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MultiCarroll dynamics

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Abstract

Unlike a single Carroll particle, a multiparticle Carroll system can move under suitable conditions, as we demonstrate it explicitly for two particles with a momentum-dependent interaction : the center-of-mass remains fixed, however relative motion is possible, confirming previous statements made by Bergshoeff, Casalbuoni, and their collaborators. Analogous results are obtained for electric dipoles with the roles of the center-of-mass and the relative position interchanged.

Key words: Carroll symmetry. (Im)mobility of Carroll particles.

PACS numbers: 04.20.-q Classical general relativity;

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Contents

I. Introduction	2
II. Immobility and Carroll boost symmetry	3
III. A two-body system of Bergshoeff et al	6
IV. Discussion	9
Acknowledgments	10
References	10

I. INTRODUCTION

Carroll symmetry, whose characteristic feature is that *Carroll* (or C-) *boosts* leave the position \mathbf{x} fixed but shift the “Carroll time” (that we denote by s to distinguish it from relativistic or Galilean time),

$$\mathbf{x} \rightarrow \mathbf{x}, \quad s \rightarrow s - \mathbf{b} \cdot \mathbf{x}, \quad (\text{I.1})$$

has a remarkable history. This unconventional boost-action was obtained by Lévy-Leblond [1] and (independently) by SenGupta [2] by the $c \rightarrow 0$ contraction of the Poincaré group [3] — to be contrasted with the familiar $c \rightarrow \infty$ contraction introduced by Wigner and Inönü [4], which yields the Galilei group.

However, apart of occasional studies [5–8], Carroll symmetry has long remained a rather confidential mathematical curiosity [7, 9, 10] for over 50 years — before reemerging triumphantly [11, 13].

How can this strange fate be explained ? The main cause may well be that Carroll symmetry, as introduced initially, did not have genuine physical applications — for the good reason that a *particle with Carroll symmetry does not move* — a fact recognized from the very beginning that has subsequently become a general wisdom.

The judgment of “uselessness” has changed dramatically, though, when Carroll symmetry was applied to general relativity, in particular to black holes and gravitational waves [8, 14–18], followed by a huge number of recent papers that we cannot list all.

Another, independent reason for interest in Carroll symmetry comes from a rather different horizon: it is precisely the “no motion” property of the quasiparticles called *fractons* that attracts considerable current attention in condensed matter physics [19–26]. Defect has become a virtue !

The similarity of fractons and Carroll particles has been noted recently [25]. The “dipole symmetry” of condensed matter physicist is indeed a sort of dual of Carroll symmetry studied in the gravitational context [14, 18, 27].

There is general agreement: a *single* Carroll particle can no move. But what about *multiparticle systems* of Carroll particles ? In [11, 28] it was argued *against* their immobility. This Note confirms and extends their statements using an approach which is rather different from the one advocated in [11, 28]. Below we present indeed a simple but explicit counter-example to the “Carroll-implies-immobility” dogma. In detail, we demonstrate that two particles which interact through the momentum-dependent potential

$$V = -\frac{1}{2}(\mathbf{v}_1 - \mathbf{v}_2)^2, \quad (\text{I.2})$$

where the \mathbf{v}_a , $a = 1, 2$ represent the momenta of particles of unit mass $m_a = 1$ (see sec.IV), *do move*. Momentum-dependent interactions are rather unconventional in high-energy physics; however they are widely used in *nuclear physics* [29–32], as we learned it from an expert [33].

II. IMMOBILITY AND CARROLL BOOST SYMMETRY

Particles with Carroll symmetry can and in fact have been [9, 10] studied systematically by the “orbit construction” of Souriau [34]: a particle in d -dimensional space is described by a $2d+1$ dimensional “evolution space” $\mathcal{E} = \{(\mathbf{x}, \mathbf{v}, s)\}$ where \mathbf{x} is the position in configuration space, \mathbf{v} a tangent vector to the latter, and s is Carrollian time [18]. For a translation-invariant system the conserved momentum is proportional to the coordinate \mathbf{v} ; therefore we shall refer to \mathbf{v} simply as “momentum”.

