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ARTICLE

Non-Hausdorff parallelized manifolds over geometric models of conservative programs

Emmanuel Haucourt 

École polytechnique, LIX - UMR 7161, Palaiseau, France
emmanuel.haucourt@polytechnique.edu

(Submitted draft)

Abstract

Every directed graph G induces a locally ordered metric space \mathcal{X}_G together with a local order $\tilde{\mathcal{X}}_G$ that is locally dihomeomorphic to the standard pospace \mathbb{R} ; both are related by a morphism $\beta_G : \tilde{\mathcal{X}}_G \rightarrow \mathcal{X}_G$ satisfying a universal property. The underlying set of $\tilde{\mathcal{X}}_G$ admits a non-Hausdorff atlas \mathcal{A}_G equipped with a non-vanishing vector field f_G ; the latter is associated to $\tilde{\mathcal{X}}_G$ through the correspondence between local orders and cone fields on manifolds. The above constructions are compatible with cartesian products, so the geometric model of a conservative program is lifted through $\beta_{G_1} \times \cdots \times \beta_{G_n}$ to a subset M of the parallelized manifold $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$. By assigning the suitable norm to each tangent space of $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ the length of every directed smooth path γ on M , i.e. $\int |\gamma'(t)|_{\gamma(t)} dt$, corresponds to the execution time of the sequence of multi-instructions associated to γ . This induces a pseudometric d_n whose restrictions to sufficiently small open sets of $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ (we refer to the manifold topology, which is strictly finer than the pseudometric topology) are isometric to open subspaces of \mathbb{R}^n with the α -norm for some $\alpha \in [1, \infty]$. The transition maps of \mathcal{A}_G are translations so the representation of a tangent vector does not depend on the chart of \mathcal{A}_G in which it is represented; consequently, differentiable maps between open subsets of $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ are handled as if they were maps between open subsets of \mathbb{R}^n . For every directed path γ on M (possibly the representation of a sequence σ of multi-instructions) there is a shorter directed smooth path on M that is arbitrarily close to γ , and that can replace γ as a representation of σ .

Keywords: Concurrency; local order; conal field; directed path; execution time; multi-instruction.

1. Introduction

The concurrent programs we consider are made of finitely many sequential processes running in parallel. The idea that such programs could be interpreted geometrically was already present (yet not explicitly) in the work of Dijkstra (1968); which was quickly followed by publications containing pictures of models that are subsets of \mathbb{R}^n (Coffman et al. (1971); Kung et al. (1979); Lipski (1981)). The «geometry of concurrency» was formalized a bit later for programs P whose processes neither have branchings or loops: their representations are subsets of \mathbb{R}^n called *progress graphs* with n being the number of processes of P (Carson & Reynolds, 1987, §4). We recall two prototypical examples, both made of two processes P_1 and P_2 . In the first one, both try to take (P(m)) and release (V(m)) the mutex m concurrently (whereas mutexes cannot be held by more than one process at the time). In the second one, both wait for each other (W(b)) behind the synchronization barrier b . The corresponding progress graphs are $\mathbb{R}^2 \setminus [1, 2]^2$ (up to subtleties at the boundary of the square) and $\mathbb{R}^2 \setminus (\{x = 1; y \neq 1\} \cup \{x \neq 1; y = 1\})$, i.e. $\{(x, y) \in \mathbb{R}^2 \mid x = 1 \Leftrightarrow y = 1\}$, they are displayed on Fig. 1. The motivation for progress graphs is the relation between their directed paths (namely the componentwise order-preserving ones) and the execution traces of the programs they

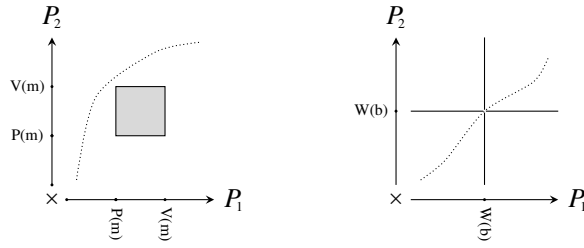
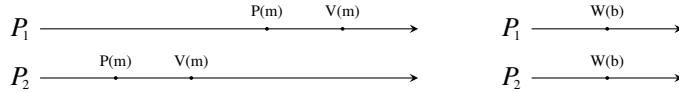


Figure 1. Progress graphs.

represent. For example, the dotted paths on Fig. 1 induce the following execution traces:



Conversely, any execution trace is induced by a directed path on the progress graph. Following this principle, and according to the postulates a progress graph X is submitted to (Carson & Reynolds, 1987, p.28), the by-product of X we are really interested in is its collection dX of directed paths.

The «geometry of concurrency» described in (Carson & Reynolds (1987)) only applies to programs without branchings or loops. At the price of some extra mathematical machinery, this limitation can be overcome (Haucourt (2018)). The key feature of a progress graph of dimension n is that its set of directed paths derives from the order and the metric inherited from \mathbb{R}^n . Let P be a program made of the processes P_1, \dots, P_n :

- i) In accordance with the standard representation of sequential programs used in compilers (Aho et al., 2007, §9.2), each P_i is given as an automaton which we associate with its underlying directed graph G_i .
- ii) Each directed graph G_i induces a *locally ordered metric graph* $|G_i|$ (Haucourt, 2018, §6.1). An execution trace of P_i is seen as a directed path on $|G_i|$, so the set of directed paths on $|G_1| \times \dots \times |G_n|$ is an overapproximation of the set of execution traces of P .

The process P_i is said to be *conservative* when for any directed path γ on $|G_i|$ the amount of resources held by P_i after the execution of γ only depends on the point where γ stops (this property is decided by a breadth first traversal of G_i given in (Haucourt, 2018, §4.1, p. 1734-1735)).

- iii) If all the processes of P are conservative, we define its *geometric model* as the complement in $|G_1| \times \dots \times |G_n|$ of the *forbidden region* of P (Carson & Reynolds, 1987, §4), (Haucourt, 2018, Def. 4.2).

Every directed path on a geometric model induces a sequence of multi-instructions that respect the constraints imposed by the synchronization mechanisms (mutexes, semaphores, and barriers) (Haucourt, 2018, Thm. 4.1). Moreover, directed paths that are metrically close to each other represent sequences of multi-instructions having the same effect on the state of the abstract machine (Haucourt, 2018, Thm. 6.1). As progress graphs, geometric models are helpful abstractions of their directed paths. In this perspective, we aim at proving that the local order and the metric of the model of P are by-products of a possibly non-Hausdorff parallelized manifold canonically defined from P (it is Hausdorff if, and only if, the program under consideration has no branchings).

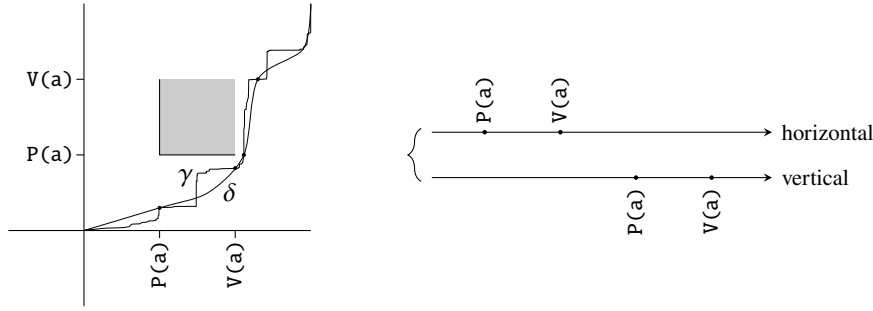
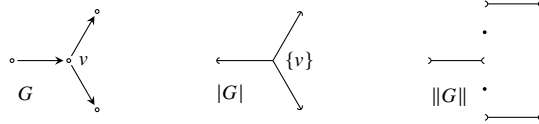


Figure 2. Replacing a directed path by a directed smooth path of lesser length.

Motivation. For now let us suppose that the geometric model X is a subset of \mathbb{R}^n (we may as well suppose that X is one of the prototypical models on Fig. 1). We give some facts about X that will be proven later in the general setting of geometric models:

- i) For any directed path γ on X , there is a directed *smooth* path δ on X that is arbitrarily close to γ , and whose length is at most that of γ , see Fig. 2.
- ii) If δ is close enough to γ , then the sequence of multi-instructions they induce have the same effect on the state of the system.
- iii) The product metric on \mathbb{R}^n can be chosen so that the lengths of γ and δ correspond to execution times, so a (directed) path of lesser length represents a faster execution trace.
- iv) A smooth path on X is directed (i.e. order-preserving in all coordinates) if, and only if, all its tangent vectors belong to \mathbb{R}_+^n .

These properties suggest that the differential structure we expect is the standard manifold \mathbb{R}^n equipped with its standard parallelization (Ex. A.16). For a taste of what lies beyond progress graphs, assume that G represents an «if-then-else» instruction (see Ex. 2.21):



The underlying set of the locally ordered metric graph $|G|$ is $V \cup A \times]0, 1[$ with A and V denoting the sets of arrows and vertices of the graph G . At every point but the four vertices of G , which form a ‘neglectable’ subset of $|G|$, the local order is locally isomorphic to \mathbb{R} . In order to get rid of these ‘singularities’, every vertex is ‘blown up’ into as many points as there are ways of going through it in respect of the local order. Consequently, the vertex v is duplicated and the extremities of G are removed. The resulting set $\|G\|$ equipped with the adhoc (non-Hausdorff) topology often appears in textbooks as an undesirable 1-dimensional smooth manifold, see (Bishop & Crittenden, 1964, p.5,6), (Lee, 2012, p.4), or (Benedetti, 2021, p.58). Such oddities are precisely what we need to consider in order to represent branchings.

Context. In geometry of concurrency, one associates every tuple of graphs (G_1, \dots, G_n) with a local order $\mathcal{X}_{G_1} \times \dots \times \mathcal{X}_{G_n}$ (Def. 5.10) on the α -product of metric spaces $|G_1| \times \dots \times |G_n|$ with $\alpha \in [1, \infty]$ (Def. 4.10). A conservative program P is a parallel composition $P_1 | \dots | P_n$ of sequential processes whose underlying graphs are G_1, \dots, G_n . The model of P is a subset $|P|$ of $|G_1| \times \dots \times |G_n|$. Every directed path γ on $|P|$ induces a sequence of multi-instructions σ which has a certain effect on the state of the abstract machine on which P is executed; by extension, the

effect of γ is that of σ . Two directed paths on $|P|$ that are sufficiently close to each other have the same effect (Haucourt, 2018, Thm. 6.1); we say that the model of P is *resilient*.

Contribution. We introduce the blowup of a graph G as a set map $\beta_G : \|G\| \rightarrow |G|$ (Def. 2.15) together with a (non-Hausdorff) atlas \mathcal{A}_G on $\|G\|$ (Def. 3.3), and a non-vanishing vector field f_G on \mathcal{A}_G (Def. 3.6). We relate the parallelized atlas $(\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}, f_{G_1}, \dots, f_{G_n})$ to the local order $\mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$ and its α -product metric by means of an intermediate local order $\tilde{\mathcal{X}}_{G_1} \times \cdots \times \tilde{\mathcal{X}}_{G_n}$ on $\|G_1\| \times \cdots \times \|G_n\|$ (Def. 5.15). More precisely, we prove that $\tilde{\mathcal{X}}_{G_1} \times \cdots \times \tilde{\mathcal{X}}_{G_n}$ is the ‘best smooth approximation’ of $\mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$ (Thm. 5.31), and that every directed path on the image of $\beta_{G_1} \times \cdots \times \beta_{G_n}$ can be lifted to a directed path on $\tilde{\mathcal{X}}_{G_1} \times \cdots \times \tilde{\mathcal{X}}_{G_n}$ (Thm. 5.27). Then we prove that the local order $\tilde{\mathcal{X}}_{G_1} \times \cdots \times \tilde{\mathcal{X}}_{G_n}$ and the conal field generated by $(f_{G_1}, \dots, f_{G_n})$ can be deduced from each other (Thm. 5.39). The vector fields f_{G_1}, \dots, f_{G_n} canonically induce a notion of length for (piecewise) smooth paths on $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ (Def. 4.16) from which we deduce the α -product metric on $|G_1| \times \cdots \times |G_n|$ (Thm. 4.21). If we take $\alpha = \infty$, then the length of a directed path on $|P|$ is the execution time of its sequence of multi-instructions (§6.1). The resilience of $|P|$ together with Thm. 6.5 and Cor. 6.6 implies that every execution trace of P can be represented by a piecewise affine directed path.

Overview. The discrete and the continuous models of a conservative program P are discussed in §2. In particular, every directed graph G comes with a set theoretic map $\beta_G : \|G\| \rightarrow |G|$ which ‘blows up’ every singularity of $|G|$ (Def. 2.15). The manifold and the local order to which the title refers are the *standard atlas* $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ with its *standard parallelization* (Def. 3.3 and 3.6), and the *standard local order* $\mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$ (Def. 5.10); their underlying sets are $\|G_1\| \times \cdots \times \|G_n\|$ and $|G_1| \times \cdots \times |G_n|$ respectively. The transition maps of $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ are translations (Lem. 3.2), which makes the derivatives of smooth maps between standard atlases easy to represent (Lem. 3.9). The standard parallelization of $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$, which we denote by (f_1, \dots, f_n) , induces a local order $\tilde{\mathcal{X}}_{G_1} \times \cdots \times \tilde{\mathcal{X}}_{G_n}$ on $\|G_1\| \times \cdots \times \|G_n\|$, and the parallelization can be recovered from the local order (Def. 5.15 and Thm. 5.39); this result is based on an equivalence of categories due to (Lawson, 1989, p. 283-284) (§5.4). The product map $\beta_{G_1} \times \cdots \times \beta_{G_n}$ satisfies a universal lifting property relating the local orders $\tilde{\mathcal{X}}_{G_1} \times \cdots \times \tilde{\mathcal{X}}_{G_n}$ and $\mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$ (Thm. 5.31).

From every map $\ell : \{\text{arrows of } G\} \rightarrow \mathbb{R}_+$ admitting a positive lower bound, one defines a metric $d_{|G|}$ on the set $|G|$; metric spaces of this form are called *metric graphs* – see (Bridson & Haefliger, 1999, §1.9, p. 6-7) and (Papadopoulos, 2013, p. 43). As in (Haucourt, 2018, §6.2, p. 1745-1748) they are meant to replace intervals of \mathbb{R} in view of modeling programs with branchings. Due to their prominent role (and because we have not found a presentation that fits our needs) we thoroughly describe these spaces in §4. For our purpose we assume that $|G_1| \times \cdots \times |G_n|$ is equipped with the α -product metric $d^{(\alpha)}$ for some $\alpha \in [1, \infty]$ (Def. B.12). If we wish α to be in accordance with the *parallel execution time principle* (§6.1) we should take $\alpha = \infty$. We transport the α -norm on \mathbb{R}^n to every tangent space of $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ through (f_1, \dots, f_n) (Def. B.14 (25)). The *smoothed α -length* $\mathcal{L}^\alpha(\gamma)$ of a piecewise smooth path γ is the sum of the lengths of its tangent vectors (Def. 4.16). Assuming that p and q are the images of p' and q' under $\beta_{G_1} \times \cdots \times \beta_{G_n}$, the distance $d^{(\alpha)}(p, q)$ is the infimum of $\mathcal{L}^\alpha(\gamma)$ for γ a piecewise smooth path from p' to q' (Thm. 4.21). So for every $\alpha \in [1, \infty]$, the metric $d^{(\alpha)}$ derives from \mathcal{L}^α which derives from (f_1, \dots, f_n) .

From the computer science point of view, for $\varepsilon > 0$ sufficiently small, the sequences of multi-instructions induced by a directed path and its ε -approximations (Def. 6.1) lead to the same result (Haucourt, 2018, Thm. 6.1). Also, every directed path γ on a tile compatible subset M of $|G_1| \times \cdots \times |G_n|$ (resp. $\|G_1\| \times \cdots \times \|G_n\|$) admits a piecewise affine ε -approximation on M (Thm. 6.5, Cor. 6.6, and Def. 2.11). Since the geometric and the smooth models of a conservative program are tile compatible (Def. 2.13 and 2.22), any execution trace of the program to model can be represented by a piecewise affine path.

2. Underlying sets of models

As we shall see in sections 3, 4, and 5, the continuous models of conservative programs inherit their structures from atlases, metrics, and local orders. We explicitly describe their underlying sets.

For the rest of the article, we write ‘graph’ to mean ‘directed (multi)graph’; hence a *graph* is a set G , whose elements are called *points*, together with three maps, namely $\dim : G \rightarrow \{0, 1\}$, and $\text{src}, \text{tgt} : G^{(1)} \rightarrow G^{(0)}$ with $G^{(\varepsilon)} = \{x \in G \mid \dim x = \varepsilon\}$ for $\varepsilon \in \{0, 1\}$. These maps are respectively called *dimension*, *source*, and *target*. A *vertex* (resp. an *arrow*) is a point of G of dimension 0 (resp. 1). Given a vertex v , an arrow a is said to be *ingoing* (resp. *outgoing*) when $\text{tgt } a = v$ (resp. $\text{src } a = v$). A vertex with no ingoing or no outgoing arrow is called an *endpoint*. We use the same denotation for the graph and its underlying set. A map $f : G \rightarrow G'$ is a *morphism of graphs* when it preserves dimensions, sources, and targets. The category of graphs is denoted by **Grph**.

The next definition is motivated by topological arguments (Haucourt, 2018, Lem. 6.1):

Definition 2.1 (Haucourt (2018), Def. 3.4). A *directed path* on a tuple of graphs (G_1, \dots, G_n) is a finite sequence p_0, \dots, p_k of points of $G_1 \times \dots \times G_n$ satisfying

$$\left\{ \begin{array}{l} \text{pr}_i(p_j) \neq \text{pr}_i(p_{j-1}) \Rightarrow \text{src}(\text{pr}_i(p_j)) = \text{pr}_i(p_{j-1}), \text{ for every } i \in \{1, \dots, n\} \\ \text{or} \\ \text{pr}_i(p_j) \neq \text{pr}_i(p_{j-1}) \Rightarrow \text{pr}_i(p_j) = \text{tgt}(\text{pr}_i(p_{j-1})), \text{ for every } i \in \{1, \dots, n\} \end{array} \right. \quad (1)$$

for every $j \in \{1, \dots, k\}$ with $\text{pr}_i : G_1 \times \dots \times G_n \rightarrow G_i$ denoting the i^{th} projection. In the first (resp. second) case of (1), we say that the j^{th} step of the directed path *raises* (resp. *lowers*) dimension.

Discrete models of conservative programs. The notion of a conservative program (Haucourt, 2018, §4.1) is based on a virtual machine allowing parallel execution of *multi-instructions* i.e. partial maps μ on $\{1, \dots, n\}$ assigning to the i^{th} process of the machine the instruction it has to execute (the i^{th} process is inactive during the execution of μ if the latter is not defined at i). Every instruction executed by μ is supposed to be *atomic* in the sense that «its execution should appear to take effect instantaneously at some moment between its invocation and response» (Herlihy et al., 2020, Principle 3.4.1, p.58).

Definition 2.2. A *sequential process* is a graph G whose arrows are labeled with atomic instructions. We suppose that every vertex of G with at least two outgoing (resp. ingoing) arrows has at least one ingoing (resp. outgoing) arrow; this assumption is used exactly once, but crucially, in the proof of Lem. 4.20. A *program* P is an n -tuple ($n \in \mathbb{N}$) of sequential processes running concurrently (if $n = 1$ then P is just a sequential process). We write $P = P_1 \mid \dots \mid P_n$ to mean that P_i is the i^{th} process of P with $i \in \{1, \dots, n\}$; in that case we denote by G_i the underlying graph of P_i . Hence P is associated with the n -tuple of graphs (G_1, \dots, G_n) .

With the notation from Def. 2.2, the vertices of G are possible positions of the instruction pointer p . At each step p moves from the source v to the target v' of the arrow a carrying the next instruction to execute. However, our setting does not allow p to jump from v to v' without going through an intermediate stage, namely the arrow a itself. The arrows of G are thus legit positions of the instruction pointer.

The virtual machine on which P is executed comes with a pool of resources shared by the processes of P . These resources are *renewable* in the sense that they can be restored after use: each process can *take* and *release* them by means of atomic instructions.

Definition 2.3. Let (G_1, \dots, G_n) be the tuple of graphs associated to P (Def. 2.2), and let p_0, \dots, p_k be a directed path on (G_1, \dots, G_n) (Def. 2.1). For $j \in \{1, \dots, k\}$, we denote by μ_j the multi-instruction that is executed when the instruction pointer moves from p_{j-1} to p_j : if the j^{th} step raises (resp. lowers) dimension (Def. 2.1), then the indices i such that $\text{pr}_i(p_j) \neq \text{pr}_i(p_{j-1})$ are those of the processes that start (resp. finish) the execution of an atomic instruction. By an abuse of language, we write ‘the execution of the directed path p_0, \dots, p_k ’ to refer to the execution of the sequence of multi-instructions μ_1, \dots, μ_k . An *execution trace* of P is a directed path on (G_1, \dots, G_n) whose execution respects conditional branchings (Haucourt, 2018, p. 1729).

Definition 2.4 (Haucourt (2018), Def. 4.1). The program P is said to be *conservative* when the amount of resources mobilized after the execution of any directed path p_0, \dots, p_k on (G_1, \dots, G_n) only depends on p_0 and p_k .

A program P is conservative if, and only if, so are its processes (Haucourt, 2018, Lem. 4.1). Whether a sequential process is conservative is decided by a breadth first traversal of its underlying graph (Haucourt, 2018, p. 1734-1735).

Definition 2.5. For the purposes of our study, we only need to know that the *discrete model* of a conservative program P (Haucourt, 2018, §4.2) is a subset $\llbracket P \rrbracket_d$ of the product of sets $G_1 \times \dots \times G_n$ satisfying the following properties:

- (1) If the virtual machine fulfills the resource requirements specified in P , then any directed path on $\llbracket P \rrbracket_d$ (Def. 2.1) induces a sequence of multi-instructions that the virtual machine can execute without exhausting its pool of resources.
- (2) For every execution trace of P , there exists a directed path on $\llbracket P \rrbracket_d$ with the same sequence of multi-instructions (Haucourt, 2018, Thm. 4.1).

In other words, the amount of resources required to execute a conservative program P is known at compile time. Moreover, according to the second point of Def. 2.5, the discrete model of P induces an overapproximation of all its possible behaviors (i.e. all the sequences of multi-instructions associated with the execution traces of P); the subtlety is that multiple directed paths on (G_1, \dots, G_n) may induce the same sequence of multi-instructions.

