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# Proximity Operators of Perspective Functions with Nonlinear Scaling

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**Abstract.** A perspective function is a construction which combines a base function defined on a given space with a nonlinear scaling function defined on another space and which yields a lower semicontinuous convex function on the product space. Since perspective functions are typically nonsmooth, their use in first-order algorithms necessitates the computation of their proximity operator. This paper establishes closed-form expressions for the proximity operator of a perspective function defined on a Hilbert space in terms of a proximity operator involving its base function and one involving its scaling function.

# 1 Introduction

Throughout,  $\mathcal{H}$  and  $\mathcal{G}$  are real Hilbert spaces and  $\Gamma_0(\mathcal{H})$  is the class of proper lower semicontinuous convex functions from  $\mathcal{H}$  to  $] -\infty, +\infty]$ . The focus of this paper is on the following construction, which arises in application such as control, mean field games, optimal transportation, information theory, and the calculus of variations; see, e.g., [1, 2, 3, 6, 10, 11, 15, 17, 18, 29, 30, 36, 37, 50, 51] (alternative constructions of nonlinearly scaled perspective functions in certain settings have been studied in [38, 39, 49]; see [13] for a discussion).

**Definition 1.1** [13] The *preperspective* of a base function  $\varphi: \mathcal{H} \rightarrow ] -\infty, +\infty]$  with respect to a *scaling* function  $s: \mathcal{G} \rightarrow ] -\infty, +\infty]$  is

$$\begin{aligned} \varphi \times s: \mathcal{H} \times \mathcal{G} &\rightarrow ] -\infty, +\infty] \\ (x, y) &\mapsto \begin{cases} s(y)\varphi\left(\frac{x}{s(y)}\right), & \text{if } 0 < s(y) < +\infty; \\ +\infty, & \text{if } -\infty \leq s(y) \leq 0 \text{ or } s(y) = +\infty, \end{cases} \end{aligned} \quad (1.1)$$

and the *perspective* of  $\varphi$  with respect to  $s$  is the largest lower semicontinuous convex function  $\varphi \star s$  minorizing  $\varphi \times s$ .

The classical notion of a perspective function, first studied in [45] and further discussed in [21, 46], corresponds to the case of a linear scaling function. Thus, if  $\varphi \in \Gamma_0(\mathcal{H})$ ,  $\mathcal{G} = \mathbb{R}$ , and  $s: y \mapsto y$  in Definition 1.1, it follows from [45, Theorem 3.E] that  $\varphi \star s$  in (1.1) reduces to

$$\tilde{\varphi}: \mathcal{H} \times \mathbb{R} \rightarrow ] -\infty, +\infty] : (x, y) \mapsto \begin{cases} y\varphi\left(\frac{x}{y}\right), & \text{if } y > 0; \\ (\text{rec } \varphi)(x), & \text{if } y = 0; \\ +\infty, & \text{if } y < 0, \end{cases} \quad (1.2)$$

where  $\text{rec } \varphi$  denotes the recession function of  $\varphi$ . A key tool in Hilbertian convex analysis to study variational problems and design solution algorithms for them is Moreau's proximity operator [41, 42]. Recall that, given  $f \in \Gamma_0(\mathcal{H})$  and  $x \in \mathcal{H}$ ,

$$\text{prox}_f x \text{ is the unique minimizer over } \mathcal{H} \text{ of the function } y \mapsto f(y) + \frac{1}{2}\|x - y\|^2. \quad (1.3)$$

This process defines the proximity operator  $\text{prox}_f: \mathcal{H} \rightarrow \mathcal{H}$  of  $f$ , which is extensively discussed in [7]. Formulas for the proximity operator of the classical perspective function  $\tilde{\varphi}$  of (1.2) were derived in [22, 23] and they have been employed to solve minimization problems arising in areas such as statistical biosciences [24], information theory [31], signal recovery [33], and machine learning [48]. Likewise, first order methods for solving variational problems involving the perspective functions of Definition 1.1, which are typically nonsmooth, require their proximity operators; see, e.g., [7, 19, 25, 26] and the references therein. It is the objective of the present paper to derive these proximity operators. The closed-form expressions we obtain for  $\text{prox}_{\varphi \star s}$  are formulated in terms of a proximity operator involving the base function  $\varphi$  and one involving its scaling function  $s$ .

In Section 2, we define our notation and provide the background necessary to our investigation. Section 3 is devoted to preliminary results. Closed-form expressions of  $\text{prox}_{\varphi \star s}$  are established in Section 4. Examples are provided in Section 5.

## 2 Notation and background

The scalar product of a Hilbert space is denoted by  $\langle \cdot | \cdot \rangle$  and the associated norm by  $\| \cdot \|$ . The closed ball with center  $x \in \mathcal{H}$  and radius  $\rho \in ]0, +\infty[$  is denoted by  $B(x; \rho)$ . The Hilbert direct sum of  $\mathcal{H}$  and  $\mathcal{G}$  is denoted by  $\mathcal{H} \oplus \mathcal{G}$ . Let  $f: \mathcal{H} \rightarrow [-\infty, +\infty]$ . Then  $\text{dom } f = \{x \in \mathcal{H} \mid f(x) < +\infty\}$  is the domain of  $f$ ,  $\text{epi } f = \{(x, \xi) \in \mathcal{H} \times \mathbb{R} \mid f(x) \leq \xi\}$  is the epigraph of  $f$ ,

$$f^*: \mathcal{H} \rightarrow [-\infty, +\infty] : x^* \mapsto \sup_{x \in \mathcal{H}} (\langle x | x^* \rangle - f(x)), \quad (2.1)$$

is the conjugate of  $f$ , and  $\partial f$  is the subdifferential of  $f$ . We declare  $f$  convex if  $\text{epi } f$  is convex, lower semicontinuous if  $\text{epi } f$  is closed, and proper if  $-\infty \notin f(\mathcal{H}) \neq \{+\infty\}$ . The recession of  $f \in \Gamma_0(\mathcal{H})$  is

$$\text{rec } f: \mathcal{H} \rightarrow [-\infty, +\infty] : x \mapsto \lim_{0 < \lambda \rightarrow +\infty} \left( \frac{f(z + \lambda x) - f(z)}{\lambda} \right), \quad (2.2)$$

where  $z \in \text{dom } f$  is arbitrary. Let  $C$  be a subset of  $\mathcal{H}$ . Then  $\iota_C$  is the indicator function of  $C$  and  $\sigma_C = \iota_C^*$  is the support function of  $C$ ; if  $C$  is nonempty, closed, and convex, then  $\text{proj}_C = \text{prox}_{\iota_C}$  is the projection operator onto  $C$ . See [7] for background on Hilbertian convex analysis and [46] for the Euclidean setting.

**Definition 2.1** Let  $f: \mathcal{H} \rightarrow [-\infty, +\infty]$ . Then

$$(\forall \xi \in [0, +\infty[) \quad \xi \odot f = \begin{cases} \iota_{\overline{\text{dom } f}}, & \text{if } \xi = 0; \\ \xi f, & \text{if } \xi > 0. \end{cases} \quad (2.3)$$

In addition,

$$f^\vee: \mathcal{H} \rightarrow ]-\infty, +\infty] : x \mapsto \begin{cases} f(x), & \text{if } -\infty < f(x) < 0; \\ +\infty, & \text{otherwise} \end{cases} \quad (2.4)$$

and the  $\blacktriangledown$  envelope of  $f$  is  $f^\blacktriangledown = f^{\vee**}$ . Furthermore,

$$f^\blacktriangle: \mathcal{H} \rightarrow ]-\infty, +\infty] : x \mapsto \begin{cases} f(x), & \text{if } 0 < f(x) < +\infty; \\ +\infty, & \text{otherwise} \end{cases} \quad (2.5)$$

and the  $\blacktriangle$  envelope of  $f$  is  $f^\blacktriangle = f^{\blacktriangle**}$ .

Let us record a few facts.

**Lemma 2.2** [7, Proposition 13.15] Let  $f: \mathcal{H} \rightarrow ]-\infty, +\infty]$  be proper, let  $x \in \mathcal{H}$ , and let  $x^* \in \mathcal{H}$ . Then  $f(x) + f^*(x^*) \geq \langle x | x^* \rangle$ .

**Lemma 2.3** [45] Let  $f \in \Gamma_0(\mathcal{H})$  and  $\gamma \in [0, +\infty[$ . Then the following hold:

- (i)  $\gamma \odot f \in \Gamma_0(\mathcal{H})$ .
- (ii)  $[\tilde{f}(\cdot, \gamma)]^* = \gamma \odot f^*$  and  $(\gamma \odot f)^* = \tilde{f}^*(\cdot, \gamma)$ .

**Lemma 2.4** [13] Let  $f \in \Gamma_0(\mathcal{H})$  be such that  $f^{-1}(]-\infty, 0]) \neq \emptyset$ . Then the following hold:

- (i)  $f^\blacktriangleright \in \Gamma_0(\mathcal{H})$ .
- (ii)  $\text{dom } f^\blacktriangleright = \overline{f^{-1}(]-\infty, 0])} = f^{-1}(]-\infty, 0])$ .
- (iii) Let  $x \in \mathcal{H}$  be such that  $f(x) \in ]-\infty, 0]$ . Then  $f^\blacktriangleright(x) = f(x)$ .

**Lemma 2.5** [13] Let  $f \in \Gamma_0(\mathcal{H})$  be such that  $f^{-1}(]0, +\infty[) \neq \emptyset$ . Then the following hold:

- (i)  $f^\blacktriangle \in \Gamma_0(\mathcal{H})$ .
- (ii)  $\text{dom } f^\blacktriangle = \overline{\text{conv}} f^{-1}(]0, +\infty[)$ .
- (iii)  $f^\blacktriangle(\text{dom } f^\blacktriangle) \subset ]0, +\infty[$ .
- (iv) Let  $x \in \mathcal{H}$  be such that  $f(x) \in ]0, +\infty[$ . Then  $f^\blacktriangle(x) = f(x)$ .

### 3 Preliminary results

We establish results on which the derivations of Section 4 will rest.