However, many physicists are not familiar with Souriau’s presymplectic geometry [34] therefore we find it more convenient to present it in a more conventional, namely Lagrangian framework. Their relation is explained in [35].

For the one particle Lagrangian $\mathcal{L}_0 = m\mathbf{v} \cdot \mathbf{x}'$ [18] the Hamiltonian action

$$\mathcal{S}_0 = \int \mathcal{L}_0 ds = \int m\mathbf{v} \cdot d\mathbf{x}, \quad (\text{II.1})$$

is invariant under C-boosts [18] and also yields the equations of motion,

$$\mathbf{x}' = 0 \quad \text{and} \quad \mathbf{v}' = 0 \quad \Rightarrow \quad \mathbf{x} = \mathbf{x}_0 \quad \text{and} \quad \mathbf{v} = \mathbf{v}_0. \quad (\text{II.2})$$

In conclusion, *a single massive Carroll particle does not move*, consistently with general wisdom [1, 2].

Let us now turn to multiparticle systems. First we note that a d -dimensional system of N Carroll particles with masses $m_a > 0$ which do not interact between themselves can be described by a generalization of the 1-particle Carroll action [18],

$$\mathcal{S}_0^N = \int \mathcal{L}_0^N ds = \int \left(\sum_{a=1}^N m_a \mathbf{v}_a \cdot \mathbf{x}'_a \right) ds = \int \sum_{a=1}^N m_a \mathbf{v}_a \cdot d\mathbf{x}_a, \quad (\text{II.3})$$

which gives us uncoupled equations of motion,

$$m_a \mathbf{x}'_a = 0, \quad \mathbf{v}'_a = 0 \quad \Rightarrow \quad \mathbf{x}_a = (\mathbf{x}_a)^0 = \text{const.} \quad \mathbf{v}_a = (\mathbf{v}_a)^0 = \text{const.} \quad (\text{II.4})$$

$a = 1, \dots, N$. None of the particles moves; their (vanishing) velocities, $\mathbf{x}'_a = 0$, are unrelated to the constant values of their “momenta” \mathbf{v}_a , which play no dynamical rôle. This is a first hint at that the system reduces to a first-order one.

We then assume that our Carroll particles do interact, namely through an a priori arbitrary *position and momentum-dependent* N -body potential $V(\mathbf{x}_a, \mathbf{v}_a, s)$,

$$\mathcal{S} = \int \mathcal{L}_{int} ds = \int \left(\sum_a^N m_a \mathbf{v}_a \cdot \mathbf{x}'_a - V \right) ds = \mathcal{S}_0 - \int V ds. \quad (\text{II.5})$$

The equations of motion which generalize (II.4) still separate,

$$m_a \mathbf{x}'_a = \frac{\partial V}{\partial \mathbf{v}_a}, \quad (\text{II.6a})$$

$$m_a \mathbf{v}'_a = -\frac{\partial V}{\partial \mathbf{x}_a}, \quad (\text{II.6b})$$

$a = 1, \dots, N$. It follows that all of our particles remain fixed, confirming the general wisdom, *unless the potential depends effectively on the momenta* \mathbf{v}_a , $\frac{\partial V}{\partial \mathbf{v}_a} \neq 0$, that we shall henceforth assume.

Then our first observation says that when the \mathbf{v} -dependent terms on the rhs of (II.6a), add up to zero,

$$\sum_a \frac{\partial V}{\partial \mathbf{v}_a} = 0 \quad (\text{II.7})$$

i.e. when the potential satisfies a sort of “*momentum dual*” of Newton’s 3rd law, then the center of mass (CoM),

$$\mathbf{X} = \frac{\sum_a m_a \mathbf{x}_a}{M}, \quad M = \sum_a m_a \quad (\text{II.8})$$

remains fixed

$$\boxed{M \mathbf{X}' = \sum_a \frac{\partial V}{\partial \mathbf{v}_a} = 0.} \quad (\text{II.9})$$

The second equation, (II.6a), is in turn irrelevant from the dynamical point of view, because the \mathbf{v}_a have no bearing on the dynamics which is of the first order. We note however that the “center of momentum”, a dual to the CoM is fixed by (II.6b), $\sum_a m_a \mathbf{v}_a = \text{const.}$ because of Newton’s usual 3rd law. The relative motion can however be non-trivial, as will be seen in Sec.III.

