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# Inner-Outer Approximation of Robust Control Invariant Sets <sup>★</sup>

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## Abstract

This work proposes an approach to replace the use of a robust control invariant set by a pair of simpler sets that provide an inner and an outer approximation of the former. In the proposed approach, the outer set plays the role of the target region and the inner set is such that the trajectories that start inside it can be kept inside the outer set and be driven back to the inner set within a finite-time horizon. We show that the existence of these two sets implies the existence of a robust control invariant set between both regions. We also provide results that allow finding an inner set from a given target outer set and we show a way of using both sets in Model Predictive Control schemes such that the target region is never abandoned in spite of the fact that neither the inner set nor the outer set are invariant. We also illustrate the ideas with an example in which the inner and outer sets are very simple notwithstanding that any robust invariant set is not convex.

*Key words:* Control Invariant Sets; Set-Based Methods; Model Predictive Control; Practical Stability.

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## 1 Introduction

In presence of disturbances and different types of non ideal assumptions, control systems are not able to drive the system state to a desired equilibrium point. In those cases, the best that can be achieved is to steer the state to some region where control goals are satisfied and to keep the trajectories bounded to it. The achievable properties are generally known under the generic term of *practical stability* [15].

A region that has the property that the state can remain in through the use of appropriate control actions defines a *positive control invariant set* (PCIS), that in presence of unknown disturbances are generally referred to as *robust positive control invariant sets* (RPCIS). In the context of this work, we shall refer to these sets simply as control invariant sets (CIS) or robust CIS (RCIS).

Set invariance plays a fundamental role in control theory [4] and there are several control design approaches

that use CIS or RCIS in order to guarantee that the state remains bounded according to the control goals. For instance, these sets are used typically as a terminal restriction in Model Predictive Control (MPC) schemes as a way of ensuring convergence to certain target region [20,1,12,16]. In fact, the non existence of a (R)CIS inside a target region for a control system implies that it is impossible to ensure that the state remains forever inside that region.

There are several procedures that allow computing CIS and RCIS [23,19,7,6,17] under different assumptions. However, in most situations, the characterization of these sets is not simple. In nonlinear or switched systems, or under the presence of finite input sets, their computation is not only complicated, but it also leads to very complex sets [22]. These sets may have hundreds of faces, they may be not even convex, and, in consequence, their usage in real control problems can become impractical or even impossible.

In this work, we propose a way of avoiding the use of control invariant sets by using two simpler sets, an outer set consisting of the target region and an inner set from which an input sequence exists such that the state does not leave the outer set and eventually (within a finite horizon of time) the state trajectory passes back by that inner set.

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We shall show that these conditions on the inner and the outer set imply the existence of a RCIS between both sets, but it is not necessary to compute it. The sole knowledge of the outer set (that we assume that is given, as it would play the role of the target region) and the computation of a simple inner set allow designing controllers that, under certain restrictions on the initial state, ensure that the outer set is never abandoned in spite of the fact that nor the outer set, neither the inner set need to be positive control invariant.

### Notation

The following notation will be used along this work. An interval of the integer numbers  $[E, F] \subset \mathbb{Z}$  is denoted as  $\mathbb{Z}_{E:F}$ . The expression  $s(k+i|k)$  denotes the value of a signal  $s$  at time  $k+i$  computed at time  $k$ .

Unless otherwise stated, we shall refer indistinctly to the input as  $u(k)$  or  $u(x(k))$  to emphasize its dependence on the state.

## 2 Background

This section introduces some previous definitions that are then used to present the main results of the work.

### 2.1 Control Invariant Sets

Consider a discrete-time perturbed system described by the following nonlinear time-invariant model:

$$x(k+1) = f(x(k), u(k), w(k)) \quad (1)$$

with  $x(k) \in X \subseteq \mathbb{R}^n$ ,  $u(k) \in U \subseteq \mathbb{R}^m$  and  $w(k) \in W \subset \mathbb{R}^p$  being the state, control input and disturbance at time  $k$ , respectively. We shall assume along this work that  $W$  is bounded thus fitting in the *persistent bounded disturbances framework*.

**Definition 1** (Robust Positive Control Invariant Set [8]). Given the system of Eq. (1) with the corresponding sets  $X$ ,  $U$  and  $W$ , a set  $\Omega \subseteq X$  is robust positive control invariant provided that for any state  $x \in \Omega$ , a control input  $u(x) \in U$  exists such that  $f(x, u(x), w) \in \Omega$  for all  $w \in W$ .

A way of verifying that a set  $\Omega$  is a RCIS is by checking that it lies inside its robust one-step set  $\mathcal{C}_1(\Omega)$  defined as follows:

**Definition 2** (Robust One-step Set [11]). Given the system of Eq. (1) with the corresponding sets  $X$ ,  $U$  and  $W$ , and a set  $\Omega \subseteq X$ , the robust one-step set to  $\Omega$  is defined as

$$\mathcal{C}_1(\Omega) \triangleq \{x \in X : \exists u(x) \in U \mid f(x, u, w) \in \Omega, \forall w \in W\}. \quad (2)$$

Notice that  $\Omega$  is a RCIS if and only if  $\Omega \subseteq \mathcal{C}_1(\Omega)$  [11].

