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On particular solutions of linear partial differential equations with polynomial right-hand-sides

Thomas G. Anderson*, Marc Bonnet[†], Luiz M. Faria[‡], and Carlos Pérez-Arancibia[‡]

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Abstract

This paper introduces general methodologies for constructing closed-form solutions to several important partial differential equations (PDEs) with polynomial right-hand sides in two and three spatial dimensions. The covered equations include the isotropic and anisotropic Poisson, Helmholtz, Stokes, and elastostatic equations, as well as the time-harmonic linear elastodynamic and Maxwell equations. Polynomial solutions have recently regained significance in the development of numerical techniques for evaluating volume integral operators and have potential applications in certain kinds of Trefftz finite element methods. Our approach to all of these PDEs relates the particular solution to polynomial solutions of the Poisson and Helmholtz polynomial particular solutions, solutions that can in turn be obtained, respectively, from expansions using homogeneous polynomials and the Neumann series expansion of the operator $(k^2 + \Delta)^{-1}$. No matrix inversion is required to compute the solution. The method naturally incorporates divergence constraints on the solution, such as in the case of Maxwell and Stokes flow equations. This work is accompanied by a freely available Julia library, `PolynomialSolutions.jl`, which implements the proposed methodology in a non-symbolic format and efficiently constructs and provides access to rapid evaluation of the desired solution.

1 Introduction

This paper shows that a combination of some simple ideas allows to obtain polynomial solutions of inhomogeneous partial differential equations (PDEs) in \mathbb{R}^d ($d = 2, 3$) with arbitrary given polynomial right-hand sides for many of the classical models arising in mathematical physics, in both the 2D and the 3D case. Such polynomial solutions hold in arbitrary regions, and are not constrained by conditions on the boundary or at infinity. The methods presented herein apply not only to the familiar scalar partial differential operators (PDOs) but also to vector and anisotropic models. We detail the construction of polynomial solutions and publish an accompanying Julia library, `PolynomialSolutions.jl`¹.

While apparently simple and obviously not satisfactory for a complete theory or as a general method, polynomial solutions of the kind considered in this work have a demonstrated usefulness as components of other numerical solution techniques, such as the method of fundamental solutions and methods that use boundary integral equations for inhomogeneous PDEs. In the former, a particular solution is straightforwardly useful to reduce the problem to a homogeneous one that the method of fundamental solutions treats [5]. In the latter approach, Green's identities are used to transform certain volume integrals to surface integrals, and in so doing particular solutions for PDEs corresponding to simple right-hand sides are introduced. One of the first methods in this vein, dating to the 1980s, is the dual reciprocity method [21] that uses a global basis of simple functions such as monomials (alternatively, radial basis functions) to approximate an inhomogeneous right-hand side (sometimes

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¹<https://github.com/WaveProp/PolynomialSolutions.jl>, version 0.1

called body force) in a linear PDE. The method thus calls for the associated (polynomial) solution to the PDE; see also [2] for another use of a global basis. Variants on the volume-to-boundary idea exist where the domain is meshed and approximation occurs on geometrically-simple regions [9, 23, 1]. Our interest arose in the course of using the latter kind of treatment as an indirect way to evaluate singular contributions to volume integral operators, and indeed the polynomial solutions given here allow the extension of the work [1] to vectorial problems such as the Stokes, elasticity, and Maxwell systems. Polynomial solutions also appear relevant to so-called “quasi-Trefftz” methods [12], wherein polynomial solutions have been observed to form a basis with favorable conditioning properties; it will be of interest if the methods presented here are useful in that context.

Given a polynomial right-hand-side, the Helmholtz equation, and many other PDEs featuring a zeroth-order derivative term, have a unique polynomial solution, whereas such solution is defined up to arbitrary harmonic polynomials for the Poisson equation (and likewise for other analogous cases such as elastostatics). Such solutions have been investigated for some time. Direct collocation approaches have long been used in the context of the method of fundamental solutions; see, e.g., [22, 15, 10]. Alternative solution methods which explicitly leverage the properties of polynomials have been developed to avoid the expense and ill-conditioning associated with certain linear systems of equations arising in collocation approaches. Recursions have been developed in [13] for polynomial solutions to scalar constant-coefficient linear problems when $d = 2$ or $d = 3$; stability challenges in the recursion are discussed. Similar formulae for Poisson solutions are given in [4] when $d = 2$, and this method is extended to the Helmholtz equation for $d = 2$ and $d = 3$ in [7]. Solutions for polyharmonic and poly-Helmholtz operators are presented in [25]. In all of these works, solutions are determined by utilizing a well-suited ansatz that relates the image of the relevant PDO to solutions corresponding to lower degree right-hand-sides.

Perhaps the work closest to the present contribution can be found in [5] for constant-coefficient second-order operators with a zeroth-order term present, as both express the solution in the form of the formal Neumann series expansion $(k^2 + \Delta)^{-1} f = \sum_{j=0}^{\infty} (-1)^j k^{-2(j+1)} \Delta^j f$, $k \neq 0$, where f denotes the polynomial right-hand-side. Such an expansion gives rise to a finite number of terms by virtue of the fact that Δ is nilpotent as an operator on polynomials. Following these ideas, our Poisson solution approach relies on expressing the right-hand-side f in terms of homogeneous polynomials and seeking $\Delta^{-1} f$ in the form $\sum_{j=0}^{\infty} c_j |\mathbf{r}|^{2(j+1)} \Delta^j f$ with \mathbf{r} denoting the position vector, from which a simple recursion relation for the finite number of non-zero coefficients c_j can be derived by applying Euler’s theorem for homogeneous functions. Our approach to the Laplace operator appears to be novel and carries some advantages; unlike certain recurrence-based methods for obtaining a (non-unique) Poisson solution, the method described yields a solution that is much more symmetric in the input variables that appear. Note that, unlike the unique polynomial solution of the Helmholtz equation, the Poisson polynomial solutions obtained by means of different methods need not necessarily coincide. Our approach for the latter is shown here to have the additional advantage of being directly generalizable to anisotropic models, i.e., PDOs involving $\text{div}(A\nabla)$ where $A \in \mathbb{C}^{d \times d}$, $d = 2, 3$; of course, such solutions are naturally asymmetric.

