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# Insights on equidistributed real spectral values in second-order delay systems: perspectives in partial pole placement\*

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## Abstract

In this work, we show that the coexistence of the maximal number of real spectral values of generic single-delay retarded second-order differential equations guarantees the realness of the rightmost spectral value. From a control theory standpoint, this entails that a delayed proportional-derivative (PD) controller can stabilize a delayed second-order differential equation. By assigning the maximum number of negative roots to the corresponding characteristic function (a quasipolynomial), we establish the conditions for asymptotic stability. If the assigned real spectral values are uniformly distributed, we specify a *necessary and sufficient condition* for the rightmost root to be negative, thus guaranteeing the exponential decay rate of the system's solutions. We illustrate the proposed design methodology in the delayed PD control of the damped harmonic oscillator. It is worth mentioning that this work represents a natural continuation of [Amrane et al., 2018] and [Bedouhene et al., 2020], addressing the problem of coexisting real spectral values for linear dynamical systems including delays in their models.

**Keywords.** Time-delay differential equation, Stability and stabilization, Spectral method, Exponential decay, Partial pole placement, Coexisting-real-root-induced-dominancy

## 1 Introduction

The purpose of this paper is to study the asymptotic stability of the solutions of a dynamical system represented by a second-order linear differential equation of the form:

$$y''(t) + ay'(t) + by(t) = u(t),$$

where  $u(t)$  is a forcing delayed proportional-derivative controller given by

$$u(t) = -\alpha y'(t - \tau) - \beta y(t - \tau), \quad \tau > 0.$$

The closed-loop system is represented by a second-order autonomous delay-differential equation (DDE) including five parameters: the parameters of the open-loop system  $a, b \in \mathbb{R}$ , the controller gains  $\alpha, \beta \in \mathbb{R}$  and the delay  $\tau \in \mathbb{R}_+^*$ .

If the interest in considering a delayed control law as above lies in its generality and simplicity (tuning only three parameters, including the *delay*) in real-time implementation [2]–[4], its main inconvenient concerns the fact that the closed-loop system becomes *infinite-dimensional* leading, in

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some cases, to unexpected dynamical behaviors with respect to changes in the system's parameters. In this frame, it should be mentioned that the idea to use the *delay* as a *controller parameter* is not new and there exists several examples in the literature exhibiting its relevance. For instance, at the end of 1970, [2] introduced the so-called "proportional-minus-delay" (PMD) controller and highlighted a few interesting properties. More precisely, similar to the PD controllers, it operates quick responses to input changes, but, surprisingly, it is less sensitive to high-frequency noise, and thus it can offer an alternative to the "classical" PD controllers by conserving its low-complexity character (i.e., three tuning parameters). Such controllers have been implemented and have demonstrated their interest in various case studies [5] including, among others, the well-known inverted pendulum [3], [4] and, more recently, human balancing [6].

From a control theory point of view, the method that we propose consists in tuning the control parameters (the "gains"  $\alpha, \beta$  and the "delay"  $\tau$ ) in such a way that the solutions of the closed-loop system are asymptotically stable with a guaranteed exponential decay rate.

To achieve this general purpose, several methodologies exist with various objectives. On the one hand, a time-domain approach based on Lyapunov functions has proven its efficiency in estimating solutions decay; see e.g. [7] and the references therein. On the other hand, in frequency-domain, recent studies [8]–[11] have illustrated that the so-called *Multiplicity-Induced-Dominancy* (MID) property can effectively be used to prescribe the decay rate of a DDE solution. In this approach, one assigns to the characteristic function of the system a single real root  $s_0$  with maximal multiplicity. Then, one shows (under necessary and sufficient conditions) that any other root should have a real part less than  $s_0$ , which proves the exponential stability for the solutions of the system. In other words,  $s_0$  represents the *spectral abscissa* of the characteristic function. The originality of the proposed approach is to assign a maximal number of simple real zeros of the characteristic function rather than a multiple zero. Then, we give necessary and sufficient conditions ensuring that the largest assigned root is *negative and dominant*, so that it is nothing but the corresponding spectral abscissa. We call this property: *Coexisting-Real-Root-Induced-Dominancy* (CRRID). The systematic analysis of the CRRID began in [8], [12], where the asymptotic stability of the solutions of an  $n$ -order linear differential equation commanded by a delayed proportional-position-controller. Notice that the MID property mentioned above [9], [10] can be seen as the limit of the CRRID property when all the simple real roots tend to the same (real) value. Moreover, it is commonly accepted that multiple roots are sensitive with respect to small parameter changes; see for instance [13] and references therein. Therefore, the CRRID property exhibits an interesting parametric robustness compared to the MID. As a matter of fact, the effectiveness of the CRRID property has already been emphasized through the problem of control of vibrations, see for instance [14].

The contribution of this work is twofold. First, we show that the CRRID property can successfully be applied to stabilize second-order linear differential equations with a delayed proportional-derivative controller. More precisely, we provide a *necessary and sufficient* condition that guarantees that the assigned real roots are negative and that the largest one is the dominant root of the characteristic function. Hence, we guarantee the exponential decay rate of the solutions of the system. To the best of the authors' knowledge, such a result represents a novelty in the literature. The proof makes use of the Stépán-Hassard formula [15], [16] applied to an appropriate transformation of the characteristic function. Second, we apply our methodology to the control of an oscillator and provide a numerical implementation of our results.

The remaining of this paper is organized as follows: some preliminary results (functional Vandermonde-type matrices, Stépán-Hassard formula) are presented in [Section 2](#). Next, the main results are derived and commented in [Section 3](#). An illustrative example is proposed in [Section 4](#) and some concluding remarks end the paper. The notations are standard.

## 2 Preliminaries

In this paper, we consider a dynamical system represented by a general second-order linear differential equation with a proportional-derivative-delay term of the form

$$y''(t) + a_1y'(t) + a_0y(t) + \alpha_1y'(t - \tau) + \alpha_0y(t - \tau) = 0, \quad (1)$$

under appropriate initial conditions belonging to the Banach space of continuous functions  $\mathcal{C}([-\tau, 0], \mathbb{R})$ . In the Laplace domain, the corresponding characteristic function is a quasipolynomial defined by  $\Delta : \mathbb{C} \times \mathbb{R}^{+*} \rightarrow \mathbb{C}$ :

$$\Delta(s, \tau) = s^2 + a_1 s + a_0 + e^{-\tau s} (\alpha_1 s + \alpha_0), \quad (2)$$

where  $(a_1, a_0, \alpha_1, \alpha_0) \in \mathbb{R}^4$ , see for instance [17]. The degree of  $\Delta$ , defined by the sum of the degrees of the polynomials plus the number of delays (one in our case), is four. The following result asserts that the maximal number of real roots counted with multiplicity is four, see [18].