### 2.2 Computation of Control Invariant Sets

While control invariant sets can take arbitrary shapes, most works in the literature are devoted to characterize ellipsoidal and polyhedral sets [5]. Ellipsoidal sets are usually obtained with procedures based on Lyapunov functions or linear matrix inequalities [14]. Polyhedral sets, in turn, are generally computed using iterative methods [10,25]. There are also constructions based on the approximations within the class of semi-algebraic sets that resort to lift and project approaches in order to benefit from the convex optimization tools [13] but remain difficult to integrate in MPC-like routines.

Among the different approaches, and closely related to the present work, there is a classic iterative procedure to obtain the maximal RCIS contained in certain set  $R \subseteq X$  first presented in [2] and then summarized in [4,5]. This procedure starts by selecting  $R_0 = R$  and then iterates to compute the sequence

$$R_{k+1} \triangleq \mathcal{C}_1(R_k) \cap R. \quad (3)$$

The sets  $R_k$  are nested, since  $R_{k+1} \subseteq R_k$ , and the maximal RCIS contained in  $R$  is shown to be  $R_\infty \triangleq \bigcap_{k=0}^{\infty} R_k$ .

The reason why this procedure is associated to polyhedral invariant sets is that, although it can be formulated for general systems, its use is mainly restricted to linear systems, and when  $R$ ,  $U$  and  $W$  are polyhedral, the same occurs with  $R_k$  (but not necessarily with  $R_\infty$ ). Some works with remarkable results for linear, polytopic and piece-wise affine systems are respectively [9,3,11].

One of the problems with more general setups is that of calculating the robust one-step set. For nonlinear systems with constraints, computing this in a useful manner that allows the implementation of the aforementioned iterative procedure is not simple. Examples of publications that address the construction of invariant sets for the nonlinear case are [6,8,7]. Furthermore, even when computing the robust one-step set may be achievable, the invariant sets obtained could be complex to such an extent that their usage in real control problems may not be feasible.

Given the complications that arise in the computation of invariant sets, it has been studied how to obtain inner and/or outer approximations to them with different properties and under various assumptions [24,7,21,18]. In this work, even though we consider inner and outer sets that could be close to some invariant, our goal is not to approximate the invariant set but to give conditions under which this set can be replaced by a pair of inner and outer sets in order to guarantee practical stability.

### 3 Main Results

This section presents the main results of the work, introducing first the notion of  $N$ -step Control Inner Sets and then studying their properties, providing some tools for its computation and usage in the context of control design.

#### 3.1 $N$ -step Control Inner Sets

The following definition provides the notion of  $N$ -step control inner set.

**Definition 3** ( $N$ -step Control Inner Set). Given the system of Eq. (1) and an outer set  $\Omega_O \subseteq X$ , we say that a set  $\Omega_I \subseteq \Omega_O$  is a  $N$ -step control inner set if for any  $x(0) \in \Omega_I$ , an input sequence  $u(x(0)), \dots, u(x(N-1)) \in U$  exists such that  $x(k) \in \Omega_O \forall k \in \mathbb{Z}_{1:N}$  and  $x(j) \in \Omega_I$  for some  $j \in \mathbb{Z}_{1:N}$  for any disturbance sequence  $w(0), \dots, w(N-1) \in W$ .

Figure 1 shows an outer set  $\Omega_O$ , a  $N$ -step control inner set  $\Omega_I$ , a RCIS  $\Omega$ , and a trajectory starting from  $x(0)$  that suggest that  $N \geq 5$ .

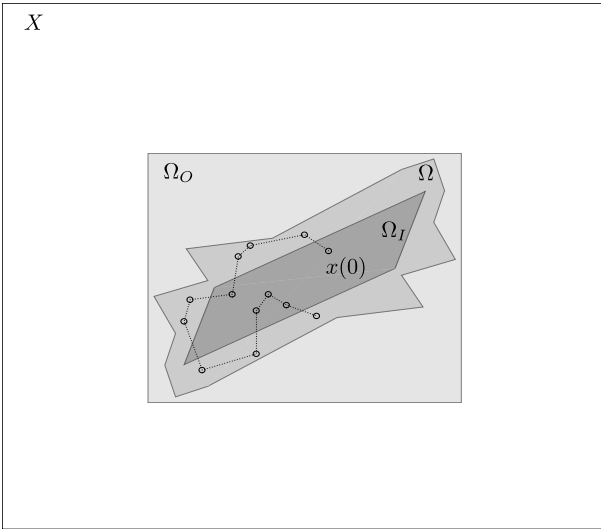


Fig. 1. A non convex RCIS  $\Omega$  with a pair of simpler outer and inner sets ( $\Omega_O$  and  $\Omega_I$ ).

We shall assume that the outer set  $\Omega_O$  is given and it will play the role of a target set to which the state trajectories must converge and remain. While this target set does not need to be control invariant, we shall assume that it contains a RCIS since otherwise it will not be possible to ensure that the trajectories can be kept inside  $\Omega_O$ . However, instead of computing a RCIS, the goal is to find a simpler  $N$ -step Control Inner Set  $\Omega_I$  verifying Definition 1 since the trajectories that pass through it can be also kept bounded to  $\Omega_O$ .

