexed by the set of integers $\mathbb{Z}$ and equipped with the scalar product

$$\langle \tilde{x}, \tilde{y} \rangle := \sum_{k\in\mathbb{Z}} \overline{\tilde{x}(k)}\tilde{y}(k), \quad \tilde{x},\, \tilde{y} \in \ell^2(\mathbb{Z})$$

and the associated norm $\|\tilde{x}\| := \sqrt{\langle \tilde{x}, \tilde{x}\rangle}$. We denote by $\{\tilde{\mathrm{e}}_m\}_{m\in\mathbb{Z}}$ the canonical basis of $\ell^2(\mathbb{Z})$, that is $\tilde{\mathrm{e}}_m(m) = 1$ and $\tilde{\mathrm{e}}_m(k) = 0$ when $k \neq m$.

The space $\ell^2(\mathbb{N})$ of square-summable complex sequences indexed by the set of nonnegative integers $\mathbb{N}$ is identified with the closed subspace of $\ell^2(\mathbb{Z})$ generated by $\{\tilde{\mathrm{e}}_m\}_{m\in\mathbb{N}}$. Thus $\ell^2(\mathbb{N})$ is a Hilbert space with the scalar product

$$\langle x, y \rangle = \sum_{k\in\mathbb{N}} \overline{x(k)}y(k), \quad x, y \in \ell^2(\mathbb{N})$$

and the canonical basis $\{\tilde{\mathrm{e}}_m\}_{m\in\mathbb{N}}$ which we also denote by $\{\mathrm{e}_m\}_{m\in\mathbb{N}}$.

If L is an operator acting in the Hilbert space $\mathcal{H}$, let $\sigma(L)$ denote its spectrum. We say that L has *discrete spectrum* if and only if every point $\lambda \in \sigma(L)$ is an isolated eigenvalue of finite multiplicity.

Assume that L has discrete spectrum. We say that $\{\lambda_j(L)\}_{j\in\mathfrak{J}}$ is *a complete eigenvalue sequence* of L if and only if any $\lambda \in \sigma(L)$ is some $\lambda_j(L)$ and has multiplicity $m_\lambda := \dim\ker(L - \lambda I) = \operatorname{card}\{j \in \mathfrak{J} : \lambda_j(L) = \lambda\}$.

Assume that a self-adjoint operator L has discrete spectrum such that $\inf\sigma(L) = -\infty$ and $\sup\sigma(L) = +\infty$. Then we can choose a complete eigenvalue sequence $\{\lambda_j(L)\}_{j\in\mathbb{Z}}$ such that $\{\lambda_j(L)\}_{j\in\mathbb{Z}}$ is a nondecreasing sequence such that $\lambda_j(L) \to \pm\infty$ as $j \to \pm\infty$. Moreover, there exists an orthogonal basis $\{f_j\}_{j\in\mathbb{Z}}$ such that $Lf_j = \lambda_j(L)f_j$ holds for all $j \in \mathbb{Z}$ and any other nondecreasing complete eigenvalue sequence is of the form $\{\lambda_{j+\kappa}(L)\}_{j\in\mathbb{Z}}$ with a shift $\kappa \in \mathbb{Z}$.

Assume now that L is a self-adjoint operator with discrete spectrum such that $\inf\sigma(L) > -\infty$ and $\sup\sigma(L) = +\infty$. Then there exists a unique nondecreasing complete eigenvalue sequence $\{\lambda_n(L)\}_{n\in\mathbb{N}}$ which we will call *the eigenvalue sequence* of L. Moreover, $\lambda_n(L) \to +\infty$ as $n \to \infty$ and there exists an orthogonal basis $\{f_n\}_{n\in\mathbb{N}}$ such that $Lf_n = \lambda_n(L)f_n$ holds for all $n \in \mathbb{N}$.

2. Formulation in terms of Jacobi matrices

2.1. **Jacobi matrices.** Let J be a real symmetric tridiagonal matrix

$$J = \begin{pmatrix} d(0) & b(0) & 0 & 0 & \dots \\ b(0) & d(1) & b(1) & 0 & \dots \\ 0 & b(1) & d(2) & b(2) & \dots \\ 0 & 0 & b(2) & d(3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2.1}$$

whose entries $\{d(k)\}_{k=0}^\infty$ and $\{b(k)\}_{k=0}^\infty$ satisfy the two conditions

$$d(k) \xrightarrow[k\to\infty]{} +\infty, \tag{2.2a}$$

$$\limsup_{k\to\infty} \frac{|b(k)| + |b(k-1)|}{d(k)} < 1. \tag{2.2b}$$

Let $D = \operatorname{diag}\{d(k)\}_{k=0}^{\infty}$ and B be the diagonal and off-diagonal parts of J:

$$J = D + B. \tag{2.3}$$

Then it is known (see [7]) that under conditions (2.2) there exist constants $C_0 > 0$ and $0 < c_0 < 1$ such that, in $\ell^2(\mathbb{N})$,

$$\pm \langle x, Bx\rangle \le c_0\langle x, Dx\rangle + C_0\|x\|^2 \tag{2.4}$$

and the Jacobi matrix (2.1) defines a self-adjoint operator $\hat{J}$ characterized by

$$\hat{J}\mathrm{e}_k = d(k)\mathrm{e}_k + b(k)\mathrm{e}_{k+1} + b(k-1)\mathrm{e}_{k-1}, \quad k = 0, 1, \dots \tag{2.5}$$

where $\hat{J}\mathrm{e}_0 = d(0)\mathrm{e}_0 + b(0)\mathrm{e}_1$ for $k = 0$.

2.2. **Jacobi matrices and the two-photon Hamiltonian.** Henceforth, $\nu = \pm 1$ and $\mu = 0, 1$. We reindex the orthogonal basis $\mathfrak{B} = \{\mathbf{e}_\nu \otimes \mathrm{e}_n\}$ where $\nu = \pm 1$ and $n = 0, 1, 2, \dots$ as follows:

$$\mathrm{e}_m^{\nu,\mu} \coloneqq \mathbf{e}_{\nu(-1)^m} \otimes \mathrm{e}_{2m+\mu}.$$

Let $\mathcal{H}_{\nu,\mu}$ denote the closed subspace of $\mathcal{H}_{\mathrm{Rabi}}$ generated by $\mathfrak{B}_{\nu,\mu} \coloneqq \{\mathrm{e}_m^{\nu,\mu}\}_{m\ge 0}$. An easy calculation shows that

$$\begin{aligned}\hat{H}_{\mathrm{Rabi}}\,\mathrm{e}_m^{\nu,\mu} = &\left(2m + \mu + \nu(-1)^m\frac{\Delta}{2}\right)\mathbf{e}_{\nu(-1)^m} \otimes \mathrm{e}_{2m+\mu}\\ &+ g\sqrt{(2m+\mu)(2m+\mu-1)}\mathbf{e}_{-\nu(-1)^m} \otimes \mathrm{e}_{2m+\mu-2}\\ &+ g\sqrt{(2m+\mu+1)(2m+\mu+2)}\mathbf{e}_{-\nu(-1)^m} \otimes \mathrm{e}_{2m+\mu+2},\end{aligned} \tag{2.6}$$

where $\mathbf{e}_{\nu(-1)^m} \otimes \mathrm{e}_{2m+\mu} = \mathrm{e}_m^{\nu,\mu}$ and $\mathbf{e}_{-\nu(-1)^m} \otimes \mathrm{e}_{2m+\mu\pm 2} = \mathrm{e}_{m\pm 1}^{\nu,\mu}$. Therefore, each subspace $\mathcal{H}_{\nu,\mu}$ is $\hat{H}_{\mathrm{Rabi}}$-invariant and the partition $\mathfrak{B} = \bigsqcup_{\nu,\mu}\mathfrak{B}_{\nu,\mu}$ defines the $\hat{H}_{\mathrm{Rabi}}$-invariant splitting (1.8):

$$\mathcal{H}_{\mathrm{Rabi}} = \mathcal{H}_{1,0} \oplus \mathcal{H}_{1,1} \oplus \mathcal{H}_{-1,0} \oplus \mathcal{H}_{-1,1}.$$

Let $\hat{H}_{\nu,\mu}$ denote the restriction of $\hat{H}_{\mathrm{Rabi}}$ to $\mathcal{H}_{\nu,\mu}$. Thus the spectrum of $\hat{H}_{\mathrm{Rabi}}$ splits into four parts:

$$\sigma(\hat{H}_{\mathrm{Rabi}}) = \sigma(\hat{H}_{1,0}) \cup \sigma(\hat{H}_{1,1}) \cup \sigma(\hat{H}_{-1,0}) \cup \sigma(\hat{H}_{-1,1}).$$

