



HAL
open science

Spectral correctness of the discontinuous Galerkin approximation of the first-order form of Maxwell's equations with discontinuous coefficients

Alexandre Ern, Jean-Luc Guermond

► **To cite this version:**

Alexandre Ern, Jean-Luc Guermond. Spectral correctness of the discontinuous Galerkin approximation of the first-order form of Maxwell's equations with discontinuous coefficients. 2023. hal-04145808v1

HAL Id: hal-04145808

<https://hal.science/hal-04145808v1>

Preprint submitted on 29 Jun 2023 (v1), last revised 9 Feb 2024 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

Spectral correctness of the discontinuous Galerkin approximation of the first-order form of Maxwell's equations with discontinuous coefficients*

Alexandre Ern[†] Jean-Luc Guermond[‡]

Draft version June 29, 2023

Abstract

The paper analyzes the discontinuous Galerkin approximation of Maxwell's equations written in first-order form and with non-homogeneous magnetic permeability and electric permittivity. Although the Sobolev smoothness index of the solution may be smaller than $\frac{1}{2}$, it is shown that the approximation is spectrally correct. The convergence proof is based on a duality argument. One essential idea is that the smoothness index of the dual solution is always larger than $\frac{1}{2}$ irrespective of the regularity of the material properties. Discrete involutions also play a key role in the analysis.

Keywords. Curl-curl problem, duality argument, involution, spectral approximation, finite elements, Maxwell's equations

MSC. 65M60, 65M12, 65N30, 35L02, 35L05, 35Q61

1 Introduction

Our objective is to approximate the Maxwell eigenvalue problem written in first-order form and posed on a bounded domain $D \subset \mathbb{R}^3$ (open, connected, Lipschitz subset of \mathbb{R}^d). The approximation is done with the discontinuous Galerkin (dG) method. For simplicity, we assume that D is a Lipschitz polyhedron, so that it can be covered exactly by a sequence of affine simplicial meshes. We address the case of a material with discontinuous properties. More precisely, we assume that the magnetic permeability, μ , and the electric permittivity, ϵ , are piecewise smooth on a polyhedral partition of D . We also assume that the meshes are conforming with the above partition. We do not make any specific assumption on the topology of D , i.e., D may be multiply connected and its boundary, ∂D , may have more than one connected component.

This work is the second part of a research program started in [24] where it is shown that the dG approximation of the Maxwell eigenvalue problem written in first-order form and with constant properties is spectrally correct. The two key ingredients of the proof therein are that

[†]CERMICS, Ecole des Ponts, 77455 Marne-la-Vallée Cedex 2, France and INRIA Paris, 75589 Paris, France

[‡]Department of Mathematics, Texas A&M University 3368 TAMU, College Station, TX 77843, USA.

*This material is based upon work supported in part by the National Science Foundation grant DMS2110868, the Air Force Office of Scientific Research, USAF, under grant/contract number FA9550-18-1-0397, the Army Research Office, under grant number W911NF-19-1-0431, and the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contracts B640889. The support of INRIA through the International Chair program is acknowledged.

the dG approximation is shown to preserve weak forms of the involutions associated with Gauss's laws of electricity and magnetism, and that the solution to the boundary-value problem has a Sobolev smoothness index larger than $\frac{1}{2}$ when the right-hand side is square integrable. But the analysis done in [24] does not hold when the magnetic permeability and the electric permittivity are discontinuous as the Sobolev smoothness index of the solution of the boundary-value problem is lower than $\frac{1}{2}$ in this case. It is the objective of the paper to address this difficulty.

The case of discontinuous properties for Maxwell's equations written in second-order form has been investigated in Buffa and Perugia [9], and the dG approximation has been shown therein to be spectrally correct. The method has been numerically tested in Buffa et al. [10]. A stabilized continuous finite element method has been studied in Bonito et al. [7] and has been shown to be spectrally correct when the Sobolev smoothness index is smaller than $\frac{1}{2}$. Conforming methods using edge elements have been investigated in Ciarlet [17] and in [22] for solving the boundary-value problem with low regularity. Very few papers have addressed the dG approximation of Maxwell's equations in first-order form. Numerical experiments performed in Hesthaven and Warburton [26, §5] and [27, p. 513-514] have revealed that the dG approximation performs well in the time domain and the approximation of the eigenvalue problem seems to be spurious free when the magnetic permeability and the electric permittivity are discontinuous. Similar observations are made in Alvarez et al. [1, §V] and Cohen and Duruflé [18, §5]. We provide here a mathematical proof of these observations.

In the paper, we extend the results from [24] and prove that the dG approximation of Maxwell's equations is spectrally correct for every positive Sobolev regularity index. The analysis significantly differs from that done in [24] as the analysis therein strongly exploits the existence of a Sobolev smoothness index for the solution that is larger than $\frac{1}{2}$. Here again, the involution properties identified in [24] play an important role, but convergence is proved by a duality argument instead of invoking a generic inf-sup condition. The first key idea is that, irrespective of the smoothness of the magnetic permeability and the electric permittivity, the solution to the dual problem always lives in a Sobolev space with a smoothness regularity index larger than $\frac{1}{2}$. Indeed, the smoothness index of the dual solution is only controlled by the smoothness of the domain D and is therefore larger than $\frac{1}{2}$ in a Lipschitz polyhedron (see Lemma 4.3). The second key idea is that although the discrete solution does not preserve the continuous involutions, the defect is small enough to be controlled by the available regularity, no matter how small (see Lemma 4.10).

The paper is organized as follows. We introduce the problem and recall useful theoretical results in §2. The discrete setting is described in §3. The important preliminary results established in this section are the commutator estimates in Lemma 3.3 and the discrete Poincaré–Steklov inequalities in Lemma 3.4. The error analysis is performed in §4. An a priori estimate on the discrete solution is proved in Lemma 4.1 as a consequence of a key deflated inf-sup condition established in Lemma 4.2. In this condition, the discrete trial space is a proper subspace of the discrete test space, and this allows us to control a stronger norm on the discrete trial space. The duality argument is introduced in §4.3 and the main convergence result of the paper is stated in Theorem 4.9. Some standard results on Helmholtz decompositions are collected in Appendix A. We point out that the above analysis can be extended to establish the spectral correctness of the dG approximation of the grad-div problem written in first-order form; details are omitted for brevity.

2 Continuous setting

In this section, we present the functional setting to formulate the exact eigenvalue problem and the associated boundary-value problem.

2.1 Domain and model parameters

We consider a material with magnetic permeability, μ , and electric permittivity, ϵ , occupying the domain $D \subset \mathbb{R}^d$, $d = 3$, which is assumed to be an open, bounded, connected polyhedron with Lipschitz boundary. The boundary of D is denoted ∂D and its unit outward normal \mathbf{n}_D . The domain D can have a general topology; in particular, D can be multiply connected and ∂D can have several connected components. The material properties μ and ϵ can be heterogeneous, and in particular, they can take discontinuous values. To fix the ideas, we assume that there is a partition of D into a finite number of disjoint Lipschitz polyhedra such that μ and ϵ are piecewise smooth on this partition.

The magnetic permeability and electric permittivity of vacuum are denoted μ_0, ϵ_0 , respectively. To simplify the presentation, we assume throughout the paper that the ratios $\mu_0^{-1}\mu$ and $\epsilon_0^{-1}\epsilon$ are of order unity, so that these ratios can be hidden in the generic constants used in the error analysis. To be dimensionally consistent, we introduce a length scale, ℓ_D , associated with D ; it can be for instance the diameter of D . Recalling that $c := (\mu_0\epsilon_0)^{-\frac{1}{2}}$ is the speed of light, we introduce the quantity $\omega := c\ell_D^{-1}$ which scales as the reciprocal of a time scale.

2.2 Functional spaces

We use standard notation for Lebesgue and Sobolev spaces. We use boldface fonts for \mathbb{R}^d -valued fields and functional spaces composed of such fields. The space $\mathbf{L}^2(D)$ is composed of Lebesgue integrable fields that are square integrable, and its canonical inner product is denoted $(\cdot, \cdot)_{\mathbf{L}^2(D)}$. We define the Hilbert space

$$\mathbf{H}(\mathbf{curl}; D) := \{\mathbf{h} \in \mathbf{L}^2(D) \mid \nabla \times \mathbf{h} \in \mathbf{L}^2(D)\}, \quad (1)$$

equipped with the natural graph norm $\|\mathbf{h}\|_{\mathbf{H}(\mathbf{curl}; D)}^2 := \|\mathbf{h}\|_{\mathbf{L}^2(D)}^2 + \ell_D^2 \|\nabla \times \mathbf{h}\|_{\mathbf{L}^2(D)}^2$. We also consider the closed subspace

$$\mathbf{H}_0(\mathbf{curl}; D) := \{\mathbf{h} \in \mathbf{H}(\mathbf{curl}; D) \mid \gamma_{\partial D}^c(\mathbf{h}) = \mathbf{0}\}, \quad (2)$$

where $\gamma_{\partial D}^c : \mathbf{H}(\mathbf{curl}; D) \rightarrow \mathbf{H}^{-\frac{1}{2}}(\partial D)$ is the extension by density of the tangent trace operator such that $\gamma_{\partial D}^c(\mathbf{h}) = \mathbf{h}|_{\partial D} \times \mathbf{n}_D$ for every smooth field $\mathbf{h} \in \mathbf{H}^r(D)$, $r > \frac{1}{2}$, with \mathbf{n}_D the unit outward normal to D . We consider the operators $\nabla \times : \mathbf{H}(\mathbf{curl}; D) \ni \mathbf{e} \mapsto \nabla \times \mathbf{e} \in \mathbf{L}^2(D)$ and $\nabla_0 \times : \mathbf{H}_0(\mathbf{curl}; D) \ni \mathbf{h} \mapsto \nabla \times \mathbf{h} \in \mathbf{L}^2(D)$. These operators are adjoint to each other since we have

$$(\nabla_0 \times \mathbf{h}, \mathbf{e})_{\mathbf{L}^2(D)} = (\mathbf{h}, \nabla \times \mathbf{e})_{\mathbf{L}^2(D)}, \quad \forall (\mathbf{h}, \mathbf{e}) \in \mathbf{H}_0(\mathbf{curl}; D) \times \mathbf{H}(\mathbf{curl}; D). \quad (3)$$

We are going to make use of the following closed subspaces:

$$\mathbf{H}(\mathbf{curl} = \mathbf{0}; D) := \{\mathbf{e} \in \mathbf{H}(\mathbf{curl}; D) \mid \nabla \times \mathbf{e} = \mathbf{0}\}, \quad (4a)$$

$$\mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D) := \{\mathbf{h} \in \mathbf{H}_0(\mathbf{curl}; D) \mid \nabla_0 \times \mathbf{h} = \mathbf{0}\}, \quad (4b)$$

together with the corresponding \mathbf{L}^2 -orthogonal projections

$$\mathbf{\Pi}^c : \mathbf{L}^2(D) \rightarrow \mathbf{H}(\mathbf{curl} = \mathbf{0}; D), \quad (5a)$$

$$\mathbf{\Pi}_0^c : \mathbf{L}^2(D) \rightarrow \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D). \quad (5b)$$

This leads to the following closed subspaces:

$$\mathbf{X}_0^c := \mathbf{H}_0(\mathbf{curl}; D) \cap \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp, \quad (6a)$$

$$\mathbf{X}^c := \mathbf{H}(\mathbf{curl}; D) \cap \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp, \quad (6b)$$

$$\mathbf{X}_{\mu,0}^c := \{\mathbf{h} \in \mathbf{H}_0(\mathbf{curl}; D) \mid \mu\mathbf{h} \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp\}, \quad (6c)$$

$$\mathbf{X}_\epsilon^c := \{\mathbf{e} \in \mathbf{H}(\mathbf{curl}; D) \mid \epsilon\mathbf{e} \in \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp\}, \quad (6d)$$

where the symbol $^\perp$ denotes the orthogonality in $\mathbf{L}^2(D)$. Using the above \mathbf{L}^2 -orthogonal projections, we can write

$$\mathbf{X}_0^c := \{\boldsymbol{\eta} \in \mathbf{H}_0(\mathbf{curl}; D) \mid \boldsymbol{\Pi}_0^c(\boldsymbol{\eta}) = \mathbf{0}\}, \quad (7a)$$

$$\mathbf{X}^c := \{\boldsymbol{\epsilon} \in \mathbf{H}(\mathbf{curl}; D) \mid \boldsymbol{\Pi}^c(\boldsymbol{\epsilon}) = \mathbf{0}\}, \quad (7b)$$

$$\mathbf{X}_{\mu,0}^c := \{\mathbf{h} \in \mathbf{H}_0(\mathbf{curl}; D) \mid \boldsymbol{\Pi}_0^c(\mu\mathbf{h}) = \mathbf{0}\}, \quad (7c)$$

$$\mathbf{X}_\epsilon^c := \{\mathbf{e} \in \mathbf{H}(\mathbf{curl}; D) \mid \boldsymbol{\Pi}^c(\epsilon\mathbf{e}) = \mathbf{0}\}. \quad (7d)$$

Lemma 2.1 (Isomorphisms). *The following operators are isomorphisms:*

$$\nabla \times : \mathbf{X}^c \rightarrow \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp, \quad \nabla_0 \times : \mathbf{X}_0^c \rightarrow \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp, \quad (8a)$$

$$\nabla \times : \mathbf{X}_\epsilon^c \rightarrow \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp, \quad \nabla_0 \times : \mathbf{X}_{\mu,0}^c \rightarrow \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp. \quad (8b)$$

Proof. See Appendix. \square

Remark 2.2 (Topology of D). *Let $\mathbf{H}(\operatorname{div}; D) := \{\mathbf{v} \in \mathbf{L}^2(D) \mid \nabla \cdot \mathbf{v} \in L^2(D)\}$ and $\mathbf{H}_0(\operatorname{div}; D) := \{\mathbf{v} \in \mathbf{H}(\operatorname{div}; D) \mid \gamma_{\partial D}^d(\mathbf{v}) = 0\}$, where $\gamma_{\partial D}^d : \mathbf{H}(\operatorname{div}; D) \rightarrow H^{-\frac{1}{2}}(\partial D)$ is the extension by density of the normal trace operator such that $\gamma_{\partial D}^d(\mathbf{v}) = \mathbf{v}|_{\partial D} \cdot \mathbf{n}_D$ for every smooth field $\mathbf{v} \in \mathbf{H}^r(D)$, $r > \frac{1}{2}$. Consider the subspaces*

$$\begin{aligned} \mathbf{H}(\operatorname{div} = 0; D) &:= \{\mathbf{v} \in \mathbf{H}(\operatorname{div}; D) \mid \nabla \cdot \mathbf{v} = 0\}, \\ \mathbf{H}_0(\operatorname{div} = 0; D) &:= \{\mathbf{v} \in \mathbf{H}_0(\operatorname{div}; D) \mid \nabla_0 \cdot \mathbf{v} = 0\}. \end{aligned}$$

Then, the identities (61) show that $\mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp = \mathbf{H}(\operatorname{div} = 0; D)$ if ∂D is connected, and $\mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp = \mathbf{H}_0(\operatorname{div} = 0; D)$ if D is simply connected.

2.3 Preliminary results

The $\nabla \times$ and $\nabla_0 \times$ operators can be extended by density to $\nabla \times : \mathbf{L}^2(D) \rightarrow (\mathbf{X}_0^c)'$ and $\nabla_0 \times : \mathbf{L}^2(D) \rightarrow (\mathbf{X}^c)'$. The following result is a straightforward consequence of Lemma 2.1.

Corollary 2.3 (Weak Poincaré–Steklov (in)equalities). *The following holds:*

$$\|\mathbf{e}\|_{\mathbf{L}^2(D)} = \ell_D \|\nabla \times \mathbf{e}\|_{(\mathbf{X}_0^c)'}, \quad \forall \mathbf{e} \in \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp, \quad (9a)$$

$$\|\mathbf{h}\|_{\mathbf{L}^2(D)} = \ell_D \|\nabla_0 \times \mathbf{h}\|_{(\mathbf{X}^c)'}, \quad \forall \mathbf{h} \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp, \quad (9b)$$

with $\|\nabla \times \mathbf{e}\|_{(\mathbf{X}_0^c)'} := \sup_{\boldsymbol{\eta} \in \mathbf{X}_0^c} \frac{|(\mathbf{e}, \nabla_0 \times \boldsymbol{\eta})_{\mathbf{L}^2(D)}|}{\ell_D \|\nabla_0 \times \boldsymbol{\eta}\|_{\mathbf{L}^2(D)}}$, $\|\nabla_0 \times \mathbf{h}\|_{(\mathbf{X}^c)'} := \sup_{\boldsymbol{\epsilon} \in \mathbf{X}^c} \frac{|(\mathbf{h}, \nabla \times \boldsymbol{\epsilon})_{\mathbf{L}^2(D)}|}{\ell_D \|\nabla \times \boldsymbol{\epsilon}\|_{\mathbf{L}^2(D)}}$.