**Proposition 1** (Pólya-Szegő bound). *Let  $\Delta$  be the quasipolynomial given by (2), and  $\alpha, \beta \in \mathbb{R}$  be such that  $\alpha \leq \beta$ . Denote by  $M_{\alpha, \beta}$  the number of roots of  $\Delta$  counted with multiplicities contained in the set  $\{s \in \mathbb{C}, \alpha \leq \Im(s) \leq \beta\}$ . Then,*

$$\frac{\tau(\beta - \alpha)}{2\pi} - 4 \leq M_{\alpha, \beta} \leq \frac{\tau(\beta - \alpha)}{2\pi} + 4.$$

As a consequence of the previous theorem, setting  $\alpha = \beta = 0$ , we conclude that the degree 4 of  $\Delta$  is a sharp bound for the number of real roots of the quasipolynomial  $\Delta$ . Our analysis of the assignment of real roots to the quasipolynomial (2) will rely on the invertibility of a certain matrix, which because of its shape and properties, is called in the sequel *functional Vandermonde-type matrix*, similar to the ones studied in [19], [20]; see Lemma 3 below. This result will be proved with the help of the multivariate functions  $G_n : \mathbb{R}^{n+1} \times \mathbb{R}^{+*} \rightarrow \mathbb{R}$  defined recursively by  $G_0(s_k, \tau) = e^{-\tau s_k}$  and

$$(s_1 - s_{n+1})G_n(s_1, \dots, s_{n+1}, \tau) = G_{n-1}(s_2, \dots, s_{n+1}, \tau) - G_{n-1}(s_1, \dots, s_n, \tau),$$

for  $n \geq 1$ . The multivariate functions  $G_n(\cdot, \tau)$  are closely related to the functions  $F_n(\cdot, \tau)$  introduced and analyzed in [8] and, thus, benefit from similar properties that we present without proof in Lemma 2 below.

**Lemma 2** (Properties of the  $G_n$  functions). *For any real numbers  $s_1, \dots, s_{n+1}$  and  $\tau > 0$ , we have*

1.  $G_n(s_1, \dots, s_{n+1}, \tau) \neq 0$ .
2.  $\lim_{\tau \rightarrow 0} \frac{1}{(-\tau)^n} G_n(s_1, \dots, s_{n+1}, \tau) = \frac{1}{n!}$ .

**Lemma 3** (Invertibility of a structured functional Vandermonde-type matrix). *Let  $\mathbf{s} = (s_1, s_2, s_3, s_4)$ , then the matrix*

$$V(\mathbf{s}, \tau) = \begin{pmatrix} s_1 & 1 & s_1 e^{-\tau s_1} & e^{-\tau s_1} \\ s_2 & 1 & s_2 e^{-\tau s_2} & e^{-\tau s_2} \\ s_3 & 1 & s_3 e^{-\tau s_3} & e^{-\tau s_3} \\ s_4 & 1 & s_4 e^{-\tau s_4} & e^{-\tau s_4} \end{pmatrix} \quad (3)$$

*is invertible for any  $\tau > 0$  and any distinct real numbers  $s_1, \dots, s_4$ .*

*Proof.* We give an explicit factorized form of the determinant  $v(\mathbf{s}, \tau)$  of  $V(\mathbf{s}, \tau)$  and then we show that  $v(\mathbf{s}, \tau) \neq 0$  for  $\tau > 0$  and  $s_i \neq s_j$ . Observe that for  $1 \leq k \leq 4$ , we have  $e^{-\tau s_k} = G_0(s_k, \tau)$  and thus  $s_k e^{-\tau s_k} = -G'_0(s_k, \tau)$ , where the prime denotes the derivative with respect to  $\tau$ . We start with the structured functional Vandermonde-type matrix  $V$  defined by (3) and we denote by  $L_i$ , for  $1 \leq i \leq 4$ , its  $i$ -th line. By replacing  $L_i$  by  $L_i - L_{i+1}$ , for  $1 \leq i < 4$  and using the properties of the  $G_n$  functions we get

$$v = (s_1 - s_2)(s_2 - s_3)(s_3 - s_4) \begin{vmatrix} 1 & 0 & G'_1(s_1, s_2, \tau) & G_1(s_1, s_2, \tau) \\ 1 & 0 & G'_1(s_2, s_3, \tau) & G_1(s_2, s_3, \tau) \\ 1 & 0 & G'_1(s_3, s_4, \tau) & G_1(s_3, s_4, \tau) \\ s_4 & 1 & -G'_0(s_4, \tau) & G_0(s_4, \tau) \end{vmatrix}.$$

We replace  $L_i$  by  $L_i - L_{i+1}$  one more time, for  $1 \leq i \leq 2$ , and factor out  $(s_1 - s_3)(s_2 - s_4)$  to obtain

$$v = (s_1 - s_2)(s_2 - s_3)(s_3 - s_4)(s_1 - s_3)(s_2 - s_4) \begin{vmatrix} 0 & 0 & G_2'(s_1, s_2, s_3, \tau) & G_2(s_1, s_2, s_3, \tau) \\ 0 & 0 & G_2'(s_2, s_3, s_4, \tau) & G_2(s_2, s_3, s_4, \tau) \\ 1 & 0 & G_1'(s_3, s_4, \tau) & G_1(s_3, s_4, \tau) \\ s_4 & 1 & -G_0'(s_4, \tau) & G_0(s_4, \tau) \end{vmatrix}.$$

To show that  $v \neq 0$ , it suffices to prove that the latter matrix is invertible. Note that we can express the determinant  $\begin{vmatrix} G_2'(s_1, s_2, s_3, \tau) & G_2(s_1, s_2, s_3, \tau) \\ G_2'(s_2, s_3, s_4, \tau) & G_2(s_2, s_3, s_4, \tau) \end{vmatrix}$  via the following quotient derivative

$$-G_2(s_1, s_2, s_3, \tau)^2 \left( \frac{G_2(s_2, s_3, s_4, \tau)}{G_2(s_1, s_2, s_3, \tau)} \right)'.$$

The above expression holds for any  $\tau > 0$ , thus its vanishing implies that there exists a real constant  $C$  (depending on  $\mathbf{s}$ ) such that

$$G_2(s_2, s_3, s_4, \tau) = CG_2(s_1, s_2, s_3, \tau).$$

By continuity, taking  $\tau \rightarrow 0$  yields that  $C = 1$ , a contradiction, since

$$G_2(s_2, s_3, s_4, \tau) - G_2(s_1, s_2, s_3, \tau) = (s_4 - s_1)G_3(\mathbf{s}, \tau) \neq 0.$$

Therefore,  $v \neq 0$  for any  $\tau > 0$  and any distinct numbers  $s_k$ ,  $k = 1, \dots, 4$ .  $\square$

To prove the dominancy of the rightmost real spectral value, we will use the Stépán-Hassard formula [15], [16] that we recall in the next theorem.

**Theorem 4** (Stépán-Hassard formula). *Let  $\Delta(x) = x^n + \sum_{k=0}^n a_k x^k + e^{-\tau x} \sum_{k=0}^m \alpha_k x^k$  be a quasipolynomial function and  $0 < \rho_1 \leq \dots \leq \rho_r$  be the positive zeros (counted with multiplicities) of the real function  $R_q(\omega) = \Re(\mathbf{i}^{-n} \Delta(\mathbf{i}\omega))$ . For each  $1 \leq j \leq r$  such that  $\Delta(\mathbf{i}\rho_j) = 0$ , assume that the multiplicity of  $\mathbf{i}\rho_j$  as a zero of  $\Delta$  is the same as the multiplicity of  $\rho_j$  as a zero of  $R_q(\omega)$ . Then the number of roots of the quasipolynomial  $\Delta(x)$  which lie in the half plane  $\{\Re(x) > 0\}$ , counted with multiplicity, is given by the formula*

$$\mathcal{Z} = \frac{n - K}{2} + \frac{1}{2}(-1)^r \operatorname{sgn} S^{(\kappa)}(0) + \sum_{j=1}^r (-1)^{j-1} \operatorname{sgn} S(\rho_j), \quad (4)$$

where  $K$  is the number of purely imaginary roots of  $\Delta$ ,  $\kappa$  is the multiplicity of 0 as a root of  $\Delta$ , and  $S(\omega) = \Im(\mathbf{i}^{-n} \Delta(\mathbf{i}\omega))$ .