Moreover, each $\mathcal{H}_{\nu,\mu}$ is isometric to $\ell^2(\mathbb{N})$ through the surjective isometry $U_{\nu,\mu}\colon \mathcal{H}_{\nu,\mu} \to \ell^2(\mathbb{N})$ defined by $U_{\nu,\mu}\mathrm{e}_m^{\nu,\mu} = \mathrm{e}_m$. This isometry transforms $\hat{H}_{\nu,\mu}$ into the operator $\hat{J}_{\nu,\mu} \coloneqq U_{\nu,\mu}\hat{H}_{\nu,\mu}U_{\nu,\mu}^{-1}$ in $\ell^2(\mathbb{N})$, so that both operators have the same spectrum:

$$\sigma(\hat{H}_{\nu,\mu}) = \sigma(\hat{J}_{\nu,\mu}).$$

Lemma 2.1. *The operator $\hat{J}_{\nu,\mu} \coloneqq U_{\nu,\mu}\hat{H}_{\nu,\mu}U_{\nu,\mu}^{-1}$ is associated with a Jacobi matrix whose diagonal and off-diagonal entries are given by $d(m) = d_{\nu,\mu}(m)$ and $b(m) = b_\mu(m)$, $m \in \mathbb{N}$, with*

$$d_{\nu,\mu}(m) \coloneqq 2m + \mu + \nu(-1)^m\frac{\Delta}{2}, \tag{2.7a}$$

$$b_\mu(m) \coloneqq g\sqrt{(2m+\mu+1)(2m+\mu+2)}. \tag{2.7b}$$

Proof. Follows from (2.6) given (2.5). □

Lemma 2.2. *Assume that $0 < g < 1/2$. Let $\hat{J}_{\nu,\mu}$, $\nu = \pm 1$, $\mu = 0, 1$ be the operator in $\ell^2(\mathbb{N})$ defined in Lemma 2.1. Then,*

(i) *$\hat{J}_{\nu,\mu}$ is self-adjoint, bounded from below, and has discrete spectrum.*
(ii) *Moreover, its eigenvalue sequence $\{\lambda_n(\hat{J}_{\nu,\mu})\}_{n=0}^{\infty}$ satisfies*

$$cn - C \le \lambda_n(\hat{J}_{\nu,\mu}) \le C(n+1), \tag{2.8}$$

where C and c are some positive constants.

Proof. (i) By the previous lemma it suffices to show that both conditions (2.2) are fulfilled when the diagonal and off-diagonal entries are given by (2.7). By (2.7a) we clearly have $d(m) \to +\infty$ if $m \to \infty$. Moreover, for $d(m)$ and $b(m)$ given by (2.7):

$$\lim_{m\to\infty} \frac{|b(m)| + |b(m-1)|}{d(m)} = 2g < 1.$$

(ii) The estimate (2.8) follows from (2.3), (2.4), and the min-max principle. □

Proposition 2.3. *Assume that $0 < g < \frac{1}{2}$. Let $\nu = \pm 1$ and $\mu = 0, 1$ be given, and let $\hat{J}_{\nu,\mu}$ be the self-adjoint operator defined in Lemma 2.1. If $\{\lambda_k(\hat{J}_{\nu,\mu})\}_{k=0}^{\infty}$ is its eigenvalue sequence we have the large n asymptotic estimate*

$$\lambda_n(\hat{J}_{\nu,\mu}) = \Big(2n + \mu + \frac{1}{2}\Big)\sqrt{1-4g^2} - \frac{1}{2} + \mathrm{O}(n^{-1/3}). \tag{2.9}$$

Proof. Lemma 3.2 will reduce the proof to that of Proposition 3.1, and the proof of Proposition will be completed in Section 6.1. □

The next lemma shows that Proposition 2.3 implies Theorem 1.1.

Lemma 2.4. *Estimates (2.9) for $\nu = \pm 1$ and $\mu = 0, 1$ imply estimate (1.4), that is, Theorem 1.1.*

Proof. We deduce (1.4) from (2.9) setting $E^{\pm}_{2k+\mu} := \lambda_k(\hat{J}_{\pm 1,\mu})$ for $\mu = 0, 1$. □

3. A new family of Jacobi matrices

In this section we introduce a new family of Jacobi matrices and we will prove Proposition 2.3, hence Theorem 1.1 in the degenerated case $\Delta = 0$.

3.1. **Jacobi matrices J^{δ}_{γ}.** For $\gamma, \delta \in \mathbb{R}$, let $\hat{J}^{\delta}_{\gamma}$ be the self-adjoint operator in $\ell^2(\mathbb{N})$ defined by the Jacobi matrix (1.9). Its diagonal and off-diagonal entries $d(m) = d^0_{\delta}(m)$ and $b(m) = b^0_{\gamma}(m)$ are defined by ($m \in \mathbb{N}$)

$$d^0_{\delta}(m) := m + (-1)^m \delta, \tag{3.1a}$$

$$b^0_{\gamma}(m) := g(m + \gamma). \tag{3.1b}$$

Thus, $\hat{J}^{\delta}_{\gamma}$ is characterized by

$$\hat{J}^{\delta}_{\gamma} \mathrm{e}_m = d^0_{\delta}(m)\mathrm{e}_m + b^0_{\gamma}(m)\mathrm{e}_{m+1} + b^0_{\gamma}(m-1)\mathrm{e}_{m-1}.$$

Proposition 3.1. *Assume $0 < g < 1/2$. The eigenvalue sequence $\{\lambda_n(\hat{J}^{\delta}_{\gamma})\}_{n=0}^{\infty}$ satisfies the large n estimate*

$$\lambda_n(\hat{J}^{\delta}_{\gamma}) = n\sqrt{1-4g^2} + \Big(\gamma - \frac{1}{2}\Big)(\sqrt{1-4g^2} - 1) + \mathrm{O}(n^{-1/3}). \tag{3.2}$$

Proof. The proof is completed in Section 6.1. □

Lemma 3.2. *Let $\nu = \pm 1$, $\mu = 0, 1$ be given. Estimate (3.2) for $\lambda_n(\hat{J}^{\delta}_{\gamma})$ with $\gamma = \frac{\mu}{2} + \frac{3}{4}$ and $\delta = \nu\Delta/4$ implies estimate (2.9) for $\lambda_n(\hat{J}_{\nu,\mu})$.*

The proof of Lemma 3.2 is given in the next section and uses the following consequence of [22, Theorem 1.1]:

Theorem 3.3 (Rozenbljum [22]). *Let L_0 be a self-adjoint operator which is bounded from below and has discrete spectrum. Let $\{\lambda_n(L_0)\}_{n=0}^{\infty}$ be its eigenvalue sequence. Assume that $0 \notin \sigma(L_0)$. Let also R be a bounded symmetric operator such that $R|L_0|^{\rho}$ is bounded for a certain $\rho > 0$.*

Then $L \coloneqq L_0 + R$ is a self-adjoint operator which is bounded from below and has discrete spectrum. Moreover, its eigenvalue sequence $\{\lambda_n(L)\}_{n=0}^{\infty}$ satisfies

$$\lambda_n(L) = \lambda_n(L_0) + \mathrm{O}(|\lambda_n(L_0)|^{-\rho}) \text{ as } n \to \infty.$$

3.2. **Proof of Lemma 3.2.** We assume we know estimate (3.2) for $\lambda_n(\hat{J}_{\gamma}^{\delta})$. The diagonal entries $d_{\nu,\mu}(m)$ and $d_{\delta}^{0}(m)$ of $J_{\nu,\mu}$ and J_{γ}^{δ} are given by (2.7a) and (3.1a), respectively. They are related by

$$\frac{1}{2}(d_{\nu,\mu}(m) - \mu) = d^{0}_{\nu\Delta/4}(m). \tag{3.3}$$

The off-diagonal entries $b_{\mu}(m)$ and $b_{\gamma}^{0}(m)$ are given by (2.7b) and (3.1b), respectively, and they are related by

$$\frac{1}{2}b_{\mu}(m) = b^{0}_{\gamma(\mu)}(m) + r_{\mu}(m), \tag{3.4}$$

where $r_{\mu}(m) = \mathrm{O}(\frac{1}{m})$ as $m \to \infty$ and

$$\gamma(\mu) \coloneqq \frac{\mu}{2} + \frac{3}{4}. \tag{3.5}$$

To prove (3.4) it suffices to observe that

$$\frac{1}{2}b_{\mu}(m) = g\sqrt{\Big(m+1+\frac{\mu}{2}\Big)\Big(m+\frac{1}{2}+\frac{\mu}{2}\Big)} = g(m+\gamma(\mu)) + \mathrm{O}\Big(\frac{1}{m}\Big)$$

holds with $\gamma(\mu)$ given by (3.5). Therefore, using (3.3) and (3.4) we can write

$$\frac{1}{2}\Big(\hat{J}_{\nu,\mu} - \mu\Big) = \hat{J}^{\nu\Delta/4}_{\gamma(\mu)} + \hat{R}_{\mu},$$

where

$$\hat{R}_{\mu}\mathrm{e}_m = r_{\mu}(m)\mathrm{e}_{m+1} + r_{\mu}(m-1)\mathrm{e}_{m-1}.$$

The operator $\hat{R}_{\mu}\hat{J}_{\nu,\mu}$ is clearly bounded due to $r_{\mu}(m) = \mathrm{O}(\frac{1}{m})$. Therefore, using the estimate (2.8) and Theorem 3.3 with $\rho = 1$, we find

$$\frac{1}{2}\Big(\lambda_n(\hat{J}_{\nu,\mu}) - \mu\Big) = \lambda_n(\hat{J}^{\nu\Delta/4}_{\gamma(\mu)}) + \mathrm{O}\Big(\frac{1}{n}\Big). \tag{3.6}$$

Combining (3.6) with (3.2) we obtain

$$\lambda_n(\hat{J}_{\nu,\mu}) - \mu = 2n\sqrt{1-4g^2} + (2\gamma(\mu)-1)(\sqrt{1-4g^2}-1) + \mathrm{O}(n^{-1/3}). \tag{3.7}$$

However the equality $2\gamma(\mu) - 1 = \mu + \frac{1}{2}$ allows us to write (3.7) in the form

$$\lambda_n(\hat{J}_{\nu,\mu}) - \mu = \Big(2n + \mu + \frac{1}{2}\Big)\sqrt{1-4g^2} - \mu - \frac{1}{2} + \mathrm{O}(n^{-1/3}),$$

which is exactly the estimate (2.9).