Continuum of states. We have already mentioned that the arrows of the graph of a sequential process are intermediate positions of the instruction pointer (Def. 2.2). Pushing this principle further, we specify *where* the pointer stands on a given arrow of G . Suppose that the map

$$\ell : G^{(1)} \rightarrow [R, \infty[$$

assigns to each arrow a *length* which is meant to be the execution time of the atomic instruction it carries (§6). The real number R is supposed to be *positive* because execution times should not be arbitrarily small. This constraint has important consequences for the mathematical properties of the models that will be defined later. We end up with a continuum of states:

Definition 2.6. The *underlying set* of the *metric graph* of G is the set

$$|G| = G^{(0)} \cup \bigcup \{ \{a\} \times]0, \ell(a)[\mid a \in G^{(1)} \} .$$

The quotient map $\pi_G : |G| \rightarrow G$ is defined by $\pi_G(v) = v$ on $G^{(0)}$ and by $\pi_G(a, t) = a$ elsewhere. The local order on $|G|$ (Def. 5.10) encodes the constraints that force the pointer p to move continuously with respect to the direction imposed by the arrows. The source and the target of an arrow a play the role of $(a, 0)$ and $(a, \ell(a))$ which are intentionally omitted. An *arrow* of $|G|$ is a subset of the form $\{a\} \times]0, \ell(a)[$ for some $a \in G^{(1)}$; by extension its source and its target are those of a .

Definition 2.7. Subsets of the form $\{a\} \times J$ are said to be *initial* (resp. *final*) when J is an *initial* (resp. *final*) subinterval of $]0, \ell(a)[$. A subset S of $|G|$ is said to be a *star* when it is of the form

$$S(v, i, f) = \bigcup_{\text{tgt } a=v} \{a\} \times]\ell(a) - f(a), \ell(a)[\cup \{v\} \cup \bigcup_{\text{src } a=v} \{a\} \times]0, i(a)[$$

with the functions $i : \text{src}^{-1}\{v\} \rightarrow \mathbb{R}_+ \setminus \{0\}$, $f : \text{tgt}^{-1}\{v\} \rightarrow \mathbb{R}_+ \setminus \{0\}$ satisfying $i(a), f(a) \leq \ell(a)$ whenever it makes sense. The vertex v is called the *center* of the star. Moreover:

- we write $S(v)$ when $i(a) = \ell(a)$ and $f(a) = \ell(a)$ whenever it makes sense.
- if both i and f are constant and equal to $r < \frac{R}{2}$, we write $S(v, r)$ instead of $S(v, i, f)$, and call $S(v, r)$ the star centered at v of radius r .
- the *canonical star* centered at v is the set $S(v, \frac{\ell}{2}, \frac{\ell}{2})$, i.e. $i(a) = \ell(a)/2$ for every $a \in \text{src}^{-1}\{v\}$ and $f(a) = \ell(a)/2$ for every $a \in \text{tgt}^{-1}\{v\}$.

Definition 2.8. A *segment* of G centered at (a, t) with $a \in G^{(1)}$ and $t \in]0, \ell(a)[$, is a subset of $|G|$ of the form $\{a\} \times]t - \varepsilon, t + \varepsilon[$ with $\varepsilon \leq \min(t, \ell(a) - t)$.

Remark 2.9. The intersection of two stars is either a star or a (possibly empty) disjoint union of segments. The intersection of a segment with a star (resp. a segment) is either a segment, or the union of two disjoint segments, or empty. Stars and segments thus form a base of a topology. The stars centered at v (resp. the segments centered at (a, t)) form a base of neighborhoods of v (resp. (a, t)). We give some immediate consequences: the topology of $|G|$ is Hausdorff; the boundary of $\{a\} \times]0, \ell(a)[$ is $\{\text{src } a, \text{tgt } a\}$; the boundary of $S(v)$ is the set $\{\text{src}(a), \text{tgt}(a) \mid a \in G^{(1)}, v \in \{\text{src}(a), \text{tgt}(a)\} \setminus \{v\}$ (for any vertex v); and the connected components of $|G| \setminus G^{(0)}$ are the segments $\{a\} \times]0, \ell(a)[$ for a arrow of G .

Definition 2.10. The *topology* of $|G|$ is the one described in Rem. 2.9; the set $|G|$ equipped with its topology is called the *geometric realization* of G .

Definition 2.11. The *tile* over $p \in G_1 \times \cdots \times G_n$ is the set $(\pi_{G_1} \times \cdots \times \pi_{G_n})^{-1}\{p\}$, i.e. $\tau_1 \times \cdots \times \tau_n$ with $\tau_i = \{p_i\}$ if p_i is a vertex, and $\tau_i = \{p_i\} \times]0, \ell(p_i)[$ if p_i is an arrow. Each τ_i admits a canonical (total) order; the product of these total orders is the canonical partial order of the tile over p . A path γ on $|G_1| \times \cdots \times |G_n|$ is said to be *directed* when for every tile τ and every interval $J \subseteq \gamma^{-1}(\tau)$, the restriction of γ to J is order preserving; we say that γ *crosses* a tile τ if there exists $[a, b] \subseteq \text{dom}(\gamma)$ such that $\gamma([a, b]) \subseteq \bar{\tau}$ (the closure of τ) and $\gamma([a, b]) \setminus \tau = \{\gamma(a), \gamma(b)\}$. A subset of $|G_1| \times \cdots \times |G_n|$ is said to be *tile compatible* when it can be written as a (necessarily disjoint) union of tiles. The *middle* of the tile over p is the point (x_1, \dots, x_n) defined by

$$x_i = \begin{cases} p_i & \text{if } p_i \text{ is a vertex,} \\ (p_i, \frac{1}{2}\ell(p_i)) & \text{if } p_i \text{ is an arrow,} \end{cases}$$

for $i \in \{1, \dots, n\}$. The *lower corner* and the *upper corner* of the tile over p are the points (x_1, \dots, x_n) and (y_1, \dots, y_n) defined, for $i \in \{1, \dots, n\}$, by

$$x_i = \begin{cases} p_i & \text{if } p_i \text{ is a vertex,} \\ \text{src}(p_i) & \text{if } p_i \text{ is an arrow} \end{cases} \quad \text{and} \quad y_i = \begin{cases} p_i & \text{if } p_i \text{ is a vertex,} \\ \text{tgt}(p_i) & \text{if } p_i \text{ is an arrow.} \end{cases}$$

Note that if one of the components of p is an arrow, then the source and the target of p do not belong to the tile over p .

Remark 2.12. With the notation from Def. 2.11, it follows from Def. 2.10 that the closure of a tile τ over p , which we denote by $\bar{\tau}$, is the set of points (x_1, \dots, x_n) such that for $i \in \{1, \dots, n\}$ we have $x_i = p_i$ if p_i is a vertex, and $x_i \in \{p_i\} \times]0, \ell(p_i)[\cup \{\text{src}(p_i), \text{tgt}(p_i)\}$ if p_i is an arrow. In particular $\bar{\tau}$ is tile compatible; this property is related to the *frontier condition* in the theory of *stratified spaces* (Pflaum, 2001, p. 15), (Nocera & Volpe, 2023, Def. 2.5).

The progress graphs shown on Fig. 1 are tile compatible, and more generally:

Definition 2.13. (Haucourt, 2018, Def. 6.3). The underlying set of the geometric model of P , which we denote by $|P|$, is the set $(\pi_{G_1} \times \dots \times \pi_{G_n})^{-1} \llbracket P \rrbracket_d$ with $\llbracket P \rrbracket_d$ denoting the discrete model of P (Def. 2.5); it is tile compatible.

The notion of tile compatibility deserves some explanations. Every $p \in \llbracket P \rrbracket_d$ is (tacitly) labeled by the multi-instruction μ that is defined at i when p_i is an arrow of G_i (the underlying graph of the i^{th} process of P), in which case the (atomic) instruction μ_i is the label of p_i – see Def. 2.2. Denote by τ_p the tile over p , and let γ be a directed path (Def. 2.11) from the lower corner to the upper corner of τ_p which only visits τ_p and its corners. Then γ represents an execution of the multi-instruction μ in which the speeds of the processes may vary independently of each other, with the only restriction that all the processes start and finish at the same time. From this point of view, the tile over p represent all the possible continuous schedules for the execution of μ (by contrast, there is only one discrete schedule for the execution of μ). A more elaborate discussion should take into account all the directed paths γ that cross τ_p (Def. 2.11). Suppose that $[a, b] \subseteq \text{dom}(\gamma)$ with $\gamma([a, b]) \subseteq \bar{\tau}_p$ and $\gamma([a, b]) \setminus \tau_p = \{\gamma(a), \gamma(b)\}$. Also suppose that γ is written as the concatenation $\gamma_1\gamma_2\gamma_3$ with γ_2 the restriction of γ to $[a, b]$. As long as one is only concerned with the execution trace associated with γ , one can ‘replace’ γ by $\delta = \gamma_1\tilde{\gamma}_2\gamma_3$ provided that $\tilde{\gamma}_2$ is a directed path defined on $[a, b]$ with $\tilde{\gamma}_2(a) = \gamma(a)$, $\tilde{\gamma}_2(b) = \gamma(b)$, and $\tilde{\gamma}_2([a, b]) \subseteq \tau_p$ – see (Haucourt, 2018, Thm. 6.1 and Cor. 6.2) for a formal statement. It may be that δ is smooth while γ is not, and the path δ may be of lesser length – see Fig. 2. If X is a tile compatible subset of $|G_1| \times \dots \times |G_n|$, then we have $\text{img}(\gamma) \subseteq X \Leftrightarrow \text{img}(\delta) \subseteq X$. Consequently, in a tile compatible set, one can choose the directed path representing a given execution trace in order to optimize certain parameters: the cases of smoothness and execution time are discussed in §6, and more specifically in Thm. 6.5 and Cor. 6.6. Tile compatible subsets also play an crucial role in the preamble of §5.

Blowing up vertices. For every arrow a of G the canonical bijection $\{a\} \times]0, \ell(a)[\rightarrow]0, \ell(a)[$ induces a *dihomomorphism* (i.e. a homeomorphism that is also an order isomorphism, see p. 19).

Definition 2.14. A *traversal* at v is an ordered pair of arrows (a, b) such that $\text{tgt } a = v = \text{src } b$, we say that v is *regular* when there is *exactly* one traversal at v in G ; otherwise v is said to be *singular*. By extension, the traversal (a, b) is said to be *regular* (resp. *singular*) when so is v .

From the topological and order theoretic points of view, the set $|G|$ is ‘almost everywhere’ like \mathbb{R} . The only possible exceptions are the singular vertices of G . As for curves in Algebraic Geometry⁽¹⁾ we ‘resolve singularities’ by ‘blowing them up’, i.e. we replace each vertex v by the traversals at v :

Definition 2.15. The *underlying set* of the *blowup* of G , which we denote by $\|G\|$, is the union of $|G| \setminus G^{(0)}$ and the set of traversals of G , that is to say

$$\|G\| = \bigcup \left\{ \{a\} \times]0, \ell(a)[\mid a \in G^{(1)} \right\} \cup \left\{ (a, b) \in G^{(1)} \times G^{(1)} \mid \text{tgt } a = \text{src } b \right\}.$$

⁽¹⁾« The effect of blowing up is thus to separate out branches of curves passing through O according to their slopes » (Hartshorne, 1977, p.30). In our context, the traversals at v play the role of the slopes of branches at O .

The *blowup* of G is the map $\beta_G : \|G\| \rightarrow |G|$ defined by $\beta_G(a, b) = \text{tgt } a$ (or $\text{src } b$) for every traversal (a, b) of G , and by $\beta_G(a, t) = (a, t)$ for every $a \in G^{(1)}$ and $t \in]0, \ell(a)[$. A subset of $\|G_1\| \times \cdots \times \|G_n\|$ is said to be *tile compatible* when it is the inverse image under $\beta_{G_1} \times \cdots \times \beta_{G_n}$ of some tile compatible set (Def. 2.11). From now on we write β_i instead of β_{G_i} .

Remark 2.16. The image of β_G is precisely $|G| \setminus \{\text{endpoints of } G\}$ (see p. 5).

Remark 2.17. The constructions $|_ |$ and $\| _ \|$ readily extend to functors from **Grph** to **Set**: given a graph morphism $f : G \rightarrow G'$, the maps $|f|$ and $\|f\|$ are defined by

$$|f|(v) = f^{(0)}(v), \quad \|f\|(a, b) = (f^{(1)}(a), f^{(1)}(b)), \quad \text{and} \quad |f|(a, t) = \|f\|(a, t) = (f^{(1)}(a), t),$$

where $v \in G^{(0)}$, $a \in G^{(1)}$, $t \in]0, \ell(a)[$, (a, b) is a traversal of G , and $f^{(0)}$, $f^{(1)}$ denote the vertex and the arrow parts of f . Moreover, the collection of blowups $\beta_G : \|G\| \rightarrow |G|$, for G running through the class of graphs, forms a natural transformation from $\| _ \|$ to $|_ |$. If $|G|$ and $|G'|$ are equipped with the topologies from Rem. 2.9, then $|f|$ is continuous.

Remark 2.18. It is natural to define the *topology of $\|G\|$* as the initial topology of the blowup β_G : a base of this topology is given by the inverse images under β_G of the stars and segments (Rem. 2.9). By observing that $\beta_G^{-1}(S(v, i, f))$ is the set

$$\bigcup_{\text{tgt } a=v=\text{src } b} \{a\} \times]\ell(a) - f(a), \ell(a)[\cup \{(a, b)\} \cup \{b\} \times]0, i(b)[$$

we conclude that if the graph G has a singular traversal τ (Def. 2.14), then the topology of $\|G\|$ is not T_1 ; indeed, the set $\{\tau\}$ is not closed. Conversely, if G has no singular traversal, then the topology of $\|G\|$ is Hausdorff (T_2).

Lemma 2.19. The image of a tile compatible set (in the sense of Def. 2.15) under $\beta_1 \times \cdots \times \beta_n$ is tile compatible (in the sense of Def. 2.11).

Proof. Suppose that $Y = (\beta_1 \times \cdots \times \beta_n)^{-1}X$ for some tile compatible set X and that $q \in Y \cap (\beta_1 \times \cdots \times \beta_n)^{-1}\tau$ for some tile τ . Let $p \in \tau$. We have $(\beta_1 \times \cdots \times \beta_n)(q) \in X$ therefore $\tau \subseteq X$ because X is tile compatible; hence $p \in X$. For $i \in \{1, \dots, n\}$ if q_i is a traversal, then put $q'_i = q_i$; otherwise $q'_i = p_i$. Then $(\beta_1 \times \cdots \times \beta_n)(q') = p$.

Example 2.20. We denote by $G_{\mathbb{Z}}$ the graph whose arrows (all of length 1) are $(n, n+1)$ with $n \in \mathbb{Z}$, the source and the target being n and $n+1$ respectively.

$$\dots \longrightarrow \overset{n}{\circ} \longrightarrow \overset{n+1}{\circ} \longrightarrow \dots$$

Every vertex of $G_{\mathbb{Z}}$ has a unique traversal so the blowup of $G_{\mathbb{Z}}$ is a bijection. We obtain a canonical bijection from $\|G_{\mathbb{Z}}\|$ to \mathbb{R} by composing the blowup of $G_{\mathbb{Z}}$ with the following bijection:

$$\begin{aligned} |G_{\mathbb{Z}}| &\longrightarrow \mathbb{R} \\ n &\longmapsto n \\ ((n, n+1), t) &\longmapsto n+t \end{aligned}$$

Example 2.21. Fig. 3 provides an exhaustive description of the blowup of a graph G whose arrows are supposed to be of length 1, and which contains a branching point v .

We will see that every set $\|G\|$ carries an atlas (Def. 3.3). Connectedness of this atlas, as well as non-Hausdorffness of its underlying topology (Def. A.8), derive from the fact that any

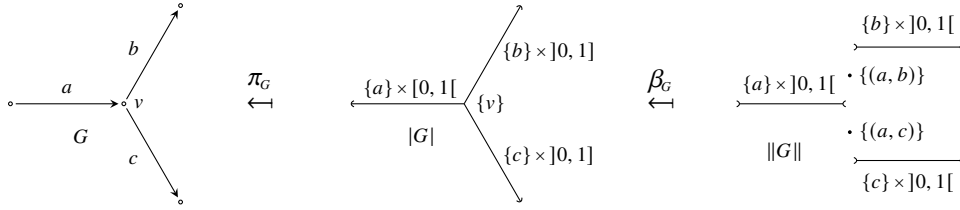


Figure 3. The blowup of a graph containing a branching point.

neighborhood of a traversal (a, b) meets any neighborhood of another traversal (a', b') precisely when $a = a'$ or $b = b'$ (see the proof of Lem. 3.2).

We have already observed that the endpoints of G are ‘forgotten’ by $\|G\|$. One may think of this as an issue since the *starting point* of the automaton associated with a process is an endpoint. We circumvent the difficulty taking advantage of the following control flow graph feature: the starting point has a unique outgoing arrow a (Def. 2.2). So we can harmlessly declare any point on $\{a\} \times]0, \ell(a)[$ to be the starting point instead of the source of a . Another (and more canonical) way to solve the problem consists of adding a fresh arrow ending at the starting point we wish to save. The same remarks hold (and the same tricks work) for the *final points* of the graph.

Definition 2.22. The *underlying set* of the *smooth model* of P , which we denote by $\|P\|$, is the set $(\beta_1 \times \cdots \times \beta_n)^{-1} |P|$ with $|P|$ as in Def. 2.13. Following Def. 2.15, the set $\|P\|$ is tile compatible.

3. Differential calculus on standard atlases

Manifolds are equivalence classes of atlases (Def. A.4). Such a class contains a maximal element (with respect to inclusion) with which the manifold is usually identified. However, each manifold we will meet in this article comes with a canonical representative that is much more tractable than the maximal one. Indeed, the representation of any tangent vector in a *standard atlas* (Def. 3.3) does not depend on the *standard chart* (Def. 3.1) in which it is represented. Consequently, smooth maps between standard atlases are handled (almost) as smooth maps between open subsets of euclidean spaces. For this reason, we will only deal with atlases – see Appendix A.

We are not concerned with global properties of manifolds, so we can harmlessly let them be non-Hausdorff.⁽²⁾ This seemingly anodyne weakening is of crucial importance here for it allows *branchings* (Ex. 2.21) which is precisely what we need to build manifolds from graphs, see Def. 3.3 and Rem. 3.5. The relation between non-Hausdorff manifolds, branching, non-determinism, and bifurcation, is well-known (Müller (2013)). Such manifolds naturally appear in the study of dynamical systems (Goel (1987)), foliations of the plane (Haefliger & Reeb (1957)), (Hirsch, 1976, 11., p.15), (Gauld, 2014, Chap.9) and general relativity (Hájíček (1971a,b); Luc & Placek (2020)). The novelty is that we use non-Hausdorff manifolds to formalize the idea that a graph is a kind of ‘discrete vector field’ (Def. 3.6).

Standard atlases. Given a graph G with $\ell : G^{(1)} \rightarrow [R, \infty[$, the set $\|G\|$ (Def. 2.15) admits a ‘natural’ atlas that we now describe. Given $x \in G^{(1)}$ we define

$$I_x =]0, \frac{\ell(x)}{2}[, \quad J_x =]\frac{\ell(x)}{2}, \ell(x)[, \quad (2)$$

and keep in mind the bijection $t \mapsto t - \ell(x)$ from J_x to $-I_x$.

⁽²⁾ « The concept of Hausdorffness is irrelevant for much of local differential geometry » (Hicks, 1965, p.3). A similar point of view is adopted in (Lang, 1999, p.23).

Definition 3.1. The *standard charts* of G are the bijections

$$\begin{aligned} \phi_a & : \{a\} \times]0, \ell(a)[\rightarrow]0, \ell(a)[, \text{ and} \\ \phi_{ab} & : \{a\} \times J_a \cup \{(a, b)\} \cup \{b\} \times I_b \rightarrow -I_a \cup \{0\} \cup I_b \\ & \text{with } (a, t) \mapsto t - \ell(a), \quad (a, b) \mapsto 0, \quad (b, t) \mapsto t \end{aligned}$$

for all arrows a and all traversals (a, b) of G . The *standard charts* of (G_1, \dots, G_n) are the products $\phi_1 \times \dots \times \phi_n$ with ϕ_i a standard chart of G_i for $i \in \{1, \dots, n\}$.

Lemma 3.2. The collection of standard charts of (G_1, \dots, G_n) is an atlas on the set $\|G_1\| \times \dots \times \|G_n\|$, and the transition maps between its charts are translations (§A.1, p. 43).

Proof. The domains of the standard charts cover $\|G\|$. Given two arrows a and a' with $a \neq a'$ the domains of ϕ_a and $\phi_{a'}$ are disjoint. By applying the distributivity of \cap over \cup , and the fact that $I_x \cap J_x = \emptyset$ – see (2), the intersection of $\text{dom } \phi_{ab}$ and $\text{dom } \phi_{a'b'}$ is the disjoint union

$$(\{a\} \cap \{a'\}) \times (J_a \cap J_{a'}) \cup (\{(a, b)\} \cap \{(a', b')\}) \cup (\{b\} \cap \{b'\}) \times (I_b \cap I_{b'}).$$

The transition map $\phi_{a'b'} \circ \phi_{ab}^{-1}$ is an identity whose domain is $\text{cod}(\phi_{ab})$ if $(a, b) = (a', b')$, the interval J_a if $a = a'$ and $b \neq b'$, the interval I_b if $a \neq a'$ and $b = b'$, empty otherwise. If $c \notin \{a, b\}$ then the domains of the charts ϕ_{ab} and ϕ_c are disjoint. So the only nontrivial case is when $c \in \{a, b\}$. We take $\{a\} \times J_a$ as witness of compatibility of ϕ_{ab} and ϕ_a since we have

$$\phi_{ab} \circ \phi_a^{-1} : t \in J_a \mapsto t - \ell(a) \in -I_a,$$

and J_b as witness of compatibility of ϕ_{ab} and ϕ_b since $\phi_{ab} \circ \phi_b^{-1}$ is just the identity map on J_b . Hence the transition maps are translations. We readily deduce the higher dimensional case from the description of the transition maps between the charts of a product of atlases (Def. A.5); in particular, the transition maps are translations as products of such maps.

Definition 3.3. The *standard atlas* of the tuple of graphs (G_1, \dots, G_n) is the collection of standard charts of (G_1, \dots, G_n) (Def. 3.1). Following Def. A.5, it is the product atlas $\mathcal{A}_{G_1} \times \dots \times \mathcal{A}_{G_n}$ of the standard atlases \mathcal{A}_{G_i} of G_i .

Example 3.4. The canonical bijection $\|G_{\mathbb{Z}}\| \cong \mathbb{R}$ (Ex. 2.20) induces a smooth diffeomorphism (Def. A.5) between the standard atlas of $G_{\mathbb{Z}}$ and $\{id_{\mathbb{R}}\}$ (Ex. A.7).

Remark 3.5. The topology of \mathcal{A}_G (Def. A.8) is Hausdorff if, and only if, the graph G has no singular traversal (Def. 2.14); in which case it is the topology of $\|G\|$ (Rem. 2.18). If G has a singular traversal, then the topology of \mathcal{A}_G (which is T_1 by Lem. A.9) differs from that of $\|G\|$ (which is not T_1 by Rem. 2.18).