Before providing a way to characterize  $\Omega_I$ , we shall first analyze the existence of a maximal  $N$ -step Control Inner

Set inside  $\Omega_O$  and relate that with the classic concepts of RCIS.

#### 3.2 Maximal $N$ -step Control Inner Set

We shall show that given an outer set, a maximal  $N$ -step control inner set exists, and, in case it is not empty, this maximal  $N$ -step control inner set turns out to be a RCIS. Moreover, we shall show that this set can be characterized as the fixed point of a simple set-iteration process.

For arriving to those results, we first provide some auxiliary propositions, lemmas and corollaries.

**Proposition 4.** Given the system of Eq. (1) and the outer set  $\Omega_O \subseteq X$ , a maximal  $N$ -step control inner set  $\Omega_I^{\max}$  exists.

*Proof.* Given two  $N$ -step control inner sets,  $\Omega_I^1, \Omega_I^2$ , it can be straightforwardly seen that their union  $\Omega_I^1 \cup \Omega_I^2$  is also a  $N$ -step control inner set. Thus, a maximal  $N$ -step control inner set exists and it consists of the union of all possible  $N$ -step control inner sets.  $\square$

Notice that  $\Omega_I^{\max}$  can be the empty set if there is no  $N$ -step control inner set verifying Definition 3.

**Lemma 5.** Let  $\Omega_I$  be a  $N$ -step control inner set. Then, the set  $\Omega_I \cup (\mathcal{C}_1(\Omega_I) \cap \Omega_O)$  is also a  $N$ -step control inner set.

*Proof.* Let  $x(0) \in \mathcal{C}_1(\Omega_I) \cap \Omega_O$ , then an input  $u(x(0)) \in U$  exists such that  $x(1) = f(x(0), u(x(0)), w(0)) \in \Omega_I$  for any disturbance  $w(0) \in W$ . Then, from Definition 3, a sequence of inputs  $u(x(1)), \dots, u(x(N)) \in U$  exists such that  $x(k) \in \Omega_O \forall k = \mathbb{Z}_{2:N+1}$  and  $x(j) \in \Omega_I$  for some  $j \in \mathbb{Z}_{2:N+1}$  for any disturbance sequence. Thus,  $x(0) \in \mathcal{C}_1(\Omega_I) \cap \Omega_O$  implies that an input sequence  $u(x(0)), \dots, u(x(N-1)) \in U$  exists such that  $x(k) \in \Omega_O \forall k = \mathbb{Z}_{1:N}$ . Moreover, this and the fact that  $x(j) \in \Omega_I$  means that  $x(j_1) \in \mathcal{C}_1(\Omega_I) \cap \Omega_O$  for  $j_1 = j - 1 \in \mathbb{Z}_{1:N}$  for any disturbance sequence. Then, it results that  $\mathcal{C}_1(\Omega_I) \cap \Omega_O$  is a  $N$ -step control inner set so it follows that  $\Omega_I \cup (\mathcal{C}_1(\Omega_I) \cap \Omega_O)$  is also a  $N$ -step control inner set.  $\square$

**Corollary 6.** The set  $\Omega_I^{\max}$  verifies  $\mathcal{C}_1(\Omega_I^{\max}) \cap \Omega_O \subseteq \Omega_I^{\max}$ .

**Lemma 7.** If the set  $\Omega_I^{\max}$  is not empty, then it is a RCIS.

*Proof.* Given any  $x(0) \in \Omega_I^{\max}$ , an input sequence exists such that  $x(k)$  remains in  $\Omega_O$  and passes inside  $\Omega_I^{\max}$  at

some time  $j \in \mathbb{Z}_{1:N}$  for any disturbance sequence. Then,  $x(j-1) \in \Omega_O$  and  $x(j-1) \in \mathcal{C}_1(\Omega_I^{\max})$ . Thus, from Corollary 6,  $x(j-1) \in \Omega_I^{\max}$ . Repeating this argument, it results that  $x(j-i) \in \Omega_I^{\max} \forall i \in \mathbb{Z}_{1:j-1}$ , and finally  $x(1) \in \Omega_I^{\max}$  proving that  $\Omega_I^{\max}$  is a RCIS.  $\square$

**Lemma 8.** *For all  $N \in \mathbb{N}$ , the maximal  $N$ -step control inner set  $\Omega_I^{\max}$  is the maximal RCIS contained in  $\Omega_O$ .*

*Proof.* Let  $\Omega^{\max}$  be the maximal RCIS contained in  $\Omega_O$ . From Definitions 1 and 3, it can be easily seen that  $\Omega^{\max}$  is a  $N$ -step control inner set for all  $N \in \mathbb{N}$ . Thus, for any  $N \in \mathbb{N}$ , it results that  $\Omega^{\max} \subseteq \Omega_I^{\max}$ .

According to Lemma 7, given  $N \in \mathbb{N}$  and the outer set  $\Omega_O$ , the maximal  $N$ -step inner set  $\Omega_I^{\max}$  is a RCIS and then  $\Omega_I^{\max} \subseteq \Omega^{\max}$ .

Then, considering that  $\Omega^{\max} \subseteq \Omega_I^{\max}$  and  $\Omega_I^{\max} \subseteq \Omega^{\max}$  it results that  $\Omega_I^{\max} = \Omega^{\max}$ .  $\square$

This last result tells that the maximal control inner set does not depend on  $N$  and that it coincides with the maximal RCIS contained in  $\Omega_O$ . A direct consequence of this result is that a trajectory that starts in any  $N$ -step control inner set can be held forever inside  $\Omega_O$ , as expressed in the following corollary.

