

GUIDO WEISS: A FEW MEMORIES OF A FRIEND AND AN INFLUENTIAL MATHEMATICIAN

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ABSTRACT. This contribution starts with an exchange between us on the way we met Guido and he influenced our mathematical lives. Then it is mainly a survey paper that illustrates this influence by describing different topics and their subsequent evolution after his seminal papers and courses. Our main thread is the notion of a space of homogeneous type. In the second section we describe how it became central in pluricomplex analysis and consider particularly the existence of weak factorization for spaces of holomorphic functions. In the last section, one revisits the construction of a basis of wavelets in a space of homogeneous type and the way it allows a Littlewood-Paley analysis.

1. INTRODUCTION

The authors of this chapter belong to two different generations of harmonic analysts. We both met Guido at a very early stage in our careers. His influence and friendship were important for both of us. We found it interesting to cross our memories of Guido. In a first section, we will tell how we met Guido and try to describe what has been his influence on our mathematical choices. This will be done through a dialog between us.

The rest of the paper will be more classical and look like a survey paper on particular points for which the influence of Guido was like a starting point for us. We will not be exhaustive on the choice of topics and will certainly not cite all the appropriate literature in the fields that we have chosen to describe in more detail. We apologize in advance for these choices, which are linked with our personal views and memories and may diverge from other choices and memories.

The main thread of this paper is the notion of space of homogeneous type that was introduced by Raphy Coifman and Guido in the early seventies. In the second section, mainly written by Aline, we will speak of its influence in pluricomplex analysis. In the third part, mainly written by Pascal, we will see how it may lead to develop all necessary tools for Littlewood-Paley theory in a very general context.

2. THE PAST, THE PAST...

Pascal. You start, of course!

Aline. For me, it begins in the academic year 1970-1971, when Guido Weiss and Raphy Coifman gave a course in Orsay, now University Paris-Saclay. Their collaboration had started a few years earlier, and at the time they were interested in developing multiplier theorems related to actions of non-commutative groups. I was in the audience, which was not so numerous: even if harmonic analysis was still one of the main topics in the maths department, there was not such a feverish atmosphere as three or four years before. Moreover, Jean-Pierre Kahane and Yves Meyer were mainly abroad. It was perhaps a chance for us. Guido and Raphy asked for volunteers to help them in writing notes, which I did in collaboration with Jean-Louis Clerc and Bernard Mischler. So I met Guido and Raphy every week. I can still see them at the blackboard, in their very different styles, Guido writing calmly with his magnificent writing long formulas for spherical harmonics, while Raphy tried to convince us of the simplicity of notions. The notes that followed, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes* [19] went much further than the initial program and, in a sense, were the first steps into a new paradigm, the world of *spaces of homogeneous type*.

Pascal. Which influence had this course on you?

Aline. The scientific life in Orsay was so intense at that time that it is difficult to sort out influences. It was my third course on real analysis given by foreigners. Eli Stein had given one in 1967-1968, just

when I started research, the course that led to his book *Singular integrals...* I had adored this course and had thought at the time that it was really what I wanted to do. But how? It seemed so difficult and the school of Chicago seemed so much in advance! At the same time I followed a course of Yves Meyer, who started to ask me questions on multipliers that led to my thesis on what is now called hypercontractivity. Working with Yves was a huge luck. At the time of the venue in Orsay of Guido and Raphy, I defended my thesis and even worked during a few months in probability theory. In the mean time, I had also attended a course of N. Riviere, who died prematurely. He talked on singular integrals related to parabolic equations. After the course of Guido and Raphy, one saw precisely how the two courses I had followed before dealt with two examples of their *Espaces de nature homogène*. One had a very clear description of the geometric objects involved. In fact their course opened the door to a lot of problems in the theory of singular integrals. Afterwards, it was obvious for me that this was the kind of mathematics I preferred.

It would be unfair to limit the influence of Guido and Raphy at Orsay that year to mathematics. From the beginning, and particularly at that time when relations between mathematicians were still somewhat formal in France, we were all stuck by their generosity and kindness, which they expressed directly to everybody.

Pascal. And what influence had this course on the harmonic analysis team in Orsay at the time?

Aline. It had a direct influence on Jean-Louis Clerc, Noël Lohoue and me, both for the content of their course itself and for the way to do mathematics, to interact between us. As I said before, relationships at work in France were then not so natural. There was still the idea one should publish alone. We started to work together, the three of us, in line with the course, trying to extend to other contexts known properties in Fourier Analysis. Inspired by their course, we read together the book of Helgason *differential geometry and symmetric manifolds*. And with Jean-Louis I studied Cesàro means for expansions in spherical harmonics. I went two months to Washington University (WU) during the spring 1972. Of course Guido and Raphy had a large influence on the harmonic analysts in Orsay, but a little later. In fact, Yves Meyer became a frequent visitor at WU and started to work with Raphy a little later, in 1974. Afterwards the interactions between WU and French harmonic analysts increased.

Again, friendship mixes with mathematics and it is impossible not to speak of this also. I visited regularly WU after 1972, even if much more rapidly. We met also frequently in Europe. To come back to the far past, I particularly remember having been at WU in 1981, at the same time as Jean-Lin Journé. We were both invited in Guido's and Barbara's summer house for a kind of country party. Conversations, canoeing on the small lake, Guido removing splinters in my hand with the same meticulousness he used to show in his classes..., there was such a charm, as in a film. Guido's jokes made a lot to create a friendly atmosphere around him. Nevertheless Guido's life had its share of mystery and suffering, starting with his childhood in Italy and the intellectual heritage of his parents. He was also a man of conviction, deeply attached to the values of human solidarity. He was a model for many of us.

Nearly twenty years later, Pascal spends two and a half academic years at WU in the context of his military service that all French men had to accomplish at that time. He is not the first student of Yves Meyer to do this: Jean-Lin Journé was there in 1981-82. Its influence on Jean-Lin may be measured when looking at the notes of the course that he gave there and which were published under the title Calderón-Zygmund Operators, Pseudo-Differential Operators and the Cauchy Integral of Calderón.

Aline. It is time for me to ask you a question. Did you know Guido before going to WU ? How was your encounter?

Pascal. No, I did not know him, nor, I must say, had read his books or articles. I was working on wavelet theory at the time, during the years 1986-87 and 1987-88. The mathematical theory of wavelets was just starting and I focused on that. For my military service, one possibility was to do it abroad in a teaching or research institution. Yves suggested that I go to WU. He contacted Guido and I recall reading his enthusiastic reply (a mailed letter at the time) dated December 31, 1987. I was just stuck by the date. Then started some exchanges with Guido (by the new email tool!) in the spring to set up things for the next fall (and for arrangements with French military office). I also read his book

with Eli Stein [74] before leaving. The encounter was made easy in all means by Guido's welcoming help. With common interests in tennis and bird-watching, it became a strong friendship.

Aline. How was the scientific life in WU at the time? Were there seminars, for instance? How did you interact with others?

Pascal. First, the mathematics department was a highly renowned place for harmonic analysis. Raphy was long gone, but Al Baernstein, Björn Dalhberg, Steven Krantz, Richard Rochberg, Mitch Taibleson were very active members. This department also attracted many doctoral students from abroad (Spain, Italy, Argentina, China...) and post-docs, some became good friends (Estella and Rodolfo Torres, Carlo Morpurgo, Marco Peloso, Maria-Jesús Carro...) and became influential mathematicians back home after having benefited from the formidable and stimulating scientific atmosphere. I was the only French person (the French system did not promote much foreign post-docs in those days). Guido wanted me to learn new maths so he proposed to Maria-Jesús, who had done her thesis on interpolation theory, and me to work on some problems on multipliers around the Riesz means, transference and other things. This led to two articles. The activity in the department was intense: regular analysis seminar, colloquiums, lots of master courses, and meetings in Guido's office. I still have the notes of the course of Guido on transference, based on his work with Raphy. All details were concisely given on the board: "too many details" was I thinking, but I realize now that this allowed the topic to profoundly print in my mind. He was always encouraging people so as to make them available for more (integrating the f_+ according to the words of A. Zygmund). I recall he brought me to visit Raphy at Yale, and this visit had tremendous impact for me, as an example of how curiosity drives mathematics. After my thesis, I was trying to understand (solve, why not?) the Kato conjecture in any dimension using wavelet methods as Ph. Tchamitchian had just reproved nicely the strongly linked $T(b)$ Theorem of David-Journé-Semmes via regular adapted wavelets [78]. When wandering in the Yale math department, I found a table with help yourself free printed (the pdf's did not yet exist and some references could be hard to find) articles that people were giving away. I discovered the article of McIntosh on bounded holomorphic calculus from the Proceedings of the Center for Mathematical Analysis of the Australian National University and also the article of Jacques-Louis Lions pursuing the interpolation theory of Kato for maximal accretive operators, which inspired me for later research.

Aline. You have a paper with Guido on Wilson bases. How was it?

Pascal. During my stay at WU, Guido was curiously NOT interested to work on wavelets, despite my attempts to suggest problems, like the one which I solved later showing that a mild sufficient condition on the wavelet generating an orthonormal basis suffices to conclude it arises from a multiresolution analysis, introducing at the same time a special series whose values are the dimensions of certain vector spaces, that was next baptised by Guido as the "dimension function" and which became a central tool as beautifully explained in Guido's reference book with Eugenio Hernandez [45]. That trip to Yale was when a new interest came in Guido's mind. Raphy explained to us the concept of local Fourier bases and Guido wanted to understand the calculations behind. We started to work out the details, trying in Guido's style to make them as conceptually simple and elegant as possible. The observation that these bases and the Meyer orthonormal wavelet basis were built from the same tricks fascinated him. Victor Wickerhauser joined the discussion and the article was ready for publication [4]. This is how he plunged into the mathematical theory of wavelets, trying to exhibit its finest structures and as always in the most simple and accessible terms. This was when I left WU at the end of 1990...

Pascal. You also have papers with Guido in the nineties. Can you say something on them?

Aline. As you said, Guido started to work on the theory of wavelets in the nineties. Just before, he was particularly interested by characterizations coming from Littlewood-Paley theory. I like very much his notes with Frazier and Jawerth in the CBMS Regional Conference Series (1991), [31]. He had all reasons to be fascinated by wavelets and had questions related to constructions. Yes, we wrote two papers, the first one [7] on band-limited wavelets. Our joint work was pursued by Gustavo Garrigós who was then his PhD student. Gustavo came later to Orléans as a post-doc, which was really great for me. But then Gustavo and I worked on Bergman spaces in tube domains, where of course we used

a lot his book with Eli Stein and the properties of Hardy spaces in tube domains given there.