3.3. **Proof of Proposition 2.3 in the case $\Delta = 0$.** It's about proving the estimate (2.9) in that case. Due to Lemma 3.2 for $\delta = \nu\Delta/4 = 0$ it suffices to prove the large n asymptotics (3.2) for $\delta = 0$. We will actually prove that

$$\lambda_n(\hat{J}^0_\gamma) = n\sqrt{1-4g^2} + \Big(\gamma - \frac{1}{2}\Big)\big(\sqrt{1-4g^2} - 1\big) + \mathrm{O}(n^{-N}) \tag{3.8}$$

holds for any integer $N \geq 1$. Let $\hat{J}^\delta_{\gamma,1}$ be the self-adjoint operator in $\ell^2(\mathbb{N})$ defined by

$$\hat{J}^\delta_{\gamma,1}\mathrm{e}_m = d^1_\delta(m)\mathrm{e}_m + b^1_\gamma(m)\mathrm{e}_{m+1} + b^1_\gamma(m-1)\mathrm{e}_{m-1},$$

where $d^1_\delta(m) = d^0_\delta(m)$ for $m \geq 1$ and $b^1_\gamma(m) = b^0_\gamma(m)$ for $m \geq 1$. Next we set $b^1_\gamma(0) \coloneqq 0$, which implies $\hat{J}^\delta_{\gamma,1}\mathrm{e}_0 = d^1_\delta(0)\mathrm{e}_0$. Then we take $d^1_\delta(0) \ll 0$ to ensure that $\lambda_0(\hat{J}^0_{\gamma,1}) = d^1_\delta(0)$. Since $\hat{R} \coloneqq \hat{J}^\delta_\gamma - \hat{J}^\delta_{\gamma,1}$ is defined by a 2×2 block, Theorem 3.3 ensures that

$$\lambda_n(\hat{J}^0_\gamma) = \lambda_n(\hat{J}^0_{\gamma,1}) + \mathrm{O}(n^{-N})$$

holds for any integer $N \geq 1$. Let $\ell^2(\mathbb{N}^*)$ be identified with the closed subspace of $\ell^2(\mathbb{N})$ generated by $\{\mathrm{e}_m\}_{m=1}^\infty$. This subspace is invariant under $\hat{J}^0_{\gamma,1}$ and we denote by $\hat{J}'_\gamma$ the restriction of $g^{-1}\hat{J}^0_{\gamma,1}$ to $\ell^2(\mathbb{N}^*)$. It turns out that this operator $\hat{J}'_\gamma$ was investigated by Janas and Malejki in [15]. The result of [15, Theorem 3.4], as completed in [15, Section 4], and applied for $\delta = g^{-1}$ and $c = \gamma$, says that the eigenvalue sequence $\{\lambda_n(\hat{J}'_\gamma)\}_{n=0}^\infty$ satisfies

$$\lambda_n(\hat{J}'_\gamma) = n\sqrt{g^{-2}-4} + \Big(\gamma + \frac{1}{2}\Big)\big(\sqrt{g^{-2}-4} - g^{-1}\big) + g^{-1} + \mathrm{O}(n^{-N})$$

for any fixed integer $N \geq 1$. Thus for $n \geq 1$ one has

$$\lambda_n(\hat{J}^0_{\gamma,1}) = g\lambda_{n-1}(\hat{J}'_\gamma) = (n-1)\sqrt{1-4g^2} + 1 + \Big(\gamma + \frac{1}{2}\Big)\big(\sqrt{1-4g^2} - 1\big) + \mathrm{O}(n^{-N}),$$

completing the proof of (3.8).

4. Extension to $\ell^2(\mathbb{Z})$

For $\gamma, \delta \in \mathbb{R}$, we consider the self-adjoint operator $\tilde{J}^\delta_\gamma$ in $\ell^2(\mathbb{Z})$ characterized by

$$\tilde{J}^\delta_\gamma\tilde{\mathrm{e}}_m = (m + (-1)^m\delta)\tilde{\mathrm{e}}_m + g(m+\gamma)\tilde{\mathrm{e}}_{m+1} + g(m-1+\gamma)\tilde{\mathrm{e}}_{m-1}, \quad m \in \mathbb{Z},$$

where $\{\tilde{\mathrm{e}}_m\}_{m\in\mathbb{Z}}$ denotes the canonical basis in $\ell^2(\mathbb{Z})$.

The following theorem is a special case of the result of [9] (see also [19])

Theorem 4.1 (see [9, 19]). *The complete eigenvalue sequence $\{\lambda_j(\tilde{J}^0_\gamma)\}_{j\in\mathbb{Z}}$ of the operator $\tilde{J}^0_\gamma$ can be ordered so that*

$$\lambda_j(\tilde{J}^0_\gamma) = \Big(j + \gamma - \frac{1}{2}\Big)\sqrt{1-4g^2} - \gamma + \frac{1}{2}.$$

In this section we present a modification of the proof given in [9]. Our purpose is to give an explicit expression of a basis of eigenvectors of $\tilde{J}^0_\gamma$.

4.1. Preliminaries.

Notations. (a) We denote by $\mathrm{L}^2(-\pi,\pi)$ the Hilbert space of Lebesgue square integrable functions $[-\pi,\pi] \to \mathbb{C}$ equipped with the scalar product

$$\langle f, g\rangle \coloneqq \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(\xi)}\, g(\xi)\, \mathrm{d}\xi$$

and with the norm $\|f\| \coloneqq \sqrt{\langle f, f\rangle}$.

(b) For $f \in \mathrm{L}^2(-\pi,\pi)$ we denote by $\langle f\rangle$ its mean value

$$\langle f\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\eta)\mathrm{d}\eta.$$

(c) We denote by $\mathcal{F}_0\colon \mathrm{L}^2(-\pi,\pi)\to \ell^2(\mathbb{Z})$ the Fourier transform defined by

$$(\mathcal{F}_0 f)(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\mathrm{e}^{-\mathrm{i}j\xi}\mathrm{d}\xi. \tag{4.1}$$

(d) We denote by $\mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$ the set of functions $u\in \mathrm{L}^2(-\pi,\pi)$ whose 2π-periodic extension is a smooth function $\mathbb{R}\to\mathbb{C}$.

(e) We denote by $\{f_j^0\}_{j\in\mathbb{Z}}$ the orthonormal basis given by $f_j^0(\eta) := \mathrm{e}^{\mathrm{i}j\eta}$.

(f) We denote by L_0 the self-adjoint operator in $\mathrm{L}^2(-\pi,\pi)$ defined by

$$L_0 f_j^0 = j f_j^0 \text{ for } j\in\mathbb{Z}.$$

We denote by $\mathcal{D}(L_0)$ the domain of L_0 and observe that L_0 is the closure of the operator $D_\eta := \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\eta}$ defined on $\mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$.

Proposition 4.2. *Let $q\in \mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$ be a real valued function and let L_q denote the self-adjoint operator defined by*

$$L_q f := L_0 f + qf, \quad f\in\mathcal{D}(L_0). \tag{4.2}$$

Then L_q is unitary equivalent to $L_0+\langle q\rangle$.