### 3 Main results

In this section, we analyze the properties of quasipolynomial functions saturating their number of real zeros, and we derive some results for the asymptotic stability of the solutions of the DDE (1). More precisely, first, in Proposition 5 we give necessary and sufficient conditions on the coefficients of the quasipolynomial (2) for the number of distinct real zeros to be maximal. For the next results, we assume that the real roots are uniformly distributed. Second, in Proposition 7 we give a necessary and sufficient condition that guarantees the negativity of the real zeros of  $\Delta$ . Third, in Theorem 8 we show that the coexistence of the maximum number of real spectral values implies the CRRID property for the quasipolynomial  $\Delta$ , i.e. the rightmost real zero is the spectral abscissa. Finally, the combination of these results allows us to prove exponential stability with a guaranteed decay rate for the solutions of (1).

### 3.1 Assigning real roots of the characteristic function

In this subsection, we show that assigning the maximal number of distinct real roots to the quasipolynomial  $\Delta$  uniquely determines the coefficients  $a_k$  and  $\alpha_k$ .

**Proposition 5** (Coexisting real roots). *For a given delay  $\tau > 0$ , the quasipolynomial (2) admits 4 distinct real spectral values  $s_1, \dots, s_4$  if, and only if, the real coefficients  $a_1, a_0$  and  $\alpha_1, \alpha_0$  are respectively given by the following functions in  $\tau$  and  $\mathbf{s} = (s_1, \dots, s_4)$ :*

$$a_1 = \frac{-1}{v(\mathbf{s}, \tau)} \det \begin{pmatrix} s_1^2 & 1 & s_1 e^{-\tau s_1} & e^{-\tau s_1} \\ s_2^2 & 1 & s_2 e^{-\tau s_2} & e^{-\tau s_2} \\ s_3^2 & 1 & s_3 e^{-\tau s_3} & e^{-\tau s_3} \\ s_4^2 & 1 & s_4 e^{-\tau s_4} & e^{-\tau s_4} \end{pmatrix}, \quad a_0 = \frac{1}{v(\mathbf{s}, \tau)} \det \begin{pmatrix} s_1^2 & s_1 & s_1 e^{-\tau s_1} & e^{-\tau s_1} \\ s_2^2 & s_2 & s_2 e^{-\tau s_2} & e^{-\tau s_2} \\ s_3^2 & s_3 & s_3 e^{-\tau s_3} & e^{-\tau s_3} \\ s_4^2 & s_4 & s_4 e^{-\tau s_4} & e^{-\tau s_4} \end{pmatrix},$$

and

$$\alpha_1 = \frac{-1}{v(\mathbf{s}, \tau)} \det \begin{pmatrix} s_1^2 & s_1 & 1 & e^{-\tau s_1} \\ s_2^2 & s_2 & 1 & e^{-\tau s_2} \\ s_3^2 & s_3 & 1 & e^{-\tau s_3} \\ s_4^2 & s_4 & 1 & e^{-\tau s_4} \end{pmatrix}, \quad \alpha_0 = \frac{1}{v(\mathbf{s}, \tau)} \det \begin{pmatrix} s_1^2 & s_1 & 1 & s_1 e^{-\tau s_1} \\ s_2^2 & s_2 & 1 & s_2 e^{-\tau s_2} \\ s_3^2 & s_3 & 1 & s_3 e^{-\tau s_3} \\ s_4^2 & s_4 & 1 & s_4 e^{-\tau s_4} \end{pmatrix},$$

where  $v(\mathbf{s}, \tau)$  is the determinant of the structured functional Vandermonde-type matrix  $V(\mathbf{s}, \tau)$ , introduced in Lemma 3.

*Proof.* Assume that (2) admits four distinct real roots  $s_1, \dots, s_4$ , which entails that the coefficients  $a_1, a_0$  and  $\alpha_1, \alpha_0$  satisfy the linear system

$$a_1 s_k + a_0 + \alpha_1 e^{-\tau s_k} s_k + \alpha_0 e^{-\tau s_k} = -s_k^2,$$

for  $1 \leq k \leq 4$ . Thanks to the invertibility of matrix  $V(\mathbf{s}, \tau)$ , as ensured by Lemma 3, one deals with a Cramer system with respect to the coefficients  $a_1, a_0$  and  $\alpha_1, \alpha_0$ . So that, we easily compute these coefficients with the standard formulas.  $\square$

The following corollary gives the closed-form expression of the coefficients  $a_1, a_0$  and  $\alpha_1, \alpha_0$ , in the case where the real zeros are equidistributed.

**Corollary 6.** *If the quasipolynomial  $\Delta$  admits the equidistant real roots  $s_k = s_1 - (k-1)d$ , for  $d > 0$  and  $1 \leq k \leq 4$ , then its coefficients are given by*

$$a_0 = \frac{6d^2}{(e^{\tau d} - 1)^2} + s_1^2 - ds_1 \left( \frac{e^{\tau d} - 5}{e^{\tau d} - 1} \right), \quad a_1 = d \left( \frac{e^{\tau d} - 5}{e^{\tau d} - 1} \right) - 2s_1,$$

$$\alpha_0 = \frac{-2de^{\tau s_1}}{(e^{\tau d} - 1)^2} \left( 3d - s_1(1 - e^{-\tau d}) \right), \quad \alpha_1 = -2de^{\tau s_1} \frac{e^{-\tau d}}{e^{\tau d} - 1}.$$

### 3.2 Assuring negativity of the real roots

The following proposition gives a necessary and sufficient condition for the coefficients of the quasipolynomial (2) to ensure that the rightmost real root is negative.

**Proposition 7.** *For a fixed delay  $\tau > 0$ , assume that the quasipolynomial  $\Delta$  admits four equidistant real roots  $s_k = s_1 - (k-1)d$ , for  $d > 0$  and  $1 \leq k \leq 4$ . Then,  $s_1$  is negative if, and only if,*

$$a_1(\mathbf{s}, \tau^*) + s_1 - d = 0, \tag{5}$$

with

$$\tau^* = \frac{1}{d} \ln \left( 1 - 4 \frac{d}{s_1} \right) > 0.$$

It should be mentioned that equation (5) is of importance for control design purposes. In particular, if the coefficient  $a_1$  is fixed by the model and one wants to assign equidistanted real roots to stabilize the system, then the distance  $d$  needs to be chosen such that  $d - a_1 < 0$ .

*Proof.* By a direct calculation using the explicit form of  $a_1$  given in Corollary 6, we obtain

$$a_1(\mathbf{s}, \tau) + s_1 - d = -s_1 - 4d \frac{1}{e^{\tau d} - 1}.$$

Then,  $a_1(\mathbf{s}, \tau^*) + s_1 - d = 0$  if, and only if,  $\tau^* = \frac{1}{d} \ln \left( 1 - 4 \frac{d}{s_1} \right)$ . Finally, one immediately observes that  $\tau^* > 0$  if, and only if,  $s_1 < 0$ .  $\square$

### 3.3 The CRRID property

In this subsection, we prove the CRRID property for the quasipolynomial  $\Delta$ . Namely, we have the following theorem.