All this follows from the reduction of the 2nd-order Newtonian dynamics to a first-order one, (II.6a).

The relation (II.9) should be compared with the discussion in [34] (pp. 141-142 and pp. 152-153), where Souriau states “Galilean relativity + the Maxwell Principle imply Newton’s 3rd law”. See also [40–42, 44].

The boost symmetry is modified in the Carroll context. The C-boosts (I.1) lift to the evolutions space,

$$\mathbf{x}_a \rightarrow \mathbf{x}_a, \quad \mathbf{v}_a \rightarrow \mathbf{v}_a + \mathbf{b}, \quad s \rightarrow s - \mathbf{b} \cdot \mathbf{X}, \quad (\text{II.10})$$

where $\mathbf{b} \in \mathbb{R}^d$, and leave the free action (II.3) invariant up to surface terms,

$$\mathcal{L}_0 \rightarrow \mathcal{L}_0 + \mathbf{b} \cdot \left(\sum_a m_a \mathbf{x}_a \right)' = \mathcal{L}_0 + \mathbf{b} \cdot M \mathbf{X}'. \quad (\text{II.11})$$

The boosts (II.10) leave both the CoM (II.8) and the relative coordinates

$$\mathbf{R}_{ab}(s) = \mathbf{x}_a(s) - \mathbf{x}_b(s) \quad (\text{II.12})$$

invariant, $\mathbf{X} \rightarrow \mathbf{X}$, $\mathbf{R}_{ab} \rightarrow \mathbf{R}_{ab}$. Then we introduce the relative momenta $\mathbf{W}_{ab} = \mathbf{v}_a - \mathbf{v}_b$ and, generalizing the momentum-dependent Ansatz (I.2) of Bergshoeff et al, limit our

investigations to potentials of the form,

$$V = V(\mathbf{v}_a) = \sum_{a>b=1}^N V_{ab}(\mathbf{W}_{ab}^2). \quad (\text{II.13})$$

All relative momenta \mathbf{W}_{ab} and thus the potential (II.13), are also left invariant.

Our assumption $\partial V/\partial \mathbf{x}_a = 0$ implies that $\mathbf{v}_a = \mathbf{v}_a^0$. Therefore $\mathbf{W}_{ab}(s) = \mathbf{W}_{ab}^0$ and thus also the potential are constants of the motion, $V = V(\mathbf{v}_a^0) = V^0 = \text{const.}$

The interaction term in (II.5) involves also the Carroll time s , but $\mathcal{L}_{int}ds$ changes again by a surface term,

$$\mathcal{L}_{int}ds = (L_0 - V) ds + dK \quad \text{where} \quad K = (M + V_0) \mathbf{b} \cdot \mathbf{X}, \quad (\text{II.14})$$

allowing us to conclude that the C-boosts span a d -parameter symmetry.

Remarkably, the V^0 terms cancel in the generated C-boost momentum $\mathcal{L}_{int}\delta s - K$, leaving us with,

$$\mathbf{D} = -M\mathbf{X}, \quad (\text{II.15})$$

whose conservation plainly implies that \mathbf{X} , the *CoM*, *can not move* — it behaves as a single particle. However no conclusion can be drawn about the relative motion, $\mathbf{R}_{ab}(s) = \mathbf{x}_a(s) - \mathbf{x}_b(s)$; the two-body examples discussed in the next section shows that \mathbf{R}_{ab} *can indeed be s -dependent*.

