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Controlled superprocesses and HJB equation in the space of finite measures

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Abstract

This paper introduces the formalism required to analyze a certain class of stochastic control problems that involve a super diffusion as the underlying controlled system. To establish the existence of these processes, we show that they are weak scaling limits of controlled branching processes. By proving their uniqueness in law, we can establish a dynamic programming principle for our stochastic control problem. This lays the groundwork for a PDE characterization of the associated value function, which is based on the concept of derivations in the space of finite positive measures. We also establish a verification theorem. To illustrate this approach, we focus on an exponential-type value function and show how a regular solution to a finite-dimensional HJB equation can be used to construct a smooth solution to the HJB equation in the space of finite measures.

MSC Classification- 93E20, 49L20, 49L12, 60J70, 35K10, 35Q93, 60H30

Keywords— Stochastic control, superprocesses, Hamilton-Jacobi-Bellman equation, dynamic programming principle.

1 Introduction

The goal of population dynamics scaling limits is to provide a simpler method for modelling large populations. When particles have the ability to divide into more particles, we refer to them as branching particle systems. These systems are called branching diffusions when their movement through space follows a diffusion process.

Branching diffusions belong to a class of measure-valued processes that have received significant attention over the past thirty years. A good introduction to this topic can be found in [11]. The class of branching diffusions, as well as their scaling limits known as superprocesses, have been extensively studied in [35, 39, 18], among others. The purpose of this article is to introduce and examine the controlled counterpart of these processes, namely the controlled superprocesses.

Branching diffusions are discrete particle systems that underlie superprocesses, and there have been several studies on their controlled versions. This was first conducted in [46] and further developed in [32]. Along the same lines as the latter paper, [9] generalized this setting to controlled

branching parameters and provided solutions to the problem using the *branching property* technique. This successful strategy involves emulating the symmetry of the problem to reduce it to a finite-dimensional optimization problem.

In particular, [9] and [32] focused on a cost of the same form. This is defined considering the product on all the particles alive at the terminal time of a continuous positive function taking values in the unit interval. Its associated value function can be seen as a cylindrical function of exponential type, which allows for the minimization of a global functional to be split into an optimization over the individual particles. Additionally, the fact that the coefficients are autonomous, meaning they only depend on the control and the position of each particle, translates the optimization into solving a finite-dimensional problem. This idea was later used in [25] to address the stochastic target problem over branching particle systems. We will utilize this technique, in the final part of the article, to provide a class of control problems for which the solution can be explicitly computed.

The concept of breaking the coupling between individual actions and global population behaviour has already appeared in the stochastic control literature. In particular, it is central to Mean Field Games (MFG) and Mean Field Control (MFC) problems. In this framework, the optimization problem for a large population is related to the control of a single participant interacting with the limit of the empirical measure of identical copies of itself. This is proven to converge to an interaction between a process and its law, satisfying a fixed point criterion. References for this topic can be found in [29, 4, 6, 7]. The existence of the controlled McKean-Vlasov dynamics is established as a weak limit of the law of interacting particle systems. In the MFC setting, for example, [27, 28, 8] use the relaxed control approach introduced in [13, 22] to show this limiting results. A similar weak reasoning method is used in [41], where the optimal stopping problem is generalized to the mean field setting using a control stopping strategy.

We aim to adopt a similar strategy for defining, weakly, the controlled limiting dynamics. Specifically, we utilize recent advancements in the analysis of controlled branching diffusions described in [33], which employs a relaxed setting that allows for a new characterization of these processes as weak controls. This weak control representation enables us to focus on the laws of these processes associated with starting condition, control, and martingale problem. By fixing the first two elements and manipulating the martingale problem, we prove that controlled superprocesses arise as a rescaling of branching processes, thereby establishing their existence. To achieve this, we extend the Aldous criterion presented in [11, 35, 39, 18] for convergence to superprocesses to a controlled setting. Furthermore, we generalize the martingale problem to a class of functionals that are convergence-determining in the space of càdlàg paths on finite measures and then use the ideas of [40], as detailed in [18, 33], to establish uniqueness in law through the duality method.

Once existence and uniqueness have been shown, we focus on the related control problem and we adopt the Dynamic Programming Principle (DPP) approach. The DPP is a powerful tool for solving control problems and we achieve it applying the methods described in [16, 17]. It has been shown (see, *e.g.*, [45]) that the DPP leads to a characterization of the problem through a nonlinear Hamilton-Jacobi-Bellmann (HJB) equation. In our setting, the HJB needs to be defined the space of finite measures. In the literature, PDEs on space of measures have already been investigated. In fact, MFG and MFC literature pushed the development of differential calculus in the space of probability measures to this tackle this problem.

This approach leads to a verification theorem, which provides the necessary conditions for proving the optimality of a controlling strategy. This is achieved by showing that the value function of the control problem is a (viscosity) solution of this equation. This point of view has been explored in several works, including those focused on Markovian controls [36], open-loop controls [1], Markovian and non-Markovian frameworks [12], closed-loop controls [44], and McKean-Vlasov mixed regular-singular control problems [21]. An example of a study that combines the branching

diffusion framework with the mean-field approach is presented in [10], where the authors introduce scaling limits that differ from the dynamics of superprocesses.

These techniques have then been extended to study dynamics on finite measures, as in [30]. In latter article, the author studies backward Kolmogorov equations associated with stochastic filtering, extending the differential calculus developed for probability measures to general finite measures. Such an extension is presented introducing flat derivative and Lions' derivative with the same strategy of [5]. In the latter paper, the flat derivative is defined as a directional derivative, while the Lions' derivative is obtained as the derivation of the flat derivative with respect to the space component. This is done since the approach used by Lions in [29] cannot be employed in this new setting, as there is no lifting of the space of finite measures to that of L^2 random variables. Theoretical studies of the intrinsic properties of these differential operators can be found in [38], where intrinsic and extrinsic differentiations are introduced and shown to coincide with the notions of flat derivative and Lions' derivative in this context.

We adopt the differential calculus developed in [30], taking advantage of its density results to extend the martingale problem used to introduce these processes. This generalization is possible since the space of finite measures is homeomorphic to a subset of the product between probability measures and the real line when far from the measure zero. This allows, in particular, to use the lifting technique by renormalizing the measures whenever we are not close to this critical measure.

We will employ the generalized martingale problem on functions with sufficient differentiability relative to the measure. With this method, we provide the HJB equation and a verification theorem. Remarkably, the resulting Dynamic Programming Equation (DPE) features a second-order flat derivative. To the best of our knowledge, such a term has not seen before in a DPE. This reveals a remarkable symmetry with the controlled diffusions, where a second-order operator appears in the PDE that describes the dynamics. This second-order operator, taken with respect to the measure, differs from what is observed in the literature, where just the first and second order Lions' derivatives of the value function are involved. Establishing the regularity of the value function poses a challenging problem in the context of solving non-linear PDEs, particularly for functions defined in measures. This question as well as the investigation of viscosity solution to our DPE is left for future research.

The paper is organized as follows: In Section 2, we introduce the model setup as well as the controlled superprocesses as a solution to a martingale problem. We prove their uniqueness in law and existence as a weak limit of rescaled branching processes in Subsection 2.2. In particular, the latter is done with the use of the martingale problem for rescaled branching diffusions. We then show its convergences to a solution to the martingale problem defining the controlled superprocesses. We also establish a non-explosion bound, with respect to the metric metrizing weak* topology. In Section 3, we present the control problem of interest and prove its measurability property and the DPP. In Section 4, we derive the HJB equation satisfied by the value function of the control problem. To this purpose, we introduce a differential calculus in the space of finite measure and generalize the initial martingale problem using the density of cylindrical functions in the space of regular functions on finite measures. Finally, in Subsection 4.4, we derive the Dynamic Programming equation and prove a verification theorem. We conclude the paper by providing a regular solution to the optimization problem.

2 Controlled superprocesses

2.1 Model setup and definitions

For a Polish space (E, d) with $\mathcal{B}(E)$ its Borelian σ -field, we write $C_b(E)$ (resp. $C_0(E)$) for the subset of the continuous functions that are bounded (resp. that vanish at infinity), and $M(E)$ (resp. $\mathcal{P}(E)$) for the set of Borel positive finite measures (resp. probability measures) on E . We equip $M(E)$ with weak* topology, *i.e.*, the weakest topology that makes continuous the maps $M(E) \ni \lambda \mapsto \int_E \varphi(x) \lambda(dx)$ for any $\varphi \in C_b(\mathbb{R}^d)$. We denote $\langle \varphi, \lambda \rangle = \int_E \varphi(x) \lambda(dx)$ for $\lambda \in M(E)$ and $\varphi \in C_b(E)$.

A family $\mathcal{F} \subseteq C_b(E)$ is said to be *separating* if, whenever $\langle \varphi, \lambda \rangle = \langle \varphi, \lambda' \rangle$ for all $\varphi \in \mathcal{F}$, and some $\lambda, \lambda' \in M(E)$, we necessarily have $\lambda = \lambda'$. Since E is Polish, from the Portmanteau theorem (see, *e.g.*, [40, Theorem 1.1.1]), the set of uniformly continuous functions, for any metric equivalent to d , is separating. From Tychonoff's embedding theorem (see, *e.g.*, [43, Theorem 17.8]), $C_b(E)$ is also separable. Therefore, there exists a countable and separating family $\mathcal{F}_E = \{\varphi_k, k \in \mathbb{N}\}$ subset of $C_b(E)$ such that the function $E \ni x \mapsto 1$ belongs to \mathcal{F}_E and $\|\varphi_k\|_\infty := \sup_E |\varphi_k| \leq 1$ for all $k \in \mathbb{N}$. We use this setting to define the following distance

$$\mathbf{d}_E(\lambda, \lambda') = \sum_{\varphi_k \in \mathcal{F}_E} \frac{1}{2^k} |\langle \varphi_k, \lambda \rangle - \langle \varphi_k, \lambda' \rangle|,$$

for $\lambda, \lambda' \in M(E)$. As in [40, Theorem 1.1.2], this distance \mathbf{d}_E induces on $M(E)$ the weak* topology. Whenever $E = \mathbb{R}^d$, we adjust this metric to take into account useful differential properties. Let $\mathcal{F}_{\mathbb{R}^d}$ be taken as a subset of $C_b^2(\mathbb{R}^d)$, the set of real functions with bounded, continuous derivatives over \mathbb{R}^d up to order two. We can take this set as separating since C^2 is dense in C^0 for local uniform convergence (application of [20, Theorem 8.14]). We define the distance

$$\mathbf{d}_{\mathbb{R}^d}(\lambda, \lambda') = \sum_{\varphi_k \in \mathcal{F}_{\mathbb{R}^d}} \frac{1}{2^k q_k} |\langle \varphi_k, \lambda \rangle - \langle \varphi_k, \lambda' \rangle|, \quad (2.1)$$

with $q_k = \max\{1, \|D\varphi_k\|_\infty, \|D^2\varphi_k\|_\infty\}$, and D and D^2 denote gradient and Hessian.

Atomic measures We write $\mathcal{N}^n[E]$ for the space of atomic measures in E where each atom has a mass multiple of $1/n$, *i.e.*,

$$\mathcal{N}^n[E] := \left\{ \frac{k_i}{n} \sum_{i \in V} \delta_{x_i} : k_i \in \mathbb{N}, x_i \in E \text{ for } i \in V, V \subseteq \mathbb{N}, |V| < \infty \right\},$$

where $|V|$ is the cardinal of the set V . For $n \geq 1$, $\mathcal{N}^n[E]$ is a weakly* closed subset of $M(E)$. If E is a Polish space, *e.g.* a Euclidean space, $\cup_{n \in \mathbb{N}} \mathcal{N}^n[E]$ is dense in $M(E)$. First, the result is shown for probability measures. For the fundamental theorem of simulation (cf [34, Theorem 1.2]), there exists a Borel function $\varphi_\lambda : [0, 1] \rightarrow E$, for any $\lambda \in \mathcal{P}(E)$, such that $\lambda = \text{Leb}_{[0,1]} \circ \varphi_\lambda^{-1}$, where $\text{Leb}_{[0,1]} \circ \varphi_\lambda^{-1}$ denotes the image measure by φ_λ^{-1} of the Lebesgue measure on the unit interval. With Glivenko–Cantelli theorem (cf [34, Theorem 4.1]) we approximate the Lebesgue measure on the unit interval by probability measures $\lambda_n \in \mathcal{N}^n[E]$. We get the final result decomposing each finite measure λ as a probability measure times its total mass $\lambda([0, 1])$ and using for the latter the approximation $\lfloor n\lambda(E) \rfloor / n$, where $\lfloor \cdot \rfloor$ denotes the integer part of a real number.

State space Fix a finite time horizon $T > 0$. Let $\mathbf{D}^d = \mathbb{D}([0, T]; M(\mathbb{R}^d))$ be the set of càdlàg functions (right continuous with left limits) from $[0, T]$ to $M(\mathbb{R}^d)$. We endow this space with Skorohod metric $d_{\mathbf{D}^d}$ associated with the metric $\mathbf{d}_{\mathbb{R}^d}$, which makes it complete (see, e.g., [3, Theorem 14.2]). For $\mathbb{P} \in \mathcal{P}(\mathbf{D}^d)$, $\mathbb{P}_t \in \mathcal{P}(M(\mathbb{R}^d))$ denotes the time- t marginal of \mathbb{P} , i.e., the image of \mathbb{P} under the map $\mathbf{D}^d \ni \mu \mapsto \mu_t \in M(\mathbb{R}^d)$. Denote $\mathbf{D}^{n,d} = \mathbb{D}([0, T]; \mathcal{N}^n[\mathbb{R}^d])$, a closed subset of \mathbf{D}^d .