Tangent vectors. For any standard charts ϕ and ψ of (G_1, \dots, G_n) , the derivative of $\psi \circ \phi^{-1}$ at any point of $\text{cod } \phi \cap \text{cod } \psi$ is the identity map $id_{\mathbb{R}^n} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (Lem. 3.2). It follows from §A.3 (22) that for all $v \in T_p(\mathcal{A}_{G_1} \times \dots \times \mathcal{A}_{G_n})$ the representation $\llbracket v \rrbracket_p^{\phi}$ does not depend on the standard chart ϕ . So we may as well decide that the tangent space of $\mathcal{A}_{G_1} \times \dots \times \mathcal{A}_{G_n}$ at p is

$$T_p(\mathcal{A}_{G_1} \times \dots \times \mathcal{A}_{G_n}) = \{p\} \times \mathbb{R}^n \quad (3)$$

with the obvious topological vector space structure.

Definition 3.6. A *standard tangent chart* of (G_1, \dots, G_n) is a product $\phi \times id_{\mathbb{R}^n}$ with ϕ a standard chart of (G_1, \dots, G_n) ; it corresponds to the chart $T\phi$ from §A.3 (23) provided that following (3)

we adopt $\|G_1\| \times \cdots \times \|G_n\| \times \mathbb{R}^n$ as the underlying set of the tangent bundle $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$:

$$\begin{aligned} T\phi : \text{dom } \phi \times \mathbb{R}^n &\rightarrow \text{cod } \phi \times \mathbb{R}^n \\ (p, t) &\mapsto (\phi p, t). \end{aligned}$$

The *standard tangent atlas* of (G_1, \dots, G_n) is the collection of its standard tangent charts. The *standard representation* of the tangent bundle of $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ is the set $\|G_1\| \times \cdots \times \|G_n\| \times \mathbb{R}^n$ equipped the standard tangent atlas. The *standard parallelization* of (G_1, \dots, G_n) is the tuple of vector fields (f_1, \dots, f_n) with:

$$\begin{aligned} f_i : \|G_1\| \times \cdots \times \|G_n\| &\rightarrow (\|G_1\| \times \cdots \times \|G_n\|) \times \mathbb{R}^n \\ p &\mapsto p, e_i \end{aligned}$$

for $i \in \{1, \dots, n\}$ and e_i the i^{th} vector of the canonical basis of \mathbb{R}^n . The *standard basis* of the tangent space at p is $(f_1(p), \dots, f_n(p))$; its *positive cone* is the set of vectors whose coordinates in the standard basis are non-negative.

Remark 3.7. Strictly speaking, even if (f_1, \dots, f_n) and (g_1, \dots, g_m) are standard parallelizations (Def. 3.6), then $(f_1, \dots, f_n, g_1, \dots, g_m)$ is not. Yet, the latter is related to a standard parallelization by the obvious smooth diffeomorphism (Def. A.5)

$$(\mathcal{A}_{G_1} \times \mathbb{R}) \times \cdots \times (\mathcal{A}_{G_n} \times \mathbb{R}) \cong (\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}) \times \mathbb{R}^n.$$

Derivatives. Let $f : \mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n} \rightarrow \mathcal{A}_{G'_1} \times \cdots \times \mathcal{A}_{G'_m}$ be a smooth map between standard atlases (Def. 3.3). The transition maps between standard charts are translations (Lem. 3.2) so we have $D(\phi_0 \circ \phi_1^{-1})_{\phi_1(p)} = id_{\mathbb{R}^n}$ and $D(\psi_1 \circ \psi_0^{-1})_{\psi_0(fp)} = id_{\mathbb{R}^m}$ for all $\phi_0, \phi_1 \in \mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ and $\psi_0, \psi_1 \in \mathcal{A}_{G'_1} \times \cdots \times \mathcal{A}_{G'_m}$, with $p \in \text{dom}(\phi_0) \cap \text{dom}(\phi_1)$ – see §A.1, p. 43. From Lem. A.6, we deduce that the representation of Df_p (Def. A.14) does not depend on the standard charts in which it is represented, so it can be identified with an $n \times m$ matrix with entries in \mathbb{R} which we just denote by $f'p$ without mentioning any chart:

Definition 3.8. The matrix $f'p$ is called the *standard representation* of Df_p and we have

$$Df_p(p, t) = Tf(p, t) = (fp, f'p \cdot t)$$

with $f'p \cdot t$ being the product of the standard representation of Df_p and the vector $t \in \mathbb{R}^n$.

Lemma 3.9. The standard representation $f'p$ (Def. 3.8) is the matrix of the linear map Df_p in the standard bases (Def. 3.6) of the tangent spaces at p and fp .

Proof. The matrix $f'p$ represents $D(\psi \circ f \circ \phi^{-1})_{\phi p}$ in the canonical bases of \mathbb{R}^n and \mathbb{R}^m regardless of the chosen standard charts ϕ and ψ provided their domains contain p and fp (Def. 3.8). We have the atlas morphisms $\phi : \{\phi\} \rightarrow \{id_{\text{cod } \phi}\}$, $\psi : \{\psi\} \rightarrow \text{cod } \{id_{\text{cod } \psi}\}$ (Ex. A.7), and $f : \{\phi\} \rightarrow \{\psi\}$, so we can write

$$D(\psi \circ f \circ \phi^{-1})_{\phi p} = D\psi_{fp} \circ Df_p \circ D\phi_p^{-1}.$$

Assume that \mathbb{R}^n and \mathbb{R}^m are equipped with their canonical bases, and that the tangent spaces at p and fp are equipped with their standard bases. Then the matrices of $D\psi_{fp}$ and $D\phi_p^{-1}$ are identities, while the matrix of Df_p is $f'p$.

Remark 3.10. If γ is a curve on the standard atlas $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$, then for every standard chart ϕ whose domain contains $\gamma(t_0)$, we have the curve $\phi \circ \gamma$ in \mathbb{R}^n . By Lem. 3.9, the coordinates of the ‘speed’ vector $(\phi \circ \gamma)'(t_0)$ in the canonical basis of \mathbb{R}^n are the coordinates of the tangent vector $\gamma'(t_0)$ in the standard basis at $\gamma(t_0)$.

Functoriality. Let \mathbf{Grph}_ℓ be the category whose objects are graphs equipped with a length map $\ell : G^{(1)} \rightarrow [R, \infty[$ and whose morphisms are 2-tuples (g, r_g) with $g : G \rightarrow G'$ a graph morphism such that $r_g = \frac{\ell'(g_1(a))}{\ell(a)}$ for every arrow a . Given two arrows a and b of G , we write $g_1 a = a'$ and $g_1 b = b'$. The map $f : \|G\| \rightarrow \|G'\|$ is defined by $f(a, t) = (a', r_g t)$ for every $t \in]0, \ell(a)[$, and by $f(a, b) = (a', b')$ when (a, b) is a traversal. Such a map f is called a *dilation*. Denoting by $\psi_{a'}$ and $\psi_{a'b'}$ the charts of the standard atlas of $\mathcal{A}_{G'}$ associated with a' and (a', b') , we have

$$\begin{aligned} \psi_{a'} \circ f \circ \phi_a^{-1} & : t \in]0, \ell(a)[& \mapsto & r_g t \in]0, \ell(a')[\\ \psi_{a'b'} \circ f \circ \phi_{ab}^{-1} & : t \in]-\frac{\ell(a)}{2}, \frac{\ell(b)}{2}[& \mapsto & r_g t \in]-\frac{\ell(a')}{2}, \frac{\ell(b')}{2}[\end{aligned}$$

so the map f is smooth, and with the denotation from Def. 3.8 we have $f'p = r_g$ for all $p \in \|G\|$. In particular the construction $G \mapsto \mathcal{A}_G$ (Def. 3.3) extends to a functor

$$\mathcal{A}_* : \mathbf{Grph}_\ell \rightarrow \mathbf{Atl}.$$

4. Metric graphs

The metric on $|G|$ (Def. 4.3) is lifted to a pseudometric on $\|G\|$ along the blowup β_G (Def. 4.7). The pseudometric on $\|G_1\| \times \cdots \times \|G_n\|$ and the standard parallelization on $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ (together with a norm on \mathbb{R}^n) induce two natural ways of defining the length of a piecewise smooth path (Def. 4.16 and B.6) which coincide (Lem. 4.19) and determine the pseudometrics on $|G_1| \times \cdots \times |G_n|$ and $\|G_1\| \times \cdots \times \|G_n\|$ (Def. 4.10, Lem. 4.14, and Thm. 4.21). The above-mentioned pseudometrics depend on the choice of a norm on \mathbb{R}^n , according to execution time consideration, we explain in §6 that the ∞ -norm should be preferred.

Given the map $\ell : G^{(1)} \rightarrow \mathbb{R}_+$ assigning a length to every arrow of a graph G , we wish to define a pseudometric space in which the distance between the points p and q is the greatest lower bound of the lengths of the paths joining p and q . Such a space is called a *metric graph*, it can be defined as the quotient of a disjoint union of intervals (Bridson & Haefliger, 1999, 5.21(3)). If ℓ has a *positive* lower bound R , the resulting pseudometric is an intrinsic metric (Bridson & Haefliger, 1999, 5.20, 5.28, 5.29). Metric graphs can be seen as 1-dimensional *metric simplicial complexes* (Bridson & Haefliger, 1999, 7.40(1)), but dealing with the latter requires a lot of material that is irrelevant in dimension 1; this is why we have chosen a more elementary approach. The pedestrian construction given here is standard (Bridson & Haefliger, 1999, §1.9, p.6), does not need any prerequisite, and allows direct proofs. We have slightly adapted the presentation to our needs.

Definition 4.1. A path γ on $|G|$ is said to be *admissible* (resp. *piecewise affine*) when it can be written as a (finite) concatenation of (resp. *affine*) steps s_1, \dots, s_n (Def. B.19); the *length* of γ is $\ell(\gamma) = \ell(s_1) + \cdots + \ell(s_n)$ (Lem. B.21). A path on $|G_1| \times \cdots \times |G_n|$ is said to be *admissible* (resp. *piecewise affine*) when so are all its components. A path on $\|G_1\| \times \cdots \times \|G_n\|$ is said to be *piecewise affine* when so is its image under the product of blowups $\beta_1 \times \cdots \times \beta_n$ (Def. 2.15).

Remark 4.2. The class of admissible paths is stable under reparametrization; the image and the length of an admissible path are left unchanged under reparametrization, every step is admissible and for every arrow a , we have $\ell(a) = \ell(\chi_a)$, see B.3 (26).

Definition 4.3. The *standard pseudometric* (one readily checks that it is a pseudometric) on $|G|$ is the map $d_{|G|} : |G| \times |G| \rightarrow \mathbb{R}_+ \cup \{\infty\}$ defined by

$$d_{|G|}(p, q) = \inf \{ \ell(\gamma) \mid \gamma : \text{admissible path from } p \text{ to } q \}$$

with the convention that $\inf \emptyset = \infty$; p and q are said to be *neighbors* when $d_{|G|}(p, q) < \frac{R}{2}$. Two neighbors are related by a unique geodesic which is rather easy to describe (Lem. B.27).

In the (discrete) context of a graph, one usually says that two vertices are neighbors when they are the extremities of an arrow; the idea is that an arrow is a geodesic between two vertices.

Remark 4.4. Let γ be an admissible path from p to q . The topology of $|G|$ (Def. 2.10) is Hausdorff (Rem. 2.9) so we have an arc α from p to q such that $\text{img}(\alpha) \subseteq \text{img}(\gamma)$ (Lem. B.3). By Lem. B.22 and B.23 we have $\ell(\alpha) \leq \ell(\gamma)$. Hence we can write ‘arc’ instead of ‘admissible path’ in Def. 4.3.

Lemma 4.5. The map $d_{|G|}$ (Def. 4.3) is an intrinsic metric.

Proof. If $d_{|G|}(p, q) = 0$, then we deduce from Lem. B.27 that $p = q$, otherwise we have

$$\begin{aligned} d_{|G|}(p, q) &= \inf \{ \ell(\gamma) \mid \gamma : \text{admissible path from } p \text{ to } q \} \quad \text{Def. 4.3} \\ &= \inf \{ \ell(\gamma) \mid \gamma : \text{arc from } p \text{ to } q \} \quad \text{Rem. 4.4} \\ &= \inf \{ L(\gamma) \mid \gamma : \text{arc from } p \text{ to } q \} \quad \text{Lem. B.22 and B.28} \\ &= \inf \{ L(\gamma) \mid \gamma : \text{path from } p \text{ to } q \} \quad \text{Lem. B.29.} \end{aligned}$$

Remark 4.6. If we allow the lengths of arrows to be taken in \mathbb{R}_+ (or even $]0, \infty[$) instead of $[R, \infty[$ there might be arbitrarily short paths joining two distinct points, resulting in a pseudometric space instead of a metric one.

We lift the distance $d_{|G|}$ along the blowup β_G (Def. 2.15):

Definition 4.7. The *standard pseudometric* on $\|G\|$ is $d_{|G|} \circ (\beta_G \times \beta_G)$, we denote it by $d_{\|G\|}$.

Remark 4.8. A map f from a topological space X to $\|G\|$ (equipped with the pseudometric topology) is continuous if, and only if, so is the composite $\beta_G \circ f$. Suppose that $\beta_G \circ f$ is continuous at $x_0 \in X$, and let $\varepsilon > 0$. We have a neighborhood V of x_0 such that $\beta_G(f(V)) \subseteq B(\beta_G(f(x_0)), \varepsilon)$. Given $x \in V$ we have $d_{\|G\|}(f(x_0), f(x)) = d_{|G|}(\beta_G(f(x_0)), \beta_G(f(x)))$ (Def. 4.7) therefore $f(V) \subseteq B(f(x_0), \varepsilon)$, and f is continuous. The converse is immediate. In particular the pseudometric $d_{\|G\|}$ is *intrinsic* because so is $d_{|G|}$ (Lem. 4.5).

Remark 4.9. The topology of the standard atlas \mathcal{A}_G (Def. 3.3 and A.8) is strictly finer than the topology of the standard pseudometric on $\|G\|$ since the distance between two traversals at the same point is null.

For the rest of §4 we fix $\alpha \in [1, \infty]$ and a tuple of graphs (G_1, \dots, G_n) .

Definition 4.10. The *standard α -distance* on (G_1, \dots, G_n) refers to both the α -product $d^{(\alpha)}$ of the standard metrics $d_{|G_i|}$ (Def. 4.3), and to the α -product $\tilde{d}^{(\alpha)}$ of the standard pseudometrics $d_{\|G_i\|}$ (Def. 4.7), with the notion of α -product given in Def. B.12.

Remark 4.11. The product of blowups $\beta_{G_1} \times \dots \times \beta_{G_n}$ preserves standard α -distances.

Definition 4.12. [Following Def. B.6]. The *α -length* of a path γ on $|G_1| \times \dots \times |G_n|$ (resp. $\|G_1\| \times \dots \times \|G_n\|$) which we denote by $L^\alpha(\gamma)$, is the least upper bound of the sums $\sum_{i=1}^k d^{(\alpha)}(\gamma(t_{i-1}), \gamma(t_i))$ for $t_0 < \dots < t_k$ subdivisions of $\text{dom } \gamma$.

Remark 4.13. Every path γ on $\|G_1\| \times \dots \times \|G_n\|$ is continuous with respect to the pseudometric from Def. 4.7, which is equivalent to have $\beta_{G_i} \circ \gamma_i$ continuous on the metric graph $|G_i|$ for every $i \in \{1, \dots, n\}$ (Rem. 4.8); in that case we have $L^\alpha(\gamma) = L^\alpha((\beta_{G_1} \times \dots \times \beta_{G_n}) \circ \gamma)$.

Lemma 4.14. The pseudometric $d^{(\alpha)}$ is intrinsic.

Proof. By Prop. B.13, Lem. 4.5, and Rem. 4.8.

Each standard chart induces a ‘local isometry’, the codomain being understood as a subspace of \mathbb{R}^n equipped with the α -norm (Def. B.12):

Proposition 4.15. For every standard chart ϕ of $G_1 \times \cdots \times G_n$ (Def. 3.1) and all $p, q \in \text{dom } \phi$, if $|\phi(p) - \phi(q)|_\alpha < \frac{R}{2}$, then $\tilde{d}^{(\alpha)}(p, q) = |\phi(p) - \phi(q)|_\alpha$.

Proof. By Def. 4.7 we have $d_{\|\cdot\|}(p, q) = d_{|\cdot|}((\beta_G \circ \phi^{-1})(\phi p), (\beta_G \circ \phi^{-1})(\phi q))$. Applying Rem. B.17, $(\beta_G \circ \phi^{-1})(\phi p)$ is equal to $\chi_a(\phi p)$ if $\phi = \phi_a$ for some arrow a ; to $\chi_a(\ell(a) - \phi p)$ or $\chi_b(\phi p)$ depending on the sign of ϕp if $\phi = \phi_{ab}$ for some traversal (a, b) . The same holds for q . We end up with 5 cases to examine, in each of which Lem. B.27 applies and gives $d_{\|\cdot\|}(p, q) = |\phi(p) - \phi(q)|$. In higher dimensions, we follow Def. 4.10 and apply the one-dimension case to each component:

$$\tilde{d}^{(\alpha)}(p, q) = |\dots, d_{G_i}(\beta_{G_i}(p_i), \beta_{G_i}(q_i)), \dots|_\alpha = |\dots, \phi_i(p_i) - \phi_i(q_i), \dots|_\alpha = |\phi(p) - \phi(q)|_\alpha.$$

Definition 4.16. The *smoothed α -length* of a piecewise smooth path γ on the atlas $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$, which we denote by $\mathcal{L}^\alpha \gamma$, is given by Def. B.14 with the standard parallelization (f_1, \dots, f_n) of $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ (Def. 3.6) and the α -norm $|\cdot|_\star$ on \mathbb{R}^n (Def. B.12). We just write \mathcal{L} when $\alpha = 1$. We denote by $d_{\mathcal{A}}^{(\alpha)}$ the metric induced by \mathcal{L}^α .

Remark 4.17. We have the standard inequality $\mathcal{L}^\alpha(\gamma) \leq \mathcal{L}(\gamma)$ as an immediate consequence of the *Jensen inequality* (Brokate & Kersting, 2015, 5.7).

The distance mapping $d : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R}_+ \cup \{\infty\}$ (Def. B.14) is a pseudometric, one may indeed have $d(p, q) = 0$ and $p \neq q$:

Example 4.18. Let \mathcal{L} be the smoothed length on some standard atlas \mathcal{A}_G with (a, b) and (c, d) two traversals at v with a, b, c , and d pairwise distinct. There is a piecewise smooth path on \mathcal{A}_G from (a, b) to (c, d) of arbitrarily small length. It suffices to concatenate the four smooth paths of length ε represented by the sequence of arrows $(a, b) \rightarrow (a, 1 - \varepsilon) \rightarrow (a, d) \rightarrow (d, \varepsilon) \rightarrow (c, d)$ and illustrated on Fig. 4. In particular, the greatest lower bound defining the distance between (a, b) and (c, d) (Def. B.14) is not reached.

Lemma 4.19. For every smooth path γ on $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ we have $\mathcal{L}^\alpha \gamma = L^\alpha \gamma$.

Proof. First observe that γ is continuous with respect to the topology of $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ which is finer than the topology of the pseudometric $d^{(\alpha)}$ (Rem. 4.9). Assume that $\text{img}(\gamma) \subseteq \text{dom}(\phi)$ for some standard chart ϕ , by Rem. 3.10 we have

$$\int_x^y |\gamma'(t)|_\alpha dt = \int_x^y |(\phi \circ \gamma)'(t)|_\alpha dt. \quad (4)$$

For a sufficiently fine subdivision $t_0 < \cdots < t_k$ of $\text{dom}(\gamma)$ we have (Prop. 4.15)

$$\sum_{i=1}^k d^{(\alpha)}(\gamma(t_i), \gamma(t_{i-1})) = \sum_{i=1}^k |\phi \circ \gamma(t_i) - \phi \circ \gamma(t_{i-1})|_\alpha. \quad (5)$$

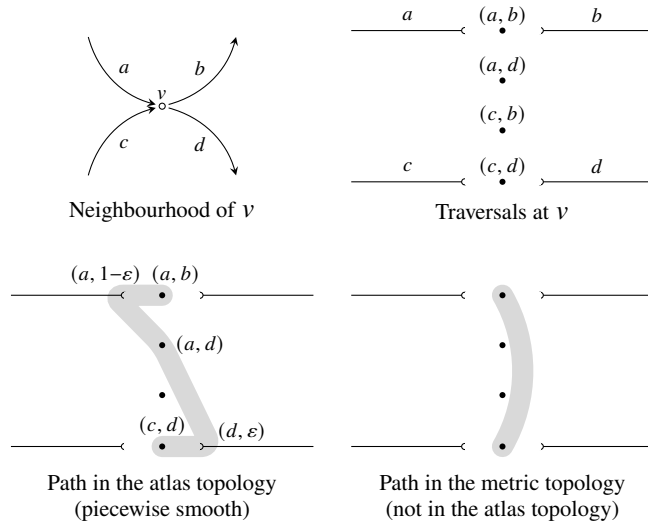


Figure 4. Illustration of Ex. 4.18.

By a standard result about smooth paths on \mathbb{R}^n , the sum $\int_x^y |(\phi \circ \gamma)'(t)|_\alpha dt$ (right member of (4)) is the least upper bound of the sums $\sum_{i=1}^k |\phi \circ \gamma(t_i) - \phi \circ \gamma(t_{i-1})|_\alpha$ (right member of (5)) for $t_0 < \dots < t_k$ running through the set of subdivisions of $dom(\gamma)$. The above-mentioned result is given in (Papadopoulos, 2013, Prop. 1.3.1) for $\alpha = 2$, although the proof is obviously valid for any $\alpha \in [1, \infty]$.

We no longer assume that $dom(\gamma)$ is contained in the domain of a standard chart. Nevertheless, since $dom(\gamma)$ is compact and the domains of the standard charts form an open covering of $\|G_1\| \times \dots \times \|G_n\|$, we have a subdivision $t_0 < \dots < t_k$ of $dom(\gamma)$ such that for all $i \in \{1, \dots, k\}$ the restriction of γ to $[t_i, t_{i-1}]$ is contained in the domain of some standard chart.

We suppose that the graphs G_1, \dots, G_n in the statements of Lem. 4.20 and Thm. 4.21 are associated with the sequential processes of some program P (Def. 2.2), which means that they all satisfy the following property: every vertex with at least two outgoing (resp. ingoing) arrows has at least one ingoing (resp. outgoing) arrow.

Lemma 4.20. For every piecewise affine map γ from p to q on $|G_1| \times \dots \times |G_n|$ (Def. 4.1) and every $\varepsilon > 0$, there exists a piecewise smooth path δ from p' to q' on $\mathcal{A}_{G_1} \times \dots \times \mathcal{A}_{G_n}$ (Def. A.7) such that the images of p and q under $\beta_{G_1} \times \dots \times \beta_{G_n}$ are p' and q' , and $\mathcal{L}^\alpha(\delta) \leq \mathcal{L}^\alpha(\gamma) + \varepsilon$. If γ is directed (Def. 2.11), then we can suppose that $\mathcal{L}^\alpha(\delta) = \mathcal{L}^\alpha(\gamma)$.

Proof. Without loss of generality, we assume that all the steps appearing in γ are non-constant.

Fix $t_0 \in dom(\gamma)$ and choose a compact interval I , neighborhood of t_0 , such that $\gamma(I \setminus \{t_0\})$ does not contain any vertex.