Proof. We define a unitary operator U_q in $\mathrm{L}^2(-\pi,\pi)$ by

$$(U_q f)(\eta) = \mathrm{e}^{\mathrm{i}\tilde{q}(\eta)-\mathrm{i}\langle q\rangle\eta} f(\eta),$$

where $\tilde{q}$ is a primitive of q. Since $\tilde{q}-\langle q\rangle$ belongs to $\mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$, the subspace $\mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$ is invariant under $U_q^{\pm 1}$. Moreover, for $f\in \mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$,

$$U_q(D_\eta+q)U_q^{-1} f = (D_\eta+\langle q\rangle)f.$$

To complete the proof it remains to observe that L_q is the closure of the operator $f\mapsto D_\eta f + qf$ defined on $\mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$. □

Corollary 4.3. *Let $\{f_j^q\}_{j\in\mathbb{Z}}$ be the orthonormal basis of $\mathrm{L}^2(-\pi,\pi)$ given by $f_j^q := U_q^{-1} f_j^0$. Then, for every $j\in\mathbb{Z}$ we have*

$$f_j^q(\eta) = \mathrm{e}^{\mathrm{i}(j+\langle q\rangle)\eta-\mathrm{i}\tilde{q}(\eta)}$$

and

$$L_q f_j^q = (j+\langle q\rangle) f_j^q.$$

4.2. **A unitary change of variable.** Assume that $p\in \mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$. Further on $p(\xi)D_\xi + \mathrm{hc}$ denotes a symmetric operator in $\mathrm{L}^2(-\pi,\pi)$ defined on $\mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$ by

$$(p(\xi)D_\xi + \mathrm{hc})f := pD_\xi f + D_\xi(\bar{p}f). \tag{4.3}$$

The right-hand side of (4.3) can be expressed in the form

$$2(\mathrm{Re}\, p)D_\xi f - (\mathrm{Im}\, p' + \mathrm{i}\,\mathrm{Re}\, p')f. \tag{4.4}$$

Assumptions. Further on we assume that $\Phi\colon[-\pi,\pi]\to[-\pi,\pi]$ is a bijection which has a derivative $\Phi'\in \mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$ with $\Phi'\geq c_0>0$ for some positive constant c_0.

These assumptions ensure the property

$$f \in \mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi]) \implies f\circ\Phi^{\pm 1} \in \mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi]).$$

Moreover, we define a unitary operator $\tilde{U}_\Phi$ in $\mathrm{L}^2(-\pi,\pi)$ by

$$(\tilde{U}_\Phi f)(\xi) = \Phi'(\xi)^{1/2} f(\Phi(\xi)),$$

and observe that the subspace $\mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$ is invariant under $\tilde{U}_\Phi^{\pm 1}$.

Proposition 4.4. *Assume that $q \in \mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$ and L_q is given by (4.2). Then for any $h \in \mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$ one has the relation*

$$\tilde{U}_\Phi L_q \tilde{U}_\Phi^{-1} h = \Big(\frac{1}{2}\Phi'(\xi)^{-1} D_\xi + \mathrm{hc}\Big)h + (q\circ\Phi)h. \tag{4.5}$$

Proof. We assume that $\eta \mapsto f(\eta)$ belongs to $\mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$ and write $\eta = \Phi(\xi)$. Then

$$\tilde{U}_\Phi(qf)(\xi) = \Phi'(\xi)^{1/2} q(\Phi(\xi)) f(\Phi(\xi)) = (q\circ\Phi)(\xi)(\tilde{U}_\Phi f)(\xi)$$

and

$$\tilde{U}_\Phi(D_\eta f)(\xi) = \Phi'(\xi)^{1/2}(D_\eta f)(\Phi(\xi)) = \Phi'(\xi)^{-1/2} D_\xi(f(\Phi(\xi))). \tag{4.6}$$

Let us consider the quantity

$$\Phi'(\xi)^{-1} D_\xi(\tilde{U}_\Phi f)(\xi) \coloneqq \Phi'(\xi)^{-1} D_\xi(\Phi'(\xi)^{1/2} f(\Phi(\xi))). \tag{4.7}$$

However, the right hand side of (4.7) can be written in the form

$$\Phi'(\xi)^{-1/2} D_\xi(f(\Phi(\xi))) + \Phi'(\xi)^{-1}\frac{(-\mathrm{i})}{2}\Phi'(\xi)^{-1/2}\Phi''(\xi) f(\Phi(\xi)) \tag{4.8}$$

and the first term of (4.8) equals $\tilde{U}_\Phi(D_\eta f)(\xi)$ due to (4.6). Thus the quantity (4.7) can be written in the form

$$\Phi'(\xi)^{-1} D_\xi(\tilde{U}_\Phi f)(\xi) = \tilde{U}_\Phi(D_\eta f)(\xi) - \frac{\mathrm{i}}{2}\Phi'(\xi)^{-2}\Phi''(\xi)(\tilde{U}_\Phi f)(\xi). \tag{4.9}$$

Then using $\Phi'(\xi)^{-2}\Phi''(\xi) = -(1/\Phi')'(\xi)$ we find that (4.9) gives

$$\tilde{U}_\Phi(D_\eta f) = (1/\Phi')D_\xi(\tilde{U}_\Phi f) - \frac{\mathrm{i}}{2}(1/\Phi')'(\tilde{U}_\Phi f).$$

Using $f = \tilde{U}_\Phi^{-1} h$ and taking into account (4.4) with $p = 1/(2\Phi')$ we obtain

$$\tilde{U}_\Phi D_\eta \tilde{U}_\Phi^{-1} h = \Big(\frac{1}{2}(1/\Phi')\, D_\xi + \mathrm{hc}\Big)h$$

for any $h \in \mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$. □

Corollary 4.5. *The operator $L_q^\Phi \coloneqq \tilde{U}_\Phi L_q \tilde{U}_\Phi^{-1}$ is essentially self-adjoint on $\mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$. If $h \in \mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$ then $L_q^\Phi h$ is given by (4.5) and one has*

$$L_q^\Phi f_{q,j}^\Phi = (j + \langle q\rangle) f_{q,j}^\Phi,$$

where $\{f_{q,j}^\Phi\}_{j\in\mathbb{Z}}$ is the orthonormal basis of $\mathrm{L}^2(-\pi,\pi)$ given by $f_{q,j}^\Phi \coloneqq \tilde{U}_\Phi f_j^q$, i.e.

$$f_{q,j}^\Phi(\xi) = \Phi'(\xi)^{1/2} \mathrm{e}^{\mathrm{i}(j+\langle q\rangle)\Phi(\xi) - \mathrm{i}\tilde{q}(\Phi(\xi))}. \tag{4.10}$$

4.3. **Proof of Theorem 4.1.** We consider the operator $L_\gamma^0 \coloneqq \mathcal{F}_0^{-1}\tilde{J}_\gamma^0\mathcal{F}_0$ introduced in (1.10). Following [9] we find that L_γ^0 is given by the formula

$$L_\gamma^0 = \Big(\Big(\frac{1}{2} + g\mathrm{e}^{\mathrm{i}\xi}\Big)D_\xi + \mathrm{hc}\Big) + \gamma g(\mathrm{e}^{\mathrm{i}\xi} + \mathrm{e}^{-\mathrm{i}\xi}).$$

Using (4.4) we find

$$L_\gamma^0 = \Big(\frac{1}{2}(1 + 2g\cos\xi)D_\xi + \mathrm{hc}\Big) + (2\gamma - 1)g\cos\xi.$$

Now we define

$$\Phi(\xi) \coloneqq \sqrt{1-4g^2}\int_0^\xi \frac{\mathrm{d}\xi'}{1 + 2g\cos\xi'} \tag{4.11}$$

and observe that an easy computation (see [9]) gives $\Phi(\pm\pi) = \pm\pi$. Thus Φ satisfies the assumptions of Section 4.2 and one has

$$1/\Phi'(\xi) = (1-4g^2)^{-1/2}(1 + 2g\cos\xi).$$

Introducing

$$q_\gamma(\eta) \coloneqq (1-4g^2)^{-1/2}\Big(\gamma - \frac{1}{2}\Big)2g\cos(\Phi^{-1}(\eta)) \tag{4.12}$$

we find that

$$L_\gamma^0 = \sqrt{1-4g^2}\,\tilde{U}_\Phi L_{q_\gamma}\tilde{U}_\Phi^{-1}.$$

Combining the last formula with Corollary 4.5 we obtain

Proposition 4.6. *Let Φ and q_γ be given by (4.11) and (4.12), respectively. For each $j \in \mathbb{Z}$ we denote $f_{\gamma,j} \coloneqq f^\Phi_{q_\gamma,j}$, where $f^\Phi_{q,j}$ is given by (4.10).*

Then $\{f_{\gamma,j}\}_{j\in\mathbb{Z}}$ is an orthonormal basis of $\mathrm{L}^2(-\pi,\pi)$ and, for every $j \in \mathbb{Z}$,

$$L_\gamma^0 f_{\gamma,j} = \sqrt{1-4g^2}(j + \langle q_\gamma\rangle)f_{\gamma,j}.$$

Due to Proposition 4.6, the proof of Theorem 4.1 will be complete if we show that

$$\langle q_\gamma\rangle = \Big(\gamma - \frac{1}{2}\Big)\Big(1 - \frac{1}{\sqrt{1-4g^2}}\Big). \tag{4.13}$$