**Theorem 8.** *If the quasipolynomial  $\Delta(\cdot, \tau)$  admits the maximal number of equidistanted real roots  $s_k = s_1 - (k - 1)d$ , for  $1 \leq k \leq 4$ , then*

$$\forall s_0 \in \mathbb{C} \setminus \{s_1\}, \quad \Delta(s_0) = 0 \quad \implies \quad \Re(s_0) < s_1.$$

Theorem 8 asserts that  $s_1$  is a strictly dominant root of  $\Delta$ , i.e. it is the spectral abscissa of the quasipolynomial. Moreover, by carefully studying the proof, we have the stronger result that the pair  $(s_1, s_2 = s_1 - d)$  is a system of dominant roots for  $\Delta$ , i.e.

$$\forall s_0 \in \mathbb{C} \setminus \{s_1, s_2\}, \quad \Delta(s_0) = 0 \quad \implies \quad \Re(s_0) \leq s_2.$$

Our strategy to prove the dominance of  $s_1$  for  $\Delta(\cdot, \tau)$  relies on the application of the Stépán-Hassard formula [16] on a transformed version of  $\Delta$ . Let us introduce the following transformation of the complex plane:

$$\begin{aligned} \phi : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto dz + s_1 - d, \end{aligned}$$

which translates the root  $s_1 - d$  to the origin and scales the distance between two consecutive real root to 1. We introduce the transformed quasipolynomial

$$Q(z, q) = \frac{1}{d^2} \Delta(\phi(z), \tau) = z^2 + b_1 z + b_0 + e^{-qz} (\beta_1 z + \beta_0),$$

where the new delay parameter is  $q = \tau d$ . The following lemma assert that the normalized quasipolynomial  $Q(\cdot, q)$  admits four distinct real roots, gives the explicit form of the coefficients  $b_1, b_0, \beta_1$ , and  $\beta_0$ , and asserts that the problem of proving that  $s_1$  is dominant for  $\Delta$  is the same as proving that  $z = 1$  is dominant for  $Q$ .

**Lemma 9** (Properties of the normalized quasipolynomial). *The normalized quasipolynomial  $Q(z, q)$  admits the real roots  $\{1, 0, -1, -2\}$ . The coefficients  $\{b_1, b_0, \beta_1, \beta_0\}$  of  $Q$  are given by*

$$\begin{aligned} b_0(q) &= 2 \frac{2e^q + 1}{(e^q - 1)^2}, & b_1(q) &= -\frac{e^q + 3}{e^q - 1}, \\ \beta_0(q) &= -2 \frac{2e^q + 1}{(e^q - 1)^2}, & \beta_1(q) &= -\frac{2}{e^q - 1}. \end{aligned}$$

*The root  $s_1$  of  $\Delta$  is dominant if, and only if,  $z = 1$  is a dominant root of  $Q(\cdot, q)$ .*

*Proof.* The map  $\phi(z)$  is an isomorphism of the complex plane, that maps  $\{1, 0, -1, -2\}$  to the real roots  $\{s_1, s_2, s_3, s_4\}$  of  $\Delta$  to. Thus, the coefficients  $\{b_1, b_0, \beta_1, \beta_0\}$  of  $Q$  satisfy the set of linear equations  $Q(z_0) = 0$  for  $z_0 \in \{1, 0, -1, -2\}$ . Therefore, using [Corollary 6](#) with  $s_1 = 1$  and  $d = 1$ , we deduce that the coefficients  $\{b_1, b_0, \beta_1, \beta_0\}$  admit the closed forms given in the statement of the Lemma.

Moreover, the linear transform  $\phi$  is the composition of a translation and a positive scaling (in particular, it does not involve a rotation). Thus, any inequality between the real parts of the roots of  $\Delta$  is preserved. Hence, the dominance of 1 for  $Q$  corresponds to the dominance of  $s_1$  for  $\Delta$ .  $\square$

To prove the dominance of 1 for  $Q$ , we apply the Stépán-Hassard formula (see [Theorem 4](#)), which relies on the properties of the following parametrized real functions

$$R_q(\omega) = \Re(-Q(i\omega, q)) \quad \text{and} \quad S_q(\omega) = \Im(-Q(i\omega, q)),$$

given by

$$\begin{aligned} R_q(\omega) &= \omega^2 - b_0(q) - \cos(q\omega)\beta_0(q) - \sin(q\omega)\beta_1(q)\omega, \\ S_q(\omega) &= \sin(q\omega)\beta_0(q) - \cos(q\omega)\omega\beta_1(q) - b_1(q)\omega. \end{aligned} \tag{6}$$

In the remaining part of this section, we compute the different numbers involved in the Stépán-Hassard formula (4).

**Number of imaginary roots of  $Q$ .** Let  $\omega \geq 0$  be such that  $Q(i\omega) = 0$ , then  $\omega$  satisfies

$$R_q(\omega) = 0 \quad \text{and} \quad S_q(\omega) = 0.$$

Using simple algebraic manipulations, we obtain that the expressions of  $\cos(q\omega)$  and  $\sin(q\omega)$  are given by

$$\begin{aligned} \cos(q\omega) &= -\frac{\omega^2(b_1\beta_1 - \beta_0) + b_0\beta_0}{\omega^2\beta_1^2 + \beta_0^2}, \\ \sin(q\omega) &= \frac{\omega(\omega^2\beta_1 - \beta_0\beta_1 + b_1\beta_0)}{\omega^2\beta_1^2 + \beta_0^2}. \end{aligned}$$

The relation  $\cos(q\omega)^2 + \sin(q\omega)^2 = 1$ , implies that  $\omega$  is necessarily given by

$$\omega_1 = 0 \quad \text{or} \quad \omega_2 = \pm\sqrt{2b_0 - b_1^2 + \beta_1^2} = \pm\mathbf{i}.$$

Therefore, the only imaginary root of  $Q(\cdot, q)$  is  $\omega = 0$ , which, by definition of  $Q$ , is of multiplicity 1. Moreover, we have

$$S'_q(0) = \frac{1}{(e^q - 1)^2} (e^{2q} - 4(q-1)e^q - 2q - 5),$$

which can be seen positive for any  $q > 0$  with the following argument. Denote by  $f(q)$  the numerator of the right-hand-side of the above equality. Then,  $f''(q) = 4e^q(e^q - (q+1))$  is positive for  $q > 0$ . Thus,  $f'(q)$  is increasing and, since  $f'(0) = 0$ , is positive. Finally, we deduce that  $f(q)$  is increasing and, since  $f(0) = 0$ , we conclude that  $f(q)$  is positive for  $q > 0$ . Hence, in the Stépán-Hassard formula we have  $K = 1$ ,  $\kappa = 1$ , and  $\text{sgn } S_q^{(\kappa)}(0) = 1$ .