If $\gamma(t_0)$ is not a vertex, then for every $t \in I$ both $|G|$ and $\|G\|$ contains $\gamma(t)$, which is the only element of $\|G\|$ satisfying $\beta_G(\gamma(t)) = \gamma(t)$. Hence the mapping $\delta : t \in I \mapsto \gamma(t) \in \|G\|$ is smooth (because it is affine) and satisfies $\beta_G \circ \delta = \gamma|_I$.

Now suppose that $\gamma(t_0)$ is a vertex. It occurs at the junction between two consecutive steps s and s' of the piecewise affine path γ . Hence the supports a and a' of s and s' are uniquely defined, and each of the steps s and s' is either directed or antidirected (Def. B.19).

If both s and s' are directed (resp. antidirected.), then $\text{tgt } a = \text{src } a'$ (resp. $\text{src } a = \text{tgt } a'$). The mapping $\delta : I \rightarrow \|G\|$ sending $t \neq t_0$ to the unique lifting of $\gamma(t)$, and t_0 to the traversal (a, a') (resp. (a', a)) is piecewise smooth. As before we have $\beta_G \circ \delta = \gamma|_I$.

If s is directed while s' is antidirected (the dual case is dealt with in the same way), then we have $\text{tgt } a = \text{tgt } s = \text{src } s' = \text{tgt } a'$. If $a = a'$ then we can shorten both s and s' so that they no longer visit $\text{tgt } a$, this shortens the path γ without changing its source or its target. If $a \neq a'$ then the vertex $\text{tgt } a$ has at least two in going arrows, and by the assumption made in Def. 2.2, it also has an outgoing arrow b . On $I_- = I \cap]-\infty, t_0]$ we define δ_1 to be the path sending t to $\gamma(t)$ for $t \neq t_0$; and t_0 to (a, b) . We have $\beta_G \circ \delta_1 = \gamma|_{I_-}$. On $I_+ = I \cap [t_0, \infty[$ we define the path δ_3 sending t to $\gamma(t)$ for $t \neq t_0$; and t_0 to (a', b) . We have $\beta_G \circ \delta_3 = \gamma|_{I_+}$. The paths δ_1 and δ_3 are smooth and we have $(\beta_G \circ \delta_1) \cdot (\beta_G \circ \delta_3) = \gamma|_I$.

As in Ex. 4.18, we have an *arbitrarily short* piecewise smooth path δ_2 joining the traversals (b, a) and (b, a') (apart from these traversals, δ_2 covers $\{b\} \times]0, \varepsilon'/2[$ with ε' being the length of δ_2). A path arising in that context is called a *patch*.

We have thus three situations: i) $\gamma(t_0)$ is not a vertex, ii) $\gamma(t_0)$ is a vertex and a and a' go in the same direction, and iii) $\gamma(t_0)$ is a vertex and a and a' go in opposite directions. *Note that if γ is directed, then the case iii) does not occur.* In cases i) and ii) we define $\delta_1 = \delta|_{I_-}$ and $\delta_3 = \delta|_{I_+}$, so the concatenation $\delta_1 \delta_3 = \delta$ makes sense. In case iii) we need a patch δ_2 to form the piecewise smooth concatenation $\delta_1 \delta_2 (\delta_3 \circ (t \in J \mapsto t - \varepsilon' \in I_+))$ with $J = I_+ + \varepsilon'$. Hence we have piecewise smooth paths δ_1 , δ_2 , and δ_3 such that

- (1) $(\beta_G \circ \delta_1) = \gamma|_{I_-}$ and $(\beta_G \circ \delta_3) = \gamma|_{I_+}$,
- (2) in cases i) and ii), $\text{dom}(\delta_2) = \{t_0\}$ and $\delta = \delta_1 \delta_3$ is piecewise smooth, and
- (3) in case iii), $\mathcal{L}^\alpha(\delta_2)$ is arbitrarily small and $\delta = \delta_1 \delta_2 (\delta_3 \circ \theta)$ is piecewise smooth with $J = I_+ + \mathcal{L}^\alpha(\delta_2)$ and $\theta : t \in J \mapsto t - \mathcal{L}^\alpha(\delta_2) \in I_+$.

In higher dimensions, we have $\gamma = \gamma_1 \times \cdots \times \gamma_n$. We can choose I so that for every $k \in \{1, \dots, n\}$ the set $\gamma_k(I \setminus \{t_0\})$ does not contain any vertex. By applying the above reasoning for every $k \in \{1, \dots, n\}$ we obtain the piecewise smooth paths $\delta_1^{(k)}$, $\delta_2^{(k)}$, $\delta_3^{(k)}$ satisfying 1), 2), and 3) with respect to γ_k . We form the products $\delta_1 = \delta_1^{(1)} \times \cdots \times \delta_1^{(n)}$, $\delta_2 = \delta_2^{(1)} \times \cdots \times \delta_2^{(n)}$, and $\delta_3 = \delta_3^{(1)} \times \cdots \times \delta_3^{(n)}$. We have $(\beta_{G_1} \times \cdots \times \beta_{G_n}) \circ \delta_1 = \gamma|_{I_-}$ and $(\beta_{G_1} \times \cdots \times \beta_{G_n}) \circ \delta_3 = \gamma|_{I_+}$ because the equalities hold componentwise. We have $\mathcal{L}^\alpha((\beta_{G_1} \times \cdots \times \beta_{G_n}) \circ \delta_j) = \mathcal{L}^\alpha(\delta_j) = \mathcal{L}^\alpha(\delta_j)$ for $j \in \{1, 2, 3\}$ (Rem. 4.13 and Lem. 4.19). Therefore $\mathcal{L}^\alpha(\gamma|_I) = \mathcal{L}^\alpha(\delta_1) + \mathcal{L}^\alpha(\delta_3)$. If γ is directed then so are all the paths γ_k , consequently the case iii) does not occur, and we have $\mathcal{L}^\alpha(\gamma|_I) = \mathcal{L}^\alpha(\delta_1 \delta_3)$. Otherwise the case iii) may occur so we cannot ignore the patch δ_2 . Yet, in each coordinate, $\mathcal{L}^\alpha(\delta_2^{(k)})$ can be made arbitrarily small. We choose a domain of definition common to all the patches $\delta_2^{(k)}$ (some of them are ‘useless’ loops if we are not in the case iii) for γ_k at t_0). In particular we have the shift $\theta = \theta_1 \times \cdots \times \theta_n$, and $\mathcal{L}^\alpha(\delta_1 \delta_2 (\delta_3 \circ \theta)) = \mathcal{L}^\alpha(\gamma|_I) + \mathcal{L}^\alpha(\delta_2)$ (keeping in mind that $\mathcal{L}^\alpha(\delta_3 \circ \theta) = \mathcal{L}^\alpha(\delta_3)$, see Rem. B.8). We have $\mathcal{L}^\alpha(\delta_2) \leq \mathcal{L}(\delta_2)$ (Rem. 4.17) so making each component $\delta_2^{(k)}$ arbitrarily small guarantees that so is $\mathcal{L}^\alpha(\delta_2)$.

Theorem 4.21. Let p and q be the images under $\beta_{G_1} \times \cdots \times \beta_{G_n}$ of p' and q' , we have

$$d^{(\alpha)}(p, q) = \inf \{ \mathcal{L}^\alpha(\delta) \mid \delta \text{ piecewise smooth path from } p' \text{ to } q' \}.$$

Proof. We have $d^{(\alpha)}(p, q) = \inf \{ \mathcal{L}^\alpha(\gamma) \mid \gamma \text{ path from } p \text{ to } q \}$ because $d^{(\alpha)}$ is intrinsic (Lem. 4.14). We have $\mathcal{L}^\alpha((\beta_{G_1} \times \cdots \times \beta_{G_n}) \circ \delta) = \mathcal{L}^\alpha(\delta) = \mathcal{L}^\alpha(\delta)$ for every piecewise smooth path δ from p' to q' (Rem. 4.13 and Lem. 4.19). Therefore $d^{(\alpha)}(p, q)$ is less than the infimum of the values $\mathcal{L}^\alpha(\delta)$; the converse inequality holds by Lem. 4.20.

5. Local orders

Local orders (§5.2) are based on ordered spaces (§5.1) as atlases on charts; see (Coursolle & Haucourt, 2024, §3) for a detailed study. They are used in concurrency theory (Fajstrup et al. (2006)) in a manner similar to that in which ‘finite causal orientations’⁽³⁾ are used in cosmology (Segal, 1976, p.23). The importance of local antisymmetry and afferent technical difficulties are coined in (Lawson, 1989, p. 277): « We will be particularly interested in the case that the conal order is antisymmetric [...] determining whether or not it [the conal order] is antisymmetric can be quite delicate», see also §5.4.

For every graph G , we equip the sets $|G|$ and $\|G\|$ (Def. 2.6 and 2.15) with the standard local orders \mathcal{X}_G and $\tilde{\mathcal{X}}_G$ (Def. 5.10 and 5.15). It follows that the *geometric model* $|P|$ of a conservative program P , which is a tile-compatible subset of $|G_1| \times \cdots \times |G_n|$ (Def. 2.13), is a sub-local order of the product $\mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$, with G_i denoting the underlying graph of the i^{th} sequential process of P (Def. 2.2). Similarly, the subset $\|P\|$ (Def. 2.22) of $\|G_1\| \times \cdots \times \|G_n\|$ is a sub-local order of the product $\tilde{\mathcal{X}}_{G_1} \times \cdots \times \tilde{\mathcal{X}}_{G_n}$.

On the one hand, the product of blowups $\beta_{G_1} \times \cdots \times \beta_{G_n}$ (Def. 2.15) is a set map from $|G_1| \times \cdots \times |G_n|$ to $\|G_1\| \times \cdots \times \|G_n\|$ inducing a local order morphism (§5.2) from $\mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$ to $\tilde{\mathcal{X}}_{G_1} \times \cdots \times \tilde{\mathcal{X}}_{G_n}$ which is actually the universal euclidean local embedding (Def. 5.5) with codomain $\mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$ (Thm. 5.31). On the other hand, the standard local order $\tilde{\mathcal{X}}_{G_1} \times \cdots \times \tilde{\mathcal{X}}_{G_n}$ is related, through Lawson’s correspondence (Thm. 5.39), to the standard atlas $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ (Def. 3.3) equipped with its standard parallelization (Def. 3.6), which we denote by (f_1, \dots, f_n) .

The models $|P|$ and $\|P\|$ are mere subsets of $|G_1| \times \cdots \times |G_n|$ and $\|G_1\| \times \cdots \times \|G_n\|$. What makes $|P|$ relevant to concurrency theory is the relation between the execution traces of P (Def. 2.3) and the directed paths on $|P|$ (Def. 5.6):

- (1) We can always assume that an execution trace p_0, \dots, p_k of P is contained in its discrete model $\|P\|_d$ (Def. 2.5 (2)).
- (2) By ‘joining’ the middles of the tiles τ_0, \dots, τ_k corresponding to p_0, \dots, p_k (Def. 2.11) we obtain γ a piecewise linear *lifting* of p_0, \dots, p_k on $|P|$ (Haucourt, 2018, Def. 6.2).

This relation extends to the directed paths on $\|P\|$ (i.e. the piecewise smooth maps σ from some compact interval to $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ with values in $\|P\|$, and whose derivative at every t has non-negative coordinates in the basis $(f_1 \circ \sigma(t), \dots, f_n \circ \sigma(t))$):

- (3) The directed path γ from (2) is lifted along $\beta_{G_1} \times \cdots \times \beta_{G_n}$ to $\tilde{\gamma}$ a piecewise linear directed path on $\tilde{\mathcal{X}}_{G_1} \times \cdots \times \tilde{\mathcal{X}}_{G_n}$.
- (4) By the Lawson’s correspondence, the path $\tilde{\gamma}$ is directed on the standard parallelized atlas $(\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}, (f_1, \dots, f_n))$.

To sum up, every execution trace of P admits a piecewise linear (therefore piecewise smooth) lifting on the subset $\|P\|$ of the non-Hausdorff parallelized manifold $(\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}, (f_1, \dots, f_n))$.

5.1 Ordered spaces and pospaces

An *ordered space* is a topological space X together with a partial order \leq on its underlying set $|X|$; the partial order \leq is said to be *closed* when so is the subset $\{(a, b) \in X \times X \mid a \leq b\}$ of the product space $X \times X$. A *chain* of (X, \leq) is a totally ordered subposet of (X, \leq) , it is said to be *unbounded* when it has neither greatest nor least element. A *Nachbin ordered space*, or *pospace*, is a topological space with a closed partial order on it; the closedness condition follows (Nachbin,

⁽³⁾In this article ‘finite causal orientations’ would be called ‘local preorders’.

1965, p. 25) although the term ‘pospace’ appeared later (Gierz et al., 1980, VI-1.1, p. 271-272). The real line \mathbb{R} with its standard topology and order is a prototypical example of a pospace. A pospace X' such that $|X'| \subseteq |X|$ is said to be a *subpospace* of X when X' is both a *subspace* and a *subposet* of X , i.e.

- the open subsets of X' are the restrictions to X' of the open subsets of X , and
- the partial order of X' is the restriction to X' of the partial order of X .

Every subset of $|X|$ induces a subpospace of X . If this subset is open in X , then the subpospace is said to be *open*. A subset C of a poset (X, \leq) is said to be *order convex* when for all $a, b \in C$, we have $\{x \in X \mid a \leq x \leq b\} \subseteq C$. A pospace (X, \leq) is said to be *locally order convex* when its underlying topology has a basis of order convex open subset, see (Nachbin, 1965, p.26), or (Gierz et al., 1980, VI-1.5, p. 273).

A *morphism of ordered spaces* is an order preserving continuous map. An isomorphism of ordered spaces is called a *dihomeomorphism*. The product of two ordered spaces X and Y is the product of their topological spaces together with the product order; if X and Y are pospaces, then so is their product.

An *embedding of ordered spaces* is a morphism of ordered spaces inducing a dihomeomorphism on its image.

Definition 5.1. The left action of the n^{th} symmetric group \mathfrak{S}_n on n -tuples is defined by

$$\sigma \cdot (t_1, \dots, t_n) = (t_{\sigma^{-1}1}, \dots, t_{\sigma^{-1}n}).$$

By extension, for every n -tuple of sets (A_1, \dots, A_n) , we define the set $\sigma \cdot (A_1 \times \dots \times A_n)$ as the product $A_{\sigma^{-1}1} \times \dots \times A_{\sigma^{-1}n}$ and the map $\pi_\sigma : A_1 \times \dots \times A_n \rightarrow \sigma \cdot (A_1 \times \dots \times A_n)$ by $\pi_\sigma(t) = \sigma \cdot t$. In particular, for every n -tuple of mappings (f_1, \dots, f_n) we have

$$\pi_\sigma \circ (f_1 \times \dots \times f_n) = (f_{\sigma^{-1}1} \times \dots \times f_{\sigma^{-1}n}) \circ \pi_\sigma \quad (6)$$

taking care that π_σ on the right hand side of (6) permutes the domains of the mappings f_k while π_σ on the left hand side permutes their codomains.

The next result is an immediate consequence of the second point of (Schröder, 2003, 10.4.10), which is about (possibly infinite) products of posets.

Lemma 5.2. Every poset automorphism of \mathbb{R}^n can be written as $(x_1 \times \dots \times x_n) \circ \pi_\sigma$ with $\sigma \in \mathfrak{S}_n$ and $x_k \in \text{Aut}(\mathbb{R})$ for $k \in \{1, \dots, n\}$.

Remark 5.3. The groups $(\text{Aut } \mathbb{R})^n$ and \mathfrak{S}_n are canonically identified with the subgroups $A = \{x_1 \times \dots \times x_n \mid x_k \in \text{Aut } \mathbb{R}; k \in \{1, \dots, n\}\}$ and $B = \{\pi_\sigma \mid \sigma \in \mathfrak{S}_n\}$ of $\text{Aut}(\mathbb{R}^n)$. Lem. 5.2 exactly states that $A \circ B = \text{Aut}(\mathbb{R}^n)$ and it is clear that $A \cap B = \{id_{\mathbb{R}^n}\}$. We deduce from (6) and (Grillet, 2007, Prop. 4.5, p. 19) that the subgroup A is normal. Hence $\text{Aut}(\mathbb{R}^n)$ is the semidirect product $(\text{Aut } \mathbb{R})^n \rtimes \mathfrak{S}_n$ (Grillet, 2007, Prop. 11.2, p. 93). In particular, the decomposition given by Lem. 5.2 is unique.

The binary greatest lower bound operator in \mathbb{R}^n is denoted by \wedge , and its dual by \vee . For every $i \in \{1, \dots, n\}$ the map $\text{pr}_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is the i^{th} projection. For every $x \in \mathbb{R}^n$ we set $\downarrow x = \{y \in \mathbb{R}^n \mid y \leq x\}$ and $\uparrow x = \{y \in \mathbb{R}^n \mid x \leq y\}$ with \leq the usual product order on \mathbb{R}^n .

Lemma 5.4. A pospace embedding $\theta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ preserves existing least upper bounds (resp. greatest lower bounds) and its image is a set product of open intervals of \mathbb{R} .

Proof. The map θ is open by invariance of domain (Hatcher, 2002, Thm. 2B.3, p.172). It follows that for every $t \in \mathbb{R}^n$ we have a tuple (J_1, \dots, J_n) of open intervals of \mathbb{R} such that $\theta(t) \in J_1 \times \dots \times J_n \subseteq \text{img}(\theta)$. Given $u \in \mathbb{R}^n$ such that $\theta(t) < u$ we have $x \in J_1 \times \dots \times J_n$ such that $\theta(t) < x < u$ (by definition of the product order on \mathbb{R}^n). Since $x \in J_1 \times \dots \times J_n \subseteq \text{img}(\theta)$ and θ is an embedding, we have $t < \theta^{-1}(x)$. We have proven that:

$$\text{for every } t, u \in \mathbb{R}^n \text{ such that } \theta(t) < u, \text{ there exists } t' > t \text{ such that } \theta(t) < \theta(t') < u. \quad (7)$$

The dual statement (obtained by exchanging the roles of $<$ and $>$) is also valid, for similar reasons.

Let \mathcal{F} be a subset of \mathbb{R}^n with a least upper bound. Then $\theta(\sup \mathcal{F})$ is an upper bound of $\theta(\mathcal{F})$ because θ is order-preserving, so $\theta(\mathcal{F})$ has a least upper bound and we have $\theta(\sup \mathcal{F}) \geq \sup \theta(\mathcal{F})$. Suppose that $\theta(\sup \mathcal{F}) \neq \sup \theta(\mathcal{F})$. By applying the dual of (7) we have $t < \sup \mathcal{F}$ such that $\theta(\sup \mathcal{F}) > \theta(t) > \sup \theta(\mathcal{F})$. From the latter inequalities and the fact that θ is a pospace embedding, we deduce that t is an upper bound of \mathcal{F} , which contradicts the fact that $t < \sup \mathcal{F}$. Similarly, we prove that θ preserves existing greatest lower bounds.

For every $k \in \{1, \dots, n\}$, let $\theta_k = \text{pr}_k \circ \theta$ and $I_k = \text{img}(\theta_k)$. Each map θ_k is continuous and open as a composite of such maps, hence each I_k is an open interval. Let $x = (x_1, \dots, x_n) \in I_1 \times \dots \times I_n$ and $\mathcal{F} = \theta^{-1}(\downarrow x)$. For every $k \in \{1, \dots, n\}$ there exists $t^{(k)} \in \mathbb{R}^n$ such that $\theta_k(t^{(k)}) = x_k$, from which we deduce that

$$\begin{cases} \theta_k(t^{(1)} \wedge \dots \wedge t^{(n)}) \leq \theta_k(t^{(1)}) \wedge \dots \wedge \theta_k(t^{(n)}) \leq x_k \\ \theta_k(t^{(1)} \vee \dots \vee t^{(n)}) \geq \theta_k(t^{(1)}) \vee \dots \vee \theta_k(t^{(n)}) \geq x_k \end{cases}$$

It follows that $t^{(1)} \vee \dots \vee t^{(n)}$ is an upper bound of \mathcal{F} which is nonempty because it contains $t^{(1)} \wedge \dots \wedge t^{(n)}$; therefore $\sup \mathcal{F}$ exists. By definition of \mathcal{F} we have $\theta(\mathcal{F}) \subseteq \downarrow x$, hence $\sup(\theta(\mathcal{F})) \leq x$. We deduce that $\theta(\sup \mathcal{F}) \leq x$ because (we have already proven that) $\theta(\sup(\mathcal{F})) = \sup(\theta(\mathcal{F}))$. Suppose that $\theta(\sup(\mathcal{F})) \neq x$. By applying (7) we have $t > \sup \mathcal{F}$ such that $\theta(\sup(\mathcal{F})) < \theta(t) < x$. From the latter inequalities and the fact that θ is a pospace embedding, we deduce that t belongs to $\theta^{-1}(\downarrow x) = \mathcal{F}$, which contradicts the fact that $t > \sup \mathcal{F}$.

5.2 Local orders

A *witness of compatibility* of the ordered spaces X and X' around $p \in X \cap X'$ is a set W containing p that is open in both X and X' , and on which both X and X' induce the same ordered space. We say that X and X' are *compatible around* the point p , which we denote by $X \mathcal{L} X'$, when $p \notin X \cap X'$ or such a witness exists. We say that X and X' are *compatible* when they are so around every point, in that case we write $X \sim X'$.

A *local order* (resp. *Nachbin local order*) \mathcal{X} is a collection of pairwise compatible ordered spaces (resp. pospaces); its *underlying set* $|\mathcal{X}|$ is the union of the underlying sets of the elements of \mathcal{X} . A local order \mathcal{X} is said to be *euclidean* of dimension n when for every $p \in |\mathcal{X}|$ there exists $X \in \mathcal{X}$ in which p has a neighbourhood that is dihomeomorphic to \mathbb{R}^n . The *canonical* local order induced by an ordered space X is the one element set $\{X\}$.

Definition 5.5. A *morphism* (resp. *local embedding*) of local orders from \mathcal{X} to \mathcal{Y} is a mapping $f : |\mathcal{X}| \rightarrow |\mathcal{Y}|$ such that for every $p \in |\mathcal{X}|$, every $Y \in \mathcal{Y}$ and every open subset V of Y containing $f(p)$, there exists $X \in \mathcal{X}$ and an open subset U of X containing p such that $f(U) \subseteq V$, and the restriction $f|_U : U \rightarrow V$ is order preserving, i.e. for all $a, b \in U$, $a \sqsubseteq_X b \Rightarrow f a \sqsubseteq_Y f b$ (resp. $f|_U : U \rightarrow V$ is an embedding). We say that f is *euclidean* when we so is \mathcal{X} . A *local dihomeomorphism* is an open local embedding. Local orders and their morphisms form the category \mathbf{L} .

The cartesian product of \mathcal{X} and \mathcal{Y} in \mathbf{L} is the local order

$$\mathcal{X} \times \mathcal{Y} = \{ X \times Y \mid X \in \mathcal{X}; Y \in \mathcal{Y} \}.$$

The *topology* (on the underlying set) of the local order \mathcal{X} is generated by the subsets U that are open in some ordered space of the collection \mathcal{X} . The category of topological spaces is denoted by **Top**, the forgetful functor $Sp : \mathbf{L} \rightarrow \mathbf{Top}$ sends every local order to its underlying topological space.