Aline. From your point of view, what is the mathematical legacy of Guido?

Pascal. Of course, the discoveries he made all along his career, some that we are going to elaborate on in this article afterwards, which paved the way to create a strong school in harmonic analysis and linked topics: interpolation, singular integrals, Hardy spaces, spaces of homogeneous type... In all of these, there was always the will to leave no black holes in his way of writing, going into every details even the simplest ones, to make the reading accessible. In that respect, he was particularly proud of the Chauvenet prize he obtained in 1967 [80], the highest American award for mathematical expository writing. For example, he told me that for his book with Eli Stein, the first to propose a synthetic exposition of the extension of harmonic analysis from one to several Euclidean dimensions that occurred in the sixties, he thought at length about notation avoiding as much as possible using coordinates, to present the computations in the most conceptual manner (and with almost no typos). This notation is still accurate as of today. The legacy is then clear. He posed the solid bricks on which many new results can be elaborated. Still, despite there are now many references on each of these topics, my first reaction when I look for a reference or a proof is to browse his masterpieces as I am sure to find what I need (and sometimes understand some new points that I left aside on earlier reading). When I write research articles, I always have in mind the words of Guido: try to stay simple and reader friendly.

Pascal. Same question.

Aline. I will only speak of my own heritage. Of course I mention first his books, primarily his book with Eli Stein and the Notes of Orsay, which played different roles in my personal Pantheon. I had the impression to have the Notes in mind, just as they were engraved during the courses of Guido and Raphy, while I came back to the book with Eli Stein to read it in all details each time I gave a course myself. The book with Eugenio reaches the same degree of perfection. May be you find elsewhere a better source for intuition, for example in Yves's books, but if you want to understand all details, the book of Eugenio and Guido makes the job remarkably. These are three books that I keep on hand in my library. Then there are the works of Guido that inspired me for long. I would mention first the paper in Annals with Raphy and Richard Rochberg on commutators and weak factorization. I have the impression to have turned around this paper part of my life. We will come back to this later. I would like also to mention some parts of his work that surprised me when I heard him speak of them and that I regret not to have understood more deeply, not to have come back to them later. You know, this kind of sensation one has that there, there is something I do not understand enough, there is some new music with which I would like to familiarize myself. For example, I recall how Guido was excited with Block spaces, that is, spaces that are defined from an atomic decomposition, but with atoms that are not of mean 0, and for which one has almost everywhere convergence results, for instance. Of course, the subject has been deepened afterwards, in particular by Fernando Soria. I have nevertheless the feeling I should have tried to understand better what was around. The same with interpolation. Being exhaustive is impossible. But I will try to show, in the next section, how the work of Guido influenced mine. And of course, as you said, there is a lot more in the legacy of Guido than a list of results. There is a way to do mathematics, to write and to talk, and to do much more than only mathematics, there is a way to be curious of life and others, there is a lot.

3. ORSAY'S COURSE AND EXCURSION INSIDE COMPLEX ANALYSIS

3.1. The Notes: a new paradigm, the one of space of homogeneous type. One of the main new notions of the course was the notion of *space of homogeneous type*. For X a topological space X , one first calls *pseudo-distance* (or *quasi-distance depending on the authors*) a function $\rho : X \times X \mapsto [0, \infty)$ such that there exists $K \geq 1$ such that, for all x, y, z in X ,

- (1) $\rho(x, y) = 0 \iff x = y$
- (2) $\rho(x, y) = \rho(y, x)$
- (3) $\rho(x, z) \leq K(\rho(x, y) + \rho(y, z))$.

The pseudo-ball (or quasi-ball, or simply ball) $B(x, r)$ with center x and radius r is the set of points $y \in X$ such that $\rho(x, y) < r$.

We say that X is a *space of homogeneous type* if X is endowed by a pseudo-distance ρ and by a Borel measure μ if

- (1) the balls B centered at x constitute a basis of neighborhoods of x
- (2) the measure μ is doubling, that is, there exists a constant A such that, for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq A\mu(B(x, r)).$$

These definitions are sufficient to be able to define the maximal function of $f \in L^1(d\mu)$ by

$$Mf(x) := \sup_{r>0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y)| d\mu(y),$$

to prove the maximal theorem in this context as a consequence of a covering lemma of Vitali type, to develop as well Whitney's covering lemmas and to make it possible to write a Calderón-Zygmund decomposition of a function $f \in L^1(d\mu)$. As in the theory of singular integrals in \mathbb{R}^n , it is then possible to prove L^p estimates for singular integrals once one assumes that one already knows them for a particular value $p_0 > 1$. To go ahead and be able to have theorems $T(1)$ or $T(b)$ asks for new tools, and this is what we will tempt to describe later.

Calderón-Zygmund theory was developed at the same time by Coifman and de Guzman in [15]. This last one studied systematically covering lemmas in relation with differentiation [36], which is always a living field of study (see for instance [71]).

It should be emphasized that, in the seventies, one had in mind some generalizations of Calderón-Zygmund theory, but not all. For example, the idea that this could be used for discrete structures appeared much later. Nevertheless, there is already this idea in the Notes that the doubling property of the measure μ can be replaced by the following one, which is weaker and is expressed only as a geometric property (named nowadays as the geometric doubling property):

- 2'. There exists a constant N such that, for all $x \in X$ and $r > 0$, the ball $B(x, 2r)$ is covered by at most N balls of radius r .

This will be seen as the right point of view for some constructions later on in this text.

There are also emerging notions in the course, which have been central in the further development of the theory. In particular, the definition of an atom of the Hardy space H^1 and the atomic decomposition of H^1 have only been given by Coifman a little later [14]. But the notion of atoms can already be found in the Notes (page 89 for instance). It is considered there for Riemannian symmetric spaces, and only from examples that are built with approximate identities. The property of singular integrals to transport them into molecules is already present. All this program is developed, this time with precise definitions and statements, in the two fundamental texts [20, 76].

In the Notes, one of the first examples of space of homogeneous type which is given is the unit sphere in \mathbb{C}^n endowed with the Euclidean measure $d\sigma$ and the distance $d_b(z, \zeta) := |1 - \langle z, \zeta \rangle|$. We note \mathbb{B}_n the unit ball in \mathbb{C}^n , $\partial\mathbb{B}^n$ its boundary, that is, the unit sphere and $\langle \cdot, \cdot \rangle$ the Hermitian inner product on \mathbb{C}^n . The distance d_b (the index b stands for the boundary) is clearly invariant under the action of the group $SU(n)$ and the distance from the point $\mathbf{1} = (1, 0, \dots, 0)$ satisfies the property

$$d(\mathbf{1}, z) \simeq |\operatorname{Im} z_1| + |z'|^2, \quad z = (z_1, z'),$$

meaning that the quotient of the two quantities lies between two uniform constants in a neighborhood of $\mathbf{1}$. Moreover, $\sigma(B(z, r)) \simeq r^n$ and

$$|z - \zeta|^2 \lesssim d_b(z, \zeta) \lesssim |z - \zeta|.$$

This metric is usually called as the Korányi's metric. It had in particular already been used by A. Korányi for generalizing Fatou Theorem of non tangential convergence for harmonic functions in the unit disc to harmonic functions in \mathbb{B}_n with respect to the Laplace-Beltrami operator for the Bergman metric [51]. Indeed, in the same way that the first question for spaces of homogeneous nature was about generalizations of the Lebesgue Differentiation theorem, the first question for holomorphic functions was about generalizations of Fatou Theorem.

At this point, let us recall definitions before going further. For Ω a smooth bounded domain in \mathbb{C}^n and $0 < p < \infty$, let $L^p(\Omega)$ denote the Lebesgue space with respect to the Lebesgue measure $dV(z)$ and $A^p(\Omega)$ be the corresponding Bergman space, that is, the closed subspace of $L^p(\Omega)$ consisting of

holomorphic functions. For $0 < p < \infty$, let $L^p(\partial\Omega)$ denote the Lebesgue space on $\partial\Omega$ with respect to the induced surface measure $d\sigma$ and $H^p(\Omega)$ the Hardy space of holomorphic functions on Ω , with norm given by

$$\|f\|_{H^p}^p := \sup_{0 < \varepsilon < \varepsilon_0} \int_{\Omega} |f(w - \varepsilon\nu_w)|^p d\sigma(w),$$

where ν_w is the unit exterior normal vector at the point w , with ν_w seen as a vector in \mathbb{C}^n . The question of generalizing non tangential convergence to higher dimension was treated in the small book of E. Stein [73] (1972), which has been a major source of inspiration for its adaptation of potential theory to this context and for its list of open questions. For such smooth domains, functions in $H^p(\partial\Omega)$ have admissible limits a.e.: the admissible regions, roughly speaking, are defined in the same way as non tangential regions in the unit disc, except that the ordinary distance on the circle is replaced by the Korányi by the pseudo-distance on the boundary $\partial\Omega$, that is,

$$d_b(z, \zeta) := |\langle \nu_z, z - \zeta \rangle| + |z - \zeta|^2.$$

It is not symmetric, but $d_b(z, \zeta) \simeq d_b(\zeta, z)$ and can be symmetrized. These admissible regions are optimal for strictly pseudo-convex domains (that is, domains that are locally diffeomorphic to strictly convex domains), but not for weakly pseudo-convex domains in which there may be a larger swelling of admissible regions in complex tangential directions and a need for a definition that takes this into account. See for instance [65]. Because of admissible limits one may consider $H^p(\Omega)$ as a subspace of $L^p(\partial\Omega)$.

The next question concerns Bergman and Szegő projections, which we denote by P_B and P_S and which are, respectively, the orthogonal projection from $L^2(\Omega)$ to $A^2(\Omega)$ and the orthogonal projection from $L^2(\partial\Omega)$ to $H^2(\Omega)$ (identified with a subspace of $L^2(\partial\Omega)$). They are respectively given by the Bergman kernel $B_\Omega(z, \zeta)$ and the Szegő kernel $S_\Omega(z, \zeta)$. For the unit ball they are given by

$$S(z, \zeta) = \frac{c_n}{(1 - \langle z, \zeta \rangle)^n}, \quad B(z, \zeta) = \frac{c_n}{(1 - \langle z, \zeta \rangle)^{n+1}}.$$

Korányi and Vági, independently of Coifman and Weiss, developed around the same time a theory of singular integrals on homogeneous spaces, which led to the L^p inequalities for the Szegő projection of the unit ball [52]. This one is now seen as an immediate example of singular integral on $\partial\mathbb{B}^n$. Indeed, it is elementary to see that

$$|S(\mathbf{1}, z) - S(\mathbf{1}, \zeta)| \lesssim \frac{d_b(z, \zeta)^{1/2}}{|1 - \mathbf{1} \cdot \zeta|^{n+1}},$$

which, by invariance by the action of $SU(n)$, gives the required inequality for a singular integral on the space of homogeneous type given by $\partial\mathbb{B}^n$, the Korányi distance and the Euclidean measure.