**Locating the real positive roots of  $R_q$ .** If  $\omega > 0$  is a root of (6), then (using the expressions of the coefficients given in [Lemma 9](#)) we get

$$\omega^2 + 2\frac{2e^q + 1}{(e^q - 1)^2} (\cos(q\omega) - 1) + \frac{2}{e^q - 1} \omega \sin(q\omega) = 0.$$



Multiplying the last equation by  $q^2$ , reduces the problem to the search of the positive zeros of the parameterized function  $F_q : \mathbb{R}^{+*} \rightarrow \mathbb{R}$  defined by

$$F_q(\rho) = \rho^2 + A(q) (\cos(\rho) - 1) + B(q)\rho \sin(\rho) = 0,$$

where  $\rho = q\omega > 0$ ,  $A(q) = 2q^2 \frac{2e^q+1}{(e^q-1)^2}$ , and  $B(q) = \frac{2q}{e^q-1}$ . The following technical lemma gives the main properties of the functions  $A$  and  $B$  that we will use the next results.

**Lemma 10** (Properties of the  $A$  and  $B$  functions). *For all  $q > 0$ , we have*

$$B(q) < 2 \quad \text{and} \quad 6 > A(q) > 3B(q) > B(q) > 0.$$

The function  $\bar{\rho}_q = \left( B(q) + \sqrt{B(q)^2 + 8A(q)} \right) / 2$  is strictly decreasing and  $\lim_{q \rightarrow 0^+} \bar{\rho}_q = 1 + \sqrt{13}$ .

*Proof.* Clearly we have  $A(q) > 0$  and  $B(q) > 0$ . Moreover,  $B'(q) = -\frac{2}{(e^q-1)^2} (e^q(q-1) + 1) < 0$ , thus  $B$  is decreasing and  $B(q) < \lim_{q \rightarrow 0^+} B(q) = 2$ . Second, we also obtain  $A'(q) < 0$ , which yields  $A(q) < \lim_{q \rightarrow 0^+} A(q) = 6$ . Third, we have  $A - 3B = \frac{2q}{(e^q-1)^2} (2qe^q - 3e^q + q + 3)$ , the second factor is seen to be positive by differentiating it twice, implying that  $A - 3B > 0$ . Consider the function  $\bar{\rho}_q$  defined in the Lemma and whose expression is  $\bar{\rho}_q = \frac{q}{e^q-1} (1 + \sqrt{5 + 8e^q})$ . To show that  $\bar{\rho}_q$  is decreasing, we show that it is the sum of two positive decreasing functions. Namely, we set  $\bar{\rho}_q = \bar{\rho}_q^1 + \bar{\rho}_q^2$ , with  $\bar{\rho}_q^1 = \frac{q}{e^q-1}$  and  $\bar{\rho}_q^2 = \frac{q}{e^q-1} \sqrt{5 + 8e^q}$ . First, we obviously have that  $\bar{\rho}_q^1$  is decreasing. Second, we have  $\frac{d}{dq}(\bar{\rho}_q^2) = -e^q \frac{4e^q+9}{(e^q-1)^2 \sqrt{8e^q+5}} < 0$ , hence  $\bar{\rho}_q^2$  is also decreasing. Thus,  $\bar{\rho}_q < \lim_{q \rightarrow 0^+} \bar{\rho}_q = 1 + \sqrt{13}$   $\square$

The results of [Lemma 10](#) are illustrated by [Figure 1](#).

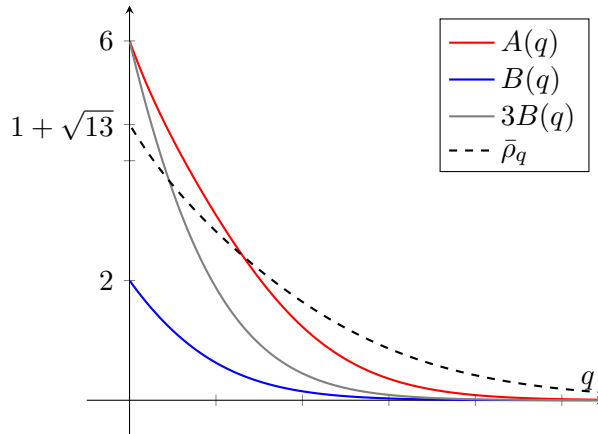


Figure 1: Qualitative behavior of the functions  $A(q)$  (red) and  $B(q)$  (blue) involved in the definition of  $F_q$ . We also illustrate the function  $\bar{\rho}_q$  (dashed).

**Lemma 11.** *The parametrized transcendental function  $F_q(\rho)$  admits at most one real positive zero  $\rho_q^*$ , and when it does, it is located within the open interval  $\mathbf{I} = ]0, \frac{3\pi}{2}[$ .*

*Proof.* First, since  $A(q) > 0$  and  $B(q) > 0$ , observe that for  $\rho > 0$  we have

$$F_q(\rho) \geq \rho^2 - 2A(q) - B(q)\rho.$$

The right hand side of the above inequality is a polynomial of degree 2 in the variable  $\rho$ , whose largest root is  $\bar{\rho}_q$  defined in [Lemma 10](#). Since  $\lim_{q \rightarrow 0} \bar{\rho}_q = 1 + \sqrt{13} < \frac{3\pi}{2}$ , we deduce that, necessarily, all roots  $\rho_q^*$  (if any) of  $F_q(\rho)$  must lie within  $\mathbf{I} = ]0, \frac{3\pi}{2}[$ . Now, by studying the first derivatives of  $F_q$  with respect to  $\rho$  we show that  $F_q$  admits at most one positive root. The third derivative of  $F_q$  with respect to  $\rho$  is given by:

$$F_q^{(3)}(\rho) = -B(q)\rho \cos(\rho) + (A(q) - 3B(q)) \sin(\rho).$$

Depending on whether the function  $C(q) = \frac{B(q)}{A(q)-3B(q)}$  is greater than 1, the equation  $\tan(\rho) = C(q)\rho$  has either one or two solutions within the interval  $\mathbf{I}$ . We separate the two cases.