**Lemma 3.1** Let  $f \in \Gamma_0(\mathcal{H})$ ,  $x \in \mathcal{H}$ ,  $p \in \mathcal{H}$ , and  $\gamma \in [0, +\infty[$ . Then the following hold:

- (i)  $\text{prox}_{\gamma \odot f} = \begin{cases} \text{proj}_{\overline{\text{dom } f}}, & \text{if } \gamma = 0; \\ \text{prox}_{\gamma f}, & \text{if } \gamma \in ]0, +\infty[. \end{cases}$
- (ii)  $\text{ran } \text{prox}_{\gamma \odot f} \subset \text{dom } (\gamma \odot f) \subset \overline{\text{dom } f}$ .
- (iii)  $p = \text{prox}_{\gamma \odot f} x \Leftrightarrow (\forall y \in \mathcal{H}) \langle y - p \mid x - p \rangle + (\gamma \odot f)(p) \leq (\gamma \odot f)(y)$ .
- (iv)  $p = \text{prox}_{\gamma \odot f} x \Leftrightarrow (\gamma \odot f)(p) + (\gamma \odot f)^*(x - p) = \langle p \mid x - p \rangle$ .
- (v) Suppose that  $\gamma > 0$ . Then  $p = \text{prox}_{\gamma f} x \Leftrightarrow f(p) + f^*((x - p)/\gamma) = \langle p \mid x - p \rangle / \gamma$ .
- (vi) Suppose that  $\gamma > 0$ . Then  $x = \text{prox}_{\gamma f} x + \gamma \text{prox}_{f^*/\gamma}(x/\gamma)$ .

*Proof.* Recall from Lemma 2.3(i) that  $\gamma \odot f \in \Gamma_0(\mathcal{H})$ .

(i): This follows from (2.3).

(ii): This follows from (1.3) and (2.3).

(iii): In view of (2.3), for  $\gamma = 0$ , this is the characterization of the projection of  $x$  onto the nonempty closed convex set  $\overline{\text{dom } f}$  [7, Theorem 3.16] while, for  $\gamma > 0$ , this is [7, Proposition 12.26].

(iv): By virtue of (ii),  $\text{dom } (\gamma \odot f) \subset \overline{\text{dom } f}$ . Hence, Lemma 2.2 and (2.1) yield

$$\langle p \mid x - p \rangle \leq (\gamma \odot f)(p) + (\gamma \odot f)^*(x - p) = \sup_{y \in \overline{\text{dom } f}} (\langle y \mid x - p \rangle + (\gamma \odot f)(p) - (\gamma \odot f)(y)). \quad (3.1)$$

On the other hand, we derive from (iii) that

$$p = \text{prox}_{\gamma \odot f} x \Leftrightarrow \sup_{y \in \overline{\text{dom } f}} (\langle y \mid x - p \rangle + (\gamma \odot f)(p) - (\gamma \odot f)(y)) \leq \langle p \mid x - p \rangle. \quad (3.2)$$

Combining (3.1) and (3.2) furnishes the desired characterization.

(v): This follows from (iv) and Lemma 2.3(ii).

(vi): See [7, Proposition 14.3(ii)].  $\square$

**Lemma 3.2** Let  $\gamma \in [0, +\infty[$ , let  $\phi \in \Gamma_0(\mathbb{R})$  be even and such that  $0 \in \text{int dom } \phi$ , set  $\varphi = \phi \circ \|\cdot\|$ , and let  $x \in \mathcal{H}$ . Then  $\varphi \in \Gamma_0(\mathcal{H})$  and the following hold:

$$(i) \quad \text{prox}_{\gamma \circ \varphi} x = \begin{cases} \frac{\text{prox}_{\gamma \circ \phi} \|x\|}{\|x\|} x, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

$$(ii) \quad \varphi(\text{prox}_{\gamma \circ \varphi} x) = \phi(\text{prox}_{\gamma \circ \phi} \|x\|).$$

*Proof.* Since (ii) follows from (i), we prove the latter. We have  $\varphi \in \Gamma_0(\mathcal{H})$ . In addition, by [7, Propositions 16.17(ii) and 16.27],  $\partial\phi(0)$  is a symmetric compact interval, say  $\partial\phi(0) = [-\tau, \tau]$ , where  $\tau \in [0, +\infty[$ . We also note that there exists  $\rho \in ]0, +\infty]$  such that

$$\overline{\text{dom } \phi} = \begin{cases} [-\rho, \rho], & \text{if } \rho < +\infty; \\ \mathbb{R}, & \text{if } \rho = +\infty \end{cases} \quad \text{and} \quad \overline{\text{dom } \varphi} = \begin{cases} B(0; \rho), & \text{if } \rho < +\infty; \\ \mathcal{H}, & \text{if } \rho = +\infty. \end{cases} \quad (3.3)$$

If  $\rho < +\infty$ , we derive from (3.3) and [7, Example 3.18] that

$$\text{proj}_{\overline{\text{dom } \varphi}} x = \frac{\rho x}{\max\{\|x\|, \rho\}} = \begin{cases} \frac{\rho \|x\|}{\max\{\|x\|, \rho\} \|x\|} x, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0 \end{cases} = \begin{cases} \frac{\text{proj}_{\overline{\text{dom } \phi}} \|x\|}{\|x\|} x, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0 \end{cases} \quad (3.4)$$

whereas, if  $\rho = +\infty$ , it is clear that  $\text{proj}_{\overline{\text{dom } \varphi}} x$  coincides with the last term above. In view of Lemma 3.1(i), this establishes the claim for  $\gamma = 0$ . Now suppose that  $\gamma > 0$ . Then it follows from [12, Proposition 2.1] that

$$\text{prox}_{\gamma \varphi} x = \begin{cases} \frac{\text{prox}_{\gamma \phi} \|x\|}{\|x\|} x, & \text{if } \|x\| > \gamma \tau; \\ 0, & \text{if } \|x\| \leq \gamma \tau. \end{cases} \quad (3.5)$$

Moreover, since, in view of (1.3),  $\|x\| \leq \gamma \tau \Leftrightarrow \|x\| \in \gamma \partial\phi(0) \Leftrightarrow \text{prox}_{\gamma \phi} \|x\| = 0$ , (3.5) reduces to

$$\text{prox}_{\gamma \varphi} x = \begin{cases} \frac{\text{prox}_{\gamma \phi} \|x\|}{\|x\|} x, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0, \end{cases} \quad (3.6)$$

as required.  $\square$

**Lemma 3.3** Let  $f \in \Gamma_0(\mathcal{H})$ , let  $x \in \mathcal{H}$ , and set  $\phi: [0, +\infty[ \rightarrow ]-\infty, +\infty] : \gamma \mapsto f(\text{prox}_{\gamma \circ f} x)$ . Then the following hold:

$$(i) \quad \text{Let } \mu \in [0, +\infty[ \text{ and } \gamma \in ]\mu, +\infty[. \text{ Then } \phi(\gamma) \leq \phi(\mu) - \|\text{prox}_{\mu \circ f} x - \text{prox}_{\gamma \circ f} x\|^2 / (\gamma - \mu).$$

$$(ii) \quad \phi \text{ is decreasing on } [0, +\infty[.$$

$$(iii) \quad \phi \text{ is continuous.}$$

*Proof.* First note that Lemma 2.3(i) guarantees that  $\text{prox}_{\gamma \circ f}$  and, therefore  $\phi$ , are well defined.

(i): Set  $p = \text{prox}_{\mu \circ f} x$  and  $q = \text{prox}_{\gamma \circ f} x$ , and note that (1.3) implies that  $q \in \text{dom } f$ . If  $\mu = 0$ , we assume that  $p = \text{proj}_{\overline{\text{dom } f}} x \in \text{dom } f$  since, otherwise,  $\phi(\mu) = +\infty$  and the inequality holds trivially. By

Lemma 3.1(iii),  $\langle q - p \mid x - p \rangle \leq \mu(f(q) - f(p))$  and  $\langle p - q \mid x - q \rangle \leq \gamma(f(p) - f(q))$ . Adding these inequalities yields

$$\|p - q\|^2 \leq (\gamma - \mu)(f(p) - f(q)) = (\gamma - \mu)(\phi(\mu) - \phi(\gamma)), \quad (3.7)$$

which is equivalent to the announced inequality.

(ii): Clear from (i).

(iii): Set  $T: [0, +\infty[ \rightarrow \mathcal{H}: \gamma \mapsto \text{prox}_{\gamma \odot f} x$ . It follows from [7, Proposition 23.31(iii)] applied to the maximally monotone operator  $\partial f$  that  $T$  is continuous on  $]0, +\infty[$  and from [7, Theorem 23.48] that it is right-continuous at 0. Now suppose that  $(\gamma_n)_{n \in \mathbb{N}}$  be a sequence in  $]0, +\infty[$  such that  $\gamma_n \rightarrow \mu \in [0, +\infty[$ . Then  $T(\gamma_n) \rightarrow T(\mu)$ . If  $\mu = 0$ , by invoking the lower semicontinuity of  $f$  and (ii), we get

$$\phi(0) = f(T(\mu)) \leq \underline{\lim} f(T(\gamma_n)) = \underline{\lim} \phi(\gamma_n) \leq \overline{\lim} \phi(\gamma_n) \leq \phi(0) \quad (3.8)$$

and therefore  $\phi(\gamma_n) \rightarrow \phi(0)$ . If  $\mu > 0$ , the continuity of  $\phi$  at  $\mu$  is established in [5, Lemma 3.27].  $\square$

The following proposition provides explicit expressions for the perspective function of Definition 1.1.