However using the change of variable $\eta = \Phi(\xi)$ in

$$\langle q_\gamma\rangle = \frac{\gamma - \frac{1}{2}}{2\pi\sqrt{1-4g^2}}\int_{-\pi}^{\pi} 2g\cos(\Phi^{-1}(\eta))\,\mathrm{d}\eta \tag{4.14}$$

we can express the quantity (4.14) in the form

$$\frac{\gamma - \frac{1}{2}}{2\pi\sqrt{1-4g^2}}\int_{-\pi}^{\pi} 2g\Phi'(\xi)\cos\xi\mathrm{d}\xi = \frac{\gamma - \frac{1}{2}}{2\pi}\int_{-\pi}^{\pi}\frac{2g\cos\xi}{1 + 2g\cos\xi}\mathrm{d}\xi \tag{4.15}$$

and the right hand side of (4.15) equals

$$\Big(\gamma - \frac{1}{2}\Big)\Big(1 - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{1 + 2g\cos\xi}\Big)\,\mathrm{d}\xi = \Big(\gamma - \frac{1}{2}\Big)\Big(1 - \frac{1}{\sqrt{1-4g^2}}\Big).$$

5. Eigenvalue asymptotics for L^δ_γ

In this section we investigate the operator L^δ_γ introduced in (1.10). We observe that we can express

$$L^\delta_\gamma = \mathcal{F}_0^{-1}\tilde{J}^\delta_\gamma\mathcal{F}_0 = L^0_\gamma + \delta T_\pi, \tag{5.1}$$

where T_π is defined in $\mathrm{L}^2(-\pi,\pi)$ by the formula

$$(T_\pi f)(\xi) = \begin{cases} f(\xi-\pi) & \text{if } 0\le\xi\le\pi, \\ f(\xi+\pi) & \text{if } -\pi\le\xi\le 0. \end{cases}$$

The operator T_π is unitary and self-adjoint because it satisfies $T_\pi^2 = I$. We will prove

Proposition 5.1. *The operator L^δ_γ has discrete spectrum and its complete eigenvalue sequence $\{\lambda_j(L^\delta_\gamma)\}_{j\in\mathbb{Z}}$ can be ordered as a nondecreasing sequence such that*

$$\lambda_j(L^\delta_\gamma) = (j+\langle q_\gamma\rangle)\sqrt{1-4g^2} + \mathrm{O}(j^{-1/3}) \text{ as } |j|\to\infty, \tag{5.2}$$

where $\langle q_\gamma\rangle$ is given by (4.13).

By (5.1) the operators $\tilde{J}^\delta_\gamma$ and L^δ_γ are unitary equivalent, hence Proposition 5.1 gives the following corollary.

Corollary 5.2. *The operator $\tilde{J}^\delta_\gamma$ has discrete spectrum and its complete eigenvalue sequence $\{\lambda_j(\tilde{J}^\delta_\gamma)\}_{j\in\mathbb{Z}}$ can be ordered as a nondecreasing sequence such that*

$$\lambda_j(\tilde{J}^\delta_\gamma) = (j+\langle q_\gamma\rangle)\sqrt{1-4g^2} + \mathrm{O}(j^{-1/3}) \text{ as } |j|\to\infty, \tag{5.3}$$

where $\langle q_\gamma\rangle$ is given by (4.13).

5.1. **Auxiliary results.** The following result was proved by Janas and Naboko (see [16, Lemma 2.1]):

Theorem 5.3 (Janas and Naboko [16])**.** *We fix $\epsilon_0>0$ and assume that $(\mu_n)_{n=0}^\infty$ is a real sequence such that $\mu_n\to+\infty$, $|\mu_n|\le|\mu_{n+1}|$ and $|\mu_n-\mu_m|\ge\epsilon_0$ if $\mu_n\ne\mu_m$ $(n,m\in\mathbb{N})$. Let $\{e^0_n\}_{n=0}^\infty$ be an orthonormal basis of the Hilbert space $\mathcal{H}$ and denote by L_0 the self-adjoint operator satisfying $L_0e^0_n=\mu_ne^0_n$ for any $n\in\mathbb{N}$.*

If R is compact in $\mathcal{H}$, then the operator $L:=L_0+R$ has discrete spectrum and its eigenvalue sequence $\{\lambda_n(L)\}_{n=0}^\infty$ can be ordered so that one has the estimate

$$|\lambda_n(L)-\mu_n|\le C\|R^*e^0_n\|,$$

where $C>0$ is a large enough constant and R^* denotes the adjoint of R.

Theorem 5.3 was used by E. A. Yanovich to prove the following result ([26, Theorem 2.2]):

Theorem 5.4 (Yanovich [26])**.** *Let $\{e^0_n\}_{n=0}^\infty$ be an orthonormal basis of the Hilbert space $\mathcal{H}$ and let L_0 be the self-adjoint operator satisfying $L_0e^0_n=ne^0_n$ for $n\in\mathbb{N}$.*

If R is a symmetric bounded operator in $\mathcal{H}$ satisfying

$$\lim_{n\to\infty}\langle e^0_{k+n}, Re^0_n\rangle = 0$$

for every $k\in\mathbb{Z}$, then the operator $L:=L_0+R$ has discrete spectrum and the eigenvalue sequence $\{\lambda_n(L)\}_{n=0}^\infty$ satisfies the estimate

$$|\lambda_n(L)-n-\langle e^0_n, Re^0_n\rangle| \le Cs_n^{1/2} \text{ for all } n\in\mathbb{N}, \tag{5.4}$$

where $C>0$ is some constant and

$$s_n := \sum_{k\ne n}\frac{|\langle e^0_k, Re^0_n\rangle|^2}{(k-n)^2}.$$

We will also use the following version of van der Corput Lemma.

Lemma 5.5. *For $h, \Psi \in \mathrm{C}^\infty_{\mathrm{per}}([-\pi,\pi])$ and $\tau \in \mathbb{R}$ denote*

$$\mathcal{J}_h^\Psi(\tau) := \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\tau\Psi(\xi)}\, h(\xi)\,\mathrm{d}\xi.$$

Assume that Ψ is real-valued and satisfies the condition

$$\Psi'(\xi) = 0 \implies \Psi''(\xi) \neq 0. \tag{5.5}$$

Then there is a constant C_0 such that for all $\tau \in \mathbb{R}^*$ one has

$$\left|\mathcal{J}_h^\Psi(\tau)\right| \leq C_0\,|\tau|^{-1/2}\left(|h(0)| + \int_{-\pi}^{\pi} |h'(t)|\,\mathrm{d}t\right).$$

5.2. **Proof of Proposition 5.1.** We begin by

Lemma 5.6. *Let $\{f_{\gamma,j}\}_{j\in\mathbb{Z}}$ be the orthonormal basis of $\mathrm{L}^2(-\pi,\pi)$ defined in Proposition 4.6 and denote $(j,k \in \mathbb{Z})$*

$$R_\gamma(j,k) := \langle f_{\gamma,j}, T_\pi f_{\gamma,k}\rangle. \tag{5.6}$$

If for every $k \in \mathbb{Z}$ one has

$$\lim_{|j|\to\infty} R_\gamma(k+j,j) = 0, \tag{5.7}$$

then L^δ_γ has discrete spectrum and its complete eigenvalue sequence $\{\lambda_j(L^\delta_\gamma)\}_{j\in\mathbb{Z}}$ can be ordered in nondecreasing order so that we have the estimate

$$|\lambda_j(L^\delta_\gamma) - \lambda_j(L^0_\gamma) - \delta R_\gamma(j,j)| \leq C s_j^{1/2} \text{ for all } j \in \mathbb{Z}, \tag{5.8}$$

where $C > 0$ is some constant and

$$s_j := \sum_{k\neq j} \frac{|R_\gamma(j,k)|^2}{(k-j)^2}. \tag{5.9}$$

Proof. Define the operators

$$\tilde{L}_0 := (1-4g^2)^{-1/2} L^0_\gamma - \langle q_\gamma\rangle,$$
$$\tilde{R} := (1-4g^2)^{-1/2}\delta T_\pi + \langle q_\gamma\rangle.$$

Then $\tilde{L}_0 f_{\gamma,j} = j f_{\gamma,j}$ for all $j \in \mathbb{Z}$ and (5.8) becomes

$$|\lambda_j(\tilde{L}_0) - j - \langle f_{\gamma,j}, \tilde{R} f_{\gamma,j}\rangle| \leq C_1 s_j^{1/2}.$$

This estimate is similar to the estimate (5.4), except that indices belong to $\mathbb{Z}$ instead of $\mathbb{N}$. However we obtain the result in the case of $\mathbb{Z}$ by reasoning as in [26] and replacing Theorem 5.3 with its generalized version proved by Malejki [18]. In fact the assumption $\mu_n \to +\infty$ from Theorem 5.3 can be replaced with $|\mu_n| \to \infty$. Therefore, choosing a bijection $\mathbb{Z} \to \mathbb{N}$, $j \mapsto n_j$, one can use the sequence $\{\mu_{n_j}\}_{j\in\mathbb{Z}}$ and the basis $\{\mathrm{e}_{n_j}\}_{j\in\mathbb{Z}}$. □

Since Proposition 5.1 follows from Lemma 5.6, it remains to show (5.7) and

$$s_j = \mathrm{O}(j^{-2/3}) \text{ as } j \to \infty, \tag{5.10}$$

where s_j is given by (5.9). However, (5.7) follows immediately from the following

Lemma 5.7. *Assume that $R_\gamma(j,k)$ is given by (5.6) and s_j by (5.9). Then there exists a positive constant C such that*

$$|R_\gamma(j,j+k)| \leq C(1+|k|)(1+|j|)^{-1/2}$$

holds for all $j,k \in \mathbb{Z}$.