We consider the canonical space \mathbf{D}^d , with μ its canonical process, and $\mathbb{F}^\mu = \{\mathcal{F}_s^\mu\}_s$ the filtration generated by μ . Let a compact subset A of \mathbb{R}^m representing the set of actions, and \mathcal{A} the set of $\{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_s\}_s$ -predictable processes from $[0, T] \times \mathbb{R}^d$ to A . Finally, for a given $\mathbb{P} \in \mathcal{P}(\mathbf{D}^d)$ and a stopping time τ , we denote $(\mathbb{P}_\omega, \omega \in \mathbf{D}^d)$ a regular conditional probability distribution of \mathbb{P} given \mathcal{F}_τ (see, e.g., [40, Chapter 1.1]).

Definition We consider the following assumptions. We are given dimensions $d, d' \in \mathbb{N}$ and the following bounded continuous functions

$$(b, \sigma, \gamma) : \mathbb{R}^d \times M(\mathbb{R}^d) \times A \rightarrow \mathbb{R}^d \times \mathbb{R}^{d \times d'} \times \mathbb{R}_+.$$

Suppose b and σ are Lipschitz uniformly in a , i.e., there exist $L > 0$ such that

$$|b(x, \lambda, a) - b(x', \lambda', a)| + |\sigma(x, \lambda, a) - \sigma(x', \lambda', a)| \leq L(|x - x'| + \mathbf{d}_{\mathbb{R}^d}(\lambda, \lambda')),$$

for any $x, x' \in \mathbb{R}^d$, $\lambda, \lambda' \in M(\mathbb{R}^d)$, and $a \in A$.

In [33], various equivalent descriptions for branching particle systems are presented. Among these, we opt to use the formalism of weak controls as it involves less cumbersome notation. We adopt the same perspective to establish a definition for controlled superprocesses, which will subsequently facilitate the proof of their existence as a weak limit of the aforementioned branching processes.

Let L be the generator defined by

$$L\varphi(x, \lambda, a) = b(x, \lambda, a)^\top D\varphi(x) + \frac{1}{2} \text{Tr} \left(\sigma \sigma^\top(x, \lambda, a) D^2 \varphi(x) \right),$$

for $\varphi \in C_b^2(\mathbb{R}^d)$. Let also \mathcal{L} be the generator defined by

$$\mathcal{L}F_\varphi(x, \lambda, a) = F'(\langle \varphi, \lambda \rangle) L\varphi(x, \lambda, a) + \frac{1}{2} F''(\langle \varphi, \lambda \rangle) \gamma(x, \lambda, a) \varphi^2(x),$$

where F_φ denotes the the cylindrical function $F_\varphi = F(\langle \varphi, \cdot \rangle)$, for $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$. For simplicity, we write $F'_\varphi(\lambda)$ for $F'(\langle \varphi, \lambda \rangle)$ and $F''_\varphi(\lambda)$ for $F''(\langle \varphi, \lambda \rangle)$.

We can now define the *controlled superprocess*.

Definition 2.1. Fix $(t, \lambda) \in [0, T] \times M(\mathbb{R}^d)$. We say that $(\mathbb{P}, \alpha) \in \mathcal{P}(\mathbf{D}^d) \times \mathcal{A}$ is a Controlled superprocess, and we denote $(\mathbb{P}, \alpha) \in \mathcal{R}_{(t, \lambda)}$, if $\mathbb{P}(\mu_t = \lambda) = 1$ and the process

$$M_s^{F_\varphi} = F_\varphi(\mu_s) - \int_t^s \int_{\mathbb{R}^d} \mathcal{L}F_\varphi(x, \mu_u, \alpha_u(x)) \mu_u(dx) du \quad (2.2)$$

is a (\mathbb{P}, \mathbb{F}) -martingale with quadratic variation

$$[M^{F_\varphi}]_s = \int_t^s (F'_\varphi(\mu_u))^2 \int_{\mathbb{R}^d} \gamma(x, \mu_u, \alpha_u(x)) \varphi^2(x) \mu_u(dx) du \quad (2.3)$$

for any $F \in C_b^2(\mathbb{R})$, $\varphi \in C_b^2(\mathbb{R}^d)$, and $s \geq t$.

2.2 Existence and uniqueness

We first focus on the uniqueness in law for the controlled superprocesses. Using *Doob's functional representation theorem* (see, e.g., [23, Lemma 1.13]), we remark that a $\{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_s\}_s$ -predictable process α from $[0, T] \times \mathbb{R}^d$ to A boils down to be a predictable map \mathbf{a} such that

$$\mathbf{a} : [0, T] \times \mathbb{R}^d \times \mathbf{D}^d \rightarrow A \quad (2.4)$$

$$\left(s, x, (\mu_u)_{u \in [0, T]} \right) \mapsto \mathbf{a} \left(s, x, (\mu_u)_{u \in [0, s]} \right) = \alpha_s(x). \quad (2.5)$$

As [33, Section 4], we generalize the martingale problem (2.2) to a domain that characterizes the law of processes in $[0, T] \times \mathbf{D}^d$. To do so, we first introduce the domain of cylindrical functions $\mathcal{D} \subseteq C^0([0, T] \times \mathbf{D}^d)$ as the set of $F_{(f_1, \dots, f_p)} : [0, T] \times \mathbf{D}^d \rightarrow \mathbb{R}$ of the form

$$F_{(f_1, \dots, f_p)}(s, \mathbf{x}) = F \left(\langle f_1(s \wedge t_1, \cdot), \mathbf{x}_{s \wedge t_1} \rangle, \dots, \langle f_p(s \wedge t_p, \cdot), \mathbf{x}_{s \wedge t_p} \rangle \right), \quad (s, \mathbf{x}) \in \mathbb{R}_+ \times \mathbf{D}^d, \quad (2.6)$$

for some $p \geq 1$, $t_1, \dots, t_p \in [0, T]$, $F \in C_b^2(\mathbb{R}^p)$, and $f_1, \dots, f_p \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$. For $f \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$, we use the notation $Lf(s, x, \lambda, a) = Lf(s, \cdot)(x, \lambda, a)$. For a measurable function $\beta : [0, T] \times \mathbb{R}^d \rightarrow A$, we then define the operator \mathbb{L}^β on \mathcal{D} by

$$\begin{aligned} \mathbb{L}^\beta F_{(f_1, \dots, f_p)}(s, \mathbf{x}) &= DF \left(\langle f_1(s \wedge t_1, \cdot), \mathbf{x}_{s \wedge t_1} \rangle, \dots, \langle f_p(s \wedge t_p, \cdot), \mathbf{x}_{s \wedge t_p} \rangle \right)^\top \mathfrak{L}^\beta \mathbf{f}(s, \mathbf{x}) \\ &\quad + \frac{1}{2} \text{Tr} \left(\left\langle \mathfrak{S}^\beta \mathbf{f} (\mathfrak{S}^\beta \mathbf{f})^\top (s, \cdot), \mathbf{x}_s \right\rangle D^2 F \left(\langle f_1(s \wedge t_1, \cdot), \mathbf{x}_{s \wedge t_1} \rangle, \dots, \langle f_p(s \wedge t_p, \cdot), \mathbf{x}_{s \wedge t_p} \rangle \right) \right) \end{aligned}$$

with $t_0 = 0$, where

$$\begin{aligned} \mathfrak{L}^\beta \mathbf{f}(s, \mathbf{x}) &:= \begin{pmatrix} \mathbb{1}_{s \leq t_1} \int_{\mathbb{R}^d} \partial_t f_1(s, x) + Lf_1(s, x, \mathbf{x}_s, \beta(s, x)) \mathbf{x}_s(dx) \\ \vdots \\ \mathbb{1}_{s \leq t_p} \int_{\mathbb{R}^d} \partial_t f_p(s, x) + Lf_p(s, x, \mathbf{x}_s, \beta(s, x)) \mathbf{x}_s(dx) \end{pmatrix}, \\ \mathfrak{S}^\beta \mathbf{f}(s, x, \mathbf{x}) &:= \begin{pmatrix} \mathbb{1}_{s \leq t_1} f_1(s, x) \sqrt{\gamma(x, \mathbf{x}_s, \beta(s, x))} \\ \vdots \\ \mathbb{1}_{s \leq t_p} f_p(s, x) \sqrt{\gamma(x, \mathbf{x}_s, \beta(s, x))} \end{pmatrix}, \end{aligned}$$

for $(s, x, \mathbf{x}) \in [0, T] \times \mathbb{R}^d \times \mathbf{D}^d$. Following the language of [19], we call the graph of \mathcal{D} the *full generator* \mathbb{G} , with

$$\mathbb{G} := \{(g, \mathbb{L}g) : g \in \mathcal{D}\}. \quad (2.7)$$

We define the domain $\mathcal{D}^T \subseteq C^0(M(\mathbb{R}^d))$ of the functions

$$F_{(f_1, \dots, f_p)}(\lambda) = F \left(\langle f_1, \lambda \rangle, \dots, \langle f_p, \lambda \rangle \right), \quad \lambda \in M(\mathbb{R}^d), \quad (2.8)$$

for some $p \geq 1$, $F \in C_b^2(\mathbb{R}^p)$, and $f_1, \dots, f_p \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$. These functions are embedded in \mathcal{D} when we consider functions as in (2.6) such that f_i does not depend on s and $t_i = T$, for $i = 1, \dots, p$. Therefore, with abuse of notation, we say that \mathbb{L} acts on \mathcal{D}^T with the obvious adjustments.

Considering the canonical process $\mu \in \mathbf{D}^d$, we have that, if $(\mathbb{P}, \alpha) \in \mathcal{R}_{(t, \lambda)}$ the process

$$\bar{M}_s^h := h(\mu_{s \wedge \cdot}) - \int_t^s \mathbb{L}^\alpha h(\mu_{u \wedge \cdot}) du, \quad t \leq u \leq T, \quad (2.9)$$

is a (\mathbb{P}, \mathbb{F}) -martingale with quadratic variation equal to

$$\begin{aligned} [\bar{M}^h]_s &:= \int_t^s \text{Tr} \left(\left\langle \mathfrak{S}^\alpha \mathbf{f} (\mathfrak{S}^\alpha \mathbf{f})^\top (s, \cdot), \mu_s \right\rangle \right. \\ &\quad \left. DF(DF)^\top \left(\langle f_1(s \wedge t_1, \cdot), \mu_{s \wedge t_1} \rangle, \dots, \langle f_p(s \wedge t_p, \cdot), \mu_{s \wedge t_p} \rangle \right) \right) du, \end{aligned} \quad (2.10)$$

for any $h = F_{(f_1, \dots, f_p)} \in \mathcal{D}$. Therefore, we can finally prove uniqueness in law for the controlled superprocesses as follows.

Proposition 2.1. *Fix $(t, \lambda) \in [0, T] \times M(\mathbb{R}^d)$ and $\alpha \in \mathcal{A}$. There exists at most one $\mathbb{P}^{t, \lambda, \alpha} \in \mathcal{P}(\mathbf{D}^d)$ such that $(\mathbb{P}^{t, \lambda, \alpha}, \alpha) \in \mathcal{R}_{(t, \lambda)}$.*

Proof. The proof is based on [33, Proposition 4.5] and [33, Theorem 4.1], whose proofs are the same as in our setting with respect to the operator \mathbb{L} . \square

We can now consider the existence problem. Existence of solutions to martingale problems is usually proven as a weak limit of solutions to well-posed problems. Superprocesses, in particular, arise as scaling limits of branching particle systems (see, e.g., [11, 18, 39]).

For $n \in \mathbb{N}$, let \mathcal{L}^n be the generator defined on the cylindrical functions $F_\varphi = F(\langle \varphi, \cdot \rangle)$, for $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$, as

$$\begin{aligned} \mathcal{L}^n F_\varphi(x, \lambda, a) &= F'(\langle \varphi, \lambda \rangle) L\varphi(x, \lambda, a) + \frac{1}{2n} F''(\langle \varphi, \lambda \rangle) |D\varphi(x)\sigma(x, \lambda, a)|^2 \\ &\quad + \gamma(x, \lambda, a) n^2 \left(F \left(\langle \varphi, \lambda \rangle - \frac{1}{n} \varphi(x) \right) \frac{1}{2} + F \left(\langle \varphi, \lambda \rangle + \frac{1}{n} \varphi(x) \right) \frac{1}{2} - F_\varphi(\lambda) \right). \end{aligned} \quad (2.11)$$

Definition 2.2. *Fix $(t, \lambda_n) \in [0, T] \times \mathcal{N}^n[\mathbb{R}^d]$. We say that $(\mathbb{P}, \alpha) \in \mathcal{P}(\mathbf{D}^d) \times \mathcal{A}$ is a n -rescaled branching diffusion, and we denote $(\mathbb{P}, \alpha) \in \mathcal{R}_{(t, \lambda_n)}^n$, if $\mathbb{P}(\mu_t = \lambda_n) = 1$ and the process*

$$M_s^{F_\varphi, n} = F_\varphi(\mu_s) - \int_t^s \int_{\mathbb{R}^d} \mathcal{L}^n F_\varphi(x, \mu_u, \alpha_u(x)) \mu_u(dx) du \quad (2.12)$$

is a (\mathbb{P}, \mathbb{F}) -martingale with quadratic variation

$$\begin{aligned} [M^{F_\varphi, n}]_s &= \int_t^s (F'_\varphi(\mu_u))^2 \int_{\mathbb{R}^d} \left(\frac{1}{n} |D\varphi(x)\sigma(x, \mu_u, \alpha_u(x))|^2 + \right. \\ &\quad \left. \gamma(x, \mu_u, \alpha_u(x)) \varphi^2(x) \right) \mu_u(dx) du \end{aligned} \quad (2.13)$$

for any $F \in C_b^2(\mathbb{R})$, $\varphi \in C_b^2(\mathbb{R}^d)$, and $s \geq t$.