Known solution techniques for general classes of vectorial problems appear to be much more limited; to the best of the authors’ knowledge, only the extension [18] of the recursive technique [13] exists for the solution of a certain class of vectorial problems. Although aimed at tackling quite general coupled vectorial PDE systems, such as the elastostatics system addressed in detail therein, the method presented in [18] does not account for some important systems that feature divergence constraints on the PDE solution. In the case of the equations for two-dimensional Stokes flow, for instance, the PDE system can be written as

$$\begin{bmatrix} \mu & 0 & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{xx} \\ v_{xx} \\ p_{xx} \end{bmatrix} + \begin{bmatrix} \mu & 0 & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{yy} \\ v_{yy} \\ p_{yy} \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_x \\ v_x \\ p_x \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_y \\ v_y \\ p_y \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix}. \quad (1)$$

using the format used in [18], where u, v denote the components of the velocity field, p is the pressure, and μ is the viscosity. Since none of the four matrices in (1) is invertible, the determinant condition in [18, Sec. 3.2] is not satisfied and thus the recursive approach developed therein is not

directly applicable to this equation. Here, by contrast, we resort to our Helmholtz or Poisson solutions together with suitable representations of solutions of vectorial PDEs by potentials, to produce polynomial solutions that satisfy the unconstrained elastodynamic or elastostatic PDE systems as well as the constrained Maxwell or Stokes systems. The polynomial solutions we provide for the latter cases represent to our knowledge the first polynomial solutions to vectorial PDEs that fulfill the physically correct constraints on the vector field divergence, namely, the charge conservation and the incompressibility conditions in the cases of Maxwell and Stokes flow equations, respectively. In addition, we show that the polynomial solutions obtained for the anisotropic Poisson equation can be used to obtain solutions for some cases of anisotropic elastostatics.

Throughout this paper, we let \mathcal{P}_N , $N \in \mathbb{N}_0$, denote the space of all d -variate polynomials of total degree at most N , with $d = 2, 3, \dots$ the ambient dimension (a curiosity being that with the exception of Maxwell all methods below, as well as the implementation, are dimension agnostic). We make use of the standard multi-index notation where, for any $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, we set $|\alpha| = \alpha_1 + \dots + \alpha_d$ and $\mathbf{r}^\alpha = r_1^{\alpha_1} \dots r_d^{\alpha_d}$ when $\mathbf{r} = (r_1, \dots, r_d)$. One simple but key fact that will be used in the sequel is that for any polynomial $p \in \mathcal{P}_N$, there exists a finite integer $m = m(p) \leq \lfloor N/2 \rfloor$ (where $\lfloor x \rfloor$ is the integral part of x) such that $\Delta^m p \neq 0$ and $\Delta^{m+1} p = 0$.

The rest of this paper is organized as follows. Polynomial solutions are derived using methods that are somewhat different depending on whether Helmholtz-like or Laplace-like equations are considered; those two main cases are addressed in Section 2 (including the elastodynamics and Maxwell systems) and Section 3 (including the elastostatic and Stokes systems and the anisotropic Poisson and elastostatic problems), respectively. Some numerical considerations are then presented in Section 4.

2 Helmholtz-like PDEs

2.1 Helmholtz

We start off by considering the Helmholtz equation with wavenumber $k \in \mathbb{C}$, $k \neq 0$. The problem at hand is thus: given $f \in \mathcal{P}_N$, find $u \in \mathcal{P}_N$ such that

$$(\Delta + k^2)u = f \quad \text{in } \mathbb{R}^d. \quad (2)$$

It is easy to show that the unique polynomial $u \in \mathcal{P}_N$ solving (2) is in fact given explicitly by

$$u = \sum_{j=0}^m (-1)^j k^{-2(j+1)} \Delta^j f = k^{-2} f - k^{-4} \Delta f + k^{-6} \Delta^2 f - \dots - (-k^2)^{-m-1} \Delta^m f. \quad (3)$$

Indeed, this explicit Neumann series-like solution can be verified by applying $(\Delta + k^2)$ to the polynomial $u \in \mathcal{P}_N$ given in (3).

Regarding uniqueness, any polynomial $u \in \mathcal{P}_N$ verifying $(\Delta + k^2)u = 0$ must have vanishing terms of degrees N and $N - 1$ (since $\Delta u \in \mathcal{P}_{N-2}$). This argument then recursively applies to $u \in \mathcal{P}_{N-2}$, $u \in \mathcal{P}_{N-4} \dots$, leaving $u \equiv 0$ as the only possibility.