Two local orders \mathcal{X} and \mathcal{Y} are said to be *equivalent*, which we denote by $\mathcal{X} \sim \mathcal{Y}$, when they have the same underlying set S and the identity map id_S induces an isomorphism between them. Assuming that \mathcal{X} and \mathcal{Y} have the same underlying set, we have $\mathcal{X} \sim \mathcal{Y}$ when $\mathcal{X} \cup \mathcal{Y}$ is still a local order, which amounts to say that every $X \in \mathcal{X}$ is compatible with every $Y \in \mathcal{Y}$ (Coursolle & Haucourt, 2024, Prop. 3.25).

Definition 5.6. A *directed path* on \mathcal{X} is a local order morphism from $\{[a, b]\}$ to \mathcal{X} , with $a \leq b$.

Standard local orders. The sets $|G|$ and $\|G\|$ respectively come with the standard local orders \mathcal{X}_G and $\tilde{\mathcal{X}}_G$ which we describe in this section.

Definition 5.7. For every vertex v the canonical star centered at v (Def. 2.7) is

$$U_v = \bigcup_{\text{tgt } a=v} \{a\} \times J_a \cup \{v\} \cup \bigcup_{\text{src } b=v} \{b\} \times I_b \quad (8)$$

with the intervals $J_a =]\frac{\ell(a)}{2}, \ell(a)[$ and $I_b =]0, \frac{\ell(b)}{2}[$ (as in §A.2 (2)). The set U_v is provided with the *greatest* topology and the *least* partial order \sqsubseteq_v making the maps

$$\begin{aligned} \gamma_a : t \in -I_a \cup \{0\} &\mapsto \begin{cases} (a, t + \ell(a)) & \text{if } t \neq 0 \\ v & \text{if } t = 0 \end{cases} \in U_v \\ \delta_b : t \in \{0\} \cup I_b &\mapsto \begin{cases} (b, t) & \text{if } t \neq 0 \\ v & \text{if } t = 0 \end{cases} \in U_v \end{aligned}$$

continuous and order-preserving for all arrows a such that $\text{tgt}(a) = v$, and all arrows b such that $\text{src}(b) = v$. In particular, the maps γ_a and δ_b are pospace embeddings. Note that for every traversal (a, b) at v the union of the mappings γ_a and δ_b is the pospace embedding

$$\xi_{ab} : t \in -I_a \cup \{0\} \cup I_b \mapsto \begin{cases} (a, t + \ell(a)) & \text{if } t < 0 \\ v & \text{if } t = 0 \\ (b, t) & \text{if } t > 0 \end{cases} \in U_v . \quad (9)$$

The set U_v comes with the map $\tau_v : U_v \rightarrow \mathbb{R}$ defined by $\tau_v(v) = 0$, and

$$\tau_v(x, t) = \begin{cases} t - \ell(x) & \text{if } t \in J_x \\ t & \text{if } t \in I_x \end{cases} .$$

We observe that $\tau_v(p) \leq 0$ (resp. ≥ 0) iff $p \sqsubseteq_v v$ (resp. $v \sqsubseteq_v p$), and that τ_v induces poset isomorphisms from $\{a\} \times I_a$ (resp. $\{a\} \times J_a$) to $\text{dom}(\delta_a) \setminus \{0\}$ (resp. $\text{dom}(\gamma_a) \setminus \{0\}$). It follows that with the map π_G from Def. 2.6, for all points p, q of U_v , we have $p \sqsubseteq_v q$ if, and only if

$$\tau_v p \leq 0 \leq \tau_v q \quad \text{or} \quad \pi_G p = \pi_G q \quad \text{and} \quad \tau_v p \leq \tau_v q . \quad (10)$$

In the first case p is before v while q is after v . In the second one, we have $p = q = v$, or both p and q belong to $\{a\} \times I_a$ or $\{a\} \times J_a$ with $a = \pi_G p$. The relation \sqsubseteq_v is a closed partial order on U_v (closedness of this partial order is proven from an easy – but tedious – case disjunction readily obtained from the negation of the formula (10)). The *standard pospace* at v is (U_v, \sqsubseteq_v) .

Remark 5.8. The stars centered at a vertex (Def. 2.7) and the sets of the form $\{a\} \times]t - \varepsilon, t + \varepsilon[$ are order convex in (U_v, \sqsubseteq_v) , which is thus locally order convex (p. 19) by Rem. 2.9.

Lemma 5.9. The collection of pospaces (U_v, \sqsubseteq_v) and $\{a\} \times]0, \ell(a)[$ (equipped with the obvious pospace structure) with $v \in G^{(0)}$ and $a \in G^{(1)}$ is a local order whose underlying set is $|G|$.

Proof. Every vertex and every middle point $(a, \ell(a)/2)$ belongs to a single element of the collection. Every point (a, t) with $t \neq \ell(a)/2$ only belongs to $\{a\} \times]0, \ell(a)[$ and U_v with $v = \text{src } \alpha$ or $v = \text{tgt } \alpha$ depending on whether $t < \ell(a)/2$ or $t > \ell(a)/2$. Hence the (nontrivial) witnesses of compatibility are the open sets $\{a\} \times]0, \ell(a)/2[$ and $\{a\} \times]\ell(a)/2, \ell(a)[$.

Definition 5.10. (Haucourt, 2018, §6.1). The *standard local order* \mathcal{X}_G on $|G|$ (resp. $|G_1| \times \cdots \times |G_n|$) is the collection \mathcal{X}_G described in Lem. 5.9 (resp. the product $\mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$ of such collections).

Remark 5.11. The notion of a directed path on $|G_1| \times \cdots \times |G_n|$ (Def. 2.11), and that of a directed path on $\mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$ (Def. 5.6), are equivalent – see (Haucourt, 2012, §6.2).

Remark 5.12. Assuming that $|G_1| \times \cdots \times |G_n|$ is equipped with the $d^{(\infty)}$ metrics (Def. 4.10) every open ball of radius $r \leq \frac{R}{4}$ is contained in some pospace $X \in \mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$.

For our purpose, the only relevant local order on $|G|$ is the standard one (Def. 5.10); the case of $\|G\|$ is a little subtler because the properties of the local order morphism induced by the blowup $\beta_G : \|G\| \rightarrow |G|$ (Def. 2.15) strongly depend on the local order on $\|G\|$.

Definition 5.13. A continuous map $f : X \rightarrow Y$ is said to be *open* when so is $f(X)$; an *embedding* when it induces a homeomorphism onto $f(X)$; a *local embedding* when for every $x \in X$, f induces an embedding of some open neighborhood U of x into Y ; a *local homeomorphism* (or *étale map*) when it is an open local embedding.

Remark 5.14. A directed path γ (Def. 5.6) is a local embedding (in the sense of Def. 5.5) if, and only if, $Sp(\gamma)$ (p. 21) is a local embedding (in the sense of Def. 5.13); moreover it is euclidean. In particular, every arc of the form $\gamma \circ j$, with j as in Lem. B.3, is a local embedding.

The standard charts ϕ_a and ϕ_{ab} (Def. 3.1) with a and (a, b) arrows and traversals, are bijections towards open subintervals of \mathbb{R} . Their domains, namely $\text{dom } \phi_a$ and $\text{dom } \phi_{ab}$, are equipped with the pospace structures so that ϕ_a and ϕ_{ab} become dihomeomorphisms. All these pospaces are locally order convex (p. 19) because so are the open intervals of \mathbb{R} .

Definition 5.15. The collection of pospaces $\text{dom } \phi_a$ and $\text{dom } \phi_{ab}$ (Def. 3.1) which we denote by $\tilde{\mathcal{X}}_G$, forms the *standard local order* on $\|G\|$ observing that the witnesses of compatibility are of the form $\{a\} \times]0, \ell(a)/2[$ and $\{a\} \times]\ell(a)/2, \ell(a)[$ with a an arrow of G . The *standard local order* on the set $\|G_1\| \times \cdots \times \|G_n\|$ is $\tilde{\mathcal{X}}_{G_1} \times \cdots \times \tilde{\mathcal{X}}_{G_n}$.

Remark 5.16. Given a traversal (a, b) at v , $\varepsilon \in]0, \ell(a)[$, and $\varepsilon' \in]0, \ell(b)[$, the blowup β_G induces the canonical pospace isomorphism

$$\begin{array}{c} \{a\} \times]\ell(a) - \varepsilon, \ell(a)[\cup \{(a, b)\} \cup \{b\} \times]0, \varepsilon'[\\ \downarrow \cong \\ \{a\} \times]\ell(a) - \varepsilon, \ell(a)[\cup \{v\} \cup \{b\} \times]0, \varepsilon'[\end{array}$$

hence a pospace embedding from $\text{dom } \phi_{ab}$ to U_v (Def. 3.1). The domains of these embeddings together with the sets $\{a\} \times I$ with I an open interval of $]0, \ell(a)[$ form a base of the topology of the local order $\tilde{\mathcal{X}}_G$, and β_G induces a euclidean local embedding of $\tilde{\mathcal{X}}_G$ into \mathcal{X}_G (Def. 5.5).

If the graph G contains a singular traversal (a, b) (Def. 2.14), then $\beta_G : \tilde{\mathcal{X}}_G \rightarrow \mathcal{X}_G$ is not open because, for $\varepsilon > 0$ sufficiently small, $\beta_G(\phi_{ab}(] \varepsilon, -\varepsilon [))$ is not open (Def. 3.1). However, we observe that for every traversal τ at a vertex v , the blowup β_G induces a bijection from $(U_v \setminus \{v\}) \cup \{\tau\}$ to U_v ; we have an analog of Lem. 5.9:

Lemma 5.17. The collection of subspaces $(U_v \setminus \{v\}) \cup \{\tau\}$ equipped with the partial orders induced by $(\beta_G \times \beta_G)^{-1}(\sqsubseteq_v)$ with τ a traversal at v , and (U_v, \sqsubseteq_v) given by Def. 5.7, along with the subspaces $\{a\} \times]0, \ell(a)[$ equipped with the obvious total orders, is a local order on $\|G\|$.

Proof. Given two traversals τ and τ' at vertices v and v' , the intersection of $V = (U_v \setminus \{v\}) \cup \{\tau\}$ and $V' = (U_{v'} \setminus \{v'\}) \cup \{\tau'\}$ is nonempty iff $v = v'$. So the partial orders of V and V' coincide because they are induced by \sqsubseteq_v and $\sqsubseteq_{v'}$. The rest of the proof follows the one of Lem. 5.9.

Definition 5.18. The *étale local order* on $\|G\|$ is the collection described in Lem. 5.17, we denote it by \mathcal{E}_G . The underlying topology of \mathcal{E}_G is called the *étale topology* on $\|G\|$, it is generated by the subsets $\{a\} \times]t - \varepsilon, t + \varepsilon[$ and $(S_v \setminus \{v\}) \cup \{\tau\}$ with $a \in G^{(1)}$, $0 < \varepsilon < \min\{t, \ell(a) - t\}$, τ a traversal at $v \in G^{(0)}$, and S_v a star centered at v (Def. 2.7). The blowup β_G induces a local dihomeomorphism $e_G : \mathcal{E}_G \rightarrow \mathcal{X}_G$ which is not euclidean (Def. 5.5); in particular the map $Sp(e_G)$ is a local homeomorphism (Def. 5.13).

Remark 5.19. The relation between local homeomorphisms and *sheaves* is explained in (Mac Lane & Moerdijk, 1994, Thm. 2, p. 89). A *section* of the local homeomorphism $Sp(e_G)$ on an open subset U of $Sp(\mathcal{X}_G)$ is a continuous map $s : U \rightarrow Sp(\mathcal{E}_G)$ such that $Sp(e_G)|_U \circ s = id_U$; the map s is entirely determined by U and a function ϕ which ‘chooses’ a traversal at v for every vertex $v \in U$ (if U contains no vertex, then ϕ is the empty map $\emptyset \rightarrow \emptyset$). The sheaf F associated with e_G assigns the set of section of e_G on U to every open subset U of $Sp(\mathcal{X}_G)$, i.e.

$$F(U) = \prod_{v \in U \cap G^{(0)}} \{\tau \text{ traversal of } G \text{ at } v\}.$$

If we equip the underlying set of G with the topology generated by the sets $\{a\}$ for a arrow of G , and $\{v\} \cup \text{src}^{-1}\{v\} \cup \text{tgt}^{-1}\{v\}$ for v vertex of G (which is an *Alexandrov discrete* topology (Arenas (1999))), then $\pi_G : |G| \rightarrow G$ (Def. 2.6) induces an open continuous map. One defines a sheaf \tilde{F} on the Alexandrov discrete space G by setting

$$\tilde{F}(A) = \prod_{v \in A \cap G^{(0)}} \{\tau \text{ traversal of } G \text{ at } v\}$$

for every open subset A of G , so we have $F(U) = \tilde{F}(\pi_G(U))$ for every open subset U of $Sp(\mathcal{X}_G)$.

Remark 5.20. Denote $\mathbf{H} \subseteq \mathbf{Top}$ (p. 21) the full subcategory of Hausdorff spaces. Denote by $H : \mathbf{Top} \rightarrow \mathbf{H}$ and $h : id_{\mathbf{Top}} \rightarrow IH$ the left adjoint to the inclusion functor $I : \mathbf{H} \hookrightarrow \mathbf{Top}$ and its unit. Given a topological space X , the Hausdorff space $H(X)$ is the quotient of X by the smallest equivalence relation \sim such that $\{(x, y) \in X \times X \mid x \sim y\}$ is closed in $X \times X$, and h_x is the associated quotient map; the latter should be understood as ‘the best Hausdorff approximation under X ’. Assume that X is the underlying topological space of $\tilde{\mathcal{X}}_G$ (resp. \mathcal{E}_G). By Def. 5.15 (resp. 5.18), for all $x, x' \in X$ with $x \neq x'$, we have $x \sim x'$ if, and only if, x and x' are traversals at the same vertex of G . Hence the corestriction of β_G to its image (Rem. 2.16) is the quotient map related to \sim , i.e. the underlying map of h_x (see Fig. 3).

5.3 The lifting properties of blowups

Lifting directed paths. Every *directed* path on the image of β_G can be lifted along it (Thm. 5.27). Nevertheless, such liftings may not be unique when the starting or the finishing point is a vertex. We begin with a simple yet useful observation:

Remark 5.21. Given a vertex v we have $x \sqsubseteq_v v$ or $v \sqsubseteq_v x$ for every $x \in U_v$ (Def. 5.7 (8)). Let C be a chain of U_v containing at least two elements. If v is an upper bound of C , then we have a unique arrow a of G such that $C \setminus \{v\} \subseteq \{a\} \times]\ell(a)/2, \ell(a)[$. Moreover $\text{tgt } a = v$. Similarly, if v is a lower bound of C , then we have a unique arrow b of G such that $C \setminus \{v\} \subseteq \{b\} \times]0, \ell(b)/2[$. Moreover $\text{src } b = v$. If v is neither a lower nor an upper bound of C , then v is both an upper bound of $\downarrow v \cap C$, and a lower bound of $\uparrow v \cap C$. Therefore we have a unique traversal (a, b) of G at v such that C is entirely contained in the image of ξ_{ab} (Def. 5.7 (9)). In addition, if C is unbounded and connected (as a subset of U_v which carries a topology), then it is isomorphic to the pospace \mathbb{R} .

Given a path γ and $t \in \text{dom } \gamma$, we say that γ is constant *before* (resp. *after*) t when $\gamma(s) = \gamma(t)$ for every $s \in \text{dom } (\gamma)$ such that $s \leq t$ (resp. $t \leq s$).

Lemma 5.22. Let γ be a directed path on \mathcal{X}_G (Def. 5.6 and 5.10) and $t \in \text{dom } \gamma$ such that $\gamma(t)$ is a vertex v . Suppose that $[t', t'']$ is the connected component of $\gamma^{-1}\{v\}$ containing t . If γ is not constant before t (i.e. t' is not the least element of $\text{dom } \gamma$) then we have a *unique* arrow a such that for every $\varepsilon \in]0, \ell(a)[$ there exists $s \in \text{dom } \gamma$ such that $s < t'$ and $\gamma[s, t] \setminus \{v\} \subseteq \{a\} \times]\ell(a) - \varepsilon, \ell(a)[$. Moreover $\text{tgt } a = v$. Similarly, if γ is not constant after t (i.e. t'' is not the greatest element of $\text{dom } \gamma$) then we have a *unique* arrow b such that for every $\varepsilon \in]0, \ell(b)[$ there exists $s \in \text{dom } \gamma$ such that $t'' < s$ and $\gamma[t, s] \setminus \{v\} \subseteq \{b\} \times]0, \varepsilon[$. Moreover $\text{src } b = v$.

Proof. We have an open interval I' containing t' such that γ induces a pospace morphism from I' to U_v because γ is a local order morphism. Let $I = \{s \in I' \mid s \leq t\}$ so v is an upper bound of the chain $\gamma(I)$ of U_v . We readily deduce from the definition of t' that $\gamma(I)$ is not reduced to $\{v\}$, so Rem. 5.21 applies and we have a unique arrow a such that $\gamma(I) \setminus \{v\} \subseteq \{a\} \times]0, \ell(a)[$, moreover $\text{tgt } a = v$. For $\varepsilon \in]0, \ell(a)/2[$ the star $S(v, \varepsilon)$ (Def. 2.7) is an open subset of U_v (Rem. 2.9). Since γ is continuous, we have a neighborhood J of t' (with $J \subseteq I'$) such that $\gamma(J)$ is contained in B , therefore in $\{a\} \times]\ell(a) - \varepsilon, \ell(a)[$. Given $s \in I \cap J$ such that $s < t'$ we have $\gamma[s, t] \subseteq \{a\} \times]\ell(a) - \varepsilon, \ell(a)[$. The case where γ is not constant after t is deduced from the previous one by reversing the local order, or by a direct similar proof.

Definition 5.23. The arrows a and b given by Lem. 5.22 are referred to as the *last* arrow visited *before* t , and the *first* arrow visited *after* t (by γ). If $\text{dom } \gamma = [t_0, t_1]$ the first arrow visited by γ is the first one it visits after t_0 . Similarly, the last arrow visited by γ is the last one it visits before t_1 .

If $\gamma(t)$ is a vertex, a traversal (a, b) at $\gamma(t)$ is said to be *compatible* with γ at t when the following are satisfied: i) if the last arrow visited before t exists, then it is a ; and ii) if the first arrow visited after t exists, then it is b .

Remark 5.24. If γ is a constant path standing on a vertex v , then any traversal at v is compatible with γ at every $t \in \text{dom } \gamma$. If γ is not constant and starts on a vertex at t_0 , then any traversal (a, b) such that b is the first arrow visited by γ is compatible at t_0 . Dually, if γ finishes on a vertex at t_1 , then any traversal (a, b) such that a is the last arrow visited by γ is compatible at t_1 .

Lemma 5.25. Suppose that $\gamma(t)$ is a vertex v with γ a directed path on \mathcal{X}_G . For any traversal (a, b) compatible with γ at t , and any $\varepsilon \in]0, R[$, there exists an interval I open in $\text{dom } \gamma$, containing t , and such that γ induces a pospace morphism

$$\tilde{\gamma} : I \rightarrow \{a\} \times]\ell(a) - \varepsilon, \ell(a)[\cup \{v\} \cup \{b\} \times]0, \varepsilon[.$$

Moreover, if γ is not constant before (resp. after) t , then $\tilde{\gamma}$ visits $\{a\} \times]\ell(a) - \varepsilon, \ell(a)[$ (resp. $\{b\} \times]0, \varepsilon[$).

Proof. Let t' be the least element of the connected component of $\gamma^{-1}\{v\}$ containing t . Note that $\gamma[t', t] = \{v\}$. If γ is constant before t then t' is the least element of $\text{dom } \gamma$. In that case we can suppose that t' is also the least element of I . Otherwise a is the last arrow visited by γ before t (Def. 5.23). By Lem. 5.22 we have $s < t'$ such that $\gamma(]s, t']) \setminus \{v\} \subseteq \{a\} \times]\ell(a) - \varepsilon, \ell(a)[$. By definition of a local order morphism (Def. 5.5) the map γ induces a pospace morphism from $]s, t]$ to $\{a\} \times]\ell(a) - \varepsilon, \ell(a)[\cup \{v\}$ for s sufficiently close to t . From the definition of t' we deduce that $\gamma(]s, t'])$ is not reduced to $\{v\}$, therefore it meets $\{a\} \times]\ell(a) - \varepsilon, \ell(a)[$. Whether γ is constant before t or not, we found an interval I_0 whose greatest element is t , that is not reduced to a single element unless t is the least element of $\text{dom } \gamma$, and such that γ induces a pospace morphism from I_0 to $\{a\} \times]\ell(a) - \varepsilon, \ell(a)[\cup \{v\}$. By similar arguments we obtain an interval I_1 whose least element is t , that is not reduced to a single element unless t is the greatest element of $\text{dom } \gamma$, and such that γ induces a pospace morphism from I_1 to $\{v\} \cup \{b\} \times]\ell(b) - \varepsilon, \ell(b)[$. The interval $I = I_0 \cup I_1$ matches the requirements.

Proposition 5.26. Let γ be a directed path on \mathcal{X}_G . Given a set map δ from $\text{dom } \gamma$ to $\|G\|$ (Def. 2.15) the following are equivalent:

- (1) The map δ is a directed path and satisfies $\beta_G \circ \delta = \gamma$.
- (2) The map δ is continuous and satisfies $\beta_G \circ \delta = \gamma$.
- (3) For every $t \in \text{dom } \gamma$, if $\gamma(t)$ is not a vertex, then $\delta(t) = \gamma(t)$, otherwise $\delta(t)$ is a traversal compatible with γ at t (Def. 5.23).

Proof. The first point implies the second one because a directed path is (in particular) a local order morphism (Def. 5.6), and every such morphism is continuous (Def. 5.5). Assume the second point is satisfied. Since $\beta_G \circ \delta = \gamma$, we have $\gamma(t) = \delta(t)$ each time $\gamma(t)$ is not a vertex (Def. 2.15). For the same reason, if $\gamma(t)$ is a vertex v , then $\delta(t)$ is a traversal (a, b) at v . Assume we are in the latter case. Since δ is continuous, and according to the topology of the local order $\tilde{\mathcal{X}}_G$ (Rem. 5.16), we have an open interval I containing t and satisfying

$$\delta(I) \subseteq (\{a\} \times]\ell(a) - \varepsilon, \ell(a)[) \cup \{(a, b)\} \cup (\{b\} \times]0, \varepsilon[) \quad .$$

If γ is not constant before t , then let a' be the last arrow visited by γ before t (Lem. 5.22). We have $s < t$ such that $\gamma[s, t] \setminus \{v\}$ is nonempty and included in $\{a'\} \times]\ell(a') - \varepsilon, \ell(a')[$. Since $\gamma(s) = \delta(s)$ when $\gamma(s)$ is not a vertex, we have $a = a'$. Similarly, if γ is not constant after t , then b is the first arrow visited by γ after t . Hence (a, b) is compatible with γ at t .