Proposition 2.2. *Fix $n \geq 1$ and $(t, \lambda_n) \in [0, T] \times \mathcal{N}^n[\mathbb{R}^d]$. For $\alpha \in \mathcal{A}$, there exists a $\mathbb{P}^{t, \lambda_n, \alpha; n} \in \mathcal{P}(\mathbf{D}^d)$ such that $(\mathbb{P}^{t, \lambda_n, \alpha; n}, \alpha) \in \mathcal{R}_{(t, \lambda_n)}^n$.*

Proof. Fix a $\alpha \in \mathcal{A}$. For $n = 1$, the existence of $\mathbb{P}^{t, \lambda_1, \alpha; 1} \in \mathcal{P}(\mathbf{D}^d)$ such that $(\mathbb{P}^{t, \lambda_1, \alpha; 1}, \alpha) \in \mathcal{R}_{(t, \lambda_1)}^1$, for any $(t, \lambda^1) \in [0, T] \times \mathcal{N}^1[\mathbb{R}^d]$, is discussed in [33, Section 4]. This is done for general horizons $T > 0$. Existence of $\mathbb{P}^{t, \lambda_n, \alpha; n} \in \mathcal{P}(\mathbf{D}^d)$ such that $(\mathbb{P}^{t, \lambda_n, \alpha; n}, \alpha) \in \mathcal{R}_{(t, \lambda_n)}^n$, for any $(t, \lambda_n) \in [0, T] \times \mathcal{N}^n[\mathbb{R}^d]$ stems from this. We denote $\bar{\mathcal{R}}_{(t, \lambda^1)}^S$ the set of 1-rescaled branching diffusions defined in the interval $[0, S]$, for $S > 0$. For $n \in \mathbb{N}$, we define, on the interval $[0, nT]$, the control α^n such that

$$\alpha_s^n = \alpha_{s/n}, \quad \text{for } s \in [0, nT].$$

Fix $(t, \lambda_n) \in [0, T] \times \mathcal{N}^n[\mathbb{R}^d]$. From the previous result, we have the existence of $\mathbb{P}^{n,1} \in \mathcal{P}(\mathbb{D}([0, nT]; M(\mathbb{R}^d)))$ such that $(\mathbb{P}^{n,1}, \alpha^n) \in \mathcal{R}_{(nt, n\lambda_n)}^{nT}$. Define the map \mathfrak{X}^n such that

$$\begin{aligned} \mathfrak{X}^n : \mathbb{D}([0, T]; \mathcal{N}^n[\mathbb{R}^d]) &\rightarrow \mathbb{D}([0, nT]; \mathcal{N}^1[\mathbb{R}^d]) \\ (\mu_s)_{s \in [0, T]} &\mapsto (n\mu_{s/n})_{s \in [0, nT]}. \end{aligned}$$

As in [11, 18, 39], we have that $\mathbb{P}^{t, \lambda_n, \alpha; n} := \mathbb{P}^{n,1} \circ (\mathfrak{X}^n)^{-1} \in \mathcal{P}(\mathbf{D}^d)$ is such that $(\mathbb{P}^n, \alpha) \in \mathcal{R}_{(t, \lambda_n)}^n$. \square

Now we have all the ingredients to give existence for controlled superprocesses.

Proposition 2.3. *Fix $(t, \lambda) \in [0, T] \times M(\mathbb{R}^d)$. For $\alpha \in \mathcal{A}$, there exists a unique $\mathbb{P} \in \mathcal{P}(\mathbf{D}^d)$ such that $(\mathbb{P}, \alpha) \in \mathcal{R}_{(t, \lambda)}$.*

Proof. Fix a $\{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_s\}_s$ -predictable process α from $[0, T] \times \mathbb{R}^d$ to A . Consider a sequence $(\lambda_n)_{n \in \mathbb{N}}$ such that $\lambda_n \rightarrow \lambda$ weakly* and $\lambda_n \in \mathcal{N}^n[\mathbb{R}^d]$. From Proposition 2.2, there exists $\mathbb{P}^n \in \mathcal{P}(\mathbf{D}^d)$ such that $(\mathbb{P}^n, \alpha) \in \mathcal{R}_{(t, \lambda_n)}^n$. Our goal is to show $(\mathbb{P}^n)_{n \in \mathbb{N}}$ converges weakly to some $\mathbb{P} \in \mathcal{P}(\mathbf{D}^d)$ and that $(\mathbb{P}, \alpha) \in \mathcal{R}_{(t, \lambda)}$.

We define the projection π_φ as

$$\pi_\varphi : M(\mathbb{R}^d) \ni \lambda \mapsto \langle \varphi, \lambda \rangle \in \mathbb{R}$$

for any $f \in C^0(\mathbb{R}^d)$. Clearly, the weak* topology is the weakest topology for which the mappings π_{φ_k} are continuous, for $\{\varphi_k\}_{k \geq 1}$ dense in $C_0(\mathbb{R}^d)$. Moreover, under \mathbb{P}^n , for the semimartingale $\langle \varphi_k, \mu_\cdot \rangle$, we define the predictable finite variation process as $V_s^n(\varphi)$ and the increasing process of the martingale part as $I_s^n(\varphi)$ for $k \geq 1$. From equations (2.12) and (2.13), we have

$$V_s^n(\varphi_k) = \int_t^s \int_{\mathbb{R}^d} L\varphi_k(x, \mu_u, \alpha_u(x)) \mu_u(dx) du, \quad (2.14)$$

$$I_s^n(\varphi_k) = \int_t^s \int_{\mathbb{R}^d} \left(\frac{1}{n} |D\varphi_k(x) \sigma(x, \mu_u, \alpha_u(x))|^2 + \gamma(x, \mu_u, \alpha_u(x)) \varphi_k^2(x) \right) \mu_u(dx) du. \quad (2.15)$$

We can now verify conditions (i) and (ii) of [39, Theorem 2.3] to prove that $\{\mathbb{P}^n\}_n$ is tight. This means proving that

- (i) $(\mathbb{P}^n \circ \pi_{\varphi_k}^{-1})_{n \geq 1}$ is tight for $k \geq 1$;
- (ii) $V_s^n(\varphi_k)$ and $I_s^n(\varphi_k)$ satisfy the following condition of Aldous for any $k \geq 1$: for each stopping time τ we can find a sequence δ_n such that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$ and such that

$$\limsup_n \mathbb{E}^{\mathbb{P}^n} [|V_{\tau+\delta_n}^n(\varphi) - V_\tau^n(\varphi)|] = 0, \quad \limsup_n \mathbb{E}^{\mathbb{P}^n} [|I_{\tau+\delta_n}^n(\varphi) - I_\tau^n(\varphi)|] = 0. \quad (2.16)$$

From (2.12), we have that $\langle 1, \mu_\cdot \rangle$ is a \mathbb{P}^n -martingale for any $n \geq 1$. Therefore, for any $n \geq 1$,

$$\mathbb{P}^n \left(\sup_{s \in [t, T]} \langle 1, \mu_s \rangle > K \right) \leq \frac{1}{K} \mathbb{E}^{\mathbb{P}^n} [\langle 1, \mu_t \rangle] = \frac{1}{K} \langle 1, \lambda_n \rangle.$$

Since $\lim_n \langle 1, \lambda_n \rangle = \langle 1, \lambda \rangle$, we obtain that $\sup_n \mathbb{P}^n \left(\sup_{s \in [t, T]} \langle 1, \mu_s \rangle > K \right)$ tends to 0 when K tends to infinity. $(\mathbb{P}^n \circ \pi_{\varphi_k}^{-1})_{n \geq 1}$ is also tight for $k \geq 1$, since each function of $C_0(\mathbb{R}^d)$ is bounded. Therefore, (i) is satisfied.

Fix $\varphi \in \{1\} \cup \{\varphi_k : k \geq 1\}$, a stopping time τ taking values in $[t, T]$, and $\delta_n > 0$. Using (2.14) and (2.15), we have

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^n} [|V_{\tau+\delta_n}^n(\varphi) - V_\tau^n(\varphi)|] &\leq \mathbb{E}^{\mathbb{P}^n} \left[\int_\tau^{\tau+\delta_n} \int_{\mathbb{R}^d} |L\varphi(x, \mu_u, \alpha_u(x))| \mu_u(dx) du \right] \\ &\leq \delta_n \mathbb{E}^{\mathbb{P}^n} [\langle 1, \mu_\tau \rangle] \|L\varphi\|_\infty = \delta_n \langle 1, \lambda_n \rangle \|L\varphi\|_\infty, \end{aligned}$$

where the last equality comes from the martingale property. By the same arguments, we also have

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^n} [|I_{\tau+\delta_n}^n(\varphi) - I_\tau^n(\varphi)|] \\ = \mathbb{E}^{\mathbb{P}^n} \left[\int_\tau^{\tau+\delta_n} \int_{\mathbb{R}^d} \left(\frac{1}{n} |D\varphi(x)\sigma(x, \mu_u, \alpha_u(x))|^2 + \gamma(x, \mu_u, \alpha_u(x))\varphi^2(x) \right) \mu_u(dx) du \right] \\ \leq \delta_n \langle 1, \lambda_n \rangle (\|D\varphi\sigma\|_\infty^2 + 2\bar{\gamma}\|\varphi\|_\infty^2). \end{aligned}$$

Therefore, if $\lim_n \delta_n = 0$, we get (2.16), which gives that $(\mathbb{P}_n)_{n \geq 1}$ is tight in \mathbf{D}^d .

To conclude, we take a sequence $(\mathbb{P}_n)_{n \geq 1}$ converging to a probability measure $\mathbb{P} \in \mathcal{P}(\mathbf{D}^d)$ and prove that $(\mathbb{P}, \alpha) \in \mathcal{R}_{(t, \lambda)}$. To do that, we focus on the convergence of \mathcal{L}_n . For $(x, \nu, a) \in \mathbb{R}^d \times M(\mathbb{R}^d)$, the third term in the expression of \mathcal{L}_n in (2.11) is equal to

$$W_n(x, \nu, a) = \gamma(x, \nu, a) n^2 \left(F \left(\langle \varphi, \nu \rangle - \frac{1}{n} \varphi(x) \right) \frac{1}{2} + F \left(\langle \varphi, \nu \rangle + \frac{1}{n} \varphi(x) \right) \frac{1}{2} - F(\langle \varphi, \nu \rangle) \right).$$

Using Taylor's development with Lagrange reminder, we have

$$W_n(x, \nu, a) = \gamma(x, \nu, a) \frac{F''(\langle \varphi, \nu \rangle + z_1^n) + F''(\langle \varphi, \nu \rangle + z_2^n)}{2},$$

with z_1^n (resp. z_2^n) a point in $\{h\langle \varphi, \nu \rangle + (1-h)\varphi(x)/n : h \in [0, 1]\}$ (resp. $\{h\langle \varphi, \nu \rangle - (1-h)\varphi(x)/n : h \in [0, 1]\}$). Since γ is bounded, we have $W(x, \nu, a) = \lim_n W_n(x, \nu, a) = F''(\nu)\gamma(x, \nu, a)\varphi^2(x)$ for any (x, ν, a) . We can now prove that $F_\varphi(\mu) - \int_t^\tau \int_{\mathbb{R}^d} \mathcal{L}F_\varphi(x, \mu_u, \alpha_u(x))\mu_u(dx)du$ is a \mathbb{P} -martingale, *i.e.*, for each stopping time τ taking value in $[t, T]$,

$$\mathbb{E}^{\mathbb{P}} \left[F_\varphi(\mu_\tau) - F_\varphi(\mu_t) - \int_t^\tau \int_{\mathbb{R}^d} \mathcal{L}F_\varphi(x, \mu_u, \alpha_u(x))\mu_u(dx)du \right] = 0.$$

We have

$$\lim_n \mathbb{E}^{\mathbb{P}^n} [F_\varphi(\mu_\tau) - F_\varphi(\mu_t)] = \mathbb{E}^{\mathbb{P}} [F_\varphi(\mu_\tau) - F_\varphi(\mu_t)],$$

and

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[\int_t^\tau \int_{\mathbb{R}^d} \mathcal{L}F_\varphi(x, \mu_u, \alpha_u(x))\mu_u(dx)du \right] &- \mathbb{E}^{\mathbb{P}^n} \left[\int_t^\tau \int_{\mathbb{R}^d} \mathcal{L}_n F_\varphi(x, \mu_u, \alpha_u(x))\mu_u(dx)du \right] \\ = \mathbb{E}^{\mathbb{P}} \left[\int_t^\tau \int_{\mathbb{R}^d} \mathcal{L}F_\varphi(x, \mu_u, \alpha_u(x))\mu_u(dx)du \right] &- \mathbb{E}^{\mathbb{P}^n} \left[\int_t^\tau \int_{\mathbb{R}^d} \mathcal{L}F_\varphi(x, \mu_u, \alpha_u(x))\mu_u(dx)du \right] \\ &+ \mathbb{E}^{\mathbb{P}^n} \left[\int_t^\tau \int_{\mathbb{R}^d} (\mathcal{L} - \mathcal{L}_n)F_\varphi(x, \mu_u, \alpha_u(x))\mu_u(dx)du \right]. \end{aligned}$$