Proof. In order to simplify the expression of

$$R_\gamma(j,k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{\gamma,k}(\xi)\,(T_\pi \bar f_{\gamma,j})(\xi)\,\mathrm{d}\xi$$

we denote

$$p_\gamma(\xi) := \mathrm{e}^{\mathrm{i}\langle q_\gamma\rangle\Phi(\xi)-\mathrm{i}\widetilde{q}_\gamma(\Phi(\xi))}(1+2g\cos\xi)^{-1/2}.$$

Then,

$$f_{\gamma,j}(\xi) = \mathrm{e}^{\mathrm{i}j\Phi(\xi)}\,p_\gamma(\xi)$$

and we can express

$$R_\gamma(j,k) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(k\Phi - jT_\pi\Phi)(\xi)}\,p_\gamma(\xi)\,(T_\pi\bar p_\gamma)(\xi)\,\mathrm{d}\xi.$$

In order to apply van der Corput Lemma 5.5 we observe that $R(j,j+k) = \mathcal{J}^{\Psi}_{h_{\gamma,k}}(j)$ holds with

$$\Psi(\xi) = \Phi(\xi) - (T_\pi\Phi)(\xi),$$
$$h_{\gamma,k}(\xi) = \mathrm{e}^{\mathrm{i}k\Phi(\xi)}p_\gamma(\xi)\,(T_\pi\bar p_\gamma)(\xi).$$

It remains to check the condition (5.5). For this purpose we observe that

$$\Psi'(\xi) = \frac{\sqrt{1-4g^2}}{1+2g\cos\xi} - \frac{\sqrt{1-4g^2}}{1-2g\cos\xi},$$

hence $\Psi'(\xi)=0 \Longleftrightarrow \cos\xi = 0$, and

$$\cos\xi = 0 \implies \Psi''(\xi) = \left(\frac{2g\sqrt{1-4g^2}}{(1+2g\cos\xi)^2} + \frac{2g\sqrt{1-4g^2}}{(1-2g\cos\xi)^2}\right)\sin\xi = \pm 4g\sqrt{1-4g^2} \neq 0. \qquad \square$$

In order to prove (5.10) we first observe that

$$\sum_{k\in\mathbb{Z}} |R_\gamma(j,j+k)|^2 = \sum_{m\in\mathbb{Z}} |\langle T_\pi f_{\gamma,j}, f_{\gamma,m}\rangle|^2 = ||T_\pi f_{\gamma,j}||^2 = 1 \tag{5.11}$$

holds due to Parseval's equality. Next we observe that using (5.11) we can estimate

$$\sum_{|k|>j^{1/3}} |k|^{-2}|R_\gamma(j,j+k)|^2 \le |j|^{-2/3}\sum_{k\in\mathbb{Z}}|R_\gamma(j,j+k)|^2 = \mathrm{O}(|j|^{-2/3})$$

and using Lemma 5.7 we get the estimate

$$\sum_{1\le|k|\le j^{1/3}} |k|^{-2}|R_\gamma(j,j+k)|^2 \le \sum_{1\le|k|\le j^{1/2}} |k|^{-2}C(1+|k|)^2(1+|j|)^{-1} = \mathrm{O}(|j|^{-2/3}).$$

5.3. **About the asymptotics conjecture.** We will show that the improved estimate (1.7) implies the validity of the three-term asymptotics (1.6). This estimate actually implies

$$\lambda_k(\hat J^\delta_\gamma) = \lambda_k(\hat J^0_\gamma) + \delta R_\gamma(k,k) + \mathrm{O}(k^{-\rho}) \text{ as } k\to\infty. \tag{5.12}$$

In Section 3.2 we proved that

$$E^\pm_{2k+\mu} = \mu + 2\lambda_k(\hat J^{\pm\Delta/4}_{\gamma(\mu)}) + \mathrm{O}(k^{-1}) \tag{5.13}$$

holds with $\gamma(\mu) = \frac{\mu}{2} + \frac{3}{4}$. Therefore, combining (5.12) and (5.13) we get

$$E^\pm_{2k+\mu} = E^0_{2k+\mu} \pm \frac{\Delta}{2}R_{\gamma(\mu)}(k,k) + \mathrm{O}(k^{-\rho})$$

and observe that the stationary phase formula allows us to express

$$R_\gamma(k,k) = \frac{\omega}{\sqrt{k}}\cos(\alpha k+\theta) + \mathrm{O}(k^{-1}),$$

where ω, α, and θ are constants depending on γ and g. Indeed, we observe that $\xi = \pm\frac{\pi}{2}$ are non-degenerated critical points of Ψ,

$$\Psi(\nu\pi/2) = \nu \int_{-\pi/2}^{\pi/2} \frac{\sqrt{1-4g^2}}{1+2g\cos\xi'} \mathrm{d}\xi' = 4\nu \arctan\left(\sqrt{\frac{1+2g}{1-2g}}\right),$$

and

$$\Psi''(\nu\pi/2) = 4g\nu\sqrt{1-4g^2}.$$

Moreover $\tilde{q}_\gamma(\Phi(\xi)) = (2\gamma-1)g(1-4g^2)^{-1/2}\sin\xi$ gives

$$\tilde{\theta} \coloneqq \tilde{q}_\gamma(\Phi(\pi/2)) - \tilde{q}_\gamma(\Phi(-\pi/2)) = 2g(2\gamma-1)(1-4g^2)^{-1/2}$$

and the stationary phase formula gives

$$\begin{aligned} R_\gamma(k,k) &= \sum_{\nu=\pm1} \frac{\mathrm{e}^{\mathrm{i}(k+\langle q_\gamma\rangle)\Psi(\nu\pi/2)+\mathrm{i}\nu\beta+\mathrm{i}\nu\pi/4}}{\sqrt{2k\pi|\Psi''(\nu\pi/2)|}} + \mathrm{O}(k^{-1}) \\ &= \frac{1}{\sqrt{k}} \frac{2\cos\big((k+\langle q_\gamma\rangle)\Psi(\pi/2)+\tilde{\theta}+\pi/4\big)}{2(2\pi g)^{1/2}(1-4g^2)^{1/4}} + \mathrm{O}(k^{-1}) \end{aligned}$$

with $\langle q_\gamma\rangle = \left(\gamma-\frac{1}{2}\right)(1-(1-4g^2)^{-1/2})$ according to (4.13).

6. End of the proof of Theorem 1.1

6.1. **Proof of Proposition 3.1.** Due to Lemma 3.2, to prove Theorem 1.1 it suffices to prove Proposition 3.1, that is, estimate (3.2) for $\lambda_n(\hat{J}_\gamma^\delta)$. We will use the following consequence of [22, Theorem 1.2]:

Theorem 6.1 (Rozenbljum [22])**.** *Let L_0 be a self-adjoint operator with discrete spectrum. We assume that $0 \notin \sigma(L_0)$, $\inf\sigma(L_0) = -\infty$, and $\sup\sigma(L_0) = \infty$. We denote by $\{\lambda_j(L_0)\}_{j\in\mathbb{Z}}$ a nondecreasing complete eigenvalue sequence of L_0. Let R be a bounded symmetric operator and denote $L \coloneqq L_0 + R$.*

If $R|L_0|^\rho$ is bounded for a certain $\rho > 0$, then L has discrete spectrum and one can order a complete eigenvalue sequence $\{\lambda_j(L)\}_{j\in\mathbb{Z}}$ so that

$$\lambda_j(L) = \lambda_j(L_0) + \mathrm{O}(|j|^{-\rho}) \quad \text{as } |j| \to \infty.$$

Let $\delta_0 > 0$ be an arbitrary fixed positive real number. Further on we assume that $\delta \in [-\delta_0, \delta_0]$. By the min-max principle, $\inf\sigma(J_\gamma^\delta) \geq -|\delta| + \sigma(J_\gamma^0)$. We choose $\rho_0 > 0$ large enough so that

$$-\rho_0 < \inf\sigma(J_\gamma^\delta) \text{ for } \delta \in [-\delta_0, \delta_0].$$