III. A TWO-BODY SYSTEM OF BERGSHOEFF ET AL

A nice illustration for multi-Carroll systems is provided by two particles with masses $m_1 = m_2 = 1$ (for simplicity) which interact through the momentum-dependent potential $V = -\frac{1}{2}\mathbf{W}^2$ (I.2), proposed by Bergshoeff et al [11]. V plainly satisfies the condition (II.9). Supplementing our previous notations

$$\mathbf{X} = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2), \quad \mathbf{R} = \mathbf{x}_1 - \mathbf{x}_2, \quad \mathbf{W} = \mathbf{v}_1 - \mathbf{v}_2, \quad (\text{III.1})$$

with $\mathbf{U} = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2)$, the Hamiltonian action (II.5) can be decomposed as

$$\mathcal{S} = \underbrace{\int 2\mathbf{U} \cdot \mathbf{X}' ds}_{\mathcal{S}_X} + \underbrace{\int \frac{1}{2}\mathbf{W} \cdot \mathbf{R}' ds + \int \frac{1}{2}\mathbf{W}^2 ds}_{\mathcal{S}_R}, \quad (\text{III.2})$$

whose variational equations,

$$\mathbf{X}' = 0, \quad \mathbf{U}' = 0, \quad \mathbf{R}' = -2\mathbf{W}, \quad \mathbf{W}' = 0, \quad (\text{III.3})$$

are solved at once by,

$$\mathbf{W} = \mathbf{W}^0, \quad \mathbf{W}^0 = \frac{1}{2}(\mathbf{v}_1^0 - \mathbf{v}_2^0) = \text{const.} \quad (\text{III.4a})$$

$$\mathbf{X}(s) = \mathbf{X}_0, \quad \mathbf{X}_0 = \frac{1}{2}(\mathbf{c}_1 + \mathbf{c}_2) = \text{const.} \quad (\text{III.4b})$$

and, even more importantly for us,

$$\boxed{\mathbf{R}(s) = \mathbf{R}(0) - 2\mathbf{W}^0 s}, \quad (\text{III.5})$$

supplemented with $\mathbf{U} = \mathbf{U}^0 = \frac{1}{2}(\mathbf{v}_1^0 + \mathbf{v}_2^0) = \text{const.}$, where the \mathbf{v}_i^a , \mathbf{c}_a , $a = 1, 2$ are integration constants.

In conclusion, while *the center of mass remains fixed the relative position, $\mathbf{R} = \mathbf{x}_1 - \mathbf{x}_2$ in (II.12) moves with constant velocity $-2\mathbf{W}^0$, (III.5)*. Therefore the position coordinates do move as shown in fig.1a, namely with constant but opposite (Carroll) velocities.

The immobility of the CoM is consistent with C-boost symmetry. Presenting the Hamiltonian action (III.2) as

$$\mathcal{S} = \int 2\mathbf{U} \cdot d\mathbf{X} + \frac{1}{2}\mathbf{W} \cdot d\mathbf{R} + \frac{1}{2}\mathbf{W}^2 ds \quad (\text{III.6})$$

shows indeed that the infinitesimal Carroll boost

$$\delta\mathbf{X} = 0, \quad \delta\mathbf{R} = 0, \quad \delta\mathbf{U} = \boldsymbol{\beta}, \quad \delta\mathbf{W} = 0, \quad \delta s = -\boldsymbol{\beta} \cdot \mathbf{X} \quad (\text{III.7})$$

changes the Cartan form of (III.2) by a surface term and the conservation of \mathbf{D} , (II.15) with $M = 2$, is recovered. $\mathbf{D}' = 0$ implies, conversely, the immobility of the CoM, \mathbf{X} , consistently with (III.4). No conclusion can be drawn from the conservation of \mathbf{D} , though, about the relative motion: unlike for the CoM, the dynamics of \mathbf{R} *does* depend on \mathbf{W} : the C-boost symmetry affects the motion of the CoM, but not the relative motion.