The last term on the right side satisfies

$$\begin{aligned} & \left| \mathbb{E}^{\mathbb{P}^n} \left[\int_t^\tau \int_{\mathbb{R}^d} (\mathcal{L} - \mathcal{L}_n) F_\varphi(x, \mu_u, \alpha_u(x)) \mu_u(dx) du \right] \right| \\ &= \left| \mathbb{E}^{\mathbb{P}^n} \left[\int_t^\tau \int_{\mathbb{R}^d} \left(\frac{1}{2n} F''(\langle \varphi, \mu_u \rangle) |D\varphi(x)\sigma(x, \mu_u, \alpha_u(x))|^2 + W(x, \mu_u, \alpha_u(x)) - W_n(x, \mu_u, \alpha_u(x)) \right) \right. \right. \\ & \quad \left. \left. \mu_u(dx) du \right] \right| \leq \frac{C}{n} (1 + T \langle 1, \lambda_n \rangle), \end{aligned}$$

for a constant C which depends only on $F''_\varphi, \sigma, D\varphi, \gamma, \varphi$. Hence,

$$\begin{aligned} & \lim_n \mathbb{E}^{\mathbb{P}^n} \left[F_\varphi(\mu_\tau) - F_\varphi(\mu_t) - \int_t^\tau \int_{\mathbb{R}^d} \mathcal{L}_n F_\varphi(x, \mu_u, \alpha_u(x)) \mu_u(dx) du \right] \\ &= \mathbb{E}^{\mathbb{P}} \left[F_\varphi(\mu_\tau) - F_\varphi(\mu_t) - \int_t^\tau \int_{\mathbb{R}^d} \mathcal{L} F_\varphi(x, \mu_u, \alpha_u(x)) \mu_u(dx) du \right] = 0. \end{aligned}$$

□

Remark 2.1. *It can be observed that the proof of the aforementioned result relies on either a compact control space or bounded coefficients. In the case of unbounded action space and linear dependence on the control in the coefficients, an integrability bound similar to the following must be established*

$$\mathbb{E}^{\mathbb{P}} \left[\int_0^T \int_{\mathbb{R}^d} |\alpha_u(x)| \mu_u(dx) du \right] < \infty.$$

Applying this condition to the rescaled problem does not guarantee its retrieval in the limit.

2.3 Moment estimates

Before defining the control problem and proving it is well posed, we need to provide moment estimates for these processes. To do that, as in [33], we give the representation of the controlled superprocesses as Stochastic Differential Equations. This makes use of martingale measures, in extensions of the original space, and lets us apply the general theory of semimartingales in a more general setting. Relevant definitions and results on these objects are concisely summarised in [15] (see, *e.g.*, [42] for a monograph on the subject). We recall briefly their definition.

Definition 2.3. *Let (G, \mathcal{G}) be a Lusin space with its σ -algebra, and $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_s\}_s)$ a filtered space satisfying the usual condition, where we define \mathcal{P} the predictable σ -field. A process \mathcal{M} on $\Omega \times [0, T] \times \mathcal{G}$ is called martingale measure on G if*

- (i) $\mathcal{M}_0(E) = 0$ a.s. for any $E \in \mathcal{G}$;
- (ii) \mathcal{M}_t is a σ -finite, $L^2(\Omega)$ -valued measure for all $t \in [0, T]$;
- (iii) $(\mathcal{M}_t(E))_{t \in [0, T]}$ is an \mathbb{F} -martingale for any $E \in \mathcal{G}$.

We say that \mathcal{M} is orthogonal if the product $\mathcal{M}_t(E)\mathcal{M}_t(E')$ is a martingale for any two disjoint sets $E, E' \in \mathcal{G}$. We also say, on one hand, that it is continuous if $(\mathcal{M}_t(E))_{t \geq 0}$ is continuous, purely discontinuous, on the other hand, if $(\mathcal{M}_t(E))_{t \geq 0}$ is a purely discontinuous martingale for any $E \in \mathcal{G}$.

Proposition 2.4. *Let $(\mathbb{P}, (\alpha_s)_s) \in \mathcal{R}_{(t,\lambda)}$. There exists an extension $(\hat{\Omega} = \mathbf{D}^d \times \tilde{\Omega}, \hat{\mathcal{F}} = \mathcal{F}_T^\mu \otimes \tilde{\mathcal{F}}, \hat{\mathbb{P}} = \mathbb{P} \otimes \tilde{\mathbb{P}}, \{\hat{\mathcal{F}}_s = \mathcal{F}_s^\mu \otimes \tilde{\mathcal{F}}_s\}_s)$ of $(\mathbf{D}^d, \mathcal{F}_T^\mu, \mathbb{P}, \mathbb{F}^\mu)$, where we naturally extend μ and α , that satisfies the following properties.*

1. $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\mathbb{P}})$ is a filtered probability space supporting a continuous $\hat{\mathbb{F}}$ -martingale measures \mathcal{M} on $\hat{\Omega} \times [0, T] \times \mathcal{B}(\mathbb{R}^d)$, with intensity measure $\mu_u(dx)du$.
2. $\hat{\mathbb{P}} \circ X_t^{-1} = \lambda$.
3. We have that

$$\begin{aligned} \langle f, \mu_s \rangle &= \langle f, \mu_t \rangle + \int_t^s \int_{\mathbb{R}^d} Lf(x, \mu_u, \alpha_u(x)) \mu_u(dx) du \\ &+ \int_t^s \int_{\mathbb{R}^d} \sqrt{\gamma(x, \mu_u, \alpha_u(x))} f(x) \mathcal{M}(dx, du). \end{aligned} \quad (2.17)$$

for all $f \in C_b^\infty(\mathbb{R}^d)$ and all $s \in [t, T]$.

Proof. The representation of these processes is grounded in representation theorems for continuous martingale measures. We follow [31] and [33, Proposition 3.2] applying their construction here. \square

We can now prove the non-explosion of these processes, which will imply the well-posedness of the optimization problem.

Proposition 2.5. *Fix $(t, \lambda) \in [0, T] \times M(\mathbb{R}^d)$ and $p \in [1, 2]$. There exists a constant $C \geq 0$, depending only on T , and the coefficient of the parameters, such that*

$$\mathbb{E}^\mathbb{P} \left[\sup_{u \in [t, T]} \mathbf{d}_{\mathbb{R}^d}(\mu_u, \mathbf{0})^p \right] \leq C \mathbf{d}_{\mathbb{R}^d}(\lambda, \mathbf{0})^p, \quad (2.18)$$

for any $(\mathbb{P}, (\alpha_s)_s) \in \mathcal{R}_{(t,\lambda)}$.

Proof. Fix $(\mathbb{P}, (\alpha_s)_s) \in \mathcal{R}_{(t,\lambda)}$. We recall that $\mathbf{d}_{\mathbb{R}^d}(\mu_u, \mathbf{0}) = \sum_{\varphi_k \in \mathcal{F}_{\mathbb{R}^d}} \frac{1}{2^k q_k} |\langle \varphi_k, \mu_u \rangle|$, for any $u \in [t, T]$. We define the stopping times τ_N as

$$\tau_N = \inf \{u \geq t : \langle \mathbf{1}, \mu_u \rangle \geq N\},$$

and denote $\mu_s^N := \mu_{\tau_N \wedge s}$, for $N \geq 1$. Proposition 2.4 implies that there exists an extension of Ω where μ can be satisfies (2.17) on the stochastic interval $[t, \tau_N]$. Such SDE is driven by \mathcal{M}^N , a orthogonal continuous martingale measure in $[0, T] \times \mathbb{R}^d$, with the intensity measure $\mu_s(dx) \mathbf{1}_{s \leq \tau_N} ds$. Applying (2.17) to φ_k , we have

$$\begin{aligned} \langle \varphi_k, \mu_s^N \rangle &= \langle \varphi_k, \lambda \rangle + \int_t^s \int_{\mathbb{R}^d} L\varphi_k(x, \mu_r, \alpha_r(x)) \mu_r(dx) \mathbf{1}_{r \leq \tau_N} dr + \\ &+ \int_t^s \int_{\mathbb{R}^d} \sqrt{\gamma(x, \mu_r, \alpha_r(x))} \varphi_k(x) \mathcal{M}^N(dx, dr). \end{aligned}$$

for $s \geq t$, and $k \in \mathbb{N}$. Applying Young's inequality, there is a constant C (which may change from line to line) such that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[\sup_{s \in [t, T]} |\langle \varphi_k, \mu_s^N \rangle|^p \right] &\leq C |\langle \varphi_k, \lambda \rangle|^p \\ &+ C \mathbb{E}^{\mathbb{P}} \left[\sup_{s \in [t, T]} \left| \int_t^s \int_{\mathbb{R}^d} L \varphi_k(x, \mu_r, \alpha_r(x)) \mu_r(dx) \mathbf{1}_{r \leq \tau_N} dr \right|^p \right] \\ &+ C \mathbb{E}^{\mathbb{P}} \left[\sup_{s \in [t, T]} \left| \int_t^s \int_{\mathbb{R}^d} \sqrt{\gamma(x, \mu_r, \alpha_r(x))} \varphi_k(x) \mathcal{M}^N(dx, dr) \right|^p \right]. \end{aligned}$$

Recalling $q_k = \max\{1, \|D\varphi_k\|_{\infty}, \|D^2\varphi_k\|_{\infty}\}$, we have

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[\sup_{s \in [t, T]} |\langle \varphi_k, \mu_s^N \rangle|^p \right] &\leq C |\langle \varphi_k, \lambda \rangle|^p + C q_k^p \mathbb{E}^{\mathbb{P}} \left[\sup_{s \in [t, T]} \left(\int_t^s |\langle \varphi_k, \mu_u^N \rangle| du \right)^p \right] \\ &+ C q_k^p \mathbb{E}^{\mathbb{P}} \left[\int_t^T \langle 1, \mu_u^N \rangle^p du \right] \\ &+ C \mathbb{E}^{\mathbb{P}} \left[\sup_{s \in [t, T]} \left| \int_t^s \int_{\mathbb{R}^d} \sqrt{\gamma(x, \mu_r, \alpha_r(x))} \varphi_k(x) \mathcal{M}^N(dx, dr) \right|^p \right]. \end{aligned}$$

From Jensen's and Burkholder-Davis-Gundy's inequalities (see, *e.g.*, cite[Chapter VII, Theorem 92]Dellacherie:Meyer:B), and recalling that $\|\varphi_k\|_{\infty} \leq 1$, we get

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[\sup_{s \in [t, T]} |\langle \varphi_k, \mu_s^N \rangle|^p \right] &\leq C |\langle \varphi_k, \lambda \rangle|^p + C q_k^p \mathbb{E}^{\mathbb{P}} \left[\int_t^T |\langle \varphi_k, \mu_u^N \rangle|^p du \right] + C q_k^p \mathbb{E}^{\mathbb{P}} \left[\int_t^T \langle 1, \mu_u^N \rangle^p du \right] \\ &\leq C q_k^p |\langle \varphi_k, \lambda \rangle|^p + C q_k^p \mathbb{E}^{\mathbb{P}} \left[\int_t^T \sup_{s \in [t, u]} |\langle \varphi_k, X_s^N \rangle|^p du \right] \\ &\quad + C q_k^p \mathbb{E}^{\mathbb{P}} \left[\int_t^T \sup_{s \in [t, u]} \langle 1, X_s^N \rangle^p du \right]. \end{aligned}$$

Finally, multiplying by $\left(\frac{1}{2^k q_k}\right)^p$, summing over $k \in \mathbb{N}$ and applying the monotone convergence theorem, in addition to the fact that function equal to 1 is in $\mathcal{F}_{\mathbb{R}^d}$, we have

$$\mathbb{E}^{\mathbb{P}} \left[\sup_{u \in [t, T]} \mathbf{d}_{\mathbb{R}^d}(X_u^N, \mathbf{O})^p \right] \leq C \mathbf{d}_{\mathbb{R}^d}(\lambda, \mathbf{O}) + C \mathbb{E}^{\mathbb{P}} \left[\int_t^T \sup_{s \in [t, u]} \mathbf{d}_{\mathbb{R}^d}(X_s^N, \mathbf{O})^p du \right].$$

Using Gronwall's lemma, we conclude that $\mathbb{E}^{\mathbb{P}} \left[\sup_{u \in [t, T]} \mathbf{d}_{\mathbb{R}^d}(X_u^N, \mathbf{O})^p \right] \leq C \mathbf{d}_{\mathbb{R}^d}(\lambda, \mathbf{O})^p$ for any $N \geq 1$. Applying Fatou's lemma, we obtain (2.18). \square

3 The control problem

We are given two continuous functions $\psi : \mathbb{R}^d \times M(\mathbb{R}^d) \times A \rightarrow \mathbb{R}$ and $\Psi : M(\mathbb{R}^d) \rightarrow \mathbb{R}$. We assume that there exists $C > 0$ such that

$$|\psi(x, \lambda, a)| \leq C (1 + \mathbf{d}_{\mathbb{R}^d}(\lambda, \mathbf{O})), \quad |\Psi(\lambda)| \leq C (1 + \mathbf{d}_{\mathbb{R}^d}(\lambda, \mathbf{O})^2) \quad (3.19)$$

for $(x, \lambda, a) \in \mathbb{R}^d \times M(\mathbb{R}^d) \times A$ with \mathbf{O} the measure 0.