2.2 Elastodynamics

Let us now consider the (linear, isotropic) time-harmonic elastodynamic system of equations corresponding to a medium endowed with mass density $\rho > 0$, shear modulus $\mu > 0$, and Poisson's ratio $\nu \in (0, \frac{1}{2})$. The problem at hand in this case is: given $\mathbf{f} \in [\mathcal{P}_N]^d$, find $\mathbf{u} \in [\mathcal{P}_N]^d$ such that

$$\Delta \mathbf{u} + \frac{1}{(1 - 2\nu)} \nabla(\text{div } \mathbf{u}) + k_2^2 \mathbf{u} = \mathbf{f} \quad \text{in } \mathbb{R}^d, \quad (4)$$

where $k_2 = \omega/c_2$ is the shear wavenumber defined in terms of the velocity $c_2 = \sqrt{\mu/\rho}$ and angular frequency $\omega > 0$.

It is well known that any time-harmonic elastodynamic displacement field \mathbf{u} generated by a given body force density $\mu\mathbf{f}$ can be expressed as [6, Sec. 5.4]

$$\mathbf{u} = 2(1 - \nu)(\Delta + k_1^2)\mathbf{g} - \nabla(\operatorname{div} \mathbf{g}) \quad (5)$$

where the *Somigliana vector* field \mathbf{g} satisfies the repeated Helmholtz equation

$$(\Delta + k_1^2)(\Delta + k_2^2)\mathbf{g} = \frac{\mathbf{f}}{2(1 - \nu)} \quad \text{in } \mathbb{R}^d, \quad (6)$$

with $k_1 = \omega/c_1$ being the compressional wavenumber with the respective velocity given by $c_1 = \sqrt{2\mu(1 - \nu)/\rho(1 - 2\nu)}$.

Given a vector-valued polynomial $\mathbf{f} \in [\mathcal{P}_N]^d$, we then seek the polynomial solution \mathbf{g} of (6) by consecutively applying the vector-valued version of (3) to the following two inhomogeneous componentwise-scalar Helmholtz problems:

$$\begin{aligned} \text{(a) find } \mathbf{q} \in [\mathcal{P}_N]^d \quad \text{such that} \quad & (\Delta + k_1^2)\mathbf{q} = \frac{\mathbf{f}}{2(1 - \nu)}, \\ \text{(b) find } \mathbf{g} \in [\mathcal{P}_N]^d \quad \text{such that} \quad & (\Delta + k_2^2)\mathbf{g} = \mathbf{q}. \end{aligned} \quad (7)$$

The polynomial displacement solution $\mathbf{u} \in [\mathcal{P}_N]^d$ is then explicitly given by (5) with the above solution $\mathbf{g} \in [\mathcal{P}_N]^d$.

2.3 Maxwell

The corresponding polynomial problem for the Maxwell equation system consists in finding the time-harmonic electromagnetic field $(\mathbf{E}, \mathbf{H}) \in [\mathcal{P}_N]^3 \times [\mathcal{P}_{N-1}]^3$, arising in a homogeneous isotropic medium (with constant scalar permittivity ε and permeability μ) and due to a given polynomial current $\mathbf{J} \in [\mathcal{P}_N]^3$ and an associated polynomial charge density $\rho \in \mathcal{P}_{N-1}$, which satisfies:

$$\begin{aligned} i\omega\varepsilon\mathbf{E} + \operatorname{rot} \mathbf{H} &= \mathbf{J}, & -i\omega\mu\mathbf{H} + \operatorname{rot} \mathbf{E} &= \mathbf{0}, \\ \varepsilon \operatorname{div} \mathbf{E} &= \rho, & \mu \operatorname{div} \mathbf{H} &= 0, \end{aligned} \quad (8)$$

in \mathbb{R}^3 , with the sources being constrained by the charge conservation equation:

$$\operatorname{div} \mathbf{J} - i\omega\rho = 0 \quad \text{in } \mathbb{R}^3. \quad (9)$$

As is well-known [24], electromagnetic fields (\mathbf{E}, \mathbf{H}) that satisfy (8) can be represented in terms of a vector potential \mathbf{A} and a scalar potential φ as

$$\mathbf{E} = i\omega\mathbf{A} - \nabla\varphi, \quad \mathbf{H} = \frac{1}{\mu} \operatorname{rot} \mathbf{A}. \quad (10)$$

Moreover, constraining the potentials through a gauge condition prevents potential indetermination in (\mathbf{A}, φ) . Here we assume (\mathbf{A}, φ) are linked by the Lorenz gauge condition:

$$\operatorname{div} \mathbf{A} - i\omega\varepsilon\mu\varphi = 0 \quad \text{in } \mathbb{R}^3, \quad (11)$$

in which case they are solutions to the following Helmholtz problems:

$$\begin{aligned} \text{(a) find } \mathbf{A} \in [\mathcal{P}_N]^3 \quad \text{such that} \quad & (\Delta + k^2)\mathbf{A} = -\mu\mathbf{J}, \\ \text{(b) find } \varphi \in \mathcal{P}_{N-1} \quad \text{such that} \quad & (\Delta + k^2)\varphi = -\rho/\varepsilon, \end{aligned} \quad (12)$$

where $k = \omega\sqrt{\varepsilon\mu}$ is the wavenumber. As shown above in Sec. 2.1, the unique polynomial potentials solving the Helmholtz equations (12) are given by

$$\mathbf{A} = -\mu \sum_{j=0}^m (-k^2)^{-j-1} \Delta^j \mathbf{J} \in [\mathcal{P}_N]^3, \quad \varphi = -1/\varepsilon \sum_{j=0}^m (-k^2)^{-j-1} \Delta^j \rho \in \mathcal{P}_{N-1}. \quad (13)$$

It is easy to show that the polynomial solution (\mathbf{A}, φ) found above satisfies the Lorenz gauge condition (11). Indeed, using the solution representations in (13) and in view of the charge conservation equation (9), we have

$$\operatorname{div} \mathbf{A} - i\omega\varepsilon\mu\varphi = -\mu \sum_{j=0}^m (-k^2)^{-j-1} \Delta^j (\operatorname{div} \mathbf{J} - i\omega\rho) = 0.$$

In summary, the unique pair of polynomial potentials solving the wave equations arising from the representation (10) constrained with the Lorenz gauge automatically satisfy that gauge (i.e. define valid electromagnetic solutions) for any (polynomial) source data verifying, as required, the charge conservation constraint (9).