Assume the third point is satisfied. One readily deduces that $\beta_G \circ \delta = \gamma$ from Def. 2.15 and 5.23. It remains to check that δ is a local order morphism. The mappings γ and δ agree on $\delta^{-1}(\|G\| \setminus G^{(0)})$ because β_G leaves unchanged any point of $\|G\|$ that is not a traversal (Def. 2.15). Let t be an element of $\text{dom } \gamma$.

If $\gamma(t)$ is not a vertex, then $\gamma(t)$ belongs to $\{a\} \times]0, \ell(a)[$ for a unique arrow a . Since γ is a local order morphism, we have an open interval I containing t such that the restriction of γ to I induces a pospace morphism from I to $\{a\} \times]0, \ell(a)[$. Hence the restriction of δ to I coincides with the restriction of γ to I , therefore it is a pospace morphism (the standard pospace $\{a\} \times]0, \ell(a)[$ belongs to both \mathcal{X}_G and $\tilde{\mathcal{X}}_G$).

If $\gamma(t)$ is a vertex v , then $\delta(t)$ is a traversal (a, b) compatible with γ at t (we have supposed that the third point is satisfied). Let $\varepsilon > 0$ be small enough so that we can apply Lem. 5.25 (with the traversal (a, b)) to obtain the pospace morphism

$$\tilde{\gamma} : I \rightarrow \{a\} \times]\ell(a) - \varepsilon, \ell(a)[\cup \{v\} \cup \{b\} \times]0, \varepsilon[$$

induced by γ with I an open interval of $\text{dom } \gamma$ containing t . We denote by $\tilde{\beta}_G$ the isomorphism of pospaces induced by the restriction of β_G to $\{a\} \times]\ell(a) - \varepsilon, \ell(a)[\cup \{(a, b)\} \cup \{b\} \times]0, \varepsilon[$ (Rem. 5.16). Then $\tilde{\beta}_G^{-1} \circ \tilde{\gamma}$ is a local order morphism defined on I satisfying $\beta_G \circ \tilde{\beta}_G^{-1} \circ \tilde{\gamma} = \gamma$. The only vertex that $\tilde{\gamma}$ reaches is v , and the maps δ and $\tilde{\beta}_G^{-1} \circ \tilde{\gamma}$ agree on $\{s \in I \mid \gamma(s) \neq v\}$. If $\tilde{\gamma}(s) = v$ then $\gamma(s) = \tilde{\gamma}(s) = v$ and $\tilde{\beta}_G^{-1}(v) = (a, b) = \delta(s)$. Hence δ and $\tilde{\beta}_G^{-1} \circ \tilde{\gamma}$ agree on I .

Theorem 5.27. Every directed path γ on the image of $\beta_{G_1} \times \cdots \times \beta_{G_n}$ (seen as a sub-local order of $\mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$) admits a lifting (i.e. a directed path δ on the local order $\tilde{\mathcal{X}}_{G_1} \times \cdots \times \tilde{\mathcal{X}}_{G_n}$ such that $(\beta_{G_1} \times \cdots \times \beta_{G_n}) \circ \delta = \gamma$).

Proof. Let $t \in \text{dom } \gamma$. If $\gamma(t)$ is not a vertex put $\delta(t) = \gamma(t)$. Otherwise, there is at least one traversal (a, b) at $\gamma(t)$ (Def. 2.15). If γ is not constant before t then we can suppose that a is the last arrow visited by γ before t (Def. 5.23). Similarly, if γ is not constant after t then we can suppose that b is the first arrow visited by γ after t . Put $\delta(t) = (a, b)$. The map δ is a lifting of γ by the third point of Prop. 5.26. The lifting of $\gamma_1 \times \cdots \times \gamma_n$ is obtained by applying the above reasoning to γ_i for every $i \in \{1, \dots, n\}$.

Universal lifting property. Let $X \in \mathcal{X}_G$ and let $x : \mathbb{R} \rightarrow X$ be a *continuous* map whose image does not contain any vertex of G (the underlying set of \mathcal{X}_G is $|G|$, see Def. 2.6). The map x thus takes its values in the space $|G| \setminus G^{(0)}$, whose connected components are the segments $\{a\} \times]0, \ell(a)[$. Consequently, there is a unique arrow a such that $\text{img}(x) \subseteq \text{dom}(\phi_a)$, see Def. 3.1. Also the blowup β_G induces the identity on $|G| \setminus G^{(0)}$ (Def. 2.15), therefore we have a unique continuous map $\tilde{x} : \mathbb{R} \rightarrow \{a\} \times]0, \ell(a)[$ such that $x(t) = \beta_G(\tilde{x}(t))$ for all $t \in \mathbb{R}$. If x is a pospace embedding, then so is \tilde{x} , and its image is an open subset of $\{a\} \times]0, \ell(a)[$. Extending the codomain of \tilde{x} to the whole domain of β_G , we obtain an embedding y such that $x = \beta_G \circ y$ and $\text{img}(y)$ is open. Note that $\xi_{ab} \circ \phi_{ab}$ is the inclusion $\text{dom}(\phi_{ab}) \hookrightarrow \beta_G^{-1}U_v$ – see Def. 3.1 and 5.7 (9). The next result deals with the case where the image of x contains a vertex:

Proposition 5.28. For every $U_v \in \mathcal{X}_G$ (with v a vertex of G) and every pospace embedding $x : \mathbb{R} \rightarrow U_v$ with $v \in \text{img}(x)$, there is a unique traversal (a, b) at v such that $\text{img}(x) \subseteq \text{img}(\xi_{ab})$. The map

$$y : t \in \mathbb{R} \mapsto \begin{cases} (a, b) & \text{if } x(t) = v \\ x(t) & \text{if } x(t) \neq v \end{cases} \in \text{dom}(\phi_{ab}) \quad (11)$$

is the only *set map* y from \mathbb{R} to $\text{dom}(\phi_{ab})$ such that

$$x = \beta_G \circ \xi_{ab} \circ \phi_{ab} \circ y. \quad (12)$$

Moreover, the mapping $\tilde{x} = \xi_{ab} \circ \phi_{ab} \circ y$ is the only *continuous map* from \mathbb{R} to $\beta_G^{-1}U_v$ satisfying $x = \beta_G \circ \tilde{x}$. Furthermore \tilde{x} is a pospace embedding whose image is open.

Proof. The map x induces a pospace isomorphism on its image therefore $\text{img}(x)$ is an unbounded connected chain of U_v . So we have a unique traversal (a, b) at v such that $\text{img}(x) \subseteq \text{img}(\xi_{ab})$ (Rem. 5.21). The map defined at line (11) readily satisfies equality (12). It is the only one because $\beta_G p = p$ for all $p \in \text{dom}(\phi_{ab}) \setminus \{(a, b)\}$ and $\beta_G(a, b) = v$. To check the uniqueness of \tilde{x} it suffices to observe that $\beta_G^{-1}U_v$ is the union of the sets $\text{dom}(\phi_{a'b'})$ for all traversals (a', b') at v , and consequently that we have $\beta_G p = p$ for all $p \in \beta_G^{-1}U_v$ that is not a traversal. However, there may be more than one traversal in $\beta_G^{-1}U_v$ and the image of any of them under β_G is v (it is actually the case when v is singular – see Def. 2.14). Denote by t the only element of \mathbb{R} such that $x(t) = v$. We already know that the image of $\mathbb{R} \setminus \{t\}$ under x is contained in $\text{dom}(\phi_{ab})$, which is a neighborhood of (a, b) not containing any other traversal. Since \tilde{x} is continuous and sends t to a traversal, this

latter must be (a, b) . Denoting by $\tilde{\xi}_{ab}$ and x' the corestrictions of the pospace embeddings ξ_{ab} and x to their images, we have

$$\tilde{x} = (\text{cod}(\phi_{ab}) \hookrightarrow \beta_G^{-1}U_v) \circ \phi_{ab}^{-1} \circ \tilde{\xi}_{ab}^{-1} \circ x'$$

with $\text{cod}(\phi_{ab}) \hookrightarrow \beta_G^{-1}U_v$ a pospace embedding, while ϕ_{ab}^{-1} , $\tilde{\xi}_{ab}^{-1}$ and x' are dihomoemorphisms; therefore \tilde{x} is a pospace embedding. Hence $\text{img}(\tilde{x})$ is an unbounded connected chain of $\text{dom}(\phi_{ab})$, so it is open in the open subset $\text{dom}(\phi_{ab})$ of $\beta_G^{-1}U_v$.

Lemma 5.29. If $\theta: \mathbb{R}^n \rightarrow X_1 \times \cdots \times X_n$ is a pospace embedding with $X_k \in \mathcal{X}_{G_k}$ for every $k \in \{1, \dots, n\}$, then $\text{img}(\theta) = C_1 \times \cdots \times C_n$ with each C_k an unbounded connected chain of X_k .

Proof. Write $\theta_k = \text{pr}_k \circ \theta$ for the composite of θ followed by the k^{th} projection. Assume that $X_k = U_v$ (Def. 5.7 (8)) for some k and that v is a vertex of G_k . Any element $x \in U_v$ with $x \neq v$ is either smaller or greater than v (Rem. 5.21). Moreover, given two elements of U_v that are strictly smaller (resp. greater) than v the following are equivalent:

- they are comparable in U_v (i.e. one of them is smaller than the other),
- they have a lower bound (resp. an upper bound) in U_v , and
- they belong to $\{a\} \times]\frac{1}{2}\ell(a), \ell(a)[$ for some arrow a of G such that $\text{tgt } a = v$ (resp. $\{b\} \times]0, \frac{1}{2}\ell(b)[$ for some arrow b of G such that $\text{src } b = v$).

Given t and t' in \mathbb{R}^n we have $\theta_k(t \wedge t') \leq \theta_k(t)$, $\theta_k(t') \leq \theta_k(t \vee t')$ because θ and the projections are order preserving. It follows that either

- $\theta_k(t)$ and $\theta_k(t')$ belong to $\{a_k\} \times]0, \ell(a_k)[$ for a unique arrow a_k of G , or
- we have $\theta_k(t) \leq v \leq \theta_k(t')$ or $\theta_k(t') \leq v \leq \theta_k(t)$.

We deduce that $\text{img}(\theta_k) \subseteq J_k$ with J_k either of the form $\{a_k\} \times]0, \ell(a_k)[$ for some arrow a_k , or $\{a_k\} \times]0, \ell(a_k)[\cup \{v\} \cup \{b_k\} \times]0, \ell(b_k)[$ for some traversal (a_k, b_k) at v . We have a dihomoemorphism $\psi: J_1 \times \cdots \times J_n \cong \mathbb{R}^n$ obtained as a product of dihomoemorphisms $\psi_k: J_k \cong \mathbb{R}$. The corestriction $\tilde{\theta}$ of θ to $\text{img}(\theta)$ is a dihomoemorphism. Hence the composite

$$\mathbb{R}^n \xrightarrow{\tilde{\theta}} \text{img}(\theta) \hookrightarrow J_1 \times \cdots \times J_n \xrightarrow{\psi} \mathbb{R}^n$$

is a pospace embedding, so its image is a product of open intervals $I_1 \times \cdots \times I_n$ (Lem. 5.4) and for every $k \in \{1, \dots, n\}$, the chain C_k is $\psi_k^{-1}(I_k)$.

Proposition 5.30. If θ is a pospace embedding as in Lem. 5.29, then there is a unique permutation $\sigma \in \mathfrak{S}_n$ and a unique tuple of pospace embeddings (x_1, \dots, x_n) with $x_k: \mathbb{R} \rightarrow X_k$ for $k \in \{1, \dots, n\}$ such that $\theta = (x_1 \times \cdots \times x_n) \circ \pi_\sigma$ – see Def. 5.1.

Proof. From Lem. 5.29 we know that the corestriction of θ to its image is a pospace isomorphism of the form $\tilde{\theta}: I_1 \times \cdots \times I_n \rightarrow C_1 \times \cdots \times C_n$ with each I_k an open interval of \mathbb{R} and each C_k a connected unbounded chain of X_k . Then for every $k \in \{1, \dots, n\}$ we have pospace isomorphisms $\phi_k: \mathbb{R} \cong I_k$ and $\psi_k: C_k \cong \mathbb{R}$. Denoting by ϕ and ψ the pospace isomorphisms $\phi_1 \times \cdots \times \phi_n$ and $\psi_1 \times \cdots \times \psi_n$, the composite $\psi \circ \tilde{\theta} \circ \phi$ is a pospace automorphism of \mathbb{R}^n . From Lem. 5.2 we deduce that

$$\psi \circ \tilde{\theta} \circ \phi = (\xi_1 \times \cdots \times \xi_n) \circ \pi_\sigma$$

for a (unique) family ξ_1, \dots, ξ_n of automorphisms of the poset \mathbb{R} and a (unique) $\sigma \in \mathfrak{S}_n$. It follows from Def. 5.1 (6) that

$$\tilde{\theta} = \left(\underbrace{(\psi_1^{-1} \circ \xi_1 \circ (\phi_{\sigma^{-1}})^{-1})}_{y_1} \times \cdots \times \underbrace{(\psi_n^{-1} \circ \xi_n \circ (\phi_{\sigma^{-1}})^{-1})}_{y_n} \right) \circ \pi_\sigma .$$

The n -tuple of mappings (y_1, \dots, y_n) is uniquely defined because π_σ is (in particular) a bijection. We conclude by setting $x_k = (C_k \hookrightarrow X_k) \circ y_k$ for every $k \in \{1, \dots, n\}$. Assume that we have $\sigma, \tau \in \mathfrak{S}_n$ and $(x_1, \dots, x_n), (y_1, \dots, y_n)$ as in the statement. If $(x_1 \times \cdots \times x_n) \circ \pi_\sigma = \theta = (y_1 \times \cdots \times y_n) \circ \pi_\tau$ then $(y_1 \times \cdots \times y_n) = (x_1 \times \cdots \times x_n) \circ \pi_{\sigma \circ \tau^{-1}}$, and therefore $\sigma = \tau$.

Theorem 5.31. For every euclidean local embedding of local orders $f : \mathcal{M} \rightarrow \mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$ of dimension n (Def. 5.5), there is a unique *continuous* map $g : \mathcal{M} \rightarrow \tilde{\mathcal{X}}_{G_1} \times \cdots \times \tilde{\mathcal{X}}_{G_n}$ such that $f = \tilde{\beta} \circ g$ with $\tilde{\beta} = \beta_{G_1} \times \cdots \times \beta_{G_n}$; moreover g is a euclidean local embedding of local orders.

Proof. Let p be a point of \mathcal{M} . We have $f_E : E \rightarrow X_1 \times \cdots \times X_n$ a restriction of f that is a pospace embedding with $p \in E$ dihomeomorphic to \mathbb{R}^n and $X_k \in \mathcal{X}_{G_k}$ (Def. 5.10) for every $k \in \{1, \dots, n\}$. Given a dihomeomorphism $\varphi_E : E \cong \mathbb{R}^n$, we apply Prop. 5.30 to the embedding $f_E \circ \varphi_E^{-1}$ to obtain

$$f_E = (x_1 \times \cdots \times x_n) \circ \pi_\sigma \circ \varphi_E$$

with $\sigma \in \mathfrak{S}_n$ and $x_k : \mathbb{R} \rightarrow X_k$ a pospace embedding for every $k \in \{1, \dots, n\}$. We can suppose that σ is the identity even if it means replacing φ_E by $\pi_\sigma \circ \varphi_E$, in which case we have

$$f_E = (x_1 \times \cdots \times x_n) \circ \varphi_E . \quad (13)$$

Let $k \in \{1, \dots, n\}$. If $X_k = \{a\} \times]0, \ell(a)[$, then let \tilde{x}_k be the identity map on $\{a\} \times]0, \ell(a)[$. If X_k is the standard pospace (U_v, \sqsubseteq_v) at a vertex v of G_k (Def. 5.7), then apply Prop. 5.28 to the pospace embedding x_k to obtain the unique continuous map $\tilde{x}_k : \mathbb{R} \rightarrow \beta_{G_k}^{-1} U_v$ such that

$$x_k = \beta_{G_k} \circ \tilde{x}_k . \quad (14)$$

The map $g_p = (\tilde{x}_1 \times \cdots \times \tilde{x}_n) \circ \varphi_E$ is a pospace embedding because each \tilde{x}_k is a pospace embedding (Prop. 5.28). As a consequence of (13) and the equalities (14) for each $k \in \{1, \dots, n\}$, we have

$$f_E = \tilde{\beta} \circ (\tilde{x}_1 \times \cdots \times \tilde{x}_n) \circ \varphi_E = \tilde{\beta} \circ g_p . \quad (15)$$

Given $k \in \{1, \dots, n\}$ we apply the k^{th} projection to the relations (13) and (15) to obtain

$$\text{pr}_k \circ f_E = x_k \circ \text{pr}_k \circ \varphi_E = \beta_{G_k} \circ \tilde{x}_k \circ \text{pr}_k \circ \varphi_E . \quad (16)$$

Since $\text{pr}_k \circ \varphi_E$ is onto, we have $\text{img}(x_k) = \text{img}(\text{pr}_k \circ f_E)$. In the light of relation (16), for $e \in E$, we deduce from Prop. 5.28 (11) that $\tilde{x}_k(\text{pr}_k(\varphi(e))) = \text{pr}_k(f_E(e))$ if $\text{pr}_k(f_E(e)) \neq v$, and $\tilde{x}_k(\text{pr}_k(\varphi(e))) = (a, b)$ if $\text{pr}_k(f_E(e)) = v$, with (a, b) being the unique traversal at v in the graph G_k such that $\text{img}(\text{pr}_k \circ f_E) \subseteq \text{img}(\xi_{ab})$ – see Def. 5.7 (9) for ξ_{ab} . What really matters here is that $\tilde{x}_k(\text{pr}_k(\varphi(e)))$ only depends on f . Its image is open because each map \tilde{x}_k is open (Prop. 5.28) and φ_E is (in particular) onto. Thus we have a family of open pospace embeddings

$$\{ g_p : E_p \rightarrow \beta_{G_1}^{-1} X_1^{(p)} \times \cdots \times \beta_{G_n}^{-1} X_n^{(p)} \mid p \in \mathcal{M} \}$$

such that:

- E_p is an open neighborhood of p that is dihomeomorphic to \mathbb{R}^n ,
- $X_k^{(p)} \in \mathcal{X}_{G_k}$ for every $k \in \{1, \dots, n\}$, and
- for all $p, q \in \mathcal{M}$, the mappings g_p and g_q coincide on $E_p \cap E_q$.

The mapping g defined by $g(p) = g_p(p)$ for every $p \in \mathcal{M}$ fulfills the requirements of the statement. It remains to see that it is the only one. To this end, assume that the continuous map $h : \mathcal{M} \rightarrow$

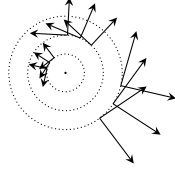


Figure 5. A conal field on the plane.

$\tilde{\mathcal{X}}_{G_1} \times \cdots \times \tilde{\mathcal{X}}_{G_n}$ satisfies $\bar{\beta} \circ h = \bar{\beta} \circ g$. Given $p \in \mathcal{M}$ we have $\bar{\beta} \circ (\tilde{x}_1 \times \cdots \times \tilde{x}_n) \circ \varphi_E = \bar{\beta} \circ h_{E_p}$ with h_{E_p} the restriction of h to E_p . We deduce that we have $\beta_{G_k} \circ \tilde{x}_k \circ \text{proj}_k \circ \varphi_E = \beta_{G_k} \circ \text{proj}_k \circ h_{E_p}$ for every $k \in \{1, \dots, n\}$. Since h is continuous, so is $\text{proj}_k \circ h_{E_p}$; we deduce from the uniqueness property in Prop. 5.28 that $\tilde{x}_k \circ \text{proj}_k \circ \varphi_E = \text{proj}_k \circ h_{E_p}$ for every $k \in \{1, \dots, n\}$. In other words we have $g_p = h_{E_p}$. Since the latter equality holds for every $p \in \mathcal{M}$ we have $h = g$.

The identity map $id_{\|\cdot\|}$ induces a 1-dimensional euclidean local embedding of local orders $\iota_G : \tilde{\mathcal{X}}_G \rightarrow \mathcal{E}_G$ which is not open (Def. 5.5, 5.15, and 5.18). Given a tuple of graphs (G_1, \dots, G_n) , we write \bar{e} and \bar{t} to denote the products $e_{G_1} \times \cdots \times e_{G_n}$ and $\iota_{G_1} \times \cdots \times \iota_{G_n}$. If $f : \mathcal{M} \rightarrow \mathcal{E}_{G_1} \times \cdots \times \mathcal{E}_{G_n}$ is a euclidean local embedding of dimension n , then so is $\bar{e} \circ f$, and we have $\bar{e} \circ f = \bar{\beta} \circ g = \bar{e} \circ \bar{t} \circ g$ with g given by Thm. 5.31; however, in general, we cannot deduce that $f = \bar{t} \circ g$:

Example 5.32. Consider the graph G on the left hand part of Fig. 3 and let f be the directed arc on the étale local order \mathcal{E}_G covering $\{a\} \times]0, 1[\cup \{(a, c)\} \cup \{b\} \times]0, 1[$, see the right hand part of Fig. 3. The map f is not even continuous on the underlying topological space of the standard local order $\tilde{\mathcal{X}}_G$. The unique continuous map g such that $\bar{e} \circ f = \bar{\beta} \circ g$ is the directed arc on $\tilde{\mathcal{X}}_G$ covering $\{a\} \times]0, 1[\cup \{(a, b)\} \cup \{b\} \times]0, 1[$; hence f and $\bar{t} \circ g$ differ in only one point. Such a pathology is possible because the underlying space of \mathcal{E}_G is not Hausdorff.

5.4 Cone fields and local orders

For any manifold M , there is an equivalence (in the categorical sense) between the local orders on (the underlying set of) M whose elements are locally order convex (p. 19), and the upper semi-continuous conal fields on M admitting sections at every point (Lawson, 1989, 2.7). The above statement involves a manifold instead of an atlas, and Lawson (1989) does not make the topological properties required on M explicit. However, this statement is local by nature, and its proof does not seem to suppose anything about the underlying topology of M ; assuming that it holds for the atlas $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$, the conal field associated to the standard parallelization and the standard local order on $\tilde{\mathcal{X}}_{G_1} \times \cdots \times \tilde{\mathcal{X}}_{G_n}$ are related by this equivalence (Prop. 5.36 and Thm. 5.39). Except for the latter results, this section is entirely based on (Lawson, 1989, §1 and §2).

A *wedge* is a topologically closed subset W of a finite dimensional real vector space such that $\mathbb{R}_+ W \subseteq W$ and $W + W \subseteq W$. A *cone* is a wedge C such that $C \cap -C = \{0\}$. A *cone field* on an atlas \mathcal{A} is a map assigning a cone $C(p)$ of the tangent space $T_p \mathcal{A}$ to every point p of \mathcal{A} , see Fig. 5 and Ex. 5.33. A *conal atlas* is an atlas together with a cone field. If C and D are cone fields on the atlases \mathcal{A} and \mathcal{B} then the mapping $C \times D$ is a cone field on $\mathcal{A} \times \mathcal{B}$.