**Corollary 9.** *If  $\Omega_I$  is a  $N$ -step control inner set, then given any  $M > 0$ , the condition  $x(0) \in \Omega_I$  implies that there exists an input sequence  $u(x(0)), \dots, u(x(M-1)) \in U$  such that  $x(k) \in \Omega_O \forall k \in \mathbb{Z}_{1:M}$  for any disturbance sequence.*

Once the structural properties of the maximal  $N$ -step control set are brought into light in Lemma 8 we can make a link with the classical control invariant constructions. The following result states that the maximal  $N$ -step control inner set can be seen as a limit set of a set-iterations of the standard procedure of Eq. (3) initialized in  $\Omega_O$ .

**Theorem 10.** *Consider the following succession of sets:*

$$\Omega_0 = \Omega_O, \quad \Omega_{k+1} = \mathcal{C}_1(\Omega_k) \cap \Omega_O. \quad (4)$$

*Then,  $\Omega_I^{\max} = \lim_{k \rightarrow \infty} \Omega_k$ .*

*Proof.* Notice first that since  $\Omega_0 = \Omega_O$ , then  $\Omega_1 = \mathcal{C}_1(\Omega_0) \cap \Omega_O \subseteq \Omega_0$ . In order to use induction, suppose that for some  $k$ ,  $\Omega_k \subseteq \Omega_{k-1}$ . Hence,  $\mathcal{C}_1(\Omega_k) \subseteq \mathcal{C}_1(\Omega_{k-1})$ . Then,  $\Omega_{k+1} = \mathcal{C}_1(\Omega_k) \cap \Omega_O \subseteq \Omega_k = \mathcal{C}_1(\Omega_{k-1}) \cap \Omega_O$ . This proves that for any  $M > 0$ ,  $\Omega_M \subseteq \Omega_{M-1} \dots \subseteq \Omega_1 \subseteq \Omega_0$ . This implies that the sequence  $\Omega_k$  converges to some set  $\Omega_\infty \triangleq \lim_{k \rightarrow \infty} \Omega_k$  (which could be the empty set).

Notice also that  $\Omega_I^{\max} \subseteq \Omega_0$  implies that  $\Omega_I^{\max} = \mathcal{C}_1(\Omega_I^{\max}) \cap \Omega_O \subseteq \mathcal{C}_1(\Omega_0) \cap \Omega_O = \Omega_1$  and then  $\Omega_I^{\max} \subseteq \Omega_1$ . Applying this recursively, we arrive to  $\Omega_I^{\max} \subseteq \Omega_k$  for all  $k$  and then  $\Omega_I^{\max} \subseteq \Omega_\infty$ . On the other hand, the fact that  $\Omega_\infty = \mathcal{C}_1(\Omega_\infty) \cap \Omega_O$  implies that  $\Omega_\infty$  is a RCIS contained in  $\Omega_O$ . Thus, it is also contained in  $\Omega_I^{\max}$  that, according to Lemma 8 is the maximal RCIS contained in  $\Omega_O$ . Hence,  $\Omega_\infty \subseteq \Omega_I^{\max}$ .

The facts that  $\Omega_I^{\max} \subseteq \Omega_\infty$  and  $\Omega_\infty \subseteq \Omega_I^{\max}$  mean that  $\Omega_I^{\max} = \Omega_\infty$  completing the proof.  $\square$

An alternative proof for Theorem 10 can be obtained using the fact that  $\Omega_I^{\max}$  is the maximal RCIS inside  $\Omega_O$  and that this set can be obtained using the procedure of Eq. (3).

**Corollary 11.** *Given an outer set  $\Omega_O$ , any  $N > 0$ , and the succession of Eq. (4), a non empty  $N$ -step control inner set exists if and only if  $\lim_{k \rightarrow \infty} \Omega_k \neq \emptyset$ .*

Another version of this corollary is

**Corollary 12.** *Given an outer set  $\Omega_O$ , for any  $N > 0$ , a non empty  $N$ -step control inner set exists if and only if  $\Omega_O$  contains a RCIS in its interior.*

### 3.3 $N$ -step Control Inner Set Characterization

The existence of the maximal  $N$ -step control inner set and its characterization as the result of an infinite iteration is of theoretical value. However, in practice, we are interested in characterizing simpler inner sets, a goal that can be achieved with the help of the following theorem.

**Theorem 13.** *Let  $S_0 \subseteq \Omega_O$  be a non-empty set and consider the succession of sets*

$$S_{k+1} = \mathcal{C}_1(S_k) \cap \Omega_O \quad (5)$$

*and the set  $T_N \triangleq \bigcup_{k=1}^N S_k$ . Then,*

- (1)  $S_0$  is a  $N$ -step control inner set if and only if  $S_0 \subseteq T_N$ .
- (2) Any set  $\Omega$  such that  $S_0 \subseteq \Omega \subseteq T_N$  is a  $N$ -step control inner set.