In order to obtain L^p estimates for the Bergman projection, one may also consider it as a singular integral for the distance that is given by the ordinary distance in the radial direction and by d_b on the boundary $\partial\mathbb{B}_n$. But $L^p(\mathbb{B}_n)$ estimates can be obtained in a much simpler way by using Schur's lemma. These kinds of estimates have been first developed by Forelli and Rudin (see [28]).

This was the beginning of a long story, which we evoke now. From 1970, a considerable work has been done on Hardy and Bergman spaces as well as Szegő and Bergman projections in smooth bounded pseudo-convex domains of \mathbb{C}^n . The link between the adequate pseudo-distance at the boundary and the two kernels, which is obvious for the unit ball, asks for long studies in general. Estimates on the Bergman and Szegő kernels have been developed in the context of strictly pseudo-convex domains, then pseudo-convex domains of finite type in \mathbb{C}^2 , then convex domains of finite type by Fefferman [26], Nagel, Rosay, Stein and Wainger [64], McNeal and Stein [56] and many others.

The work of Henkin [44] on strictly pseudo-convex domains play a fundamental role for having explicit representation formulas. They lead to the construction of support functions $H \in C^\infty(\Omega \times U)$ with U a neighborhood of the boundary, such that $H(\cdot, \zeta)$ is holomorphic on Ω for all $\zeta \in U$ and

$$C^{-1}d(z, \zeta) \leq |H(z, \zeta)| \leq Cd(z, \zeta).$$

Here d is defined in a neighborhood of the boundary by

$$(3.2) \quad d(z, \zeta) = \delta(z) + \delta(\zeta) + d_b(\pi(z), \pi(\zeta))$$

where $\delta(z)$ is the distance of z to $\partial\Omega$ and $\pi(z)$ is the closest point of $\partial\Omega$, which is well-defined in U for U small enough and d_b is the adequate distance on the boundary. When Ω is the unit ball, one can take simply $H(z, \zeta) = 1 - \langle z, \zeta \rangle$. It is well adapted to Szegő or Bergman spaces, which have only

singularities at the boundary. For convex domains of finite type (with the adequate pseudo-distance on the boundary), such a construction has been given by Diederich and Fornæss [25]. We will use these support functions later on. Unlike what happens in the unit ball, they are not found in connection with the Bergman kernel.

3.3. Weak factorization, commutators and Hankel operators. In this subsection we will concentrate on one of the other fundamental papers of Guido, namely the one on commutators [18]. It was written in 1975 with Richard Rochberg and Raphy and, again, had a large influence on problems that have been considered later. Some of them have been solved recently, some are still open. It is among the top cited papers of Guido and has been generalized in many directly during the nearly passed fifty years. One can only find here a personal point of view, which leaves many developments aside.

The first question that is treated in this paper is the characterization of $BMO(\mathbb{R}^n)$ as the class of functions b for which the commutator $[M_b, R_j]$ are bounded in $L^2(\mathbb{R}^n)$. Here M_b stands for the multiplication by b and R_j is the j -th Riesz transform, that is, the convolution by the distribution $c_n p.v. \frac{x_j}{|x|^{n+1}}$. This has been fundamental and, almost twenty years later, revealed itself to be one of the ways to understand the div-curl lemma in relation with compensated compactness [16].

We will consider in more detail the second part of the article, which deals with holomorphic functions and factorization. Let us recall that in dimension 1 every function $f \in H^p(\mathbb{B}_1)$ may be written as the product gh , with $g \in H^{p_1}(\mathbb{B}_1)$, $h \in H^{p_2}(\mathbb{B}_1)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. This is false in higher dimension (see [34]). In this second part one finds for the first time the idea to replace factorization by weak factorization. Namely, they prove that any function $f \in H^1(\mathbb{B}_n)$ can be written as

$$f = \sum_{j=1}^{\infty} g_j h_j, \quad \sum_j \|g_j\|_{H^p} \|g_j\|_{H^{p'}} \simeq \|f\|_{H^1}.$$

Their proof uses atomic decomposition of Hardy spaces $H^1(\mathbb{B}_n)$. Namely, a holomorphic atom A is the Szegő projection of an atom a , that is, $A = P_S a$ with a a bounded function of mean zero, which is supported in a pseudo-ball $B \subset \mathbb{B}_n$, and such that $\|a\|_{\infty} \leq |B|^{-1}$. They prove that H^1 has an atomic decomposition, that is, $f \in H^1(\mathbb{B}_n)$ can be written in terms of atoms a_j related to balls $B_j = B(w_j, r_j)$

$$f = c + \sum_{j=1}^{\infty} \lambda_j a_j, \quad \sum_j |\lambda_j| \simeq \|f\|_{H^1}.$$

It is then sufficient to factorize an atom. At this point their proof may be in some way simplified and can be seen as an easy consequence of the existence of a support function that satisfies (3.2), which makes it easier to find generalizations to other domains. One uses an idea that emerged later on: it is possible to ask for the atoms to have more moments that vanish, so that $|P_S a_j|$, which is not supported in B_j , decreases rapidly outside B_j . Then, one just takes $g_j = H(z, \zeta_j)^{-l}$, where ζ_j is at distance r_j of $\partial\mathbb{B}_n$ and $\pi(z_j) = w_j$. So for $l > n$ the norm of g_j in H^p is easily estimated, as well as the one of $H(z, \zeta_j)^l P a$ in $H^{p'}$ when the atoms has sufficiently vanishing moments.

This kind of proof extends easily to other domains, see [9]. It allows one to generalize weak factorization to H^p for $p < 1$ but does not work when there is no atomic decomposition of H^p , that is, for $p > 1$, even in the unit ball \mathbb{B}^n . Up to our knowledge, the existence of weak factorization of H^p is still an open problem in dimension $n > 1$:

For $p > 1$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, is there weak factorization for $H^p(\mathbb{B}^n)$ with products of functions in $H^{p_1}(\mathbb{B}^n)$ and $H^{p_2}(\mathbb{B}^n)$ when $n \geq 2$?

Another way to prove weak factorization consists in proving boundedness of the Hankel operator, which is the anti-linear operator on H^p defined by $H_b(f) = P_S(b\bar{f})$. More precisely, the fact that H^1 may be weakly factorized with products of functions in H^p and $H^{p'}$ is equivalent to the fact that H_b is bounded on H^p if and only if b belongs to the dual of H^1 . The same considerations prove that the open problem given above is equivalent to the fact that the Hankel operator H_b maps H^{p_1} into H^{p_2} if and only if b belongs to $H^{p'}$.

In \mathbb{R}^n , such characterizations have been proved recently by Hytönen [47] for commutators $[M_b, R_j]$. It is straightforward that these commutators map L^p into L^q for $b \in L^r$, with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. The challenge was to prove the converse, which is quite intricate. Unfortunately, this result, even if proved for the Szegő projection on $\partial\mathbb{B}_n$, is not sufficient to find weak factorization for $H^p(\mathbb{B}_n)$ for $p > 1$.

Weak factorization for the Hardy space H^1 has also been studied in the polydisc, where there is no atomic decomposition. It was announced in a celebrated paper of Ferguson and Lacey but at this moment there is a gap in their proof as proved in [46]. It is quite curious that the weak characterization of H^1 , if true, implies, again on the bidisc, a weak factorization of H^p by products of functions in H^{2p} when $p < 3/2$, see [10].

The paper of Coifman, Rochberg and Weiss finishes with the weak factorization of Bergman spaces. This is by now also completely classical and has been generalized in many contexts. The classical method, now, (see for instance [81]) goes through the atomic decomposition that has been developed by Coifman and Rochberg [17] a little later: for $p > 0$ given, every function in $f \in A^p(\mathbb{B}_n)$ may be written as

$$f(z) = \sum_j \lambda_j \frac{\omega_j (1 - |w_j|)^N}{(1 - \langle z, w_j \rangle)^{n+N+1}}, \quad \sum |\lambda_j|^p \simeq \|f\|_{A^p}^p,$$

where the points z_j do not depend of the function f but constitute what is called an η -net: they are at distance at most η for the Bergman distance, but balls of radius $\eta/2$ centered at those points are disjoint while balls of radius $2r$ are almost disjoint. The quantities that appear here, $\frac{\omega_j (1 - |w_j|)^N}{(1 - \langle z, w_j \rangle)^{n+N+1}}$, are the values at (z, w_j) of the Bergman kernel when the Lebesgue measure dV is replaced by the weighted measure $(1 - |z|)^N dV(z)$. The constant ω_j is a normalization factor so that its A^p -norm is 1. One has not only the atomic decomposition, but the weak factorization of Bergman spaces for $p \leq 1$. Generalizations are valid in the same context as for Hardy spaces, as well as estimates for related Hankel operators, now defined as $H_b(f) = P_B(b\bar{f})$. But one can go much further in many directions.

First, the fact that such decompositions are valid for $p > 1$ and a tricky use of Rademacher coefficients has allowed Pau and Zhao [70] to get estimates with loss for the Hankel operator, and so to obtain weak factorization of Bergman spaces of $A^p(\mathbb{B}^n)$ for all values of p .

Secondly, the validity of atomic decomposition and weak factorization of Bergman spaces can be proved far beyond the contexts for which one can conclude for Hardy spaces, as outlined in [17]: one can find such decompositions in tube domains over symmetric cones, for instance, but with restrictions that are due to the lack of validity of L^p inequalities for the Bergman projection for all $p > 1$. In particular, if $\Omega = \mathbb{R}^n + i\Gamma$, where Γ is a symmetric cone (think of the forward light cone $y_0 > \sqrt{y_1^2 + \dots + y_n^2}$), results related to the Bergman space may be adapted (see [66]), but with new constraints that are linked to the validity of L^p inequalities for the Bergman projection. The constraints for these L^p inequalities given in [6] are the best possible in the case of the forward light cone as a consequence of the decoupling inequalities of Bourgain and Demeter [11].

We are then far from the geometry of spaces of homogeneous type and singular integrals, but all this finds its origin in the seminal papers written from the years 1970 by Guido and those researchers that gathered around him.

4. LITTLEWOOD-PALEY DECOMPOSITIONS AND WAVELETS

4.1. The context. The tools based on covering arguments that only need balls allow, in principle, to forget about algebraic contents to focus more on geometry or distributions of points. Still, familiar tool boxes such as convolution and the Fourier transform need to be replaced by something else.