The following result is a consequence of the elliptic regularity theory and is proved in Bonito et al. [6], Jochmann [29]. It generalizes a result due to Costabel [19, Thm. 2] (see also Birman and Solomyak [4, Thm. 3.1]) for either constant or smooth properties, and in this case the smoothness

index is $s = \frac{1}{2}$ in a Lipschitz domain. One has $s \in (\frac{1}{2}, 1]$ if D is a Lipschitz polyhedron (see Amrouche et al. [2, Prop. 3.7]) and $s = 1$ if D is convex (see [2, Thm. 2.17]). The reader is also referred to Ciarlet [16, Thm. 16] for particular situations with heterogeneous properties for which smoothness is established with $s > \frac{1}{2}$. In what follows, the symbol C_D denotes a generic positive constant whose value can change at each occurrence as long as it only depends on D and, whenever relevant, the material properties.

Lemma 2.4 (Regularity shift). *The following holds:*

(i) *Constant properties: There is $s' \in (\frac{1}{2}, 1]$ such that, for all $(\boldsymbol{\eta}, \boldsymbol{\varepsilon}) \in \mathbf{X}_0^c \times \mathbf{X}^c$,*

$$|\boldsymbol{\eta}|_{\mathbf{H}^{s'}(D)} \leq C_D \ell_D^{1-s'} \|\nabla_0 \times \boldsymbol{\eta}\|_{\mathbf{L}^2(D)}, \quad |\boldsymbol{\varepsilon}|_{\mathbf{H}^{s'}(D)} \leq C_D \ell_D^{1-s'} \|\nabla \times \boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)}. \quad (10)$$

(ii) *Heterogeneous properties: There is $s \in (0, \frac{1}{2}]$ such that, for all $(\mathbf{h}, \mathbf{e}) \in \mathbf{X}_{\mu,0}^c \times \mathbf{X}_\varepsilon^c$,*

$$|\mathbf{h}|_{\mathbf{H}^s(D)} \leq C_D \ell_D^{1-s} \|\nabla_0 \times \mathbf{h}\|_{\mathbf{L}^2(D)}, \quad |\mathbf{e}|_{\mathbf{H}^s(D)} \leq C_D \ell_D^{1-s} \|\nabla \times \mathbf{e}\|_{\mathbf{L}^2(D)}. \quad (11)$$

2.4 Boundary-value and eigenvalue problems

We define the space $L^c := \mathbf{L}^2(D) \times \mathbf{L}^2(D)$ which we equip with the norm

$$\|(\mathbf{f}, \mathbf{g})\|_{L^c} := \left\{ \|\mu^{\frac{1}{2}} \mathbf{f}\|_{\mathbf{L}^2(D)}^2 + \|\varepsilon^{\frac{1}{2}} \mathbf{g}\|_{\mathbf{L}^2(D)}^2 \right\}^{\frac{1}{2}}, \quad \forall (\mathbf{f}, \mathbf{g}) \in L^c. \quad (12)$$

We define the operator $T : L^c \rightarrow L^c$ such that, for all $(\mathbf{f}, \mathbf{g}) \in L^c$, the pair $(\mathbf{H}, \mathbf{E}) := T(\mathbf{f}, \mathbf{g}) \in \mathbf{H}_0(\mathbf{curl}; D) \times \mathbf{H}(\mathbf{curl}; D)$ solves the following boundary-value problem:

$$\omega \boldsymbol{\Pi}_0^c(\mu \mathbf{H}) - \nabla \times \mathbf{E} = \omega(\mathbf{I} - \boldsymbol{\Pi}_0^c)(\mu \mathbf{f}) =: \mathbf{f}', \quad (13a)$$

$$\omega \boldsymbol{\Pi}^c(\varepsilon \mathbf{E}) + \nabla_0 \times \mathbf{H} = \omega(\mathbf{I} - \boldsymbol{\Pi}^c)(\varepsilon \mathbf{g}) =: \mathbf{g}'. \quad (13b)$$

We notice that $\boldsymbol{\Pi}_0^c(\mathbf{f}') = \boldsymbol{\Pi}^c(\mathbf{g}') = \mathbf{0}$. Moreover, (3) readily shows that

$$\boldsymbol{\Pi}_0^c(\nabla \times \mathbf{E}) = \boldsymbol{\Pi}^c(\nabla_0 \times \mathbf{H}) = \mathbf{0}, \quad \forall (\mathbf{H}, \mathbf{E}) \in \mathbf{H}_0(\mathbf{curl}; D) \times \mathbf{H}(\mathbf{curl}; D). \quad (14)$$

Hence, the problem (13) can be equivalently rewritten as follows: Find $(\mathbf{H}, \mathbf{E}) \in \mathbf{H}_0(\mathbf{curl}; D) \times \mathbf{H}(\mathbf{curl}; D)$ so that

$$-\nabla \times \mathbf{E} = \mathbf{f}', \quad \boldsymbol{\Pi}^c(\varepsilon \mathbf{E}) = \mathbf{0}, \quad (15a)$$

$$\nabla_0 \times \mathbf{H} = \mathbf{g}', \quad \boldsymbol{\Pi}_0^c(\mu \mathbf{H}) = \mathbf{0}. \quad (15b)$$

The following result confirms that the definition of the operator T makes sense.

Lemma 2.5 (Well-posedness and stability). (i) *The boundary-value problem (13) has a unique solution and its solution sits in $\mathbf{X}_{\mu,0}^c \times \mathbf{X}_\varepsilon^c$.* (ii) *The solution (\mathbf{H}, \mathbf{E}) satisfies the a priori estimate*

$$\ell_D (\mu_0^{\frac{1}{2}} \|\nabla_0 \times \mathbf{H}\|_{\mathbf{L}^2(D)} + \varepsilon_0^{\frac{1}{2}} \|\nabla \times \mathbf{E}\|_{\mathbf{L}^2(D)}) \leq \|(\mathbf{f}, \mathbf{g})\|_{L^c}, \quad (16a)$$

and there is $s \in (0, \frac{1}{2}]$ such that

$$|\mathbf{H}|_{\mathbf{H}^s(D)} \leq C_D \ell_D^{1-s} \|\nabla_0 \times \mathbf{H}\|_{\mathbf{L}^2(D)}, \quad |\mathbf{E}|_{\mathbf{H}^s(D)} \leq C_D \ell_D^{1-s} \|\nabla \times \mathbf{E}\|_{\mathbf{L}^2(D)}. \quad (16b)$$

(iii) *The operator $T : L^c \rightarrow L^c$ is compact.*

Proof. (i) The existence and uniqueness of a solution to (15) is a direct consequence of (8b). This proves the first assertion because the problems (13) and (15) are equivalent.

(ii) As $\|\nabla \times \mathbf{E}\|_{\mathbf{L}^2(D)} \leq \omega \mu_0^{\frac{1}{2}} \|\mu^{\frac{1}{2}} \mathbf{f}\|_{\mathbf{L}^2(D)}$ and $\|\nabla_0 \times \mathbf{H}\|_{\mathbf{L}^2(D)} \leq \omega \varepsilon_0^{\frac{1}{2}} \|\varepsilon^{\frac{1}{2}} \mathbf{g}\|_{\mathbf{L}^2(D)}$, the estimate (16a) is obtained by observing that $\omega \mu_0^{\frac{1}{2}} \varepsilon_0^{\frac{1}{2}} = \ell_D^{-1}$. Moreover, the estimates (16b) come from (11).

(iii) This is a direct consequence of Lemma 2.4(ii) and the Rellich–Kondrachov embedding theorem. \square

We say that $\lambda \in \mathbb{C}$ and $(\mathbf{H}, \mathbf{E}) \in L^c$ form an eigenpair of T if $T(\mathbf{H}, \mathbf{E}) = \lambda(\mathbf{H}, \mathbf{E})$ (notice that the eigenvalue λ is nondimensional). Whenever $\lambda \neq 0$, we obtain $(\mathbf{H}, \mathbf{E}) \in \mathbf{X}_{\mu,0}^c \times \mathbf{X}_\varepsilon^c$ and

$$-\nabla \times \mathbf{E} = \frac{\omega}{\lambda} \mu \mathbf{H}, \quad \nabla_0 \times \mathbf{H} = \frac{\omega}{\lambda} \varepsilon \mathbf{E}. \quad (17)$$

The goal of the paper is to construct a discontinuous Galerkin approximation of T that is spectrally correct. This is done by proving that the discontinuous Galerkin approximation is strongly convergent in the operator norm induced by the norm (12).

Remark 2.6 (Involutions). *We refer to the orthogonality properties $\mathbf{\Pi}^c(\varepsilon \mathbf{E}) = \mathbf{0}$ and $\mathbf{\Pi}_0^c(\mu \mathbf{H}) = \mathbf{0}$ as involutions since these properties are involutive for the time-dependent Maxwell equations. These conditions imply Gauss’s laws, i.e., $\nabla \cdot (\varepsilon \mathbf{E}) = 0$ (in the absence of free charges) and $\nabla \cdot (\mu \mathbf{H}) = 0$, but they are not equivalent to Gauss’s laws when the topology of D is nontrivial (see Remark 2.2). We emphasize that the above involutions are essential to prove Lemma 2.1 (see, e.g., Hiptmair [28, Cor. 4.4] for a related statement formulated as a Poincaré–Steklov inequality).*

3 Discrete setting

In this section, we introduce the discrete setting.

3.1 Mesh, polynomial spaces, and orthogonal projections

Let $(\mathcal{T}_h)_{h \in \mathcal{H}}$ be a shape-regular family of affine simplicial meshes such that each mesh covers D exactly and is compatible with the partition of D associated with the discontinuities of the material properties. For simplicity, we assume in the paper that the meshes are quasi-uniform. A generic mesh cell is denoted K , its diameter h_K and its outward unit normal \mathbf{n}_K . We define \tilde{h} as the piecewise constant function on \mathcal{T}_h such that $\tilde{h}|_K = h_K$ for all $K \in \mathcal{T}_h$; we set $h := \|\tilde{h}\|_{L^\infty(D)}$. The set of mesh faces, \mathcal{F}_h , is split into the subset of mesh interfaces, say \mathcal{F}_h° , and the subset of mesh boundary faces, say \mathcal{F}_h^∂ . Each interface is shared by two distinct mesh cells which we denote K_l, K_r . Each boundary face is shared by one mesh cell, K_l , and the boundary, ∂D . For every mesh face $F \in \mathcal{F}_h$, h_F denotes the diameter of F . Every mesh interface $F \in \mathcal{F}_h^\circ$ is oriented by the unit normal, \mathbf{n}_F , pointing from K_l to K_r . Every boundary face $F \in \mathcal{F}_h^\partial$ is oriented by the unit normal $\mathbf{n}_F := \mathbf{n}_D$.

In what follows, for positive real numbers A, B , we abbreviate as $A \lesssim B$ the inequality $A \leq CB$ where C is a generic constant whose value can change at each occurrence as long as it is independent of $h \in \mathcal{H}$, the parameters $\mu_0, \varepsilon_0, \ell_D$ and any fields involved in the inequality. For simplicity, C can depend on the ratios $\operatorname{ess\,inf} \frac{\mu}{\mu_0}, \operatorname{ess\,sup} \frac{\mu}{\mu_0}, \operatorname{ess\,inf} \frac{\varepsilon}{\varepsilon_0}$, and $\operatorname{ess\,sup} \frac{\varepsilon}{\varepsilon_0}$.

Let $k \geq 0$ be the polynomial degree. Let $\mathbb{P}_{k,d}$ be the space composed of d -variate polynomials of total degree at most k and set $\mathbf{P}_{k,d} := [\mathbb{P}_{k,d}]^d$. Consider the vector-valued broken polynomial spaces

$$\mathbf{P}_k^b(\mathcal{T}_h) := \{\mathbf{w}_h \in \mathbf{L}^\infty(D) \mid \mathbf{w}_h|_K \in \mathbf{P}_{k,d}, \forall K \in \mathcal{T}_h\}. \quad (18)$$

We shall also consider the $\mathbf{H}(\mathbf{curl}; D)$ -conforming finite element space, $\mathbf{P}_k^c(\mathcal{T}_h)$, composed of piecewise Nédélec polynomials of order $k \geq 0$, and its subspace, $\mathbf{P}_{k0}^c(\mathcal{T}_h)$, composed of discrete fields in $\mathbf{P}_k^c(\mathcal{T}_h)$ whose tangential trace vanishes on ∂D . We set

$$\mathbf{P}_k^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) := \{\mathbf{e}_h \in \mathbf{P}_k^c(\mathcal{T}_h) \mid \nabla \times \mathbf{e}_h = \mathbf{0}\}, \quad (19a)$$

$$\mathbf{P}_{k0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) := \{\mathbf{h}_h \in \mathbf{P}_{k0}^c(\mathcal{T}_h) \mid \nabla_0 \times \mathbf{h}_h = \mathbf{0}\}. \quad (19b)$$

Notice that $\mathbf{P}_k^c(\mathcal{T}_h)$ is not a subspace of $\mathbf{P}_k^b(\mathcal{T}_h)$ (actually $\mathbf{P}_k^c(\mathcal{T}_h) \subsetneq \mathbf{P}_{k+1}^b(\mathcal{T}_h)$), but we are going to make use of the following well-known result (see, e.g., [23, Chap. 15&18]).

Lemma 3.1 (Curl-free subspaces). *The following identities hold:*

$$\mathbf{P}_k^b(\mathcal{T}_h) \cap \mathbf{H}(\mathbf{curl} = \mathbf{0}; D) = \mathbf{P}_k^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h), \quad (20a)$$

$$\mathbf{P}_k^b(\mathcal{T}_h) \cap \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D) = \mathbf{P}_{k0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h). \quad (20b)$$

We define the \mathbf{L}^2 -orthogonal projection onto the broken polynomial space $\mathbf{P}_k^b(\mathcal{T}_h)$,

$$\mathbf{\Pi}_h^b : \mathbf{L}^2(D) \rightarrow \mathbf{P}_k^b(\mathcal{T}_h), \quad (21)$$

and the \mathbf{L}^2 -orthogonal projections onto the curl-free subspaces defined in (19),

$$\mathbf{\Pi}_h^c : \mathbf{L}^2(D) \rightarrow \mathbf{P}_k^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h), \quad (22a)$$

$$\mathbf{\Pi}_{h0}^c : \mathbf{L}^2(D) \rightarrow \mathbf{P}_{k0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h). \quad (22b)$$

The above discrete projection operators play a key role to establish that the involutions satisfied by the dG approximation are strong enough to guarantee the strong convergence of the approximation in the operator norm induced by that of L^c .

Lemma 3.2 (Discrete projections). *We have*

$$\mathbf{\Pi}_{h0}^c \circ \mathbf{\Pi}_0^c = \mathbf{\Pi}_{h0}^c, \quad \mathbf{\Pi}_h^c \circ \mathbf{\Pi}^c = \mathbf{\Pi}_h^c. \quad (23)$$

The above lemma, whose proof is omitted for brevity, allows us to assert that if $\mathbf{\Pi}^c(\mathbf{e}) = \mathbf{0}$, then $\mathbf{\Pi}_h^c(\mathbf{e}) = \mathbf{0}$; similarly, if $\mathbf{\Pi}_0^c(\mathbf{h}) = \mathbf{0}$, then $\mathbf{\Pi}_{h0}^c(\mathbf{h}) = \mathbf{0}$.