2.4 General linear PDO having a zeroth-order term

Finally, in this section we generalize the procedure of Section 2.1. Let B denote a d -variate polynomial whose zeroth-degree (i.e. constant) term is equal to zero, and let the PDO \mathcal{B} be defined as $\mathcal{B} = B(\partial)$ (with $\partial = (\partial_1, \dots, \partial_d)$); we thus assume that \mathcal{B} does not have zeroth-order terms. For a given scalar $\alpha \neq 0$, the equation

$$\mathcal{B}u + \alpha u = f \quad \text{in } \mathbb{R}^d, \quad (14)$$

has for a polynomial right-hand side f a unique² polynomial solution u given by

$$u = \alpha^{-1}f - \alpha^{-2}\mathcal{B}f + \alpha^{-3}\mathcal{B}^2f - \alpha^{-4}\mathcal{B}^3f \dots, \quad (15)$$

where the sum is finite by virtue of f being polynomial and \mathcal{B} containing no zeroth-order term (so that $\mathcal{B} + \alpha$ does). This method has a straightforward extension to vector-valued PDOs of the form $\mathcal{B}\mathbf{u} + \mathbf{K}\mathbf{u}$, where \mathcal{B} is a matrix-valued PDO with no zeroth-order term and $\mathbf{K} \in \mathbb{C}^{d \times d}$ is an invertible matrix. This includes e.g. the elastodynamics operator for general anisotropic elastic media, and in particular provides an alternative to the method of Section 2.2 for isotropic elastodynamics.

3 Laplace-like PDEs

3.1 Poisson/Laplace

In this section we focus on the polynomial Poisson problem: given $f \in \mathcal{P}_N$, find a polynomial $u \in \mathcal{P}_{N+2}$ such that

$$\Delta u = f \quad \text{in } \mathbb{R}^d. \quad (16)$$

We propose a generic solution method for (16), valid for the 2D and 3D cases, based on the following observations:

- Any $f \in \mathcal{P}_N$ is a finite sum of homogeneous polynomials of degree at most N . By linearity, we can then simplify the problem (16) by assuming that $f \in \mathcal{H}_n$, where here and in the sequel \mathcal{H}_n denotes the space of d -variate homogeneous polynomials of total degree n . Note that $\dim(\mathcal{H}_n) = n + 1$ if $d = 2$ and $\dim(\mathcal{H}_n) = (n + 1)(n + 2)/2$ if $d = 3$. For example, the Maclaurin series

$$f(\mathbf{r}) = \sum_{|\alpha|=0}^N \frac{D^\alpha f(\mathbf{0})}{\alpha!} \mathbf{r}^\alpha, \quad \mathbf{r} \in \mathbb{R}^d$$

provides the expansion of a given polynomial $f \in \mathcal{P}_N$ on the basis of homogeneous polynomials $\mathbf{r}^\alpha \in \mathcal{H}_{|\alpha|}$ ($0 \leq |\alpha| \leq N$).

²To show uniqueness, suppose that $\mathcal{B}u + \alpha u = 0$ has a non-trivial polynomial solution $u \in \mathcal{P}_n$. Then, u can be expressed as $u = h_n + u_{n-1}$, $n \in \mathbb{N}$, where $h_n \in \mathcal{H}_n$, with $\mathcal{H}_n \subset \mathcal{P}_n$ denoting the homogeneous polynomials of total degree n , and $u_{n-1} \in \mathcal{P}_{n-1}$. Then, $\mathcal{B}u + \alpha u = \mathcal{B}u + \alpha h_n + \alpha u_{n-1}$, where $\mathcal{B}u$ has at most degree $n - 1$ by assumption on \mathcal{B} . Therefore $h_n \equiv 0$ and $u \in \mathcal{P}_{n-1}$. Repeating n times the foregoing argument, decreasing the degree of u one unit at a time, leads to $u \equiv 0$.

- For some $\ell \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$, let $v = r^{2\ell+2}h_n$ with $h_n \in \mathcal{H}_n$ and $r = |\mathbf{r}|$. Then:

$$\Delta v = \gamma_\ell^n r^{2\ell} h_n + r^{2\ell+2} \Delta h_n \quad \text{and} \quad \gamma_\ell^n := 2(\ell+1)(2\ell+2n+d). \quad (17)$$

Indeed, elementary computations yield

$$\nabla(r^{2\ell+2}) = (2\ell+2)r^{2\ell}\mathbf{r} \quad \text{and} \quad \Delta(r^{2\ell+2}) = 2(\ell+1)(2\ell+d)r^{2\ell}. \quad (18)$$

Therefore,

$$\begin{aligned} \Delta v &= (\Delta r^{2\ell+2})h_n + r^{2\ell+2}(\Delta h_n) + 2\nabla(r^{2\ell+2}) \cdot \nabla h_n \\ &= 2(\ell+1)(2\ell+d)h_n + 4(\ell+1)r^{2\ell}\mathbf{r} \cdot \nabla h_n + r^{2\ell+2}\Delta h_n \\ &= \gamma_\ell^n r^{2\ell} h_n + r^{2\ell+2}\Delta h_n, \end{aligned}$$

where we have used the fact that $\mathbf{r} \cdot \nabla h_n = nh_n$, i.e., Euler's theorem for homogeneous functions.