*Proof.* 1.  $\implies$  ) Being  $x(0) \in \Omega_O$ , the condition  $x(0) \notin T_N$  implies that  $x(0) \notin \bigcup_{j=1}^N S_j = \bigcup_{j=1}^N \mathcal{C}_1(S_{j-1}) \cap \Omega_O$ . Then, an input  $u(0) \in U$  cannot be found such that  $x(1) \in \bigcup_{j=0}^{N-1} S_j$  for all possible disturbances  $w(0) \in W$ . Thus, selecting any input  $u(0) \in U$ , a disturbance  $w(0) = w(x(0), u(0)) \in W$  exists such that  $x(1) \notin \bigcup_{j=0}^{N-1} S_j$ .

In order to proceed by induction, assuming that  $x(k) \in \Omega_O$ , the condition  $x(k) \notin \bigcup_{j=1}^{N-k} S_j = \bigcup_{j=1}^{N-k} \mathcal{C}_1(S_{j-1}) \cap \Omega_O$  implies that selecting any input  $u(k) \in U$ , a disturbance  $w(k) = w(x(k), u(k)) \in W$  exists such that  $x(k+1) \notin \bigcup_{j=0}^{N-k-1} S_j$ .

Then, for any input sequence  $u(0), \dots, u(N-1) \in U$ , a disturbance sequence  $w(0), \dots, w(N-1) \in W$  exists such that  $x(k) \notin S_0 \subseteq \bigcup_{j=0}^{N-k-1} S_j$  for all  $k \in \mathbb{Z}_{1:N}$  assuming that  $x(k) \in \Omega_O$ . This implies that when we can chose  $x(0)$  such that  $x(0) \in S_0 \subseteq \Omega_O$  and  $x(0) \notin T_N$ , then  $S_0$  is not a  $N$ -step control inner set either because  $x(k) \notin S_0$  for any  $k \in \mathbb{Z}_{1:N}$  or because  $x(k) \notin \Omega_O$  for some  $k \in \mathbb{Z}_{1:N}$ .

Thus,  $S_0 \subseteq T_N$  is a necessary condition for  $S_0$  to be a  $N$ -step control inner set.

$\Leftarrow$ ) The condition  $S_0 \subseteq T_N$  implies that given any  $x(0) \in S_0, x(0) \in S_j = \mathcal{C}_1(S_{j-1}) \cap \Omega_O$  for some  $j \in \mathbb{Z}_{1:N}$ . Thus, an input  $u(0) \in U$  exists such that  $x(1) \in S_{j-1}$  for any disturbance  $w(0) \in W$ . Then, by induction an input sequence  $u(0), \dots, u(j-1) \in U$  exists such that  $x(1) \in S_{j-1}, \dots, x(j) \in S_0$  for any disturbance sequence. This implies that  $S_0$  is a  $N$ -step control inner set.

2. Take  $R_0 = \Omega$  and consider the succession  $R_{k+1} = \mathcal{C}_1(R_k) \cap \Omega_O$  for  $k \in \mathbb{Z}_{1:N-1}$ . Notice that  $S_0 \subseteq R_0$  implies that  $S_k \subseteq R_k$  for all  $k \in \mathbb{Z}_{1:N}$  and then  $T_N \subseteq \bigcup_{k=1}^N R_k$ . Thus, the condition  $\Omega = R_0 \subseteq T_N$  implies that  $R_0 \subseteq \bigcup_{k=1}^N R_k$  and according to the first part of this Theorem, this means that  $\Omega = R_0$  is a  $N$ -step inner set.  $\square$

This last result allows computing simple inner sets, by starting with some candidate inner set and iterating until it is covered by their successive controllable sets. If after  $N$  iterations the candidate set is not covered by  $T_N$ , then the chosen set was not a  $N$ -step control inner set. In that case, a possible workaround is to keep iterating until  $S_0$  is covered, or until  $T_N$  converges without covering  $S_0$  so we can ensure that  $S_0$  is not a  $N$ -step control inner set for any  $N > 0$ .

If we find that  $S_0$  is a  $N$ -step control inner set but  $N$  is too large, then the following result can be used for finding an inner set with less steps.

**Theorem 14.** *Let  $S_0$  be a  $N$ -step control inner set. Then the set  $V_M = \bigcup_{k=0}^M S_k$  for  $M \in \mathbb{Z}_{0:N-1}$  with  $S_k$  defined in Eq. (5) is a  $(N-M)$ -step control inner set.*

*Proof.* Let  $Z_0 = V_M$  and consider the succession  $Z_{k+1} = \mathcal{C}_1(Z_k) \cap \Omega_O$ . We shall prove first that

$$Z_k = \bigcup_{j=k}^{M+k} S_j, \quad \forall k \in \mathbb{Z}_{0:N-M}. \quad (6)$$

Notice that  $Z_0 = V_M = \bigcup_{j=0}^M S_j$ . Then, in order to proceed by induction, assuming that  $Z_k = \bigcup_{j=k}^{M+k} S_j$  for some  $k \in \mathbb{Z}_{0:N-M}$ , it results

$$\begin{aligned} Z_{k+1} &= \mathcal{C}_1(Z_k) \cap \Omega_O = \mathcal{C}_1\left(\bigcup_{j=k}^{M+k} S_j\right) \cap \Omega_O \\ &= \bigcup_{j=k}^{M+k} \mathcal{C}_1(S_j) \cap \Omega_O = \bigcup_{j=k}^{M+k} S_{j+1} = \bigcup_{j=k+1}^{M+k+1} S_j, \end{aligned}$$

showing that Eq. (6) holds. Then, it results

$$\hat{T}_{N-M} \triangleq \bigcup_{k=1}^{N-M} Z_k = \bigcup_{j=1}^N S_j = T_N.$$