First one needs approximations to the identity in relation to scales replacing usual mollifiers. Second to develop a rich function space theory and the action of operators on them (such as Calderón-Zygmund operators), one needs Littlewood-Paley decompositions. It took some time to develop those tools in the optimal generality offered by the definition of a space of homogeneous type, that is, without extra hypotheses. The aim of this section is to explain the evolution of the ideas and a solution. We use material from Auscher-Hytönen [3] and also the introduction of Han-Li-Ward [37]. The latter reference contains a rich bibliography as well.

Recall that the definition of a space of homogeneous type involves a set X equipped with a quasi-distance d and a positive doubling Borel measure μ (Borel measure is defined in several ways in the literature: either it is a measure on the Borel σ -algebra and one works only with Borel measurable functions or it is an outer measure for which Borel sets are measurable (in the sense of Carathéodory): in the latter case, we add the regularity property that every measurable set is contained in a Borel set with same measure). As said, extra hypotheses had to be imposed in the proofs. However, it seemed that this would not limit the range of applicability thanks to an article of Macias-Segovia that

describes the structure of spaces of homogeneous type [55]. This explains why removing these extra conditions remained unexplored for some time.

A popular assumption is that the volume of a ball $B(x, r) = \{y \in X : d(x, y) < r\}$ is comparable to a power r^α of its radius, uniformly in its center (this is called the Ahlfors-David regularity property for the measure). Another one is that the volume of a ball $B(x, Cr)$ exceeds $(1 + \varepsilon)$ times that the volume of $B(x, r)$ for some constants $1 < C < \infty$, $\varepsilon > 0$ (this is called the reverse doubling property). One or the other are satisfied in many cases. Nevertheless, if one wants one or the other for all $0 < r$ less than a fraction of the diameter of X , this excludes atomic measures, and for example \mathbb{Z} equipped with the induced absolute value and counting measure. Considering that Hardy-Littlewood maximal theorem's original setup was on p -summable sequences, this would be paradoxical not to have a theory covering all situations.

The balls $B(x, r)$ define the topology on X but they might not be open sets nor even Borel sets when d is not a distance. This can be remedied by replacing them by their interiors in the definition of doubling. Alternately, one result of Macias-Segovia [55] says one can change the quasi-distance to a metrically equivalent one (actually, a power of a genuine distance) for which the balls are open. So this is not a major issue as this only affects the quasi-distance and the doubling constants.

A more serious issue is again due to the difference between the metric case and the quasi-metric case: a distance is always Lipschitz-continuous (with respect to itself) from the triangular inequality; it is not necessarily the case of a quasi-distance which might not even be Hölder-continuous. Another result of Macias-Segovia [55] says that one can change the balls so that it becomes true and even the measure becomes Ahlfors-David regular. This is why most articles afterwards claimed that their results were proved in a "general" space of homogeneous type. But this is not quite correct. First, this change preserves the topology but it is not metrically equivalent. Next, it changes the classes of Hölder-continuous functions and of Calderón-Zygmund kernels. An example in [37] shows that doing this change might not be desirable if the quasi-distance has a special interpretation in terms of an explicit operator as one may lose the connection. In complex analysis as above, the quasi-distance is defined according to the geometric context and one does not want to change it. This is, however, in this context (with a new quasi-distance from [55]) that the proof of the T(b) Theorem by David-Journé-Semmes was given [22]. They were able to implement an unpublished method of Coifman to produce approximations to the identity and Littlewood-Paley decompositions. More precisely, the approximations to the identities are families of operators $(S_k)_{k \in \mathbb{Z}}$ ($k \geq k_0$ if X is bounded) with kernels $S_k(x, y)$ enjoying the following properties for some $C < \infty$, $\varepsilon > 0$ and all k and points

$$\begin{aligned} S_k(x, y) &= 0 \text{ if } d'(x, y) \geq C2^{-k}, \text{ and } |S_k(x, y)| \leq C2^k \\ |S_k(x, y) - S_k(x, y')| &\leq C2^{k(1+\varepsilon)} d'(y, y')^\varepsilon \\ |S_k(x, y) - S_k(x', y)| &\leq C2^{k(1+\varepsilon)} d'(x, x')^\varepsilon \\ \int_X S_k(x, y) d\mu(y) &= 1 = \int_X S_k(x, y) d\mu(x) \end{aligned}$$

Here the quasi-distance d' is the one provided by the Macias-Segovia result. If $D_k := S_{k+1} - S_k$, then for all $1 < p < \infty$ and $f \in L^p(X, \mu)$ ($k \geq k_0$ and $\int f d\mu = 0$ when X is bounded)

$$C^{-1} \|f\|_p \leq \left\| \left(\sum_k |D_k f|^2 \right)^{1/2} \right\|_p \leq C \|f\|_p.$$

Hence the map $f \mapsto (D_k f)$ acts as a Littlewood-Paley decomposition. From sometime on, these methods have been streamlined and generalized (see [39, 24]) in the same context.

The next development has been to work on spaces with the reverse doubling property. This case is of much interest for example on Lie groups where the volumes of balls may obey two different power laws whether we assume small or large radii (and the exponents are often called local and global dimensions respectively). This covers also the case of some complete Riemannian manifolds. The reverse doubling condition is indeed a quantitative expression of connectedness of the space: annuli between two co-centered balls can never be empty and non-emptiness self improves to having mass at the correct scale. There, the technology of Coifman can be developed nicely (see Han-Müller-Yang, [38]) in a sense that estimates on the kernel family $S_k(x, y)$ go through with slight modification in their formulations. A decisive inequality for the treatment of singular integrals in reverse doubling spaces

is that

$$(4.2) \quad \sum_{k \geq 0} (\mu(B(x, C^k r))^{-1} \lesssim (\mu(B(x, r))^{-1}$$

as the series is controlled in a geometric manner. Again, such conditions exclude spaces of homogeneous types of "discrete" nature and more. By the early 2010, about 40 years after the seminal definition of a space of homogeneous type, although the covering arguments and maximal theorem worked beautifully, it was still not known how to develop Littlewood-Paley theory in full generality (again, despite what most authors wrote) and thus the function spaces and operator theory of singular integrals that go with. We shall see that the solution came from the need to work in absence of the doubling condition. Let us explain how.

In parallel to the development of "regular" decompositions, dyadic analysis was made possible thanks to the introduction of "dyadic cubes" on spaces of homogeneous type, first for Ahlfors-David regular measures by G. David [21] and then by M. Christ [13] in full generality. The idea is that containment into a unique parent dyadic cube and having a bounded number of children dyadic subcubes can be obtained from a carefully defined partial order on a family of selected points with conditions only involving the quasi-distance. These points become the "centers" of the cubes, which are sets contained in balls and containing balls with those centers and radii comparable to their diameters. This clever construction allowed a local formulation of the T(b) theorem that was best designed toward an analytic capacity result in mind.

In \mathbb{R}^n , the dyadic cubes are linked to the Haar wavelets which form an orthonormal basis of $L^2(dx)$ where dx is Lebesgue measure. The construction of Christ naturally leads to an orthonormal basis with respect to the doubling measure at hand, of course with non regular functions. The beautiful theory of orthonormal wavelets on \mathbb{R}^n , launched by Meyer [59] after the first unnoticed discoveries of Strömberg [75], allows one to build orthonormal bases of localized and regular functions: not only they have decay away from a dyadic cube (or even have compact support) but also they enjoy Hölder regularity. Some can even be infinitely smooth. Explicitly or implicitly, Fourier transform is used. In fact, translation invariance and dilation invariance play a crucial role and indeed regular wavelet bases can be built on stratified nilpotent Lie groups [54].

Coming back to Haar wavelets, it can be shown they provide unconditional bases of $L^p(dx)$ when $1 < p < \infty$. The situation is different when $p = 1$ and $p = \infty$. When $p = 1$, they are unconditional families in the real Hardy space $H^1(\mathbb{R}^n)$ but span a strict subspace called the dyadic Hardy space $H_{dyad}^1(\mathbb{R}^n)$. It had become a challenge to prove that $H^1(\mathbb{R}^n)$ also admits an unconditional basis. B. Maurey [58] solved the question by showing the two spaces are isomorphic, but without providing an explicit basis. Strömberg's orthonormal basis of spline functions (that turn to fit the wavelet theory and were rediscovered by Battle and Lemarié [5, 53]) precisely yields an explicit construction. In the dual range, $BMO(\mathbb{R}^n)$, the dual of $H^1(\mathbb{R}^n)$ from the theorem of C. Fefferman [27], was known to be strictly contained in its dyadic version $BMO_{dyad}(\mathbb{R}^n)$. It has to do with the fact that Euclidean distance $|x - y|$ and the distance induced by the dyadic cubes (the diameter of the smallest dyadic cube containing x and y) are not equivalent. In his celebrated book ([33], p.417), J. Garnett implicitly mentioned that classical $BMO(\mathbb{R})$ is the intersection of three of shifted copies of $BMO_{dyad}(\mathbb{R})$. By shifted, it is meant that one considers dyadic grids $2^k Q_a + \ell$, $k \in \mathbb{Z}, \ell \in \mathbb{Z}$, where Q_a is the shifted interval $a + [0, 1[$ by $a \in [0, 1[$. More than an amusing observation, this shifting property can be thought as a random process among dyadic grids and this became a key tool to develop the singular integral theory on subsets X of Euclidean space equipped with a Radon measure satisfying the power law upper estimate $\mu(B(x, r)) \leq Cr^\beta$ for all $x \in X$ and $r > 0$ as evidenced in the work of Nazarov-Treil-Volberg, [68]. The construction of random families of dyadic grids on other contexts would then allow to develop singular integral theory there. But randomness cannot come from translation and dilation invariance anymore. A first construction of T. Hytönen and H. Martikainen on quasi-metric sets with the geometric doubling property [49] was designed to prove a T(b) theorem on such sets equipped with a measure satisfying the upper doubling condition, that is $\mu(B(x, r)) \leq \lambda(x, r)$ where $\lambda(x, 2r) \leq C\lambda(x, r)$ (a condition that covers both the doubling condition for the measure itself and the power law upper estimate). Another possible construction of random grids was proposed in [67] and T. Hytönen and A. Kairema [48] streamlined the construction in [49].

4.3. A regular wavelet basis after all. As mentioned, while dyadic decompositions occurred in full generality, neither Littlewood-Paley decomposition, nor regular wavelet bases were available in 2011

in a "general" space of homogeneous spaces. The existence of an orthonormal basis of regular wavelets was too much of a dream to have been conjectured anywhere. In [24], which was one of the most advanced text on the topic at this time, it was even written that they are "out of reach" and frames (a substitute notion where redundancy is allowed) were obtained but using again the Macias-Segovia trick. Still, when Tuomas Hytönen came to Orsay in 2011, I (Pascal) asked him the problem. How to overcome the lack of any algebraic structure? How even can we make sure that for a quasi-distance that is not Hölder-continuous, one can obtain profusion (dense classes) of Hölder-continuous functions? Should we restrict to the metric case? It turned out that the idea of random dyadic grids opened the door to a solution in [3] that required five steps.