3.2 Jumps, averages, and discrete curl operators

For all $K \in \mathcal{T}_h$, all $F \in \mathcal{F}_h$ with $F \subset \partial K$, and all $\mathbf{w}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$, we define the local trace operators such that $\gamma_{K,F}^g(\mathbf{w}_h)(\mathbf{x}) := \mathbf{w}_h|_K(\mathbf{x})$, $\gamma_{K,F}^c(\mathbf{w}_h)(\mathbf{x}) := \mathbf{w}_h|_K(\mathbf{x}) \times \mathbf{n}_F$ for a.e. $\mathbf{x} \in F$. Then, for all $F \in \mathcal{F}_h^\circ$ and $\mathbf{x} \in \{g, c\}$, we define the jump and average operators such that

$$[[\mathbf{w}_h]]_F^{\mathbf{x}} := \gamma_{K_l, F}^{\mathbf{x}}(\mathbf{w}_h) - \gamma_{K_r, F}^{\mathbf{x}}(\mathbf{w}_h), \quad \{\{\mathbf{w}_h\}\}_F^{\mathbf{x}} := \frac{1}{2}(\gamma_{K_l, F}^{\mathbf{x}}(\mathbf{w}_h) + \gamma_{K_r, F}^{\mathbf{x}}(\mathbf{w}_h)). \quad (24a)$$

To allow for more compact expressions, we also set $[[\mathbf{w}_h]]_F^{\mathbf{x}} := \{\{\mathbf{w}_h\}\}_F^{\mathbf{x}} := \gamma_{K_l, F}^{\mathbf{x}}(\mathbf{w}_h)$ for all $F \in \mathcal{F}_h^\partial$. Notice that $[[\mathbf{h}_h]]_F^c = \mathbf{0}$ for all $\mathbf{h}_h \in \mathbf{P}_{k0}^c(\mathcal{T}_h)$ and all $F \in \mathcal{F}_h$, whereas $[[\mathbf{e}_h]]_F^c = \mathbf{0}$ for all $\mathbf{e}_h \in \mathbf{P}_k^c(\mathcal{T}_h)$ and all $F \in \mathcal{F}_h^\circ$. We define the jump sesquilinear forms such that for all $\mathbf{H}_h, \mathbf{h}_h, \mathbf{E}_h, \mathbf{e}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$,

$$s_h^c(\mathbf{H}_h, \mathbf{h}_h) := \sum_{F \in \mathcal{F}_h} ([[\mathbf{H}_h]]_F^c, [[\mathbf{h}_h]]_F^c)_{\mathbf{L}^2(F)}, \quad s_h^{c,\circ}(\mathbf{E}_h, \mathbf{e}_h) := \sum_{F \in \mathcal{F}_h^\circ} ([[\mathbf{E}_h]]_F^c, [[\mathbf{e}_h]]_F^c)_{\mathbf{L}^2(F)}.$$

These sesquilinear forms induce the following seminorms which we henceforth call jump seminorms:

$$|\mathbf{h}_h|_J^c := s_h^c(\mathbf{h}_h, \mathbf{h}_h)^{\frac{1}{2}}, \quad |\mathbf{e}_h|_J^{c,\circ} := s_h^{c,\circ}(\mathbf{e}_h, \mathbf{e}_h)^{\frac{1}{2}}. \quad (25)$$

The discrete curl operators

$$\mathbf{C}_{h0} : \mathbf{P}_k^b(\mathcal{T}_h) \rightarrow \mathbf{P}_k^b(\mathcal{T}_h), \quad \mathbf{C}_h : \mathbf{P}_k^b(\mathcal{T}_h) \rightarrow \mathbf{P}_k^b(\mathcal{T}_h) \quad (26)$$

are defined as follows: For all $(\mathbf{h}_h, \mathbf{e}_h) \in \mathbf{P}_k^b(\mathcal{T}_h) \times \mathbf{P}_k^b(\mathcal{T}_h)$, $\mathbf{C}_{h0}(\mathbf{h}_h)$ and $\mathbf{C}_h(\mathbf{e}_h)$ are the unique members of $\mathbf{P}_k^b(\mathcal{T}_h)$ such that the following identities hold true for all $\mathbf{e}'_h, \mathbf{h}'_h \in \mathbf{P}_k^b(\mathcal{T}_h)$:

$$(\mathbf{C}_{h0}(\mathbf{h}_h), \mathbf{e}'_h)_{L^2(D)} := (\nabla_h \times \mathbf{h}_h, \mathbf{e}'_h)_{L^2(D)} + \sum_{F \in \mathcal{F}_h} ([\mathbf{h}_h]_F^c, \llbracket \mathbf{e}'_h \rrbracket_F^g)_{L^2(F)}, \quad (27a)$$

$$(\mathbf{C}_h(\mathbf{e}_h), \mathbf{h}'_h)_{L^2(D)} := (\nabla_h \times \mathbf{e}_h, \mathbf{h}'_h)_{L^2(D)} + \sum_{F \in \mathcal{F}_h^\circ} ([\mathbf{e}_h]_F^c, \llbracket \mathbf{h}'_h \rrbracket_F^g)_{L^2(F)}, \quad (27b)$$

where $\nabla_h \times$ denotes the broken curl operator (evaluated cellwise). The following integration by parts formula holds (compare with (3)):

$$(\mathbf{C}_{h0}(\mathbf{h}_h), \mathbf{e}_h)_{L^2(D)} = (\mathbf{h}_h, \mathbf{C}_h(\mathbf{e}_h))_{L^2(D)}, \quad \forall (\mathbf{h}_h, \mathbf{e}_h) \in \mathbf{P}_k^b(\mathcal{T}_h) \times \mathbf{P}_k^b(\mathcal{T}_h). \quad (28a)$$

Since $\mathbf{P}_k^b(\mathcal{T}_h)$ is not a subset of $\mathbf{H}(\mathbf{curl}; D)$, it is useful to define the spaces

$$\mathbf{V}_\#^H := \mathbf{H}_0(\mathbf{curl}; D) + \mathbf{P}_k^b(\mathcal{T}_h), \quad \mathbf{V}_\#^E := \mathbf{H}(\mathbf{curl}; D) + \mathbf{P}_k^b(\mathcal{T}_h). \quad (29)$$

If the pair $(\mathbf{H}, \mathbf{E}) \in \mathbf{H}_0(\mathbf{curl}; D) \times \mathbf{H}(\mathbf{curl}; D)$ solves the continuous problem (13) and if the pair $(\mathbf{H}_h, \mathbf{E}_h) \in \mathbf{P}_k^b(\mathcal{T}_h) \times \mathbf{P}_k^b(\mathcal{T}_h)$ solves the discrete problem resulting from the dG approximation (to be defined more precisely in Section 4), the errors satisfy

$$\mathbf{H} - \mathbf{H}_h \in \mathbf{V}_\#^H, \quad \mathbf{E} - \mathbf{E}_h \in \mathbf{V}_\#^E. \quad (30)$$

It is then convenient to extend the jump sesquilinear forms and the discrete curl operators defined above to the spaces $\mathbf{V}_\#^H$ and $\mathbf{V}_\#^E$. To this purpose, we notice that although the sums in (29) are not direct, every field \mathbf{h}_h in $\mathbf{H}_0(\mathbf{curl}; D) \cap \mathbf{P}_k^b(\mathcal{T}_h)$ satisfies $[\mathbf{h}_h]_F^c = \mathbf{0}$ for all $F \in \mathcal{F}_h$, and $\mathbf{C}_{h0}(\mathbf{h}_h) = \nabla_0 \times \mathbf{h}_h$ because $\nabla_0 \times (\mathbf{H}_0(\mathbf{curl}; D) \cap \mathbf{P}_k^b(\mathcal{T}_h)) \subset \mathbf{P}_k^b(\mathcal{T}_h)$. Similarly, every field \mathbf{e}_h in $\mathbf{H}(\mathbf{curl}; D) \cap \mathbf{P}_k^b(\mathcal{T}_h)$ satisfies $[\mathbf{e}_h]_F^c = \mathbf{0}$ for all $F \in \mathcal{F}_h^\circ$, and $\mathbf{C}_h(\mathbf{e}_h) = \nabla \times \mathbf{e}_h$. It is therefore legitimate to set, for all $\mathbf{h} = \tilde{\mathbf{h}} + \mathbf{h}_h \in \mathbf{V}_\#^H$ with $\tilde{\mathbf{h}} \in \mathbf{H}_0(\mathbf{curl}; D)$ and $\mathbf{h}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$, and for all $\mathbf{e} = \tilde{\mathbf{e}} + \mathbf{e}_h \in \mathbf{V}_\#^E$ with $\tilde{\mathbf{e}} \in \mathbf{H}(\mathbf{curl}; D)$ and $\mathbf{e}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$,

$$\mathbf{C}_{h0}(\mathbf{h}) := \nabla_0 \times \tilde{\mathbf{h}} + \mathbf{C}_{h0}(\mathbf{h}_h), \quad [\mathbf{h}]_F^c := [\mathbf{h}_h]_F^c, \quad \forall F \in \mathcal{F}_h, \quad (31a)$$

$$\mathbf{C}_h(\mathbf{e}) := \nabla \times \tilde{\mathbf{e}} + \mathbf{C}_h(\mathbf{e}_h), \quad [\mathbf{e}]_F^c := [\mathbf{e}_h]_F^c, \quad \forall F \in \mathcal{F}_h^\circ. \quad (31b)$$

For simplicity, we use the same symbols for the extended operators; in particular, we now have $\mathbf{C}_{h0} : \mathbf{V}_\#^H \rightarrow L^2(D)$ and $\mathbf{C}_h : \mathbf{V}_\#^E \rightarrow L^2(D)$. The following commutators are useful to estimate the defects introduced by the above extensions:

$$\delta_h(\mathbf{h}_h, \mathbf{e}) := (\mathbf{h}_h, \nabla \times \mathbf{e})_{L^2(D)} - (\mathbf{C}_{h0}(\mathbf{h}_h), \mathbf{e})_{L^2(D)}, \quad (32a)$$

$$\delta_h^\circ(\mathbf{e}_h, \mathbf{h}) := (\mathbf{e}_h, \nabla_0 \times \mathbf{h})_{L^2(D)} - (\mathbf{C}_h(\mathbf{e}_h), \mathbf{h})_{L^2(D)}, \quad (32b)$$

for all $\mathbf{h}_h, \mathbf{e}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$, all $\mathbf{e} \in \mathbf{H}(\mathbf{curl}; D)$, and all $\mathbf{h} \in \mathbf{H}_0(\mathbf{curl}; D)$. The following result gives a control on the defects whenever the continuous fields are smooth enough, for instance, when they are in $\mathbf{X}_0^c \times \mathbf{X}^c$.

Lemma 3.3 (Bound on commutators). *The following inequalities hold for all $(\mathbf{h}_h, \mathbf{e}_h) \in \mathbf{P}_k^b(\mathcal{T}_h) \times \mathbf{P}_k^b(\mathcal{T}_h)$ and all $(\boldsymbol{\eta}, \boldsymbol{\varepsilon}) \in \mathbf{X}_0^c \times \mathbf{X}^c$,*

$$|\delta_h(\mathbf{h}_h, \boldsymbol{\varepsilon})| \lesssim (h/\ell_D)^{s' - \frac{1}{2}} \ell_D^{\frac{1}{2}} |\mathbf{h}_h|_J^c \|\nabla \times \boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)}, \quad (33a)$$

$$|\delta_h^\circ(\mathbf{e}_h, \boldsymbol{\eta})| \lesssim (h/\ell_D)^{s' - \frac{1}{2}} \ell_D^{\frac{1}{2}} |\mathbf{e}_h|_J^{c;\circ} \|\nabla_0 \times \boldsymbol{\eta}\|_{\mathbf{L}^2(D)}, \quad (33b)$$

where $s' \in (\frac{1}{2}, 1]$ is the regularity pickup introduced in (10).

Proof. We only prove (33a) since the proof of the other estimate is similar. Let $\mathbf{h}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$ and $\boldsymbol{\varepsilon} \in \mathbf{X}^c$. Using the definition of the commutator (32a) and integrating by parts, we obtain

$$\begin{aligned} \delta_h(\mathbf{h}_h, \boldsymbol{\varepsilon}) &= (\mathbf{h}_h, \nabla \times \boldsymbol{\varepsilon})_{\mathbf{L}^2(D)} - (\mathbf{C}_{h0}(\mathbf{h}_h), \boldsymbol{\varepsilon})_{\mathbf{L}^2(D)} \\ &= (\nabla_h \times \mathbf{h}_h, \boldsymbol{\varepsilon})_{\mathbf{L}^2(D)} + \sum_{F \in \mathcal{F}_h} ([\mathbf{h}_h]_F^c, \{\{\boldsymbol{\varepsilon}\}_F^g\})_{\mathbf{L}^2(F)} - (\mathbf{C}_{h0}(\mathbf{h}_h), \boldsymbol{\varepsilon})_{\mathbf{L}^2(D)}. \end{aligned}$$

Notice that the summation over the mesh faces is meaningful since the regularity estimate (10) implies that $\boldsymbol{\varepsilon} \in \mathbf{H}^{s'}(D)$ with $s' > \frac{1}{2}$. Moreover, we have $(\mathbf{C}_{h0}(\mathbf{h}_h), \boldsymbol{\varepsilon})_{\mathbf{L}^2(D)} = (\mathbf{C}_{h0}(\mathbf{h}_h), \boldsymbol{\Pi}_h^b(\boldsymbol{\varepsilon}))_{\mathbf{L}^2(D)}$. Thus, using the definition (27a) of the operator \mathbf{C}_{h0} , we infer that

$$\begin{aligned} \delta_h(\mathbf{h}_h, \boldsymbol{\varepsilon}) &= (\nabla_h \times \mathbf{h}_h, \boldsymbol{\varepsilon})_{\mathbf{L}^2(D)} + \sum_{F \in \mathcal{F}_h} ([\mathbf{h}_h]_F^c, \{\{\boldsymbol{\varepsilon}\}_F^g\})_{\mathbf{L}^2(F)} - (\mathbf{C}_{h0}(\mathbf{h}_h), \boldsymbol{\Pi}_h^b(\boldsymbol{\varepsilon}))_{\mathbf{L}^2(D)} \\ &= (\nabla_h \times \mathbf{h}_h, \boldsymbol{\varepsilon} - \boldsymbol{\Pi}_h^b(\boldsymbol{\varepsilon}))_{\mathbf{L}^2(D)} + \sum_{F \in \mathcal{F}_h} ([\mathbf{h}_h]_F^c, \{\{\boldsymbol{\varepsilon} - \boldsymbol{\Pi}_h^b(\boldsymbol{\varepsilon})\}_F^g\})_{\mathbf{L}^2(F)} \\ &= \sum_{F \in \mathcal{F}_h} ([\mathbf{h}_h]_F^c, \{\{\boldsymbol{\varepsilon} - \boldsymbol{\Pi}_h^b(\boldsymbol{\varepsilon})\}_F^g\})_{\mathbf{L}^2(F)}. \end{aligned}$$

Hence, $|\delta_h(\mathbf{h}_h, \boldsymbol{\varepsilon})| \lesssim h^{s' - \frac{1}{2}} |\mathbf{h}_h|_J^c |\boldsymbol{\varepsilon}|_{\mathbf{H}^{s'}(D)}$. The assertion follows by invoking the regularity estimate (10). \square

3.3 Quasi-interpolation operators

In the paper, we invoke two types of quasi-interpolation operators. The first type of operators, considered in the proof of Lemma 3.4 and Lemma 4.10, are the averaging operators $\mathcal{I}_{h0}^{c,av} : \mathbf{P}_k^b(\mathcal{T}_h) \rightarrow \mathbf{P}_{k0}^c(\mathcal{T}_h)$ and $\mathcal{I}_h^{c,av} : \mathbf{P}_k^b(\mathcal{T}_h) \rightarrow \mathbf{P}_k^c(\mathcal{T}_h)$ analyzed in [21, §6]. The key approximation property of these operators is that, for all $\mathbf{v}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$,

$$\|\mathbf{v}_h - \mathcal{I}_{h0}^{c,av}(\mathbf{v}_h)\|_{\mathbf{L}^2(D)} \lesssim h^{\frac{1}{2}} |\mathbf{v}_h|_J^c, \quad \|\mathbf{v}_h - \mathcal{I}_h^{c,av}(\mathbf{v}_h)\|_{\mathbf{L}^2(D)} \lesssim h^{\frac{1}{2}} |\mathbf{v}_h|_J^{c;\circ}. \quad (34)$$

The second type of operators, invoked in the proof of Lemma 3.4, Lemma 4.2, and Lemma 4.10, are the commuting approximation operators $\mathcal{J}_{h0}^c : \mathbf{L}^2(D) \rightarrow \mathbf{P}_{k0}^c(\mathcal{T}_h)$ $\mathcal{J}_h^c : \mathbf{L}^2(D) \rightarrow \mathbf{P}_k^c(\mathcal{T}_h)$ devised in [23, §23.3] following the seminal ideas in Arnold et al. [3], Christiansen [14], Christiansen and Winther [15], Schöberl [31]. The required properties of these operators are the following: (i) they are projections; (ii) they map curl-free fields to curl-free fields; (iii) they are stable in \mathbf{L}^2 and enjoy optimal approximation properties in the \mathbf{L}^2 -norm for fields in $\mathbf{H}^r(D)$, $r \in (0, 1]$. As these operators are not locally defined, their approximation properties involve the global mesh-size. This is not an issue here since we have assumed the meshes to be quasi-uniform. We notice that an interesting extension of our work is to lift this restriction by considering the operators devised in Chaumont-Frelet and Vohralík [13].