Let then $f \in \mathcal{H}_n$ (e.g. a monomial). We seek a solution $u \in \mathcal{P}_{n+2}$ of $\Delta u = f$ of the form

$$u = \sum_{\ell=0}^m c_\ell r^{2\ell+2} \Delta^\ell f,$$

where the coefficients $\{c_\ell\}_{\ell=0}^m$ are to be determined and where $m = m(f)$ is defined, as before, as the smallest integer such that $\Delta^{m+1}f \equiv 0$. To this end, we first observe that $\Delta^\ell f \in \mathcal{H}_{n-2\ell}$. Therefore, by virtue of (17), we have

$$\begin{aligned} \Delta u &= \sum_{\ell=0}^m c_\ell \gamma_\ell^{n-2\ell} r^{2\ell} \Delta^\ell f + \sum_{\ell=0}^{m-1} c_\ell r^{2\ell+2} \Delta^{\ell+1} f \\ &= c_0 \gamma_0^n f + \sum_{\ell=1}^m (\gamma_\ell^{n-2\ell} c_\ell + c_{\ell-1}) r^{2\ell} \Delta^\ell f. \end{aligned}$$

The polynomial u thus satisfies $\Delta u = f$ if we set

$$c_0 = \frac{1}{\gamma_0^n} = \frac{1}{2(2n+d)} \quad \text{and} \quad c_\ell = -\frac{c_{\ell-1}}{\gamma_\ell^{n-2\ell}} = -\frac{c_{\ell-1}}{2(\ell+1)(2n-2\ell+d)}, \quad 1 \leq \ell \leq m. \quad (19)$$

3.2 Bilaplacian

We next consider the case of the bilaplacian PDO. The polynomial solutions developed in this section will become important in the sequel to construct solutions to the inhomogeneous elastostatic and Stokes equations. In detail, we seek a polynomial solution $u \in \mathcal{P}_{N+4}$ of

$$\Delta^2 u = f \quad \text{in} \quad \mathbb{R}^d, \quad (20)$$

where $f \in \mathcal{P}_N$. Following the construction of the Poisson equation solutions presented above in Section 3.1, we note that it suffices to restrict ourselves to the case when the source is a homogeneous polynomial $f \in \mathcal{H}_n$, $n \in \mathbb{N}_0$. Finding a particular $u \in \mathcal{H}_{n+2}$ that solves (20) for an $f \in \mathcal{H}_n$ is a straightforward task using either of the following two approaches.

The **first approach** simply consists in applying twice the procedure for the Poisson equation, i.e.,

- (a) find $g \in \mathcal{H}_{n+2}$ such that $\Delta g = f$, (b) find $u \in \mathcal{H}_{n+4}$ such that $\Delta u = g$.

The **second approach** consists in directly seeking $u \in \mathcal{H}_{n+4}$ as the sum

$$u = \sum_{\ell=0}^m c_\ell r^{2\ell+4} \Delta^\ell f,$$

so that it satisfies $\Delta^2 u = f$. Then, in view of the identity

$$\Delta^2 r^{2\ell+4} h_n = \gamma_{\ell+1}^n \gamma_\ell^n r^{2\ell} h_n + 2\gamma_{\ell+1}^n r^{2\ell+2} \Delta h_n + r^{2\ell+4} \Delta^2 h_n,$$

obtained by applying Δ to $\Delta r^{2\ell+4} h_n$ using twice the formula (17) and exploiting the fact that $\Delta h_n \in \mathcal{H}_{n-2}$ for $h_n \in \mathcal{H}_n$, we arrive at

$$\begin{aligned} \Delta^2 u &= \sum_{\ell=0}^m c_\ell \gamma_\ell^{n-2\ell} \gamma_{\ell+1}^{n-2\ell} r^{2\ell} \Delta^\ell f + \sum_{\ell=0}^{m-1} 2c_\ell \gamma_{\ell+1}^{n-2\ell} r^{2\ell+2} \Delta^{\ell+1} f + \sum_{\ell=0}^{m-2} c_\ell r^{2\ell+4} \Delta^{\ell+2} f \\ &= c_0 \gamma_0^n \gamma_1^n f + (c_1 \gamma_1^{n-2} \gamma_2^{n-2} + 2c_0 \gamma_1^n) r^2 \Delta f + \sum_{\ell=2}^m (c_\ell \gamma_\ell^{n-2\ell} \gamma_{\ell+1}^{n-2\ell} + 2c_{\ell-1} \gamma_\ell^{n+2-2\ell} + c_{\ell-2}) r^{2\ell} \Delta^\ell f. \end{aligned}$$

The polynomial $u \in \mathcal{H}_{n+4}$ is thus made to satisfy $\Delta^2 u = f \in \mathcal{H}_n$ by recursively defining the coefficients $\{c_\ell\}_{\ell=0}^m$ as

$$c_0 = \frac{1}{\gamma_0^n \gamma_1^n}, \quad c_1 = -\frac{2\gamma_1^n c_0}{\gamma_1^{n-2} \gamma_2^{n-2}}, \quad c_\ell = -\frac{2\gamma_\ell^{n+2-2\ell} c_{\ell-1}}{\gamma_\ell^{n-2\ell} \gamma_{\ell+1}^{n-2\ell}} - \frac{c_{\ell-2}}{\gamma_\ell^{n-2\ell} \gamma_{\ell+1}^{n-2\ell}}, \quad \ell \geq 2.$$

The solutions $u \in \mathcal{H}_{n+4}$ yielded by the two proposed approaches for solving $\Delta^2 u = f$ are not identical. For simplicity, the first approach for computing u is implemented in `PolynomialSolutions.jl`.