- (1) The first step is the construction of appropriate random dyadic grids using **nested** collections of separated and dense subsets of X at each given scale.
- (2) The second one is the construction of **spline** functions that are Hölder-continuous of (some) order in $(0, 1)$ depending on the space.
- (3) In the third step, once the doubling measure is given, the spline functions provide a **multiresolution analysis** on $L^2(\mu)$ (without the translation and dilation requirements, of course).
- (4) In the fourth step, **construct the wavelets** by transposing Hilbertian algorithms of Y. Meyer [61, 60] for the construction of spline wavelets on \mathbb{R}^n to spaces of homogeneous type.
- (5) In the fifth step, the **localisation** of the wavelets is obtained by extending from the metric case to the quasi-metric case existing lemmas due to Demko [23] to estimate the entries of the inverse of a band matrix.

Steps 1 and 2 do not require measure information but only the geometric doubling property. The spline functions have interesting properties of interpolation and reproduction which permit to construct the wavelets through a multiresolution structure. The Hölder regularity of the wavelets is the one of the spline functions. The construction of random grids was slightly modified later by Hytönen-Tapiola [50] in the metric case to allow arbitrary Hölder exponent in $(0, 1)$.

This in the end yields the following result (gathering both [3, 50])

4.4. Theorem. *Let (X, d, μ) be any space of homogeneous type and call A_0 its quasi-triangle constant. Let $a := (1 + 2 \log_2 A_0)^{-1}$ or $a := 1$ if d is Lipschitz-continuous. There exist $\eta \in (0, 1)$ depending only on A_0 (any $\eta \in (0, 1)$ works when $A_0 = 1$), constants $\delta > 0$ and $\gamma > 0$ depending only on A_0 , and $0 < C < \infty$ depending on A_0 and the doubling constant, one has the following.*

There are non-decreasing sets \mathcal{X}^k , $k \in \mathbb{Z}$ (and $k \geq k_0$ if X is bounded) of δ^k -separated and $2A_0\delta^k$ -dense points in X , such that if $\mathcal{Y}^k := \mathcal{X}^{k+1} \setminus \mathcal{X}^k$, labelled as $\mathcal{Y}^k = \{y_\beta^k\}_\beta$, there exists an orthonormal basis of real-valued functions ψ_β^k , $k \in \mathbb{Z}$ (and $k \geq k_0$ if X is bounded), corresponding to each $y_\beta^k \in \mathcal{Y}^k$, of $L^2(\mu)$ (or the orthogonal space to constants if X is bounded) having exponential decay

$$|\psi_\beta^k(x)| \leq \frac{C}{\sqrt{\mu(B(y_\beta^k, \delta^k))}} \exp(-\gamma(\delta^{-k}d(y_\beta^k, x))^a),$$

Hölder-regularity

$$|\psi_\beta^k(x) - \psi_\beta^k(y)| \leq \frac{C}{\sqrt{\mu(B(y_\beta^k, \delta^k))}} \left(\frac{d(x, y)}{\delta^k}\right)^\eta \exp(-\gamma(\delta^{-k}d(y_\beta^k, x))^a), \quad d(x, y) \leq \delta^k,$$

and vanishing mean

$$\int_X \psi_\beta^k(x) d\mu(x) = 0, \quad k \in \mathbb{Z}, k \geq k_0, y_\beta^k \in \mathcal{Y}^k.$$

A subset S of X is η -separated if two distinct elements of S have distance at least η , it is η -dense if any point in X has distance to S not exceeding η . This theorem and its argument provide us with the desired Littlewood-Paley decomposition.

4.5. Corollary. *Fix $1 < p < \infty$. The orthogonal projections Q_k on the closed linear span W_k of the ψ_β^k form a Littlewood-Paley decomposition. More precisely, there is a constant $0 < C < \infty$ such that for all $f \in L^p(\mu)$, ($k \geq k_0$ and $\int_X f d\mu = 0$ if X is bounded)*

$$C^{-1} \|f\|_p \leq \left\| \left(\sum_{k \in \mathbb{Z}} |Q_k f|^2 \right)^{1/2} \right\|_p \leq C \|f\|_p.$$

Moreover, the family ψ_β^k (together with the indicator of X if X is bounded) forms an unconditional basis of $L^p(\mu)$ and $P_k = \sum_{j < k} Q_j$ ($j \geq k_0$ and add the orthogonal projection onto constants if X is bounded) is a family of approximation to the identity with kernels $P_k(x, y)$, symmetric in x, y , having size estimates

$$(4.6) \quad |P_k(x, y)| \leq \frac{C}{\sqrt{\mu(B(x, \delta^k))\mu(B(y, \delta^k))}} \exp(-\gamma(\delta^{-k}d(x, y))^s),$$

regularity estimates

$$|P_k(x, y) - P_k(x, y')| \leq C \left(\frac{d(y, y')}{\delta^k} \right)^\eta \left(\frac{\exp(-\gamma(\delta^{-k}d(x, y))^s)}{\sqrt{\mu(B(x, \delta^k))\mu(B(y, \delta^k))}} + \frac{\exp(-\gamma(\delta^{-k}d(x, y'))^s)}{\sqrt{\mu(B(x, \delta^k))\mu(B(y', \delta^k))}} \right)$$

for some C, γ, s and all $x, y, y' \in X$ and $k \in \mathbb{Z}$ (with $k \geq k_0$ if X is bounded). Moreover

$$\int_X P_k(x, y) d\mu(x) = 1.$$

The kernel $Q_k(x, y)$ of Q_k has similar estimates with s changed to a and the additional multiplicative exponential factor in the size estimate

$$(4.7) \quad \exp(-\gamma((\delta^{-k}d(x, \mathcal{I}^k))^a + (\delta^{-k}d(y, \mathcal{I}^k))^a)),$$

and the cancellation condition

$$\int_X Q_k(x, y) d\mu(x) = 0.$$

The exponential factor in (4.7) is the key new object quantifying the geometry of X . We shall come back to this.

The decay of wavelets is only exponential and an open question is whether one can obtain bounded support up to relaxing orthonormality to bi-orthogonality if necessary.

We shall now indicate the idea of the construction of the splines: the guiding principle is that their interpolation properties, their reproducing properties and their regularity are obtained from a probabilistic argument allowed from the axioms in the construction of random dyadic cubes.

4.8. The key ideas of the construction of the spline functions. The classical piece-wise linear splines on \mathbb{R} are generated by translation and dilation of the function

$$s(x) = x1_{(0,1]}(x) + (2-x)1_{(1,2)}(x).$$

Another formula is $s(x) = 1_{[0,1]} * 1_{[0,1]}(x)$ and writing out this convolution yields

$$s(x) = \int_0^1 1_{[0,1]}(x-u) du = \int_0^1 1_{[u, u+1]}(x) du.$$

Random dyadic intervals of sidelength 1, in the sense of Nazarov, Treil and Volberg [68, Sec. 9.1], are precisely defined by translating the standard intervals $[k, k+1)$ by a random number $u \in [0, 1)$. Thus the unit cube with left end at the origin, $[0, 1)$, is translated to $[u, u+1)$. So one can think of $s(x)$ as an average of the indicators of random dyadic intervals, or in probabilistic notation

$$s(x) = \mathbb{P}_u(x \in [u, u+1)),$$

where \mathbb{P}_u is a fancy notation for the uniform probability measure on $[0, 1[$ (i.e. the Lebesgue measure). This is the basic idea which will guide us in constructing splines for a quasi-metric space X with the geometric doubling property. We next provide some details and key points of the constructions, especially of the splines. Proofs can be consulted in [3].

The **reference dyadic points** are chosen with the additional requirement of **nestedness** compared to the earlier references [13, 48, 49]. For every $k \in \mathbb{Z}$, we choose a set of *reference dyadic points* x_α^k as follows: For $k = 0$, let $\mathcal{X}^0 := \{x_\alpha^0\}_\alpha$ be a maximal collection of 1-separated points. Inductively, for $k \in \mathbb{Z}_+$, let $\mathcal{X}^k := \{x_\alpha^k\}_\alpha \supseteq \mathcal{X}^{k-1}$ and $\mathcal{X}^{-k} := \{x_\alpha^{-k}\}_\alpha \subseteq \mathcal{X}^{-(k-1)}$ be maximal δ^k - and δ^{-k} -separated collections in X and in $\mathcal{X}^{-(k-1)}$, respectively. It is easy to show that for all $k \in \mathbb{Z}$ and $x \in X$, the reference dyadic points satisfy

$$d(x_\alpha^k, x_\beta^k) \geq \delta^k \quad (\alpha \neq \beta), \quad d(x, \mathcal{X}^k) = \min_\alpha d(x, x_\alpha^k) < 2A_0\delta^k.$$

The second property is the $2A_0\delta^k$ -density of the set \mathcal{X}^k . Note that $\mathcal{X}^k \subseteq \mathcal{X}^{k+1}$, so that every x_α^k is also a point of the form x_β^{k+1} , and thus of all the finer levels. Note also that \mathcal{X}^k reduces to one point

for $k \leq k_0$ for some k_0 if X is bounded. Still we can continue with the full chain of sets for $k \in \mathbb{Z}$. We denote $\mathcal{Y}^k := \mathcal{X}^{k+1} \setminus \mathcal{X}^k$, and relabel these points as $\mathcal{Y}^k = \{y_\beta^k\}_\beta$. These points will be the parameter set of our wavelets.

The **reference partial order** among the pairs (k, α) is the same as in M. Christ [13]. For $\delta > 0$ small enough (the smallness is fixed at the end to make every step work, and depends only on A_0), each $(k+1, \beta)$ satisfies $(k+1, \beta) \leq (k, \alpha)$ for exactly one (k, α) , in such a way that

$$(4.9) \quad d(x_\beta^{k+1}, x_\alpha^k) < \frac{1}{2A_0}\delta^k \implies (k+1, \beta) \leq (k, \alpha) \implies d(x_\beta^{k+1}, x_\alpha^k) < 2A_0\delta^k.$$

The pairs $(k+1, \beta)$ with $(k+1, \beta) \leq (k, \alpha)$ are called the children of (k, α) . Geometric doubling implies that their number is uniformly bounded.

