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A NOTE ON THE LONG TIME BEHAVIOUR OF STOCHASTIC MCKEAN-VLASOV EQUATIONS WITH COMMON NOISE

RAPHAEL MAILLET

ABSTRACT. This paper presents an investigation into the long-term behaviour of solutions to a non-linear stochastic McKean-Vlasov equation with common noise. The equation arises naturally in the mean-field limit of systems composed of interacting particles subject to both idiosyncratic and common noise. Initially, we demonstrate that the addition of common diffusion in each particle's dynamics does not disrupt the established stability results observed in the absence of common noise. However, our main objective is to understand how the presence of common noise can restore the uniqueness of equilibria. Specifically, in a non-convex landscape, we establish uniqueness and convergence towards equilibria under two specific conditions: (1) when the dimension of the ambient space equals 1, and (2) in the absence of idiosyncratic noise in the system.

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1. INTRODUCTION

We consider the following non-linear Stochastic Partial Differential Equation on $[0, +\infty) \times \mathbb{R}^d$,

$$(1) \quad d_t m_t = \nabla \cdot \left(\frac{\sigma^2 + \sigma_0^2}{2} \nabla m_t + m_t b(t, \cdot, m_t) \right) dt - \sigma_0 Dm_t \cdot dB_t^0.$$

This SPDE is posed on a filtered probability space $(\Omega_0, \mathcal{F}^0, \mathbb{F}^0, \mathbb{P}_0)$, B^0 is a d -dimensional \mathbb{F}^0 -Brownian motion, the drift $b : [0, +\infty) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ depends in time, space and measure, σ and σ_0 are two non-negative constants. This paper aims to study the long-term behavior of solutions of Equation (1), and more precisely, to understand the effect of the common noise on the asymptotic stability. We assume that the drift term b has a specific linear structure with respect to the measure variable, with two continuously differentiable functions V and W such that $b(t, x, \mu) = -\nabla V(x) - \nabla W * \mu(x)$, where $*$ stands for the convolution operator. This assumption is typical for studying long-term behavior in McKean-Vlasov type equations, see *e.g.* [1], [3], [16], [33] or [46]. In the following, we are interested in getting existence and uniqueness results for the invariant measure of the probability measure valued process (m_t) . As the problem under consideration falls within the realm of McKean-Vlasov type, one may expect existence of an invariant measure and uniqueness at least in some specific cases. In this paper, we verify whether, in the case where W is convex and V is uniformly convex, the introduction of common noise does not compromise the classical uniqueness results of [2]. When V is not convex, the matter becomes considerably more intricate, studied so far without the presence of common noise, only partial results are known. There exist cases in which the uniqueness of the invariant measure is not satisfied. Unlike linear elliptic equations, the presence of nonlinearity leads to the existence of multiple invariant measures. Specifically, it has been proven in [33] that when the confinement potential uniformly convex outside of a ball centred in the origine, admits a double-well and the diffusion coefficient σ is sufficiently small, there exist exactly three invariant solutions of the following equation:

$$\partial_t m_t = \frac{\sigma^2}{2} \Delta m_t + \nabla \cdot (m_t (\nabla V + \nabla W * m_t)).$$

Since 2019, several papers [17], [18] or [19] investigate the restoration of uniqueness in mean-field games derived from deterministic differential games with a large number of players by introducing an external noise. In a similar manner, this paper explores the restoration of uniqueness of the invariant measure by introducing common noise to the system. More precisely, we prove existence and uniqueness of the invariant measure for process (m_t) in the following cases: (1) when the confinement potential V is uniformly convex and the interaction potential is convex; (2) when the confinement potential V is not convex and the dimension of the ambient space $d = 1$; (3) when the potential V is not convex and there is no idiosyncratic noise in the system, *i.e.* $\sigma = 0$.

Without uniform convexity assumptions and in higher dimension $d \geq 2$, we can get uniqueness of the invariant measure for solutions of

$$(2) \quad d_t m_t = \nabla \cdot \left(\frac{\sigma_0^2}{2} \nabla m_t + m_t b(t, \cdot, m_t) \right) dt - \sigma_0 Dm_t \cdot dB_t^0,$$

whenever $\sigma = 0$. In this case, we even get exponential rates of convergence to the invariant measure. More precisely, we show the existence of $\bar{P} \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$, and a constant $\eta > 0$, such that

for all initial condition P_0 , we get the existence of a constant $C > 0$, such that for each time $t > 0$,

$$d_1^{\mathcal{P}(\mathbb{R}^d)}(P_t, \bar{P}) \leq C e^{-\eta t},$$

where $P_t = \mathcal{L}(m_t)$, and for any $p \geq 1$, the Wasserstein- p distance on $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ is defined by

$$(3) \quad d_p^{\mathcal{P}(\mathbb{R}^d)}(P, Q) = \inf_{\pi \in \Pi(P, Q)} \int_{\mathcal{P}(\mathbb{R}^d)} d_p^{\mathbb{R}^d}(\mu, \nu) \pi(d\mu, d\nu),$$

for any $P, Q \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$, and where d_p stands for the classical Wasserstein- p distance on \mathbb{R}^d . Obtaining uniqueness of the invariant measure in the general case of Equation (1) without strong convexity assumptions on the confinement potential appears to be a challenging problem which is not solved in this paper. In fact, the long-term behaviour of solutions to SPDEs with nonlinear drifts is still not fully understood and therefore, the uniqueness results obtained in this paper represent a step towards understanding the long-term behaviour of solutions to Equation (1), but are far from giving a complete understanding of the asymptotic stability for that kind of Stochastic McKean-Vlasov Equations with common noise.

Probabilistic setting & Motivation. Let us consider a filtered probability space $(\Omega^1, \mathcal{F}^1, \mathbb{F}^1, \mathbb{P}^1)$. Then, we define the following product structure

$$\Omega = \Omega^0 \times \Omega^1, \quad \mathcal{F}, \mathbb{F}, \mathbb{P},$$

where $(\mathcal{F}, \mathbb{P})$ is the completion of the set $(\mathcal{F}^0 \otimes \mathcal{F}^1, \mathbb{P}^0 \otimes \mathbb{P}^1)$ and \mathbb{F} is the right continuous augmentation of $(\mathcal{F}_t^0 \otimes \mathcal{F}_t^1)_{t \geq 0}$. We also consider a d -dimensional Brownian motion B^0 supported by $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$, adapted to \mathbb{F}^0 and another Brownian motion B supported by $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$, adapted to \mathbb{F}^1 and independent of \mathbb{F}^0 . Let us now consider a probability measure on the space of probability measures $P_0 \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$, we are able to define m_0 , a \mathcal{F}_0^0 -measurable random variable with value in the space of probability measure $\mathcal{P}(\mathbb{R}^d)$ and such that $\mathcal{L}(m_0) = P_0$, in the sense that for any bounded measurable function $F : \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}$, $\mathbb{E}_{\mathbb{P}^0}[F(m_0)] = \langle P_0; F \rangle$. We can now define on the whole probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a random variable X_0 such that $\mathcal{L}(X_0 | \mathcal{F}_0^0) = m_0$ almost surely. Let us define the stochastic process X evolving in \mathbb{R}^d , supported by $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, which dynamic is given by

$$(4) \quad \begin{cases} dX_t = -\nabla V(X_t)dt - \nabla W * m_t(X_t)dt + \sigma dB_t + \sigma_0 dB_t^0 \\ X_{|t=0} = X_0, \end{cases}$$

where m_t stands for the conditional law of the random variable X_t , with respect to the σ -algebra \mathcal{F}_t^0 . Precisely, $m_t = \mathcal{L}(X_t | \mathcal{F}_t^0)$ almost surely, and B is a d -dimensional \mathbb{F}^1 -Brownian motion independent of \mathbb{F}^0 . The dynamic of the process $(m_t)_{t \geq 0}$ is well known and given by the following Lemma (see for example [8] among other references).

Lemma 1. *The measure valued process $(m_t)_{t \geq 0}$ is solution in the weak sense of the following Stochastic Partial Differential Equation:*

$$d_t m_t = \nabla \cdot \left(\frac{\sigma^2 + \sigma_0^2}{2} \nabla m_t + m_t b(t, \cdot, m_t) \right) dt - \sigma_0 Dm_t \cdot dB_t^0,$$

with initial condition $\mathcal{L}(m_0) = P_0$.

Equation (1) connects closely to the McKean-Vlasov Equation with common noise, a common model in stochastic dynamics. It shows how large systems of interacting particles evolve. In mathematical finance, this model is particularly useful for situations like inter-bank borrowing and lending systems (see [9] or [30]). Studying the long-term behavior of solutions to Equation

(1) is important because it gives us insight of the behavior of the solution to Equation (4).

Literature. Stochastic Partial Differential Equations in the more general form,

$$(5) \quad d_t m_t = \left[\sum_{i,j} D_{ij}^2 (a_{ij}(t, \cdot, m_t) m_t) + \operatorname{div} (m_t b(t, \cdot, m_t)) \right] dt - \sigma_0(t, \cdot, m_t) D m_t \cdot dB_t^0,$$

have been extensively studied in recent decades as it naturally arises in several applications. Equation (5) is linked to the stochastic scalar conservations law of the form

$$(6) \quad d_t m_t + \nabla \cdot (\sigma_0(\cdot, u_t) u_t) \circ dW_t = 0,$$

where \circ stands for the Stratonovich stochastic integral. In the case where $\sigma_0(x, \mu) = \sigma_0(\mu(x))$, meaning that the diffusion coefficient depends in the measure in a local way, this class of equations has been introduced in [39] paving the way to several papers dealing with well-posedness of solutions of (6) in various frameworks [40, 28, 26, 29, 25]. Uniqueness of the solutions to (1) is a well known result in the class of solutions admitting a square integrable density with respect to the Lebesgue measure, see [36], and has been shown recently without any further moments assumptions in [13].

In a slightly different context, a series of papers demonstrated the well-posedness for a large class of Stochastic Differential Equations similar to (1), called *Mean Reflected Stochastic Differential Equations*, see [5], [6] and [4]. More precisely, [6] and [4] state conditional propagation of chaos under regularity conditions on the drift and diffusion terms. This equations naturally appears when considering interacting particle systems with constraint on the empirical measure of the systems and then the study of such equations is particularly important for example for applications to Mean Field Games.

In this paper, we focus on the following specific Stochastic McKean-Vlasov Equation with common noise:

$$(7) \quad d_t m_t = \nabla \cdot \left(\frac{\sigma_0^2}{2} \nabla m_t + m_t b(t, \cdot, m_t) \right) dt - \sigma_0 D m_t \cdot dB_t^0.$$

As mentioned before, extensive research has been conducted on the equation in question, and recent studies have made notable contributions to understanding its properties. For example [32] explores the existence and uniqueness of solutions for McKean-Vlasov Stochastic Differential Equations (SDEs) with common noise. Similarly, [42] proposes a regularization approach in an infinite-dimensional setting for the McKean-Vlasov equation with Wasserstein diffusion, enhancing the understanding of solutions' regularity properties. Additionally, [35] investigate the well-posedness and numerical methods for McKean-Vlasov equations with common noise, providing valuable insights on the stability and convergence of computational approaches for solving these equations.

However, to the best of our knowledge, little is known about the asymptotic behavior of the solutions of equations of the form (7). In the case without common noise, $\sigma_0 = 0$, where m is a deterministic flow of measures, past research has focused on various aspects of the solutions of (4), including existence, uniqueness ([43], [27], [31]), and stability. Over the past two decades, significant advancements have been made in understanding the convergence to equilibrium for solutions of the deterministic McKean-Vlasov equation. For example, see [10] or [11] for proofs of an exponential convergence rate to equilibrium under strict convexity conditions on the potentials V and W . The case without strict convexity assumptions is more intricate. Nevertheless, through a thorough examination of the dissipation of the Wasserstein distance, [3] showed an exponential convergence to equilibrium in a weakly-convex case. Recently, involving a coupling method

issued from [38], it has been shown using nice concentration properties from [22] that the convergence to equilibrium holds with an exponential speed in the case of a confinement potential that is only convex far from the origin, as seen in [21]. The latter shows uniform in time propagation of chaos property, as introduced in [34] and [45], allowing one to conclude the uniqueness of the invariant measure and provide a rate of convergence to equilibrium.

Organisation of the paper. This paper has three main parts and primarily uses a probabilistic approach. In Section 2, we study the existence of an invariant measure for the process $(m_t)_{t \geq 0}$, controlled by Equation (1), and provide conditions on potentials V and W for this existence. We also give moment estimate for the invariant measures. Section 3 examines the uniqueness of invariant measures under a uniformly convex confinement potential V , adapting known results without common noise. We also discuss the Ornstein-Uhlenbeck process with common noise, where the invariant measure can be explicitly described. In both Sections 2 and 3, we show uniform-in-time propagation of chaos and convergence to equilibrium. Section 4 explores the same topic for non-convex potential V when $d = 1$ or $\sigma = 0$. Technical proofs are provided in the Appendix.