3.3 Isotropic elastostatics

Let us now consider a (linearly elastic, isotropic) medium endowed with shear modulus $\mu > 0$ and Poisson's ratio $\nu \in (0, \frac{1}{2})$. The problem at hand is then to find a elastostatic displacement field $\mathbf{u} \in [\mathcal{P}_{N+2}]^d$ generated by a given body force density $\mu \mathbf{f} \in [\mathcal{P}_N]^d$, which satisfies the PDE system

$$\Delta \mathbf{u} + \frac{1}{(1-2\nu)} \nabla(\operatorname{div} \mathbf{u}) = \mathbf{f} \quad \text{in } \mathbb{R}^d. \quad (21)$$

As is well known [17, Sec. 4.1.7], any displacement field solving (21) can be expressed as

$$\mathbf{u} = 2(1-\nu) \Delta \mathbf{g} - \nabla(\operatorname{div} \mathbf{g}), \quad (22)$$

in terms of a *Galerkin vector potential* \mathbf{g} , which solves

$$\Delta^2 \mathbf{g} = \frac{\mathbf{f}}{2(1-\nu)} \quad \text{in } \mathbb{R}^d. \quad (23)$$

A particular polynomial solution $\mathbf{g} \in \mathcal{P}_{N+4}$ of (23) can then be obtained for any given $\mathbf{f} \in \mathcal{P}_N$ by solving the vectorial bilaplacian equation (23) component-wise applying either of the approaches presented above in Section 3.2, whereupon plugging \mathbf{g} into (22) yields a particular elastostatic displacement \mathbf{u} generated by the force density $\mu \mathbf{f}$.

3.4 Stokes flows

As it turns out, the Galerkin vector potential representation (22) applies also to the stationary inhomogeneous Stokes equation [14], which is formally identical to that of incompressible isotropic elasticity for which $\nu = 1/2$ [11, Sec. 2.2.4]. Indeed, a steady velocity field $\mathbf{u} \in [\mathcal{P}_{N+2}]^d$ and pressure field $p \in \mathcal{P}_{N+1}$ solving

$$\mu \Delta \mathbf{u} - \nabla p = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \mathbb{R}^d, \quad (24)$$

(where μ is here the dynamic viscosity of the fluid material) for a given body force density $\mathbf{f} \in [\mathcal{P}_N]^d$, can be expressed as

$$\mathbf{u} = \Delta \mathbf{g} - \nabla(\operatorname{div} \mathbf{g}), \quad p = -\mu \Delta(\operatorname{div} \mathbf{g}), \quad (25)$$

where the Galerkin vector $\mathbf{g} \in [\mathcal{P}_{N+4}]^d$ solves

$$\Delta^2 \mathbf{g} = \frac{\mathbf{f}}{\mu} \quad \text{in } \mathbb{R}^3. \quad (26)$$

In particular, the representation (25) automatically satisfies the incompressibility constraint. Then, particular (polynomial) solutions can again be obtained for (\mathbf{u}, p) by solving (e.g. componentwise) the bilaplacian equation (26) for \mathbf{g} and using that solution in (25).

3.5 Anisotropic Laplacian

We now focus on the anisotropic Laplacian Δ_A defined as $\Delta_A u = \text{div}(\mathbf{A}\nabla u) = A_{ij}\partial_{ij}u$, where \mathbf{A} is a symmetric positive definite $d \times d$ matrix describing anisotropic conductivity properties; in particular we have $\Delta_I = \Delta$. To look for polynomial solutions $u \in \mathcal{P}_{N+2}$ of the anisotropic Poisson equation

$$\Delta_A u = f \quad \text{in } \mathbb{R}^d, \quad (27)$$

with $f \in \mathcal{P}_N$, we define the anisotropic length r_A of $\mathbf{r} \in \mathbb{R}^d$ by

$$r_A^2 := \mathbf{r}^T \mathbf{A}^{-1} \mathbf{r}. \quad (28)$$

Generalizing formulae (18), it can be shown that

$$\nabla(r_A^{2\ell}) = 2\ell r_A^{2\ell-2} \mathbf{A}^{-1} \mathbf{r}, \quad \text{and} \quad \Delta_A(r_A^{2\ell}) = 2\ell(2\ell + d - 2)r_A^{2\ell-2}. \quad (29)$$

Therefore, letting $v = r_A^{2\ell+2} h_n$ with $h_n \in \mathcal{H}_n$ for some $\ell \in \mathbb{N}_0$ and $p \in \mathbb{N}_0$, we have

$$\Delta_A v = \gamma_\ell^n r_A^{2\ell} h_n \quad \text{in } \mathbb{R}^d, \quad (30)$$

with the coefficients γ_ℓ^n again defined in (17). The proof of (30) is essentially identical to that of (17), so it is omitted for conciseness.

The aforementioned properties then yield the solution

$$u = \sum_{\ell=0}^m c_\ell r_A^{2\ell+2} \Delta_A^\ell f,$$

of (27) for $f \in \mathcal{H}_n$ with the coefficients $\{c_\ell\}_{\ell=0}^m$ again recursively defined by (19). As in the case of the isotropic Poisson equation (16), the general solution of (27) for an arbitrary $f \in \mathcal{P}_N$ can be obtained by expanding f in a basis of homogeneous polynomials.