Since  $S_0$  is a  $N$ -step control inner set, according to Theorem 13, it results that  $S_0 \subseteq T_N$ . Then,

$$\hat{T}_{N-M} = T_N = \bigcup_{j=1}^N S_j = \bigcup_{j=0}^N S_j = \bigcup_{j=0}^{N-M} Z_j$$

meaning that  $Z_0 \subseteq \hat{T}_{N-M}$ . This, according to Theorem 13, implies that  $Z_0 = V_M$  is a  $(N-M)$ -step control inner set completing the proof.  $\square$

While the results of Theorems 13-14 are useful for checking that a candidate set  $S_0$  is a  $N$ -step control inner set and for finding a set with a smaller value of  $N$ , they do not tell how to choose the candidate set  $S_0$ .

We shall not provide a general answer to this problem since the solution may depend on the different assumptions made: the system may be linear or nonlinear, it can include one or multiple equilibria or limit sets and these can be open loop stable or unstable, the input set may be convex or finite, etc. Anyway, there are some general considerations that can be taken into account which are valid for all cases:

- Since every inner set verifies  $\Omega_I \subseteq \Omega_I^{\max}$ , then the candidate set  $S_0$  must be inside any set  $\Omega_k$  in the succession of Eq. (4). Thus, a possible procedure may consist in performing some iterations of that succession and then taking a simple set contained in  $\Omega_k$  as candidate.
- While  $S_0$  may not be covered by  $T_N = \bigcup_{k=1}^N S_k$ , it could happen that some  $S_k$  is covered by selecting a sufficiently large value of  $N$ . Thus, we can then adopt  $\Omega_I = S_k$ . In case  $S_k$  has a complex shape, we can also exploit the second part of Theorem 13 and take  $\Omega_I$  such that  $S_k \subseteq \Omega_I \subseteq T_N$ .
- In several occasions, choosing a small candidate set  $S_0$  works. However, as we shall see in Section 3.4, we may want  $\Omega_I$  to be as large as possible. Thus, as before, we can exploit the second part of Theorem 13

to construct a larger inner set by placing some simple set  $\Omega$  between the candidate set  $S_0$  and the set  $T_N$ . Moreover, we can use this idea recursively by starting back the procedure with  $S_0 = \Omega$  and enlarge further the inner set.

- Taking  $S_0 = \Omega$  where  $\Omega$  is a non-maximal RCIS contained in  $\Omega_O$  will always work since it results that  $S_0 \subseteq S_1 \subseteq T_N$  in Theorem 13. Then, after computing  $T_N$  and provided that  $T_N \neq S_0$ , a simpler set  $\Omega_I$  can be adopted satisfying  $S_0 \subseteq \Omega_I \subseteq T_N$  and proceeding recursively like in the previous item. A way of finding a non-maximal RCIS  $\Omega$  is using the procedure of Theorem 10 replacing  $\Omega_O$  by some smaller set  $\tilde{\Omega}_O \subset \Omega_O$  verifying  $\Omega_I^{\max} \not\subseteq \tilde{\Omega}_O$ . Provided that  $\tilde{\Omega}_O$  contains a RCIS, that procedure will compute the maximal RCIS contained on it.
- In any case, using  $\Omega_I = \Omega_I^{\max}$  will work and Theorem 10 provides a procedure to compute that maximal inner set. However, the goal of this work is to use simpler sets than those that are the result of an iteration that may lead to very complex shapes. We are assuming in this work that  $\Omega_I^{\max}$  results too complex to be used in practice, what justifies the need for using inner and outer approximations.

### 3.4 Application to Control Design

The existence of a RCIS like  $\Omega_I^{\max}$  inside a target region  $\Omega_O$  is a necessary condition for the existence of a control law that can keep the state inside this region. Moreover, in several MPC schemes the RCIS need to be explicitly computed since they are used as terminal constraints for the predicted states in order to ensure recursive feasibility and practical stability.

However, as it was already mentioned, in many situations the computation of the control invariant set can be very difficult or can lead to a complex set that cannot be used in practice. In such case, this complex set can be replaced by the target region itself  $\Omega_O$  and an inner approximation  $\Omega_I$  that verifies Definition 3. In order to show how to proceed with this replacement, we shall consider a MPC scheme where the predicted states for an input sequence  $\hat{u}(k|k), \dots, \hat{u}(k+N-1|k)$  and a disturbance sequence  $\hat{w}(k|k), \dots, \hat{w}(k+N-1|k)$  are given by

$$\hat{x}(k+j+1|k) = f(\hat{x}(k+j|k), \hat{u}(k+j|k), \hat{w}(k+j|k)) \quad (7)$$

with  $\hat{x}(k|k) = x(k)$ .

The following lemma then shows that using a pass-through constraint on the inner set  $\Omega_I$  ensures recursive feasibility of the MPC scheme.