**Proposition 3.4** [13] *Let  $\varphi \in \Gamma_0(\mathcal{H})$  and let  $s: \mathcal{G} \rightarrow [-\infty, +\infty]$  be such that  $S = s^{-1}(]0, +\infty[) \neq \emptyset$ . Let  $x \in \mathcal{H}$  and  $y \in \mathcal{G}$ . Then the following hold:*

(i) *Suppose that  $\varphi^*(\mathcal{H}) \subset [0, +\infty]$ ,  $(\varphi^*)^{-1}(]0, +\infty[) \neq \emptyset$ , and  $-s \in \Gamma_0(\mathcal{G})$ . Then*

$$(\varphi \blacktriangleright s)(x, y) = \begin{cases} s(y)\varphi\left(\frac{x}{s(y)}\right), & \text{if } 0 < s(y) < +\infty; \\ (\text{rec } \varphi)(x), & \text{if } s(y) = 0; \\ +\infty, & \text{otherwise.} \end{cases} \quad (3.9)$$

(ii) *Suppose that  $\varphi^*(\mathcal{H}) \subset \{0, +\infty\}$ . Then  $(\varphi \blacktriangleright s)(x, y) = \varphi(x) + \iota_{\overline{\text{conv}} S}(y)$ .*

(iii) *Suppose that  $\varphi^*(\mathcal{H}) \subset ]-\infty, 0] \cup \{+\infty\}$ ,  $(\varphi^*)^{-1}(]-\infty, 0[) \neq \emptyset$ , and  $s \in \Gamma_0(\mathcal{G})$ . Then*

$$(\varphi \blacktriangleright s)(x, y) = \begin{cases} s(y)\varphi\left(\frac{x}{s(y)}\right), & \text{if } 0 < s(y) < +\infty; \\ (\text{rec } \varphi)(x), & \text{if } y \in \overline{\text{conv}} S \text{ and } s(y) \leq 0; \\ +\infty, & \text{otherwise.} \end{cases} \quad (3.10)$$

Additionally, in each case,  $\varphi \blacktriangleright s \in \Gamma_0(\mathcal{H} \oplus \mathcal{G})$ .

Our derivation of the proximity operator of perspective functions will rely on the following properties.

**Proposition 3.5** [13] *Let  $\varphi: \mathcal{H} \rightarrow ]-\infty, +\infty]$  be proper, let  $s: \mathcal{G} \rightarrow [-\infty, +\infty]$  be such that  $S = s^{-1}(]0, +\infty[) \neq \emptyset$ , let  $x^* \in \mathcal{H}$ , and let  $y^* \in \mathcal{G}$ . Then the following hold:*

(i) *Suppose that  $\varphi^*(\mathcal{H}) \subset [0, +\infty]$  and  $(\varphi^*)^{-1}(]0, +\infty[) \neq \emptyset$ . Then*

$$(\varphi \blacktriangleright s)^*(x^*, y^*) = \begin{cases} \varphi^*(x^*)(-s)^{\blacktriangleright*}\left(\frac{y^*}{\varphi^*(x^*)}\right), & \text{if } 0 < \varphi^*(x^*) < +\infty; \\ \sigma_{\overline{\text{conv}} S}(y^*), & \text{if } \varphi^*(x^*) = 0; \\ +\infty, & \text{if } \varphi^*(x^*) = +\infty. \end{cases} \quad (3.11)$$

(ii) Suppose that  $\varphi^*(\mathcal{H}) \subset \{0, +\infty\}$ . Then

$$(\varphi \blacktriangleright s)^*(x^*, y^*) = \iota_{(\varphi^*)^{-1}(\{0\})}(x^*) + \sigma_{\text{conv} S}(y^*). \quad (3.12)$$

(iii) Suppose that  $\varphi^*(\mathcal{H}) \subset ]-\infty, 0] \cup \{+\infty\}$  and  $(\varphi^*)^{-1}(]-\infty, 0]) \neq \emptyset$ . Then

$$(\varphi \blacktriangleright s)^*(x^*, y^*) = \begin{cases} -\varphi^*(x^*) s^{\blacktriangleleft*} \left( \frac{y^*}{-\varphi^*(x^*)} \right), & \text{if } -\infty < \varphi^*(x^*) < 0; \\ \sigma_{\text{conv} S}(y^*), & \text{if } \varphi^*(x^*) = 0; \\ +\infty, & \text{if } \varphi^*(x^*) = +\infty. \end{cases} \quad (3.13)$$

## 4 Computation of the proximity operator

We are now ready to present our main result.

**Theorem 4.1** Let  $\varphi \in \Gamma_0(\mathcal{H})$  and let  $s : \mathcal{G} \rightarrow [-\infty, +\infty]$  be such that  $S = s^{-1}(]0, +\infty[) \neq \emptyset$ . Let  $x \in \mathcal{H}$ ,  $y \in \mathcal{G}$ , and  $\gamma \in ]0, +\infty[$ . Then the following hold:

(i) Suppose that  $\varphi^*(\mathcal{H}) \subset [0, +\infty]$ ,  $(\varphi^*)^{-1}(]0, +\infty[) \neq \emptyset$ , and  $-s \in \Gamma_0(\mathcal{G})$ . Then there exists a unique  $\eta \in [0, +\infty[$  such that

$$(-s)^\blacktriangleright \left( \text{prox}_{\gamma\varphi^*} \left( \text{prox}_{\frac{\eta}{\gamma} \odot \varphi^*} \left( \frac{x}{\gamma} \right) \right) \odot (-s)^\blacktriangleright y \right) + \eta = 0. \quad (4.1)$$

Furthermore,

$$\text{prox}_{\gamma(\varphi \blacktriangleright s)}(x, y) = \left( x - \gamma \text{prox}_{\frac{\eta}{\gamma} \odot \varphi^*} \left( \frac{x}{\gamma} \right), \text{prox}_{\gamma\varphi^*} \left( \text{prox}_{\frac{\eta}{\gamma} \odot \varphi^*} \left( \frac{x}{\gamma} \right) \right) \odot (-s)^\blacktriangleright y \right). \quad (4.2)$$

(ii) Suppose that  $\varphi^*(\mathcal{H}) \subset \{0, +\infty\}$ . Then  $\text{prox}_{\gamma(\varphi \blacktriangleright s)}(x, y) = (\text{prox}_{\gamma\varphi} x, \text{proj}_{\text{conv} S} y)$ .

(iii) Suppose that  $\varphi^*(\mathcal{H}) \subset ]-\infty, 0] \cup \{+\infty\}$ ,  $(\varphi^*)^{-1}(]-\infty, 0]) \neq \emptyset$ , and  $s \in \Gamma_0(\mathcal{G})$ . Then there exists a unique  $\eta \in [0, +\infty[$  such that

$$\varphi^* \left( \text{prox}_{\frac{1}{\gamma} s^{\blacktriangleleft*}} \left( \text{prox}_{\gamma\eta \odot s^{\blacktriangleleft*}} y \right) \odot \varphi^* \left( \frac{x}{\gamma} \right) \right) + \eta = 0. \quad (4.3)$$

Furthermore,

$$\text{prox}_{\gamma(\varphi \blacktriangleright s)}(x, y) = \left( x - \gamma \text{prox}_{\frac{1}{\gamma} s^{\blacktriangleleft*}} \left( \text{prox}_{\gamma\eta \odot s^{\blacktriangleleft*}} y \right) \odot \varphi^* \left( \frac{x}{\gamma} \right), \text{prox}_{\gamma\eta \odot s^{\blacktriangleleft*}} y \right). \quad (4.4)$$

*Proof.* Set  $(p, q) = \text{prox}_{\gamma(\varphi \blacktriangleright s)}(x, y)$ , which is well defined by virtue of Proposition 3.4. It follows from Lemma 3.1(v) that

$$(\varphi \blacktriangleright s)(p, q) + (\varphi \blacktriangleright s)^* \left( \frac{x-p}{\gamma}, \frac{y-q}{\gamma} \right) = \left\langle p \left| \frac{x-p}{\gamma} \right. \right\rangle + \left\langle q \left| \frac{y-q}{\gamma} \right. \right\rangle \quad (4.5)$$



and, furthermore, that

$$(p, q) \in \text{dom}(\varphi \ltimes s) \quad (4.6)$$

and

$$\left( \frac{x-p}{\gamma}, \frac{y-q}{\gamma} \right) \in \text{dom}(\varphi \ltimes s)^*. \quad (4.7)$$

(i): We deduce from (4.7) and Proposition 3.5(i) that

$$\varphi^* \left( \frac{x-p}{\gamma} \right) \in [0, +\infty[ \quad (4.8)$$

and from (4.6) and Proposition 3.4(i) that

$$s(q) \in [0, +\infty[. \quad (4.9)$$

Since  $\varphi \in \Gamma_0(\mathcal{H})$ , we have  $\varphi^{**} = \varphi$  [7, Corollary 13.38]. Hence, it follows from Proposition 3.4(i), (1.2), Lemma 2.3(ii), and (4.9) that

$$(\varphi \ltimes s)(p, q) = \tilde{\varphi}(p, s(q)) = (s(q) \odot \varphi^*)^*(p). \quad (4.10)$$

Next, since Lemma 2.4(i) asserts that  $(-s)^\blacktriangledown \in \Gamma_0(\mathcal{G})$ , we have  $(-s)^{\blacktriangledown*} \in \Gamma_0(\mathcal{G})$  and hence deduce from Lemma 2.4(ii) and [7, Proposition 13.49] that

$$\sigma_{\overline{\text{conv}} S} = \sigma_{\overline{S}} = \sigma_{\text{dom}(-s)^\blacktriangledown} = \text{rec}(-s)^{\blacktriangledown*}. \quad (4.11)$$

Thus, it follows from (4.8), Proposition 3.5(i), (1.2), and Lemma 2.3(ii) that

$$\begin{aligned} & (\varphi \ltimes s)^* \left( \frac{x-p}{\gamma}, \frac{y-q}{\gamma} \right) \\ &= \begin{cases} \varphi^* \left( \frac{x-p}{\gamma} \right) (-s)^{\blacktriangledown*} \left( \frac{(y-q)/\gamma}{\varphi^*((x-p)/\gamma)} \right), & \text{if } 0 < \varphi^* \left( \frac{x-p}{\gamma} \right) < +\infty; \\ \left( \text{rec}(-s)^{\blacktriangledown*} \right) \left( \frac{y-q}{\gamma} \right), & \text{if } \varphi^* \left( \frac{x-p}{\gamma} \right) = 0 \end{cases} \end{aligned} \quad (4.12)$$

$$\begin{aligned} &= \widetilde{(-s)^{\blacktriangledown*}} \left( \frac{y-q}{\gamma}, \varphi^* \left( \frac{x-p}{\gamma} \right) \right) \\ &= \left( \varphi^* \left( \frac{x-p}{\gamma} \right) \odot (-s)^{\blacktriangledown} \right)^* \left( \frac{y-q}{\gamma} \right). \end{aligned} \quad (4.13)$$