Let J and v be respectively the cost and the value functions, defined as

$$J(t, \lambda, \alpha) = \mathbb{E}^{\mathbb{P}^{t, \lambda, \alpha}} \left[\int_t^T \int_{\mathbb{R}^d} \psi(x, \mu_s, \alpha_s(x)) \mu_s(dx) ds + \Psi(\mu_T) \right], \quad (3.20)$$

$$v(t, \lambda) = \inf_{\alpha \in \mathcal{A}} J(t, \lambda, \alpha), \quad (3.21)$$

for $(t, \lambda) \in [0, T] \times M(\mathbb{R}^d)$, and $\alpha \in \mathcal{A}$. From Proposition 2.5, the cost function J is finite for any control $\alpha \in \mathcal{A}$. Moreover, using (3.19), J is uniformly bounded from below, therefore the optimization problem that defines v is well-posed.

3.1 Weak formulation

Before establishing the Dynamic Programming Principle (DPP), we give a new description of the control problem (3.20)-(3.21). As described in [33, Section 5], we interpret the control set as a subset of finite measures. This is the so-called *weak formulation*, introduced in [14], and it allows for dealing with the control space and its topology more flexibly.

Consider $[0, T] \times \mathbb{R}^d \times A$ equipped with the σ -algebra $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A)$. Let $\mathfrak{A}^{\text{Leb}} \subseteq M([0, T] \times \mathbb{R}^d \times A)$ be the set of measures, whose projection on $[0, T]$ is the Lebesgue measure. Each $\alpha \in \mathfrak{A}^{\text{Leb}}$ can be identified with its disintegration (see, *e.g.*, [24, Corollary 1.26, Chapter 1]). In particular, we have $\bar{\alpha}(ds, dx, da) = ds \mathbf{y}_s(dx) \bar{\alpha}_s(x, da)$, for a process $(\mathbf{y}_s(dx))_s$ (resp. $(\bar{\alpha}_s(x, da))_s$) taking values in the set of functions from $[0, T]$ (resp. $[0, T] \times \mathbb{R}^d$) to $M(\mathbb{R}^d)$ (resp. $M(A)$).

We denote $\bar{\Omega} := \mathbf{D}^d \times \mathfrak{A}^{\text{Leb}}$. On $\bar{\Omega}$, let (μ, β) be the projection maps (or canonical processes), and $\mathbb{F}^{\mu, \beta} = \left\{ \mathcal{F}_s^{\mu, \beta} \right\}_s$ the filtration generated by these maps, *i.e.*,

$$\mathcal{F}_s^{\mu, \beta} = \sigma \left(\mu_s(B), \beta([0, r] \times B' \times C), \text{ for } s, r \in [0, T], B, B' \in \mathcal{B}(\mathbb{R}^d), C \in \mathcal{B}(A) \right).$$

Moreover, define the following map

$$\begin{aligned} \pi_{\mathcal{A}} : \mathbf{D}^d \times \mathcal{A} &\rightarrow \mathfrak{A}^{\text{Leb}} \\ (\mathbf{x}, \alpha) &\mapsto ds \mathbf{x}_s(dx) \delta_{\alpha_s(x)}(da). \end{aligned}$$

Definition 3.4. Fix $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$. We say that $\mathbb{P} \in \mathcal{P}(\bar{\Omega})$ is a weak control rule, and we denote $\mathbb{P} \in \mathcal{C}_{(t, \lambda)}$, if $\mathbb{P}(\mu_t = \lambda) = 1$, there exists $\alpha^{\mathbb{P}} \in \mathcal{A}$ such that $\mathbb{P}(\pi_{\mathcal{A}}(\mu, \alpha^{\mathbb{P}}) = \beta) = 1$, and the process

$$M_s^{F\varphi} = F_\varphi(\mu_s) - \int_t^s \int_{\mathbb{R}^d \times A} \mathcal{L}F_\varphi(x, \mu_u, a) \beta_s(x, da) \mu_u(dx) du$$

is a $(\mathbb{P}, \mathbb{F}^{\mu, \beta})$ -martingale with quadratic variation

$$[M^{F\varphi}]_s = \int_t^s (F'_\varphi(\mu_u))^2 \int_{\mathbb{R}^d \times A} \gamma(x, \mu_u, a) \beta_s(x, da) \varphi^2(x) \mu_u(dx) du$$

for any $F \in C_b^2(\mathbb{R})$, $\varphi \in C_b^2(\mathbb{R}^d)$, and $s \geq t$.

It is clear that each element of $\mathcal{C}_{(t, \lambda)}$ can be identified to an element of $\mathcal{R}_{(t, \lambda)}$, and viceversa. With abuse of notation, we write $J(t, \lambda, \mathbb{P})$ for $\mathbb{P} \in \mathcal{C}_{(t, \lambda)}$ to denote $J(t, \lambda, \alpha^{\mathbb{P}})$. With this description, we have

$$v(t, \lambda) = \inf_{\alpha \in \mathcal{A}} J(t, \lambda, \alpha) = \inf_{\mathbb{P} \in \mathcal{C}_{(t, \lambda)}} J(t, \lambda, \mathbb{P}).$$

In this framework, we can consider the notion of conditioning as well as concatenation on $\bar{\Omega}$. For $(t, \bar{w}) \in [0, T] \times \bar{\Omega}$, we denote

$$\begin{aligned}\mathfrak{P}_{\bar{w}}^t &:= \{\bar{\omega} : \mu_t(\bar{\omega}) = \mu_t(\bar{w})\}, \\ \mathfrak{P}_{t, \bar{w}} &:= \left\{ \bar{\omega} : (\mu_s, \mathcal{M}_s(\phi))(\bar{\omega}) = (\mu_s, \mathcal{M}_s(\phi))(\bar{w}), \text{ for } s \in [0, t], \phi \in C_b([0, T] \times \mathbb{R}^d \times A) \right\},\end{aligned}$$

where

$$\mathcal{M}_s(\phi) := \int_0^s \int_{\mathbb{R}^d \times A} \phi(s, x, a) \beta(ds, dx, da).$$

Then, for all $\bar{\omega} \in \mathfrak{P}_{\bar{w}}^t$, we define the concatenated path $\bar{w} \otimes_t \bar{\omega}$ by

$$(\mu_s, \mathcal{M}_s(\phi))(\bar{w} \otimes_t \bar{\omega}) = \begin{cases} (\mu_s, \mathcal{M}_s(\phi))(\bar{w}), & \text{for } s \in [0, t], \\ (\mu_s, \mathcal{M}_s(\phi) - \mathcal{M}_t(\phi))(\bar{\omega}) + (\mu_s, \mathcal{M}_t(\phi))(\bar{w}), & \text{for } s \in [t, T], \end{cases}$$

for all $\phi \in C_b([0, T] \times \mathbb{R}^d \times A)$.

Fix $\mathbb{P} \in \mathcal{P}(\bar{\Omega})$, and τ a $\mathbb{F}^{\mu, \beta}$ -stopping time. From [45, Proposition 1.9, Chapter 1], there is a family of regular conditional probability distribution (r.c.p.d.) $(\mathbb{P}_{\bar{\omega}})_{\bar{\omega} \in \bar{\Omega}}$ w.r.t. $\mathcal{F}_{\tau}^{\mu, \beta}$ such that the $\mathcal{F}_{\tau}^{\mu, \beta}$ -measurable probability kernel $(\mathbb{P}_{\bar{\omega}})_{\bar{\omega} \in \bar{\Omega}}$ satisfies

$$\mathbb{P}_{\bar{\omega}}(\mathfrak{P}_{\tau(\bar{\omega}), \bar{\omega}}) = 1 \quad \text{for } \mathbb{P} - \text{a.e. } \bar{\omega} \in \bar{\Omega}.$$

On the other hand, take a probability measure \mathbb{P} defined on $(\bar{\Omega}, \mathcal{F}_{\tau}^{\mu, \beta})$ and a family of probability measures $(\mathbb{Q}_{\bar{\omega}})_{\bar{\omega} \in \bar{\Omega}}$ such that $\bar{\omega} \mapsto \mathbb{Q}_{\bar{\omega}}$ is $\mathcal{F}_{\tau}^{\mu, \beta}$ -measurable and

$$\mathbb{Q}_{\bar{\omega}}(\mathfrak{P}_{\bar{\omega}}^{\tau(\bar{\omega})}) = 1 \quad \text{for } \mathbb{P} - \text{a.e. } \bar{\omega} \in \bar{\Omega}.$$

There is a unique concatenated probability measure that we denote $\mathbb{P} \otimes_{\tau} \mathbb{Q}$. defined by

$$\mathbb{P} \otimes_{\tau} \mathbb{Q}(C) := \int_{\bar{\Omega}} \mathbb{P}(d\bar{w}) \int_{\bar{\Omega}} \mathbf{1}_C(\bar{w} \otimes_{\tau(\bar{w})} \bar{\omega}) \mathbb{Q}_{\bar{\omega}}(d\bar{\omega}) \quad \text{for } C \in \mathcal{F}_T^{\mu, \beta}.$$

3.2 Measurable selection and DPP

This weak formulation has the advantage to simplify the proof of the DPP. We follow the path detailed in [16] and [17], which clarify [2, Chapter 7] in the context of stochastic control theory, generalizing it to our setting. In particular, to reach the DPP, as in [16, Theorem 4.10] and [17, Theorem 3.1], we need to show that the setting so far presented satisfies [17, Assumption 2.2], which in our setting reads as follows.

It is clear that the *full* generator \mathbb{G} as defined in (2.7) is countably generated. Combining Proposition 2.3 with the weak formulation description, $\mathcal{C}_{(t, \lambda)}$ is nonempty for all $(t, \lambda) \in [0, T] \times M(\mathbb{R}^d)$. Fix $(t, \lambda) \in [0, T] \times M(\mathbb{R}^d)$, $\mathbb{P} \in \mathcal{C}_{(t, \lambda)}$ and τ a $\mathbb{F}^{\mu, \beta}$ -stopping time taking value in $[t, T]$. Using [17, Lemma 3.2] and [17, Lemma 3.3], we obtained

- *Stability by conditioning:* There is a family of r.c.p.d. $(\mathbb{P}_{\bar{\omega}})_{\bar{\omega} \in \bar{\Omega}}$ w.r.t. $\mathcal{F}_{\tau}^{\mu, \beta}$ such that $\mathbb{P}_{\bar{\omega}} \in \mathcal{C}_{(\tau(\bar{\omega}), \mu(\bar{\omega}))}$ for \mathbb{P} -a.e. $\bar{\omega} \in \bar{\Omega}$.
- *Stability by concatenation:* Let $(\mathbb{Q}_{\bar{\omega}})_{\bar{\omega} \in \bar{\Omega}}$ be a probability kernel from $\mathcal{F}_{\tau}^{\mu, \beta}$ into $(\bar{\Omega}, \mathcal{F}_T^{\mu, \beta})$ such that $\bar{\omega} \mapsto \mathbb{Q}_{\bar{\omega}}$ is $\mathcal{F}_{\tau}^{\mu, \beta}$ -measurable, and $\mathbb{Q}_{\bar{\omega}} \in \mathcal{C}_{(\tau(\bar{\omega}), \mu(\bar{\omega}))}$ for \mathbb{P} -a.e. $\bar{\omega} \in \bar{\Omega}$. Then, $\mathbb{P} \otimes_{\tau} \mathbb{Q} \in \mathcal{C}_{(t, \lambda)}$.

These two conditions are those of [17, Assumption 2.2]. This allows to prove the following DPP.

Theorem 3.1. *We have*

$$\begin{aligned} v(t, \lambda) &= \inf_{\mathbb{P} \in \mathcal{C}(t, \lambda)} \mathbb{E}^{\mathbb{P}} \left[\int_t^\tau \int_{\mathbb{R}^d \times A} \psi(x, \mu_s, a) \beta_s(x, da) \mu_s(dx) ds + v(\tau, \mu_\tau) \right] \\ &= \inf_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^{t, \lambda, \alpha}} \left[\int_t^\tau \int_{\mathbb{R}^d} \psi(x, \mu_s, \alpha_s(x)) \mu_s(dx) ds + v(\tau, \mu_\tau) \right], \end{aligned} \quad (3.22)$$

for any $(t, \lambda) \in [0, T] \times M(\mathbb{R}^d)$, and τ stopping time taking value in $[t, T]$.