**Lemma 15.** *Consider the system of Eq. (1), an outer set  $\Omega_O$  and an associated  $N$ -step control inner set  $\Omega_I$ . Suppose that in an MPC scheme the admissible input*

*sequences  $\hat{u}(k|k), \dots, \hat{u}(k+N-1|k) \in U$  are those that ensure that, for any disturbance sequence, the predicted states verify  $\hat{x}(k+j|k) \in \Omega_I$  for some  $j \in \mathbb{Z}_{1:N}$  and  $\hat{x}(k+i|k) \in \Omega_O \quad \forall i \in \mathbb{Z}_{j:N}$ . If the state  $x(k)$  is such that an admissible input sequence exists, then an admissible input sequence exists also for  $x(k+1)$ .*

*Proof.* Let  $\hat{u}(k|k), \dots, \hat{u}(k+N-1|k) \in U$  be the admissible input sequence chosen by the MPC scheme such that  $u(k) = \hat{u}(k|k)$ . Let  $w(k)$  be the actual disturbance and let  $x(k+1) = f(x(k), u(k), w(k))$  be the state at time  $k+1$ .

Take  $j \in \mathbb{Z}_{1:N}$  as the minimum value such that  $\hat{x}(k+j|k) \in \Omega_I$  for any disturbance sequence. Then, if  $j=1$  it results that  $\hat{x}(k+1|k) \in \Omega_I$  for any disturbance  $\hat{w}(k|k)$ . Thus,  $x(k+1) \in \Omega_I$  and, according to Definition 3, an admissible input sequence exists for  $x(k+1)$ .

In case  $j > 1$ , consider the input sequence  $\hat{u}(k+i|k+1) = \hat{u}(k+i|k)$  for  $i \in \mathbb{Z}_{1:j-1}$ . Then, according to the definition of  $j$ , it results that the predicted state satisfies  $\hat{x}(k+j|k+1) \in \Omega_I$ . Thus, from Definition 3, an input sequence  $\hat{u}(k+j|k+1), \dots, \hat{u}(k+N|k+1) \in U$  exists such that  $\hat{x}(k+i|k+1) \in \Omega_O$  for all  $i \in \mathbb{Z}_{j+1:N+1}$  for any disturbance sequence.

This implies that starting from  $x(k+1)$  an admissible input sequence exists  $\hat{u}(k+1|k+1), \dots, \hat{u}(k+N|k+1) \in U$  such that the predicted states pass through  $\Omega_I$  and after that never abandon  $\Omega_O$  within the prediction horizon.  $\square$

The next result shows that admitting only input sequences that keep the state inside  $\Omega_O$  and pass through  $\Omega_I$  allows keeping forever the state inside the target set  $\Omega_O$  ensuring practical stability.

**Lemma 16.** *Consider the system of Eq. (1), an outer set  $\Omega_O$  and an associated  $N$ -step control inner set  $\Omega_I$ . Suppose that in a MPC scheme the admissible input sequences  $\hat{u}(k|k), \dots, \hat{u}(k+N-1|k) \in U$  are those ensuring that, for any disturbance sequence, the predicted states verify  $\hat{x}(k+j|k) \in \Omega_I$  for some  $j \in \mathbb{Z}_{1:N}$  and  $\hat{x}(k+i|k) \in \Omega_O \quad \forall i \in \mathbb{Z}_{1:N}$ . Then, if the state  $x(k) \in \Omega_O$  is such that an admissible input sequence exists, then  $x(k+1) \in \Omega_O$  and an admissible input sequence exists also for  $x(k+1)$ .*

This lemma, whose proof is very similar to that of the previous one, tells that given a target region  $\Omega_O$ , a predictive control scheme can be formulated such that the state remains bounded to that region even when it is not invariant. Moreover, it can be easily seen that the state is in fact bounded to the maximal control invariant set  $\Omega_I^{\max}$  without actually knowing the latter set.

In order to exploit these features in the design of a controller, given the target region  $\Omega_O$  we need to compute an inner set  $\Omega_I$  for which Theorem 13 can be used. In practice, the  $N$ -step control inner set  $\Omega_I$  should be *large*. Otherwise, the horizon length  $N$  could result too long or the feasibility region may result too small. The reason is that  $\Omega_I$  must be reached in up to  $N$ -steps from the initial state, so using a small inner set may limit the region from which it is reached within the prediction horizon.

#### 4 Application Example

This section presents a numerical example that illustrates the advantages of replacing a RCIS by an outer target region and a  $N$ -step control inner set.

We consider in particular a linear discrete-time uncertain system with a finite input set, given by:

$$x(k+1) = \begin{bmatrix} 0.9 & 0 \\ 0.65 & 0.8 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) + w(k), \quad (8)$$

with  $x(k) = [x_1(k), x_2(k)]^T \in X = \mathbb{R}^2$ ,  $u(k) \in U = \{-1, 1\}$  and  $w(k) \in W = \{w \in \mathbb{R}^2 : \|w\|_\infty \leq 0.02\}$ .

We shall suppose that the target region is the outer set  $\Omega_O = \{x = [x_1, x_2]^T \in X : |x_1| \leq 0.75, |x_2| \leq 1.5\}$ . In this simple case, it is possible to compute the maximal RCIS  $\Omega_I^{\max}$  contained in  $\Omega_O$  since the succession of Eq. (4) of Theorem 10 converges after two iterations. However, the resulting set, depicted in Figure 2, is not only non-convex but it is also non-connected. Moreover, it can be easily shown that any RCIS inside  $\Omega_O$  results non-connected.