Randomness is made possible by distinguishing the reference points using **labels** as in [48]. Points (k, α) and (k, β) are called *neighbours*, if they have children $(k+1, \gamma) \leq (k, \alpha)$ and $(k+1, \eta) \leq (k, \beta)$ such that $d(x_\gamma^{k+1}, x_\eta^{k+1}) < (2A_0)^{-1}\delta^k$. In this case, by the quasi- A_0 -triangle inequality,

$$\begin{aligned} d(x_\alpha^k, x_\beta^k) &\leq A_0 d(x_\alpha^k, x_\gamma^{k+1}) + A_0^2 d(x_\gamma^{k+1}, x_\eta^{k+1}) + A_0^2 d(x_\eta^{k+1}, x_\beta^k) \\ &< 2A_0^2\delta^k + \frac{1}{2}A_0\delta^k + 2A_0^3\delta^k < 5A_0^3\delta^k. \end{aligned}$$

The number of neighbours that any point can have is also uniformly bounded.

Each pair (k, α) is equipped with two *labels*. The primary label, denoted by $\text{label}_1(k, \alpha) \in \{0, 1, \dots, L\}$, where L is the maximal number of neighbours, is chosen in such a way that any two neighbours have a different label. The secondary label, denoted by $\text{label}_2(k, \alpha) \in \{1, \dots, M\}$, where M is the maximal number of children, is chosen in such a way that no two children of the same parent have the same label.

As described above, we now want to perform a perturbation of the original x_α^k and \leq so as to obtain a **parametrized family** of similar objects, on which probabilistic statements can later be made. The parameter space is

$$\Omega = \left(\{0, 1, \dots, L\} \times \{1, \dots, M\} \right)^\mathbb{Z},$$

with a typical point denoted by $\omega = (\omega_k)_{k \in \mathbb{Z}}$, where $\omega_k = (\ell_k, m_k) \in \{0, 1, \dots, L\} \times \{1, \dots, M\}$.

The *random new dyadic points* $z_\alpha^k(\omega)$ are defined by

$$z_\alpha^k(\omega) := \begin{cases} x_\beta^{k+1} & \text{if } \text{label}_1(k, \alpha) = \ell_k, \text{ and } (k+1, \beta) \leq (k, \alpha), \text{ and } \text{label}_2(k+1, \beta) = m_k, \\ x_\alpha^k & \text{if } \text{label}_1(k, \alpha) \neq \ell_k, \text{ or } \nexists (k+1, \beta) \leq (k, \alpha) \text{ such that } \text{label}_2(k+1, \beta) = m_k. \end{cases}$$

Clearly, this point depends only on ω_k so we write $z_\alpha^k(\omega) = z_\alpha^k(\omega_k) = z_\alpha^k$ and use the most convenient notational choice.

Equip Ω with the natural probability measure \mathbb{P}_ω , which makes all coordinates $\omega_k = (\ell_k, m_k)$ independent of each other and uniformly distributed over the finite set $\{0, 1, \dots, L\} \times \{1, \dots, M\}$. For $(k+1, \beta) \leq (k, \alpha)$ fixed, then

$$\mathbb{P}_\omega(z_\alpha^k(\omega) = x_\beta^{k+1}) \geq \frac{1}{(L+1)M}.$$

In other words, every old point on the level $k+1$ has a positive (and bounded from below) probability of being a new point on the level k . The new points behave qualitatively like the reference points, only with slightly weaker constants in the separation and density:

$$d(z_\alpha^k, z_\beta^k) \geq \frac{1}{2A_0}\delta^k, \quad \min_\alpha d(x, z_\alpha^k) < 4A_0^2\delta^k.$$

The *new partial order* \leq_ω , $\omega = (\omega_k)_{k \in \mathbb{Z}}$, is set up as follows, declaring that

$$(4.10) \quad (k+1, \beta) \leq_\omega (k, \alpha) \stackrel{\text{def}}{\iff} \begin{cases} d(x_\beta^{k+1}, z_\alpha^k(\omega)) < \frac{1}{4}A_0^{-2}\delta^k, & \text{or} \\ (k+1, \beta) \leq (k, \alpha) \text{ and } \nexists \gamma : d(x_\beta^{k+1}, z_\gamma^k(\omega)) < \frac{1}{4}A_0^{-2}\delta^k. \end{cases}$$

In other words, to find the new parent of $(k+1, \beta)$ for the new partial order \leq_ω , we first check whether the reference point x_β^{k+1} is close (within distance $\frac{1}{4}A_0^{-2}\delta^k$) to some new dyadic point $z_\alpha^k(\omega)$. If yes, then the corresponding (k, α) will be the new parent of $(k+1, \beta)$. If no such close point exists, then we simply use the original partial order \leq to decide the parent of $(k+1, \beta)$. With this definition, one can observe that for any given k, α, β , the truth or falsity of the relation $(k+1, \beta) \leq_\omega (k, \alpha)$ depends only on the component ω_k of ω . Thus, it is independent of any event that occurs at level $k+1$ and

higher. This explicit definition of the original partial order \leq and of \leq_ω only depends of the geometric configuration of the points, and a condition analogous to (4.9) is a *consequence* of the definition:

$$d(z_\beta^{k+1}, z_\alpha^k) < \frac{1}{5}A_0^{-3}\delta^k \implies (k+1, \beta) \leq_\omega (k, \alpha) \implies d(z_\beta^{k+1}, z_\alpha^k) < 5A_0^3\delta^k.$$

Iteration yields that for all $\ell \geq k$,

$$d(z_\beta^\ell, z_\alpha^k) < \frac{1}{6}A_0^{-4}\delta^k \implies (\ell, \beta) \leq_\omega (k, \alpha) \implies d(z_\beta^\ell, z_\alpha^k) < 6A_0^4\delta^k.$$

With the auxiliary objects at hand, the **random dyadic cubes** are easy to define. As in [48], introduce three families of these cubes—the preliminary, the closed, and the open:

$$\hat{Q}_\alpha^k(\omega) := \{z_\beta^\ell(\omega) : (\ell, \beta) \leq_\omega (k, \alpha)\},$$

$$\bar{Q}_\alpha^k(\omega) := \overline{\hat{Q}_\alpha^k(\omega)}, \quad \tilde{Q}_\alpha^k(\omega) := \text{interior } \bar{Q}_\alpha^k(\omega).$$

Note that $\hat{Q}_\alpha^k(\omega)$, and hence $\bar{Q}_\alpha^k(\omega)$ and $\tilde{Q}_\alpha^k(\omega)$, only depends on ω_ℓ for $\ell \geq k$. Of course, the name "cube" solely refers to the analogous situation in Euclidean space without any other geometric meaning for such sets (that same construction in Euclidean space would not necessarily end up with a cube).

The following theorem summarizes the above properties of the random dyadic cubes for a fixed parameter ω (that is, they separately form dyadic grids), and supplements the key statement about their probabilistic behaviour under the random choice of $\omega \in \Omega$.

4.11. Theorem. *For any fixed $\omega \in \Omega := (\{0, 1, \dots, L\} \times \{1, \dots, M\})^{\mathbb{Z}}$, the cubes satisfy the following relations of a dyadic grid: the covering properties*

$$X = \bigcup_{\alpha} \bar{Q}_\alpha^k(\omega), \quad \bar{Q}_\alpha^k(\omega) = \bigcup_{\beta: (k+1, \beta) \leq_\omega (k, \alpha)} \bar{Q}_\beta^{k+1}(\omega);$$

the mutual disjointness property

$$\bar{Q}_\alpha^k(\omega) \cap \bar{Q}_\beta^k(\omega) = \emptyset \quad (\alpha \neq \beta);$$

and the comparability with balls:

$$B(z_\alpha^k(\omega), \frac{1}{6}A_0^{-5}\delta^k) \subseteq \tilde{Q}_\alpha^k(\omega) \subseteq \bar{Q}_\alpha^k(\omega) \subseteq B(z_\alpha^k(\omega), 6A_0^4\delta^k).$$

Moreover, when Ω is equipped with the natural probability measure \mathbb{P}_ω , we have for some $\eta \in (0, 1]$ the small boundary layer property:

$$(4.12) \quad \mathbb{P}_\omega \left(x \in \bigcup_{\alpha} \partial_\varepsilon Q_\alpha^k(\omega) \right) \leq C\varepsilon^\eta \quad \left(\partial_\varepsilon Q_\alpha^k(\omega) := \{y \in \bar{Q}_\alpha^k(\omega) : d(y, {}^c\bar{Q}_\alpha^k(\omega)) < \varepsilon\delta^k \} \right);$$

and in particular the negligible boundary property:

$$\mathbb{P}_\omega \left(x \in \bigcup_{k, \alpha} \partial Q_\alpha^k(\omega) \right) = 0 \quad \left(\partial Q_\alpha^k(\omega) := \bar{Q}_\alpha^k(\omega) \setminus \tilde{Q}_\alpha^k(\omega) \right).$$

Eventually, the original dyadic point x_α^k may also be viewed as a ‘centre’ of the random dyadic cubes $\bar{Q}_\alpha^k(\omega)$ for all $\omega \in \Omega$:

$$B(x_\alpha^k, \frac{1}{8}A_0^{-3}\delta^k) \subseteq \bar{Q}_\alpha^k(\omega) \subseteq \bar{B}(x_\alpha^k, 8A_0^5\delta^k).$$

The small boundary property may not be true for balls in general. It was already observed by G. David (without randomness) and used by M. Christ for singular integrals that dyadic cubes should have this property and that this can be built-in from the partial order. It will be the key to obtain the Hölder-regularity exponent of our splines and wavelets. The **construction of splines** on X , and the proof of their basic properties, is indeed an easy consequence of the preparations above. For every (k, α) , we define the **spline function** s_α^k by

$$(4.13) \quad s_\alpha^k(x) := \mathbb{P}_\omega \left(x \in \bar{Q}_\alpha^k(\omega) \right).$$

4.14. Theorem. *The splines (4.13) satisfy the following properties: bounded support*

$$(4.15) \quad 1_{B(x_\alpha^k, \frac{1}{8}A_0^{-3}\delta^k)}(x) \leq s_\alpha^k(x) \leq 1_{B(x_\alpha^k, 8A_0^5\delta^k)}(x);$$

the interpolation and reproducing properties

$$(4.16) \quad s_\alpha^k(x_\beta^k) = \delta_{\alpha\beta}, \quad \sum_{\alpha} s_\alpha^k(x) = 1, \quad s_\alpha^k(x) = \sum_{\beta} p_{\alpha\beta}^k \cdot s_\beta^{k+1}(x)$$

where $p_{\alpha\beta}^k = \mathbb{P}_\omega((k+1, \beta) \leq_\omega (k, \alpha))$ so that $\sum_\beta p_{\alpha\beta}^k = 1$; and Hölder-continuity

$$|s_\alpha^k(x) - s_\alpha^k(y)| \leq C \left(\frac{d(x, y)}{\delta^k} \right)^\eta.$$