3.4 Discrete Poincaré–Steklov inequalities

We prove in this section discrete versions of the Poincaré–Steklov (in)equalities stated in Corollary 2.3. We consider the following discrete subspaces:

$$\mathbf{X}_{\mu,h_0}^c := \{\mathbf{h}_h \in \mathbf{P}_k^b(\mathcal{T}_h) \mid \mu \mathbf{h}_h \in \mathbf{P}_{k_0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp\}, \quad (35a)$$

$$\mathbf{X}_{\epsilon,h}^c := \{\mathbf{e}_h \in \mathbf{P}_k^b(\mathcal{T}_h) \mid \epsilon \mathbf{h}_h \in \mathbf{P}_k^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp\}. \quad (35b)$$

Lemma 3.4 (Discrete Poincaré–Steklov inequalities). *The following holds:*

$$\|\mathbf{h}_h\|_{\mathbf{L}^2(D)} \lesssim \ell_D \|\nabla_0 \times \mathbf{h}_h\|_{(\mathbf{X}^c)'} + h^{\frac{1}{2}} |\mathbf{h}_h|_J^c, \quad \forall \mathbf{h}_h \in \mathbf{X}_{\mu,h_0}^c, \quad (36a)$$

$$\|\mathbf{e}_h\|_{\mathbf{L}^2(D)} \lesssim \ell_D \|\nabla \times \mathbf{e}_h\|_{(\mathbf{X}_\epsilon^c)'} + h^{\frac{1}{2}} |\mathbf{e}_h|_J^{c,\circ}, \quad \forall \mathbf{e}_h \in \mathbf{X}_{\epsilon,h}^c. \quad (36b)$$

Proof. We only prove (36a) since the proof of (36b) is similar. We revisit the proof of Lemma 3.2 in [24] to account for the presence of discontinuous material properties. Let $\mathbf{h}_h \in \mathbf{X}_{\mu,h_0}^c$. We define

$$\mathbf{h}_h^c := \mathcal{I}_{h_0}^{c,\text{av}}(\mathbf{h}_h), \quad \boldsymbol{\xi} := \mathbf{h}_h^c - \boldsymbol{\Pi}_0^c(\mathbf{h}_h^c).$$

As $\boldsymbol{\xi} \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$, the weak Poincaré–Steklov (in)equality (9b) gives

$$\|\boldsymbol{\xi}\|_{\mathbf{L}^2(D)} = \ell_D \|\nabla_0 \times \boldsymbol{\xi}\|_{(\mathbf{X}^c)'} = \ell_D \|\nabla_0 \times \mathbf{h}_h^c\|_{(\mathbf{X}^c)'}. \quad (37)$$

Since $\mathbf{h}_h^c \in \mathbf{P}_{k_0}^c(\mathcal{T}_h)$, we have $\mathcal{J}_{h_0}^c(\mathbf{h}_h^c) = \mathbf{h}_h^c$, whence

$$\mathbf{h}_h^c - \mathcal{J}_{h_0}^c(\boldsymbol{\xi}) = \mathcal{J}_{h_0}^c(\mathbf{h}_h^c - \boldsymbol{\xi}) = \mathcal{J}_{h_0}^c(\boldsymbol{\Pi}_0^c(\mathbf{h}_h^c)).$$

The commuting property of $\mathcal{J}_{h_0}^c$ implies that

$$\nabla_0 \times (\mathbf{h}_h^c - \mathcal{J}_{h_0}^c(\boldsymbol{\xi})) = \nabla_0 \times (\mathcal{J}_{h_0}^c(\boldsymbol{\Pi}_0^c(\mathbf{h}_h^c))) = \mathcal{J}_{h_0}^d(\nabla_0 \times (\boldsymbol{\Pi}_0^c(\mathbf{h}_h^c))) = \mathcal{J}_{h_0}^d(\mathbf{0}) = \mathbf{0}.$$

Hence, $\mathbf{h}_h^c - \mathcal{J}_{h_0}^c(\boldsymbol{\xi}) \in \mathbf{P}_{k_0}^c(\mathcal{T}_h) \cap \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D) = \mathbf{P}_{k_0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$. As $\mathbf{h}_h \in \mathbf{X}_{\mu,h_0}^c$, we have $\mu \mathbf{h}_h \in \mathbf{P}_{k_0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp$, so that

$$\begin{aligned} \|\mu^{\frac{1}{2}} \mathbf{h}_h\|_{\mathbf{L}^2(D)}^2 &= (\mu \mathbf{h}_h, \mathbf{h}_h - \mathbf{h}_h^c)_{\mathbf{L}^2(D)} + (\mu \mathbf{h}_h, \mathbf{h}_h^c - \mathcal{J}_{h_0}^c(\boldsymbol{\xi}))_{\mathbf{L}^2(D)} + (\mu \mathbf{h}_h, \mathcal{J}_{h_0}^c(\boldsymbol{\xi}))_{\mathbf{L}^2(D)} \\ &= (\mu \mathbf{h}_h, \mathbf{h}_h - \mathbf{h}_h^c)_{\mathbf{L}^2(D)} + (\mu \mathbf{h}_h, \mathcal{J}_{h_0}^c(\boldsymbol{\xi}))_{\mathbf{L}^2(D)}. \end{aligned}$$

Invoking the Cauchy–Schwarz inequality, the \mathbf{L}^2 -stability of $\mathcal{J}_{h_0}^c$, and (37), we obtain

$$\begin{aligned} \mu_0^{-\frac{1}{2}} \|\mu^{\frac{1}{2}} \mathbf{h}_h\|_{\mathbf{L}^2(D)} &\lesssim \|\mathbf{h}_h - \mathbf{h}_h^c\|_{\mathbf{L}^2(D)} + \|\boldsymbol{\xi}\|_{\mathbf{L}^2(D)} \\ &\lesssim \|\mathbf{h}_h - \mathbf{h}_h^c\|_{\mathbf{L}^2(D)} + \ell_D \|\nabla_0 \times \mathbf{h}_h^c\|_{(\mathbf{X}^c)'}. \end{aligned}$$

Using the triangle inequality, the inequality $\ell_D \|\nabla_0 \times \boldsymbol{\phi}\|_{(\mathbf{X}^c)'} \leq \|\boldsymbol{\phi}\|_{\mathbf{L}^2(D)}$ which holds for all $\boldsymbol{\phi} \in \mathbf{L}^2(D)$, and the approximation properties of $\mathcal{I}_{h_0}^{c,\text{av}}$, we infer that

$$\begin{aligned} \mu_0^{-\frac{1}{2}} \|\mu^{\frac{1}{2}} \mathbf{h}_h\|_{\mathbf{L}^2(D)} &\lesssim \|\mathbf{h}_h - \mathbf{h}_h^c\|_{\mathbf{L}^2(D)} + \ell_D \|\nabla_0 \times \mathbf{h}_h\|_{(\mathbf{X}^c)'} \\ &\lesssim h^{\frac{1}{2}} |\mathbf{h}_h|_J^c + \ell_D \|\nabla_0 \times \mathbf{h}_h\|_{(\mathbf{X}^c)'}. \end{aligned}$$

This proves (36a). □

4 Discontinuous Galerkin approximation

This section deals with the analysis of the dG approximation of the eigenvalue problem associated with the operator $T : L^c \rightarrow L^c$ defined in (13). Recall that $L^c := \mathbf{L}^2(D) \times \mathbf{L}^2(D)$ is equipped with the norm defined in (12), i.e., $\|(\mathbf{f}, \mathbf{g})\|_{L^c}^2 := \|\mu^{\frac{1}{2}} \mathbf{f}\|_{\mathbf{L}^2(D)}^2 + \|\epsilon^{\frac{1}{2}} \mathbf{g}\|_{\mathbf{L}^2(D)}^2$.

4.1 Definitions

We define the discrete space $L_h^c := \mathbf{P}_k^b(\mathcal{T}_h) \times \mathbf{P}_k^b(\mathcal{T}_h)$. The dG sesquilinear form $a_h : L_h^c \times L_h^c \rightarrow \mathbb{C}$ is defined as follows:

$$\begin{aligned} a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{h}_h, \mathbf{e}_h)) &:= \omega(\Pi_{h0}^c(\mu \mathbf{H}_h), \mathbf{h}_h)_{\mathbf{L}^2(D)} + \omega(\Pi_h^c(\epsilon \mathbf{E}_h), \mathbf{e}_h)_{\mathbf{L}^2(D)} \\ &\quad - (\mathbf{C}_h(\mathbf{E}_h), \mathbf{h}_h)_{\mathbf{L}^2(D)} + (\mathbf{C}_{h0}(\mathbf{H}_h), \mathbf{e}_h)_{\mathbf{L}^2(D)} \\ &\quad + \kappa_H s_h^c(\mathbf{H}_h, \mathbf{h}_h) + \kappa_E s_h^{c,o}(\mathbf{E}_h, \mathbf{e}_h), \end{aligned} \quad (38)$$

with $\kappa_H := \mu_0 \omega \ell_D = (\mu_0 / \epsilon_0)^{\frac{1}{2}}$ and $\kappa_E := \epsilon_0 \omega \ell_D = (\epsilon_0 / \mu_0)^{\frac{1}{2}}$; notice that $\kappa_H \kappa_E = 1$.

We now define $T_h : L^c \rightarrow L_h^c \subset L^c$, the discrete counterpart of the operator T . For all $(\mathbf{f}, \mathbf{g}) \in L^c$, $T_h(\mathbf{f}, \mathbf{g}) := (\mathbf{H}_h, \mathbf{E}_h) \in L_h^c$ is the unique pair in L_h^c so that the following equality holds for all $(\mathbf{h}_h, \mathbf{e}_h) \in L_h^c$,

$$a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{h}_h, \mathbf{e}_h)) = \omega((\mathbf{I} - \Pi_{h0}^c)(\mu \mathbf{f}), \mathbf{h}_h)_{\mathbf{L}^2(D)} + \omega((\mathbf{I} - \Pi_h^c)(\epsilon \mathbf{g}), \mathbf{e}_h)_{\mathbf{L}^2(D)}. \quad (39)$$

The definition of T_h makes sense owing to Lemma 4.1.

The goal of this section is to prove that $\lim_{h \in \mathcal{H} \rightarrow 0} \|T - T_h\|_{\mathcal{L}(L^c; L^c)} = 0$. To this purpose, we are going to derive an a priori estimate on the discrete solution $(\mathbf{H}_h, \mathbf{E}_h)$ by establishing a deflated inf-sup condition restricted to the involution-preserving space $\mathbf{X}_{\mu, h0}^c \times \mathbf{X}_{\epsilon, h}^c$ (see (35) for the definition of these spaces). Then we prove convergence by using a duality argument à la Aubin–Nitsche. Invoking a duality argument to establish convergence in a low-regularity context is rather natural. The idea has been employed to analyze the convergence of the dG approximation to the Helmholtz problem in Chaumont-Frelet [11] and that of Maxwell's equations in the frequency domain in Chaumont-Frelet and Ern [12].

4.2 Discrete involutions, deflated inf-sup condition and a priori estimate

We equip the space L_h^c with the norm $\|\cdot\|_{L^c}$ (induced by L^c). We also introduce the mesh-dependent norm

$$\begin{aligned} \|(\mathbf{h}_h, \mathbf{e}_h)\|_{b,h} &:= \omega^{\frac{1}{2}} \|(\mathbf{h}_h, \mathbf{e}_h)\|_{L^c} \\ &\quad + \kappa_H^{\frac{1}{2}} \{ \|\tilde{h}^{\frac{1}{2}} \mathbf{C}_{h0}(\mathbf{h}_h)\|_{\mathbf{L}^2(D)} + |\mathbf{h}_h|_J^c \} + \kappa_E^{\frac{1}{2}} \{ \|\tilde{h}^{\frac{1}{2}} \mathbf{C}_h(\mathbf{e}_h)\|_{\mathbf{L}^2(D)} + |\mathbf{e}_h|_J^{c,o} \}. \end{aligned} \quad (40)$$

Lemma 4.1 (Discrete involutions and a priori estimate). *The discrete boundary-value problem (39) is well-posed, its solution $(\mathbf{H}_h, \mathbf{E}_h)$ sits in $\mathbf{X}_{\mu, h0}^c \times \mathbf{X}_{\epsilon, h}^c$ and satisfies the a priori estimate*

$$\|(\mathbf{H}_h, \mathbf{E}_h)\|_{b,h} \lesssim \omega^{\frac{1}{2}} \|(\mathbf{f}, \mathbf{g})\|_{L^c}. \quad (41)$$

Proof. Notice that

$$(\mathbf{C}_h(\mathbf{E}_h), \Pi_{h0}^c(\mu \mathbf{H}_h))_{\mathbf{L}^2(D)} = (\mathbf{E}_h, \nabla_0 \times \Pi_{h0}^c(\mu \mathbf{H}_h))_{\mathbf{L}^2(D)} = 0.$$

Similarly, $(\mathbf{C}_{h0}(\mathbf{H}_h), \mathbf{\Pi}_h^c(\epsilon \mathbf{E}_h))_{L^2(D)} = 0$. Moreover, since $\mathbf{\Pi}_{h0}^c(\mu \mathbf{H}_h)$ has zero tangential jumps and zero tangential traces, and since $\mathbf{\Pi}_h^c(\epsilon \mathbf{E}_h)$ has zero tangential jumps, we have $s_h^c(\cdot, \mathbf{\Pi}_{h0}^c(\mu \mathbf{H}_h)) = s_h^{c,\circ}(\cdot, \mathbf{\Pi}_h^c(\epsilon \mathbf{E}_h)) = 0$. Hence, testing (39) with $(\mathbf{h}_h, \mathbf{e}_h) := (\mathbf{\Pi}_{h0}^c(\mu \mathbf{H}_h), \mathbf{\Pi}_h^c(\epsilon \mathbf{E}_h))$ gives

$$\mathbf{\Pi}_h^c(\mu \mathbf{H}_h) = \mathbf{0}, \quad \mathbf{\Pi}_h^c(\epsilon \mathbf{E}_h) = \mathbf{0}. \quad (42)$$

This proves that every solution to (39) satisfies $(\mathbf{H}_h, \mathbf{E}_h) \in \mathbf{X}_{\mu, h0}^c \times \mathbf{X}_{\epsilon, h}^c$. The deflated inf-sup condition established in Lemma 4.2 then implies the uniqueness of the solution to (39). Since (39) amounts to a square linear system, this also proves the existence of the solution. Finally, the a priori estimate (41) follows from the Cauchy–Schwarz inequality and the deflated inf-sup inequality (43). \square

Lemma 4.2 (Deflated Inf-sup condition). *The following holds for every discrete pair $(\mathbf{H}_h, \mathbf{E}_h)$ in $\mathbf{X}_{\mu, h0}^c \times \mathbf{X}_{\epsilon, h}^c$:*

$$\omega^{\frac{1}{2}} \|(\mathbf{H}_h, \mathbf{E}_h)\|_{b, h} \lesssim \sup_{(\mathbf{h}_h, \mathbf{e}_h) \in L_h^c} \frac{|a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{h}_h, \mathbf{e}_h))|}{\|(\mathbf{h}_h, \mathbf{e}_h)\|_{L^c}}. \quad (43)$$

Proof. Let \mathbb{S} denote the supremum on the right-hand side of (43).