Definition and notation. Throughout the paper, for a Polish space E we write $\mathcal{P}(E)$ for the space of Borel probability measures on E equipped with the topology of weak convergence and the corresponding Borel σ -algebra. In this paper, we consider a stochastic process $(m_t)_{t \geq 0}$ with value in the space of probability measures $\mathcal{P}(\mathbb{R}^d)$. We denote by P_t the law $\mathcal{L}(m_t)$ of m_t , for $t \geq 0$, which is a probability measure on the space of probability measure. Then $(m_t)_{t \geq 0}$ is a continuous $\mathcal{P}(\mathbb{R}^d)$ -valued process, and $P = (P_t)_{t \geq 0}$ belongs to $C([0, +\infty[; \mathcal{P}(\mathcal{P}(\mathbb{R}^d)))$, the space of continuous functions from $[0, +\infty[$ to $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$. Moreover, for $P, Q \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$, we denote by $\Pi(P, Q)$ the set of transport plans between P and Q , and for any distance $d : \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \rightarrow [0, +\infty)$, we define:

$$(8) \quad d_d^{\mathcal{P}(\mathbb{R}^d)}(P, Q) = \inf_{\Lambda \in \Pi(P, Q)} \int_{\mathcal{P}(\mathbb{R}^d)} d(\mu, \nu) \Lambda(d\mu, d\nu).$$

In this paper, we mainly use a probability-based approach, often switching between the measure-valued stochastic process m and its probabilistic counterpart X , which solves Equation (4). At this point, it is worth noting that we are concerned with a stochastic process $(m_t)_{t \geq 0}$ that takes values in the space of probability measures $\mathcal{P}(\mathbb{R}^d)$. This means that at each time $t > 0$, we are dealing with measures on the space of probability, rather than on the underlying space \mathbb{R}^d . To study this process, we use a probability-based approach and introduce several definitions that are specific to this setting. For instance, we define the notion of an invariant measure in $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$, which is a probability measure that remains invariant under the evolution of the stochastic process. We also define the notion of a probability measure in $L_p(\mathcal{P}(\mathbb{R}^d))$, which is a measure that satisfies a certain integrability condition. These definitions are essential for our analysis and detailed below.

Definition 1. *Let us consider a random variable X defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and $P \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$. According to Lemma 2.4 in [8], for \mathbb{P}^0 -a.e. $\omega_0 \in \Omega^0$, $X(\omega_0, \cdot)$ is a random variable on $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$. By defining $\mathcal{L}^1(X) : \Omega^0 \ni \omega_0 \mapsto \mathcal{L}(X(\omega_0, \cdot))$, we get a random variable from $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ into $\mathcal{P}(\mathbb{R}^d)$, providing a conditional law of X given \mathcal{F}^0 . Finally, we say that $\mathcal{L}(X) = P$ whenever $\mathcal{L}^1(X)$ is distributed with respect to P .*

Definition 2 (Invariant measure). • We say that $\bar{P} \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$, is an invariant measure for the process $(m_t)_{t \geq 0}$, if whenever m_0 is distributed according to \bar{P} , then at each time, the law of m_t is

independent of t . More precisely, we say that \bar{P} is an invariant measure if and only if

$$\mathcal{L}(m_0) = \bar{P} \quad \Rightarrow \quad \mathcal{L}(m_t) = \bar{P}, \quad \forall t > 0.$$

- We say that a stochastic process X on the probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ admits an invariant measure in $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ if and only if, the measure valued stochastic process $m_t = \mathcal{L}(X_t | \mathcal{F}_t^0)$ admits an invariant measure.

Definition 3. We say that a probability measure $P \in L_p(\mathcal{P}(\mathbb{R}^d))$ for $p \geq 1$, whenever

$$\int_{\mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |x|^p m(dx) P(dm) < +\infty.$$

Definition 4. For any increasing, continuous and concave function such that $f(0) = 0$, such that $(x, y) \mapsto f(|x - y|)$ defines a distance on \mathbb{R}^d , we define the following distance on $\mathcal{P}(\mathbb{R}^d)$

$$d_f^{\mathcal{P}(\mathbb{R}^d)}(P, Q) = \inf_{\Gamma \in \Pi(P, Q)} \int_{\mathcal{P}_2(\mathbb{R}^d)} d_f^{\mathbb{R}^d}(\mu, \nu) \Gamma(d\mu, d\nu),$$

with $d_f^{\mathbb{R}^d}(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d} f(|x - y|) \pi(dx, dy)$.

1.1. The process m as a mean field limit. Thanks to the definition of the previous subsection, we are now ready to give an interpretation of the process (m_t) in terms of mean field limit for interacting particle system. Let us consider $P_0 \in L_2(\mathcal{P}(\mathbb{R}^d))$, let also $N \geq 1$ be an integer, and $(X_0^{1,N}, \dots, X_0^{N,N})$, N random variables which are conditionally independent and identically distributed with respect to \mathcal{F}_0^0 , such that $\mathcal{L}(X_0^{i,N}) = P_0$ for all $i \in \{1, \dots, N\}$. We now define the following interacting particle system

$$\begin{cases} dX_t^{i,N} = -\nabla V(X_t^{i,N}) - N^{-1} \sum_{j=1}^N \nabla W(X_t^{i,N} - X_t^{j,N}) dt + \sigma dB_t^i + \sigma_0 dB_t^0, \\ X_{|t=0}^{i,N} = X_0^{i,N}, \quad \forall i \in \{1, \dots, N\}, \end{cases}$$

where the B^i are independent d -dimensional \mathbb{F}^1 -Brownian motion which are independent of \mathbb{F}^0 . Then, we consider the mean-field limit system $(\bar{X}^1, \dots, \bar{X}^N)$ driven by

$$\begin{cases} d\bar{X}_t^i = -\nabla V(\bar{X}_t^i) - \nabla W * m_t(\bar{X}_t^i) dt + \sigma dB_t^i + \sigma_0 dB_t^0, \\ \bar{X}_{|t=0}^i = X_0^i, \quad \forall i \in \{1, \dots, N\}, \end{cases}$$

where $(\bar{X}_0^i)_i$ are conditionally *iid* random variables with respect to \mathcal{F}_0^0 , such that $\mathcal{L}(\bar{X}_0^i) = P_0$, for all $i \in \{1, \dots, N\}$. Our framework is exactly the same as the classical one for mean field games system with common noise. However here, the law of the initial conditions is random. Then, conditioning with respect to the σ -algebra \mathcal{F}_0^0 , we get back to a more classical framework where the initial condition is a deterministic measure. More precisely, as stated in [8], under sufficient regularity conditions on the transport part b mainly Lipschitz continuity with respect to the space and measure variables, we get that for any fixed $t \geq 0$,

$$\lim_{N \rightarrow +\infty} \mathbb{E}[|\bar{X}_t^i - X_t^{i,N}|^2] + \mathbb{E}[d_2^{\mathbb{R}^d}(m_t^N, m_t)] = 0,$$

where $d_2^{\mathbb{R}^d}(\cdot, \cdot)$ stands for the classical Wasserstein distance on \mathbb{R}^d , and $m_t^N := N^{-1} \sum_{i=1}^N \delta_{X_t^{i,N}}$ is the empirical measure of the interacting particle system.

2. EXISTENCE OF AN INVARIANT MEASURE FOR THE STOCHASTIC FLOW OF MEASURES.

Let us consider a stochastic process $(m_t)_{t \geq 0}$ with value in the space of probability measures $\mathcal{P}(\mathbb{R}^d)$, and with dynamic given by (1). Then, m is a weak solution of

$$d_t m_t = \nabla \cdot \left(\frac{\sigma^2 + \sigma_0^2}{2} \nabla m_t + m_t (\nabla V + \nabla W * m_t) \right) dt - \sigma_0 Dm_t \cdot dB_t^0.$$

More precisely, for all $t \geq 0$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$d \langle m_t, \varphi \rangle = \langle m_t, L_{m_t} \varphi \rangle dt + \sigma_0 \langle m_t, (\nabla \varphi)^\top \rangle dB_t^0,$$

where for any probability measure $m \in \mathcal{P}(\mathbb{R}^d)$, the operator L_m acts on a smooth function φ of compact support by

$$L_m \varphi = -(\nabla V + \nabla W * m) \cdot \nabla \varphi + \frac{\sigma_0^2 + \sigma^2}{2} \Delta \varphi,$$

where ∇, Δ respectively stands for the gradient and laplacian operator, while \cdot denotes the usual inner product in \mathbb{R}^d , and for any $\varphi \in C_c^\infty(\mathbb{R}^d)$ and any probability measure m ,

$$\langle m; \varphi \rangle = \int_{\mathbb{R}^d} \varphi dm.$$

In this section, we aim at giving conditions on the potentials V and W to ensure existence of an invariant measure for the process (m_t) . Let us now consider the following assumptions:

Assumption (A1). (1) *There exists a continuous function $\kappa : [0, +\infty) \rightarrow \mathbb{R}$, such that*

$$\liminf_{r \rightarrow +\infty} \kappa(r) > 0,$$

and

$$(\nabla V(x) - \nabla V(y)) \cdot (x - y) \geq \kappa(|x - y|) |x - y|^2.$$

(2) ∇V is L_V -Lipschitz continuous.

Assumption (A2). (1) *W is symmetric, i.e., $W(x) = W(-x)$ for all $x \in \mathbb{R}^d$.*

(2) ∇W is L_W -Lipschitz continuous.

In the following, we work under the set of assumptions (A1)&(A2). Under this set of assumptions, both equations (4) and (1) are well-posed, as shown in [15]. The assumption regarding confinement potential V primarily ensures convexity at infinity, which helps keep the process within a compact set with a high probability. We can moreover note that this implies the existence of $m_V > 0$ and $M_V \geq 0$, such that

$$(\nabla V(x) - \nabla V(y)) \cdot (x - y) \geq m_V |x - y|^2 - M_V.$$

In this section, we start with a result regarding uniform-in-time control of the process $(X_t)_{t \geq 0}$ with initial condition in $L_2(\mathcal{P}(\mathbb{R}^d))$ dynamic given by (4). To prove the existence of an invariant measure for $(m_t)_{t \geq 0}$ solution of (1), we will use the concept of intrinsic derivative for a functional defined on a space of measure, as seen in [7].

Definition 5. *Let us define $\mathcal{C}_b^2(\mathcal{P}(\mathbb{R}^d))$ as the collection of continuous, bounded functions $F : \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}$ with the following properties:*

- *There exists a unique continuous and bounded function $\partial_m F : \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that*

$$\lim_{h \rightarrow 0} \frac{F(m + h(m' - m)) - F(m)}{h} = \int_{\mathbb{R}^d} \partial_m F(m, v) (m' - m)(dv),$$

for all $m, m' \in \mathcal{P}(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} \partial_m F(m, v) m(dv) = 0, \quad m \in \mathcal{P}(\mathbb{R}^d);$$

- The mapping $x \mapsto \partial_m F(m, x)$ is continuously differentiable with uniformly bounded gradient $D_m F(m, x)$ in (m, x) ;
- For any fixed $x \in \mathbb{R}^d$, every component of the \mathbb{R}^d -valued function $m \mapsto D_m F(m, x)$ satisfies the same conditions as the first two bullet points, resulting in a continuous and bounded $D_m^2 F(m, x, y) \in \mathbb{R}^{d \times d}$;
- For $m \in \mathcal{P}(\mathbb{R}^d)$, we use $D_x D_m F(m, v)$ to denote the Jacobian of the function $x \mapsto D_m F(m, x)$, which is assumed to be continuous and bounded in (m, x) .

Proposition 1. *Considering a measure valued process (m_t) , which dynamic is given by (1), and defining $P_t = \mathcal{L}(m_t)$, we have that for any bounded and twice differentiable function $F \in \mathcal{C}_b^2(\mathcal{P}(\mathbb{R}^d), \mathbb{R})$:*

$$(9) \quad \langle P_t - P_0, F \rangle = \int_0^t \langle P_s, \mathcal{M}F \rangle ds, \quad \forall t > 0,$$

where we define, for $m \in \mathcal{P}(\mathbb{R}^d)$,

$$\begin{aligned} \mathcal{M}F(m) &:= \int_{\mathbb{R}^d} \left[D_m F(m, x) \cdot b(x, m) + \frac{\sigma^2 + \sigma_0^2}{2} \nabla \cdot (D_m F(m, x)) \right] m(dx) \\ &\quad + \frac{\sigma_0^2}{2} \int_{\mathbb{R}^{2d}} \text{Tr} [D_{mm}^2 F(m, x, y)] m(dx) m(dy), \end{aligned}$$

and where for all measurable and bounded function $\Phi : \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}$, and all $P \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$,

$$\langle P; \Phi \rangle = \int_{\mathcal{P}(\mathbb{R}^d)} \Phi(m) P(dm).$$

Moreover, \bar{P} is an invariant measure if and only if

$$(10) \quad \langle \bar{P}, \mathcal{M}F \rangle = 0, \quad \forall F \in \mathcal{C}_b^2(\mathcal{P}(\mathbb{R}^d), \mathbb{R}).$$

The proof of Proposition 1 is based on Section 1.2 of [37] and then postponed to the Appendix A.1.

2.1. The existence result. We now present a result about the existence of an invariant measure for the equation of interest, adapting classical results for McKean-Vlasov equation without common noise where the flow of probability measures, represented as $(m_t)_{t \geq 0}$ is deterministic. We begin this section with the following Lemma, the proof of which is classical and postponed to Appendix A.2.

Lemma 2. *Under Assumptions (A1) & (A2), let us consider $P_0 \in L_2(\mathcal{P}(\mathbb{R}^d))$. Then, denoting by $(X_t)_t$ the associated stochastic process with initial condition $\mathcal{L}(X_0) = P_0$ in the sense of Definition 1, and dynamic given by (4), we have the following uniform in time moment control:*

$$\sup_{t > 0} \mathbb{E} [|X_t|^2] < +\infty.$$

Now, we are ready to state the following Proposition:

Proposition 2. *Under Assumptions (A1) & (A2), the dynamical system given by (1) admits at least an invariant measure $\bar{P} \in L_2(\mathcal{P}(\mathbb{R}^d))$, i.e such that*

$$\int_{\mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |x|^2 m(dx) \bar{P}(dm) < +\infty.$$

Proof of Proposition 2. Let us fix $P_0 \in L_2(\mathcal{P}(\mathbb{R}^d))$, i.e such that:

$$\int_{\mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |x|^2 m(dx) P_0(dm) < +\infty.$$

Step 1. Let us now consider a random flow of probability measures $(m_t)_{t \geq 0}$ with dynamic given by (1) and initial condition $P_0 \in L_2(\mathcal{P}(\mathbb{R}^d))$. At each time $t \geq 0$, we denote by P_t the law of this measure valued process. For $T > 0$, we define the process $(Q_T)_{T > 0}$ with value in $L_2(\mathcal{P}(\mathbb{R}^d))$ by

$$Q_T = T^{-1} \int_0^T P_t dt.$$

This defines a sequence of probability measures on $\mathcal{P}(\mathbb{R}^d)$. Let us show that $(Q_T)_{T > 0}$ admits at least a convergent subsequence. Thanks to Prohorov theorem, we only need to show that this sequence is tight. For $R > 0$, let us consider the set

$$K_R = \left\{ m \in \mathcal{P}(\mathbb{R}^d), \int_{\mathbb{R}^d} |x|^2 m(dx) \leq R \right\}.$$

This set is compact for the topology of weak convergence in $\mathcal{P}(\mathbb{R}^d)$. Now, for $T > 0$,

$$\begin{aligned} Q_T(K_R) &= T^{-1} \int_0^T P_t(K_R) dt \\ &= T^{-1} \int_0^T \int_{\mathcal{P}(\mathbb{R}^d)} \mathbb{1}_{\{m \in K_R\}} P_t(dm) dt \\ &= T^{-1} \int_0^T \mathbb{E} [\mathbb{1}_{\{m_t \in K_R\}}] dt \\ &\geq 1 - \frac{c}{R}, \end{aligned}$$

where $c = \sup_{t > 0} \mathbb{E} [|X_t|^2] < +\infty$, thanks to Lemma 2. Hence, for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $Q_T(K_{R_\varepsilon}) > 1 - \varepsilon$, for all $T > 0$. This gives tightness of the sequence and then existence of a converging subsequence that we keep denoting by $(Q_T)_{T > 0}$ in the following.