Further on $\ell^2(\mathbb{Z}\setminus\mathbb{N})$ denotes the orthogonal complement of $\ell^2(\mathbb{N})$ in $\ell^2(\mathbb{Z})$. Then for $n_0 \in \mathbb{N}$ we denote by $\hat{J}_{\gamma,n_0}^\delta$ the self-adjoint operator in $\ell^2(\mathbb{Z}\setminus\mathbb{N})$ satisfying

$$\hat{J}_{\gamma,n_0}^\delta \tilde{\mathrm{e}}_m = d_\delta^{n_0}(m)\tilde{\mathrm{e}}_m + b_\gamma^{n_0}(m)\tilde{\mathrm{e}}_{m+1} + b_\gamma^{n_0}(m-1)\tilde{\mathrm{e}}_{m-1} \text{ for } m \in \mathbb{Z},\ m < 0,$$

with

$$d_\delta^{n_0}(m) \coloneqq \begin{cases} m + (-1)^m\delta & \text{if } m < -n_0, \\ -\rho_0 & \text{if } -n_0 \leq m < 0, \end{cases}$$

$$b_\gamma^{n_0}(m) \coloneqq \begin{cases} (m+\gamma)g & \text{if } m < -n_0, \\ 0 & \text{if } -n_0 \leq m < 0. \end{cases}$$

Then it is clear that we can fix n_0 large enough to ensure

$$\sup\sigma(\hat{J}_{\gamma,n_0}^\delta) = -\rho_0 \text{ for } \delta \in [-\delta_0, \delta_0].$$

Let $\tilde{J}^{\delta}_{\gamma,n_0}$ be the self-adjoint operator in $\ell^2(\mathbb{Z})$ defined as the direct sum

$$\tilde{J}^{\delta}_{\gamma,n_0} := \hat{J}^{\delta}_{\gamma,n_0} \oplus \hat{J}^{\delta}_{\gamma}$$

and let $\{\lambda_j(\tilde{J}^{\delta}_{\gamma,n_0})\}_{j\in\mathbb{Z}}$ be a nondecreasing complete eigenvalue sequence of $\tilde{J}^{\delta}_{\gamma,n_0}$. Since all eigenvalues of $\hat{J}^{\delta}_{\gamma,n_0}$ are smaller than $\inf\sigma(\hat{J}^{\delta}_{\gamma})$ we can order the sequence $\{\lambda_j(\tilde{J}^{\delta}_{\gamma,n_0})\}_{j\in\mathbb{Z}}$ so that

$$\lambda_n(\tilde{J}^{\delta}_{\gamma,n_0}) = \lambda_n(\hat{J}^{\delta}_{\gamma}) \text{ for } n \in \mathbb{N}. \tag{6.1}$$

Next we observe that $R := \tilde{J}^{\delta}_{\gamma} - \tilde{J}^{\delta}_{\gamma,n_0}$ is given by a finite block matrix. Thus, Theorem 6.1 can be applied for any $\rho > 0$. Therefore there exists $\kappa_\gamma(\delta) \in \mathbb{Z}$ such that

$$\lambda_j(\tilde{J}^{\delta}_{\gamma,n_0}) = \lambda_{j+\kappa_\gamma(\delta)}(\tilde{J}^{\delta}_{\gamma}) + \mathrm{O}(|j|^{-\rho}) \text{ as } |j| \to \infty. \tag{6.2}$$

However, using (6.1) and (5.3) in (6.2) we obtain

$$\lambda_n(\hat{J}^{\delta}_{\gamma}) = (n + \kappa_\gamma(\delta) + \langle q_\gamma\rangle)\sqrt{1-4g^2} + \mathrm{O}(n^{-1/3}) \text{ as } n \to \infty. \tag{6.3}$$

Therefore we can express

$$\kappa_\gamma(\delta) = \lim_{n\to\infty}\Big((1-4g^2)^{-1/2}\lambda_n(\hat{J}^{\delta}_{\gamma}) - n - \langle q_\gamma\rangle\Big)$$

and we claim that the function $\delta \mapsto \kappa_\gamma(\delta)$ is continuous. Indeed,

$$|\lambda_n(\hat{J}^{\delta_1}_{\gamma}) - \lambda_n(\hat{J}^{\delta_2}_{\gamma})| \le |\delta_1 - \delta_2|$$

holds due to the min-max principle. Using the fact that the function $\kappa_\gamma\colon [-\delta_0,\delta_0] \to \mathbb{Z}$ is continuous, we deduce that the function κ_γ is constant. Since we have proved that in the case $\delta = 0$ the formula (6.3) holds with $\kappa_\gamma(0) = 0$, we conclude that (6.3) holds with $\kappa_\gamma(\delta) = 0$ for every $\delta \in \mathbb{R}$.

6.2. **Proof of (1.5).** In the case $\Delta = 0$ the two-photon Rabi Hamiltonian $\hat{H}_{\mathrm{Rabi}}$ is unitary equivalent to

$$\hat{H}'_{\mathrm{Rabi}} = \hat{a}^\dagger a + g\sigma_z \otimes \big(\hat{a}^2 + (\hat{a}^\dagger)^2\big)\,.$$

The decomposition $\mathfrak{B}_{\mathrm{Rabi}} = \mathfrak{B}_1 \cup \mathfrak{B}_{-1}$ with $\mathfrak{B}_\nu := \{\mathbf{e}_\nu \otimes \mathrm{e}_m\}_{m=0}^{\infty}$ shows that $\hat{H}'_{\mathrm{Rabi}}$ is unitary equivalent to the direct sum $\hat{H}'_g \oplus \hat{H}'_{-g}$, where

$$\hat{H}'_{\pm g} := \hat{a}^\dagger a \pm g\big(\hat{a}^2 + (\hat{a}^\dagger)^2\big).$$

If $0 < g < \frac{1}{2}$, then $\hat{H}'_{\pm g}$ is self-adjoint in $\mathcal{H}_{\mathrm{field}} = \ell^2(\mathbb{N})$ and its spectrum is discrete and bounded from below. Let $\{\lambda_n(\hat{H}'_{\pm g})\}_{n=0}^{\infty}$ be the eigenvalue sequence of $\hat{H}'_{\pm g}$. It remains to prove

$$\lambda_n(\hat{H}'_{\pm g}) = \Big(n + \frac{1}{2}\Big)\sqrt{1-4g^2} - \frac{1}{2}. \tag{6.4}$$

The proof is based on the following result.

Lemma 6.2 (Emary and Bishop [10]). *Denote $\alpha := (1-\sqrt{1-4g^2})/(\pm 2g)$ and define*

$$\hat{z}^\dagger := \frac{\hat{a}^\dagger + \alpha\hat{a}}{\sqrt{1-\alpha^2}}, \qquad \hat{z} := \frac{\hat{a} + \alpha\hat{a}^\dagger}{\sqrt{1-\alpha^2}}, \tag{6.5}$$

Then $[\hat{z},\hat{z}^\dagger] = 1$ and

$$\hat{H}'_{\pm g} + \frac{1}{2} = \sqrt{1-4g^2}\Big(\hat{z}^\dagger z + \frac{1}{2}\Big).$$

To complete the proof of (6.4) it suffices to check that

$$\lambda_n(\hat{H}_{\pm g}) = n + \frac{1}{2} \tag{6.6}$$

where

$$\hat{H}_{\pm g} \coloneqq (1-4g^2)^{-1/2}\Big(\hat{H}'_{\pm g} + \frac{1}{2}\Big).$$

The idea is well known, but we present the details below.

For simplicity let $H \coloneqq \hat{H}_{\pm g}$. We first observe that $\langle Hx, x\rangle \geq \frac{1}{2}||x||^2$ implies $\sigma(H) \subset [\frac{1}{2}, \infty)$. We denote by $\mathcal{D}(H)$ the domain of H equipped with the graph norm. Let $z^\dagger$ and z denote bounded operators $\mathcal{D}(H) \to \mathcal{H}_{\rm field}$ acting on $\{\mathrm{e}_m\}_{m\in\mathbb{N}}$ according to (6.5). Since $H\hat{z}^\dagger - \hat{z}^\dagger H = \hat{z}^\dagger$ holds on the subspace generated by $\{\mathrm{e}_m\}_{m=0}^\infty$, we deduce $z^\dagger H^{-1} - H^{-1}z^\dagger = H^{-1}z^\dagger H^{-1}$. Assume now that $\lambda \in \sigma(H)$. Then $H^{-1}\lambda x = x$ holds for a certain $x \in \mathcal{D}(H) \setminus \{0\}$ and the equality $z^\dagger H^{-1}\lambda x - H^{-1}z^\dagger \lambda x = H^{-1}z^\dagger H^{-1}\lambda x$ implies $H^{-1}z^\dagger\lambda x = z^\dagger x - H^{-1}z^\dagger x$. We conclude that $z^\dagger x \in \mathcal{D}(H)$ and $(\lambda+1)H^{-1}z^\dagger x = z^\dagger x$, hence $\lambda+1 \in \sigma(H)$ if $z^\dagger x \neq 0$. However, if $z^\dagger x = 0$, then for every y in the subspace generated by $\{\mathrm{e}_m\}_{m=0}^\infty$ one has $\langle z^\dagger x, \hat{z}^\dagger y\rangle = \langle x, \hat{z}\hat{z}^\dagger y\rangle = \langle x, (\frac{1}{2}+H)y\rangle = \langle(\frac{1}{2}+H)x, y\rangle = 0$, and $(\frac{1}{2}+H)x = 0$ is in contradiction with $x \neq 0$. Thus $\lambda \in \sigma(H)$ implies $\lambda+1 \in \sigma(H)$.