Let us prove the **Hölder-continuity of the splines** to show how the probabilistic smallness of the boundary regions, as expressed by (4.12), comes into play. Indeed,

$$\begin{aligned} |s_\alpha^k(x) - s_\alpha^k(y)| &= \left| \mathbb{P}_\omega(x \in \bar{Q}_\alpha^k(\omega)) - \mathbb{P}_\omega(y \in \bar{Q}_\alpha^k(\omega)) \right| \\ &= \left| \int_\Omega (1_{\omega: x \in \bar{Q}_\alpha^k(\omega)} - 1_{\omega: y \in \bar{Q}_\alpha^k(\omega)}) d\mathbb{P}_\omega \right| \\ &\leq \int_\Omega (1_{\omega: x \in \bar{Q}_\alpha^k(\omega), y \notin \bar{Q}_\alpha^k(\omega)} + 1_{\omega: y \in \bar{Q}_\alpha^k(\omega), x \notin \bar{Q}_\alpha^k(\omega)}) d\mathbb{P}_\omega \\ &= \mathbb{P}_\omega(x \in \bar{Q}_\alpha^k(\omega), y \notin \bar{Q}_\alpha^k(\omega)) + \mathbb{P}_\omega(y \in \bar{Q}_\alpha^k(\omega), x \notin \bar{Q}_\alpha^k(\omega)) \\ &\leq \mathbb{P}_\omega(x \in \partial_{d(x, y)\delta^{-k}} \bar{Q}_\alpha^k(\omega)) + \mathbb{P}_\omega(y \in \partial_{d(x, y)\delta^{-k}} \bar{Q}_\alpha^k(\omega)) \\ &\leq C \left(\frac{d(x, y)}{\delta^k} \right)^\eta. \end{aligned}$$

So far, there is no measure on X attached to this construction.

4.17. Proposition. *Let μ be a non-trivial positive Borel measure on a quasi-metric space with the geometric doubling property X , finite on bounded Borel sets. Let $1 \leq p < \infty$. Then, the linear span of the spline functions and in particular the space of Hölder- η -continuous functions with bounded support where η is the Hölder exponent of the splines are dense in $L^p(\mu)$.*

Recall that there is a precaution to take in the definition of a Borel measure. Of course, density of Lipschitz-continuous functions with bounded support is well-known in the metric case. Density of Hölder-continuous functions to some order with bounded support has been proved independently in [63] at about the same time by a different argument. It is not clear it had been proved before.

4.18. The Meyer algorithm. We continue the development of the spline theory in the presence of a non-trivial Borel measure μ on (X, d) so that (X, d, μ) is a space of homogeneous type in the most general sense with the exception that we assume that balls are Borel sets. If not, we may replace them by their interior and this only affects constants.

The splines provide a multiresolution analysis of $L^2(d\mu)$. This consists of all properties of a classical multiresolution analysis of Meyer [62, Definition 2.1] and Mallat [57], to the extent that this definition is meaningful in a quasi-metric space context: the classical postulates dealing with translations and dilations, specific to the Euclidean space and the Lebesgue measure, are now meaningless. Let V_k be the closed linear span of $\{s_\alpha^k\}_\alpha$ in $L^2(d\mu)$. Then $V_k \subseteq V_{k+1}$, and

$$\overline{\bigcup_{k \in \mathbb{Z}} V_k} = L^2(d\mu), \quad \bigcap_{k \in \mathbb{Z}} V_k = \begin{cases} \{0\}, & \text{if } X \text{ is unbounded,} \\ V_{k_0} = \{\text{constants}\}, & \text{if } X \text{ is bounded,} \end{cases}$$

where k_0 is some integer. Moreover, the functions $s_\alpha^k / \sqrt{\mu_\alpha^k}$ form a Riesz basis of V_k : for all sequences of numbers λ_α , we have the two-sided estimate

$$\left\| \sum_\alpha \lambda_\alpha s_\alpha^k \right\|_{L^2(d\mu)} \approx \left(\sum_\alpha |\lambda_\alpha|^2 \mu_\alpha^k \right)^{1/2},$$

with $\mu_\alpha^k := \mu(B(x_\alpha^k, \delta^k))$. With this at hand, the algorithm is as follows.

- (1) Define the closed linear span Y_k of the splines s_β^{k+1} that vanish at all the points in \mathcal{X}^k . Then, $V_{k+1} = V_k \oplus Y_k$, where the sum is topological and let $S_k : V_{k+1} \rightarrow Y_k$ be the bounded projection corresponding to this decomposition.
- (2) If Q_k is the orthogonal projection of V_{k+1} onto W_k , the orthogonal complement of V_k in V_{k+1} , then $S_k : W_k \rightarrow Y_k$ is an isomorphism, with inverse $Q_k : Y_k \rightarrow W_k$.
- (3) Define the L^∞ -normalized pre-wavelets

$$\tilde{\psi}_\beta^k := Q_k s_\beta^{k+1}$$

where the s_β^{k+1} are the splines in Y_k of item 2.

(4) Orthonormalize the pre-wavelets.

This algorithm comes with explicit constructions:

(1) Construct the bi-orthogonal Riesz basis $\{\tilde{s}_\alpha^k\}_\alpha$ to $\{s_\alpha^k\}_\alpha$ in V_k , where bi-orthogonal means

$$(4.19) \quad \langle s_\alpha^k, \tilde{s}_\beta^k \rangle_{L^2(\mu)} = \delta_{\alpha,\beta}.$$

One has

$$\sqrt{\mu_\alpha^k} \tilde{s}_\alpha^k(x) = \sum_{\beta \in \mathcal{X}^k} M_k^{-1}(\alpha, \beta) \frac{\tilde{s}_\beta^k(x)}{\sqrt{\mu_\beta^k}},$$

with M_k being the infinite matrix indexed by $\mathcal{X}^k \times \mathcal{X}^k$ (write by abuse $\alpha \in \mathcal{X}^k$ for $x_\alpha^k \in \mathcal{X}^k$) with entries

$$M_k(\alpha, \beta) = \frac{\langle s_\alpha^k, \tilde{s}_\beta^k \rangle_{L^2(\mu)}}{\sqrt{\mu_\alpha^k \mu_\beta^k}}.$$

(2) Write

$$(4.20) \quad Q_k f = f - \sum_{\alpha \in \mathcal{X}^k} \langle f, \tilde{s}_\alpha^k \rangle_{L^2(\mu)} s_\alpha^k = f - \sum_{\alpha \in \mathcal{X}^k} \langle f, s_\alpha^k \rangle_{L^2(d\mu)} \tilde{s}_\alpha^k,$$

because the sum is the orthogonal projection of f onto V_k . Compute the pre-wavelets.

(3) Observe that one can use \mathcal{Y}^k (write by abuse $\beta \in \mathcal{Y}^k$ for $y_\beta^k \in \mathcal{Y}^k$) as a label set for the pre-wavelets. Then

$$\psi_\alpha^k(x) := \sum_{\beta \in \mathcal{Y}^k} \widetilde{M}_k^{-1/2}(\alpha, \beta) \frac{\tilde{\psi}_\beta^k(x)}{\sqrt{\mu_\beta^{k+1}}},$$

where $\widetilde{M}_k^{-1/2}(\alpha, \beta)$ are the entries of the inverse of the square root of the positive self-adjoint matrix

$$\widetilde{M}_k(\alpha, \beta) := \frac{\langle \tilde{\psi}_\alpha^k, \tilde{\psi}_\beta^k \rangle_{L^2(\mu)}}{\sqrt{\mu_\alpha^{k+1} \mu_\beta^{k+1}}}$$

indexed by $\mathcal{Y}^k \times \mathcal{Y}^k$. Here, $\mu_\alpha^{k+1} = \mu(B(y_\alpha^k, \delta^{k+1})) \sim \mu(B(y_\alpha^k, \delta^k))$ and recall that the points y_α^k are the x_α^{k+1} that do not belong to \mathcal{X}^k .

4.21. Matrices with exponential decay on a quasi-metric discrete set. The algorithm requires to invert matrices in certain classes. The following result is well-known in the metric case. The extension to the quasi-metric case comes with an interesting argument having the spirit of the Macias-Segovia work (see also [69] for a very neat argument). We give the proof to illustrate this.

4.22. Lemma. *Let Ξ be a 1-separated set in a quasi-metric space (X, d) with quasi-triangle constant A_0 having the geometric doubling property with constant N . Consider a matrix $M = (M(\alpha, \beta))$ indexed by $\Xi \times \Xi$ such that there exists $c > 0$ for which*

$$C = \sup_{(\alpha, \beta)} \exp(cd(\alpha, \beta)) |M(\alpha, \beta)| < \infty.$$

Then M is bounded on $\ell^2(\Xi)$. If M is invertible, then there exists $c' > 0$ such that

$$\sup_{(\alpha, \beta)} \exp(c'(d(\alpha, \beta))^s) |M^{-1}(\alpha, \beta)| < \infty$$

with $s = (1 + \log_2 A_0)^{-1}$ or $s = 1$ if d is a Lipschitz-continuous quasi-distance. If, in addition, M is positive self-adjoint, then the same conclusion holds for $M^{-1/2}$.

If d is a genuine distance or a Lipschitz-continuous quasi-distance, then this follows from Theorem 5 in [54] and the remark that follows it, which extends an earlier result in [23] for band-limited matrices, and $s = 1$. The exponent s is best possible in the general case.

Proof. That M is bounded on $\ell^2(\Xi)$ follows from a simple application the Schur lemma and we skip it.

We continue with the following observation. For $n \geq 1$, let κ_n be the best constant in the inequality

$$d(\alpha_0, \alpha_n) \leq \kappa_n (d(\alpha_0, \alpha_1) + d(\alpha_1, \alpha_2) + \dots + d(\alpha_{n-1}, \alpha_n))$$

for every chain $(\alpha_0, \alpha_1, \dots, \alpha_n)$ of $n+1$ elements (not necessarily distinct) of Ξ . It is clear that (κ_n) is non-decreasing and $\kappa_1 = 1$, $\kappa_2 \leq A_0$. Moreover, using $d(\alpha_0, \alpha_{m+n}) \leq A_0(d(\alpha_0, \alpha_m) + d(\alpha_m, \alpha_{m+n}))$, it follows that $\kappa_{m+n} \leq A_0 \cdot \max\{\kappa_m, \kappa_n\}$. Thus $\kappa_{2^n} \leq A_0 \kappa_n$ and therefore, $\kappa_{2^j} \leq A_0^j$. We conclude that $\kappa_n \leq A_0^{1+\log_2 n} = A_0 n^{\log_2 A_0}$. Note also that if d is L -Lipschitz, then $d(\alpha_0, \alpha_n) \leq d(\alpha_0, \alpha_{n-1}) + Ld(\alpha_{n-1}, \alpha_n)$, which by iteration gives $\kappa_n \leq L$ for all n .