Proof. Fix $(t, \lambda) \in [0, T] \times M(\mathbb{R}^d)$, and τ to be a stopping time taking values in $[t, T]$. We have that the cost function (3.20) is continuous, thus a fortiori upper semi-analytic. Following the stability by conditioning, for any $\mathbb{P} \in \mathcal{C}(t, \lambda)$, there is $(\mathbb{P}_{\bar{\omega}})_{\bar{\omega} \in \bar{\Omega}}$ a family of r.c.p.d. w.r.t. $\mathcal{F}_\tau^{\mu, \beta}$ such that $\mathbb{P}_{\bar{\omega}} \in \mathcal{C}(\tau(\bar{\omega}), \mu(\bar{\omega}))$ for \mathbb{P} -a.e. $\bar{\omega} \in \bar{\Omega}$. Therefore, we get

$$J(\tau(\bar{\omega}), \mu_{\tau(\bar{\omega})}(\bar{\omega}), \mathbb{P}_{\bar{\omega}}) = \mathbb{E}^{\mathbb{P}_{\bar{\omega}}} \left[\int_\tau^T \int_{\mathbb{R}^d \times A} \psi(x, \mu_s, a) \beta_s(x, da) \mu_s(dx) ds + \Psi(\mu_T) \right], \quad \text{for } \mathbb{P} - \text{a.e. } \bar{\omega} \in \bar{\Omega}.$$

Since, by definition, $v(\tau(\bar{\omega}), \mu_{\tau(\bar{\omega})}(\bar{\omega})) \leq J(\tau(\bar{\omega}), \mu_{\tau(\bar{\omega})}(\bar{\omega}), \mathbb{P}_{\bar{\omega}})$, it follows from the tower property of conditional expectations that

$$\begin{aligned} J(t, \lambda, \mathbb{P}) &= \int_{\bar{\Omega}} \left(J(\tau(\bar{\omega}), \mu_{\tau(\bar{\omega})}(\bar{\omega}), \mathbb{P}_{\bar{\omega}}) + \int_t^\tau \int_{\mathbb{R}^d \times A} \psi(x, \mu_s, a) \beta_s(x, da) \mu_s(dx) ds \right) \mathbb{P}(d\bar{\omega}) \\ &\geq \mathbb{E}^{\mathbb{P}} \left[v(\tau, \mu_\tau) + \int_t^\tau \int_{\mathbb{R}^d \times A} \psi(x, \mu_s, a) \beta_s(x, da) \mu_s(dx) ds \right]. \end{aligned}$$

which provides $v(t, \lambda) \geq \inf_{\mathbb{P} \in \mathcal{C}(t, \lambda)} \mathbb{E}^{\mathbb{P}} \left[\int_t^\tau \int_{\mathbb{R}^d \times A} \psi(x, \mu_s, a) \beta_s(x, da) \mu_s(dx) ds + v(\tau, \mu_\tau) \right]$ by the arbitrariness of \mathbb{P} .

We now turn to the reverse inequality. Fix some arbitrary $\mathbb{P} \in \mathcal{C}(t, \lambda)$ and $\varepsilon > 0$. Consider the set $\mathcal{C}_{(t', \lambda')}^\varepsilon$ defined as follows

$$\mathcal{C}_{(t', \lambda')}^\varepsilon := \{ \mathbb{Q} \in \mathcal{C}(t', \lambda') : v(t', \lambda') + \varepsilon \geq J(t', \lambda', \mathbb{Q}) \}, \quad \text{for } (t', \lambda') \in [0, T] \times M(\mathbb{R}^d).$$

From the [16, Proposition 2.21], there exists a family of probability $(\mathbb{Q}_{\bar{\omega}}^\varepsilon)_{\bar{\omega} \in \bar{\Omega}}$ from $\mathcal{F}_\tau^{\mu, \beta}$ into $(\bar{\Omega}, \mathcal{F}_T^{\mu, \beta})$ such that $\bar{\omega} \mapsto \mathbb{Q}_{\bar{\omega}}^\varepsilon$ is $\mathcal{F}_\tau^{\mu, \beta}$ -measurable, and $\mathbb{Q}_{\bar{\omega}}^\varepsilon \in \mathcal{C}_{(\tau(\bar{\omega}), \mu(\bar{\omega}))}^\varepsilon$ for \mathbb{P} -a.e. $\bar{\omega} \in \bar{\Omega}$. Then, $\mathbb{P} \otimes_\tau \mathbb{Q}^\varepsilon \in \mathcal{C}(t, \lambda)$ by the stability by concatenation condition. This implies that

$$\begin{aligned} J(t, \lambda, \mathbb{P} \otimes_\tau \mathbb{Q}^\varepsilon) &= \mathbb{E}^{\mathbb{P} \otimes_\tau \mathbb{Q}^\varepsilon} \left[\int_t^T \int_{\mathbb{R}^d \times A} \psi(x, \mu_s, a) \beta_s(x, da) \mu_s(dx) ds + \Psi(\mu_T) \right] \\ &= \int \left(\int_t^{\tau(\bar{\omega})} \int_{\mathbb{R}^d \times A} \psi(x, \mu_s, a) \beta_s(x, da) \mu_s(dx) ds \right. \\ &\quad \left. + \mathbb{E}^{\mathbb{Q}_{\bar{\omega}}^\varepsilon} \left[\int_\tau^T \int_{\mathbb{R}^d \times A} \psi(x, \mu_s, a) \beta_s(x, da) \mu_s(dx) ds + \Psi(\mu_T) \right] \right) \mathbb{P}(d\bar{\omega}) \\ &= \mathbb{E}^{\mathbb{P}} \left[\int_t^{\tau(\bar{\omega})} \int_{\mathbb{R}^d \times A} \psi(x, \mu_s, a) \beta_s(x, da) \mu_s(dx) ds + J(\tau(\bar{\omega}), \mu_{\tau(\bar{\omega})}(\bar{\omega}), \mathbb{Q}_{\bar{\omega}}^\varepsilon) \right] \\ &\leq \mathbb{E}^{\mathbb{P}} \left[\int_t^\tau \int_{\mathbb{R}^d \times A} \psi(x, \mu_s, a) \beta_s(x, da) \mu_s(dx) ds + v(\tau, \mu_\tau) \right] + \varepsilon. \end{aligned}$$

From the arbitrariness of $\mathbb{P} \in \mathcal{C}_{(t,\lambda)}$ and $\varepsilon > 0$, we obtain the inequality

$$v(t, \lambda) \leq \inf_{\mathbb{P} \in \mathcal{C}_{(t,\lambda)}} \mathbb{E}^{\mathbb{P}} \left[\int_t^\tau \int_{\mathbb{R}^d \times A} \psi(x, \mu_s, a) \beta_s(x, da) \mu_s(dx) ds + v(\tau, \mu_\tau) \right].$$

□

4 Dynamic Programming Equation

The DPP opens the way to the characterization of the value function as a (viscosity) solution to nonlinear PDE. This approach links the (optimal) controlled dynamics of the processes under analysis in a "weak" way. This means that we are interested in the behaviour of $s \mapsto u(\mu_s)$ for a certain class of test functions u .

For this purpose, first, we need to analyse the differential property of the space of finite measure. There exists a growing literature about differential calculus in the space of probability measures. This is due to the development of the mean field games theory. The two main objects discussed in this context are the linear functional derivative (also called flat derivative or extrinsic derivative) and the L -derivative (also called intrinsic derivative). The first is defined directly in $\mathcal{P}(\mathbb{R}^d)$, while the second relies on the lifting on a Hilbert space. It is found that one is the spatial gradient of the previous one, coinciding with the notion of derivative of [29]. Therefore, sometimes this is the definition used for the L -derivatives, like in [37]. Detailed discussions of this topic can be found for example in [6, 5, 37].

Readjusting these concepts to $M(\mathbb{R}^d)$, we present the same two notions, as introduced also in [30]. A survey on how these notions of derivatives intertwine is [38], where their properties are studied in a more general setting.

4.1 Differential properties

Definition 4.5 (Linear derivative). *A function $u : M(\mathbb{R}^d) \rightarrow \mathbb{R}$ is said to have linear derivative if it is continuous, bounded and if there exists a function*

$$\delta_\lambda u : M(\mathbb{R}^d) \times \mathbb{R}^d \ni (\lambda, x) \mapsto \delta_\lambda u(\lambda, x) \in \mathbb{R},$$

that is bounded, and continuous for the product topology, such that

$$u(\lambda) - u(\lambda') = \int_0^1 \int_{\mathbb{R}^d} \delta_\lambda u(t\lambda + (1-t)\lambda', x) (\lambda - \lambda')(dx) dt,$$

for $\lambda, \lambda' \in M(\mathbb{R}^d)$. We call $C^1(M(\mathbb{R}^d))$ the class of functions from $M(\mathbb{R}^d)$ to \mathbb{R} that are differentiable in linear functional sense.

Notice that $\delta_\lambda u$ is uniquely defined up to a constant. We take

$$\int_{\mathbb{R}^d} \delta_\lambda u(\lambda, x) \lambda(dx) = 0$$

as a convention in this paper. Moreover, second-order derivatives are introduced for $u \in C^1(M(\mathbb{R}^d))$ imposing that $\lambda \mapsto \delta_\lambda u(\lambda, x)$ is differentiable in linear functional sense for every x and that $(\lambda, x, y) \mapsto \delta_\lambda^2 u(\lambda, x, y)$ is bounded and continuous. We call $C^2(M(\mathbb{R}^d))$ this class of functions.

Finite positive measures could not rely on lifting procedure. For this reason, the notion of *intrinsic* derivative is introduced deriving with respect to the x component the flat derivative, as done in [5, Definition 2.2].

Definition 4.6 (Intrinsic derivative). Fix $u \in C^1(M(\mathbb{R}^d))$. If $\delta_\lambda u$ is of class C^1 with respect to the second variable, the intrinsic derivative $D_\lambda u : M(\mathbb{R}^d) \times \mathbb{R}^d \rightarrow \mathbb{R}$ is

$$D_\lambda u(\lambda, x) = \partial_x \delta_\lambda u(\lambda, x).$$

We denote with $C^{1,1}(M(\mathbb{R}^d))$ this class of functions.

Deriving with respect to the measure or the space component are two different operations. We denote $C^{k,\ell}(M(\mathbb{R}^d))$ with $k \in \mathbb{N}$ to be the collection of functions u that are differentiable k times with respect to the measure and such that the k -th derivative with respect to the measure is ℓ -th times continuously differentiable with respect to its spatial components.

Remark 4.2. As in [30, Example 2.9], we have that $\mathcal{D}^T \subseteq C^{2,2}(M(\mathbb{R}^d))$, where \mathcal{D}^T is the domain of cylindrical functions as in (2.8). In particular, it holds that

$$\begin{aligned} \delta_\lambda h(\lambda, x) &= DF(\langle f_1, \lambda \rangle, \dots, \langle f_p, \lambda \rangle)^\top \mathbf{f}(x), \\ \delta_\lambda^2 h(\lambda, x, y) &= \mathbf{f}(y)^\top DF^2(\langle f_1, \lambda \rangle, \dots, \langle f_p, \lambda \rangle) \mathbf{f}(x), \\ D_\lambda h(\lambda, x) &= DF(\langle f_1, \lambda \rangle, \dots, \langle f_p, \lambda \rangle)^\top D\mathbf{f}(x), \end{aligned}$$

with $\mathbf{f}(x) := (f_1, \dots, f_p)(x)^\top$, and $D\mathbf{f}(x) = (Df_1, \dots, Df_p)(x)^\top$, for $h = F_{(f_1, \dots, f_p)} \in \mathcal{D}^T$.

4.2 Density properties

To get the Dynamic Programming Equation associated with the value function v we first need to generalize the martingale problem (2.2). As in [21] and [30], to do this we first restrict ourselves in a compact space, prove the density of \mathcal{D}^T in $C^{2,2}(M(\mathbb{R}^d))$ and conclude with a localization argument.

First, we restrict the space we work into a compact set. This is done to apply the Stone-Weierstrass theorem and prove the density of cylindrical functions. Consider the set of compact rectangles $\{K_N := [-N, N]^d\}_{N \geq 1} \subseteq \mathbb{R}^d$. For every $k, N \in \mathbb{N}$, we define the

$$\begin{aligned} M_k(\mathbb{R}^d) &:= \left\{ \lambda \in M(\mathbb{R}^d) : \lambda(\mathbb{R}^d) \in \left[\frac{1}{k}, k \right] \right\}, \\ \mathcal{K}_N^k &:= \left\{ \lambda \in M_k(\mathbb{R}^d) : \text{supp}(\lambda) \subseteq K_N \right\}. \end{aligned}$$

These sets are non-empty. In particular, \mathcal{K}_N^k is compact for the weak* topology for any $k, N \in \mathbb{N}$. Indeed, this space is homeomorphic to $\mathcal{K}_N^{k,1} \times [\frac{1}{k}, k]$ with $\mathcal{K}_N^{k,1} := \mathcal{K}_N^k \cap \mathcal{P}(\mathbb{R}^d)$ with the homeomorphism

$$\begin{aligned} \mathfrak{H} : M(\mathbb{R}^d) \setminus \{\mathbf{0}\} &\rightarrow \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}_+ \\ \lambda &\mapsto \left(\frac{1}{\lambda(\mathbb{R}^d)} \lambda, \lambda(\mathbb{R}^d) \right). \end{aligned}$$

The set $\mathcal{K}_N^{k,1}$ is weakly* precompact, using Prokhorov's theorem (see *e.g.* [3, Theorem 1.6.1]), and is closed as any limit point of the sequence in $\mathcal{K}_N^{k,1}$ also has support contained in K_N . Therefore, \mathcal{K}_N^k is compact as homeomorphic to the product of two compact sets.