Step 2. Let us denote by Q the limit of this converging subsequence, and show that Q is an invariant measure. Let $T > 0$, and $F \in C_b^2(\mathcal{P}(\mathbb{R}^d), \mathbb{R})$, thanks to Proposition 1, we get that

$$\begin{aligned} \langle Q_T, \mathcal{M}F \rangle &= \int_{\mathcal{P}(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} [D_m F(m, x) \cdot b(x, m) + \frac{\sigma^2 + \sigma_0^2}{2} \nabla \cdot (D_m F(m, x))] m(dx) \right. \\ &\quad \left. + \frac{\sigma_0^2}{2} \int_{\mathbb{R}^{2d}} \text{Tr}[D_{mm}^2 F(m, x, y)] m(dx) m(dy) \right) Q_T(dm) \\ &= T^{-1} \int_0^T \int_{\mathcal{P}(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} [D_m F(m, x) \cdot b(x, m) + \frac{\sigma^2 + \sigma_0^2}{2} \nabla \cdot (D_m F(m, x))] m(dx) \right. \\ &\quad \left. + \frac{\sigma_0^2}{2} \int_{\mathbb{R}^{2d}} \text{Tr}[D_{mm}^2 F(m, x, y)] m(dx) m(dy) \right) P_t(dm) \\ &= T^{-1} \langle P_T - P_0, F \rangle. \end{aligned}$$

Hence, for all $F \in C_b^2(\mathcal{P}(\mathbb{R}^d))$, $\langle Q, \mathcal{M}F \rangle = 0$. This ensures that Q is an invariant measure.

Step 3. Finally we move on to the moment estimate. We know that there exists a subsequence of Q_T which converges weakly to \bar{P} . Moreover,

$$\int_{\mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |x|^2 m(dx) Q_T(dm) = \frac{1}{T} \int_0^T \int_{\mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |x|^2 m(dx) P_t(dm) dt,$$

and

$$\sup_{T>0} \frac{1}{T} \int_0^T \int_{\mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |x|^2 m(dx) P_t(dm) dt < +\infty,$$

as $\sup_{t>0} \mathbb{E}[|X_t|^2] < +\infty$. Moreover, the function $m \in \mathcal{P}_2(\mathbb{R}^d) \mapsto \int_{\mathbb{R}^d} |x|^2 m(dx)$ is lower semi continuous, and then,

$$\int_{\mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |x|^2 m(dx) \bar{P}(dm) \leq \liminf_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \int_{\mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |x|^2 m(dx) P_t(dm) dt < +\infty.$$

□

From the previous result we get the following Corollary

Corollary 1. *For $P \in L_2(\mathcal{P}(\mathbb{R}^d))$, we have that $d_2^{\mathcal{P}(\mathbb{R}^d)}(P, \bar{P}) < +\infty$, where $d_2^{\mathcal{P}(\mathbb{R}^d)}(\cdot, \cdot)$ is defined in 3.*

3. THE CASE OF A UNIFORMLY CONVEX CONFINEMENT POTENTIAL

In this section, we show that the process $(m_t)_{t \geq 0}$, driven by equation (1), has a unique invariant measure under strong convexity assumptions on the confinement potential. Moreover, our method allows us to find exponential rates of convergence toward the invariant measure for a specific set of initial conditions. We again consider $P_0 \in L_2(\mathcal{P}(\mathbb{R}^d))$ and a random variable X_0 such that $\mathcal{L}(X_0) = P_0$, following Definition 1. Next, we study the stochastic process (X_t) driven by equation (4) with the initial condition X_0 . In this part of the paper, we consider strict convexity assumptions on the confinement potential V . To be specific, we adopt the following assumptions throughout this section:

Assumption (A4). • V is uniformly convex, more precisely, there exists $\beta > 0$ such that:

$$\nabla^2 V \geq \beta \text{Id}.$$

- ∇V is Lipschitz continuous.
- W is even, convex, and ∇W is globally Lipschitz continuous, with Lipschitz constant L_W .

Thanks to the previous section, under Assumption (A4), a process X driven by equation (4) has an invariant measure. Specifically, there exists $\bar{P} \in L_2(\mathcal{P}(\mathbb{R}^d))$ such that the process $(m_t)_{t \geq 0}$, governed by the dynamic in Equation (1) and with an initial condition of $\mathcal{L}(m_0) = \bar{P}$, is invariant. This section starts with a key result that states uniform propagation of chaos uniformly in time. This result will then help us establish the uniqueness of the invariant measure and to give a rate of convergence to equilibrium.

3.1. Uniform in time propagation of chaos. In the case without common noise, the uniqueness of the invariant measure for the process X has been already been established see *e.g.* [12], [41], and [3]. In this section, our aim is to adapt this result to the case with common noise and obtain the uniqueness of the invariant measure for the probability measure valued stochastic process (m_t) , and exponentially fast convergence to the equilibria.

Theorem 1. *Let us consider $P, Q \in L_2(\mathcal{P}(\mathbb{R}^d))$, and two particle systems $\mathbf{X} = (X^1, \dots, X^N)$ and $\mathbf{X}^N = (X^{1,N}, \dots, X^{N,N})$ with dynamics,*

$$(11) \quad dX_t^i = -\nabla V(X_t^i)dt - \nabla W * m_t(X_t^i)dt + \sigma dB_t^i + \sigma_0 dB_t^0, \quad \forall i \in \{1, \dots, N\},$$

and

$$(12) \quad dX_t^{i,N} = -\nabla V(X_t^{i,N})dt - \frac{1}{N} \sum_{j=1}^N \nabla W(X_t^{i,N} - X_t^{j,N})dt + \sigma dB_t^i + \sigma_0 dB_t^0, \quad \forall i \in \{1, \dots, N\}.$$

where $(X_0^i)_{i \in \{1, \dots, N\}}$ are independent and identically distributed such that $\mathcal{L}(\bar{X}_0^1) = P$ in the sense of Definition 1, and where the same holds for the second system with $\mathcal{L}(X_0^{1,N}) = Q$. Then, under assumptions (A4), there exists a constant $C > 0$ depending only on the dimension d and the probability measures P and Q , such that

$$\mathbb{E} \left[d_2^{\mathbb{R}^d} (m_{\mathbf{X}_t}^{P,N}, m_{\mathbf{X}_t^N}^{Q,N}) \right] \leq C \left(e^{-\beta t} + \frac{1}{2\beta\sqrt{N-1}} \right),$$

where $m_{\mathbf{X}_t}^{P,N} = \frac{1}{N} \sum_{i=1}^N \delta_{X_t^i}$ and $m_{\mathbf{X}_t^N}^{Q,N} = \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i,N}}$.

The proof of this result closely follows the approach presented in [2] and [41], which shows the propagation of chaos for particle systems in cases without common noise. However, in our situation, we need to be careful with the interaction term and its dependency on the common noise.

Proof. We only sketch the proof, as it follows closely the proof of [41], Theorem 3.3. Let $i \in \{1, \dots, N\}$, using Itô formula, we get

$$\begin{aligned} \frac{d}{dt} |X_t^{i,N} - X_t^i|^2 &= -2(X_t^{i,N} - X_t^i) \cdot (\nabla V(X_t^{i,N}) - \nabla V(X_t^i)) \\ &\quad - \frac{2}{N} \sum_{j=1}^N (X_t^{i,N} - X_t^i) \cdot (\nabla W(X_t^{i,N} - X_t^{j,N}) - \nabla W * m_t(X_t^i)). \end{aligned}$$

In order to control the second term, we make the following decomposition:

$$\begin{aligned} &-2(X_t^{i,N} - X_t^i) \left(N^{-1} \sum_{j=1}^N \nabla W(X_t^{i,N} - X_t^{j,N}) - \nabla W * m_t(X_t^i) \right) \\ &= -2(X_t^{i,N} - X_t^i) \left(N^{-1} \sum_{j=1}^N \nabla W(X_t^{i,N} - X_t^{j,N}) - N^{-1} \sum_{j=1}^N \nabla W(X_t^i - X_t^j) \right) \\ &\quad - 2(X_t^{i,N} - X_t^i) \left(N^{-1} \sum_{j=1}^N \nabla W(X_t^i - X_t^j) - \nabla W * m_t(X_t^i) \right) \\ &= \Xi_t^{i,N} + \Upsilon_t^{i,N}. \end{aligned}$$

Summing the first term over i shows and using the convexity of the interaction potential, we get

$$\frac{1}{N} \sum_{i=1}^N \Xi_t^{i,N} = -2N^{-2} \sum_{i,j=1}^N (X_t^{i,N} - X_t^{j,N} - X_t^i + X_t^j) (\nabla W(X_t^i - X_t^j) - \nabla W(X_t^i - X_t^j)) \leq 0.$$

Moreover, the second term can be decomposed into two terms

$$\begin{aligned}\Upsilon_t^{i,N} &= -2(X_t^{i,N} - X_t^i) \cdot \left((N-1)^{-1} \sum_{j=1}^N \nabla W(X_t^i - X_t^j) - \nabla W * m_t(X_t^i) \right) \\ &\quad - 2(X_t^{i,N} - X_t^i) \cdot \left(\left(\frac{1}{N} - \frac{1}{N-1} \right) \sum_{j=1}^N \nabla W(X_t^i - X_t^j) \right)\end{aligned}$$

For the first one

$$\begin{aligned}\mathbb{E}[|\nabla W * m_t(X_t^i) - \frac{1}{N-1} \sum_{j=1}^N \nabla W(X_t^i - X_t^j)|^2] &= \mathbb{E}\left[\mathbb{E}\left[\left|\nabla W * m_t(X_t^i) - \frac{1}{N-1} \sum_{j=1}^N \nabla W(X_t^i - X_t^j)\right|^2 \middle| X_t^i, \mathcal{F}_t^0\right]\right] \\ &= \mathbb{E}\left[\text{Var}\left[\frac{1}{N-1} \sum_{j=1}^N \nabla W(X_t^i - X_t^j) \middle| X_t^i, \mathcal{F}_t^0\right]\right] \\ &\leq \frac{1}{N-1} \mathbb{E}[\mathbb{E}[|\nabla W(X_t^i - X_t^j)|^2 | X_t^i, \mathcal{F}_t^0]], \quad \text{for some } j \neq i \\ &\leq \frac{L_W^2}{N-1} \mathbb{E}[\mathbb{E}[|X_t^i - X_t^j|^2 | X_t^i, \mathcal{F}_t^0]] \\ &\leq \frac{2L_W^2}{N-1} \mathbb{E}[|X_t^1|^2],\end{aligned}$$

where the first inequality comes from the fact that $\mathbb{E}[\nabla W(X_t^i - X_t^j) | X_t^i, \mathcal{F}_t^0] = \nabla W * m_t(X_t^i)$. Then, thanks to Lemma 2, there exists a constant $C > 0$, such that

$$\begin{aligned}\mathbb{E}\left[(X_t^{i,N} - X_t^i) \cdot \left((N-1)^{-1} \sum_{j=1}^N \nabla W(X_t^i - X_t^j) - \nabla W * m_t(X_t^i) \right)\right] &\leq \mathbb{E}[|X_t^{i,N} - X_t^i|^2]^{1/2} \mathbb{E}\left[\left|\nabla W * m_t(X_t^i) - \frac{1}{N-1} \sum_{j=1}^N \nabla W(X_t^i - X_t^j)\right|^2\right]^{1/2} \\ &\leq \frac{C}{\sqrt{N-1}} \mathbb{E}[|X_t^{i,N} - X_t^i|^2]^{1/2}.\end{aligned}$$

Now,

$$\mathbb{E}\left[\left|\frac{1}{N-1} \cdot \frac{1}{N} \sum_{j=1}^N \nabla W(X_t^i - X_t^j)\right|^2\right] \leq \frac{2L_W^2}{(N-1)^2} \mathbb{E}[|X_t^1|^2].$$

Finally, using once again Lemma 2 we get the existence of a constant C , such that:

$$\mathbb{E}[|\Upsilon_t^{i,N}|] \leq C \left(\frac{1}{N-1} + \frac{1}{\sqrt{N-1}} \right) \mathbb{E}[|X_t^{i,N} - X_t^i|^2]^{1/2}.$$

Finally, we get that

$$\frac{1}{N} \sum_{i=1}^N \frac{d}{dt} \mathbb{E}[|X_t^i - X_t^{i,N}|^2] \leq -\frac{2\beta}{N} \sum_{i=1}^N \mathbb{E}[|X_t^i - X_t^{i,N}|^2] + \frac{C}{\sqrt{N-1}} \left(N^{-1} \sum_{i=1}^N \mathbb{E}[|X_t^{i,N} - X_t^i|^2] \right)^{1/2},$$

for some constant $C > 0$. Let us now denote $v_N(t) = N^{-1} \sum_{i=1}^N \mathbb{E}[|X_t^i - X_t^{i,N}|^2]$, then we have

$$v'_N(t) \leq -2\beta v_N(t) + \frac{C}{\sqrt{N-1}} v_N(t)^{1/2}.$$

This gives, using Grönwall Lemma:

$$(13) \quad v_N(t)^{1/2} \leq e^{-\beta t} v_N(0) + \frac{C}{2\beta\sqrt{N-1}}.$$

Moreover, $v_N(0) = N^{-1} \sum_{i=1}^N \mathbb{E}[|X_0^i - X_0^{i,N}|^2] \leq \mathbb{E}[|X_0^i|^2 + |X_0^{i,N}|^2]$ is bounded uniformly in N , and as $\mathbb{E}[d_2^{\mathbb{R}^d}(m_{X_t}^N, m_{X_t^N}^N)] \leq v_N(t)^{1/2}$, we conclude the proof of Theorem 1. \square

3.2. Uniqueness of the invariant measure. The main consequence of the previous result is the uniqueness of the invariant measure for the process $(m_t)_{t \geq 0}$ driven by (1). As recalled at the beginning of the section, we have already shown that under Assumptions (A1) and (A2), there exists an invariant measure \bar{P} . From Theorem 1, we get the following Corollary

Corollary 2. *Under Assumptions (A4), the stochastic process (m_t) admits a unique invariant measure $\bar{P} \in \mathcal{P}_2(\mathcal{P}(\mathbb{R}^d))$. Moreover, for each $P_0 \in L_2(\mathcal{P}(\mathbb{R}^d))$, there is an exponential convergence to the invariant measure:*

$$d_2^{\mathcal{P}(\mathbb{R}^d)}(P_t, \bar{P}) \leq e^{-\beta t} d_2^{\mathcal{P}(\mathbb{R}^d)}(P_0, \bar{P})^2,$$

where we recall that the Wasserstein-2 distance on $\mathcal{P}(\mathbb{R}^d)$ is defined by in (3).