Now assume that M is positive self-adjoint and invertible. In this case, one can write $M = h(I - A)$ with $h = (\|M\| + \|M^{-1}\|)/2$ a positive real number and A a matrix with norm $r = (\|M\| - \|M^{-1}\|)/(\|M\| + \|M^{-1}\|) < 1$. Moreover, the coefficients of A have the same decay as those of M . Without loss of generality, we normalize $h = 1$. Develop $(I - A)^{-1}$ in the Neumann series $\sum A^n$ and estimate the coefficients $A^n(\alpha, \beta)$, $n \geq 1$, $\alpha \neq \beta$, in two ways. First $|A^n(\alpha, \beta)| \leq r^n$. Second, we have

$$\begin{aligned} |A^n(\alpha, \beta)| &\leq \sum_{(\alpha_1, \dots, \alpha_{n-1}) \in \Xi^{n-1}} C^n \exp(-c(d(\alpha, \alpha_1) + d(\alpha_1, \alpha_2) + \dots + d(\alpha_{n-1}, \beta))) \\ &\leq C^n \exp\left(-\frac{c}{2\kappa_n} d(\alpha, \beta)\right) \sum_{(\alpha_1, \dots, \alpha_{n-1}) \in \Xi^{n-1}} \exp\left(-\frac{c}{2}(d(\alpha_1, \alpha_2) + \dots + d(\alpha_{n-1}, \beta))\right) \\ &\leq \tilde{C}^n \exp\left(-\frac{c}{2\kappa_n} d(\alpha, \beta)\right), \end{aligned}$$

where we applied $n-1$ times the inequality with $\varepsilon = c/2$,

$$\sup_{\alpha \in X} \exp(\varepsilon d(\alpha, \Xi)/A_0) \sum_{\beta \in \Xi} \exp(-\varepsilon d(\alpha, \beta)) \leq c(\varepsilon, A_0, N) < \infty.$$

As κ_n is non decreasing, we have for any integer n_0 using the second estimate for $0 \leq n \leq n_0$ and the first for $n > n_0$,

$$|M^{-1}(\alpha, \beta)| \leq (n_0 + 1)\tilde{C}^{n_0} \exp\left(-\frac{c}{2\kappa_{n_0}} d(\alpha, \beta)\right) + r^{n_0+1}(1-r)^{-1}$$

and $(n_0 + 1)\tilde{C}^{n_0} \leq D^{n_0}$ for some large constant $D > 0$. Choosing n_0 as the first integer such that the first term dominates, we see that $d(\alpha, \beta) \approx n_0 \cdot \kappa_{n_0} \lesssim n_0^{1/s}$, where $s = 1/(1 + \log_2 A_0)$ (or $s = 1$ if d is Lipschitz-continuous, recalling that $\kappa_{n_0} \leq L$ in this case). Hence, for some constant $c' > 0$,

$$|M^{-1}(\alpha, \beta)| \lesssim r^{n_0} \lesssim \exp(-c' d(\alpha, \beta)^s).$$

For $M^{-1/2}$, we use the power series $(I - A)^{-1/2} = \sum c_n A^n$. As $0 \leq c_n \lesssim n^{1/2}$, the argument is the same. Finally, if M is not positive self-adjoint, then we use $M^{-1} = M^*(MM^*)^{-1}$, the remark that on a 1-separated set $d \geq d^s$ and that the product of matrices having the exponential decay with factor c as in the statement has exponential decay with factor $c' \in (0, c/A_0)$. \square

The above lemma applies a first time to the matrix $M_k(\alpha, \beta)$ on the 1-separated set \mathcal{X}^k for the renormalized quasi-distance d_k given by $d_k(x, y) := \delta^{-k} d(x, y)$ on $X \times X$ and then a second time to the matrix $\tilde{M}_k(\alpha, \beta)$ on the 1-separated set \mathcal{Y}^k for the renormalized quasi-distance d_{k+1}^s . The constants are therefore uniform over all k 's. This provides us with the parameters s and a that one can see in the Corollary. This insures the exponential decay in the statements and the preservation of the Hölder exponent throughout the construction.

4.23. The completion of Guido and Raphy's program. We come back to the importance of the distance to the point set \mathcal{Y}^k in the estimates for the kernel $Q_k(x, y)$ of the projection Q_k . The absence of the reverse doubling property is linked to the presence of holes at certain scales in X . It could be very well be that \mathcal{Y}^k is empty (in which case the distance to it is $-\infty$ by convention and there is no corresponding wavelets) or that \mathcal{Y}^k contains no points in some regions of X where there is an absence of nearby points at scale exactly δ^{k+1} . Therefore the distance to \mathcal{Y}^k is larger than δ^{k+1} in those regions. In other words, the set \mathcal{Y}^k may not be dense at the scale δ^{k+1} . This can be quantified in some organisation for the growth of the measure of balls: there exists $\varepsilon > 0$ depending only on A_0 and the doubling constant such that for every $x \in X$ and $r > 0$, there exists a decreasing sequence, finite or infinite, of integers $\{k_j\}_{j=0}^J$ such that $r \leq \delta^{k_0} < \delta^{k_1} < \dots$ and

$$\mu(B(x, \delta^k)) \gtrsim (1 + \varepsilon)^j \mu(B(x, r)) \quad \text{and} \quad d(x, \mathcal{Y}^k) + \delta^k \gtrsim \delta^{k_{j+1}} \quad \text{if } k_j \geq k > k_{j+1},$$

where we interpret $k_{J+1} := -\infty$ if $J < \infty$. As one cannot tell the growth of this sequence k_j , there is no reason that (4.2) still converges. But with this organisation, one can show the substitute inequality

to (4.2): For all $x \in X$ and all positive numbers r, ν, a, γ , we have

$$\sum_{k: \delta^k \geq r} \mu(B(x, \delta^k))^{-\nu} \exp(-\gamma(\delta^{-k}d(x, \mathcal{Y}^k))^a) \lesssim \mu(B(x, r))^{-\nu}$$

with implicit constant independent of x and r . Such estimates apply to check Calderón-Zygmund kernel estimates. For example, one can deduce

$$\sum_{k, \alpha} |\psi_\alpha^k(x) \psi_\alpha^k(y)| \leq \frac{C}{\mu(B(x, d(x, y)))}.$$

The operators defined by $\psi_\alpha^k \mapsto \varepsilon_\alpha^k \psi_\alpha^k$ where $\varepsilon_\alpha^k = \pm 1$, have kernels formally given by $K(x, y) = \sum_{k, \alpha} \varepsilon_\alpha^k \psi_\alpha^k(x) \psi_\alpha^k(y)$. They are bounded on $L^2(d\mu)$ and, from the estimate above and similar ones for the regularity of $K(x, y)$ from the regularity of the wavelets, are Calderón-Zygmund operators. One applies the extrapolation result on spaces of homogeneous type in full generality to obtain their $L^p(d\mu)$, $1 < p < \infty$, boundedness. As a consequence, the family of wavelets is an unconditional basis in $L^p(d\mu)$. This was done in [3]. Also the wavelet decomposition of BMO via a Carleson measure characterisation and a proof of the T(1) theorem were presented there.

It was clear that this would also open the way to develop the functional spaces and operator theory in this context: Hardy, Besov, Triebel-Lizorkin spaces with various definitions and descriptions. Here is a non-exhaustive list of references where these aspects are treated [12, 29, 37, 32, 41, 43, 42]. As said, the construction of regular wavelets in full generality offers the first regular Littlewood-Paley decomposition at the same time. The Corollary above also points out the difference in the kernel estimates for the approximation to the identity ($P_k(x, y)$) and the dyadic block ($Q_k(x, y)$). In the first one the scale is given by \mathcal{X}^k and size estimates (4.6) come with somehow classical (or expected) decay; in the second, in addition to the scale and the (expected) decay, there is an extra term (4.7) measuring the decay in the distance to $\mathcal{Y}^k = \mathcal{X}^{k+1} \setminus \mathcal{X}^k$. Nor this extra decay, neither the role of the difference set \mathcal{Y}^k could be detected under reverse doubling. This tells us how should the operators of a Littlewood-Paley decomposition look like and what should the Calderón reproducing formula be in full generality (see in particular [37, 41, 42, 43]). Its convergence in an appropriate distributional sense can be established. Next, the functional spaces can be defined with any such Littlewood-Paley decomposition (the orthogonality being not required) and they are independent of this choice as it gives an equivalent norm with any other choice. For example, this shows the intrinsic character of the spaces; they are independent of the choice of the wavelet basis (Each time we select nested sets \mathcal{X}^k one can construct a wavelet basis. Hence, there are many different choices). Using another point of view (as the one described by Aline above), the set of points $(\delta^{k+1}, y_\alpha^k) \in \mathbb{R}_+ \times X$ furnishes a representation by a hyperbolic network structure in which one should work. From there, one can imagine developing tent space theory and more.

In conclusion, the intuition of Guido and Raphy behind their definition of a space of homogeneous type, continuing a long line of development since Fourier series (see for example the preface by Y. Meyer in [24]) has proved to be fully correct, in the sense that one can build **all** the tool boxes that harmonic analysts rely on in Euclidean spaces only using geometry and distribution of points. This also makes a bridge between continuous and discrete situations where *ad hoc* methods were developed independently. Indeed, the same tools can be **universally** applied to familiar (for analysts) situations such as Ahlfors-David or reverse doubling measures but also to basic examples that were ruled out by these hypotheses like discrete groups \mathbb{Z} or $\mathbb{Z}/p\mathbb{Z}$ and the multidimensional analogs (see, e.g., [72]), the typical discrete metric structures arising in theoretical computer science (trees, graphs, or strings from a finite alphabet), \mathbb{Q}_p from arithmetic [2], or discrete approximations (like those constructed in [1]). The missing ingredients came from the need to develop appropriate technology to work in non-doubling spaces. We would not call it the fall of the doubling condition as did J. Verdera in 2002 [79], but rather the completion of the doubling theory.

And last but not least, one can do this using orthonormal wavelets (which are special molecules, another invention of Guido and Raphy).

The completion of the program launched by Guido and Raphy makes the ground ready for more. For example, another challenge is: can one construct regular wavelets or Littlewood-Paley decompositions that would apply universally in those non-doubling spaces?

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