On the other hand, (4.9) yields  $q \in \text{dom } s$ , while (4.7) and Proposition 3.5(i) imply that  $(x-p)/\gamma \in (\varphi^*)^{-1}([0, +\infty[)$ . Therefore, since (4.9) and Lemma 2.4(iii) yield

$$0 \leq s(q) = -(-s)^\blacktriangledown(q), \quad (4.14)$$

we deduce from (2.3) that

$$\left( s(q) \odot \varphi^* \right) \left( \frac{x-p}{\gamma} \right) + \left( \varphi^* \left( \frac{x-p}{\gamma} \right) \odot (-s)^\blacktriangledown \right) (q) = 0. \quad (4.15)$$

Consequently, it results from (4.10), (4.13), and (4.5) that

$$\begin{aligned}
& (s(q) \odot \varphi^*)^*(p) + (s(q) \odot \varphi^*) \left( \frac{x-p}{\gamma} \right) + \left( \varphi^* \left( \frac{x-p}{\gamma} \right) \odot (-s)^\blacktriangledown \right) (q) + \left( \varphi^* \left( \frac{x-p}{\gamma} \right) \odot (-s)^\blacktriangledown \right)^* \left( \frac{y-q}{\gamma} \right) \\
&= (s(q) \odot \varphi^*)^*(p) + \left( \varphi^* \left( \frac{x-p}{\gamma} \right) \odot (-s)^\blacktriangledown \right)^* \left( \frac{y-q}{\gamma} \right) \\
&= (\varphi \blacktriangleright s)(p, q) + (\varphi \blacktriangleright s)^* \left( \frac{x-p}{\gamma}, \frac{y-q}{\gamma} \right) \\
&= \left\langle p \mid \frac{x-p}{\gamma} \right\rangle + \left\langle q \mid \frac{y-q}{\gamma} \right\rangle.
\end{aligned} \tag{4.16}$$

We therefore derive from Lemma 2.2 and Lemma 2.3(i) that

$$(s(q) \odot \varphi^*)^*(p) + (s(q) \odot \varphi^*)^{**} \left( \frac{x-p}{\gamma} \right) = \left\langle p \mid \frac{x-p}{\gamma} \right\rangle \tag{4.17}$$

and

$$\left( \varphi^* \left( \frac{x-p}{\gamma} \right) \odot (-s)^\blacktriangledown \right) (q) + \left( \varphi^* \left( \frac{x-p}{\gamma} \right) \odot (-s)^\blacktriangledown \right)^* \left( \frac{y-q}{\gamma} \right) = \left\langle q \mid \frac{y-q}{\gamma} \right\rangle. \tag{4.18}$$

In turn, (4.17) and Lemma 3.1(v)–(vi) yield

$$p = \text{prox}_{\gamma(s(q) \odot \varphi^*)^*} x = x - \gamma \text{prox}_{\frac{s(q)}{\gamma} \odot \varphi^*} \left( \frac{x}{\gamma} \right), \tag{4.19}$$

while (4.18) and Lemma 3.1(v) yield

$$q = \text{prox}_{\gamma \varphi^* \left( (x-p)/\gamma \right) \odot (-s)^\blacktriangledown} y. \tag{4.20}$$

Upon combining (4.19) and (4.20), we obtain

$$q = \text{prox}_{\gamma \varphi^* \left( \text{prox}_{\frac{s(q)}{\gamma} \odot \varphi^*} \left( \frac{x}{\gamma} \right) \right) \odot (-s)^\blacktriangledown} y. \tag{4.21}$$

Consequently, we deduce from (4.14) that  $\eta = s(q) \in [0, +\infty[$  solves (4.1), from which (4.2) follows. To establish the uniqueness of the solution to (4.1), let us define

$$\begin{cases} \phi_1: [0, +\infty[ \rightarrow ]-\infty, 0] : \mu \mapsto (-s)^\blacktriangledown \left( \text{prox}_{\gamma \mu \odot (-s)^\blacktriangledown} y \right) \\ \phi_2: [0, +\infty[ \rightarrow [0, +\infty[ : \eta \mapsto \varphi^* \left( \text{prox}_{\frac{\eta}{\gamma} \odot \varphi^*} (x/\gamma) \right). \end{cases} \tag{4.22}$$

Lemma 3.3(ii) asserts that  $\phi_1$  and  $\phi_2$  are decreasing, which renders  $\phi_1 \circ \phi_2$  increasing. In turn,  $\phi_1 \circ \phi_2 + \text{Id}$  is strictly increasing and therefore the equation  $(\phi_1 \circ \phi_2)(\eta) + \eta = 0$ , which is precisely (4.1), has at most one solution in  $[0, +\infty[$ .

(ii): This follows from Proposition 3.4(ii) and [7, Proposition 24.11].

(iii): A noteworthy consequence of Lemma 3.1(vi) is that

$$(p, q) = (x, y) - \gamma \text{prox}_{(\varphi \blacktriangleright s)^*/\gamma} \left( \frac{x}{\gamma}, \frac{y}{\gamma} \right). \tag{4.23}$$

Since Lemma 2.5(i)–(ii) yields  $\overline{\text{conv}} S = \text{dom } s^\blacktriangle = \text{dom } s^{\blacktriangle**}$ , we invoke [7, Proposition 13.49] to get

$$\text{rec}(s^{\blacktriangle**}) = \sigma_{\text{dom } s^\blacktriangle} = \sigma_{\overline{\text{conv}} S}. \quad (4.24)$$

In turn, by virtue of Proposition 3.5(iii) and Proposition 3.4(i), we obtain

$$\begin{aligned} (\varphi \blacktriangleright s)^*: (x^*, y^*) &\mapsto \begin{cases} -\varphi^*(x^*) s^{\blacktriangle**} \left( \frac{y^*}{-\varphi^*(x^*)} \right), & \text{if } -\infty < \varphi^*(x^*) < 0; \\ \sigma_{\overline{\text{conv}} S}(y^*), & \text{if } \varphi^*(x^*) = 0 \end{cases} \\ &= (s^{\blacktriangle**} \blacktriangleright (-\varphi^*)) (y^*, x^*). \end{aligned} \quad (4.25)$$

Now set  $(r, t) = \text{prox}_{(\varphi \blacktriangleright s)^*/\gamma}(x/\gamma, y/\gamma)$ . Then we derive from (4.25) and [7, Proposition 24.8(iv)] that

$$(t, r) = \text{prox}_{(s^{\blacktriangle**} \blacktriangleright (-\varphi^*))/\gamma} \left( \frac{y}{\gamma}, \frac{x}{\gamma} \right). \quad (4.26)$$

Therefore, since  $0 \leq s^\blacktriangle \in \Gamma_0(\mathcal{G})$  by Lemma 2.5(i)&(iii),  $(s^\blacktriangle)^{-1}(]0, +\infty[) = S \neq \emptyset$  by Lemma 2.5(iv), and  $-(-\varphi^*)^\blacktriangleright = -(-\varphi^*) = \varphi^* \in \Gamma_0(\mathcal{H})$ , it follows from (i) that

$$(t, r) = \left( \frac{y}{\gamma} - \frac{1}{\gamma} \text{prox}_{\gamma \odot s^\blacktriangle} y, \text{prox}_{\frac{1}{\gamma} s^\blacktriangle} (\text{prox}_{\gamma \odot s^\blacktriangle} y) \odot \varphi^* \left( \frac{x}{\gamma} \right) \right), \quad (4.27)$$

where  $\eta$  is the unique solution in  $]0, +\infty[$  to (4.3). The conclusion then comes from (4.23).  $\square$

Next, we provide explicit formulas for  $\text{prox}_{\gamma(\varphi \blacktriangleright s)}(x, y)$  in items (i) and (iii) of Theorem 4.1 (item (ii) is already explicit).

**Proposition 4.2** *Consider the assumptions and notation of Theorem 4.1(i), and set*

$$\begin{cases} \Omega_1 = \left\{ (u, v) \in \mathcal{H} \times \mathcal{G} \mid \varphi^* \left( \text{proj}_{\overline{\text{dom}} \varphi^*} \left( \frac{u}{\gamma} \right) \right) = 0 \text{ and } s(\text{proj}_{\overline{S}} v) = 0 \right\} \\ \Omega_2 = \left\{ (u, v) \in \mathcal{H} \times \mathcal{G} \mid \varphi^* \left( \text{proj}_{\overline{\text{dom}} \varphi^*} \left( \frac{u}{\gamma} \right) \right) \in ]0, +\infty[ \text{ and } s \left( \text{prox}_{\gamma \varphi^*} \left( \text{proj}_{\overline{\text{dom}} \varphi^*} \left( \frac{u}{\gamma} \right) \right) (-s)^\blacktriangleright v \right) = 0 \right\} \\ \Omega_3 = \left\{ (u, v) \in \mathcal{H} \times \mathcal{G} \mid \varphi^* \left( \text{prox}_{\frac{s(\text{proj}_{\overline{S}} v)}{\gamma} \varphi^*} \left( \frac{u}{\gamma} \right) \right) = 0 \text{ and } s(\text{proj}_{\overline{S}} v) \in ]0, +\infty[ \right\} \\ \Omega_4 = (\mathcal{H} \times \mathcal{G}) \setminus (\Omega_1 \cup \Omega_2 \cup \Omega_3). \end{cases} \quad (4.28)$$

Then exactly one of the following holds:

- (i)  $(x, y) \in \Omega_1$ ,  $\eta = 0$ , and  $\text{prox}_{\gamma(\varphi \blacktriangleright s)}(x, y) = \left( x - \gamma \text{proj}_{\overline{\text{dom}} \varphi^*}(x/\gamma), \text{proj}_{\overline{S}} y \right)$ .
- (ii)  $(x, y) \in \Omega_2$ ,  $\eta = 0$ , and  $\text{prox}_{\gamma(\varphi \blacktriangleright s)}(x, y) = \left( x - \gamma \text{proj}_{\overline{\text{dom}} \varphi^*}(x/\gamma), \text{prox}_{\gamma \varphi^*} \left( \text{proj}_{\overline{\text{dom}} \varphi^*}(x/\gamma) \right) (-s)^\blacktriangleright y \right)$ .
- (iii)  $(x, y) \in \Omega_3$ ,  $\eta = s(\text{proj}_{\overline{S}} y) > 0$ , and  $\text{prox}_{\gamma(\varphi \blacktriangleright s)}(x, y) = \left( x - \gamma \text{prox}_{\frac{s(\text{proj}_{\overline{S}} y)}{\gamma} \varphi^*}(x/\gamma), \text{proj}_{\overline{S}} y \right)$ .