This implies uniqueness of the invariant measure and the convergence to this equilibria for a large class of initial conditions P_0 . In order to prove the previous result, we begin with a technical Lemma:

Lemma 3. *Let m and ρ be two probability measures valued random variables which are \mathcal{F}_0^0 -measurable. Then, there exists a random variable ξ defined on the space $(\Omega, \mathcal{F}, \mathbb{P})$ and with value in $\mathcal{P}(\mathbb{R}^d)$, such that almost surely:*

$$\xi \in \arg \min_{\pi \in \Pi(m, \rho)} \int_{\mathbb{R}^d} |x - y|^2 \pi(dx, dy).$$

The proof of this Lemma is postponed to Appendix A.3 and relies on mesurability arguments for set valued functions issued from [44].

Proof of Corollary 2. The proof of this result relies on the result and the proof of Theorem 1, but the important difference is the choice of the initial conditions. More precisely, for $P_0 \in L_2(\mathcal{P}(\mathbb{R}^d))$, we pick $\Gamma \in \Pi(P_0, \bar{P})$ which is not empty. Let us consider a couple of probability measure valued random variables (m_0, \bar{m}_0) , and such that $\mathcal{L}((m_0, \bar{m}_0)) = \Gamma$. It means that $\mathcal{L}(m_0) = P_0$ and $\mathcal{L}(\bar{m}_0) = \bar{P}_0$. Thanks to Lemma 3, we know that there exists ξ random variable such that almost surely,

$$\xi \in \arg \min_{\pi \in \Pi(m_0, \bar{m}_0)} \int_{\mathbb{R}^d} |x - y|^2 \pi(dx, dy).$$

We consider once again the particle system:

$$dX_t^i = -\nabla(X_t^i)dt - \nabla W * m_t(X_t^i)dt + \sigma dB_t^i + \sigma_0 dB_t^0, \quad \forall i \in \{1, \dots, N\},$$

and

$$dX_t^{i,N} = -\nabla(X_t^{i,N})dt - \frac{1}{N} \sum_{j=1}^N \nabla W(X_t^{i,N} - X_t^{j,N})dt + \sigma dB_t^i + \sigma_0 dB_t^0, \quad \forall i \in \{1, \dots, N\},$$

where the $(X_0^i, X_0^{i,N})$ for $i \in \{1, \dots, N\}$ are independent and such that $\mathcal{L}((X_0^i, X_0^{i,N}) | \mathcal{F}_0^0) = \xi$. From Theorem 1, and more precisely Equation (13), we know that there exists a constant $C > 0$, such that:

$$(14) \quad \mathbb{E}[\mathfrak{d}_2^{\mathbb{R}^d}(m_{\mathbf{X}_t}^N, m_{\mathbf{X}_t^N}^N)^2] \leq e^{-\beta t} \mathbb{E}[|X_0^1 - X_0^{1,N}|^2] + CN^{-1/2}.$$

Moreover, with the particular choice we made for the initial conditions, we get that

$$\begin{aligned} \mathbb{E}[|X_0^1 - X_0^{1,N}|^2] &= \mathbb{E}[\mathbb{E}[|X_0^1 - X_0^{1,N}|^2 | \mathcal{F}_0^0]] \\ &= \mathbb{E}\left[\int_{\mathbb{R}^d} |x - y|^2 \xi(\mathrm{d}x, \mathrm{d}y)\right] \\ &= \int_{\mathcal{P}(\mathbb{R}^d)} \mathfrak{d}_2^{\mathbb{R}^d}(\mu, \nu) \Gamma(\mathrm{d}\mu, \mathrm{d}\nu). \end{aligned}$$

Taking the infimum over all the transport plan Γ between P_0 and \bar{P} in (14), we get that

$$(15) \quad \mathbb{E}\left[\mathfrak{d}_2^{\mathbb{R}^d}(m_{\mathbf{X}_t}^N, m_{\mathbf{X}_t^N}^N)^2\right] \leq e^{-\beta t} \mathfrak{d}_2^{\mathcal{P}(\mathbb{R}^d)}(P_0, \bar{P}) + \frac{C}{\sqrt{N}}.$$

Moreover,

$$(16) \quad \mathfrak{d}_2^{\mathcal{P}(\mathbb{R}^d)}(P_t, \bar{P}) \leq \mathbb{E}\left[\mathfrak{d}_2^{\mathbb{R}^d}(\bar{m}_t, m_t)^2\right],$$

where (\bar{m}_t) is the stochastic flow of measure driven by (1) with initial condition \bar{P} and (m_t) has the same dynamic with initial condition P_0 . Now combining (15) and (16), taking the limit N to infinity, we get

$$\mathfrak{d}_2^{\mathcal{P}(\mathbb{R}^d)}(P_t, \bar{P}) \leq e^{-\beta t} \mathfrak{d}_2^{\mathcal{P}(\mathbb{R}^d)}(P_0, \bar{P}).$$

□

Remarks 1. *Some remarks are in order: (1) We recall that thanks to Proposition 1, the distance $\mathfrak{d}_2^{\mathcal{P}(\mathbb{R}^d)}(P_0, \bar{P})$ is finite because $P_0 \in L_2(\mathcal{P}(\mathbb{R}^d))$ in the sense of Definition 3; (2) Common noise doesn't ruin the usual convergence results. It makes sense because the common noise mainly acts like a drift term in Equation (1) and should not make the usual convergence results go haywire.*

3.3. Example. In this section, we consider an Ornstein-Uhlenbeck process with common noise, that is taking $V : x \mapsto |x|^2/2$ and $W = 0$. Let X be a real valued stochastic process evolving in \mathbb{R}^d , driven by the following stochastic differential equation:

$$(17) \quad \mathrm{d}X_t = -X_t \mathrm{d}t + \sigma \mathrm{d}W_t + \sigma_0 \mathrm{d}B_t^0.$$

Then, the process associated process $(m_t)_{t \geq 0}$ is solution of

$$(18) \quad \mathrm{d}_t m_t = \nabla \cdot \left[\frac{\sigma^2 + \sigma_0^2}{2} \nabla m_t + m_t x \right] \mathrm{d}t - \sigma_0 D m_t \cdot \mathrm{d}B_t^0,$$

Moreover, we consider the following initial condition $\mathcal{L}(X_0) = P_0$ (in the sense of Definition 1), for some $P_0 \in L_2(\mathcal{P}(\mathbb{R}^d))$. In this particular setting, we are able to explicitly describe the invariant measure.

Proposition 3. *The unique invariant measure \bar{P} of the process on the space $\mathcal{P}(\mathbb{R}^d)$ is the image, by the function $x \in \mathbb{R}^d \mapsto \mathcal{N}_d(-x, \sigma^2 \mathrm{Id}) \in \mathcal{P}(\mathbb{R}^d)$ of the measure γ_{σ_0} ; where $\gamma_{\sigma_0}(\mathrm{d}x) = (2\sigma_0^2)^{-1/2} e^{-|x|^2/2\sigma_0^2} \mathrm{d}x$, where for all $\mu \in \mathbb{R}^d, \Sigma \in \mathcal{M}_d(\mathbb{R}), \mathcal{N}_d(\mu, \Sigma)$ denotes a gaussian distribution in dimension d centered in μ and with variance-covariance matrix Σ .*

The fact that in this case we are able to exhibit the invariant measure comes from the linearity of the equation and the linearity of the confinement forces which derives from a quadratic potential, as shown in the proof. The convergence to the equilibria holds at an exponential thanks to Theorem 1.

Proof of Proposition 3. Let be the process X^0 defined dynamic given by:

$$\begin{cases} dX_t^0 = -X_t^0 dt - \sigma_0 dB_t^0 \\ \mathcal{L}(X_0^0) = \mathcal{N}(0, \sigma_0^2). \end{cases}$$

The initial condition is such that the process X^0 is stationary. We now define the process $(\bar{m}_t)_t$ with \bar{m}_0 , such that $\mathcal{L}(\bar{m}_0) = P$ as initial condition and staying

$$(19) \quad d_t \bar{m}_t = \nabla \cdot \left[\frac{\sigma^2 + \sigma_0^2}{2} \nabla \bar{m}_t + \bar{m}_t x \right] dt - \sigma_0 D \bar{m}_t \cdot dB_t^0.$$

Let us define $\tilde{m}_t := (Id - \sigma^0 B_t^0) \# \bar{m}_t$, where $\#$ stands for the pushforward operator. Then, applying Itô-Wentzell formula, we get that \tilde{m}_t satisfies

$$(20) \quad \begin{cases} \partial_t \tilde{m}_t = \nabla \cdot \left[\frac{\sigma^2}{2} \nabla \tilde{m}_t + \tilde{m}_t (x + B_t^0) \right], \\ \tilde{m}_0 = \bar{m}_0. \end{cases}$$

Moreover, we get that $m_t := \mathcal{N}(-(X_t^0 + \sigma_0 B_t^0), \sigma^2 Id)$ satisfies,

$$(21) \quad d_t m_t(x) = -\sigma^{-2} (x + X_t^0 + \sigma_0 B_t^0) m_t(x) d(X_t^0 + \sigma_0 B_t^0) = \sigma^{-2} (x + X_t^0 + \sigma_0 B_t^0) \cdot X_t^0 m_t(x) dt.$$

Then, as

it shows that $(m_t)_t$ satisfies (20) with $m_0 = \bar{m}_0$ and then $((Id + \sigma^0 B_t^0) \# m_t)_t$ is solution of (19) with the same initial condition. Finally,

$$\bar{m}_t(dx) = (Id + \sigma^0 B_t^0) \# m_t(dx) = c \exp\{-|x + X_t^0|^2 / 2\sigma^2\} dx.$$

As $(X_t^0)_t$ is stationary in \mathbb{R}^d , \bar{m}_t is also in $\mathcal{P}(\mathbb{R}^d)$, which shows the first part of the Proposition. The uniqueness of this invariant measure is then given by Corollary 2. \square

4. THE ONE DIMENSIONAL CASE

To get uniqueness of the invariant measure when the confinement potential is not convex, we need to establish a more advanced coupling than the one used in previous section. To do this, we follow [21], [23], [22], or [14]. For the sake of clarity, we recall here the assumptions in dimension $d = 1$:

Assumption (A1). (1) *There exists a continuous function $\kappa : [0, +\infty) \rightarrow \mathbb{R}$, such that*

$$\liminf_{r \rightarrow +\infty} \kappa(r) > 0,$$

and

$$(V'(x) - V'(y))(x - y) \geq \kappa(|x - y|)|x - y|^2.$$

(2) *V' is L_V -Lipschitz continuous.*

Assumption (A2). (1) *W is symmetric, i.e., $W(x) = W(-x)$ for all $x \in \mathbb{R}^d$.*

(2) *W' is L_W -Lipschitz continuous.*

Remark 1. *As the function κ exhibits positive values far from the origin, its continuity implies that it possesses a lower bound. Hence, we can ensure the boundedness of the function $\kappa_- = \max(-f, 0)$.*

This framework allows multi-well interactions potentials for which uniqueness of the invariant measure does not hold without common noise, see [33].

4.1. Uniform in time propagation of chaos in dimension 1. This section introduces a quantitative study of the uniform propagation of chaos. Because of the non-convexity of the confinement potential, we cannot use the classical propagation of chaos approach from Section 3. Instead, we will apply a more advanced approximation method. However, this method does not produce satisfactory results in dimensions $d \geq 2$ with common noise. For this reason, in this section, we focus on the case where $d = 1$, and will explain why in more detail later.