1. If  $F_q^{(3)}(\rho)$  has only one zero  $\rho_1 \in \mathbb{I}$ . Using that  $F_q^{(3)}(0) = 0$  and  $F_q^{(3)}\left(\frac{3\pi}{2}\right) = 3B(q) - A(q) < 0$ , we deduce that  $F_q''(\rho)$  is increasing on  $]0, \rho_1[$  and decreasing on  $]\rho_1, \frac{3\pi}{2}[$ . Observe that  $F_q''\left(\frac{3\pi}{2}\right) = 2 + \frac{3\pi}{2}B(q) > 0$  and denote  $f_2(q) = F_q''(0) = 2 - A(q) + 2B(q)$ .
  - (a) If  $f_2(q) \geq 0$ , then  $F_q''(\rho)$  is positive on  $\mathbb{I}$ . Thus,  $F_q'(\rho)$  is increasing and since  $F_q'(0) = 0$ , we conclude that  $F_q'$  is positive. Finally,  $F_q(\rho)$  is increasing on  $\mathbb{I}$  and, since  $F_q(0) = 0$ , we conclude that  $F_q$  does not admit a zero on  $\mathbb{I}$ .
  - (b) If  $f_2(q) < 0$ , then  $F_q''(\rho)$  admits a zero  $\rho_2 \in ]0, \rho_1[$ . Thus,  $F_q'(\rho)$  is decreasing on  $]0, \rho_2[$  and increasing on  $]\rho_2, \frac{3\pi}{2}[$ . Using that  $F_q'(0) = 0$  and  $F_q'\left(\frac{3\pi}{2}\right) = A - B + 3\pi > 0$ , we conclude that  $F_q'(\rho)$  admits a zero  $\rho_3 \in ]\rho_2, \frac{3\pi}{2}[$ . Therefore,  $F_q$  is decreasing on  $]0, \rho_3[$  and increasing on  $]\rho_3, \frac{3\pi}{2}[$ . Since  $F_q(0) = 0$  and  $F_q\left(\frac{3\pi}{2}\right) = \frac{9\pi^2}{4} - \frac{3\pi}{2}B(q) - A(q) > 0$ , we conclude that  $F_q$  admits a unique zero  $\rho^* \in ]\rho_3, \frac{3\pi}{2}[$ .
2. If  $F_q^{(3)}(\rho)$  has two zeros  $\rho_1, \rho_2 \in \mathbb{I}$ . Then,  $F_q''(\rho)$  is decreasing on  $]0, \rho_1[$ , increasing on  $]\rho_1, \rho_2[$ , and decreasing on  $]\rho_2, \frac{3\pi}{2}[$ . In this case we necessarily have  $F_q''(0) < 0$  and, using  $F_q''\left(\frac{3\pi}{2}\right) > 0$ , we conclude that  $F_q''$  has a unique zero  $\rho_3 \in ]\rho_1, \rho_2[$ . So  $F_q'(\rho)$  is decreasing on  $]0, \rho_3[$  and increasing on  $]\rho_3, 0[$ . Since  $F_q'(0) = 0$  and  $F_q'\left(\frac{3\pi}{2}\right) > 0$ , the function  $F_q'$  admits a unique zero  $\rho_4 \in ]\rho_3, \frac{3\pi}{2}[$ . Finally,  $F_q$  is decreasing on  $]0, \rho_4[$  and increasing on  $]\rho_4, \frac{3\pi}{2}[$ . Since,  $F_q(0) = 0$  and  $F_q\left(\frac{3\pi}{2}\right) > 0$ , we conclude that  $F_q$  admits a unique zero  $\rho^* \in ]\rho_4, \frac{3\pi}{2}[$ .

To summarize the case distinction made above, we have proved that for any value of the parameter  $q$ , the function  $F_q(\rho)$  admits at most a root  $\rho^* > 0$ .  $\square$

Therefore, for a given delay  $q^*$ , there exists at most one  $\omega^* > 0$ , defined by  $\rho^* = q^*\omega^*$ , such that  $R_q(\omega^*) = 0$ . We now show that  $S_q(\omega^*) > 0$ . In the same fashion as for  $R_q$ , we introduce the parametrized univariate function  $H_q(\rho) = qS_q\left(\frac{\rho}{q}\right)$ , which can be expressed as

$$H_q(\rho) = C(q)\rho + D(q)\rho \cos(\rho) + E(q) \sin(\rho),$$

with  $C(q) = \frac{e^q+3}{e^q-1}$ ,  $D(q) = \frac{2}{e^q-1}$ , and  $E(q) = -2q\frac{2e^q+1}{(e^q-1)^2}$ . The following lemma details the properties of the  $C, D, E$  functions that we will use in the next results.

**Lemma 12** (Properties of the  $C, D, E$  functions). *For all  $q > 0$ , we have*

$$C(q) > D(q) > 0 \quad \text{and} \quad 2D(q) + E(q) < 0.$$

*Proof.* It is immediate to obtain  $C(q) > 0$  and  $D(q) > 0$ ; moreover,  $C(q) - D(q) = \frac{e^q+1}{e^q-1} > 0$ . Next, we have  $2D(q) + E(q) = -\frac{2}{(e^q-1)^2} (2e^q(q-1) + q + 2)$ , which can be seen to be positive since the second factor is positive (proved using its second derivative).  $\square$

The results of [Lemma 12](#) are illustrated by [Figure 2](#).

**Lemma 13.** *For any  $q > 0$  and  $\rho \in ]0, \frac{3\pi}{2}[$ , we have  $H_q(\rho) > 0$ .*

*Proof.* By a direct calculation, we obtain

$$H_q''(\rho) = -D(q)\rho \cos(\rho) - (2D(q) + E(q)) \sin(\rho).$$

The function  $J(q) = -\frac{D(q)}{2D(q)+E(q)}$  satisfies  $0 < J(q) < 1$ , so we conclude that the equation  $\tan(\rho) = J(q)\rho$  admits a unique solution in the interval  $\mathbb{I}$ . Using  $H_q''(0) = 0$  and  $H_q''\left(\frac{3\pi}{2}\right) = 2D(q) + E(q) < 0$ , it follows that  $H_q'$  is increasing on  $]0, \rho_1[$  and decreasing on  $]\rho_1, \frac{3\pi}{2}[$ . Since,  $H_q'(0) = C(q) + D(q) + E(q) > 0$  and  $H_q'\left(\frac{3\pi}{2}\right) = \frac{3\pi}{2}D(q) + C(q) > 0$ , we conclude that  $H_q'$  is positive, and  $H_q$  is increasing, since  $H_q(0) = 0$ , it follows that  $H_q(\rho) > 0$ .  $\square$

Hence, in the Stépán-Hassard formula we either have  $r = 0$  or  $r = 1$  and  $\text{sgn } S_q(\rho^*) = 1$ .

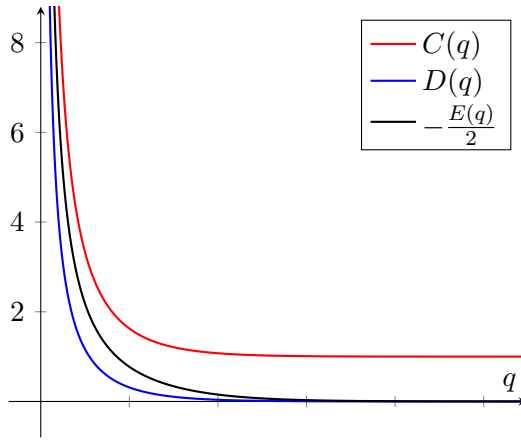


Figure 2: Qualitative behavior of the functions  $C(q)$  (red),  $D(q)$  (blue), and  $E(q)$  (black), involved in the definition of  $H_q$ .

**Applying the Stépán-Hassard formula.** The previous preliminary results allow to compute the number of zeros of  $Q$  located in the right-half plane.

**Theorem 14.** *The root 1 is dominant for  $Q$ .*

*Proof.* We apply the Stépán-Hassard formula (4) to obtain the number of zeros of  $Q$  in  $\{\Re(z) > 0\}$ . If  $r = 0$ , then

$$\mathcal{Z}_0 = \frac{2-1}{2} + \frac{1}{2}(-1)^0(1) = 1.$$

If  $r = 1$ , then

$$\mathcal{Z}_1 = \frac{2-1}{2} + \frac{1}{2}(-1)^1(1) + (-1)^0(1) = 1.$$

Thus, in both cases, the quasipolynomial  $Q$  admits only one root in the right-half plane. But, we already know that  $z = 1$  is a root of  $Q$ . Therefore any other root  $z_0 \in \mathbb{C}$  of  $Q$  satisfies  $\Re(z_0) \leq 0$  and we conclude the dominance of 1 for  $Q$ .  $\square$

As a consequence of [Theorem 14](#) and of [Lemma 9](#), we conclude that  $s_1$  is dominant for  $\Delta(s, \tau)$  and that  $\Delta$  satisfies the CRRID property.