Example 5.33. Any parallelization (f_1, \dots, f_n) (§A.3, p. 46) canonically induces a cone field:

$$C(p) = \left\{ \lambda_1 f_1(p) + \cdots + \lambda_n f_n(p) \mid \lambda_i \geq 0 \text{ for all } i \in \{1, \dots, n\} \right\}.$$

Such cone fields are said to be *cartesian*. If C and D are the cartesian cones induced by the parallelizations (f_1, \dots, f_n) and (g_1, \dots, g_m) , then $C \times D$ is the cartesian cone induced by the parallelization $(f_1, \dots, f_n, g_1, \dots, g_m)$.

Definition 5.34. The *standard cone field* C_G on \mathcal{A}_G is the cartesian cone field induced by its standard parallelization (Def. 3.6), that is to say:

$$C_G : p \in \|G\| \mapsto \{(p, t) \mid t \in \mathbb{R}_+\} \text{ cone of } \{p\} \times \mathbb{R}.$$

Conal preorder. For every curve c extending a smooth path γ (Def. A.7) and every $t \in \text{dom}(\gamma)$ we have the *left hand* and the *right hand* derivatives of γ at t , namely

$$\lim_{x \rightarrow t^-} (D\gamma)_x = Dc_t \quad \text{and} \quad \lim_{x \rightarrow t^+} (D\gamma)_x = Dc_t;$$

which we denote by $D\gamma_{t^-}$ and $D\gamma_{t^+}$. A piecewise smooth path admits derivatives on both sides at every t , although there are finitely many t 's at which they may not coincide.

A piecewise smooth path γ is said to be *conal* on (\mathcal{A}, C) if $D\gamma_{t^+} \in C(\gamma(t))$ for every $t \in \text{dom}(\gamma)$. Given the points p and q of an open subset U of \mathcal{A} , we write $p \preceq_v q$ when there exists a conal curve on U from p to q . The relation \preceq_v thus defined is the *conal preorder* on U .

Infinitesimal preorder. A *partial curve* on an atlas \mathcal{A} is a map $\gamma : D \rightarrow |\mathcal{A}|$ with $0 \in D \subseteq \mathbb{R}_+$ and 0 a cluster point of D (Munkres, 2000, §17, p. 97). Given $X \subseteq |\mathcal{A}|$, we say that $v \in T_p \mathcal{A}$ is a *subtangent vector* of X at p when there exists a partial curve $\gamma : D \rightarrow |\mathcal{A}|$ with $\gamma(0) = p$ and $\gamma(D \setminus \{0\}) \subseteq X$, and a chart $\phi \in \mathcal{A}$, such that

$$\llbracket v \rrbracket_p^\phi = \lim_{0 \leftarrow t \in D} \frac{1}{t} (\phi(\gamma(t)) - \phi(p)).$$

Given $\psi \in \mathcal{A}$ with $p \in \text{dom} \psi$, it suffices to write $\psi \circ \gamma = (\psi \circ \phi^{-1}) \circ (\phi \circ \gamma)$ and apply the chain rule (i.e. a form of the chain rule adapted to partial curves) to check that the above equality still holds with ψ instead of ϕ .

Assume that the local order \mathcal{X} and the atlas \mathcal{A} are based on the same topological space. The *infinitesimal preorder* of \mathcal{X} is the map assigning to each point $p \in \mathcal{A}$ the smallest wedge W_p of $T_p \mathcal{A}$ containing all the subtangent vectors of the set $\{p \preceq_u u \mid u \in U\}$ at p for some pospace $(U, \preceq) \in \mathcal{X}$ such that $p \in U$. Because of the coherence condition satisfied by a local order, the wedge W_p does not depend on (U, \preceq) . In the context of this article, W_p is roughly the set of speed vectors $\gamma'(t)$ for curves γ on \mathcal{A} that are directed on \mathcal{X} and satisfy $\gamma(t) = p$ with $t \in \text{dom}(\gamma)$.

Example 5.35. Assume that \mathcal{X} is the local order on the punctured plane (at the origin) whose directed paths are of the form $t \in J \mapsto \rho(t)e^{i\theta(t)} \in \mathbb{R}^2 \setminus \{0\}$ with J interval of \mathbb{R} , and $\rho : J \rightarrow]0, \infty[$, $\theta : J \rightarrow \mathbb{R}$ non decreasing continuous maps. The infinitesimal preorder of \mathcal{X} is the one shown on Fig. 5, i.e. $W_{(x,y)} = \{a(x, y) + b(-y, x) \mid a, b \in \mathbb{R}_+\}$.

The correspondence between conal preorders and infinitesimal preorders (which is the subject of the next paragraph) is dealt with in (Lawson (1989)) by means of partial curves.

Lawson correspondence. We say that the cone C' *surrounds* the cone C when $C \setminus \{0\}$ is contained in the interior of C' ; see (Hilgert et al., 1989, IV.6.4, p.342) for a more conceptual definition of surrounding cones. Given a chart $\phi \in \mathcal{A}$ and $p, q \in \text{dom} \phi$, the composite $(\llbracket _ \rrbracket_q^\phi)^{-1} \circ \llbracket _ \rrbracket_p^\phi : T_p \mathcal{A} \rightarrow T_q \mathcal{A}$ is a tvs isomorphism (Def. A.12) which we denote by $\llbracket _ \rrbracket_{pq}^\phi$; its inverse is $\llbracket _ \rrbracket_{qp}^\phi$, and $\llbracket _ \rrbracket_{pp}^\phi$ is the identity map.

A cone field C on \mathcal{A} is said to be *upper semi-continuous* at p when for every chart $\phi \in \mathcal{A}$ around p , and every cone C' that surrounds $C(p)$, there exists a neighborhood U of p such that C' surrounds $\llbracket C(q) \rrbracket_{qp}^\phi$ for every $q \in U$. We say that C *admits sections* at p when for every $v \in C(p)$ there exists a vector field f on \mathcal{A} such that $f(p) = v$.

An open cover \mathcal{U} of the atlas \mathcal{A} is said to be *admissible* for the cone field C on \mathcal{A} when for every $U \in \mathcal{U}$ the conal preorder \preceq_v is antisymmetric, the pospace (U, \preceq_v) is locally order convex (§5.1, p. 19), and the collection $\{(U, \preceq_v) \mid U \in \mathcal{U}\}$ is a local order. Given an atlas \mathcal{A} , Lawson's result (Lawson, 1989, 2.7) states that:

- (1) For every upper semi-continuous cone field C on \mathcal{A} admitting sections at every point, there exists an open cover \mathcal{U} admissible for C . Moreover, for every open cover \mathcal{U} admissible for C , the infinitesimal preorder of $\{(U, \preceq_U) \mid U \in \mathcal{U}\}$ is C .
- (2) For every local order \mathcal{X} made of locally convex pospaces, the infinitesimal preorder of \mathcal{X} is an upper semi-continuous cone field C on \mathcal{A} admitting sections at every point. Moreover, the local order \mathcal{X} is equivalent (in the sense of §5.2, p. 21) to $\{(U, \preceq_U) \mid U \in \mathcal{U}\}$ for every open cover \mathcal{U} admissible for C .

Proposition 5.36. The standard local order $\tilde{\mathcal{X}}_G$ on $\|G\|$ (Def. 5.15) and the standard cone field C_G on \mathcal{A}_G (Def. 5.34) are deduced from each other along Lawson's correspondence.

Proof. The fact that C_G is upper semi-continuous and admits sections at every point is immediate. Denote by \sqsubseteq_{ab} the partial order on $\text{dom}(\phi_{ab})$ that comes from $\phi_{ab} \in \tilde{\mathcal{X}}_G$. Denote by \preceq_{ab} the conal preorder on $\text{dom}(\phi_{ab})$ induced by C_G . Let $p, q \in \text{dom} \phi_{ab}$. The inequality $p \sqsubseteq_{ab} q$ amounts to $\phi_{ab} p \leq \phi_{ab} q$ in \mathbb{R} . The restriction of ϕ_{ab}^{-1} to the segment $[\phi_{ab} p, \phi_{ab} q]$ is a conal curve on $\text{dom} \phi_{ab}$, which implies that $p \preceq_{ab} q$. The other way round, if $p \preceq_{ab} q$, then we have a piecewise smooth conal curve γ on $\text{dom} \phi_{ab}$ from p to q . The fact that γ is conal means that for every $t \in \text{dom}(\gamma)$ we have $D\gamma_t(1) = rC_G(p)$ with $r > 0$, and therefore the map $\phi_{ab}^{-1} \circ \gamma$ is a piecewise derivative map between intervals of \mathbb{R} whose derivative is non-negative; it is thus non-decreasing. The map ϕ_{ab} (Def. 3.1) is a dihomoemorphism (Def. 5.15 and its preamble), hence γ is a pospace morphism; we deduce that $p \sqsubseteq_{ab} q$. Given $\phi_a \in \tilde{\mathcal{X}}_G$ the same reasoning holds for the partial order \sqsubseteq_a on $\text{dom}(\phi_a)$, and \preceq_a the conal preorder on $\text{dom}(\phi_a)$ induced by C_G . So we have proven that \sqsubseteq_{ab} (resp. \sqsubseteq_a) is the conal preorder on $\text{dom}(\phi_{ab})$ (resp. $\text{dom}(\phi_a)$) induced by C_G . It follows that the collection

$$\mathcal{U} = \{ \text{dom}(\phi_a) \mid a \text{ arrow of } G \} \cup \{ \text{dom}(\phi_{ab}) \mid (a, b) \text{ traversal of } G \}$$

is admissible for C_G , and that the local order $\{(U, \preceq_U) \mid U \in \mathcal{U}\}$ is $\tilde{\mathcal{X}}_G$. From the first point of Lawson's result, we deduce that C_G is the infinitesimal preorder of $\tilde{\mathcal{X}}_G$.

The following lemmas derive from the fact that given atlases \mathcal{A} and \mathcal{B} , partial curves (resp. piecewise smooth paths) on $\mathcal{A} \times \mathcal{B}$ are of the form $\alpha \times \beta$ with α and β partial curves (resp. piecewise smooth paths) on \mathcal{A} and \mathcal{B} .

Lemma 5.37. If C and D are the infinitesimal preorders of the local orders \mathcal{X} and \mathcal{Y} , then $C \times D$ is the infinitesimal preorder of the local order $\mathcal{X} \times \mathcal{Y}$.

Proof. Let $U \in \mathcal{X}$, $V \in \mathcal{Y}$, and $(p, q) \in U \times V$. A tangent vector $v \in T_{(p,q)}(\mathcal{A} \times \mathcal{B})$ is a subtangent vector of $\{(p, q) \preceq_U \times \preceq_V (u, v) \mid (u, v) \in U \times V\}$ when there exists a partial curve $\gamma : D \rightarrow |\mathcal{A}| \times |\mathcal{B}|$ with $\gamma(0) = (p, q)$, $\gamma(0) \preceq_U \times \preceq_V \gamma(t)$ for every $t \in D \setminus \{0\}$ and

$$\llbracket v \rrbracket_{(p,q)}^{\phi \times \psi} = \lim_{0 \leftarrow t \in D} \frac{1}{t} ((\phi \times \psi)(\gamma(t)) - (\phi \times \psi)(p, q)).$$

The vector space $T_{(p,q)}(\mathcal{A} \times \mathcal{B})$ is isomorphic to $T_p \mathcal{A} \times T_q \mathcal{B}$ so we can suppose that $\llbracket _ \rrbracket_{(p,q)}^{\phi \times \psi} = \llbracket _ \rrbracket_p^\phi \times \llbracket _ \rrbracket_q^\psi$ and $v = (a, b)$ with $a \in T_p \mathcal{A}$ and $b \in T_q \mathcal{B}$. It follows that $\gamma = \alpha \times \beta$ with $\alpha : D \rightarrow |\mathcal{A}|$ and $\beta : D \rightarrow |\mathcal{B}|$ partial curves satisfying $\alpha(0) = p$, $\beta(0) = q$, and $\alpha(0) \preceq_U \alpha(t)$ and $\beta(0) \preceq_V \beta(t)$ for every $t \in D \setminus \{0\}$, and

$$\llbracket a \rrbracket_p^\phi = \lim_{0 \leftarrow t \in D} \frac{1}{t} (\phi(\alpha(t)) - \phi(p)) \quad \text{and} \quad \llbracket b \rrbracket_q^\psi = \lim_{0 \leftarrow t \in D} \frac{1}{t} (\psi(\beta(t)) - \psi(q)).$$

This is equivalent to having a and b subtangent vectors of $\{p \preceq_U u \mid u \in U\}$ and $\{q \preceq_V v \mid v \in V\}$.

Lemma 5.38. If \mathcal{X} and \mathcal{Y} are the local orders associated with the upper semi-continuous cone fields C and D , then the local order $\mathcal{X} \times \mathcal{Y}$ is associated with the cone field $C \times D$ (which is upper semi-continuous).

Proof. Having a smooth path γ on $\mathcal{A} \times \mathcal{B}$ from (p, q) to (u, v) satisfying $d(\gamma)(t) \in C(\alpha(t)) \times D(\beta(t))$ for every $t \in \text{dom}(\gamma)$, is equivalent to having smooth paths α and β (with the same domain of definition J) on \mathcal{A} and \mathcal{B} , respectively from p to u and from q to v , such that $d\alpha(t) \in C(\alpha(t))$ and $d\beta(t) \in D(\beta(t))$ for every $t \in J$. In other words we have $(p, q) \preceq_{u,v} (u, v)$ if, and only if, $p \preceq_u u$ and $q \preceq_v v$, i.e. $(p, q) \preceq_{u \times v} (u, v)$.

Let (G_1, \dots, G_n) be a tuple of graphs. For each $k \in \{1, \dots, n\}$ we have the standard local order $\tilde{\mathcal{X}}_{G_k}$ (Def. 5.15) and the standard cone field C_{G_k} (Def. 5.34) on the set $\|G_k\|$. As an immediate consequence of Prop. 5.36 and Lem. 5.37 and 5.38, we have:

Theorem 5.39. The cone field $C_{G_1} \times \dots \times C_{G_n}$ and the local order $\tilde{\mathcal{X}}_{G_1} \times \dots \times \tilde{\mathcal{X}}_{G_n}$ are deduced from each other along Lawson's correspondence.

6. Execution time

The smooth model $\|P\|$ of a program P (Def. 2.22) is equipped with a pseudometric inherited from that of $\|G_1\| \times \dots \times \|G_n\|$; while the pseudometric of each $\|G_i\|$ is derived from $\ell_i : G_i^{(1)} \rightarrow [R, \infty[$ (Def. 4.1, 4.3, and 4.7). The pseudometric of $\|G_1\| \times \dots \times \|G_n\|$ also depends on a parameter $\alpha \in [1, \infty]$ (Def. 4.10 and 4.16, and Thm. 4.21). In order that the length of a directed path on $\|P\|$ coincides with the execution time of its sequence of multi-instructions, we should take $\alpha = \infty$. We explain why in §6.1 and prove in §6.2 that directed paths admit piecewise affine approximations (Thm. 6.5 and Cor. 6.6). A similar discussion can be found in (Dang & Gerner, 2004, §3.4).

6.1 The parallel execution time principle

If several tasks that have started at the same time are executed in parallel (i.e. simultaneously and independently of each other) then the time required to complete all of them is the time required to complete the longest one; we refer to this claim as the *parallel execution time principle*. As a consequence of it, the time required to execute a multi-instruction μ on a machine allowing parallel execution (we call it the *parallel execution time* of μ) is

$$\max \{ T_i \mid i \in \text{dom } \mu \} \quad (17)$$

with T_i being the time required by the i^{th} process to execute the instruction $\mu(i)$ (for $i \in \text{dom } \mu$).

In comparison, if several tasks are performed sequentially (i.e. one task starts the moment the previous one ends) then the time required to complete all of them is the sum of the times required to complete each of them; we refer to this claim as the *sequential execution time principle*. Consequently, the time required to execute a multi-instruction μ on a sequential machine (we call it the *sequential execution time* of μ) is

$$\sum_{i \in \text{dom } \mu} T_i . \quad (18)$$

Following the preamble of §5, consider a directed smooth path γ (Def. 5.6 and A.7) representing an execution trace (in accordance with the discussion that follows Def. 2.13, γ should be thought of as a schedule). On an infinitesimal scale, the path γ at t is identified with the segment from $\gamma(t)$ to $\gamma(t) + \gamma'(t)$ which represents the execution of an ‘infinitesimal multi-instruction’ $d\mu$. Put $\gamma'(t) = (\gamma'_1(t), \dots, \gamma'_n(t))$ and assume that $|\gamma'_i(t)|$ is the time spent by the i^{th} process to execute the i^{th} component of $d\mu$ (with the convention that $|\gamma'_i(t)| = 0$ when $i \notin \text{dom}(d\mu)$). As a consequence of (17) and (18), the parallel execution time of $d\mu$ is $|\gamma'(t)|_\infty = \max \{ |\gamma'_i(t)| \mid i \in \{1, \dots, n\} \}$, while the sequential execution time of $d\mu$ is $|\gamma'(t)|_1 = |\gamma'_1(t)| + \dots + |\gamma'_n(t)|$. Hence the parallel and the

sequential execution times of the sequence of multi-instructions represented by γ are respectively

$$\mathcal{L}^\infty(\gamma) = \int_x^y |\gamma'(t)|_\infty dt \quad \text{and} \quad \mathcal{L}^1(\gamma) = \int_x^y |\gamma'(t)|_1 dt$$

with $[x, y] = \text{dom}(\gamma)$ – see Def. 4.16. We observe that

$$\int_x^y |\gamma'(t)|_1 dt = \int_x^y \left(\sum_{i=1}^n |\gamma'_i(t)| dt \right) = \sum_{i=1}^n \left(\int_x^y |\gamma'_i(t)| dt \right)$$

which confirms that the sequential execution time does not depend on the schedule in the sense that we have $\mathcal{L}^1(\gamma) = \mathcal{L}^1(\gamma \circ \theta)$ for any surjective increasing smooth path θ from $[x, y]$ to $[x, y]$. This is wrong for the parallel execution time principle: by considering the directed (piecewise) smooth paths $\gamma : t \in [0, 1] \mapsto (t, t) \in [0, 1]^2$ and

$$\theta : t \in [0, 1] \mapsto \begin{cases} (0, 2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ (2t - 1, 1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

we have $\mathcal{L}^\infty(\gamma) = 1$ and $\mathcal{L}^\infty(\gamma \circ \theta) = 2$. The ratio $\frac{\mathcal{L}^1(\gamma)}{\mathcal{L}^\infty(\gamma)}$ indicates the efficiency of the schedule γ . See (Cormen et al., 2022, p. 757-759) and (Herlihy et al., 2020, §16.2, p. 384-386) for related discussions.

Let M be the model of a program P . We denote by dM the set of directed paths on M . Two points of M are said to be *equivalent* when they belong to the same tile (Def. 2.11); two elements of dM are said to be *equivalent* when so are their extremities and their sequences of multi-instructions have the same effect on the system – see §1 or (Haucourt, 2018, §6). We have just seen that M admits a metric such that the length of a directed path on M is the execution time of its sequence of multi-instructions, so it seems natural to minimize length in order to optimize execution time. However, a constant path and a directed loop on M with the same extremities are not equivalent in general. Therefore, assuming that σ is the sequence of multi-instructions of $\gamma \in dM$, when it comes to replacing σ by a faster sequence of multi-instructions doing the same job, we cannot just replace γ by a shorter directed path with equivalent extremities. Nevertheless, assuming that dM is equipped with the distance defined by

$$d(\delta_0, \delta_1) = \max\{d(\delta_0(t), \delta_1(t)) \mid t \in J\}$$

for all $\delta_0, \delta_1 \in dM$ such that $\text{dom}(\delta_0) = \text{dom}(\delta_1) = J$, there is an open ball B centered at γ such that for every $\delta \in B$, the directed paths γ and δ are equivalent when so are their extremities (Haucourt, 2018, Thm. 6.1). We easily obtain a possible radius for B as follows. We define the set of atomic instructions appearing in a multi-instruction μ as $A(\mu) = \{\mu_i \mid i \in \text{dom}(\mu)\}$, and the set of atomic instructions appearing in a sequence of multi-instructions μ_1, \dots, μ_k as the union $A(\mu_1) \cup \dots \cup A(\mu_k)$. The smallest execution time of the atomic instructions appearing in σ can be taken as the radius of B . This result allows local optimizations based on purely geometric considerations, a topic which is discussed further in §7.

6.2 Piecewise affine approximations

As long as we are only interested in minimizing the length of directed paths between two points of a tile compatible set (Def. 2.11 and 2.15) we can restrict our attention to piecewise affine paths (Def. 4.1): this claim is formalized in Thm. 6.5 and Cor. 6.6.

The sets $|G_1| \times \dots \times |G_n|$ and $\|G_1\| \times \dots \times \|G_n\|$ are equipped with their standard local orders (Def. 5.10 and 5.15) from which their topologies are inherited (§5.2, p. 21). The pseudometrics d_i on $|G_i|$ and $\|G_i\|$ are given by Def. 4.3 and 4.7. Following execution time considerations (§6.1)

the chosen product metric is

$$d^{(\infty)}((p_1, \dots, p_n), (q_1, \dots, q_n)) = \max \{d_i(p_i, q_i) \mid i \in \{1, \dots, n\}\}.$$

Let $\gamma : I \rightarrow X$ be a path with X a subset of $|G_1| \times \dots \times |G_n|$ or $\|G_1\| \times \dots \times \|G_n\|$.

Definition 6.1. An ε -approximation of γ on X is a path $\alpha : I \rightarrow X$ such that:

- α and γ have the same starting and the same finishing points,
- α and γ are ε -close, i.e. $d^{(\infty)}(\alpha(t), \gamma(t)) < \varepsilon$ for every $t \in I$,
- α is not longer than γ , and
- if γ is directed, then so is α .

Of course any path is its own ε -approximation whatever ε and X are. From now on we assume that X is tile-compatible; our goal is to prove that any directed path γ on X has a piecewise affine ε -approximation for arbitrarily small $\varepsilon > 0$. We first deal with the case where $X \subseteq |G_1| \times \dots \times |G_n|$, which requires more technicalities.

Definition 6.2. An open ball B of radius r in the metric space $(|G_1| \times \dots \times |G_n|, d^{(\infty)})$ is said to be *admissible* when $r \leq \frac{R}{4}$ and for every $k \in \{1, \dots, n\}$, if the k^{th} projection of the center of B is not a vertex, then the k^{th} projection of B contains no vertex.

Definition 6.3. Given $\varepsilon > 0$, an ε -discretization of a path $\gamma : [x, y] \rightarrow |G_1| \times \dots \times |G_n|$ is a sequence $x = t_0 < \dots < t_k \leq y$ with admissible open balls B_0, \dots, B_k such that i) each B_i is centered at $\gamma(t_i)$ with radius at most ε , ii) for every $i \in \{1, \dots, k\}$, $]t_{i-1}, t_i[\cap \gamma^{-1}(B_{i-1} \cap B_i) \neq \emptyset$, and iii) $\text{img}(\gamma) \subseteq B_0 \cup \dots \cup B_k$.