(1) Testing (39) with $(\mathbf{h}_h, \mathbf{e}_h) := (\mathbf{H}_h, \mathbf{E}_h)$, using that $\mathbf{\Pi}_h^c(\mu \mathbf{H}_h) = \mathbf{\Pi}_h^c(\epsilon \mathbf{E}_h) = \mathbf{0}$ (by assumption) and the integration by parts property (28a) gives

$$\kappa_H (|\mathbf{H}_h|_J^c)^2 + \kappa_E (|\mathbf{E}_h|_J^c)^2 = a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{H}_h, \mathbf{E}_h)) \leq \mathbb{S} \|(\mathbf{H}_h, \mathbf{E}_h)\|_{L^c}. \quad (44)$$

(2) Testing (39) with $(\mathbf{h}_h, \mathbf{e}_h) := (-\kappa_E \tilde{h} \mathbf{C}_h(\mathbf{E}_h), \mathbf{0})$ and using that $\kappa_H \kappa_E = 1$ gives

$$\begin{aligned} \kappa_E \|\tilde{h}^{\frac{1}{2}} \mathbf{C}_h(\mathbf{E}_h)\|_{L^2(D)}^2 &= a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{h}_h, \mathbf{e}_h)) - \kappa_H s_h^c(\mathbf{H}_h, \mathbf{h}_h) \\ &= \mathbb{S} \|\mu^{\frac{1}{2}} \mathbf{h}_h\|_{L^2(D)} + s_h^c(\mathbf{H}_h, \tilde{h} \mathbf{C}_h(\mathbf{E}_h)). \end{aligned}$$

On the one hand, we have

$$\|\mu^{\frac{1}{2}} \mathbf{h}_h\|_{L^2(D)} \lesssim (\mu_0 \ell_D \kappa_E)^{\frac{1}{2}} \kappa_E^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \mathbf{C}_h(\mathbf{E}_h)\|_{L^2(D)} = \omega^{-\frac{1}{2}} \kappa_E^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \mathbf{C}_h(\mathbf{E}_h)\|_{L^2(D)},$$

where we used that $h \leq \ell_D$ and $\ell_D \mu_0 \kappa_E = \omega^{-1}$. On the other hand, invoking a discrete trace inequality shows that $|\tilde{h} \mathbf{C}_h(\mathbf{E}_h)|_J^c \lesssim \|\tilde{h}^{\frac{1}{2}} \mathbf{C}_h(\mathbf{E}_h)\|_{L^2(D)}$. Using the Cauchy–Schwarz inequality and using again that $\kappa_H \kappa_E = 1$, this gives

$$\kappa_E \|\tilde{h}^{\frac{1}{2}} \mathbf{C}_h(\mathbf{E}_h)\|_{L^2(D)}^2 \lesssim (\omega^{-\frac{1}{2}} \mathbb{S} + \kappa_H^{\frac{1}{2}} |\mathbf{H}_h|_J^c) \kappa_E^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \mathbf{C}_h(\mathbf{E}_h)\|_{L^2(D)}.$$

Therefore, we conclude that

$$\kappa_E^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \mathbf{C}_h(\mathbf{E}_h)\|_{L^2(D)} \lesssim \omega^{-\frac{1}{2}} \mathbb{S} + \kappa_H^{\frac{1}{2}} |\mathbf{H}_h|_J^c.$$

Similarly, testing (39) with $(\mathbf{h}_h, \mathbf{e}_h) := (\mathbf{0}, \kappa_H \tilde{h} \mathbf{C}_{h0}(\mathbf{H}_h))$ gives

$$\kappa_H^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \mathbf{C}_{h0}(\mathbf{H}_h)\|_{L^2(D)} \lesssim \omega^{-\frac{1}{2}} \mathbb{S} + \kappa_E^{\frac{1}{2}} |\mathbf{E}_h|_J^{c,\circ}.$$

Gathering the above two bounds and invoking (44) gives

$$\kappa_H^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \mathbf{C}_{h0}(\mathbf{H}_h)\|_{L^2(D)} + \kappa_E^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \mathbf{C}_h(\mathbf{E}_h)\|_{L^2(D)} \lesssim \omega^{-\frac{1}{2}} \mathbb{S} + \mathbb{S}^{\frac{1}{2}} \|(\mathbf{H}_h, \mathbf{E}_h)\|_{L^c}^{\frac{1}{2}}. \quad (45)$$

(3) We now estimate $\|\nabla_0 \times \mathbf{H}_h\|_{(\mathbf{X}^c)^\prime}$ and $\|\nabla \times \mathbf{E}_h\|_{(\mathbf{X}_0^c)^\prime}$. Let $\boldsymbol{\varepsilon} \in \mathbf{X}^c$. Recalling the definition of the commutator δ_h in (32a) and setting $\boldsymbol{\varepsilon}_h := \mathcal{J}_h^c(\boldsymbol{\varepsilon})$, we have

$$\begin{aligned} (\mathbf{H}_h, \nabla \times \boldsymbol{\varepsilon})_{L^2(D)} &= (\mathbf{C}_{h0}(\mathbf{H}_h), \boldsymbol{\varepsilon})_{L^2(D)} + \delta_h(\mathbf{H}_h, \boldsymbol{\varepsilon}) \\ &= (\mathbf{C}_{h0}(\mathbf{H}_h), \boldsymbol{\varepsilon}_h)_{L^2(D)} + (\mathbf{C}_{h0}(\mathbf{H}_h), \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h)_{L^2(D)} + \delta_h(\mathbf{H}_h, \boldsymbol{\varepsilon}) \\ &= a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{0}, \boldsymbol{\varepsilon}_h)) + (\mathbf{C}_{h0}(\mathbf{H}_h), \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h)_{L^2(D)} + \delta_h(\mathbf{H}_h, \boldsymbol{\varepsilon}), \end{aligned}$$

where the last equality follows from the fact that $\boldsymbol{\varepsilon}_h$ has zero tangential jumps. Using the approximation properties of \mathcal{J}_h^c , the commutator estimate (33a), and the estimate $\|\boldsymbol{\varepsilon}_h\|_{L^2(D)} \lesssim \|\boldsymbol{\varepsilon}\|_{L^2(D)} + h^{s'}|\boldsymbol{\varepsilon}|_{\mathbf{H}^{s'}(D)} \lesssim \ell_D \|\nabla \times \boldsymbol{\varepsilon}\|_{L^2(D)}$ (which follows from the Poincaré–Steklov inequality $\|\boldsymbol{\varepsilon}\|_{L^2(D)} \lesssim \ell_D \|\nabla \times \boldsymbol{\varepsilon}\|_{L^2(D)}$ and the regularity property (10) since $\boldsymbol{\varepsilon} \in \mathbf{X}^c$), we obtain

$$\begin{aligned} |(\mathbf{H}_h, \nabla \times \boldsymbol{\varepsilon})_{L^2(D)}| &\lesssim \mathbb{S} \epsilon_0^{\frac{1}{2}} \|\boldsymbol{\varepsilon}_h\|_{L^2(D)} + h^{s'-\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \mathbf{C}_{h0}(\mathbf{H}_h)\|_{L^2(D)} |\boldsymbol{\varepsilon}|_{\mathbf{H}^{s'}(D)} \\ &\quad + (h/\ell_D)^{s'-\frac{1}{2}} \ell_D^{\frac{1}{2}} |\mathbf{H}_h|_J^c \|\nabla \times \boldsymbol{\varepsilon}\|_{L^2(D)} \\ &\lesssim \{\epsilon_0^{\frac{1}{2}} \mathbb{S} + (h/\ell_D)^{s'-\frac{1}{2}} \ell_D^{-\frac{1}{2}} (\|\tilde{h}^{\frac{1}{2}} \mathbf{C}_{h0}(\mathbf{H}_h)\|_{L^2(D)} + |\mathbf{H}_h|_J^c)\} \\ &\quad \times \ell_D \|\nabla \times \boldsymbol{\varepsilon}\|_{L^2(D)}. \end{aligned}$$

Hence, recalling the definition of $\|\nabla_0 \times \mathbf{H}_h\|_{(\mathbf{X}^c)^\prime}$ introduced in Corollary 2.3, we have

$$\|\nabla_0 \times \mathbf{H}_h\|_{(\mathbf{X}^c)^\prime} \lesssim \epsilon_0^{\frac{1}{2}} \mathbb{S} + (h/\ell_D)^{s'-\frac{1}{2}} \ell_D^{-\frac{1}{2}} (\|\tilde{h}^{\frac{1}{2}} \mathbf{C}_{h0}(\mathbf{H}_h)\|_{L^2(D)} + |\mathbf{H}_h|_J^c).$$

We proceed similarly to establish the estimate

$$\|\nabla \times \mathbf{E}_h\|_{(\mathbf{X}_0^c)^\prime} \lesssim \mu_0^{\frac{1}{2}} \mathbb{S} + (h/\ell_D)^{s'-\frac{1}{2}} \ell_D^{-\frac{1}{2}} (\|\tilde{h}^{\frac{1}{2}} \mathbf{C}_h(\mathbf{E}_h)\|_{L^2(D)} + |\mathbf{E}_h|_J^{c,\circ}).$$

(4) Since $\mathbf{H}_h \in \mathbf{X}_{\mu, h0}^c$, we use the discrete Poincaré–Steklov inequality (36a) in Lemma 3.4 and the above estimate on $\|\nabla_0 \times \mathbf{H}_h\|_{(\mathbf{X}^c)^\prime}$ to infer that

$$\begin{aligned} \|\mu^{\frac{1}{2}} \mathbf{H}_h\|_{L^2(D)} &\lesssim \mu_0^{\frac{1}{2}} \|\mathbf{H}_h\|_{L^2(D)} \\ &\lesssim \mu_0^{\frac{1}{2}} (\ell_D \|\nabla_0 \times \mathbf{H}_h\|_{(\mathbf{X}^c)^\prime} + h^{\frac{1}{2}} |\mathbf{H}_h|_J^c) \\ &\leq \mu_0^{\frac{1}{2}} (\ell_D \|\nabla_0 \times \mathbf{H}_h\|_{(\mathbf{X}^c)^\prime} + (h/\ell_D)^{s'-\frac{1}{2}} \ell_D^{\frac{1}{2}} |\mathbf{H}_h|_J^c) \\ &\lesssim \omega^{-1} \{\mathbb{S} + (h/\ell_D)^{s'-\frac{1}{2}} \omega^{\frac{1}{2}} \kappa_H^{\frac{1}{2}} (\|\tilde{h}^{\frac{1}{2}} \mathbf{C}_{h0}(\mathbf{H}_h)\|_{L^2(D)} + |\mathbf{H}_h|_J^c)\}, \end{aligned}$$

where we used that $h^{\frac{1}{2}} \leq (h/\ell_D)^{s'-\frac{1}{2}} \ell_D^{\frac{1}{2}}$ on the third line, and $\ell_D (\mu_0 \epsilon_0)^{\frac{1}{2}} = \omega^{-1}$ and $\mu_0 \ell_D \kappa_H^{-1} = \omega^{-1}$ on the fourth line. We proceed similarly to establish that

$$\|\epsilon^{\frac{1}{2}} \mathbf{E}_h\|_{L^2(D)} \lesssim \omega^{-1} \{\mathbb{S} + (h/\ell_D)^{s'-\frac{1}{2}} \omega^{\frac{1}{2}} \kappa_E^{\frac{1}{2}} (\|\tilde{h}^{\frac{1}{2}} \mathbf{C}_h(\mathbf{E}_h)\|_{L^2(D)} + |\mathbf{E}_h|_J^{c,\circ})\}.$$

Gathering the above two bounds, invoking (44) and (45), and using $h \leq \ell_D$ gives

$$\|(\mathbf{H}_h, \mathbf{E}_h)\|_{L^c} \lesssim \omega^{-1} \mathbb{S} + \omega^{-\frac{1}{2}} \mathbb{S}^{\frac{1}{2}} \|(\mathbf{H}_h, \mathbf{E}_h)\|_{L^c}^{\frac{1}{2}}.$$

Invoking Young's inequality, we conclude that $\|(\mathbf{H}_h, \mathbf{E}_h)\|_{L^c} \lesssim \omega^{-1} \mathbb{S}$. Substituting this bound into (44) and (45) and putting everything together proves (43). \square

4.3 Duality argument

The main result of the paper is established in this section, namely in Theorem 4.9 where we prove that $\lim_{h \in \mathcal{H} \rightarrow 0} \|T - T_h\|_{\mathcal{L}(L^c; L^c)} = 0$. This is done by invoking a duality argument. Recall that the space L^c is equipped with the norm $\|(\mathbf{f}, \mathbf{g})\|_{L^c} := \{\|\mu^{\frac{1}{2}} \mathbf{f}\|_{L^2(D)}^2 + \|\epsilon^{\frac{1}{2}} \mathbf{g}\|_{L^2(D)}^2\}^{\frac{1}{2}}$.

Let (\mathbf{H}, \mathbf{E}) be the solution to the continuous problem (13) and let $(\mathbf{H}_h, \mathbf{E}_h)$ be the solution to the discrete problem (40). We define the errors

$$\delta \mathbf{h} := \mathbf{H} - \mathbf{H}_h, \quad \delta \mathbf{e} := \mathbf{E} - \mathbf{E}_h. \quad (46)$$

We consider the following dual problem: Find $(\boldsymbol{\eta}, \boldsymbol{\varepsilon})$ in $\mathbf{H}_0(\mathbf{curl}; D) \times \mathbf{H}(\mathbf{curl}; D)$ such that

$$\omega \mu \mathbf{\Pi}_0^c(\boldsymbol{\eta}) + \nabla \times \boldsymbol{\varepsilon} = \omega(\mathbf{I} - \mathbf{\Pi}_0^c)(\mu \delta \mathbf{h}), \quad (47a)$$

$$\omega \epsilon \mathbf{\Pi}^c(\boldsymbol{\varepsilon}) - \nabla_0 \times \boldsymbol{\eta} = \omega(\mathbf{I} - \mathbf{\Pi}^c)(\epsilon \delta \mathbf{e}). \quad (47b)$$

The following regularity estimate on the dual solution plays a key role in the analysis.

Lemma 4.3 (Well-posedness and stability). (i) *The dual problem (47) has a unique solution $(\boldsymbol{\eta}, \boldsymbol{\varepsilon})$, and this solution sits in $\mathbf{X}_0^c \times \mathbf{X}^c$.* (ii) *The solution $(\boldsymbol{\eta}, \boldsymbol{\varepsilon})$ satisfies the following a priori estimates with $s' \in (\frac{1}{2}, 1]$:*

$$\ell_D(\mu_0^{\frac{1}{2}} \|\nabla_0 \times \boldsymbol{\eta}\|_{L^2(D)} + \epsilon_0^{\frac{1}{2}} \|\nabla \times \boldsymbol{\varepsilon}\|_{L^2(D)}) \lesssim \|(\delta \mathbf{h}, \delta \mathbf{e})\|_{L^c}, \quad (48a)$$

$$|\boldsymbol{\eta}|_{\mathbf{H}^{s'}(D)} \lesssim \ell_D^{1-s'} \|\nabla_0 \times \boldsymbol{\eta}\|_{L^2(D)}, \quad |\boldsymbol{\varepsilon}|_{\mathbf{H}^{s'}(D)} \lesssim \ell_D^{1-s'} \|\nabla \times \boldsymbol{\varepsilon}\|_{L^2(D)}. \quad (48b)$$

Proof. (i) Testing (47a) with $\mathbf{\Pi}_0^c(\boldsymbol{\eta})$ and testing (47b) with $\mathbf{\Pi}^c(\boldsymbol{\varepsilon})$ readily shows that $\|\mu^{\frac{1}{2}} \mathbf{\Pi}_0^c(\boldsymbol{\eta})\|_{L^2(D)} = \|\epsilon^{\frac{1}{2}} \mathbf{\Pi}^c(\boldsymbol{\varepsilon})\|_{L^2(D)} = 0$. This proves that

$$\mathbf{\Pi}_0^c(\boldsymbol{\eta}) = \mathbf{\Pi}^c(\boldsymbol{\varepsilon}) = \mathbf{0}. \quad (49)$$

The existence and uniqueness of a solution to (47) is then a direct consequence of (8a) in Lemma 2.1. Moreover, (49) means that $(\boldsymbol{\eta}, \boldsymbol{\varepsilon}) \in \mathbf{X}_0^c \times \mathbf{X}^c$.