### 3.4 Asymptotic stability of the DDE

Collecting the results of the previous subsections give the following theorem.

**Theorem 15.** *If the quasipolynomial  $\Delta$  given by (2) admits four equidistributed real zeros and (5) is satisfied, then the trivial solution of (1) is asymptotically stable with a decay rate equal to  $s_1$ .*

## 4 Illustrative example

To illustrate the potential of our approach we show that we can improve the decay rate of a stable second-order control system without instantaneous velocity and state observations. We consider the control problem

$$y''(t) + 2\zeta\omega y'(t) + \omega^2 y(t) = u(t), \quad (7)$$

where  $\zeta > 0$  is the damping ratio and  $\omega > 0$  is the oscillator's natural frequency. In absence of a controller, i.e.  $u(t) = 0$ , the characteristic polynomial is  $\Delta(s) = s^2 + 2\zeta\omega s + \omega^2$ , the roots of which

are  $s_{\pm} = \omega \left( -\zeta \pm \mathbf{i}\sqrt{1 - \zeta^2} \right)$  if  $0 < \zeta < 1$ , and  $s_{\pm} = \omega \left( -\zeta \pm \sqrt{\zeta^2 - 1} \right)$  if  $\zeta \geq 1$ . Hence, the spectral abscissa of  $\Delta$  is given by

$$\gamma_0 = \begin{cases} -\zeta\omega & \text{if } 0 < \zeta < 1, \\ \omega \left( -\zeta + \sqrt{\zeta^2 - 1} \right) & \text{if } \zeta \geq 1. \end{cases}$$

In particular, we have  $\gamma_0 < 0$  and the system is asymptotically stable. For many applications, we need a delay  $\tau > 0$  to access a good approximation of the velocity and state of the system. This suggests to use a controller  $u$  having a *delayed proportional-minus-derivative* structure, i.e.

$$u(t) = -\alpha_1 y'(t - \tau) - \alpha_0 y(t - \tau), \quad \tau > 0. \quad (8)$$

The closed-loop system defined by (7) and (8) has a characteristic function given by the four-degree quasipolynomial

$$\Delta(s) = s^2 + 2\zeta\omega s + \omega^2 + e^{-\tau s}(\alpha_1 s + \alpha_0). \quad (9)$$

To illustrate the results of Section 3.3, we consider the case where  $\Delta$  admits equidistributed real roots  $s_k = s_1 - (k - 1)d$ , with  $d > 0$  and  $1 \leq k \leq 4$ . One solves the system of transcendental equations  $\Delta(s_k) = 0$  for the control parameters  $(\alpha_1, \alpha_0, \tau)$  in terms of the system's physical parameters  $\zeta$  and  $\omega$ , as well as the assigned root  $s_1$  and the distance between two consecutive roots "d". More precisely, we obtain the following expressions:

$$\begin{aligned} \tau &= \frac{\sigma}{d}, \\ \alpha_1 &= -\frac{1}{2} (d - 2s_1 - 2\zeta\omega) e^{-\tau(d-s_1)}, \\ \alpha_0 &= -\frac{15}{8} (d - 2s_1 - 2\zeta\omega) \left( d - \frac{2}{3}s_1 - \frac{2}{5}\zeta\omega \right) e^{-\tau(d-s_1)}, \end{aligned} \quad (10)$$

with

$$\sigma = \ln \left( \frac{5d - 2s_1 - 2\zeta\omega}{d - 2s_1 - 2\zeta\omega} \right).$$

Moreover, we also get the following relation that links the distance  $d$ , the spectral value  $s_1$ , and the parameters  $\zeta$  and  $\omega$

$$\frac{1}{2}d^2 - 2(\omega\zeta + s_1)d + \frac{2}{3}(3\zeta^2 - 2)\omega^2 + \frac{4}{3}(\zeta\omega s_1) + \frac{1}{3}(2s_1^2) = 0. \quad (11)$$

Now, to achieve asymptotic stability, we need to choose the positive distance  $d$  and the negative root  $s_1$  such that the positivity of the delay  $\tau$ , given by (10), is guaranteed and such that the constraint (11) is satisfied. First, note that  $\tau > 0$  corresponds to choosing  $s_1$  and  $d$  such that

$$\frac{5d - 2s_1 - 2\zeta\omega}{d - 2s_1 - 2\zeta\omega} = 1 + \frac{4d}{d - 2s_1 - 2\zeta\omega} > 1,$$

which is equivalent to the condition

$$d - 2s_1 - 2\zeta\omega > 0. \quad (12)$$

Second, solving (11) we infer that  $d$  is necessarily given by

$$d = 2s_1 + 2\zeta\omega + \frac{2}{3}\sqrt{6s_1^2 + 6\omega^2 + 12\zeta\omega s_1},$$

which is real if,

$$s_1 < \omega \left( -\zeta - \sqrt{\zeta^2 - 1} \right) \quad \text{or} \quad s_1 > \omega \left( -\zeta + \sqrt{\zeta^2 - 1} \right),$$

when  $\zeta > 1$  or  $s_1$  is arbitrary when  $0 < \zeta \leq 1$ . Furthermore, we have  $d > 0$  if

$$\begin{aligned} s_1 &> \omega \left( -\zeta - \sqrt{2}\sqrt{\zeta^2 - 1} \right) & \text{if } 0 < \zeta \leq 1, \\ s_1 &> \omega \left( -\zeta - \sqrt{\zeta^2 - 1} \right) & \text{if } \zeta \geq 1, \end{aligned}$$

Summing up all the conditions to ensure the realness and the positivity of  $d$ , we obtain the assignment region for the spectral abscissa  $s_1$  that is depicted in [Figure 3](#). In particular, we observe that we can improve the negative spectral abscissa  $\gamma_0$  only in the case  $\zeta < 1$ .

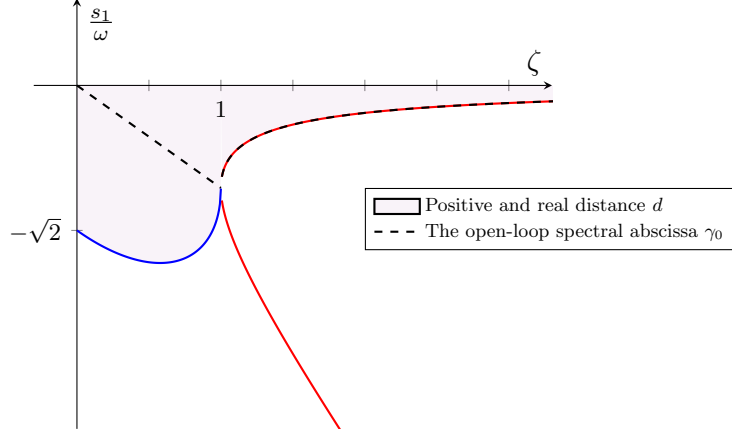


Figure 3: Assignment region for  $\frac{s_1}{\omega}$  to ensure the positivity (blue) and realness (red) of the distance  $d$ .