The approximation method. Following the framework of [22], we now define the constants:

$$\begin{aligned} R_0 &= \inf \{s \geq 0, \kappa(r) \geq 0, \forall r \geq s\}, \\ R_1 &= \inf \{s \geq R_0, s(s - R_0)\kappa(r) \geq 4\sigma_0^2, \forall r \geq s\}. \end{aligned}$$

Moreover, we consider $\varphi, \Phi, g : [0, +\infty) \rightarrow [0, +\infty)$ defined by

$$\begin{aligned} \varphi(r) &= \exp\left(-\frac{1}{2\sigma_0^2} \int_0^r s\kappa_-(s)ds\right), \\ \Phi(r) &= \int_0^r \varphi(s)ds, \\ g(r) &= 1 - \frac{\ell}{2} \int_0^{r \wedge R_1} \Phi(s)/\varphi(s)ds, \end{aligned}$$

where $\kappa_- = \max(0, -\kappa)$ and $\ell = \left(\int_0^{R_1} \Phi(s)\varphi(s)^{-1}ds\right)^{-1}$. We now define an increasing function $f : [0, +\infty) \rightarrow [0, +\infty)$ by:

$$f(r) = \int_0^r \varphi(s)g(s)ds.$$

The function f that has been constructed is clearly positive, non-decreasing and concave. Moreover, it satisfies

$$(22) \quad \varphi(R_0)r/2 \leq f(r) \leq r.$$

This ensures that $(x, y) \mapsto f(|x - y|)$ defines a distance which is equivalent to the Euclidean one. Below, we will use contraction properties in Wasserstein-1 distance based on the underlying distance $f(|x - y|)$. These contraction property is a consequence of Proposition 4.

Proposition 4. *The following inequalities holds for all $r > 0$:*

$$f''(r) - \frac{1}{2\sigma_0^2} r\kappa(r)f'(r) \leq -\ell f(r)/2.$$

The proof of this Proposition is found in Appendix A.4. However, it is essential to emphasize that the function f was constructed to achieve a contraction inequality, ensuring uniform propagation of chaos over time. These techniques were greatly inspired by [38] and further developed in [22]. In that sense, let us consider $P, Q \in L_4(\mathcal{P}(\mathbb{R}^d))$. Let us also consider $\delta > 0$, and once again two particle systems $\mathbf{X}^\delta = (X^{1,\delta}, \dots, X^{N,\delta})$ and $\mathbf{X}^{N,\delta} = (X^{1,N,\delta}, \dots, X^{N,N,\delta})$ with dynamic given by:

$$(23) \quad dX_t^{i,\delta} = -V'(X_t^{i,\delta})dt - W' * m_t^\delta(X_t^{i,\delta}) dt + \sigma dB_t^i + \sigma_0 \left\{ \pi_\delta(E_t^N) dB_t^0 + \lambda_\delta(E_t^N) d\tilde{B}_t^0 \right\},$$

and

$$(24) \quad dX_t^{i,N,\delta} = -V' \left(X_t^{i,N,\delta} \right) dt - \frac{1}{N} \sum_{j=1}^N W' \left(X_t^{i,N,\delta} - X_t^{j,N,\delta} \right) dt + \sigma dB_t^i \\ + \sigma_0 \left\{ -\pi_\delta \left(E_t^{N,\delta} \right) dB_t^0 + \lambda_\delta \left(E_t^{N,\delta} \right) d\tilde{B}_t^0 \right\},$$

where:

- The B^i are independent Wiener processes in \mathbb{R} , which all are independent of \mathbb{F}^0 ;
- B^0 is a Wiener process, adapted to the filtration \mathbb{F}^0 and independent of \mathbb{F}^1 ;
- m_t^δ stands for the conditional law of each process $X^{i,\delta}$ with respect to the σ -field \mathcal{F}_t^0 , at each time $t > 0$;
- $E_t^{N,\delta}$ stands for the vector of differences at time t , more precisely, for $i \in \{1, \dots, N\}$,

$$E_t^{i,N,\delta} = X_t^{i,\delta} - X_t^{i,N,\delta}.$$

- The pairs $(X_0^{i,N,\delta}, X_0^{i,\delta})$ are conditionally independent and identically distributed with respect to the σ -algebra \mathcal{F}_0^0 . Furthermore, for every $i \in \{1, \dots, N\}$, the law of X_0^i is P according to Definition 1, and the law of $X_0^{i,N}$ is Q , in the same manner.
- We define a non-decreasing and continuous function π , such that for $x \in \mathbb{R}^N$,

$$\pi(x) = \begin{cases} 1 & \text{if } N^{-1} \sum_{i=1}^N |x^i| \geq 1 \\ 0 & \text{if } N^{-1} \sum_{i=1}^N |x^i| \leq 1/2, \end{cases}$$

and, consider a non negative function λ such that

$$\pi(x)^2 + \lambda(x)^2 = 1, \quad \forall x \in \mathbb{R}^N.$$

Moreover, we extend π on the whole space, with the constraint that this remains a non decreasing and Lipschitz continuous function. Finally, we define $\pi_\delta : x \mapsto \pi(x/\delta)$.

The condition on π and λ implies in particular that $-\int_0^t \pi_\delta(E_s^{N,\delta}) dB_s^0 + \int_0^t \lambda_\delta(E_s^{N,\delta}) d\tilde{B}_s^0$ indeed defines a Brownian motion thanks to Levy's characterisation of Brownian motion. Then, (23) and (24) defines a coupling of (11) and (12). Before giving the first main result of the section, let us introduce the quantity

$$(25) \quad c(V, W, \sigma_0) = \ell \sigma_0^2 - 4L_W \varphi(R_0)^{-1}.$$

This quantity naturally appears in the result of uniform in time propagation of chaos. In order to ensure the uniformity in time, let us denote for all V and W , the space

$$\mathfrak{D}_{V,W} = \{ \sigma_0 \geq 0, c(V, W, \sigma_0) > 0 \}.$$

Proposition 5. *There exists $\bar{L} > 0$, $\bar{\sigma}_0 > 0$ such that for all $\sigma \geq 0$, for any W which is L -Lipschitz continuous with $L < \bar{L}$, $[\bar{\sigma}_0; +\infty) \subset \mathfrak{D}_{V,W}$.*

The proof of this Proposition is postponed to Appendix A.6. We are now ready to give the main result of this section

Theorem 2. *Under Assumptions (A1) and (A2) there exist two systems $(\mathbf{X}, \mathbf{X}^N)$, couplings of (11) and (12), which are subssequential weak limit as $\delta \rightarrow 0$ of $(\mathbf{X}^\delta, \mathbf{X}^{N,\delta})$. Let $\sigma \geq 0$, W such that $L_W < \bar{L}$, and $\sigma_0 \geq \bar{\sigma}_0$, where $\bar{L}, \bar{\sigma}_0$ are defined in Proposition 5. Then, there exists a constant $C > 0$ which does not depend on t and N , such that:*

$$\mathbb{E} \left[d_f^{\mathbb{R}^d} (m_{\mathbf{X}_t}^N, m_{\mathbf{X}_t^N}^N) \right] \leq C \left(e^{-c(V,W,\sigma_0)t} + \frac{1}{\sqrt{N}} \right),$$

where $d_f^{\mathbb{R}^d}(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d} f(|x - y|) \pi(dx, dy)$, and $m_{\mathbf{X}_t}^N$ (resp. $m_{\mathbf{X}_t^N}^N$) stands for the empirical measures of all the particles of the system \mathbf{X} (resp. \mathbf{X}^N) at time t .

Remark 2. Some remarks are in order: (1) As in [21], we need to consider a slight distortion of the Wasserstein-1 distance. This allows for easy handling of the residual terms, due to the non-convexity of the confinement potential, through a functional inequality. (2) The Lipschitz continuity assumption on the confinement force ∇V does not help convergence toward the invariant measure and mainly comes from technical needs. It is not a sharp condition and similar outcomes might be reached by assuming for example that ∇V has at most a polynomial growth. (3) The first drawback of this result is the impossibility to take advantage of the potential convexity structure of the interaction potential W , as strong interactions within the system should ideally help in converging to equilibrium. This limit is shown by the term $-4L\varphi(R_0)^{-1}$ in $c(V, W, \sigma_0)$, which penalizes the convergence rate. (4) The main issue with this method is that it does not work in higher dimensions. Using the same coupling techniques in dimensions $d \geq 2$ requires a reflection with the orthogonal matrix $Id - 2e_t^{i,N}(e_t^{i,N})^t$ in order to ensure that both process are remaining close in large time, as explained in [38]. However, this is not possible in our situation as we cannot use a reflection matrix dependent on a single particle for the common noise. In the following section, we will discuss a specific case where we can overcome this difficulty.

In order to prove the previous quantitative and uniform in time propagation of chaos result, we begin with the following Proposition.

Proposition 6. Under Assumptions (A1) and (A2), we have the following decomposition for all $i \in \{1, \dots, N\}$:

$$d|E_t^{i,N,\delta}| = -e_t^{i,N,\delta} \left(V'(X_t^{i,\delta}) - V'(X_t^{i,N,\delta}) \right) dt + A_t^{i,N,\delta} dt + 2\sigma_0 \pi_\delta(E_t^{N,\delta}) (e_t^{i,N,\delta})^T dB_t^0,$$

where

$$e_t^{i,N,\delta} = \begin{cases} E_t^{i,N,\delta} / |E_t^{i,N,\delta}| & \text{if } E_t^{i,N,\delta} \neq 0 \\ 0 & \text{otherwise} \end{cases},$$

and

$$A_t^{i,N,\delta} \leq \left| W' * m_t^\delta(X_t^{i,\delta}) - \frac{1}{N} \sum_{j=1}^N W'(X_t^{i,N,\delta} - X_t^{j,N,\delta}) \right|.$$

The result above adapts Lemma 7 from [21] for the common noise setting, and thus, the proof is postponed to Appendix A.5. This Proposition is important as under Assumptions (A1) and (A2) the main term helps with uniform control over time, while the secondary term is handled as a perturbation. The two martingale terms are centered, and the other terms guarantee concentration, even when the confinement potential is not convex. Now, let us present the following Lemma.

Lemma 4. Under assumptions (A1) and (A2), for all $T > 0$, there exists a positive constant C independent of δ , such that for all $0 < s < t < T$,

$$\mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N |X_t^{i,\delta} - X_s^{i,\delta}|^4 + \frac{1}{N} \sum_{i=1}^N |X_t^{i,N,\delta} - X_s^{i,N,\delta}|^4 \right] \leq C|t - s|^2.$$

Proof of Lemma 4. Step 1. Let us begin this proof by considering the existence of a finite moment of order 4 for every particle in both systems at all times. Let us do it only for the mean-field

limit system as the proof is exactly the same for the interacting particle system. Then, from Itô's formula, we get for each $i \in \{1, \dots, N\}$

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[|X_t^{i,\delta}|^4] &= -p \left(\mathbb{E}[(X_t^{i,\delta})^3 V'(X_t^{i,\delta})] + \mathbb{E}[(X_t^{i,\delta})^3 W' * m_t^\delta(X_t^{i,\delta})] \right) \\ &\quad + 6(\sigma^2 + \sigma_0^2) \mathbb{E}[(X_t^{i,\delta})^2] \\ &\leq -4m_V \mathbb{E}[(X_t^{i,\delta})^4] + 4M_V \mathbb{E}[(X_t^{i,\delta})^3] + 4\mathbb{E}[(X_t^{i,\delta})^3] V'(0) - 4\mathbb{E}[(X_t^{i,\delta})^3 W' * m_t^\delta(X_t^{i,\delta})] \\ &\quad + 6(\sigma^2 + \sigma_0^2) \mathbb{E}[(X_t^{i,\delta})^2] + 6(\sigma^2 + \sigma_0^2). \end{aligned}$$

Furthermore, we have knowledge that W' is Lipschitz continuous, which implies that the convolution $W' * m_t^\delta$ also is. Then, we get

$$\frac{d}{dt} \mathbb{E}[|X_t^{i,\delta}|^4] \leq c_1 \mathbb{E}[|X_t^{i,\delta}|^4] + c_2,$$

where c_1 and c_2 are two constants independent of δ and t . Using Grönwall Lemma, this gives:

$$\mathbb{E}[|X_t^{i,\delta}|^4] \leq e^{c_2 t} \mathbb{E}[|X_0^{i,\delta}|^4] + c_2 c_1^{-1} e^{c_1 t}.$$

Then, as $P_0, Q_0 \in L_4(\mathcal{P}(\mathbb{R}^d))$ we get that for each $T > 0$, $\sup_{t \in [0, T]} \mathbb{E}[|X_t^{i,\delta}|^4] < +\infty$.

Step 2. Let us now consider $T > 0$, and $0 < s < t < T$.

$$\begin{aligned} \mathbb{E}[|X_t^{i,\delta} - X_s^{i,\delta}|^4] &= \mathbb{E} \left[\left| \int_s^t -V'(X_r^{i,\delta}) - W' * m_r^\delta(X_r^{i,\delta}) dr + \sigma(B_t^i - B_s^i) \right. \right. \\ &\quad \left. \left. + \sigma_0 \left\{ \int_s^t \pi_\delta(E_r^N) d\tilde{B}_r^0 + \int_s^t \lambda_\delta(E_r^N) dB_r^0 \right\} \right|^4 \right] \\ &\leq \mathbb{E} \left[\left| \int_s^t -V'(X_r^{i,\delta}) - W' * m_r^\delta(X_r^{i,\delta}) dr \right|^4 \right] + C(\sigma, \sigma_0) |t - s|^2, \end{aligned}$$

where we used the Burkholder-Davis-Gundy inequality to obtain the constant $C(\sigma, \sigma_0)$, which does not depend on δ . Controlling the residual term is a straightforward process that involves using the result of *Step 1* with the Lipschitz continuity of V' and W' . Everything works exactly the same way for the interacting particle system. \square

This result mainly ensures the existence of the coupling (X, X^N) of Theorem 2, which we are now ready to prove.