(iv)  $(x, y) \in \Omega_4$ ,  $\eta > 0$  solves

$$\eta = s \left( \text{prox}_{\gamma\varphi^*} \left( \text{prox}_{\frac{\eta}{\gamma}\varphi^*} \left( \frac{x}{\gamma} \right) \right)_{(-s)\blacktriangledown} y \right), \quad (4.29)$$

and

$$\text{prox}_{\gamma(\varphi \blacktriangleright s)}(x, y) = \left( x - \gamma \text{prox}_{\frac{\eta}{\gamma}\varphi^*} \left( \frac{x}{\gamma} \right), \text{prox}_{\gamma\varphi^*} \left( \text{prox}_{\frac{\eta}{\gamma}\varphi^*} \left( \frac{x}{\gamma} \right) \right)_{(-s)\blacktriangledown} y \right). \quad (4.30)$$

*Proof.* Lemma 2.4(ii) yields

$$\text{dom } (-s)\blacktriangledown = \overline{(-s)^{-1}(\cdot - \infty, 0]} = \bar{S} = s^{-1}([0, +\infty]). \quad (4.31)$$

Hence, it follows from Lemma 3.1(ii) that

$$(\forall \mu \in [0, +\infty]) \quad \text{prox}_{\mu \odot (-s)\blacktriangledown} y \in s^{-1}([0, +\infty]). \quad (4.32)$$

Therefore, Lemma 2.4(iii) implies that (4.1) is equivalent to

$$\eta = s \left( \text{prox}_{\gamma\varphi^*} \left( \text{prox}_{\frac{\eta}{\gamma}\odot\varphi^*} \left( \frac{x}{\gamma} \right) \right)_{\odot(-s)\blacktriangledown} y \right). \quad (4.33)$$

(i): Since Lemma 3.1(i) and (4.31) yield

$$s \left( \text{prox}_{\gamma\varphi^*} \left( \text{proj}_{\overline{\text{dom } \varphi^*}} \left( \frac{x}{\gamma} \right) \right)_{\odot(-s)\blacktriangledown} y \right) = s(\text{proj}_{\bar{S}} y) = 0, \quad (4.34)$$

we deduce from (4.33) and Lemma 3.1(i) that  $\eta = 0$ . The claim therefore follows from (4.2).

(ii): Lemma 3.1(i) yields

$$s \left( \text{prox}_{\gamma\varphi^*} \left( \text{proj}_{\overline{\text{dom } \varphi^*}} \left( \frac{x}{\gamma} \right) \right)_{\odot(-s)\blacktriangledown} y \right) = s \left( \text{prox}_{\gamma\varphi^*} \left( \text{proj}_{\overline{\text{dom } \varphi^*}} \left( \frac{x}{\gamma} \right) \right)_{(-s)\blacktriangledown} y \right) = 0, \quad (4.35)$$

and we deduce from (4.33) and Lemma 3.1(i) that  $\eta = 0$ . Therefore, the claim follows from (4.2).

(iii): Since  $s(\text{proj}_{\bar{S}} y) > 0$ , Lemma 3.1(i) and (4.31) yield

$$\begin{aligned} s \left( \text{prox}_{\gamma\varphi^*} \left( \text{prox}_{\frac{s(\text{proj}_{\bar{S}} y)}{\gamma}\odot\varphi^*} \left( \frac{x}{\gamma} \right) \right)_{\odot(-s)\blacktriangledown} y \right) &= s \left( \text{prox}_{\gamma\varphi^*} \left( \text{prox}_{\frac{s(\text{proj}_{\bar{S}} y)}{\gamma}\varphi^*} \left( \frac{x}{\gamma} \right) \right)_{\odot(-s)\blacktriangledown} y \right) \\ &= s(\text{proj}_{\bar{S}} y), \end{aligned} \quad (4.36)$$

and we deduce from (4.33) that  $\eta = s(\text{proj}_{\bar{S}} y) > 0$ . Therefore, the claim follows from (4.2).

(iv): Suppose that  $\eta = 0$ . Then it follows from (4.33) that

$$0 = s \left( \text{prox}_{\gamma\varphi^*} \left( \text{proj}_{\overline{\text{dom } \varphi^*}} \left( \frac{x}{\gamma} \right) \right)_{\odot(-s)\blacktriangledown} y \right). \quad (4.37)$$

Therefore, if  $\varphi^*(\text{proj}_{\overline{\text{dom } \varphi^*}}(x/\gamma)) = 0$ , then (4.37) yields  $0 = s(\text{proj}_{\bar{S}} y)$ , which implies that  $(x, y) \in \Omega_1$ . On the other hand, if  $\varphi^*(\text{proj}_{\overline{\text{dom } \varphi^*}}(x/\gamma)) > 0$ , then (4.37) yields

$$0 = s \left( \text{prox}_{\gamma\varphi^*} \left( \text{proj}_{\overline{\text{dom } \varphi^*}} \left( \frac{x}{\gamma} \right) \right)_{(-s)\blacktriangledown} y \right) \quad (4.38)$$

and thus  $(x, y) \in \Omega_2$ . However, since  $(x, y) \in \Omega_4$ , we have  $(x, y) \notin \Omega_1 \cup \Omega_2$  and obtain a contradiction. This shows that  $\eta > 0$ . In turn, (4.33) reduces to

$$\eta = s \left( \text{prox}_{\gamma\varphi^*} \left( \text{prox}_{\frac{\eta}{\gamma}\varphi^*} \left( \frac{x}{\gamma} \right) \right) \ominus (-s) \blacktriangleright y \right). \quad (4.39)$$

Hence, if  $\varphi^*(\text{prox}_{\frac{\eta}{\gamma}\varphi^*}(x/\gamma)) = 0$ , we deduce from (4.31) that  $0 < \eta = s(\text{proj}_{\overline{S}}y)$ , which yields  $(x, y) \in \Omega_3$ . However, since  $(x, y) \in \Omega_4$ , we have  $\varphi^*(\text{prox}_{\frac{\eta}{\gamma}\varphi^*}(x/\gamma)) > 0$ . Consequently, the claim follows from Lemma 3.1(i).

Finally, it is clear from (4.28) that  $\Omega_1 \cap \Omega_2 = \emptyset$  and  $\Omega_1 \cap \Omega_3 = \emptyset$ . Moreover, we infer from (ii) and (iii) that  $\Omega_2 \cap \Omega_3 = \emptyset$ . Altogether,  $(\Omega_i)_{1 \leq i \leq 4}$  is a partition of  $\mathcal{H} \times \mathcal{G}$  and the proof is complete.  $\square$

**Proposition 4.3** Consider the assumptions and notation of Theorem 4.1(iii), and set

$$\begin{cases} \Xi_1 = \left\{ (u, v) \in \mathcal{H} \times \mathcal{G} \mid s^\blacktriangleleft(\text{proj}_{\overline{\text{conv}} S}v) = 0 \text{ and } \varphi^* \left( \text{proj}_{\overline{\text{dom}} \varphi^*} \left( \frac{u}{\gamma} \right) \right) = 0 \right\} \\ \Xi_2 = \left\{ (u, v) \in \mathcal{H} \times \mathcal{G} \mid s^\blacktriangleleft(\text{proj}_{\overline{\text{conv}} S}v) \in ]0, +\infty[ \text{ and } \varphi^* \left( \text{prox}_{\frac{1}{\gamma}s^\blacktriangleleft(\text{proj}_{\overline{\text{conv}} S}v)} \varphi^* \left( \frac{u}{\gamma} \right) \right) = 0 \right\} \\ \Xi_3 = \left\{ (u, v) \in \mathcal{H} \times \mathcal{G} \mid s^\blacktriangleleft \left( \text{prox}_{\gamma(-\varphi^*(\text{proj}_{\overline{\text{dom}} \varphi^*}(\frac{u}{\gamma}))} s^\blacktriangleleft v) \right) = 0 \text{ and } \varphi^* \left( \text{proj}_{\overline{\text{dom}} \varphi^*} \left( \frac{u}{\gamma} \right) \right) < 0 \right\} \\ \Xi_4 = (\mathcal{H} \times \mathcal{G}) \setminus (\Xi_1 \cup \Xi_2 \cup \Xi_3). \end{cases} \quad (4.40)$$

Then exactly one of the following holds:

- (i)  $(x, y) \in \Xi_1$ ,  $\eta = 0$ , and  $\text{prox}_{\gamma(\varphi \blacktriangleright s)}(x, y) = \left( x - \gamma \text{proj}_{\overline{\text{dom}} \varphi^*}(x/\gamma), \text{proj}_{\overline{\text{conv}} S}y \right)$ .
- (ii)  $(x, y) \in \Xi_2$ ,  $\eta = 0$ , and  $\text{prox}_{\gamma(\varphi \blacktriangleright s)}(x, y) = \left( x - \gamma \text{prox}_{\frac{1}{\gamma}s^\blacktriangleleft(\text{proj}_{\overline{\text{conv}} S}y)} \varphi^*(x/\gamma), \text{proj}_{\overline{\text{conv}} S}y \right)$ .
- (iii)  $(x, y) \in \Xi_3$ ,  $\eta = -\varphi^*(\text{proj}_{\overline{\text{dom}} \varphi^*}(x/\gamma)) > 0$ , and

$$\text{prox}_{\gamma(\varphi \blacktriangleright s)}(x, y) = \left( x - \gamma \text{proj}_{\overline{\text{dom}} \varphi^*} \left( \frac{x}{\gamma} \right), \text{prox}_{(-\gamma\varphi^*(\text{proj}_{\overline{\text{dom}} \varphi^*}(\frac{x}{\gamma}))} s^\blacktriangleleft y) \right). \quad (4.41)$$

- (iv)  $(x, y) \in \Xi_4$ ,  $\eta > 0$  solves

$$\varphi^* \left( \text{prox}_{\frac{1}{\gamma}s^\blacktriangleleft(\text{prox}_{\gamma\eta s^\blacktriangleleft} y)} \varphi^* \left( \frac{x}{\gamma} \right) \right) + \eta = 0 \quad (4.42)$$

and

$$\text{prox}_{\gamma(\varphi \blacktriangleright s)}(x, y) = \left( x - \gamma \text{prox}_{\frac{1}{\gamma}s^\blacktriangleleft(\text{prox}_{\gamma\eta s^\blacktriangleleft} y)} \varphi^* \left( \frac{x}{\gamma} \right), \text{prox}_{\gamma\eta s^\blacktriangleleft} y \right). \quad (4.43)$$

*Proof.* Lemma 2.5(ii) yields

$$\text{dom } s^\blacktriangleleft = \overline{\text{conv}} S. \quad (4.44)$$

(i): Since Lemma 3.1(i) and (4.44) yield

$$\varphi^* \left( \text{prox}_{\frac{1}{\gamma} s^\blacktriangle} (\text{proj}_{\overline{\text{conv}} S} y) \odot \varphi^* \left( \frac{x}{\gamma} \right) \right) = \varphi^* \left( \text{proj}_{\overline{\text{dom}} \varphi^*} \left( \frac{x}{\gamma} \right) \right) = 0, \quad (4.45)$$

we deduce from (4.3) and Lemma 3.1(i) that  $\eta = 0$ . The claim therefore follows from (4.4).