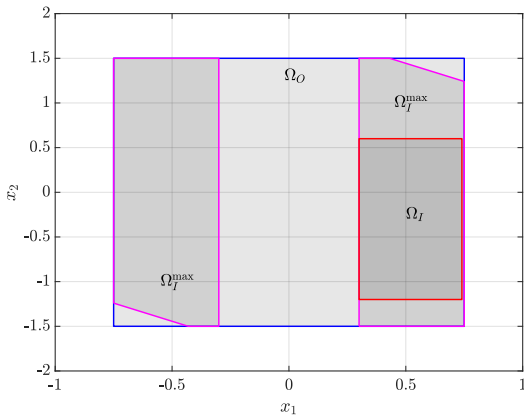


Fig. 2.  $\Omega_O$ ,  $\Omega_I$ , and the non-connected RCIS  $\Omega_I^{\max}$  for the system of Eq. (8).

In spite of the fact that any RCIS is non-connected, a very simple control inner set can be found. For instance, taking  $\Omega_I = \{x = [x_1, x_2]^T \in X : 0.3 \leq x_1 \leq$

$0.74, -1.2 \leq x_2 \leq 0.6\}$ , the succession of Eq. (5) in Theorem 13 shows that it is indeed a 4-step control inner set. The inner and the outer sets are also shown in Figure 2 and they are clearly simpler than the non-convex RCIS.

In order to show the usage of these sets in the context of MPC, we designed a very simple controller, where the input trajectories within the prediction horizon (of length  $H = N = 4$ ) were restricted to satisfy the constraints of Lemma 15 before reaching  $\Omega_I$ . This way, recursive feasibility was ensured. After  $\Omega_I$  was reached, the constraints were those of Lemma 16 ensuring that the trajectories never leave the outer region. Besides verifying the constraints, the input trajectories were chosen such that the cost function

$$J(\hat{\mathbf{u}}(k)) = \sum_{i=1}^H \|\hat{x}(k+i|k)\|_2 \quad (9)$$

is minimized. Here, the feasible input sequences are written as  $\hat{\mathbf{u}}(k) = [\hat{u}(k|k), \hat{u}(k+1|k), \dots, \hat{u}(k+N-1|k)]$ , and the predicted states are computed as

$$\hat{x}(k+i+1|k) = A\hat{x}(k+i|k) + B\hat{u}(k+i|k)$$

for  $i \in \mathbb{Z}_{0:N-1}$  with  $\hat{x}(k|k) = x(k)$ . Then, the input at time  $k$  is selected as  $u(k) = \hat{u}^*(k|k)$  where the sequence  $\hat{\mathbf{u}}^*(k)$  is the one that minimizes Eq. (9).

Figure 3 shows the state trajectories that were obtained in the simulation of the resulting scheme from different initial states that verify the aforementioned restrictions. These state trajectories verify that, as stated in Lemmas 15 and 16, recursive feasibility and practical stability are achieved using only the simple inner and outer sets rather than the complicated non-connected RCIS.

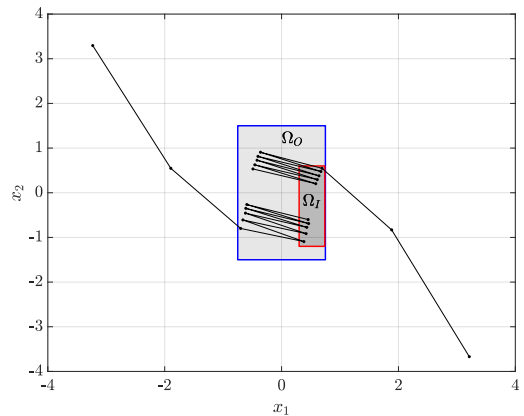


Fig. 3. MPC state trajectories.

Notice that while  $\Omega_I$  is a 4-step control inner set, the trajectories of Figure 3 enter back that set after only two steps. However, for initial states very close to the vertices of  $\Omega_I$ , there are possible disturbance trajectories



that do not allow the state to be back in that set before four steps.

## 5 Conclusions

We introduced a methodology to replace the use of complicated robust positive control invariant sets by simpler outer and inner approximations for control design purposes. We showed that given a target outer set  $\Omega_O$ , instead of computing and using a complicated RCIS contained in its interior, we may use a simpler  $N$ -step control inner set  $\Omega_I$  and still ensure that the state trajectories will not leave the target region. We also provided theoretical results that relate  $N$ -step control inner sets with the existence of robust positive control invariant sets and some more practical results that can help to construct and characterize these inner sets. In addition, we showed how these inner and outer approximations can be used in the context of model predictive controllers as a way of replacing the usual terminal restriction to a RCIS that ensures practical stability, illustrating the ideas with a numerical example.

Regarding future work, we are currently working on specific MPC schemes that exploit the presence of inner and outer sets to ensure practical stabilization and finite time convergence to the target region. In addition, we are working in the design of systematic procedures to compute inner set approximations in some particular cases (linear systems with finite input sets, for instance).

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