Proof of Theorem 2. From Proposition 3 and using Itô's formula applied to the previously defined function f , we get

$$(26) \quad \begin{aligned} df(|E_t^{i,N,\delta}|) &= -f'(|E_t^{i,N,\delta}|) e_t^{i,N,\delta} (V'(X_t^{i,\delta}) - V'(X_t^{i,N,\delta})) dt + A_t^{i,N,\delta} f'(|E_t^{i,N,\delta}|) dt \\ &\quad + 2\sigma_0^2 f''(|E_t^{i,N,\delta}|) \pi_\delta(E_t^{N,\delta})^2 dt + 2\sigma_0 e_t^{i,N,\delta} f'(|E_t^{i,N,\delta}|) \pi_\delta(E_t^{N,\delta}) dB_t^0. \end{aligned}$$

Moreover,

$$\begin{aligned} &-f'(|E_t^{i,N,\delta}|) e_t^{i,N,\delta} (V'(X_t^{i,\delta}) - V'(X_t^{i,N,\delta})) + 2\sigma_0^2 f''(|E_t^{i,N,\delta}|) \pi_\delta(E_t^{N,\delta})^2 \\ &= 2\sigma_0^2 \left(f''(|E_t^{i,N,\delta}|) \pi_\delta(E_t^{N,\delta})^2 - \frac{1}{2\sigma_0^2} f'(|E_t^{i,N,\delta}|) e_t^{i,N,\delta} (V'(X_t^{i,\delta}) - V'(X_t^{i,N,\delta})) \right) \\ &\leq 2\sigma_0^2 \left(f''(|E_t^{i,N,\delta}|) \pi_\delta(E_t^{N,\delta})^2 - \frac{1}{2\sigma_0^2} |E_t^{i,N,\delta}| \kappa(|E_t^{i,N,\delta}|) f'(|E_t^{i,N,\delta}|) \right) \\ &\leq 2\sigma_0^2 \left(-\frac{\ell}{2} f(|E_t^{i,N,\delta}|) \pi_\delta(E_t^{N,\delta})^2 + \frac{1}{2\sigma_0^2} |E_t^{i,N,\delta}| \kappa_- (|E_t^{i,N,\delta}|) \lambda_\delta(E_t^{N,\delta})^2 \right). \end{aligned}$$

thanks to Proposition 4 and the fact that $\pi^2(x) + \lambda^2(x) = 1$, for all $x \in \mathbb{R}^N$. Then, averaging over all the particles, and using the fact that κ_- is bounded as a consequence of Remark 1, we get:

$$\begin{aligned} & \frac{1}{N} \sum_{i=1}^N -f'(|E_t^{i,N,\delta}|) e_t^{i,N,\delta} (V'(X_t^{i,\delta}) - V'(X_t^{i,N,\delta})) + 2\sigma_0^2 f''(|E_t^{i,N,\delta}|) \pi_\delta(E_t^{i,N,\delta})^2 \\ & \leq -\frac{\ell\sigma_0^2}{N} \sum_{i=1}^N f(|E_t^{i,N,\delta}|) \pi_\delta(E_t^{i,N,\delta})^2 + \frac{|\kappa_-|_\infty}{N} \sum_{i=1}^N |E_t^{i,N,\delta}| \lambda_\delta(E_t^{i,N,\delta})^2 \end{aligned}$$

Plugging the last inequality into (26), we have

$$\frac{1}{N} \sum_{i=1}^N \frac{d}{dt} \mathbb{E}[f(|E_t^{i,N,\delta}|)] \leq -\frac{\ell\sigma_0^2}{N} \sum_{i=1}^N \mathbb{E}[f(|E_t^{i,N,\delta}|)] + \frac{1}{N} \sum_{i=1}^N \mathbb{E}[A_t^{i,N,\delta}] + \ell\sigma_0^2\delta + |\kappa_-|_\infty\delta.$$

We now need to control one last term $N^{-1} \sum_{i=1}^N \mathbb{E}[A_t^{i,N,\delta}]$. This part of the proof follows the same lines as the one of [21] and we get

$$\frac{1}{N} \sum_{i=1}^N \mathbb{E}[|A_t^{i,N,\delta}|] \leq \frac{4L_W\varphi(R_0)^{-1}}{N} \sum_{i=1}^N \mathbb{E}[f(|E_t^{i,N,\delta}|)] + ML_W N^{-1/2},$$

for some $M > 0$. The last inequality is not unexpected, as it is anticipated that the interaction term in Equation 12 will converge at a rate of $N^{1/2}$ towards the conditional expectation $\mathbb{E}[\nabla W(X_t^i - \tilde{X}_t^i) | \mathcal{F}_t^0, X_t^i]$ for any independent replication \tilde{X} of X^i . Our objective now is to consider the limit as δ tends to zero. Let us define $\mathbb{P}_T^{N,\delta}$ as the law of $(\{X^{i,\delta}, X^{i,N,\delta}\}_{i=1,\dots,N})$ on $\mathcal{C}([0, T], \mathbb{R}^{2N})$. With Lemma 4 and Kolmogorov's continuity criterion, we establish the tightness of the sequence $(\mathbb{P}_T^{N,\delta})_\delta$, leading to the existence of a subsequence $(\delta_n)_n$ that tends to zero, such that $\mathbb{P}_T^{N,\delta_n}$ converges to \mathbb{P}_T^N , which is defined on $\mathcal{C}([0, T], \mathbb{R}^{2N})$. By a diagonalization argument and consistency of the family of probability $\{\mathbb{P}_T^N, T > 0\}$, we can extend this probability measure and define \mathbb{P}^N on $\mathcal{C}([0, +\infty), \mathbb{R}^{2N})$. Now we can define two systems of particle $\mathbf{X} = (X^1, \dots, X^N)$ and $\mathbf{X}^N = (X^{1,N}, \dots, X^{N,N})$ with law \mathbb{P}^N on $\mathcal{C}([0, +\infty), \mathbb{R}^{2N})$. This argument that justifies the existence of the coupling is classical, see for example [20] or [24]. Moreover, by uniqueness in law of the solutions, we get that $(\mathbf{X}, \mathbf{X}^N)$ is a coupling of (11) and (12). Then, defining:

$$E_t^{i,N} = X_t^i - X_t^{i,N},$$

and

$$e_t^{i,N} = \begin{cases} E_t^{i,N}/|E_t^{i,N}| & \text{if } E_t^{i,N} \neq 0 \\ 0 & \text{otherwise} \end{cases},$$

we get

$$\frac{1}{N} \sum_{i=1}^N \frac{d}{dt} \mathbb{E}[f(|E_t^{i,N}|)] \leq -\left(\frac{\ell\sigma_0^2}{N} - 4L_W\varphi(R_0)^{-1}\right) \sum_{i=1}^N \mathbb{E}[f(|E_t^{i,N}|)] + ML_W N^{-1/2}$$

Now, as $\sigma_0 \in \mathcal{D}_{V,W}$, we can use the Grönwall lemma, and get that

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N \mathbb{E}[f(|E_t^{i,N}|)] & \leq \exp\left(-(\ell\sigma_0^2 - 4L_W\varphi(R_0)^{-1})t\right) \frac{1}{N} \sum_{i=1}^N \mathbb{E}[f(|E_0^{i,N}|)] \\ & \quad + \left(\ell\sigma_0^2 - 4L_W\varphi(R_0)^{-1}\right)^{-1} ML_W N^{-1/2} \end{aligned}$$

Moreover, we know that

$$\mathbb{E}[\mathfrak{d}_f^{\mathbb{R}}(m_{\mathbf{X}_t}^N, m_{\mathbf{X}_t^N}^N)] \leq \frac{1}{N} \sum_{i=1}^N \mathbb{E}[f(|E_t^{i,N}|)].$$

Then,

$$\mathbb{E}[\mathfrak{d}_f^{\mathbb{R}}(m_{\mathbf{X}_t}^N, m_{\mathbf{X}_t^N}^N)] \leq e^{-c(V,W,\sigma_0)t} \frac{1}{N} \sum_{i=1}^N \mathbb{E}[f(|E_0^{i,N}|)] + c(V,W,\sigma_0)^{-1} M L_W N^{-1/2}.$$

and finally

$$(27) \quad \begin{aligned} \mathbb{E}[f(|E_0^{1,N}|)] &= \mathbb{E}[\mathbb{E}[f(|X_0^1 - X_0^{1,N}|) | \mathcal{F}_0^0]] \\ &\leq C \mathbb{E}[|X_0^{1,N}| + |X_0^1|] < +\infty, \end{aligned}$$

thanks to Assumption Lemma 2, which concludes the proof. \square

4.2. Uniqueness of the invariant measure. As in the previous section with convex potential, we can deduce from the propagation of chaos result the uniqueness of the invariant measure for the process $(m_t)_{t \geq 0}$ driven by (1), whenever the intensity of the common noise is large enough.

Corollary 3. *Whenever $d = 1$ and Assumptions (A1) and (A2) hold, assuming that W is such that $L < \bar{L}$, and $\sigma_0 \geq \bar{\sigma}_0$, where \bar{L} and $\bar{\sigma}_0$ are defined in Proposition 5, the stochastic process (m_t) admits a unique invariant measure $\bar{P} \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$.*

Moreover, for each $P_0 \in L_4(\mathcal{P}(\mathbb{R}))$, we get

$$\mathfrak{d}_f^{\mathcal{P}(\mathbb{R})}(P_t, \bar{P}) \leq e^{-c(V,W,\sigma_0)t} \mathfrak{d}_f^{\mathcal{P}(\mathbb{R})}(P_0, \bar{P}),$$

where $\mathfrak{d}_f^{\mathcal{P}(\mathbb{R})}(\cdot, \cdot)$ is defined in Definition 5.

It is interesting to highlight here that the presence of common noise allows to get uniqueness, in fact in the case without common noise, when considering non-convex confinement potential, it has been shown that uniqueness of the invariant measure (in $\mathcal{P}(\mathbb{R}^d)$) does not hold. More precisely, considering the deterministic flow of measure $(m_t)_t$ which is weak solution of

$$(28) \quad \partial_t m_t = \frac{\sigma^2}{2} \partial_{xx}^2 m_t + \partial_x (m_t (V' + W' * m_t)),$$

it is known, see e.g [33], that whenever σ is small enough, there is exactly three invariant measures. This is roughly due to the fact that whenever the intensity of the noise is not important enough, the associated process X , solution of

$$dX_t = -V'(X_t)dt - W' * m_t(X_t)dt + \sigma dB_t$$

stays stuck in the minimum of V due to the lack of convexity. Here the presence of a strong enough common noise allows to recover uniqueness but in the space $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$.

5. MULTIDIMENSIONAL NON-CONVEX CASE WITH $\sigma = 0$

In this section, we return to the general dimension case where $d \geq 2$. It turns out that in some specific situations, we can use the presence of interaction to adapt the coupling introduced in Section 4 and achieve an exponential rate of convergence to the unique invariant measure when there is no idiosyncratic noise ($\sigma = 0$). More precisely, let us consider m with dynamic given by

$$d_t m_t = \nabla \cdot \left(\frac{\sigma_0^2}{2} \nabla m_t + m_t (\nabla V + \nabla W * m_t) \right) dt - \sigma_0 Dm_t \cdot dB_t^0.$$

In the following of the section, let us consider the following Assumption:

Assumption (A6). *The potential V satisfies Assumption (A1) and there exists $\alpha > 0$, such that $W(x) = \alpha|x|^2/2$.*

5.1. Existence of an invariant measure. We begin this section with a non quantitative proof of existence and uniqueness of the invariant measure.

Proposition 7. *Under Assumption (A6), for α large enough, there exists a unique invariant measure \bar{P} in the sense of Definition 1, and this measure is supported by Dirac masses. More precisely,*

$$\bar{P}(dm) = \int_{\mathbb{R}^d} \delta_{\delta_a} m_0(da)$$

where m_0 is the probability measure on \mathbb{R}^d solution of $-\frac{\sigma_0^2}{2}\Delta m_0 - \nabla \cdot (\nabla V m_0) = 0$.

Proof of Proposition 7. If $\bar{P}(dm) = \int_{\mathbb{R}^d} \delta_{\delta_a} m_0(da)$, then for all twice differentiable function in the sense of Lions derivatives $F \in \mathcal{C}^2(\mathcal{P}(\mathbb{R}^d), \mathbb{R})$

(29)

$$\begin{aligned} I(F) &:= \int_{\mathcal{P}_2(\mathbb{R}^d)} \left[\int_{\mathbb{R}^d} \left(D_m F(m, x) \cdot (-\nabla V(x) - \nabla W * m(x)) + \frac{\sigma_0^2}{2} \operatorname{div}_x D_m F(m, x) \right) m(dx) \right. \\ &\quad \left. + \frac{\sigma_0^2}{2} \int_{\mathbb{R}^{2d}} \operatorname{Tr} [D_{mm}^2 F(m, x, y)] m(dx) m(dy) \right] \bar{P}(dm) \\ &= \int_{\mathbb{R}^d} \left[D_m F(\delta_a, a) \cdot (-\nabla V(a)) + \frac{\sigma_0^2}{2} \operatorname{div}_x D_m F(\delta_a, a) + \frac{\sigma_0^2}{2} \operatorname{Tr} [D_{mm}^2 F(\delta_a, a, a)] \right] m_0(da), \end{aligned}$$

where, if we define $\varphi(a) = F(\delta_a)$, then

$$\nabla \varphi(a) = D_m F(\delta_a, a), \quad \Delta \varphi(a) = \operatorname{Tr} [D_{mm}^2 F(\delta_a, a, a) + D_{xm}^2 F(\delta_a, a)].$$

Then,

$$(30) \quad I(F) = \int_{\mathbb{R}^d} \left[-\nabla \varphi(a) \cdot \nabla V(a) + \frac{\sigma_0^2}{2} \Delta \varphi(a) \right] m_0(da) = 0,$$

as m_0 is solution of $\frac{\sigma_0^2}{2}\Delta m_0 + \nabla \cdot (\nabla V m_0) = 0$. \square

5.2. Uniqueness and convergence to the equilibrium. When $\sigma = 0$, we can explicitly find the unique invariant measure. Moreover, we can use the strong interaction structure to achieve exponential rates of convergence to this invariant measure. To do this, we build a coupling similar to the one in the previous section, taking advantage of the fact that in this setting, each particle is close to the average of all particles in the system. Consequently, we get the following result

Theorem 3. *Whenever $\sigma = 0$ and under Assumptions (A4), for any initial conditions $P_0, Q_0 \in L_4(\mathcal{P}(\mathbb{R}^d))$, there exists a constant C that does not depend on t such that*

$$d_f^{\mathcal{P}(\mathbb{R}^d)}(P_t, Q_t) \leq C \left(e^{-\ell \sigma_0^2 t} + e^{-(\alpha - 2L_V)t} \right),$$

where f and ℓ are defined in Section 4, and the distance $d_f^{\mathcal{P}(\mathbb{R}^d)}(\cdot, \cdot)$ is defined in (8).