(ii): Lemma 3.1(i) yields

$$\varphi^* \left( \text{prox}_{\frac{1}{\gamma} s^\blacktriangle} (\text{proj}_{\overline{\text{conv}} S} y) \odot \varphi^* \left( \frac{x}{\gamma} \right) \right) = \varphi^* \left( \text{prox}_{\frac{1}{\gamma} s^\blacktriangle} (\text{proj}_{\overline{\text{conv}} S} y) \varphi^* \left( \frac{x}{\gamma} \right) \right) = 0, \quad (4.46)$$

and we deduce from (4.3) and Lemma 3.1(i) that  $\eta = 0$ . Therefore, the claim follows from (4.4).

(iii): Since  $\varphi^* (\text{proj}_{\overline{\text{dom}} \varphi^*} (\frac{x}{\gamma})) < 0$ , Lemma 3.1(i) yields

$$\begin{aligned} \varphi^* \left( \text{prox}_{\frac{1}{\gamma} s^\blacktriangle} (\text{prox}_{\gamma (-\varphi^* (\text{proj}_{\overline{\text{dom}} \varphi^*} (\frac{x}{\gamma})))} s^\blacktriangle y) \odot \varphi^* \left( \frac{x}{\gamma} \right) \right) \\ = \varphi^* \left( \text{prox}_{\frac{1}{\gamma} s^\blacktriangle} (\text{prox}_{\gamma (-\varphi^* (\text{proj}_{\overline{\text{dom}} \varphi^*} (\frac{x}{\gamma})))} s^\blacktriangle y) \odot \varphi^* \left( \frac{x}{\gamma} \right) \right) \\ = \varphi^* \left( \text{proj}_{\overline{\text{dom}} \varphi^*} \left( \frac{x}{\gamma} \right) \right) \end{aligned} \quad (4.47)$$

and we deduce from (4.3) that  $\eta = -\varphi^* (\text{proj}_{\overline{\text{dom}} \varphi^*} (x/\gamma)) > 0$ . Therefore, the claim follows from (4.4).

(iv): Suppose that  $\eta = 0$ . Then it follows from (4.3) that

$$0 = \varphi^* \left( \text{prox}_{\frac{1}{\gamma} s^\blacktriangle} (\text{proj}_{\overline{\text{conv}} S} y) \odot \varphi^* \left( \frac{x}{\gamma} \right) \right). \quad (4.48)$$

Therefore, if  $s^\blacktriangle (\text{proj}_{\overline{\text{conv}} S} y) = 0$ , then (4.48) yields  $0 = \varphi^* (\text{proj}_{\overline{\text{dom}} \varphi^*} (x/\gamma))$ , which implies that  $(x, y) \in \Xi_1$ . On the other hand, if  $s^\blacktriangle (\text{proj}_{\overline{\text{conv}} S} y) > 0$ , then (4.48) yields

$$0 = \varphi^* \left( \text{prox}_{\frac{1}{\gamma} s^\blacktriangle} (\text{proj}_{\overline{\text{conv}} S} y) \varphi^* \left( \frac{x}{\gamma} \right) \right) \quad (4.49)$$

and thus  $(x, y) \in \Xi_2$ . At the same time, since  $(x, y) \in \Xi_4$ , we have  $(x, y) \notin \Xi_1 \cup \Xi_2$ . This contradiction shows that  $\eta > 0$ . In turn, (4.3) reduces to

$$-\eta = \varphi^* \left( \text{prox}_{\frac{1}{\gamma} s^\blacktriangle} (\text{prox}_{\gamma \eta s^\blacktriangle} y) \odot \varphi^* \left( \frac{x}{\gamma} \right) \right). \quad (4.50)$$

Hence, if  $s^\blacktriangle (\text{prox}_{\gamma \eta s^\blacktriangle} y) = 0$ ,  $0 > -\eta = \varphi^* (\text{proj}_{\overline{\text{dom}} \varphi^*} (x/\gamma))$ , which yields  $(x, y) \in \Xi_3$ . However, since  $(x, y) \in \Xi_4$ , we have  $s^\blacktriangle (\text{prox}_{\gamma \eta s^\blacktriangle} y) > 0$ . Consequently, the claim follows from Lemma 3.1(i).

Finally, it is clear from (4.40) that  $\Xi_1 \cap \Xi_2 = \emptyset$  and  $\Xi_1 \cap \Xi_3 = \emptyset$ . Moreover, we infer from (ii) and (iii) that  $\Xi_2 \cap \Xi_3 = \emptyset$ . Altogether,  $(\Xi_i)_{1 \leq i \leq 4}$  is a partition of  $\mathcal{H} \times \mathcal{G}$  and the proof is complete.  $\square$

**Remark 4.4** In cases (i)–(iii) of Proposition 4.2, the computation of  $\text{prox}_{\gamma(\varphi \blacktriangleright s)}(x, y)$  requires only the ability to compute the projection operators onto  $\overline{\text{dom}} \varphi^*$  and  $\overline{S}$ , as well as the proximity operators of  $\varphi^*$  and  $(-s)^\blacktriangledown$ . Examples of explicit formulas for these operators can be found in [7, 20]. The case (iv) requires additionally the solution  $\eta \in ]0, +\infty[$  to (4.29). To determine  $\eta$ , let us define  $\phi_1$  and  $\phi_2$  as in (4.22) and note that it is the root of  $T = \phi_1 \circ \phi_2 + \text{Id}$ . Since Lemma 3.3 implies that  $T$  is strictly increasing and continuous on  $]0, +\infty[$ ,  $\eta$  can be found via standard one-dimensional root-finding routines [44, Chapter 9]. A similar observation can be made for Proposition 4.3.

## 5 Examples

We provide four examples of computation of the proximity operator of perspective functions.

**Example 5.1** Let  $\phi \in \Gamma_0(\mathbb{R})$  be an even coercive function such that  $\phi^*(\mathbb{R}) \subset [0, +\infty]$  and  $(\phi^*)^{-1}(]0, +\infty[) \neq \emptyset$ , and set  $\varphi = \phi \circ \|\cdot\|$ . Then,  $\phi^* \in \Gamma_0(\mathbb{R})$  is even,  $0 \in \text{int dom } \phi^*$  by [7, Proposition 14.16],  $\varphi \in \Gamma_0(\mathcal{H})$ , and [7, Example 13.8] implies that  $\varphi^* = \phi^* \circ \|\cdot\|$ . In turn,  $\varphi^*(\mathcal{H}) \subset [0, +\infty]$  and  $(\varphi^*)^{-1}(]0, +\infty[) \neq \emptyset$ . Furthermore, let  $-s \in \Gamma_0(\mathcal{G})$ , let  $(x, y) \in \mathcal{H} \times \mathcal{G}$ , and note that Proposition 3.4(i) asserts that

$$(\varphi \star s)(x, y) = \begin{cases} s(y)\phi\left(\frac{\|x\|}{s(y)}\right), & \text{if } 0 < s(y) < +\infty; \\ (\text{rec } \phi)(\|x\|), & \text{if } s(y) = 0; \\ +\infty, & \text{otherwise.} \end{cases} \quad (5.1)$$

Now let  $\gamma \in ]0, +\infty[$ . It follows from Lemma 3.2(ii) and Proposition 4.2 that the sets

$$\begin{cases} \Omega_1 = \left\{ (u, v) \in \mathcal{H} \times \mathcal{G} \mid \phi^*\left(\text{proj}_{\overline{\text{dom } \phi^*}}\left(\frac{\|u\|}{\gamma}\right)\right) = 0 \text{ and } s(\text{proj}_{\overline{\mathcal{G}}}v) = 0 \right\} \\ \Omega_2 = \left\{ (u, v) \in \mathcal{H} \times \mathcal{G} \mid \phi^*\left(\text{proj}_{\overline{\text{dom } \phi^*}}\left(\frac{\|u\|}{\gamma}\right)\right) \in ]0, +\infty[ \text{ and } s\left(\text{prox}_{\gamma\phi^*\left(\text{proj}_{\overline{\text{dom } \phi^*}}\left(\frac{\|u\|}{\gamma}\right)\right)}(-s)^\blacktriangledown v\right) = 0 \right\} \\ \Omega_3 = \left\{ (u, v) \in \mathcal{H} \times \mathcal{G} \mid \phi^*\left(\text{prox}_{\frac{s(\text{proj}_{\overline{\mathcal{G}}}v)}{\gamma}}\phi^*\left(\frac{\|u\|}{\gamma}\right)\right) = 0 \text{ and } s(\text{proj}_{\overline{\mathcal{G}}}v) \in ]0, +\infty[ \right\} \\ \Omega_4 = (\mathcal{H} \times \mathcal{G}) \setminus (\Omega_1 \cup \Omega_2 \cup \Omega_3) \end{cases} \quad (5.2)$$

form a partition of  $\mathcal{H} \times \mathcal{G}$ , which brings up four cases for consideration:

- $(x, y) \in \Omega_1$ : We derive from Lemma 3.1(i), Lemma 3.2(i), and Proposition 4.2(i) that

$$\text{prox}_{\gamma(\varphi \star s)}(x, y) = \begin{cases} \left( \left( \left( 1 - \frac{\gamma}{\|x\|} \text{proj}_{\overline{\text{dom } \phi^*}}\left(\frac{\|x\|}{\gamma}\right) \right) x, \text{proj}_{\overline{\mathcal{G}}}y \right), & \text{if } x \neq 0; \\ (0, \text{proj}_{\overline{\mathcal{G}}}y), & \text{if } x = 0. \end{cases} \quad (5.3)$$

- $(x, y) \in \Omega_2$ : We derive from Lemma 3.1(i), Lemma 3.2(i), and Proposition 4.2(ii) that

$$\text{prox}_{\gamma(\varphi \star s)}(x, y) = \begin{cases} \left( \left( \left( 1 - \frac{\gamma}{\|x\|} \text{proj}_{\overline{\text{dom } \phi^*}}\left(\frac{\|x\|}{\gamma}\right) \right) x, \text{prox}_{\gamma\phi^*\left(\text{proj}_{\overline{\text{dom } \phi^*}}\left(\frac{\|x\|}{\gamma}\right)\right)}(-s)^\blacktriangledown y \right), & \text{if } x \neq 0; \\ (0, \text{prox}_{\gamma\phi^*(0)}(-s)^\blacktriangledown y), & \text{if } x = 0. \end{cases} \quad (5.4)$$

- $(x, y) \in \Omega_3$ : We derive from Lemma 3.1(i), Lemma 3.2(i), and Proposition 4.2(iii) that

$$\text{prox}_{\gamma(\varphi \star s)}(x, y) = \begin{cases} \left( \left( \left( 1 - \frac{\gamma}{\|x\|} \text{prox}_{\frac{s(\text{proj}_{\overline{\mathcal{G}}}y)}{\gamma}}\phi^*\left(\frac{\|x\|}{\gamma}\right) \right) x, \text{proj}_{\overline{\mathcal{G}}}y \right), & \text{if } x \neq 0; \\ (0, \text{proj}_{\overline{\mathcal{G}}}y), & \text{if } x = 0. \end{cases} \quad (5.5)$$

- $(x, y) \in \Omega_4$ : In view of Lemma 3.2(i) and Lemma 3.1(i), Theorem 4.1(i) and Proposition 4.2(iv) guarantee the existence of a unique solution  $\eta \in ]0, +\infty[$  to

$$\eta = s \left( \text{prox}_{\gamma\phi^*} \left( \text{prox}_{\frac{\eta}{\gamma}\phi^*} \left( \frac{\|x\|}{\gamma} \right) \right)_{(-s)\blacktriangledown} y \right) \quad (5.6)$$

and

$$\text{prox}_{\gamma(\varphi\blacktriangleright s)}(x, y) = \begin{cases} \left( \left( 1 - \frac{\gamma}{\|x\|} \text{prox}_{\frac{\eta}{\gamma}\phi^*} \left( \frac{\|x\|}{\gamma} \right) \right) x, \text{prox}_{\gamma\phi^*} \left( \text{prox}_{\frac{\eta}{\gamma}\phi^*} \left( \frac{\|x\|}{\gamma} \right) \right)_{(-s)\blacktriangledown} y \right), & \text{if } x \neq 0; \\ \left( 0, \text{prox}_{\gamma\phi^*(0)}_{(-s)\blacktriangledown} y \right), & \text{if } x = 0. \end{cases} \quad (5.7)$$

Our next example addresses the counterpart of the previous one in which the sign of  $\phi^*$  is flipped.

**Example 5.2** Let  $\phi \in \Gamma_0(\mathbb{R})$  be an even coercive function such that  $\phi^*(\mathbb{R}) \subset ]-\infty, 0] \cup \{+\infty\}$  and  $(\phi^*)^{-1}(]-\infty, 0]) \neq \emptyset$ , and set  $\varphi = \phi \circ \|\cdot\|$ . As in Example 5.1,  $\phi^* \in \Gamma_0(\mathbb{R})$  is even,  $0 \in \text{int dom } \phi^*$ ,  $\varphi \in \Gamma_0(\mathcal{H})$ , and  $\varphi^* = \phi^* \circ \|\cdot\|$ . In turn,  $\varphi^*(\mathcal{H}) \subset ]-\infty, 0] \cup \{+\infty\}$  and  $(\varphi^*)^{-1}(]-\infty, 0]) \neq \emptyset$ . In addition, let  $s \in \Gamma_0(\mathcal{G})$  and  $(x, y) \in \mathcal{H} \times \mathcal{G}$ . Then, by Proposition 3.4(iii),

$$(\varphi\blacktriangleright s)(x, y) = \begin{cases} s(y)\phi\left(\frac{\|x\|}{s(y)}\right), & \text{if } 0 < s(y) < +\infty; \\ (\text{rec } \phi)(\|x\|), & \text{if } y \in \overline{\text{conv}} S \text{ and } s(y) \leq 0; \\ +\infty, & \text{otherwise.} \end{cases} \quad (5.8)$$

Now let  $\gamma \in ]0, +\infty[$ . It follows from Lemma 3.2(ii) and Proposition 4.3 that the sets

$$\begin{cases} \Xi_1 = \left\{ (u, v) \in \mathcal{H} \times \mathcal{G} \mid s^\blacktriangleleft(\text{proj}_{\overline{\text{conv}} S} v) = 0 \text{ and } \phi^*\left(\text{proj}_{\text{dom } \phi^*} \left( \frac{\|u\|}{\gamma} \right)\right) = 0 \right\} \\ \Xi_2 = \left\{ (u, v) \in \mathcal{H} \times \mathcal{G} \mid s^\blacktriangleleft(\text{proj}_{\overline{\text{conv}} S} v) \in ]0, +\infty[ \text{ and } \phi^*\left(\text{prox}_{\frac{1}{\gamma}s^\blacktriangleleft(\text{proj}_{\overline{\text{conv}} S} v)} \phi^* \left( \frac{\|u\|}{\gamma} \right)\right) = 0 \right\} \\ \Xi_3 = \left\{ (u, v) \in \mathcal{H} \times \mathcal{G} \mid s^\blacktriangleleft\left(\text{prox}_{\gamma(-\phi^*(\text{proj}_{\text{dom } \phi^*} \left( \frac{\|u\|}{\gamma} \right))}\right)_{s^\blacktriangleleft} v\right) = 0 \text{ and } \phi^*\left(\text{proj}_{\text{dom } \phi^*} \left( \frac{\|u\|}{\gamma} \right)\right) < 0 \right\} \\ \Xi_4 = (\mathcal{H} \times \mathcal{G}) \setminus (\Xi_1 \cup \Xi_2 \cup \Xi_3) \end{cases} \quad (5.9)$$

form a partition of  $\mathcal{H} \times \mathcal{G}$ . This leads us to consider the following cases:

- $(x, y) \in \Xi_1$ : We derive from Lemma 3.1(i), Lemma 3.2(i), and Proposition 4.3(i) that

$$\text{prox}_{\gamma(\varphi\blacktriangleright s)}(x, y) = \begin{cases} \left( \left( 1 - \frac{\gamma}{\|x\|} \text{proj}_{\text{dom } \phi^*} \left( \frac{\|x\|}{\gamma} \right) \right) x, \text{proj}_{\overline{\text{conv}} S} y \right), & \text{if } x \neq 0; \\ \left( 0, \text{proj}_{\overline{\text{conv}} S} y \right), & \text{if } x = 0. \end{cases} \quad (5.10)$$

- $(x, y) \in \Xi_2$ : We derive from Lemma 3.1(i), Lemma 3.2(i), and Proposition 4.3(ii) that

$$\text{prox}_{\gamma(\varphi\blacktriangleright s)}(x, y) = \begin{cases} \left( \left( 1 - \frac{\gamma}{\|x\|} \text{prox}_{\frac{1}{\gamma}s^\blacktriangleleft(\text{proj}_{\overline{\text{conv}} S} y)} \phi^* \left( \frac{\|x\|}{\gamma} \right) \right) x, \text{proj}_{\overline{\text{conv}} S} y \right), & \text{if } x \neq 0; \\ \left( 0, \text{proj}_{\overline{\text{conv}} S} y \right), & \text{if } x = 0. \end{cases} \quad (5.11)$$



- $(x, y) \in \Xi_3$ : We derive from Lemma 3.1(i), Lemma 3.2(i), and Proposition 4.3(iii) that

$$\text{prox}_{\gamma(\varphi \star s)}(x, y) = \begin{cases} \left( \left( 1 - \frac{\gamma}{\|x\|} \text{proj}_{\text{dom } \phi^*} \left( \frac{\|x\|}{\gamma} \right) \right) x, \text{prox}_{(-\gamma \phi^* (\text{proj}_{\text{dom } \phi^*} (\frac{\|x\|}{\gamma})))_{s \blacktriangle} y} \right), & \text{if } x \neq 0; \\ (0, \text{prox}_{(-\gamma \phi^*(0))_{s \blacktriangle} y}), & \text{if } x = 0. \end{cases} \quad (5.12)$$

- $(x, y) \in \Xi_4$ : By virtue of Lemma 3.2(i) and Lemma 3.1(i), it follows from Theorem 4.1(iii) and Proposition 4.3(iv) that there exists a unique solution  $\eta \in ]0, +\infty[$  to

$$\eta = -\phi^* \left( \text{prox}_{\frac{1}{\gamma} s \blacktriangle (\text{prox}_{\gamma \eta s \blacktriangle} y) \phi^*} \left( \frac{\|x\|}{\gamma} \right) \right) \quad (5.13)$$

and

$$\text{prox}_{\gamma(\varphi \star s)}(x, y) = \begin{cases} \left( \left( 1 - \frac{\gamma}{\|x\|} \text{prox}_{\frac{1}{\gamma} s \blacktriangle (\text{prox}_{\gamma \eta s \blacktriangle} y) \phi^*} \left( \frac{\|x\|}{\gamma} \right) \right) x, \text{prox}_{\gamma \eta s \blacktriangle} y \right), & \text{if } x \neq 0; \\ (0, \text{prox}_{(-\gamma \phi^*(0))_{s \blacktriangle} y}), & \text{if } x = 0. \end{cases} \quad (5.14)$$

Starting with the work [8], convex optimization problems involving the perspective function with linear scaling (1.2) appear in optimal transport theory and in mean field games [1, 9, 35, 47]. In this context, numerical methods employing its proximity operator are investigated in [14]. Extensions to variational models with  $q$ th root scaling functions have been proposed to address optimal control of McKean–Vlasov systems with congestion [2, 3], as well as optimal transport with nonlinear mobilities [29]. In the following example, we compute the proximity operator of perspective functions with such scaling functions and incorporate a scale constraint which can be used, in particular, to model density constraints [16, 27, 28, 40].