(ii) The a priori estimate (48a) follows from (47), (49), and $\ell_D \omega(\mu_0 \epsilon_0)^{\frac{1}{2}} = 1$. The a priori estimate (48b) is a consequence of the regularity estimate (10). \square

Lemma 4.4 (L^2 -error representation). *The following holds:*

$$\omega \|(\delta \mathbf{h}, \delta \mathbf{e})\|_{L^c}^2 = \theta_{\text{app}} + \theta_{\text{gal}} + \theta_{\text{curl}} + \theta_{\text{div}}, \quad (50)$$

with the approximation error, the Galerkin orthogonality error, the curl commuting error, and the divergence conformity error defined as follows:

$$\theta_{\text{app}} := a_h((\delta \mathbf{h}, \delta \mathbf{e}), (\boldsymbol{\eta} - \boldsymbol{\eta}_h, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h)), \quad (51a)$$

$$\theta_{\text{gal}} := a_h((\delta \mathbf{h}, \delta \mathbf{e}), (\boldsymbol{\eta}_h, \boldsymbol{\varepsilon}_h)), \quad (51b)$$

$$\theta_{\text{curl}} := \delta_h^\circ(\delta \mathbf{h}, \boldsymbol{\varepsilon}) - \delta_h^\circ(\delta \mathbf{e}, \boldsymbol{\eta}), \quad (51c)$$

$$\theta_{\text{div}} := \omega \{(\delta \mathbf{h}, \mathbf{\Pi}_0^c(\mu \delta \mathbf{h}))_{L^2(D)} + (\delta \mathbf{e}, \mathbf{\Pi}^c(\epsilon \delta \mathbf{e}))_{L^2(D)}\}, \quad (51d)$$

with $\boldsymbol{\eta}_h := \mathbf{\Pi}_h^b(\boldsymbol{\eta})$, $\boldsymbol{\varepsilon}_h := \mathbf{\Pi}_h^b(\boldsymbol{\varepsilon})$, $\mathbf{\Pi}_h^b$ defined in (21), and the commutators δ_h and δ_h° defined in (32).

Proof. Since $\mathbf{\Pi}_{h0}^c(\mu\mathbf{H}_h) = \mathbf{0}$ and $\mathbf{\Pi}_{h0}^c(\mu\mathbf{H}) = \mathbf{\Pi}_{h0}^c(\mathbf{\Pi}_0^c(\mu\mathbf{H})) = \mathbf{0}$ (see (23)), we infer that $\mathbf{\Pi}_{h0}^c(\mu\delta\mathbf{h}) = \mathbf{0}$. Similarly, we have $\mathbf{\Pi}_h^c(\epsilon\delta\mathbf{e}) = \mathbf{0}$. We also have $s_h^c(\cdot, \boldsymbol{\eta}) = s_h^{c,\circ}(\cdot, \boldsymbol{\varepsilon}) = 0$. Recalling the definition of the sesquilinear form a_h (see (38)), the definition of the commutators (see (32)), and using that $\nabla \times \boldsymbol{\varepsilon} = \omega(\mathbf{I} - \mathbf{\Pi}_0^c)(\mu\delta\mathbf{h})$ and $-\nabla_0 \times \boldsymbol{\eta} = \omega(\mathbf{I} - \mathbf{\Pi}^c)(\epsilon\delta\mathbf{e})$, we infer that

$$\begin{aligned} a_h((\delta\mathbf{h}, \delta\mathbf{e}), (\boldsymbol{\eta}, \boldsymbol{\varepsilon})) &= -(\mathbf{C}_h(\delta\mathbf{e}), \boldsymbol{\eta})_{L^2(D)} + (\mathbf{C}_{h0}(\delta\mathbf{h}), \boldsymbol{\varepsilon})_{L^2(D)} \\ &= -(\delta\mathbf{e}, \nabla_0 \times \boldsymbol{\eta})_{L^2(D)} + \delta_h^{\circ}(\delta\mathbf{e}, \boldsymbol{\eta}) + (\delta\mathbf{h}, \nabla \times \boldsymbol{\varepsilon})_{L^2(D)} - \delta_h(\delta\mathbf{h}, \boldsymbol{\varepsilon}) \\ &= \omega\{(\delta\mathbf{h}, (\mathbf{I} - \mathbf{\Pi}_0^c)(\mu\delta\mathbf{h}))_{L^2(D)} + (\delta\mathbf{e}, (\mathbf{I} - \mathbf{\Pi}^c)(\epsilon\delta\mathbf{e}))_{L^2(D)}\} - \theta_{\text{cr1}} \\ &= \omega\{\|\mu^{\frac{1}{2}}\delta\mathbf{h}\|_{L^2(D)}^2 + \|\epsilon^{\frac{1}{2}}\delta\mathbf{e}\|_{L^2(D)}^2\} - \theta_{\text{div}} - \theta_{\text{cr1}} \\ &= \omega\|(\delta\mathbf{h}, \delta\mathbf{e})\|_{L^c}^2 - \theta_{\text{div}} - \theta_{\text{cr1}}. \end{aligned}$$

Since $a_h((\delta\mathbf{h}, \delta\mathbf{e}), (\boldsymbol{\eta}, \boldsymbol{\varepsilon})) = \theta_{\text{app}} + \theta_{\text{gal}}$, re-organizing the terms proves (50). \square

We shall use the following a priori estimates which follow from Lemma 2.5 and Lemma 4.1, respectively:

$$\ell_D(\mu_0^{\frac{1}{2}}\|\nabla_0 \times \mathbf{H}\|_{L^2(D)} + \epsilon_0^{\frac{1}{2}}\|\nabla \times \mathbf{E}\|_{L^2(D)}) \leq \|(\mathbf{f}, \mathbf{g})\|_{L^c}, \quad (52a)$$

$$\|(\mathbf{H}_h, \mathbf{E}_h)\|_{b,h} \lesssim \omega^{\frac{1}{2}}\|(\mathbf{f}, \mathbf{g})\|_{L^c}. \quad (52b)$$

To alleviate the notation, we also set

$$c_{\text{rot}}(\boldsymbol{\eta}, \boldsymbol{\varepsilon}) := \ell_D(\mu_0^{\frac{1}{2}}\|\nabla_0 \times \boldsymbol{\eta}\|_{L^2(D)} + \epsilon_0^{\frac{1}{2}}\|\nabla \times \boldsymbol{\varepsilon}\|_{L^2(D)}), \quad (53a)$$

and observe that $c_{\text{rot}}(\boldsymbol{\eta}, \boldsymbol{\varepsilon}) \lesssim \|(\delta\mathbf{h}, \delta\mathbf{e})\|_{L^c}$ owing to (48a).

Lemma 4.5 (Bound on approximation error). *The following holds:*

$$|\theta_{\text{app}}| \lesssim (h/\ell_D)^{s' - \frac{1}{2}} \omega\|(\mathbf{f}, \mathbf{g})\|_{L^c} c_{\text{rot}}(\boldsymbol{\eta}, \boldsymbol{\varepsilon}). \quad (54)$$

Proof. Recalling that $\mathbf{\Pi}_{h0}^c(\mu\delta\mathbf{h}) = \mathbf{\Pi}_h^c(\epsilon\delta\mathbf{e}) = \mathbf{0}$, we have

$$\begin{aligned} \theta_{\text{app}} &= -(\mathbf{C}_h(\delta\mathbf{e}), \boldsymbol{\eta} - \boldsymbol{\eta}_h)_{L^2(D)} + (\mathbf{C}_{h0}(\delta\mathbf{h}), \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h)_{L^2(D)} \\ &\quad + \kappa_H s_h^c(\delta\mathbf{h}, \boldsymbol{\eta} - \boldsymbol{\eta}_h) + \kappa_E s_h^{c,\circ}(\delta\mathbf{e}, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h) \\ &= -(\nabla \times \mathbf{E}, \boldsymbol{\eta} - \boldsymbol{\eta}_h)_{L^2(D)} + (\nabla_0 \times \mathbf{H}, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h)_{L^2(D)} \\ &\quad - \kappa_H s_h^c(\mathbf{H}_h, \boldsymbol{\eta} - \boldsymbol{\eta}_h) - \kappa_E s_h^{c,\circ}(\mathbf{E}_h, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h), \end{aligned}$$

since $(\mathbf{C}_h(\mathbf{E}_h), \boldsymbol{\eta} - \boldsymbol{\eta}_h)_{L^2(D)} = (\mathbf{C}_{h0}(\mathbf{H}_h), \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h)_{L^2(D)} = 0$ by definition of the L^2 -orthogonal projection $\mathbf{\Pi}_h^b$ and since $s_h^c(\mathbf{H}, \cdot) = s_h^{c,\circ}(\mathbf{E}, \cdot) = 0$. Invoking the Cauchy–Schwarz inequality and the approximation properties of $\mathbf{\Pi}_h^b$, we obtain

$$\begin{aligned} &| -(\nabla \times \mathbf{E}, \boldsymbol{\eta} - \boldsymbol{\eta}_h)_{L^2(D)} + (\nabla_0 \times \mathbf{H}, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h)_{L^2(D)} | \\ &\lesssim \|\nabla \times \mathbf{E}\|_{L^2(D)} h^{s'} |\boldsymbol{\eta}|_{H^{s'}(D)} + \|\nabla_0 \times \mathbf{H}\|_{L^2(D)} h^{s'} |\boldsymbol{\varepsilon}|_{H^{s'}(D)}. \end{aligned}$$

Invoking the a priori estimate (52a) on (\mathbf{H}, \mathbf{E}) and the regularity estimate (48b) on $(\boldsymbol{\eta}, \boldsymbol{\varepsilon})$ gives

$$| -(\nabla \times \mathbf{E}, \boldsymbol{\eta} - \boldsymbol{\eta}_h)_{L^2(D)} + (\nabla_0 \times \mathbf{H}, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h)_{L^2(D)} | \lesssim (h/\ell_D)^{s'} \omega\|(\mathbf{f}, \mathbf{g})\|_{L^c} c_{\text{rot}}(\boldsymbol{\eta}, \boldsymbol{\varepsilon}).$$

Moreover, since $|\boldsymbol{\eta} - \boldsymbol{\eta}_h|_{\mathbf{J}}^c \lesssim h^{s' - \frac{1}{2}} |\boldsymbol{\eta}|_{\mathbf{H}^{s'}(D)} \lesssim (h/\ell_D)^{s' - \frac{1}{2}} \ell_D^{\frac{1}{2}} \|\nabla_0 \times \boldsymbol{\eta}\|_{\mathbf{L}^2(D)}$ and since $|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h|_{\mathbf{J}}^{c,\circ} \lesssim (h/\ell_D)^{s' - \frac{1}{2}} \ell_D^{\frac{1}{2}} \|\nabla \times \boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)}$, we infer from the a priori estimate (52b) that

$$\begin{aligned} & |\kappa_{\mathbf{H}} s_h^c(\mathbf{H}_h, \boldsymbol{\eta} - \boldsymbol{\eta}_h) + \kappa_{\mathbf{E}} s_h^{c,\circ}(\mathbf{E}_h, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h)| \\ & \lesssim \kappa_{\mathbf{H}}^{\frac{1}{2}} |\mathbf{H}_h|_{\mathbf{J}}^c \kappa_{\mathbf{H}}^{\frac{1}{2}} |\boldsymbol{\eta} - \boldsymbol{\eta}_h|_{\mathbf{J}}^c + \kappa_{\mathbf{E}}^{\frac{1}{2}} |\mathbf{E}_h|_{\mathbf{J}}^{c,\circ} \kappa_{\mathbf{E}}^{\frac{1}{2}} |\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h|_{\mathbf{J}}^{c,\circ} \\ & \lesssim \omega^{\frac{1}{2}} \|(\mathbf{f}, \mathbf{g})\|_{L^c} (h/\ell_D)^{s' - \frac{1}{2}} \left(\kappa_{\mathbf{H}}^{\frac{1}{2}} \ell_D^{\frac{1}{2}} \|\nabla_0 \times \boldsymbol{\eta}\|_{\mathbf{L}^2(D)} + \kappa_{\mathbf{E}}^{\frac{1}{2}} \ell_D^{\frac{1}{2}} \|\nabla \times \boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)} \right) \\ & = (h/\ell_D)^{s' - \frac{1}{2}} \omega \|(\mathbf{f}, \mathbf{g})\|_{L^c C_{\text{rot}}}(\boldsymbol{\eta}, \boldsymbol{\varepsilon}), \end{aligned}$$

since $\kappa_{\mathbf{H}}^{\frac{1}{2}} \ell_D^{\frac{1}{2}} = \omega^{\frac{1}{2}} \ell_D \mu_0^{\frac{1}{2}}$ and $\kappa_{\mathbf{E}}^{\frac{1}{2}} \ell_D^{\frac{1}{2}} = \omega^{\frac{1}{2}} \ell_D \epsilon_0^{\frac{1}{2}}$. Combining the above two estimates and using $h \leq \ell_D$ proves the assertion. \square

Lemma 4.6 (Bound on Galerkin orthogonality error). *The following holds:*

$$|\theta_{\text{gal}}| \lesssim (h/\ell_D)^{s'} \omega \|(\mathbf{f}, \mathbf{g})\|_{L^c C_{\text{rot}}}(\boldsymbol{\eta}, \boldsymbol{\varepsilon}). \quad (55)$$

Proof. We observe that

$$\theta_{\text{gal}} = a_h((\mathbf{H}, \mathbf{E}), (\boldsymbol{\eta}_h, \boldsymbol{\varepsilon}_h)) - a_h((\mathbf{H}_h, \mathbf{E}_h), (\boldsymbol{\eta}_h, \boldsymbol{\varepsilon}_h)).$$

Since $\boldsymbol{\Pi}_{h_0}^c(\mu \mathbf{H}) = \boldsymbol{\Pi}_h^c(\epsilon \mathbf{E}) = \mathbf{0}$, $s_h^c(\mathbf{H}, \cdot) = s_h^{c,\circ}(\mathbf{E}, \cdot) = 0$, and $\mathbf{C}_{h_0}(\mathbf{H}) = \nabla_0 \times \mathbf{H}$, $\mathbf{C}_h(\mathbf{E}) = \nabla \times \mathbf{E}$, we have

$$\begin{aligned} a_h((\mathbf{H}, \mathbf{E}), (\boldsymbol{\eta}_h, \boldsymbol{\varepsilon}_h)) & = -(\nabla \times \mathbf{E}, \boldsymbol{\eta}_h)_{\mathbf{L}^2(D)} + (\nabla_0 \times \mathbf{H}, \boldsymbol{\varepsilon}_h)_{\mathbf{L}^2(D)} \\ & = \omega \{ ((\mathbf{I} - \boldsymbol{\Pi}_0^c)(\mu \mathbf{f}), \boldsymbol{\eta}_h)_{\mathbf{L}^2(D)} + ((\mathbf{I} - \boldsymbol{\Pi}^c)(\epsilon \mathbf{g}), \boldsymbol{\varepsilon}_h)_{\mathbf{L}^2(D)} \}. \end{aligned}$$

Moreover, by definition of the discrete solution $(\mathbf{H}_h, \mathbf{E}_h)$, we have

$$a_h((\mathbf{H}_h, \mathbf{E}_h), (\boldsymbol{\eta}_h, \boldsymbol{\varepsilon}_h)) = \omega \{ ((\mathbf{I} - \boldsymbol{\Pi}_{h_0}^c)(\mu \mathbf{f}), \boldsymbol{\eta}_h)_{\mathbf{L}^2(D)} + ((\mathbf{I} - \boldsymbol{\Pi}_h^c)(\epsilon \mathbf{g}), \boldsymbol{\varepsilon}_h)_{\mathbf{L}^2(D)} \}.$$

Hence, we have

$$\theta_{\text{gal}} = \omega \{ ((\boldsymbol{\Pi}_{h_0}^c - \boldsymbol{\Pi}_0^c)(\mu \mathbf{f}), \boldsymbol{\eta}_h)_{\mathbf{L}^2(D)} + ((\boldsymbol{\Pi}_h^c - \boldsymbol{\Pi}_h^c)(\epsilon \mathbf{g}), \boldsymbol{\varepsilon}_h)_{\mathbf{L}^2(D)} \}.$$