3.6 Anisotropic elastostatics

Finally, we consider a general anisotropic elastic material, whose constitutive behavior is described by the 4th-order elasticity tensor \mathcal{C} . The Cartesian components \mathcal{C}_{ijkl} of \mathcal{C} satisfy the usual major and minor symmetries $\mathcal{C}_{ijkl} = \mathcal{C}_{klij} = \mathcal{C}_{jikl}$ ($i, j, k, \ell = 1, \dots, d$). Any elastostatic displacement field \mathbf{u} generated in such medium by a given body force density \mathbf{f} satisfies

$$-\mathcal{C}_{ijkl}\partial_{j\ell}u_k = f_i \quad \text{in } \mathbb{R}^d, \quad i = 1, \dots, d, \quad (31)$$

where Einstein's implicit summation convention on repeated indices is used. Following classical solution methods for anisotropic elasticity (see e.g. [20, Chap. 1]), let the *Christoffel matrix* $K_{ik} = K_{ik}(\boldsymbol{\xi})$ be defined for any $\boldsymbol{\xi} \in \mathbb{R}^d$ by $K_{ik}(\boldsymbol{\xi}) = \mathcal{C}_{ijkl}\xi_j\xi_\ell$ (so that $K_{ik}(-i\partial)$ is the differential operator on the left-hand side of (31)). Since the Christoffel matrix is homogeneous in $\boldsymbol{\xi}$ with degree 2, there exist a scalar function $E(\boldsymbol{\xi})$ and a matrix-valued function $N_{ik}(\boldsymbol{\xi})$, which are homogeneous with respective degrees $2d$ and $2d - 2$ in $\boldsymbol{\xi}$ and verify

$$K_{ik}(\boldsymbol{\xi})N_{kj}(\boldsymbol{\xi}) = \delta_{ij}E(\boldsymbol{\xi}). \quad (32)$$

Then, let the displacement \mathbf{u} be represented in terms of a vector field \mathbf{g} as

$$u_i = N_{ij}(-i\partial)g_j \quad (33)$$

Writing (31) in the Fourier domain and using (33), we find that \mathbf{g} satisfies the following componentwise-scalar differential equation of order $2d$:

$$E(-i\partial)\mathbf{g} = \mathbf{f} \quad (34)$$

In analogy with the Galerkin representation of isotropic elastostatics, particular solutions $\mathbf{u} \in [\mathcal{P}_{N+2}]^d$ of (31) for given (polynomial) right-hand sides $\mathbf{f} \in [\mathcal{P}_N]^d$ are therefore obtained by solving equation (34) for \mathbf{g} and substituting \mathbf{g} into (33). For general anisotropic materials, this task amounts to solving $2d$ -th order inhomogeneous scalar PDEs instead of the second-order d -dimensional PDE (31). However, there are classes of anisotropic elastic materials for which an explicit factorization of $E(\boldsymbol{\xi})$ is known, in which case solving (34) reduces to sequentially solving lower-order scalar PDEs with polynomial right-hand sides. In particular, for three-dimensional transversely isotropic materials (characterized by five independent elastic constants), we have [26]

$$E(\boldsymbol{\xi}) = \prod_{i=1}^3 (A_i(\xi_1^2 + \xi_2^2) + \xi_3^2), \quad (35)$$

where the (real, positive) constants A_1, A_2, A_3 are known in terms of the material elastic constants. The above factorization thus implies that equation (34) leads to sequentially solving three inhomogeneous anisotropic-Laplace equations (with three different tensors \mathbf{A}), a task to which the method of Sec. 3.5 applies. Such factorizations are also available for other cases of elastic anisotropy, e.g. hexagonal crystals [16] and two-dimensional cubic crystals.

For the special case of isotropic elasticity, $N_{ij}(\boldsymbol{\xi})$ and $E(\boldsymbol{\xi})$ are found to be homogeneous with respective degrees 2 and 4 (for both $d = 2$ and $d = 3$), and their known closed-form expressions [20, Chap. 1] show that the foregoing method reduces to the Galerkin representation method used in Sec. 3.3.

4 Implementation details

The methods proposed in the previous sections for constructing polynomial solutions have been implemented in the Julia [3] library `PolynomialSolutions.jl`, made available under an MIT license on GitHub. The library is self-contained (i.e., it has no dependencies other than the Julia language itself), and consists of a few hundred lines of code; in particular, no symbolic computations are performed so that both the computation and the evaluation of the polynomial solutions are fully numerical procedures (i.e. the polynomial coefficients are regular numeric types such as single or double precision floating point numbers, long integers, etc).

In our implementation, polynomials are represented as a dictionary mapping a d -tuple α of exponents (the keys) to the corresponding coefficient c_α (the values). To allow for flexibility on both the spatial dimension and on the type of the coefficients, a generic `Polynomial{N,T}` type is defined, templated on both the ambient dimension N and on the numerical type of coefficient T ; in Julia parlance, the `Polynomial` structure is said to be of parametric type. A bi-variate polynomial with double precision coefficients corresponds for instance to a `Polynomial{2,Float64}` object.