Remark 3. *In the light of Remark 2, we can now observe that the rate of convergence is not penalised by the presence of a strong interaction. In fact, in this case we are able to fully take advantage of the quadratic structure of the interaction potential.*

The key observation to prove the previous result is that when considering an interacting particle system with each particle's dynamic given by dynamic (23), the interaction acts like an attractor gathering all the particles together. The coupling of Section 4 fails due to the dependence of the reflection matrix on a single particle. To address this dependence, we replace it with a dependence on the empirical mean of all system particles. By showing that the resulting error becomes negligible as the number of particles and time increase, we show the expected result.

Proof of Theorem 3. Let us consider two measures P_0, Q_0 on $\mathcal{P}(\mathbb{R}^d)$, which admit a moment of order four. In the light of the proof of Proposition 2, and Remark 2, let us introduce the two following particle systems for $\delta > 0$, $\mathbf{X}^\delta = (X^{1,\delta}, \dots, X^{N,\delta})$ and $\mathbf{Y}^{N,\delta} = (Y^{1,N,\delta}, \dots, Y^{N,N,\delta})$, with dynamic

$$(31) \quad dX_t^{i,\delta} = -\nabla V(X_t^{i,\delta})dt - \nabla W * m_t^\delta(X_t^{i,\delta})dt + \sigma dB_t^i + \sigma_0 \{ \pi_\delta(\bar{E}_t^{N,\delta})dB_t^0 + \lambda_\delta(\bar{E}_t^{N,\delta})d\tilde{B}_t^0 \},$$

and

$$dY_t^{i,N,\delta} = -\nabla V(Y_t^{i,N,\delta})dt - \frac{1}{N} \sum_{j=1}^N \nabla W(Y_t^{i,N,\delta} - Y_t^{j,N,\delta})dt + \sigma dB_t^i + \sigma_0 \{ \pi_\delta(\bar{E}_t^{N,\delta})(Id - 2\bar{e}_t^{N,\delta}(\bar{e}_t^{N,\delta})^t)dB_t^0 + \lambda_\delta(\bar{E}_t^{N,\delta})d\tilde{B}_t^0 \},$$

where $\bar{E}_t^{N,\delta} = N^{-1} \sum_{i=1}^N X_t^{i,\delta} - N^{-1} \sum_{i=1}^N Y_t^{i,N,\delta}$, and

$$\bar{e}_t^{N,\delta} = \begin{cases} \bar{E}_t^{N,\delta} / |\bar{E}_t^{N,\delta}| & \text{if } |\bar{E}_t^{N,\delta}| \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Moreover, the functions π and λ are defined in Section 4. Finally we assume that the $X_0^{i,\delta}$ are conditionally *iid* with respect to the common noise with law P_0 . Analogously, the same is assumed for the $Y^{i,N,\delta}$ with law Q_0 .

Proposition 8. *Under Assumption (A4), we get:*

$$\mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N |X_t^{i,\delta} - \bar{X}_t^{N,\delta}|^2 \right] \leq \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N |X_0^{i,\delta} - \bar{X}_0^{N,\delta}|^2 \right] e^{-2(\alpha - 2L_V)t},$$

and

$$\mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N |Y_t^{i,N,\delta} - \bar{Y}_t^{N,\delta}|^2 \right] \leq \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N |Y_0^{i,N,\delta} - \bar{Y}_0^{N,\delta}|^2 \right] e^{-2(\alpha - 2L_V)t},$$

for $\alpha > 2L_V$ and where $\bar{X}_t^{N,\delta} = N^{-1} \sum_{i=1}^N X_t^{i,\delta}$ and $\bar{Y}_t^{N,\delta} = N^{-1} \sum_{i=1}^N Y_t^{i,N,\delta}$ stand for the empirical means of each system at time t .

The proof is straightforward using the Lipschitz continuity of ∇V and the quadratic structure of the interacting potential.

Proof. Let us only perform this calculation for the mean field limit system of N particles $\mathbf{X}^\delta = (X^{1,\delta}, \dots, X^{N,\delta})$. Let now $t > 0$ and $i \in \{1, \dots, N\}$,

$$\begin{aligned} d|X_t^{i,\delta} - \bar{X}_t^{N,\delta}|^2 &= -2(X_t^{i,\delta} - \bar{X}_t^{N,\delta}) \cdot \left(\nabla V(X_t^{i,\delta}) - \frac{1}{N} \sum_{i=1}^N \nabla V(X_t^{i,\delta}) \right) dt \\ &\quad - 2(X_t^{i,\delta} - \bar{X}_t^{N,\delta}) \cdot \left(\nabla W * m_t^\delta(X_t^{i,\delta}) - \frac{1}{N} \sum_{i=1}^N \nabla W * m_t^\delta(X_t^{i,\delta}) \right) dt \\ &\leq 2L_V |X_t^{i,\delta} - \bar{X}_t^{N,\delta}|^2 dt + 2 \frac{L_V}{N} \sum_{j=1}^N |X_t^{j,\delta} - \bar{X}_t^{N,\delta}| |X_t^{i,\delta} - \bar{X}_t^{N,\delta}| dt \\ &\quad - 2\alpha |X_t^{i,\delta} - \bar{X}_t^{N,\delta}|^2 dt. \end{aligned}$$

Then, averaging over i and using Cauchy-Schwarz inequality, we get that

$$\frac{d}{dt} \frac{1}{N} \sum_{i=1}^N \mathbb{E}[|X_t^{i,\delta} - \bar{X}_t^{N,\delta}|^2] \leq 2(2L_V - \alpha) \frac{1}{N} \sum_{i=1}^N \mathbb{E}[|X_t^{i,\delta} - \bar{X}_t^{N,\delta}|^2].$$

Then, one can conclude using Grönwall Lemma. \square

Considering the function f introduced in Section 4, we can write:

$$(32) \quad f(|X_t^{i,\delta} - Y_t^{i,N,\delta}|) \leq f(|\bar{Y}_t^{N,\delta} - Y_t^{i,N,\delta}|) + f(|X_t^{i,\delta} - \bar{X}_t^{N,\delta}|) + f(|\bar{X}_t^{N,\delta} - \bar{Y}_t^{N,\delta}|).$$

By taking the expectation and averaging, we can control the first two terms using Proposition 8. We now need to tackle the third term. To deal with this problem, we use an approach similar to the one employed in the proof of Theorem 2. Taking advantage of the presence of π_δ , we can again apply the stochastic dominated convergence theorem, as shown in the proof of Theorem 2. Consequently, we obtain

$$(33) \quad \begin{aligned} d|\bar{X}_t^{N,\delta} - \bar{Y}_t^{N,\delta}| &= -\bar{e}_t^{N,\delta} \cdot \left(\frac{1}{N} \sum_{i=1}^N \nabla V(X_t^{i,\delta}) - \nabla V(Y_t^{i,N,\delta}) \right) dt \\ &\quad - \bar{e}_t^{N,\delta} \cdot \left(\frac{1}{N} \sum_{i=1}^N \nabla W * m_t(X_t^{i,\delta}) \right) dt \\ &\quad + 2\sigma_0 \pi_\delta(\bar{E}_t^{N,\delta}) (\bar{e}_t^{N,\delta})^t dB_t^0. \end{aligned}$$

Now, we can write

$$\begin{aligned} df(|\bar{X}_t^{N,\delta} - \bar{Y}_t^{N,\delta}|) &= -f'(|\bar{E}_t^{N,\delta}|) \bar{e}_t^{N,\delta} \cdot \left(\frac{1}{N} \sum_{i=1}^N \nabla V(X_t^{i,\delta}) - \nabla V(Y_t^{i,N,\delta}) \right) dt \\ &\quad - f'(|\bar{E}_t^{N,\delta}|) \bar{e}_t^{N,\delta} \cdot \left(\frac{1}{N} \sum_{i=1}^N \nabla W * m_t^\delta(X_t^{i,\delta}) \right) dt \\ &\quad + 2\sigma_0 f'(|\bar{E}_t^{N,\delta}|) \pi_\delta(\bar{E}_t^{N,\delta}) (\bar{e}_t^{N,\delta})^t dB_t^0 \\ &\quad + 2\sigma_0^2 f''(|\bar{E}_t^{N,\delta}|) \pi_\delta(\bar{E}_t^{N,\delta})^2 dt. \end{aligned}$$

To control the first term, we carry out the following decomposition

$$\begin{aligned}
& -f'(|\bar{E}_t^{N,\delta}|)\bar{e}_t^{N,\delta} \cdot \left(\frac{1}{N} \sum_{i=1}^N \nabla V(X_t^{i,\delta}) - \nabla V(Y_t^{i,N,\delta})\right) \\
& = -f'(|\bar{E}_t^{N,\delta}|)\bar{e}_t^{N,\delta} \cdot \left(\frac{1}{N} \sum_{i=1}^N \nabla V(X_t^{i,\delta}) - \nabla V(\bar{X}_t^{N,\delta})\right) \\
& \quad - f'(|\bar{E}_t^{N,\delta}|)\bar{e}_t^{N,\delta} \cdot \left(\frac{1}{N} \sum_{i=1}^N \nabla V(\bar{Y}_t^{N,\delta}) - \nabla V(Y_t^{i,N,\delta})\right) \\
& \quad - f'(|\bar{E}_t^{N,\delta}|)\bar{e}_t^{N,\delta} \cdot (\nabla V(\bar{X}_t^{N,\delta}) - \nabla V(\bar{Y}_t^{N,\delta})).
\end{aligned}$$

Using the fact that f' is globally bounded and Proposition 8, we get the existence of $C_1 > 0$, independent of N, t and δ , such that

$$\begin{aligned}
& -\mathbb{E}\left[f'(|\bar{E}_t^{N,\delta}|)\bar{e}_t^{N,\delta} \cdot \left(\frac{1}{N} \sum_{i=1}^N \nabla V(X_t^{i,\delta}) - \nabla V(Y_t^{i,N,\delta})\right)\right] \\
& \leq C_1 e^{-(\alpha-2L_V)t} - \mathbb{E}\left[f'(|\bar{E}_t^{N,\delta}|)\bar{e}_t^{N,\delta} \cdot (\nabla V(\bar{X}_t^{N,\delta}) - \nabla V(\bar{Y}_t^{N,\delta}))\right].
\end{aligned}$$

Finally, thanks to the law of large number, we get the existence of a constant C_2 such that

$$\mathbb{E}\left[\left|\frac{1}{N} \sum_{i=1}^N \nabla W * m_t(X_t^{i,\delta})\right|\right] \leq C_2 N^{-1/2}.$$

We finally get,

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}[f(|\bar{X}_t^{N,\delta} - \bar{Y}_t^{N,\delta}|)] & \leq C_1 e^{-(\alpha-2L_V)t} + C_2 N^{-1/2} \\
& \quad + \mathbb{E}\left[-f'(|\bar{E}_t^{N,\delta}|)\bar{e}_t^{N,\delta} \cdot (\nabla V(\bar{X}_t) - \nabla V(\bar{Y}_t))\right] \\
& \quad + 2\sigma_0^2 \left[f''(|\bar{E}_t^{N,\delta}|)\pi_\delta(\bar{E}_t^{N,\delta})^2\right].
\end{aligned}$$

Finally, using Proposition 4, we conclude that

$$\frac{d}{dt}\mathbb{E}[f(|\bar{X}_t^{N,\delta} - \bar{Y}_t^{N,\delta}|)] \leq C_1 e^{-(\alpha-2L_V)t} + C_2 N^{-1/2} - \ell\sigma_0^2 f(|\bar{E}_t^{N,\delta}|) + \ell\sigma_0^2 \delta + |\kappa_-|_\infty \delta,$$

Then, taking the limit $\delta \rightarrow 0$ exactly as in the proof of Theorem 2, we get the existence of two systems \mathbf{X} and \mathbf{X}^N which are couplings of (11) and (12), such that

$$\frac{d}{dt}\mathbb{E}[f(|\bar{X}_t^N - \bar{Y}_t^N|)] \leq C_1 e^{-(\alpha-2L_V)t} + C_2 N^{-1/2} - \ell\sigma_0^2 f(|\bar{E}_t^{N,\delta}|).$$

Then, using Grönwall lemma, we get that for α large enough,

$$\mathbb{E}[f(|\bar{X}_t^N - \bar{Y}_t^N|)] \leq C_3 \left(e^{-(\alpha-2L_V)t} + e^{-\ell\sigma_0^2 t} + N^{-1/2}\right),$$

for some constant C_3 that does not depends on t and N . Plugging this into Equation (32) gives

$$\mathbb{E}\left[\frac{1}{N} \sum_{i=1}^N f(|X_t^i - Y_t^{i,N}|)\right] \leq C \left(e^{-(\alpha-2L_V)t} + e^{-\ell\sigma_0^2 t} + N^{-1/2}\right).$$

Finally, using the exact same methodology as in Corollary 2, we conclude to the expected result. \square

APPENDIX A.