Since $\boldsymbol{\Pi}_0^c(\boldsymbol{\eta}) = \mathbf{0}$ and $\boldsymbol{\Pi}_{h_0}^c(\boldsymbol{\eta}) = \mathbf{0}$, invoking the Cauchy–Schwarz inequality and the approximation properties of $\boldsymbol{\Pi}_h^c$, we infer that

$$\begin{aligned} \omega |((\boldsymbol{\Pi}_{h_0}^c - \boldsymbol{\Pi}_0^c)(\mu \mathbf{f}), \boldsymbol{\eta}_h)_{\mathbf{L}^2(D)}| & = \omega |((\boldsymbol{\Pi}_{h_0}^c - \boldsymbol{\Pi}_0^c)(\mu \mathbf{f}), \boldsymbol{\eta}_h - \boldsymbol{\eta})_{\mathbf{L}^2(D)}| \\ & \lesssim \omega \mu_0^{\frac{1}{2}} \|\mu^{\frac{1}{2}} \mathbf{f}\|_{\mathbf{L}^2(D)} h^{s'} |\boldsymbol{\eta}|_{\mathbf{H}^{s'}(D)} \\ & \lesssim \omega \mu_0^{\frac{1}{2}} \|\mu^{\frac{1}{2}} \mathbf{f}\|_{\mathbf{L}^2(D)} (h/\ell_D)^{s'} \ell_D \|\nabla_0 \times \boldsymbol{\eta}\|_{\mathbf{L}^2(D)} \\ & \lesssim (h/\ell_D)^{s'} \omega \|(\mathbf{f}, \mathbf{g})\|_{L^c C_{\text{rot}}}(\boldsymbol{\eta}, \boldsymbol{\varepsilon}). \end{aligned}$$

A similar bound holds for $\omega |((\boldsymbol{\Pi}_h^c - \boldsymbol{\Pi}^c)(\epsilon \mathbf{g}), \boldsymbol{\varepsilon}_h)_{\mathbf{L}^2(D)}|$. \square

Lemma 4.7 (Bound on curl commuting error). *The following holds:*

$$|\theta_{\text{crl}}| \lesssim (h/\ell_D)^{s' - \frac{1}{2}} \omega \|(\mathbf{f}, \mathbf{g})\|_{L^c C_{\text{rot}}}(\boldsymbol{\eta}, \boldsymbol{\varepsilon}). \quad (56)$$

Proof. We observe that

$$\theta_{\text{crl}} = -\delta_h(\mathbf{H}_h, \boldsymbol{\varepsilon}) + \delta_h^\circ(\mathbf{E}_h, \boldsymbol{\eta}),$$

since $\delta_h(\mathbf{H}, \boldsymbol{\varepsilon}) = \delta_h^\circ(\mathbf{E}, \boldsymbol{\eta}) = 0$. Invoking Lemma 3.3 then gives

$$|\theta_{\text{crl}}| \lesssim (h/\ell_D)^{s'-\frac{1}{2}} \ell_D^{\frac{1}{2}} \{ |\mathbf{H}_h|_{\text{J}}^{\text{c}} \|\nabla \times \boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)} + |\mathbf{E}_h|_{\text{J}}^{\text{c},\circ} \|\nabla_0 \times \boldsymbol{\eta}\|_{\mathbf{L}^2(D)} \}.$$

Using that $\ell_D \kappa_{\text{H}} \epsilon_0 = \omega^{-1}$, we obtain

$$\begin{aligned} \ell_D^{\frac{1}{2}} |\mathbf{H}_h|_{\text{J}}^{\text{c}} \|\nabla \times \boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)} &= \kappa_{\text{H}}^{\frac{1}{2}} |\mathbf{H}_h|_{\text{J}}^{\text{c}} (\ell_D \kappa_{\text{H}} \epsilon_0)^{-\frac{1}{2}} \ell_D \epsilon_0^{\frac{1}{2}} \|\nabla \times \boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)} \\ &\lesssim \omega^{\frac{1}{2}} \|(\mathbf{f}, \mathbf{g})\|_{L^c} \omega^{\frac{1}{2}} c_{\text{rot}}(\boldsymbol{\eta}, \boldsymbol{\varepsilon}) \\ &= \omega \|(\mathbf{f}, \mathbf{g})\|_{L^c} c_{\text{rot}}(\boldsymbol{\eta}, \boldsymbol{\varepsilon}). \end{aligned}$$

A similar bound holds true for $\ell_D^{\frac{1}{2}} |\mathbf{E}_h|_{\text{J}}^{\text{c},\circ} \|\nabla_0 \times \boldsymbol{\eta}\|_{\mathbf{L}^2(D)}$. This completes the proof. \square

Lemma 4.8 (Bound on divergence conformity error). *The following estimate holds with $\sigma := \min(s, s' - \frac{1}{2}) > 0$:*

$$|\theta_{\text{div}}| \lesssim (h/\ell_D)^\sigma \omega \|(\mathbf{f}, \mathbf{g})\|_{L^c} \|(\boldsymbol{\delta h}, \boldsymbol{\delta e})\|_{L^c}. \quad (57)$$

Proof. The proof crucially relies on Lemma 4.10 below. Since $\Pi_{h_0}^{\text{c}}(\mu \boldsymbol{\delta h}) = \mathbf{0}$, we can apply the estimate (58a) with $\mathbf{B} := \mu \boldsymbol{\delta h}$, which gives

$$\begin{aligned} |(\boldsymbol{\delta h}, \Pi_0^{\text{c}}(\mu \boldsymbol{\delta h}))_{\mathbf{L}^2(D)}| &\lesssim \left\{ (h/\ell_D)^s \ell_D^s |\mathbf{H}|_{\mathbf{H}^s(D)} + (h/\ell_D)^{s'} \ell_D \|\nabla_0 \times \mathbf{H}\|_{\mathbf{L}^2(D)} \right. \\ &\quad \left. + (h/\ell_D)^{s'-\frac{1}{2}} \ell_D^{\frac{1}{2}} (\|\tilde{h}^{\frac{1}{2}} \mathbf{C}_{h_0}(\mathbf{H}_h)\|_{\mathbf{L}^2(D)} + |\mathbf{H}_h|_{\text{J}}^{\text{c}}) \right\} \|\Pi_0^{\text{c}}(\mu \boldsymbol{\delta h})\|_{\mathbf{L}^2(D)}. \end{aligned}$$

Since $\ell_D^s |\mathbf{H}|_{\mathbf{H}^s(D)} \lesssim \ell_D \|\nabla_0 \times \mathbf{H}\|_{\mathbf{L}^2(D)} \leq \mu_0^{-\frac{1}{2}} \|(\mathbf{f}, \mathbf{g})\|_{L^c}$ owing to the continuous a priori estimate (52a) and also invoking the discrete a priori estimate (52b), we obtain since $h \leq \ell_D$ and $s \leq s'$,

$$\begin{aligned} |(\boldsymbol{\delta h}, \Pi_0^{\text{c}}(\mu \boldsymbol{\delta h}))_{\mathbf{L}^2(D)}| &\lesssim (h/\ell_D)^\sigma \|(\mathbf{f}, \mathbf{g})\|_{L^c} \mu_0^{-\frac{1}{2}} \|\Pi_0^{\text{c}}(\mu \boldsymbol{\delta h})\|_{\mathbf{L}^2(D)} \\ &\lesssim (h/\ell_D)^\sigma \|(\mathbf{f}, \mathbf{g})\|_{L^c} \|\mu^{\frac{1}{2}} \boldsymbol{\delta h}\|_{\mathbf{L}^2(D)}. \end{aligned}$$

Proceeding similarly, one proves that

$$|(\boldsymbol{\delta e}, \Pi^{\text{c}}(\epsilon \boldsymbol{\delta e}))_{\mathbf{L}^2(D)}| \lesssim (h/\ell_D)^\sigma \|(\mathbf{f}, \mathbf{g})\|_{L^c} \|\epsilon^{\frac{1}{2}} \boldsymbol{\delta e}\|_{\mathbf{L}^2(D)}.$$

Putting the above two bounds together proves the claim. \square

We are now ready to state the main result of the paper which, owing to standard spectral approximation results (see, e.g., Bramble and Osborn [8, Lem. 2.2], Osborn [30, Thm. 3&4], Boffi [5, Prop. 7.4]), proves the spectral correctness of the dG approximation. Recall that $\sigma := \min(s, s' - \frac{1}{2}) > 0$.

theorem 4.9 (Convergence). *We have $\|T - T_h\|_{\mathcal{L}(L^c; L^c)} \lesssim (h/\ell_D)^\sigma$.*

Proof. Combining the L^2 -error representation formula (50) together with the bounds from Lemmas 4.5–4.8 and since $c_{\text{rot}}(\boldsymbol{\eta}, \boldsymbol{\varepsilon}) \lesssim \|(\boldsymbol{\delta h}, \boldsymbol{\delta e})\|_{L^c}$ owing to (48a), we infer that

$$\omega \|(\boldsymbol{\delta h}, \boldsymbol{\delta e})\|_{L^c}^2 \lesssim (h/\ell_D)^\sigma \omega \|(\mathbf{f}, \mathbf{g})\|_{L^c} \|(\boldsymbol{\delta h}, \boldsymbol{\delta e})\|_{L^c},$$

where we used that $\sigma \leq s' - \frac{1}{2} < s'$ and $h \leq \ell_D$. This implies that $\|(\boldsymbol{\delta h}, \boldsymbol{\delta e})\|_{L^c} \lesssim (h/\ell_D)^\sigma \|(\mathbf{f}, \mathbf{g})\|_{L^c}$, whence the claim. \square

4.4 Technical lemma

In this section, we establish a result that plays a central role in the analysis to estimate the divergence conformity error.

Lemma 4.10 (Divergence conformity). *Let $s \in (0, \frac{1}{2}]$. For all $\mathbf{H} \in \mathbf{H}_0(\mathbf{curl}; D) \cap \mathbf{H}^s(D)$, all $\mathbf{H}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$, and all $\mathbf{B} \in \mathbf{L}^2(D)$ satisfying $\mathbf{\Pi}_{h_0}^c(\mathbf{B}) = \mathbf{0}$, we have*

$$\begin{aligned} |(\mathbf{H} - \mathbf{H}_h, \mathbf{\Pi}_0^c(\mathbf{B}))_{\mathbf{L}^2(D)}| &\lesssim \left\{ (h/\ell_D)^s \ell_D^s |\mathbf{H}|_{\mathbf{H}^s(D)} + (h/\ell_D)^{s'} \ell_D \|\nabla_0 \times \mathbf{H}\|_{\mathbf{L}^2(D)} \right. \\ &\quad \left. + (h/\ell_D)^{s' - \frac{1}{2}} \ell_D^{\frac{1}{2}} (\|\tilde{h}^{\frac{1}{2}} \mathbf{C}_{h_0}(\mathbf{H}_h)\|_{\mathbf{L}^2(D)} + |\mathbf{H}_h|_{\mathcal{J}}^c) \right\} \|\mathbf{\Pi}_0^c(\mathbf{B})\|_{\mathbf{L}^2(D)}, \end{aligned} \quad (58a)$$

and for all $\mathbf{E} \in \mathbf{H}(\mathbf{curl}; D) \cap \mathbf{H}^s(D)$, all $\mathbf{E}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$, and all $\mathbf{D} \in \mathbf{L}^2(D)$ satisfying $\mathbf{\Pi}_h^c(\mathbf{D}) = \mathbf{0}$, we have

$$\begin{aligned} |(\mathbf{E} - \mathbf{E}_h, \mathbf{\Pi}^c(\mathbf{D}))_{\mathbf{L}^2(D)}| &\lesssim \left\{ (h/\ell_D)^s \ell_D^s |\mathbf{E}|_{\mathbf{H}^s(D)} + (h/\ell_D)^{s'} \ell_D \|\nabla \times \mathbf{E}\|_{\mathbf{L}^2(D)} \right. \\ &\quad \left. + (h/\ell_D)^{s' - \frac{1}{2}} \ell_D^{\frac{1}{2}} (\|\tilde{h}^{\frac{1}{2}} \mathbf{C}_h(\mathbf{E}_h)\|_{\mathbf{L}^2(D)} + |\mathbf{E}_h|_{\mathcal{J}^c}^{\circ}) \right\} \|\mathbf{\Pi}^c(\mathbf{D})\|_{\mathbf{L}^2(D)}, \end{aligned} \quad (58b)$$

where $s' \in (\frac{1}{2}, 1]$ results from the regularity estimate (10).

Proof. We only prove the estimate (58a), since the proof of (58b) is similar.

(1) Using the commuting property of $\mathcal{J}_{h_0}^c$, we obtain $\nabla \times \mathcal{J}_{h_0}^c(\mathbf{\Pi}_0^c(\mathbf{H} - \mathbf{H}_h)) = \mathbf{0}$. This implies that $\mathcal{J}_{h_0}^c(\mathbf{\Pi}_0^c(\mathbf{H} - \mathbf{H}_h))$ is a member of $\mathbf{P}_{k_0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$. Moreover, invoking Lemma 3.2, we have $\mathbf{\Pi}_{h_0}^c(\mathbf{\Pi}_0^c(\mathbf{B})) = \mathbf{\Pi}_{h_0}^c(\mathbf{B}) = \mathbf{0}$ by assumption. Hence, $\mathbf{\Pi}_0^c(\mathbf{B}) \in \mathbf{P}_{k_0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp$. This, in turn, implies the following identity:

$$(\mathbf{H} - \mathbf{H}_h, \mathbf{\Pi}_0^c(\mathbf{B}))_{\mathbf{L}^2(D)} = (\mathbf{H} - \mathbf{H}_h - \mathcal{J}_{h_0}^c(\mathbf{\Pi}_0^c(\mathbf{H} - \mathbf{H}_h)), \mathbf{\Pi}_0^c(\mathbf{B}))_{\mathbf{L}^2(D)}.$$

Using the \mathbf{L}^2 -orthogonality of $\mathbf{\Pi}_0^c$ followed by the Cauchy–Schwarz inequality gives

$$|(\mathbf{H} - \mathbf{H}_h, \mathbf{\Pi}_0^c(\mathbf{B}))_{\mathbf{L}^2(D)}| \leq \|(\mathbf{I} - \mathcal{J}_{h_0}^c)(\mathbf{\Pi}_0^c(\mathbf{H} - \mathbf{H}_h))\|_{\mathbf{L}^2(D)} \|\mathbf{\Pi}_0^c(\mathbf{B})\|_{\mathbf{L}^2(D)}.$$

It remains to bound $\|(\mathbf{I} - \mathcal{J}_{h_0}^c)(\mathbf{\Pi}_0^c(\mathbf{H} - \mathbf{H}_h))\|_{\mathbf{L}^2(D)}$.

(2) Let us set $\mathbf{H}_h^c := \mathcal{I}_{h_0}^{c, \text{av}}(\mathbf{H}_h)$ and $\mathbf{w} := \mathbf{H} - \mathbf{H}_h^c - \mathbf{\Pi}_0^c(\mathbf{H} - \mathbf{H}_h^c)$. Then $\mathbf{w} \in \mathbf{H}_0(\mathbf{curl}; D)$ with $\nabla_0 \times \mathbf{w} = \nabla_0 \times (\mathbf{H} - \mathbf{H}_h^c)$, and $\mathbf{\Pi}_0^c(\mathbf{w}) = \mathbf{0}$. Hence, \mathbf{w} is a member of \mathbf{X}_0^c . The regularity estimate (10) followed by the triangle inequality give

$$|\mathbf{w}|_{\mathbf{H}^{s'}} \lesssim \ell_D^{1-s'} \|\nabla_0 \times (\mathbf{H} - \mathbf{H}_h^c)\|_{\mathbf{L}^2(D)} \leq \ell_D^{1-s'} (\|\nabla_0 \times \mathbf{H}\|_{\mathbf{L}^2(D)} + \|\nabla_0 \times \mathbf{H}_h^c\|_{\mathbf{L}^2(D)}).$$

Invoking the triangle inequality, the approximation properties of $\mathcal{J}_{h_0}^c$, and the fact that $(\mathbf{I} - \mathcal{J}_{h_0}^c)(\mathbf{H}_h^c) = \mathbf{0}$ (since $\mathbf{H}_h^c \in \mathbf{P}_{k_0}^c(\mathcal{T}_h)$), this implies that

$$\begin{aligned} \|(\mathbf{I} - \mathcal{J}_{h_0}^c)(\mathbf{\Pi}_0^c(\mathbf{H} - \mathbf{H}_h^c))\|_{\mathbf{L}^2(D)} &\lesssim \|(\mathbf{I} - \mathcal{J}_{h_0}^c)(\mathbf{w})\|_{\mathbf{L}^2(D)} + \|(\mathbf{I} - \mathcal{J}_{h_0}^c)(\mathbf{H} - \mathbf{H}_h^c)\|_{\mathbf{L}^2(D)} \\ &\lesssim h^{s'} |\mathbf{w}|_{\mathbf{H}^{s'}(D)} + \|(\mathbf{I} - \mathcal{J}_{h_0}^c)(\mathbf{H})\|_{\mathbf{L}^2(D)} \\ &\lesssim (h/\ell_D)^{s'} \ell_D (\|\nabla_0 \times \mathbf{H}\|_{\mathbf{L}^2(D)} + \|\nabla_0 \times \mathbf{H}_h^c\|_{\mathbf{L}^2(D)}) + (h/\ell_D)^s \ell_D^s |\mathbf{H}|_{\mathbf{H}^s(D)}. \end{aligned}$$