It is interesting to compare our approach with the MID one. Adapting the results of [10], [11] to our context, we see that the MID can be applied with a controller of the form (8) only when  $0 < \zeta < 1$ , in which case the assigned root of maximal multiplicity four is  $s_0 = \omega \left( -\zeta - \sqrt{2}\sqrt{1 - \zeta^2} \right)$ . This case illustrates the property that the MID is the limiting case of the CRRID when the distance  $d$  approaches 0. To apply the CRRID, we need to choose the spectral abscissa  $s_1$  in such a way that the corresponding distance  $d$  is not zero. Hence, the spectral abscissa given by the CRRID is larger than the one given by the MID. However, the CRRID approach involves choosing the spectral abscissa in an open interval, whereas the MID approach assigns a unique value for the spectral abscissa given in terms of the corresponding delay, illustrating the robustness property of the CRRID. Moreover, the CRRID approach can be applied to systems with  $\zeta > 1$ , even though the exponential decay rate will be worse than the one of the open-loop system.

For a numerical illustration of our results (dominancy of  $s_1$  and stability), we set  $\omega = 1$  and  $\zeta = \frac{1}{5}$  and choose initial conditions  $y(t) = 1$  and  $y'(t) = 0$ , for  $t \in [-\tau, 0]$ .

First, [Figure 4](#) shows the dependence of the distance  $d$  and the delay  $\tau$  on the choice of the root  $s_1$ . In particular, we observe the bound  $s_1 > -\frac{1}{5} - \frac{4\sqrt{3}}{5}$  for the positivity of  $d$  and we notice that  $\tau > 0$  for any choice of  $s_1$ . We now set  $s_1 = -1$ , which yields the following values of the other parameters

$$\begin{cases} d = -\frac{8}{5} + \frac{8}{\sqrt{15}} \approx 0.47, \\ \tau = \frac{30 \ln(5\sqrt{5} - 5\sqrt{3}) - 15 \ln(5)}{16\sqrt{15} - 48} \approx 1.38, \\ \alpha_1 = -4 \cdot 15 \frac{15}{16\sqrt{15} - 48} (5\sqrt{15} - 12) \frac{9 - 8\sqrt{15}}{8\sqrt{15} - 24} \approx -0.14, \\ \alpha_0 = -\frac{4}{75} (10\sqrt{15} - 19) 15 \frac{16\sqrt{15} - 33}{16\sqrt{15} - 48} (5\sqrt{15} - 12) \frac{9 - 8\sqrt{15}}{8\sqrt{15} - 24} \approx -0.54. \end{cases} \quad (13)$$

In [Figure 5a](#), we give the spectrum of the characteristic function (9) with the above parameters and we illustrate the dominancy of the root  $s_1$  with respect to all the other complex roots. Finally, [Figure 5b](#) shows the time response of the closed-loop system, given by equations (7) and (8), and emphasizes the exponential stability of the solutions with decay rate  $s_1$ .

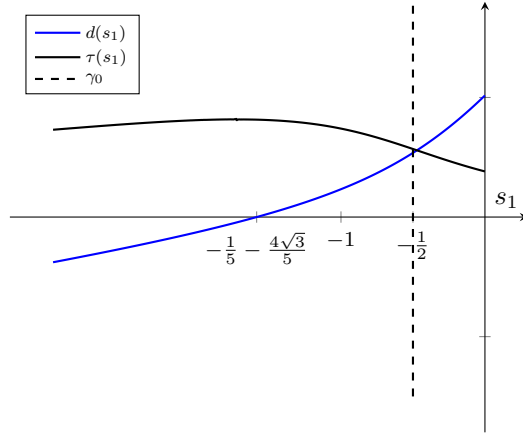
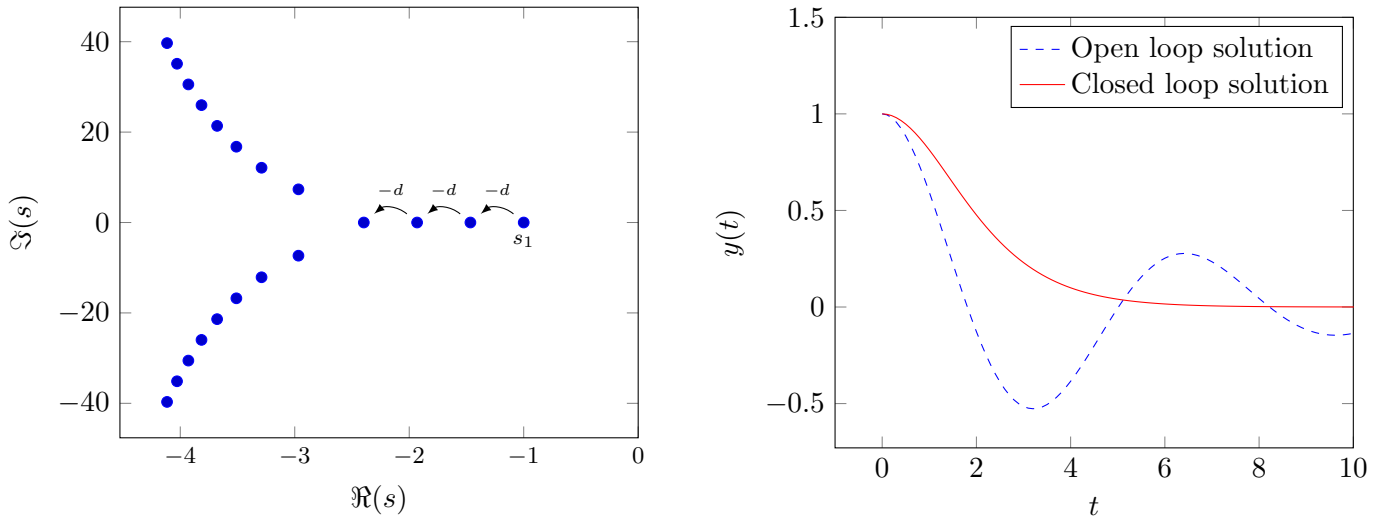


Figure 4: Illustration of the dependence of the parameters  $d$  and  $\tau$  on the root  $s_1$ .



(a) Numerical spectrum of the characteristic function (9) of the closed-loop system given by (7) and (8), with parameters given in (13).

(b) Time response of the closed-loop system given by (7) and (8).

Figure 5: Numerical simulation of the closed-loop system given by (7) and (8), with parameters given in (13).

## 5 P3 $\delta$ Software

*Partial pole placement via delay action* (P3 $\delta$ ) is an intuitive Python software application that facilitates the design of stabilizing feedback laws incorporating time-delays. The software makes use of quasipolynomial properties (in particular, the MID property as studied in [9], [11]) to compute efficient control laws. Based on the results of this work, as well as reference [8], the software has been enhanced to include the CRRID property for design control strategies. P3 $\delta$  is available for free download at <https://cutt.ly/p3delta>, where installation instructions, video demonstrations, and the user manual can also be found. Interested readers may also contact directly any of the authors of the paper.

## 6 Concluding remarks

This work focused on the stabilization of a generic linear differential equation of order 2 with a delayed PD controller via the assignment of the maximum number of real roots of the corresponding characteristic function. We provided the expression of the differential equation for which the corresponding quasipolynomial admits the maximal number of real roots (which is, in our case, equal to four). In the case where the real roots are equidistant, we determined a necessary and sufficient condition on those

coefficients for the largest real root to be negative. And we showed that this root is always the spectral abscissa of the quasipolynomial. Therefore, we proved exponential stability of the second-order linear differential equation and thus, we provided a partial poles placement alternative to the MID property.

## References

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