Electric dipoles. The conservation of the center of mass is reminiscent of that of the dipole moment of fractonic systems [20–26] which can easily be incorporated into the multi-Carrollian picture by noticing that the masses m_a can in fact also be viewed as *charges*

which then can take both positive and negative values, $m_a \rightsquigarrow q_a$. Then the definition of the center of mass reduces to the familiar expression of the total dipole moment,

$$\mathbf{D} = \sum_a q_a \mathbf{x}_a. \quad (\text{III.8})$$

Our general theory then implies that the motion of interacting Carroll particles conserves the total dipole moment (III.8) provided the interaction is realized via a momentum-dependent two-particle potential (II.13) – and then the conservation of (III.8) follows from the C-boost symmetry via Noether’s theorem.

Let us spell this out for an electric dipole composed of two opposite electric charges with equal masses, $m_1 = m_2 = 1$ but opposite charges, $q_1 = q$, $q_2 = -q$. Their total charge is thus zero and no “center of charge” analogous to the center of mass can be defined. The Hamiltonian action is the integral of

$$\mathcal{L}_{dip} ds = q(\mathbf{v}_1 d\mathbf{x}_1 - \mathbf{v}_2 d\mathbf{x}_2) - V ds, \quad (\text{III.9})$$

where the potential is independent of the charges; we have chosen $V = -\frac{1}{2}\mathbf{W}^2$ as before. The motions can be determined directly: the equations of motion are

$$\mathbf{X}' = -\frac{\mathbf{W}}{q}, \quad \mathbf{R}' = 0. \quad (\text{III.10})$$

Comparison with (III.3) then shows that the new casting amounts simply to *interchanging the CoM and the relative position*, $\mathbf{X} \leftrightarrow \frac{1}{2}\mathbf{R}$. We still have $\mathbf{W} = \mathbf{W}^0 = \text{const.}$ but the positions now *move* with *identical* Carroll velocities, $\mathbf{x}'_1 = \mathbf{x}'_2 = -\mathbf{W}^0$, as shown in fig.1b.

Due to the relative sign change the no-motion-for-the-CoM condition (II.9) becomes now a “no-relative-motion condition”, $\mathbf{R}' = 0$. Remarkably, the CoM of an *isolated* dipole is in turn *not fixed*, consistently with the violation of the Kohn condition [39–42, 44],

$$q_a/m_a = \text{const.} \quad \text{for } a = 1, \dots, N, \quad (\text{III.11})$$

which is indeed necessary for the center-of-mass decomposition [34]. However for a dipole we have instead $q_1/m_1 = -q_2/m_2$.

The dipole is symmetric w.r.t. (II.10) which changes the dipole Lagrangian (III.9) by a surface term ; the associated conserved quantity is the *electric dipole momentum*,

$$\mathbf{D}_{dip} = q(\mathbf{x}_1 - \mathbf{x}_2) = q \mathbf{R}, \quad (\text{III.12})$$

whose conservation implies $\dot{\mathbf{R}} = 0$, consistently with the relative position \mathbf{R} being shifted rigidly with the displacement of the CoM, $\mathbf{X}(s) = -\mathbf{W} s + \mathbf{X}_0$, as shown in fig.1b.

identified with s -based Carroll symmetry [18].

Interactions break the t -based symmetry to mere t -translations, however the first, linear term of the s -based one does survive — and then yields dipole symmetry.

Returning to immobility, the Carroll symmetry effects only the CoM and implies, for a two-particle Carroll system with the BGL potential (I.2), the immobility of the latter. The relative coordinates are not affected by the Carroll action (III.7) and they can indeed move, as illustrated in sec.III, cf. fig.1a.

Replacing the masses by electric charges, the rôles of center of mass, \mathbf{X} , and of the relative coordinate, \mathbf{R} , are interchanged; Carroll boosts act on \mathbf{R} but not on \mathbf{X} , cf. fig.1b.

We note in conclusion that immobility is related to Kohn’s theorem on decomposing a particle system into center-of-mass and relative coordinates [39–44], which requires that the charge/mass ratio of all particles be the same, (III.11). Kohn’s theorem is rooted in the very structure of the Galilei group [34]. In this note we argue, in the Carroll context, in favor of a similar the rôle played by C-boosts, (I.1).

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