Given a polynomial Q , a polynomial solution P is obtained by means of the appropriate invocation `solve_pde(Q,parameters)`, where `pde` corresponds to one of the supported partial differential equations (currently available choices for `pde` are `helmholtz`, `elastodynamics`, `maxwell`, `laplace`, `bilaplace`, `elastostatics`, `stokes`), and `parameters` are the numerical values of the physical parameters. For example, solving $\Delta P + 4P = x^2y^3z$ is accomplished through the code in Listing 1. For this example, the construction of P takes around a microsecond, and the returned polynomial can be evaluated at three-dimensional points in a few nanoseconds (on a 2022 MacBook Pro with a 2.3 GHz 8-core Intel Core i9 processor), making the library sufficiently fast for our main application of interest.

```

julia> f = Polynomial((2,3,1)=>1)
x2y3z
julia> u = solve_helmholtz(f,k=2)
0.375yz - 0.125y3z - 0.375x2yz + 0.25x2y3z

```

Listing 1: Helmholtz with floating point coefficients

Because we avoid symbolic algebra, the coefficients of the polynomial solutions are subject to truncation errors if the intermediate stages of the computation cannot be represented exactly. While this is not necessarily a problem in itself (we did not observe catastrophic accumulation of truncation errors in the tests we performed using double precision floating point numbers), it may be convenient to either avoid truncation error altogether, or to obtain rigorous interval bounds on the coefficients which are computed. Both means of providing guarantees on the correctness of the computed coefficients are naturally supported by the `PolynomialSolutions.jl` library.

To illustrate how rational numbers can be used instead of floating point numbers, we consider again Listing 1, but modify k to be of `Rational` type (Julia provides native support for rational numbers). The code snippet shown in Listing 2 illustrates how this is accomplished. Note that because (3) involves only iterated Laplacians of \mathbb{Q} and divisions by powers of k^2 , the intermediate stages of the computation involve only rational coefficients provided both k and the coefficients of Q are rational³. The same is true for the other Helmholtz-like problems presented in this paper.

When the problem parameters are not rational numbers, we may either approximate them by rationals to the desired precision and proceed as shown before, or use interval arithmetic [19] to propagate error bounds on floating point operations. Using the `IntervalArithmetic.jl` package, we can easily construct an `Interval` representation for the problem parameters (e.g. `k = Interval(pi)`), and then pass it to our solver as shown in Listing 3. Interestingly, due to the generic nature of our

```

julia> f = Polynomial((2,3,1)=>1)
x2y3z
julia> u = solve_helmholtz(f,k=Rational(2))
3//8yz - 1//8y3z - 3//8x2yz + 1//4x2y3z

```

Listing 2: Helmholtz with rational coefficients. The double-slash notation `a//b` denotes the fraction $\frac{a}{b}$.

```

julia> using IntervalArithmetic
julia> f = Polynomial((2,3,1)=>1)
x2y3z
julia> u = solve_helmholtz(f,k=Interval(pi))
[0.0249638, 0.0249639]yz - [0.0205319, 0.020532]y3z -
[0.0615958, 0.0615959]x2yz + [0.101321, 0.101322]x2y3z

```

Listing 3: Helmholtz with interval coefficients. The square brackets in the coefficients of P provide a lower and upper bound on the value.

³For large polynomial orders, the 64-bit integer types used by default in Julia may overflow. Since Julia provides support to multiple precision arithmetic by wrapping the GNU MP library [8], a simple fix is to use the `BigInt` type if needed (e.g. use `k=Rational{BigInt}(2)` instead of `k=Rational(2)`).

```

julia> f = (Polynomial((1,1)=>1),Polynomial((1,0)=>1))
(xy, x)
julia> u,p = solve_stokes(f,μ=Rational(2));
julia> u
-1//96y3 - 1//32x2y + 1//64x3y + 5//192xy3
1//32xy2 + 5//96x3 - 5//768x4 - 3//128x2y2 - 5//768y4
julia> p
-1//4xy - 1//12y3 - 1//4x2y

```

Listing 4: Stokes with rational coefficients

```

julia> J = (Polynomial((2,1,0)=>1),Polynomial((1,0,0)=>1),Polynomial((0,0,0)=>1))
(x2y, x, 1)
julia> E,H = solve_maxwell(J,μ=2);
julia> E
((-0.0 - 1.0im)x2y, (-0.0 - 2.0im)x, (-0.0 - 1.0im))
julia> H
(0, 0, -1.0 + 0.5x2)

```

Listing 5: Maxwell system. Additional keyword arguments for ϵ and ω can be passed to `solve_maxwell`; by default their value is one.

Polynomial type, the code works as is even when the coefficients of the polynomials are `Interval` objects. The computed solution has coefficients which are intervals instead of numbers, thus providing error bounds on the coefficient values due to truncation.

Although we have used the Helmholtz equation to illustrate some of the functionality and possible pitfalls, the library supports many other PDEs under a very similar API. An example illustrating how one can solve the (vectorial) Stokes system, where the solution is much less trivial to obtain manually, is shown in Listing 4. Finally, in Listing 5 we show an example for the (three-dimensional) Maxwell system, where the coefficients of the solution are complex numbers.

5 Conclusions

We presented a general methodology for finding polynomial solutions to various linear, constant coefficient PDEs, in the presence of a polynomial source term. While for Helmholtz-like problems the proposed technique, based on a formal Neumann series of the differential operator, has been successfully employed in the past (e.g. [5]), the approach we employ in the case that no zeroth order term appears (i.e. Laplace-like problems) is seemingly novel. Furthermore, to the best of our knowledge, we propose the first general method for obtaining polynomial solutions of equations incorporating an arbitrary polynomial source term and a divergence constraint, such as the Stokes and Maxwell systems. We expect that the presented methods and the accompanying Julia library will be applicable to other PDE models not treated here, such as the poroelasticity system, and prove useful to others for developing PDE solution methods that require particular polynomial solutions.

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