**Example 5.3** Let  $p \in ]1, +\infty[$  and  $q \in ]0, 1[$ . Set  $p^* = p/(p-1)$ , and  $\phi = |\cdot|^p/p$ . Let  $I \subset [0, +\infty[$  be a closed interval such that  $0 \in I$  and define

$$\psi: \mathbb{R} \rightarrow \{-\infty\} \cup [0, +\infty[ : y \mapsto \begin{cases} y^q, & \text{if } y \geq 0; \\ -\infty, & \text{if } y < 0, \end{cases} \quad \text{and set } s = \psi - \iota_I. \quad (5.15)$$

Then  $\phi^* = |\cdot|^{p^*}/p^*$ ,  $(-s)^\blacktriangleright = -s$ ,  $\text{dom } \phi^* = \mathbb{R}$ , and  $\bar{S} = I$ . This places us in the framework of Example 5.1 with  $\mathcal{G} = \mathbb{R}$ , (5.1) reduces to

$$(\varphi \star s)(x, y) = \begin{cases} \frac{\|x\|^p}{p y^{q(p-1)}}, & \text{if } 0 < y \in I; \\ 0, & \text{if } x = 0 \text{ and } y = 0; \\ +\infty, & \text{otherwise,} \end{cases} \quad (5.16)$$

and (5.2) reduces to

$$\begin{cases} \Omega_1 = \{0\} \times ]-\infty, 0] \\ \Omega_2 = \left\{ (u, v) \in \mathcal{H} \times \mathbb{R} \mid u \neq 0 \text{ and } \text{prox}_{\frac{\gamma}{p^*} \left| \frac{\|u\|}{\gamma} \right|^{p^*} (-s)} v = 0 \right\} \\ \Omega_3 = \{0\} \times ]0, +\infty[ \\ \Omega_4 = ((\mathcal{H} \setminus \{0\}) \times \mathbb{R}) \setminus \Omega_2. \end{cases} \quad (5.17)$$

Since, for every  $\mu \in ]0, +\infty[$  and every  $v \in \mathbb{R}$ , Lemma 3.1(iii) yields  $\text{prox}_{-\mu s} v \in ]0, +\infty[$ , we have  $\Omega_2 = \emptyset$  and  $\Omega_4 = (\mathcal{H} \setminus \{0\}) \times \mathbb{R}$ . Therefore, Example 5.1 and [7, Proposition 24.47] yield

$$\text{prox}_{\gamma(\varphi \star s)}(x, y) = \begin{cases} (0, \text{proj}_I y), & \text{if } x = 0; \\ \left( \left( \left( 1 - \frac{\gamma}{\|x\|} \text{prox}_{\frac{\eta}{\gamma} \phi^*} \left( \frac{\|x\|}{\gamma} \right) \right) x, \text{proj}_I \left( \text{prox}_{\frac{\gamma}{p^*} \left| \text{prox}_{\frac{\eta}{\gamma} \phi^*} \left( \frac{\|x\|}{\gamma} \right) \right|^{p^*} (-\psi)} y \right) \right) \right), & \text{if } x \neq 0, \end{cases} \quad (5.18)$$

where, if  $x \neq 0$ ,  $\eta$  is the unique solution in  $]0, +\infty[$  to

$$\eta = \left| \text{proj}_I \left( \text{prox}_{\frac{\gamma}{p^*} \left| \text{prox}_{\frac{\eta}{\gamma} \phi^*} \left( \frac{\|x\|}{\gamma} \right) \right|^{p^*} (-\psi)} y \right) \right|^q. \quad (5.19)$$

Note that, in view of (5.19), (5.18) can be written as

$$\text{prox}_{\gamma(\varphi \star s)}(x, y) = \begin{cases} (0, \text{proj}_I y), & \text{if } x = 0; \\ \left( \left( \left( 1 - \frac{\gamma}{\|x\|} \text{prox}_{\frac{\eta}{\gamma} \phi^*} \left( \frac{\|x\|}{\gamma} \right) \right) x, \eta^{1/q} \right) \right), & \text{if } x \neq 0. \end{cases} \quad (5.20)$$

In the case when  $x \neq 0$ , let us point out that, given  $\xi \in ]0, +\infty[$ , [7, Example 24.38] asserts that  $\rho(\xi) = \text{prox}_{\frac{\xi}{\gamma} \phi^*}(\|x\|/\gamma)$  is the unique solution to

$$\|x\| = \rho\gamma + \xi\rho^{p^*-1}. \quad (5.21)$$

On the other hand, for every  $\mu \in ]0, +\infty[$ , in view of Lemma 3.1(iii),  $z(\mu) = \text{prox}_{\gamma\mu(-\psi)} y \in ]0, +\infty[$  is the unique solution to

$$y = z - q\gamma\mu z^{q-1}. \quad (5.22)$$

Therefore, finding  $\eta \in ]0, +\infty[$  such that (5.19) holds amounts to solving  $\eta = |\text{proj}_I(z(\rho(\eta)^{p^*}/p^*))|^q$ , that is,

$$\eta = \left| \min \left\{ z \left( \frac{\rho(\eta)^{p^*}}{p^*} \right), \sup I \right\} \right|^q, \quad (5.23)$$

which can be handled by one-dimensional root-finding methods.

A fundamental objective in statistical inference is the estimation of both location  $x$  (i.e., the regression vector) and scale  $y$  (e.g., the standard deviation of the noise or some other parameter) of the statistical model from the data. In robust statistics, the maximum likelihood-type estimator (M-estimator) for location with concomitant scale [32, p. 179] couples both parameters via a convex objective function in the form of a standard perspective (1.2); see [4, 23, 24, 34, 43] for further instances of this model. Most of these models involve the perspective function of the Huber function with a scalar scale. Our analysis allows us to extend it to nonlinear scales. An illustration of Example 5.2 in this context is provided in the following example, where the proximity operator of the resulting function, a central piece in algorithms for solving concomitant estimation problems [22, 23, 24], is computed.

**Example 5.4** Let  $\phi$  is the Huber function with parameter  $\alpha \in ]0, +\infty[$ , that is,

$$\phi: \xi \mapsto \begin{cases} \alpha|\xi|, & \text{if } |\xi| > \alpha; \\ \frac{|\xi|^2 + \alpha^2}{2}, & \text{if } |\xi| \leq \alpha. \end{cases} \quad (5.24)$$

It follows from [7, Example 13.7] that

$$\phi^*: \xi^* \mapsto \begin{cases} +\infty, & \text{if } |\xi^*| > \alpha; \\ \frac{|\xi^*|^2 - \alpha^2}{2}, & \text{if } |\xi^*| \leq \alpha. \end{cases} \quad (5.25)$$

Therefore  $\text{dom } \phi^* = [-\alpha, \alpha]$  and we deduce from [7, Example 24.9] and Lemma 3.1(vi) that

$$\text{prox}_{\gamma\phi^*}: \xi \mapsto \begin{cases} \alpha \text{ sign}(\xi), & \text{if } |\xi| > (\gamma + 1)\alpha; \\ \xi/(\gamma + 1), & \text{if } |\xi| \leq (\gamma + 1)\alpha. \end{cases} \quad (5.26)$$

Furthermore, let  $\beta \in ]0, +\infty[$  and set

$$s: \mathbb{R} \rightarrow [0, +\infty[ : y \mapsto \sqrt{\beta + |y|^2}. \quad (5.27)$$

Then  $s^\star = s$  and  $\overline{\text{conv}} S = \mathbb{R}$ . Altogether, we are in the framework of Example 5.2 with  $\mathcal{G} = \mathbb{R}$ , (5.8) reduces to

$$(\varphi \star s)(x, y) = \begin{cases} \alpha\|x\|, & \text{if } \|x\| > \alpha\sqrt{\beta + |y|^2}; \\ \frac{\|x\|^2 + \alpha^2(\beta + y^2)}{2\sqrt{\beta + |y|^2}}, & \text{if } \|x\| \leq \alpha\sqrt{\beta + |y|^2}, \end{cases} \quad (5.28)$$

and (5.9) reduces to

$$\begin{cases} \Xi_1 = \Xi_3 = \emptyset \\ \Xi_2 = \{(u, v) \in \mathcal{H} \times \mathbb{R} \mid \|u\| \geq \alpha(\sqrt{\beta + v^2} + \gamma)\} \\ \Xi_4 = \{(u, v) \in \mathcal{H} \times \mathbb{R} \mid \|u\| < \alpha(\sqrt{\beta + v^2} + \gamma)\}. \end{cases} \quad (5.29)$$

Therefore, we deduce from Example 5.2 that

$$\text{prox}_{\gamma(\varphi \star s)}(x, y) = \begin{cases} \left( \left( 1 - \frac{\alpha\gamma}{\|x\|} \right) x, y \right), & \text{if } \|x\| \geq \alpha(\sqrt{\beta + y^2} + \gamma); \\ \left( \left( \frac{\sqrt{\beta + q(\eta, y)^2}}{\gamma + \sqrt{\beta + q(\eta, y)^2}} \right) x, q(\eta, y) \right), & \text{if } \|x\| < \alpha(\sqrt{\beta + y^2} + \gamma), \end{cases} \quad (5.30)$$

where  $\eta$  is the unique solution in  $]0, +\infty[$  to

$$\eta = \frac{\alpha^2|\gamma + \sqrt{\beta + q(\eta, y)^2}|^2 - \|x\|^2}{2|\gamma + \sqrt{\beta + q(\eta, y)^2}|^2} \quad (5.31)$$

and  $q(\eta, y)$  is the unique solution to the quartic equation

$$q^4 - 2yq^3 + (y^2 + \beta - \gamma^2\mu^2)q^2 - 2\beta yq + \beta y^2 = 0, \quad (5.32)$$

in  $[0, y]$  if  $y \geq 0$  and in  $[y, 0]$  if  $y < 0$ . Such an equation can be solved explicitly [52, Section 2.2.3].

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