Given $\lambda \in M_k(\mathbb{R}^d)$, we denote with $\rho^N \mu$ the measure such that $\frac{d\rho^N \lambda}{d\lambda} = \rho^N$, with ρ^N a positive and smooth cut-off function equal to 1 in K_N and identically zero outside K_{N+1} . We observe that $\rho^N \lambda$ is always in \mathcal{K}_{N+1}^k . Thus, we set

$$u^N(\lambda) := u(\rho^N \lambda), \quad \text{for } \lambda \in M_k(\mathbb{R}^d), N \geq 1. \quad (4.23)$$

With these notations, we give [30, Lemma 3.3] that proves the first approximation theorem for functions on $M_k(\mathbb{R}^d)$.

Lemma 4.1. *Fix $k \geq 1$ and $u \in C^{2,2}(M_k(\mathbb{R}^d))$. Let $\{u^N\}_{N \geq 1}$ be the sequence defined by (4.23). Then, for every $\lambda \in M_k(\mathbb{R}^d)$, $u^N(\lambda) \rightarrow u(\lambda)$ as $n \rightarrow \infty$ and $\{\delta_\lambda u^N\}_{N \geq 1}$, $\{\delta_\lambda^2 u^N\}_{N \geq 1}$, $\{D_\lambda u^N\}_{N \geq 1}$, and $\{\partial_x D_\lambda u^N\}_{N \geq 1}$ pointwise converge to the respective derivatives of u . Moreover, $\|u^N\|_\infty \leq \|u\|_\infty$, and there exists $C > 0$ independent of u , N , and k such that*

$$\begin{aligned} \|\delta_\lambda u^N\|_\infty &\leq \|\delta_\lambda u\|_\infty, \\ \|\delta_\lambda^2 u^N\|_\infty &\leq \|\delta_\lambda^2 u\|_\infty, \\ \|D_\lambda u^N\|_\infty &\leq C (\|D_\lambda u\|_\infty + \|\delta_\lambda u\|_\infty), \\ \|\partial_x D_\lambda u^N\|_\infty &\leq C (\|D_\lambda u\|_\infty + \|\delta_\lambda u\|_\infty + \|\partial_x D_\lambda u\|_\infty). \end{aligned}$$

This result allows us to approximate functions in $C^{2,2}(M_k(\mathbb{R}^d))$ with functions in $C^{2,2}(\mathcal{K}_N^k)$, for $k, N \in \mathbb{N}$. We can now adapt [30, Lemma 3.4] showing that the domain of cylindrical functions \mathcal{D}^T is dense in $C^{2,2}(\mathcal{K}_N^k)$.

Lemma 4.2. *Fix $k \geq 1$ and $N \geq 1$. Let u be in $C^{2,2}(\mathcal{K}_N^k)$. There exists a sequence of cylindrical functions $\{u_n\}_{n \geq 1} \subseteq \mathcal{D}^T$ such that $u_n(\lambda) \rightarrow u(\lambda)$ as $n \rightarrow \infty$ and $\{\delta_\lambda u_n\}_{n \geq 1}$, $\{\delta_\lambda^2 u_n\}_{n \geq 1}$, $\{D_\lambda u_n\}_{n \geq 1}$, and $\{\partial_x D_\lambda u_n\}_{n \geq 1}$ converge pointwise to the respective derivatives of u for any $\lambda \in \mathcal{K}_N^k$. Moreover, $\|u_n\|_\infty \leq \|u\|_\infty$, and the same holds for the derivatives, up to a multiplicative constant independent of u , N and k .*

Since \mathcal{K}_N^k is compact in $M(\mathbb{R}^d)$, the previous lemma proves

$$\|u_n - u\|_{C_b^{2,2}(M(\mathbb{R}^d))} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

where

$$\begin{aligned} \|u\|_{C_b^{2,2}(M(\mathbb{R}^d))} &:= \sup_{\lambda \in \mathcal{K}_N^k, x, y \in K_N} \left\{ |u(\lambda)| + |\delta_\lambda u(\lambda, x)| + |\delta_\lambda^2 u(\lambda, x, y)| + |D_\lambda u(\lambda, x)| \right. \\ &\quad \left. + |\partial_x D_\lambda u(\lambda, x)| \right\}, \end{aligned} \quad (4.24)$$

for u in $C^{2,2}(M(\mathbb{R}^d))$ which is bounded with bounded derivatives. A stronger norm could be used by adding in the supremum also the terms depending on $\partial_x \delta_\lambda^2 u$ and $D_\lambda^2 u$ as in [30]. For our scope, norm (4.24) is enough to generalize the martingale problem (2.2).

Remark 4.3. *A different approach could have been taken to prove that \mathcal{D}^T is dense in $C^{2,2}(\mathcal{K}_N^k)$. As proven in [21], this domain separates points in \mathcal{K}_N^k and vanishes at no point, therefore using Stone-Weierstrass theorem, it is dense in the $C^0(\mathcal{K}_N^k)$ with the topology of the strong convergence. Then, using the definition for the Linear derivative and the intrinsic derivative, the convergences of the different derivatives could be established, as in [21, Lemma 3.12].*

4.3 Generalized martingale problem

We define the operator \mathbf{L} on $u \in C_b^{2,2}(M(\mathbb{R}^d))$ by

$$\begin{aligned} \mathbf{L}u(\lambda, x, a) &= b(x, \lambda, a)^\top D_\lambda u(\lambda, x) + \frac{1}{2} \text{Tr} \left(\sigma \sigma^\top(x, \lambda, a) \partial_x D_\lambda u(\lambda, x) \right) \\ &\quad + \frac{1}{2} \gamma(x, \lambda, a) \delta_\lambda^2 u(\mu, x, x) \end{aligned}$$

for $(x, \lambda, a) \in \mathbb{R}^d \times M(\mathbb{R}^d) \times A$.

Proposition 4.6. For $(t, \lambda) \in [0, T] \times M(\mathbb{R}^d)$ and $\alpha \in \mathcal{A}$, the following are equivalent:

(i) $(\mathbb{P}^{t, \lambda, \alpha}, \alpha) \in \mathcal{R}_{(t, \lambda)}$;

(ii) the process

$$M_s^u = u(\mu_s) - \int_t^s \int_{\mathbb{R}^d} \mathbf{L}u(x, \mu_u, \alpha_u(x)) \mu_u(dx) du \quad (4.25)$$

is a (\mathbb{P}, \mathbb{F}) -martingale with quadratic variation

$$[M^u]_s = \int_t^s \gamma(x, \mu_u, \alpha_u(x)) |\delta_\lambda u(\mu_u, x)|^2 \mu_u(dx) du \quad (4.26)$$

for any $u \in C_b^{2,2}(\mathbb{R}^d)$, and $s \geq t$.

Proof. (ii) \implies (i): From Remark 4.2, $F_\varphi \in C_b^{2,2}(\mathbb{R}^d)$ for any $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$ and equations (4.25) and (4.26) becomes (2.2) and (2.3) for $u = F_\varphi$.

(i) \implies (ii): If $\lambda = \mathbb{O}$, it is clear that M^u is constant in time, thus a martingale with a null quadratic variation. Consider a starting condition (t, λ) , with $\langle 1, \lambda \rangle > 0$, a control $\alpha \in \mathcal{A}$ and the sequence of stopping times $\{\tau_k\}_{k \geq 1}$ as

$$\tau_k := \inf \left\{ s \geq t : \langle 1, \mu_s \rangle > k \right\} \wedge \inf \left\{ s \geq t : \langle 1, \mu_s \rangle < \frac{1}{k} \right\}.$$

Defining $\mu^k := \mu_{\cdot \wedge \tau_k}$, for $\langle 1, \lambda \rangle \in [1/k, k]$, we have that under $\mathbb{P}^{(t, \lambda, \alpha)}$, this process lives in $M_k(\mathbb{R}^d)$ by construction. Thus, applying Lemma 4.1 and Lemma 4.2, there exists a sequence $u_n \in \mathcal{D}^T$ such that $u_n \rightarrow u$ as $n \rightarrow \infty$ pointwise as well as their derivatives and there exists $C > 0$ such that $\|u_n\|_{C_b^{2,2}(M(\mathbb{R}^d))} \leq C \|u\|_{C_b^{2,2}(M(\mathbb{R}^d))}$.

Remark 4.2 shows how derivatives operate on cylindrical functions. Looking at (2.9) and (2.10), we see that for $h \in \mathcal{D}^T$ equations (4.25) and (4.26) are satisfied if applied to μ^k . We prove now that $u(\mu^k) - \int_t^\theta \int_{\mathbb{R}^d} \mathbf{L}u(x, \mu_u, \alpha_u(x)) \mu_u(dx) du$ is a $\mathbb{P}^{(t, \lambda, \alpha)}$ -martingale, i.e., for each stopping time θ in $[t, T]$,

$$\mathbb{E}^{\mathbb{P}^{(t, \lambda, \alpha)}} \left[u(\mu_\theta^k) - u(\lambda) - \int_t^\theta \int_{\mathbb{R}^d} \mathbf{L}u(x, \mu_u^k, \alpha_u(x)) \mu_u^k(dx) du \right] = 0.$$

Since (4.25) is satisfied for $h \in \mathcal{D}^T$, and from the bounds on the derivatives and on the coefficients b , σ and γ , we can apply the Dominated Convergence Theorem and obtain

$$\begin{aligned} 0 &= \lim_n \mathbb{E}^{\mathbb{P}^{(t, \lambda, \alpha)}} \left[u_n(\mu_\theta^k) - u_n(\lambda) - \int_t^\theta \int_{\mathbb{R}^d} \mathbf{L}u_n(x, \mu_u^k, \alpha_u(x)) \mu_u^k(dx) du \right] \\ &= \mathbb{E}^{\mathbb{P}^{(t, \lambda, \alpha)}} \left[u(\mu_\theta^k) - u(\lambda) - \int_t^\theta \int_{\mathbb{R}^d} \mathbf{L}u(x, \mu_u^k, \alpha_u(x)) \mu_u^k(dx) du \right]. \end{aligned}$$

By definition of the quadratic variation and (4.26) applied to $u_n \in \mathcal{D}^T$, we have for $n \in \mathbb{N}$ that

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}^{(t, \lambda, \alpha)}} \left[\left(u_n(\mu_s^k) - u_n(\lambda) - \int_t^s \int_{\mathbb{R}^d} \mathbf{L}u_n(x, \mu_u^k, \alpha_u(x)) \mu_u^k(dx) du \right)^2 \right] \\ &= \mathbb{E}^{\mathbb{P}^{(t, \lambda, \alpha)}} \left[\int_t^s \gamma(x, \mu_u^k, \alpha_u(x)) |\delta_\lambda u_n(\mu_u^k, x)|^2 \mu_u^k(dx) du \right] \end{aligned} \quad (4.27)$$

Therefore, we apply again Dominated Convergence Theorem and obtain (4.27) with respect to u .

Finally, we can remove the localization using Dominated Convergence Theorem since $u \in C_b^{2,2}(\mathbb{R}^d)$ and the bound (2.18). \square

4.4 HJB Equation

We are ready to introduce the HJB equation associated with this control problem. Looking at (4.25), define an operator H on $\mathbb{R}^d \times M(\mathbb{R}^d) \times A \times \mathbb{R}^d \times \mathbb{R}^{d \times d} \times \mathbb{R}$ such that

$$H(x, \lambda, a, p, M, r) = b(x, \lambda, a)^\top p + \frac{1}{2} \text{Tr} \left(\sigma \sigma^\top(x, \lambda, a) M \right) + \frac{1}{2} \gamma(x, \lambda, a) r + \psi(x, \lambda, a) \quad (4.28)$$

Then, if the value function (3.21) is sufficiently smooth, generalizing Proposition 4.6 to function depending in time and measure yields the following HJB equation

$$\begin{cases} \partial_t v(t, \lambda) + \int_{\mathbb{R}^d} \inf_{a \in A} H \left(x, \lambda, a, D_\lambda v(t, \lambda, x), \right. \\ \left. \partial_x D_\lambda v(t, \lambda, x), \delta_\lambda^2 v(t, \lambda, x, x) \right) \lambda(dx) = 0 \\ v(T, \lambda) = \Psi(\lambda) \end{cases}$$

The form of this HJB equation looks like the one for mean field control (see, *e.g.*, [21], [36], [1], [12], [44]), where the infimum (or the supremum if maximizing) is taken inside the integral. The major differences here are that we consider the space of finite measures and not only probability measures and that the second-order flat derivative is explicitly involved in the Hamiltonian.

We have the following result in the regular case.

Theorem 4.2 (Verification Theorem). *Let $V : [0, T] \times M(\mathbb{R}^d) \rightarrow \mathbb{R}$ be a function living in $C_b^{1,(2,2)}([0, T] \times M(\mathbb{R}^d)) \cap C^0([0, T] \times M(\mathbb{R}^d))$.*

(i) *Suppose that V satisfies*

$$\begin{cases} \partial_t V(t, \lambda) + \int_{\mathbb{R}^d} \inf_{a \in A} H \left(x, \lambda, a, D_\lambda V(t, \lambda, x), \right. \\ \left. \partial_x D_\lambda V(t, \lambda, x), \delta_\lambda^2 V(t, \lambda, x, x) \right) \lambda(dx) \leq 0 \\ V(T, \lambda) \leq \Psi(\lambda). \end{cases} \quad (4.29)$$

Then $V(t, \lambda) \leq v(t, \lambda)$ for any $(t, \lambda) \in [0, T] \times M(\mathbb{R}^d)$, with v as in (3.21).