A.1. Proof of Proposition 1. Equation (9) comes from Section 1.5 in [37] (Equation (1.15)). Hence, if (m_t) is an invariant measure in the sense of Definition 2, then \bar{P} straightforwardly is a solution of Equation (10). Conversely, let us consider a probability measure \bar{P} , solution of Equation (10). More precisely, for all $F \in \mathcal{C}_b^2(\mathcal{P}(\mathbb{R}^d))$

$$\langle \bar{P}, \mathcal{M}F \rangle = 0.$$

Let us consider $T > 0$, then

$$\begin{aligned} & \int_{\mathcal{P}(\mathbb{R}^d)} \|\nabla V + \nabla W * m\|_{L^2(m)}^2 \bar{P}(dm) \\ & \leq C \left(1 + \int_{\mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |\nabla V(x)|^2 m(dx) \bar{P}(dm) + \int_{\mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |\nabla W * m(x)|^2 m(dx) \bar{P}(dm) \right) \\ & \leq C \left(1 + \int_{\mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |x|^2 m(dx) \bar{P}(dm) + \int_{\mathcal{P}(\mathbb{R}^d)} |\nabla W * m(0)|^2 \bar{P}(dm) \right) \\ & \leq C \left(1 + \int_{\mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |x|^2 m(dx) \bar{P}(dm) \right), \end{aligned}$$

for some constant C that may change from line to line and depends only on the Lipschitz constant of ∇V and ∇W . Then using Proposition 2, we get that

$$\int_{\mathcal{P}(\mathbb{R}^d)} \|\nabla V + \nabla W * m\|_{L^2(m)}^2 \bar{P}(dm) < +\infty.$$

Applying Theorem 1.5 in [37], we get the existence of a process $(\mu_t) \in \mathcal{C}([0, T], \mathcal{P}(\mathbb{R}^d))$ such that μ has dynamic given by (1) and for all $t \in [0, T]$, $\mathcal{L}(\mu_t) = \bar{P}$. Then by weak uniqueness of the solutions of (1), we get that \bar{P} is an invariant measure for m in the sense of Definition 2.

A.2. Proof of Lemma 2. We consider the process (X_t) , driven by the dynamic (4) and with initial condition X_0 such that $\mathcal{L}(X_0) = P_0$, in the sense of Definition 1. Let us denote in the following $m_2(t) = \mathbb{E}[|X_t|^2]$. Then expanding using Ito formula and taking the time derivative, gives:

$$\begin{aligned} m_2'(t) &= -2\mathbb{E}[X_t \cdot (\nabla V(X_t) + \nabla W * m_t(X_t))] + (\sigma^2 + \sigma_0^2)d \\ &= -2\mathbb{E}[X_t \cdot (\nabla V(X_t) - \nabla V(0))] - 2\nabla V(0)\mathbb{E}[X_t] - 2\mathbb{E}[X_t \cdot \nabla W * m_t(X_t)] + (\sigma_0^2 + \sigma^2)d. \end{aligned}$$

Moreover, if \tilde{X}_t an independent copy of X_t , then:

$$\begin{aligned} \nabla W * m_t(X_t) &= \int_{\mathbb{R}^d} \nabla W(X_t - y) m_t(dy) \\ &= \mathbb{E}[\nabla W(X_t - \tilde{X}_t) | X_t, \mathcal{F}_0^0]. \end{aligned}$$

Now, the fact that W is even gives $2\mathbb{E}[X_t \nabla W * m_t(X_t)] = \mathbb{E}[(X_t - \tilde{X}_t) \cdot \nabla W(X_t - \tilde{X}_t)]$. This decomposition is the key, the end of the proof is straightforward using Assumption made on the potential W (see Assumptions (A2)).

A.3. A measurable selection result. In this part, we will mainly prove Lemma 3:

Lemma 5. *Let m and ρ be two probability measure valued random variables which are \mathcal{F}_0^0 -measurable. Assuming that both random measures m and ρ admit a moment of order two almost surely, then there exists a random variable ξ , such that almost surely:*

$$\xi \in \arg \min_{\pi \in \Pi(m, \rho)} \int_{\mathbb{R}^d} |x - y|^2 \pi(dx, dy).$$

Proof. The purpose of the proof is to show that there exists a measurable function $\phi : \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathcal{P}(\mathbb{R}^d)$, such that for all $(m, \rho) \in \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathcal{P}(\mathbb{R}^d)$, $\phi(m, \rho) \in \Pi_{\text{opt}}(m, \rho)$, the set of minimizers for the transport problem. First of all, it is straightforward that Π_{opt} is never empty. Then, we need to show that the set valued function

$$\Phi : \begin{array}{l} \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \rightarrow 2^{\mathcal{P}(\mathbb{R}^d)} \\ (m, \rho) \mapsto \Pi(m, \rho), \end{array}$$

is measurable, where 2^A stands for the set of subsets of A . Considering the graph Γ_Φ , of Φ :

$$\Gamma_\Phi = \{((m, \rho), \xi), \xi \in \Phi(m, \rho)\},$$

it is clear that it is a closed set, and then measurable. In particular, the multi-application Φ is measurable. Moreover, by Lemma 12.1.7 in [44], the application Ψ which associate to any compact set $K \in \mathcal{P}(\mathbb{R})^2$ the set

$$\Psi(K) = \arg \inf_{\pi \in K} \int_{\mathbb{R}^2} f(|x - y|) \pi(dx, dy)$$

is measurable. Then the function

$$\Lambda : \begin{array}{l} \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \rightarrow \text{Ens}(\mathcal{P}(\mathbb{R}^d)) \\ (m, \rho) \mapsto \Pi_{\text{opt}}(m, \rho), \end{array}$$

is a compound of two mesurables functions. Finally by the measurable selection theorem, there exists a measurable function ϕ such that for all $(m, \rho) \in \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d)$, $\phi(m, \rho) \in \Pi_{\text{opt}}(m, \rho)$. \square

A.4. Proof of Proposition 4. This proof is given in [22], we repeat it here for the reader convenience. We begin with the easy case, which is whenever $r < R_1$. In fact in this case we have:

$$\begin{aligned} f''(r) &= \varphi'(r)g(r) + \varphi(r)g'(r) \\ &= -\frac{1}{2(\sigma^2 + \sigma_0^2)} r \kappa_-(r) f(r) - \frac{\ell}{2} \Phi(r) \\ &\leq -\frac{1}{2(\sigma^2 + \sigma_0^2)} r \kappa_-(r) f(r) - \frac{\ell}{2} f(r). \end{aligned}$$

This gives:

$$\begin{aligned} f''(r) - \frac{1}{2(\sigma^2 + \sigma_0^2)} r \kappa(r) f'(r) &\leq -\frac{1}{2(\sigma^2 + \sigma_0^2)} r f(r) (\kappa_-(r) + \kappa(r)) - \frac{\ell}{2} f(r) \\ &\leq -\frac{\ell}{2} f(r), \end{aligned}$$

because $\kappa_-(r) - \kappa(r) \geq 0$. The case where $r \geq R_1$ is more intricate. We can easily see that $f''(r) = 0$, and

$$\begin{aligned} (34) \quad -\frac{1}{2(\sigma^2 + \sigma_0^2)} r \kappa(r) f'(r) &\leq -\frac{1}{4(\sigma^2 + \sigma_0^2)} r \kappa(r) \varphi(R_0) \\ &\leq -r \varphi(R_0) (R_1(R_1 - R_0))^{-1}. \end{aligned}$$

We easily show that the function $r \mapsto r/\Phi(r)$ is non-decreasing on $[R_1, +\infty[$. Then, it comes that $rR_1^{-1} \leq \Phi(r)\Phi(R_1)^{-1}$, which gives:

$$-\frac{1}{2(\sigma^2 + \sigma_0^2)} r \kappa(r) f'(r) \leq -\varphi(R_0) \Phi(r) (\Phi(R_1) (R_1 - R_0))^{-1}.$$

Moreover,

$$\Phi(r) = \int_0^r \varphi(s) ds = (r - R_0) \varphi(R_0) + \Phi(R_0),$$

and

$$\begin{aligned} \int_{R_0}^{R_1} \Phi(s) \varphi(s)^{-1} ds &= \Phi(R_0) \varphi^{-1}(R_0) (R_1 - R_0) + \frac{1}{2} (R_1 - R_0)^2 \\ &= (R_1 - R_0) \varphi(R_0)^{-1} \left(\Phi(R_0) + \frac{1}{2} (R_1 - R_0) \varphi(R_0) \right) \\ &\geq (R_1 - R_0) \Phi(R_1) \varphi(R_0)^{-1/2}. \end{aligned}$$

Finally,

$$\begin{aligned} -\varphi(R_0) \Phi(r) (\Phi(R_1) (R_1 - R_0))^{-1} &\leq -\frac{1}{2} \Phi(r) \left(\int_{R_0}^{R_1} \Phi(s) \varphi(s)^{-1} ds \right)^{-1} \\ &\leq -\frac{\ell}{2} f(r), \end{aligned}$$

because $\Phi(r) \geq f(r)$ for all $r \geq 0$;

A.5. Proof of Proposition 6. We begin this proof with the Itô decomposition of $|E_t^{i,N}|^2$:

$$\begin{aligned} d|E_t^{i,N}|^2 &= -2E_t^{i,N,\delta} (V'(X_t^{i,\delta}) - V'(X_t^{i,N,\delta})) dt \\ &\quad - 2E_t^{i,N,\delta} \left(W' * m_t^\delta(X_t^{i,\delta}) - N^{-1} \sum_{j=1}^N W'(X_t^{i,N,\delta} - X_t^{j,N,\delta}) \right) dt \\ &\quad + 4\sigma_0 \pi_\delta(E_t^{N,\delta}) E_t^{i,N,\delta} dB_t^0 + 4\sigma_0^2 \pi_\delta(E_t^{N,\delta})^2 dt. \end{aligned}$$

Let us now consider $\psi_a : x \mapsto (x + a)^{1/2}$, using once again Itô formula, we get:

$$\begin{aligned} d\psi_a(|E_t^{i,N,\delta}|^2) &= -2\psi'_a(|E_t^{i,N,\delta}|^2) E_t^{i,N,\delta} (V'(X_t^{i,\delta}) - V'(X_t^{i,N,\delta})) dt \\ &\quad - 2\psi'_a(|E_t^{i,N,\delta}|^2) E_t^{i,N,\delta} \left(W' * m_t^\delta(X_t^{i,\delta}) - N^{-1} \sum_{j=1}^N W'(X_t^{i,N,\delta} - X_t^{j,N,\delta}) \right) dt \\ &\quad + 4\sigma_0 E_t^{i,N,\delta} \psi'_a(|E_t^{i,N,\delta}|^2) \pi_\delta(E_t^{N,\delta}) dB_t^0 + 4\sigma_0^2 \psi'_a(|E_t^{i,N,\delta}|^2) dt \\ &\quad + 8\sigma_0^2 |E_t^{i,N,\delta}|^2 \psi''_a(|E_t^{i,N,\delta}|^2) \pi_\delta(E_t^{N,\delta})^2 dt. \end{aligned}$$

As $|2r\psi_a(r^2)| \leq 1/2$, we can use the dominated convergence theorem and get

$$\lim_{a \rightarrow 0} \int_0^T 2\psi'_a(|E_t^{i,N,\delta}|^2) (V'(X_t^{i,\delta}) - V'(X_t^{i,N,\delta})) E_t^{i,N,\delta} dt = \int_0^T (V'(X_t^{i,\delta}) - V'(X_t^{i,N,\delta})) e_t^{i,N} dt.$$

Moreover, once again using dominated convergence theorem and the definition of π_δ , we get:

$$\lim_{a \rightarrow 0} \int_0^T \pi_\delta(E_t^{i,N,\delta})^2 \left\{ 4\sigma^2 \psi'_a(|E_t^{i,N,\delta}|^2) + 8\sigma^2 |E_t^{i,N,\delta}|^2 \psi''_a(|E_t^{i,N,\delta}|^2) \right\} dt = 0,$$

and the same holds for the term that is issued from the common noise. The observed outcomes may initially seem unexpected; however, the presence of π_δ helps such approximations by inhibiting the differential process from staying close to the origin, which explains the absence of Local time at 0.

A.6. Proof of Proposition 5. We know that:

$$\gamma := \liminf_{r \rightarrow +\infty} \kappa(r) > 0,$$

then, there exists $r_0 > 0$, such that $\kappa(r) \geq \gamma/2, \forall r \geq r_0$. It is clear that $R_0 \leq r_0$, and then

$$R_1 \leq \inf \{s \geq r_0, s(s - R_0)\kappa(r) \geq 4(\sigma^2 + \sigma_0^2)\}.$$

Now, using the definition of r_0 , we get $R_1 \leq \inf \{s \geq r_0, s(s - R_0) \geq \frac{8}{\gamma}(\sigma^2 + \sigma_0^2)\}$. This gives us that

$$R_1 \leq \frac{R_0 + \sqrt{R_0^2 + 32(\sigma^2 + \sigma_0^2)/\gamma}}{2} \vee r_0.$$

Then, as $\Phi(r) \geq r\varphi(R_0), \forall r \geq 0$ and $\varphi(s) \leq 1, \forall s \geq 0$, we can write that

$$(35) \quad \ell = \left(\int_0^{R_1} \Phi(s)\varphi(s)^{-1} ds \right)^{-1} \geq \frac{2\varphi(R_0)}{R_1^2}.$$

Moreover, for σ_0 large enough, one can claim that

$$(36) \quad R_1^2 \leq 2R_0^2 + 32(\sigma^2 + \sigma_0^2)/\gamma.$$

Finally, combing (35) and (36), we have the following bound which is valid for σ_0 large enough:

$$c(V, W, \sigma_0) \geq \frac{\gamma(\sigma^2 + \sigma_0^2)}{2\gamma R_0 + 32(\sigma^2 + \sigma_0^2)}\varphi(R_0) - 4L_W\varphi(R_0)^{-1}.$$

The right term appears to be (for each fixed values of σ) a non decreasing function of σ_0^2 . Then, as the right term converges to $\gamma/32\varphi(R_0) - 4L_W\varphi(R_0)^{-1}$, there exists a $\bar{\sigma}_0$, such that for all $\sigma_0 \geq \bar{\sigma}_0$, $c(V, W, \sigma_0) \geq \gamma/64\varphi(R_0) - 2L_W\varphi(R_0)^{-1}$. This allows to conclude that for $L_W < \bar{L} := \gamma\varphi(R_0)^2/128$, and $\sigma_0 > \bar{\sigma}_0$, $c(V, W, \sigma_0) > 0$. Hence, $\sigma_0 \in \mathfrak{D}_{V,W}$.

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