Assume now that $\lambda > \frac{1}{2}$. Using $H\hat{z} - \hat{z}H = -\hat{z}$ in similar computations we find that $Hx = \lambda x$ implies $(\lambda-1)H^{-1}zx = zx$, hence $\lambda - 1 \in \sigma(H)$ or $zx = 0$. However, if $zx = 0$, then for every y in the subspace generated by $\{\mathrm{e}_m\}_{m=0}^\infty$ one has $\langle zx, \hat{z}y\rangle = \langle x, \hat{z}^\dagger\hat{z}y\rangle = \langle x, (H-\frac{1}{2})y\rangle = \langle(H-\frac{1}{2})x, y\rangle = 0$ and $(H-\frac{1}{2})x = 0$ implies $\lambda = \frac{1}{2}$. Thus $\lambda - 1 \in \sigma(H)$ holds if $\lambda > \frac{1}{2}$.

Assume now that there is $k \in \mathbb{N}$ such that $k+\frac{1}{2} < \lambda < k+\frac{3}{2}$ and $\lambda \in \sigma(H)$. Then $\lambda - m \in \sigma(H)$ holds for $m = 1, \ldots, k+1$. In particular we obtain $\lambda - k - 1 \in \sigma(H)$ and $\lambda - k - 1 < \frac{1}{2}$ is in contradiction with $\sigma(H) \subset [\frac{1}{2}, \infty)$.

To complete the proof of (6.6) we take $\lambda \in \sigma(H)$ and observe that there exists $k \in \mathbb{N}$, such that $\lambda = k + \frac{1}{2}$. Then we have $\lambda - m \in \sigma(H)$ for $m = 1, \ldots, k$, hence $\frac{1}{2} \in \sigma(H)$ and consequently $\frac{1}{2} + m \in \sigma(H)$ for every $m \in \mathbb{N}$. It remains to show that each eigenvalue of H has multiplicity 1. However the family $g \mapsto H_g$ is analytic in the sense of Kato, hence the multiplicity of each $\lambda \in \mathbb{Z} + \frac{1}{2}$ is constant. Thus the multiplicity is 1 as in the case $g = 0$.

References

[1] A. Boutet de Monvel, S. Naboko, and L. O. Silva, *The asymptotic behavior of eigenvalues of a modified Jaynes–Cummings model*, Asymptot. Anal. **47** (2006), no. 3-4, 291–315. ↑1

[2] A. Boutet de Monvel and L. Zielinski, *Asymptotic behavior of large eigenvalues of a modified Jaynes–Cummings model*, Spectral Theory and Differential Equations, Amer. Math. Soc. Transl. Ser. 2, vol. 233, Amer. Math. Soc., Providence, RI, 2014, pp. 77–93. ↑1

[3] ———, *Asymptotic behavior of large eigenvalues for Jaynes–Cummings type models*, J. Spectr. Theory **7** (2017), no. 2, 559–631. ↑1

[4] ———, *Oscillatory behavior of large eigenvalues in quantum Rabi models*, Int. Math. Res. Not. IMRN (2020), 59 pp., to appear, available at https://arXiv.org/abs/1711.03366v2. ↑1, 2, 3

[5] ———, *On the spectrum of the quantum Rabi model*, Analysis as a Tool in Mathematical Physics: in Memory of Boris Pavlov, Oper. Theory Adv. Appl., Birkhäuser/Springer, Cham, pp. 9, to appear. ↑1

[6] D. Braak, Q.-H. Chen, M. T. Batchelor, and E. Solano, *Semi-classical and quantum Rabi models: in celebration of 80 years [Preface]*, J. Phys. A **49** (2016), no. 30, 300301, 4. ↑1

[7] P. A. Cojuhari and J. Janas, *Discreteness of the spectrum for some unbounded Jacobi matrices*, Acta Sci. Math. (Szeged) **73** (2007), no. 3-4, 649–667. ↑5

[8] L. Duan, Y.-F. Xie, D. Braak, and Q.-H. Chen, *Two-photon Rabi model: analytic solutions and spectral collapse*, J. Phys. A **49** (2016), no. 46, 464002, 13. ↑1

[9] J. Edward, *Spectra of Jacobi matrices, differential equations on the circle, and the* su(1, 1) *Lie algebra*, SIAM J. Math. Anal. **24** (1993), no. 3, 824–831. ↑3, 8, 11

[10] C. Emary and R. F. Bishop, *Bogoliubov transformations and exact isolated solutions for simple nonadiabatic Hamiltonians*, J. Math. Phys. **43** (2002), no. 8, 3916–3926. ↑3, 16

[11] ———, *Exact isolated solutions for the two-photon Rabi Hamiltonian*, J. Phys. A **35** (2002), no. 39, 8231–8241. ↑1
[12] I. D. Feranchuk, L. I. Komarov, and A. P. Ulyanenkov, *Two-level system in a one-mode quantum field: numerical solution on the basis of the operator method*, J. Phys. A: Math. Gen. **29** (1996), no. 14, 4035–4047. ↑3
[13] C. C. Gerry, *Two-photon Jaynes–Cummings model interacting with the squeezed vacuum*, Phys. Rev. A (3) **37** (1988), no. 7, 2683–2686. ↑1
[14] E. K. Irish, *Generalized Rotating-Wave Approximation for Arbitrarily Large Coupling*, Phys. Rev. Lett. **99** (2007), no. 17, 173601. ↑3
[15] J. Janas and M. Malejki, *Alternative approaches to asymptotic behavior of eigenvalues of some unbounded Jacobi matrices*, J. Comput. Appl. Math. **200** (2007), no. 1, 342–356. ↑1, 8
[16] J. Janas and S. Naboko, *Infinite Jacobi matrices with unbounded entries: asymptotics of eigenvalues and the transformation operator approach*, SIAM J. Math. Anal. **36** (2004), no. 2, 643–658 (electronic). ↑1, 12
[17] E. T. Jaynes and F. W. Cummings, *Comparison of quantum and semiclassical radiation theories with application to the beam maser*, Proc. IEEE **51** (1963), no. 1, 89–109. ↑1
[18] M. Malejki, *Asymptotics of the discrete spectrum for complex Jacobi matrices*, Opuscula Mathematica **34** (2014), no. 1, 139-160. ↑1, 13
[19] D. R. Masson and J. Repka, *Spectral theory of Jacobi matrices in* $l^2(\mathbf{Z})$ *and the* $\mathrm{su}(1,1)$ *Lie algebra*, SIAM J. Math. Anal. **22** (1991), no. 4, 1131–1146. ↑3, 8
[20] I. I. Rabi, *On the process of space quantization*, Phys. Rev. **49** (1936), no. 4, 324. ↑1
[21] ———, *Space quantization in a gyrating magnetic field*, Phys. Rev. **51** (1937), no. 8, 652. ↑1
[22] G. V. Rozenbljum, *Near-similarity of operators and the spectral asymptotic behavior of pseudodifferential operators on the circle (Russian)*, Trudy Maskov. Mat. Obshch **36** (1978), 59-84. ↑6, 7, 15
[23] M. O. Scully and M. S. Zubairy, *Quantum Optics*, Cambridge University Press, Cambridge, 1997. ↑1
[24] È. A. Tur, *Jaynes–Cummings model: solution without rotating wave approximation*, Optics and Spectroscopy **89** (2000), no. 4, 574–588. ↑1
[25] E. A. Tur, *Jaynes–Cummings model without rotating wave approximation. Asymptotics of eigenvalues* (2002), 12 pp., available at https://arXiv.org/abs/math-ph/0211055. ↑1
[26] E. A. Yanovich, *Asymptotics of eigenvalues of an energy operator in a problem of quantum physics*, Operator Methods in Mathematical Physics, Oper. Theory Adv. Appl., vol. 227, Birkhäuser/Springer Basel AG, Basel, 2013, pp. 165–177. ↑1, 4, 12, 13
[27] Q. Xie, H. Zhong, M. T. Batchelor, and C. Lee, *The quantum Rabi model: solution and dynamics*, J. Phys. A **50** (2017), no. 11, 113001, 40. ↑1

[1]Institut de Mathématiques de Jussieu-PRG, Université de Paris, bâtiment Sophie Germain, case 7012, 75205 Paris Cedex 13, France, E-mail: anne.boutet-de-monvel@imj-prg.fr

[2]Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville EA 2597, Université du Littoral Côte d'Opale, F-62228 Calais, France, E-mail: Lech.Zielinski@lmpa.univ-littoral.fr