(ii) *Moreover, suppose that V satisfies (4.29) with equality and there exists a continuous function $\hat{a}(t, x, \lambda)$ for $(t, x, \lambda) \in [0, T] \times \mathbb{R}^d \times M(\mathbb{R}^d)$, valued in A such that*

$$\hat{a}(t, x, \lambda) \in \arg \min_{a \in A} H \left(x, \lambda, a, D_\lambda v(t, \lambda, x), \partial_x D_\lambda v(t, \lambda, x), \delta_\lambda^2 v(t, \lambda, x, x) \right). \quad (4.30)$$

Suppose also that the corresponding control $\alpha^ = \{\alpha_s^*(x) := \hat{a}(s, x, \mu_s), s \in [t, T]\} \in \mathcal{A}$. Then $V = v$, with v as in (3.21), and α^* is an optimal Markovian control.*

Proof. (i) Since $V \in C_b^{1,(2,2)}([0, T] \times M(\mathbb{R}^d))$, we have for all $(t, \lambda) \in [0, T] \times M(\mathbb{R}^d)$, and $\alpha \in \mathcal{A}$, by (4.25), the process

$$V(s, \mu_s) - V(t, \lambda) - \int_t^s \partial_t V(u, \mu_u) + \int_{\mathbb{R}^d} \mathbf{L}V(u, x, \mu_u, \alpha_u(x)) \mu_u(dx) du$$

is a martingale under $\mathbb{P}^{t, \lambda, \alpha}$. By taking the expectation, we get

$$\mathbb{E}^{\mathbb{P}^{t, \lambda, \alpha}} [V(s, \mu_s)] = V(t, \lambda) + \mathbb{E}^{\mathbb{P}^{t, \lambda, \alpha}} \left[\int_t^s \partial_t V(u, \mu_u) + \int_{\mathbb{R}^d} \mathbf{L}V(u, x, \mu_u, \alpha_u(x)) \mu_u(dx) du \right]$$

Since w satisfies (4.29), we have

$$\partial_t V(u, \mu_u) + \int_{\mathbb{R}^d} \mathbf{L}V(u, x, \mu_u, \alpha_u(x)) + \psi(x, \mu_u, \alpha_u(x)) \mu_u(dx) \leq 0, \quad \mathbb{P}^{t, \lambda, \alpha} - \text{a.s.}$$

for any $\alpha \in \mathcal{A}$. Therefore,

$$\mathbb{E}^{\mathbb{P}^{t, \lambda, \alpha}} [V(s, \mu_s)] \leq V(t, \lambda) - \mathbb{E}^{\mathbb{P}^{t, \lambda, \alpha}} \left[\int_t^s \int_{\mathbb{R}^d} \psi(x, \mu_u, \alpha_u(x)) \mu_u(dx) du \right], \quad \mathbb{P}^{t, \lambda, \alpha} - \text{a.s.}$$

for any $\alpha \in \mathcal{A}$. Since V is continuous on $[0, T] \times M(\mathbb{R}^d)$, we obtain by the dominated convergence theorem and by (4.29)

$$\mathbb{E}^{\mathbb{P}^{t, \lambda, \alpha}} [\Psi(\mu_T)] \leq V(t, \lambda) - \mathbb{E}^{\mathbb{P}^{t, \lambda, \alpha}} \left[\int_t^T \int_{\mathbb{R}^d} \psi(x, \mu_u, \alpha_u(x)) \mu_u(dx) du \right], \quad \mathbb{P}^{t, \lambda, \alpha} - \text{a.s.}$$

for any $\alpha \in \mathcal{A}$. From the arbitrariness of the control, we deduce that $V(t, \lambda) \leq v(t, \lambda)$, for all $(t, \lambda) \in [0, T] \times M(\mathbb{R}^d)$.

(ii) By (4.25),

$$\mathbb{E}^{\mathbb{P}^{t, \lambda, \alpha}} [V(s, \mu_s)] = V(t, \lambda) + \mathbb{E}^{\mathbb{P}^{t, \lambda, \alpha}} \left[\int_t^s \partial_t V(u, \mu_u) + \int_{\mathbb{R}^d} \mathbf{L}V(u, x, \mu_u, \alpha_u(x)) \mu_u(dx) du \right]$$

By definition of $\hat{a}(t, x, \lambda)$, we have

$$\partial_t V(t, \lambda) + \int_{\mathbb{R}^d} \mathbf{L}V(t, \lambda, x, \hat{a}(t, x, \lambda)) + \psi(x, \lambda, \hat{a}(t, x, \lambda)) \lambda(dx) = 0,$$

and so

$$\mathbb{E}^{\mathbb{P}^{t, \lambda, \alpha^*}} [V(s, \mu_s)] = V(t, \lambda) - \mathbb{E}^{\mathbb{P}^{t, \lambda, \alpha^*}} \left[\int_t^s \int_{\mathbb{R}^d} \psi(x, \lambda, \alpha_u^*(x, \mu_u)) \mu_u(dx) du \right].$$

By sending s to T , we then obtain

$$V(t, \lambda) = \mathbb{E}^{\mathbb{P}^{t, \lambda, \alpha^*}} \left[\Psi(\mu_T) + \int_t^T \int_{\mathbb{R}^d} \psi(x, \lambda, \alpha_u^*(x, \mu_u)) \mu_u(dx) du \right] = J(t, \lambda, \alpha^*),$$

which shows that $V(t, \lambda) = J(t, \lambda, \alpha^*) \geq v(t, \lambda)$. Therefore, $V = v$ and α^* is an optimal Markovian control. \square

4.5 Example of regular solution

For the following part of the paper, we suppose that there is no dependence on the measure for b , σ , and γ . With abuse of notation, we denote $b(x, a)$ (resp. $\sigma(x, a)$, $\gamma(x, a)$) instead of $b(x, \lambda, a)$ (resp. $\sigma(x, \lambda, a)$, $\gamma(x, \lambda, a)$). Fix $h \in C_b(\mathbb{R}^d)$ with $h(x) \geq 0$ for any $x \in \mathbb{R}^d$, and let $\Psi(\lambda) := \exp(-\langle h, \lambda \rangle)$ and $\psi = 0$. Therefore, the cost function J writes as

$$J(t, \lambda, \alpha) = \mathbb{E}^{\mathbb{P}^{t, \lambda, \alpha}} [\exp(-\langle h, \mu_T \rangle)], \quad (4.31)$$

for $(t, \lambda) \in [0, T] \times M(\mathbb{R}^d)$, and $\alpha \in \mathcal{A}$.

The following assumption will ensure that there exists a smooth solution to the HJB equation (4.28) associated with this cost function.

Assumption A1. Assume that the following conditions hold:

(i) $h \in C_b^3(\mathbb{R}^d)$;

(ii) $(b, \sigma, \gamma)(\cdot, a) \in C^2(\mathbb{R}^d)$ for any $a \in A$, and b, σ , and γ and their partial derivatives are bounded on $\mathbb{R}^d \times A$;

(iii) there exists $C_\sigma > 0$ such that

$$\sigma\sigma^\top(x, a) \geq C_\sigma \mathbf{I}_d, \quad \text{for } (x, a) \in \mathbb{R}^d \times A.$$

Proposition 4.7. Under Assumption A1, there exists a function $w \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$, such that

$$\begin{cases} -\partial_t w(t, x) - \sup_{a \in A} \left\{ b(x, a)^\top Dw(t, x) + \frac{1}{2} \text{Tr} \left(\sigma\sigma^\top(x, a) D^2 w(t, x) \right) - \frac{1}{2} \gamma(x, a) w(t, x)^2 \right\} = 0 \\ w(T, x) = h(x). \end{cases} \quad (4.32)$$

Moreover, we have

$$v(t, \lambda) = \exp(\langle w(t, \cdot), \lambda \rangle)$$

for any $(t, \lambda) \in [0, T] \times M(\mathbb{R}^d)$, with v as in (3.21).

Proof. Our goal is to determine the function w using [26, Theorem 6.4.4], which guarantees the existence of smooth solutions for a certain category of fully nonlinear partial differential equations. First, we need to modify (4.32) to fall into this class. If w satisfies (4.32), we see that the function $\tilde{w}(t, x) := e^{-t} w(t, x)$ satisfies the following nonlinear PDE

$$\begin{cases} -\partial_t \tilde{w}(t, x) - \sup_{a \in A} \left\{ b(x, a)^\top D\tilde{w}(t, x) + \frac{1}{2} \text{Tr} \left(\sigma\sigma^\top(x, a) D^2 \tilde{w}(t, x) \right) \right. \\ \left. - \frac{1}{2} \gamma(x, a) e^t \tilde{w}(t, x)^2 + \tilde{w}(t, x) \right\} = 0 \\ \tilde{w}(T, x) = e^{-T} h(x). \end{cases} \quad (4.33)$$

Let $C_\gamma > 0$ be a constant such that

$$\gamma(x, a) \geq C_\gamma, \quad \text{for all } (x, a) \in \mathbb{R}^d \times A.$$

Without loss of generalities, we can take C_γ to be such that $\frac{C_\gamma e^T}{4} > 1$. This means that

$$\begin{aligned} \frac{1}{2} \gamma(x, a) e^t M_0^2 - M_0 &\leq \frac{C_\gamma e^T}{2} M_0^2 - M_0 \leq -\delta_0, \\ \frac{1}{2} \gamma(x, a) e^t M_0^2 + M_0 &\geq M_0 \geq \delta_0, \end{aligned}$$

for all $(x, a) \in \mathbb{R}^d \times A$, with $M_0 := \frac{4}{C_\gamma e^T} > 0$ and $\delta_0 := M_0^2$. Combining these inequalities with Assumption A1, the property that equation (4.33) belongs to the class of [26, Theorem 6.4.4] follows from [26, Example 6.1.8]. Therefore, this theorem ensures that there exists $\tilde{w} \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$ solution to (4.33). Thus, $w(t, x) := e^t \tilde{w}(t, x)$ is a bounded solution of (4.32) belonging to $C_b^{1,2}([0, T] \times \mathbb{R}^d)$.

We conclude this proposition by applying Theorem 4.2. Define $V(t, \lambda) := \exp(\langle w(t, \cdot), \lambda \rangle)$. Using the terminal condition of w , we see that $V(t, \lambda) = \Psi(\lambda)$. Moreover, (4.28) in this setting writes as

$$\exp(\langle w(t, \cdot), \lambda \rangle) \int_{\mathbb{R}^d} \left(-\partial_t w(t, x) - \sup_{a \in A} \left\{ b(x, a)^\top Dw(t, x) + \frac{1}{2} \text{Tr} \left(\sigma \sigma^\top(x, a) D^2 w(t, x) \right) - \frac{1}{2} \gamma(x, a) w(t, x)^2 \right\} \right) \lambda(dx) = 0.$$

It is now clear that V satisfies (4.28) since w satisfies (4.32). Moreover, the optimal control \hat{a} , defined as in (4.30), is the point that reaches the maximum over a compact set of a continuous function. This means that \hat{a} can be chosen continuously, thus predictably. Therefore, its associated optimal control belongs to \mathcal{A} . We can now apply Theorem 4.2 and conclude. \square

Remark 4.4. *When we are not under Assumption A1, (4.32) can be solved in a viscosity sense. In particular, one can follow [9, Section 5] to prove a comparison principle and an approximation procedure to give a solution in a viscosity sense to this equation. This translates directly to (4.28) and is a way to entail viscosity properties reducing the dimensionality.*

5 Conclusion

This research centres on controlled superprocesses, which is, to the best of our knowledge, a novel category of processes. The first part of our study is devoted to introducing the formalism, which we present in a weak form through a controlled martingale problem. Following the definition, we prove their existence and uniqueness in law. Uniqueness uses the same method as in [33], where the initial martingale problem is generalized to define càdlàg processes with values in $M(\mathbb{R}^d)$. Subsequently, we employ the duality method to establish that there exists at most one probability satisfying the martingale problem once control and starting condition are fixed. In the second part, we prove the existence of these processes as weak limits of rescaled branching processes. Thus, we introduce this new class of processes, whose existence has been proved in [33]. We prove the weak limit using the Aldous criterion (see, *e.g.*, [11, 18, 39]), while adapting these ideas to this new setting. We notice that there are several choices for the branching parameters to build the class of superprocesses as limit points. We refer to [11, 35, 18, 39] for more details.

Once we establish the non-explosion property with respect to the chosen distance that metricizes weak* topology, we introduce the control problem. We define a weak formulation for controlled superprocesses as in [33]. This allows to extend the methods in [16, 17] to this setting and prove the DPP.

In the final section, we derived an HJB equation on the space of measures. Adopting the differential calculus developed in [30], we generalized the original martingale problem to a larger class of functions. This enabled us to derive the HJB equation and provide a verification theorem. Finally, we used this result to introduce a class of solvable problems. Utilizing the branching property technique, we translated the problem on finite measures to a finite-dimensional nonlinear PDE, reducing the dimensionality. We solved the latter using results from [26], and, with the help of the verification theorem, we provide an explicit description of the value function.

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