Moreover, invoking the triangle inequality, an inverse inequality, and the approximation properties of $\mathcal{I}_{h_0}^{c, \text{av}}$, we infer that

$$\begin{aligned} \|\nabla_0 \times \mathbf{H}_h^c\|_{\mathbf{L}^2(D)} &\lesssim \|\mathbf{C}_{h_0}(\mathbf{H}_h^c - \mathbf{H}_h)\|_{\mathbf{L}^2(D)} + \|\mathbf{C}_{h_0}(\mathbf{H}_h)\|_{\mathbf{L}^2(D)} \\ &\lesssim h^{-\frac{1}{2}} (\|\tilde{h}^{\frac{1}{2}} \mathbf{C}_{h_0}(\mathbf{H}_h)\|_{\mathbf{L}^2(D)} + |\mathbf{H}_h|_{\mathcal{J}}^c). \end{aligned}$$

We conclude that

$$\begin{aligned} \|(\mathbf{I} - \mathcal{J}_{h_0}^c)(\mathbf{\Pi}_0^c(\mathbf{H} - \mathbf{H}_h^c))\|_{\mathbf{L}^2(D)} &\lesssim (h/\ell_D)^{s'} \ell_D \|\nabla_0 \times \mathbf{H}\|_{\mathbf{L}^2(D)} + (h/\ell_D)^s \ell_D^s |\mathbf{H}|_{\mathbf{H}^s(D)} \\ &\quad + (h/\ell_D)^{s' - \frac{1}{2}} \ell_D^{\frac{1}{2}} (\|\tilde{h}^{\frac{1}{2}} \mathbf{C}_{h_0}(\mathbf{H}_h)\|_{\mathbf{L}^2(D)} + |\mathbf{H}_h|_J^c). \end{aligned}$$

Finally, using the triangle inequality, the \mathbf{L}^2 -stability of $\mathcal{J}_{h_0}^c$ and that of $\mathbf{\Pi}_0^c$, and the approximation properties of $\mathcal{I}_{h_0}^{c,av}$, we obtain

$$\begin{aligned} \|(\mathbf{I} - \mathcal{J}_{h_0}^c)(\mathbf{\Pi}_0^c(\mathbf{H} - \mathbf{H}_h))\|_{\mathbf{L}^2(D)} &\lesssim \|(\mathbf{I} - \mathcal{J}_{h_0}^c)(\mathbf{\Pi}_0^c(\mathbf{H} - \mathbf{H}_h^c))\|_{\mathbf{L}^2(D)} \\ &\quad + \|(\mathbf{I} - \mathcal{J}_{h_0}^c)(\mathbf{\Pi}_0^c(\mathbf{H}_h^c - \mathbf{H}_h))\|_{\mathbf{L}^2(D)} \\ &\lesssim \|(\mathbf{I} - \mathcal{J}_{h_0}^c)(\mathbf{\Pi}_0^c(\mathbf{H} - \mathbf{H}_h^c))\|_{\mathbf{L}^2(D)} + h^{\frac{1}{2}} |\mathbf{H}_h|_J^c. \end{aligned}$$

The assertion follows readily. \square

A Helmholtz decompositions

In this appendix, we recall some useful results on Helmholtz decompositions; these results are mostly drawn from Amrouche et al. [2], Dautray and Lions [20], and Girault and Raviart [25]. We also give a short proof of Lemma 2.1.

A.1 Topology of D

Recall that D is an open, bounded, Lipschitz polyhedron of \mathbb{R}^3 . We denote Γ_0 the boundary of the only unbounded connected component of $\mathbb{R}^d \setminus \overline{D}$. If ∂D is not connected, i.e., $\partial D \neq \Gamma_0$, we denote $\{\Gamma_i\}_{i \in \{1:J\}}$ the connected components of ∂D that are different from Γ_0 (see, e.g., [25, p. 37], [2, p. 835], [20, p. 217]). If D is not simply connected, we assume that there exist J cuts ($(d-1)$ -dimensional smooth manifolds) $\{\Sigma_j\}_{j \in \{1:J\}}$ that make the open set $D^\Sigma := D \setminus \bigcup_{j \in \{1:J\}} \Sigma_j$ simply connected. Additional regularity assumptions on these cuts as stated in [2, Hyp. 3.3, p. 836] are assumed to hold true.

A.2 Helmholtz decompositions

We consider the subspaces

$$\mathbf{H}^\Gamma(\operatorname{div} = 0; D) := \{\mathbf{v} \in \mathbf{H}(\operatorname{div} = 0; D) \mid \int_{\Gamma_i} \mathbf{v} \cdot \mathbf{n} \, ds = 0, \forall i \in \{1:J\}\}, \quad (59a)$$

$$\mathbf{H}_0^\Sigma(\operatorname{div} = 0; D) := \{\mathbf{v} \in \mathbf{H}_0(\operatorname{div} = 0; D) \mid \int_{\Sigma_j} \mathbf{v} \cdot \mathbf{n} \, ds = 0, \forall j \in \{1:J\}\}. \quad (59b)$$

The following \mathbf{L}^2 -orthogonal decompositions hold true:

$$\mathbf{L}^2(D) = \mathbf{H}_0(\operatorname{curl} = \mathbf{0}; D) \overset{\perp}{\oplus} \mathbf{H}^\Gamma(\operatorname{div} = 0; D), \quad (60a)$$

$$\mathbf{L}^2(D) = \mathbf{H}(\operatorname{curl} = \mathbf{0}; D) \overset{\perp}{\oplus} \mathbf{H}_0^\Sigma(\operatorname{div} = 0; D). \quad (60b)$$

In other words, we have

$$\mathbf{H}_0(\operatorname{curl} = \mathbf{0}; D)^\perp = \mathbf{H}^\Gamma(\operatorname{div} = 0; D), \quad (61a)$$

$$\mathbf{H}(\operatorname{curl} = \mathbf{0}; D)^\perp = \mathbf{H}_0^\Sigma(\operatorname{div} = 0; D). \quad (61b)$$

For all $q \in L^2(D)$ such that $q|_{D^\Sigma} \in H^1(D^\Sigma)$, we denote by $\nabla_\Sigma q$ the broken gradient of q such that $(\nabla_\Sigma q)(\mathbf{x}) = (\nabla q|_{D^\Sigma})(\mathbf{x})$ for a.e. $\mathbf{x} \in D$. For all $i \in \mathbb{N}$, let c_i denote any real number. We define

$$H_\Gamma^1(D) := \{q \in H^1(D) \mid q|_{\Gamma_0} = 0, q|_{\Gamma_i} = c_i, \forall i \in \{1:I\}\}, \quad (62a)$$

$$H_\Sigma^1(D) := \{q \in L^2(D) \mid q|_{D^\Sigma} \in H^1(D^\Sigma), \llbracket q \rrbracket|_{\Sigma_j} = c_j, \forall j \in \{1:J\}\}. \quad (62b)$$

Then, the following L^2 -orthogonal decompositions also hold true:

$$L^2(D) = \nabla H_\Gamma^1(D) \overset{\perp}{\oplus} \nabla \times \mathbf{H}(\mathbf{curl}; D), \quad (63a)$$

$$L^2(D) = \nabla_\Sigma H_\Sigma^1(D) \overset{\perp}{\oplus} \nabla_0 \times \mathbf{H}_0(\mathbf{curl}; D). \quad (63b)$$

A.3 Proof of Lemma 2.1

We only prove that the operators $\nabla \times : \mathbf{X}_\epsilon^c \rightarrow \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$ and $\nabla_0 \times : \mathbf{X}_{\mu,0}^c \rightarrow \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp$ are isomorphisms, since the other proof is similar. (Recall that the spaces \mathbf{X}_ϵ^c and $\mathbf{X}_{\mu,0}^c$ are defined in (6).)

(1) Integration by parts readily shows that $\nabla \times \mathbf{e} \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$ for all $\mathbf{e} \in \mathbf{X}_\epsilon^c$ and $\nabla_0 \times \mathbf{h} \in \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp$ for all $\mathbf{h} \in \mathbf{X}_{\mu,0}^c$.

(2) Injectivity. Let $\mathbf{e} \in \mathbf{X}_\epsilon^c$ be such that $\nabla \times \mathbf{e} = \mathbf{0}$. Then, $\mathbf{e} \in \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)$, and by definition of \mathbf{X}_ϵ^c , we infer that $(\epsilon \mathbf{e}, \mathbf{e})_{L^2(D)} = 0$. This proves that $\mathbf{e} = \mathbf{0}$. Hence, the operator $\nabla \times : \mathbf{X}_\epsilon^c \rightarrow \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$ is injective. The proof that the operator $\nabla_0 \times : \mathbf{X}_{\mu,0}^c \rightarrow \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp$ is injective is similar.

(3) Surjectivity. Let $\boldsymbol{\theta} \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$. Since every $q \in H_\Gamma^1(D)$ satisfies $\nabla q \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)$, we have $\boldsymbol{\theta} \in (\nabla H_\Gamma^1(D))^\perp$. Owing to (63a), we infer that there is $\mathbf{e}' \in \mathbf{H}(\mathbf{curl}; D)$ such that $\nabla \times \mathbf{e}' = \boldsymbol{\theta}$. Let $p \in H_\Sigma^1(D)$ be the unique function solving $(\epsilon \nabla_\Sigma p, \nabla_\Sigma q)_{L^2(D)} = (\epsilon \mathbf{e}', \nabla_\Sigma q)_{L^2(D)}$ for all $q \in H_\Sigma^1(D)$. Set $\mathbf{e} := \mathbf{e}' - \nabla_\Sigma p$. We have $\nabla \times \mathbf{e} = \nabla \times \mathbf{e}' = \boldsymbol{\theta}$, and $\epsilon \mathbf{e} \in (\nabla_\Sigma H_\Sigma^1(D))^\perp$. Owing to (63b), we infer that $\epsilon \mathbf{e}$ is in the range of the $\nabla_0 \times$ operator. Integration by parts readily implies that $\epsilon \mathbf{e} \in \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp$. In summary, we have shown that $\mathbf{e} \in \mathbf{X}_\epsilon^c$ and that $\nabla \times \mathbf{e} = \boldsymbol{\theta}$. This proves the surjectivity of the operator $\nabla \times : \mathbf{X}_\epsilon^c \rightarrow \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$. The proof that the operator $\nabla_0 \times : \mathbf{X}_{\mu,0}^c \rightarrow \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp$ is surjective is similar.

References

- [1] J. Alvarez, L. Angulo, A. Rubio Bretones, and S. G. Garcia. A spurious-free discontinuous Galerkin time-domain method for the accurate modeling of microwave filters. *IEEE Transactions on Microwave Theory and Techniques*, 60(8):2359–2369, 2012.
- [2] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault. Vector potentials in three-dimensional non-smooth domains. *Math. Methods Appl. Sci.*, 21(9):823–864, 1998.
- [3] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numer.*, 15:1–155, 2006.
- [4] M. S. Birman and M. Z. Solomyak. L_2 -theory of the Maxwell operator in arbitrary domains. *Russian Mathematical Surveys*, 42(6):75, 1987.
- [5] D. Boffi. Finite element approximation of eigenvalue problems. *Acta Numer.*, 19:1–120, 2010.

- [6] A. Bonito, J.-L. Guermond, and F. Luddens. Regularity of the Maxwell equations in heterogeneous media and Lipschitz domains. *J. Math. Anal. Appl.*, 408(2):498–512, 2013.
- [7] A. Bonito, J.-L. Guermond, and F. Luddens. An interior penalty method with C^0 finite elements for the approximation of the Maxwell equations in heterogeneous media: convergence analysis with minimal regularity. *ESAIM Math. Model. Numer. Anal.*, 50(5):1457–1489, 2016.
- [8] J. H. Bramble and J. E. Osborn. Rate of convergence estimates for nonselfadjoint eigenvalue approximations. *Math. Comp.*, 27:525–549, 1973.
- [9] A. Buffa and I. Perugia. Discontinuous Galerkin approximation of the Maxwell eigenproblem. *SIAM J. Numer. Anal.*, 44(5):2198–2226, 2006.
- [10] A. Buffa, P. Houston, and I. Perugia. Discontinuous Galerkin computation of the Maxwell eigenvalues on simplicial meshes. *J. Comput. Appl. Math.*, 204(2):317–333, 2007.
- [11] T. Chaumont-Frelet. Duality analysis of interior penalty discontinuous galerkin methods under minimal regularity and application to the a priori and a posteriori error analysis of helmholtz problems. Technical report, INRIA, 2023. <https://inria.hal.science/hal-03765207>.
- [12] T. Chaumont-Frelet and A. Ern. Duality argument of the discontinuous galerkin approximation of maxwell’s equations in the frequency domain. Technical report, INRIA, 2023.
- [13] T. Chaumont-Frelet and M. Vohralík. A stable local commuting projector and optimal hp approximation estimates in $H(\text{curl})$. Technical report, INRIA, 2023. <https://inria.hal.science/hal-03817302>.
- [14] S. H. Christiansen. Stability of Hodge decompositions in finite element spaces of differential forms in arbitrary dimension. *Numer. Math.*, 107(1):87–106, 2007.
- [15] S. H. Christiansen and R. Winther. Smoothed projections in finite element exterior calculus. *Math. Comp.*, 77(262):813–829, 2008.
- [16] P. Ciarlet, Jr. On the approximation of electromagnetic fields by edge finite elements. Part 1: Sharp interpolation results for low-regularity fields. *Comput. Math. Appl.*, 71(1):85–104, 2016.
- [17] P. Ciarlet, Jr. On the approximation of electromagnetic fields by edge finite elements. Part 1: Sharp interpolation results for low-regularity fields. *Comput. Math. Appl.*, 71(1):85–104, 2016.
- [18] G. Cohen and M. Duruflé. Non spurious spectral-like element methods for maxwell’s equations. *Journal of Computational Mathematics*, 25(3):282–304, 2007.
- [19] M. Costabel. A remark on the regularity of solutions of Maxwell’s equations on Lipschitz domains. *Math. Methods Appl. Sci.*, 12(4):365–368, 1990.
- [20] R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology. Vol. 3. Spectral theory and applications*. Springer-Verlag, Berlin, Germany, 1990.
- [21] A. Ern and J.-L. Guermond. Finite element quasi-interpolation and best approximation. *ESAIM Math. Model. Numer. Anal.*, 51(4):1367–1385, 2017.
- [22] A. Ern and J.-L. Guermond. Analysis of the edge finite element approximation of the Maxwell equations with low regularity solutions. *Comput. Math. Appl.*, 75(3):918–932, 2018.

- [23] A. Ern and J.-L. Guermond. *Finite Elements I: Approximation and Interpolation*, volume 72 of *Texts in Applied Mathematics*. Springer Nature, Cham, Switzerland, 2021.
- [24] A. Ern and J.-L. Guermond. The discontinuous galerkin approximation of the grad-div and curl-curl operators in first-order form is involution-preserving and spectrally correct. *SIAM J. Numer. Anal.*, ??(??):??, 2023.
- [25] V. Girault and P.-A. Raviart. *Finite element methods for Navier–Stokes equations. Theory and algorithms*. Springer Series in Computational Mathematics. Springer-Verlag, Berlin, Germany, 1986.
- [26] J. S. Hesthaven and T. Warburton. Nodal high-order methods on unstructured grids. I. Time-domain solution of Maxwell’s equations. *J. Comput. Phys.*, 181(1):186–221, 2002.
- [27] J. S. Hesthaven and T. Warburton. High-order nodal discontinuous Galerkin methods for the Maxwell eigenvalue problem. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 362(1816):493–524, 2004.
- [28] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numer.*, 11:237–339, 2002.
- [29] F. Jochmann. Regularity of weak solutions of Maxwell’s equations with mixed boundary-conditions. *Math. Methods Appl. Sci.*, 22(14):1255–1274, 1999.
- [30] J. E. Osborn. Spectral approximation for compact operators. *Math. Comp.*, 29:712–725, 1975.
- [31] J. Schöberl. Commuting quasi-interpolation operators for mixed finite elements. Technical Report ISC-01-10-MATH, Texas A&M University, 2001. URL www.isc.tamu.edu/publications-reports/tr/0110.pdf.