Lemma 6.4. For every $\varepsilon > 0$, every path γ on $|G_1| \times \dots \times |G_n|$ admits an ε -discretization.

Proof. Suppose that $\text{dom}(\gamma) = [x, y]$ and denote by t_∞ the least upper bound of the set of elements $t \in [x, y]$ such that the restriction $\gamma|_{[x, t]}$ has an ε -discretization. Choose an admissible open ball B_∞ centered at $\gamma(t_\infty)$ with radius less than ε . Let $t < t_\infty$ such that $\gamma[t, t_\infty] \subseteq B_\infty$. Let $t_0 < \dots < t_k \leq t$ with open balls B_0, \dots, B_k be an ε -discretization of the restriction $\gamma|_{[x, t]}$. Even if it means taking t a bit bigger, we can suppose that $t_k < t < t_\infty$ with $\gamma(t) \in B_k \cap B_\infty$. Then $t_0 < \dots < t_k < t_\infty \leq t_\infty$ with open balls $B_0, \dots, B_k, B_\infty$ is an ε -discretization of the restriction $\gamma|_{[x, t_\infty]}$. If we had $t_\infty < y$ then we would have $t' \in]t_\infty, y[$ such that $\gamma[t_\infty, t'] \subseteq B_\infty$ so the previous ε -discretization of $\gamma|_{[x, t_\infty]}$ would also be an ε -discretization of $\gamma|_{[x, t']}$, which would contradict the definition of t_∞ .

We deduce the approximation result:

Theorem 6.5. For every $\varepsilon > 0$, every path γ on X admits a piecewise affine ε -approximation.

Proof. Let $\varepsilon > 0$ and suppose that $x = t_0 < \dots < t_k \leq y$ with open balls B_0, \dots, B_k is an ε -discretization of γ defined on $[x, y]$. For every $i \in \{1, \dots, k\}$ choose $t'_i \in]t_{i-1}, t_i[$ such that $\gamma(t'_i) \in B_{i-1} \cap B_i$ and define $t'_{k+1} = y$. We have the sequence

$$\gamma(t_0), \gamma(t'_1), \gamma(t_1), \dots, \gamma(t'_k), \gamma(t_k), \gamma(y)$$

and for every $i \in \{0, \dots, k\}$ we apply Prop. B.31 from $\gamma(t'_i)$ to $\gamma(t_i)$ and from $\gamma(t_i)$ to $\gamma(t'_{i+1})$ in the ball B_i (omitting $\gamma(t'_0)$ which is not defined). We obtain a sequence of dashes (Def. B.24)

$$\gamma(t_0) \rightarrow \gamma(t'_1) \rightarrow \gamma(t_1) \rightarrow \dots \rightarrow \gamma(t'_k) \rightarrow \gamma(t_k) \rightarrow \gamma(y)$$

whose concatenation is the expected ε -approximation.

Corollary 6.6. For every $\varepsilon > 0$, every directed path γ on a tile compatible subset X of $\|G_1\| \times \cdots \times \|G_n\|$ admits a piecewise affine ε -approximation on X .

Proof. The directed path $(\beta_1 \times \cdots \times \beta_n) \circ \gamma$ on the tile compatible set $(\beta_1 \times \cdots \times \beta_n)(X)$ (Lem. 2.19) admits a (directed) piecewise affine ε -approximation α (Thm. 6.5). The expected approximation is the lifting of α (Thm. 5.27).

7. Afterword

A glimpse of Finsler geometry. Finsler metrics are motivated by the observation that measuring the length of a curve on a manifold \mathcal{M} only requires a way of measuring the length of tangent vectors, see (Papadopoulos, 2013, p.40), (Chern & Shen, 2005, p.1). In this context, we call (*geometric*) *action functional* any map F from $T\mathcal{M}$ to \mathbb{R}_+ such that $F(x+y) \leq F(x) + F(y)$ and $F(\lambda x) = \lambda F(x)$ for all $p \in \mathcal{M}$, $x, y \in T_p\mathcal{M}$, and $\lambda > 0$. As suggested by the physical motivations described in (Bao et al., 2000, p.1,2), the action functional F can be chosen so that the length of every curve γ with respect to F has a physical meaning. Hence minimizing length amounts to minimizing action. Any map arising from Def. B.14 (25) is an action functional because $|\cdot|_*$ is a norm. When the latter is $|\cdot|_\infty$ the length of γ is the time required to execute the sequence of multi-instructions that γ represents; see §6.1 and compare to the navigation problem (Chern & Shen, 2005, 1.4).

Following this idea, a *Finsler metric* is a family of Minkowski norms on $T_p\mathcal{M}$ smoothly varying with $p \in \mathcal{M}$ (Bao et al., 2000, p.5). More formally, it is a map $F : T\mathcal{M} \rightarrow \mathbb{R}_+$ which is smooth on the *slit tangent bundle* (i.e. $T\mathcal{M} \setminus \{\text{zero tangent vectors}\}$) and whose restrictions to tangent spaces are Minkowski norms (Chern & Shen, 2005, 1.2.1). These axioms are motivated by the notion of *curvature*, which plays a central role in the study of both Riemannian and Finsler geometries. Indeed, the latter are meant to generalize the former, for which the importance of curvature is well-known (Bao et al., 2000, Introduction).

Although our mathematical framework arises from the same motivations, technicalities rule it out of the scope of standard Finsler geometry. First, manifolds of the form $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ are non-Hausdorff (in all relevant cases) so their topologies are not induced by their (pseudo)metrics. This defect is related to the non-determinism specific to our topic, and it seems impossible to get rid of it. The other issue is about the regularity of the action functional introduced in Def. B.14 (25): it fails to be smooth on the slit tangent bundle when it is built from the norm we are interested in, namely $|\cdot|_\infty$ (§6.1). The reason is that the norm $|\cdot|_\infty$ is derivative at (x_1, \dots, x_n) if, and only if, $x_i \neq x_j$ for all $i \neq j$. About this last point, one may argue that ‘ill-behaved’ points form a neglectable closed subset of \mathbb{R}^n (with respect to Lebesgue measure), so they may be manageable. Indeed, the problem of finding minimum action curves in degenerate (i.e. non-smooth) Finsler metrics is studied in (Heymann (2015)) whose preface indicates concrete situations in which they naturally occur. Heymann’s theory is designed for an action functional defined on \mathbb{R}^n , yet we are confident that this limitation can be easily overcome, at least for manifolds of the form $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ (although we will have to renounce uniqueness of geodesics, which is not prohibitive in computer science). Heymann theory satisfies many desirable properties (Heymann, 2015, p. 6), in particular it applies to any action functional F on \mathbb{R}^n that follows Def. B.14 (25) (Heymann, 2015, Def. 2.4). In view of applications to geometric models of concurrency, we would like to adapt it to find *directed* minimizing curves. Such curves would indeed correspond to optimal schedules: this should be compared to the work of Dang & Gerner (2004).

Dealing with the endpoints of a graph. We have observed (after Def. 2.15) that the blowup $\beta_G : \|G\| \rightarrow |G|$ may not be surjective; indeed, the elements of the set $|G| \setminus \beta_G(\|G\|)$ are the endpoints of G , i.e. the vertices of G without ingoing arrow or without outgoing arrow. One can remedy this by means of manifolds with boundary (Benedetti, 2021, Def. 4.20, p. 78), (Bröcker and Jänich, 1982, Chap. 13, p.129-131), (Gauld, 1982, p. 131), (Guillemin & Pollack, 1974, §1,

p. 57), (Hirsch, 1976, §4, p.29-30), (Lee, 2012, p.25) (Kosinski, 1993, p. 2), (Mukherjee, 2015, p. 39), (Shastri, 2011, Def. 3.2.1, p. 80):

Definition 7.1. A *chart with boundary* of dimension $n \in \mathbb{N}$, or *n-chart with boundary*, is a *bijection* ϕ whose codomain is an open subset of $\mathbb{R}_+ \times \mathbb{R}^{n-1}$, with $\mathbb{R}_+ = [0, \infty[$. A map $f : U \rightarrow \mathbb{R}^m$ with U an open subset of $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ is said to be smooth if it has a smooth extension to an open subset of \mathbb{R}^n (i.e. there exists an open subset V of \mathbb{R}^n and a smooth map $\tilde{f} : V \rightarrow \mathbb{R}^m$ such that $U \subseteq V$ and f is the restriction of \tilde{f} to U). Compatibility between charts with boundary is defined as for charts except that the second item of Def. A.2 refers to smooth maps defined on open subsets of $\mathbb{R}_+ \times \mathbb{R}^{n-1}$. An *atlas with boundary* is a collection \mathcal{A} of pairwise compatible charts with boundary (compare with Def. A.4). The n -dimensional atlas (with boundary) \mathcal{A} induces an $(n-1)$ -dimensional atlas (without boundary) $\partial\mathcal{A}$ on the union of the sets $\phi^{-1}(\{0\} \times \mathbb{R}^{n-1})$ for ϕ ranging through \mathcal{A} (given $\phi, \psi \in \mathcal{A}$, for every point $p \in \text{dom}(\phi) \cap \text{dom}(\psi)$, we have $p \in \phi^{-1}(\{0\} \times \mathbb{R}^{n-1})$ if, and only if, $p \in \psi^{-1}(\{0\} \times \mathbb{R}^{n-1})$). The atlas $\partial\mathcal{A}$ is called the *boundary* of \mathcal{A} .⁽⁴⁾

Definition 7.2. The *endpoint traversals* of G are the elements of the set

$$\partial\|G\| = \{(a, v), (v, b) \mid v \text{ endpoint of } G; \text{tgt}(a) = v; \text{src}(b) = v\},$$

and the *standard charts with boundary* of G are the bijections

$$\begin{aligned} \phi_{av} : \{a\} \times J_a \cup \{(a, v)\} &\rightarrow]\frac{-\ell(a)}{2}, 0] & \phi_{vb} : \{(v, b)\} \cup \{b\} \times I_b &\rightarrow [0, \frac{\ell(b)}{2}[\\ \text{with } (a, t) \mapsto t - \ell(a), & (a, v) \mapsto 0 & \text{with } (v, b) \mapsto 0, & (b, t) \mapsto t \end{aligned}$$

with $J_a =]\frac{\ell(a)}{2}, \ell(a)[$, $I_b =]0, \frac{\ell(b)}{2}[$, and $(a, v), (v, b) \in \partial\|G\|$. We define $\bar{\mathcal{A}}_G$, the *standard atlas with boundary* of G , as the collection of all standard charts of G , with or without boundary. We denote by $\bar{\beta}_G$ the extension of the blowup map β_G to $\|G\| \cup \partial\|G\|$ defined by $\bar{\beta}_G(a, v) = \bar{\beta}_G(v, b) = v$.

Remark 7.3. If X is the underlying topology of the standard atlas with boundary of G , then h_x the unit at X of the adjunction between **H** and **Top** (Rem. 5.20) is the map $\bar{\beta}_G$.

The map $\bar{\beta}_G$ should satisfy a universal property similar to that of Thm. 5.31, see Fig. 6, nevertheless we have not checked the details. The higher dimensional version of this universal property, which concerns $\bar{\beta}_1 \times \cdots \times \bar{\beta}_n$, raises a technical issue: the product of manifolds with boundary is not a manifold with boundary; it is actually a *manifold with corners*. Roughly speaking, the latter are defined as manifold with boundary, except that $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ is replaced by \mathbb{R}_+^n – see (Lee, 2012, p. 415). One of the subtleties one has to deal with is that for $k_0, k_1 \in \{1, \dots, n\}$ with $k_0 \neq k_1$, $\mathbb{R}_+^{k_0} \times \mathbb{R}^{n-k_0}$ and $\mathbb{R}_+^{k_1} \times \mathbb{R}^{n-k_1}$ are homeomorphic but not diffeomorphic (Lee, 2012, p. 416).

Hausdorff vs Non-Hausdorff manifolds. Apart from a few marginal observations here and there, references dealing with non-Hausdorff manifolds seems to be quite rare; (Gauld, 2014, Chapter 9) and (Mardani, 2014, §2, p. 61–85) are among the few exceptions. Here are some of the reasons why non-Hausdorff manifolds are ignored: «Non-Hausdorff manifolds occasionally turn up, but it is hard to prove anything interesting about them» (Hirsch, 1976, pp. 32-33), «Most of the interesting results about manifolds do in fact require these properties.⁽⁵⁾» (Lee, 2012, p. 4), and «... we are not defining manifolds as sets supplied with an atlas [...] because [...] this [...] definition would put us under the obligation to bestow the noble title of manifold upon certain ungainly objects.» (Nestruev, 2020, §5.6, p. 57). We illustrate these claims with some basic observations which prove that standard facts of differential topology are either wrong or not relevant for non-Hausdorff manifolds:

⁽⁴⁾Every atlas in the sense of Def. A.4 is an atlas with boundary ... whose boundary is empty!

⁽⁵⁾The quotation refers to the Hausdorff and second-countability properties.

1) There can be more than one integral curve passing through a given point on a non-Hausdorff manifold; compare with (Lang, 1999, Thm. 2.1, p. 90). Such a pathology occurs for every standard parallelization (Def. 3.6) whose underlying topology is non-Hausdorff; this is related to the notion of *bifurcate curves of the second kind* in physics (Hájíček, 1971a, p. 158).

2) The Whitney embedding theorem is an emblematic result of differential topology which asserts that every Hausdorff, second countable, smooth manifold \mathcal{M} can be smoothly embedded into \mathbb{R}^{2n} with n denoting the dimension of \mathcal{M} , see (Whitney, 1944, Thm. 5), or (Adachi, 1993, Thm. 2.11, p. 67), (Benedetti, 2021, Thm. 7.17, p. 146) for the compact case; most textbooks provide a smooth embedding of \mathcal{M} into \mathbb{R}^{2n+1} , see (Milnor, 1958, Cor. 1.32), (Auslander & MacKenzie, 1963, Thm. 6-3, p. 116), (Guillemin & Pollack, 1974, p. 53), (Hirsch, 1976, Thm. 2.14, p. 55), (Bröcker and Jänich, 1982, 7.10, p. 71), (Shastri, 2011, Thm. 5.7.6, p. 148), (Lee, 2012, Thm. 6.15, p. 134), (Mukherjee, 2015, Thm. 2.5.1, p. 62). Conversely, since the collection of Hausdorff (resp. second countable) spaces is stable under subspaces, every manifold \mathcal{M} that can be embedded in \mathbb{R}^N for some $N \in \mathbb{N}$ is Hausdorff and second countable. Hence the Whitney embedding theorem does not hold for non-Hausdorff manifolds.

3) Given a smooth embedding $\phi : \mathcal{M} \rightarrow \mathbb{R}^N$, the subspace $\phi(\mathcal{M})$ is actually a smooth submanifold of \mathbb{R}^N so it inherits a Riemannian metric g from the euclidean structure of \mathbb{R}^n ; the Riemannian metric g is transferred to \mathcal{M} via the embedding ϕ (Lafontaine, 2015, p. 223). One can also build a Riemannian metric on \mathcal{M} from a partition of unity (Bishop & Crittenden, 1964, Thm. 2, p. 126); indeed, a connected manifold \mathcal{M} admits a partition of unity if, and only if, it is Hausdorff and second countable (Brickell & Clark, 1970, Prop. 3.4.4, p. 51).⁽⁶⁾ Anyway, the topology of a Hausdorff second countable smooth manifold \mathcal{M} is induced by any Riemannian metric g on \mathcal{M} (Bishop & Crittenden, 1964, Thm. 1, p. 125). These considerations are often summarized by a concise statement like «every smooth manifold admits a Riemannian metric» (do Carmo, 1992, Prop. 2.10, p. 43), or (Lee, 2018, Prop. 2.4, p. 11), being assumed that the manifolds under consideration are Hausdorff and second countable. A non-Hausdorff manifold is not metrizable, nevertheless there may be a Riemannian metric on it. For example, every standard atlas (Def. 3.3) can be equipped with a Riemannian metric for which the standard parallelization (Def. 3.6) is orthonormal. As in Def. B.14, the distance between two points p, q is the infimum of the lengths of the (smooth) paths from p to q , we denote it by $d(p, q)$. Then we may have $p \neq q$ and $d(p, q) = 0$ (Ex. 4.18) so the corresponding topology is not T_0 , while the topology of the atlas is T_1 (Lem. A.9). This suggests that the standard tools of differential geometry are of little help as soon as we have to deal with the topology of non-Hausdorff manifolds.

4) The connected subsets of the circle are the only 1-dimensional connected metrizable manifolds with boundary, see (Milnor, 1965, Appendix) or (Gale (1987)), while the collection of 1-dimensional standard atlases is infinite (up to diffeomorphism). To see this we define $N(G, a, b)$ as the number of vertices of G with a ingoing arrows and b outgoing arrows, for every finite graph G and every $(a, b) \in \mathbb{N} \times \mathbb{N}$ such that $ab \neq 0$ and $a + b \neq 2$. If $N(G, a, b) \neq N(G', a, b)$ then the standard atlases \mathcal{A}_G and $\mathcal{A}_{G'}$ (Def. 3.3) are not diffeomorphic. The classification of standard atlases remains rather tractable because it only depends on a rather simple classification of graphs (which we do not provide here); however «a reasonable classification of non-Hausdorff manifolds seems infeasible even in dimension 1» (Gauld, 2014, p. 153).⁽⁷⁾ This claim is to be compared with the classification theorems for metrizable manifolds in dimension 2 – see (Hirsch, 1976, Thm. 3.5, p. 204), (Gauld, 1982, Chapter 14), or (Massey, 1991, Thm. 5.1, p. 9)), in dimension 3 – see (Hempel, 1976, Thm. 3.21, p. 35) or (Jaco, 1980, II.4, p. 20), and in dimension $n \geq 5$ – see (Kreck

⁽⁶⁾Take care that in (Brickell & Clark, 1970, p. 51), ‘a paracompact manifold’ is defined as ‘a manifold whose connected components are Hausdorff and second countable’.

⁽⁷⁾Non-Hausdorff 1-manifolds are related to foliations of the plane (Gauld, 2014, Thm. 9.14, p. 165).

(1999)).⁽⁸⁾ The branching topology of a manifold (which will be defined below), gives a concrete insight of how complicated a non-Hausdorff manifold might be.

5) Every point p of a manifold \mathcal{M} is contained in an open Hausdorff submanifold V of \mathcal{M} : it suffices to consider a chart of \mathcal{M} whose domain contains p . One can even suppose that V is maximal with respect to inclusion (Hájíček, 1971b, Thm. 1). The collection of finite intersections of maximal open Hausdorff submanifolds of \mathcal{M} is a \cap -semilattice (i.e. the greatest lower bound of two elements is their intersection) which we call the *branching semilattice* of \mathcal{M} ; the latter is in particular a base of a topology (on the underlying set of \mathcal{M}) which we call the *branching topology* of \mathcal{M} . By definition, the branching topology of every Hausdorff manifold is the coarse one. The branching semilattice of a standard atlas \mathcal{A}_G satisfies the descending chain condition (i.e. there is no infinite sequence $X_1 \supseteq X_2 \supseteq \dots \supseteq X_n \supseteq \dots$ of elements of the branching semilattice). Hence for every point p of \mathcal{A}_G , there is a smallest element of the branching semilattice containing p ; equivalently, the branching topology is Alexandroff-discrete (Arenas (1999)). To see this, let $\|G\|_s$ be the set of singular traversals of G (Def. 2.14). Then $\|G\| \setminus \|G\|_s$ is a subspace of (the underlying topological space of) \mathcal{A}_G (Def. A.8). The connected component of $p \in \|G\| \setminus \|G\|_s$ is the smallest neighborhood of p in the branching topology of \mathcal{A}_G . For every arrow a we denote by C_a the connected component of $\|G\| \setminus \|G\|_s$ containing $\{a\} \times]0, \ell(a)[$. Let (a, b) be a traversal of G . If (a, b) is regular then $C_a = C_b$ and it is the smallest neighbourhood of (a, b) in the branching topology of \mathcal{A}_G . If (a, b) is singular, then $C_a \cup \{(a, b)\} \cup C_b$ is the smallest neighbourhood of (a, b) in the branching topology of \mathcal{A}_G .

We now provide an example of a manifold whose branching semilattice does not satisfy the descending chain condition; it is a special instance of (Mardani, 2014, Ex. 2.4.2, p. 66). The underlying set is $\{0, 1\} \times \mathbb{R}$. For every $x \in \mathbb{R}$ consider the chart ϕ_x defined on the set $\{(\varepsilon, y) \in \{0, 1\} \times \mathbb{R} \mid \varepsilon = 1 \Leftrightarrow y = x\}$ by $\phi_x(\varepsilon, y) = y$. Observe that for all $x, x' \in \mathbb{R}$ with $x \neq x'$, we have $\text{dom}(\phi_x) \cap \text{dom}(\phi_{x'}) = \{0\} \times (\mathbb{R} \setminus \{x, x'\})$, which we denote by $D_{x,x'}$, and the transition map $\phi_{x'} \circ \phi_x^{-1}$ is the identity on $D_{x,x'}$. Hence the collection $\{\phi_x \mid x \in \mathbb{R}\}$ is an atlas whose corresponding manifold is denoted by \mathcal{M} . Every element of the branching semilattice is of the form $(\{0\} \times (\mathbb{R} \setminus F_0)) \cup (\{1\} \times F_1)$ with $F_1 \subseteq F_0$ finite subsets of \mathbb{R} ; it is a maximal element when $F_1 = F_0$. In particular, the branching topology of \mathcal{M} is strictly coarser than its manifold topology (e.g. the subset $\{0\} \times]0, 1[$ is open in the latter, but not in the former), and the branching semilattice of \mathcal{M} does not satisfy the descending chain condition. Moreover, given two distinct points p and q of \mathcal{M} , there are \mathcal{H}_1 and \mathcal{H}_2 , two maximal Hausdorff submanifolds of \mathcal{M} , such that $\{p, q\} \subseteq (\mathcal{H}_1 \setminus \mathcal{H}_2) \cup (\mathcal{H}_2 \setminus \mathcal{H}_1)$; in other words every point p of \mathcal{M} is characterized by the family of maximal Hausdorff submanifolds \mathcal{H} of \mathcal{M} such that $p \in \mathcal{H}$. As a consequence, the branching topology of \mathcal{M} is T_1 . See (Kent et. al. (2009)) and (Mardani, 2014, §2, p. 61–84) for more topological facts about non-Hausdorff manifolds.

6) The ‘cut and glue’ operation on manifolds is of crucial importance in standard differential topology: «A favorite method of studying smooth manifolds consists in observing how they are put together from smaller pieces. The pieces are not, however, smooth manifolds, they are manifolds with boundary.» (Kosinski, 1993, p.2). The underlying idea is quite simple: given a submanifold B of a manifold M and a manifold C such that $\partial B \cong \partial C \neq \emptyset$ (i.e. the boundaries of B and C are nonempty and diffeomorphic; see Def. 7.1), one obtains a new manifold M' by ‘replacing’ B by C in M (for technical reasons, M' may also depend on the choice of the diffeomorphism $\phi : \partial B \cong \partial C$, we can safely ignore this problem here). Informally speaking, we cut B and glue C . A crucial property of this construction is that if M and C are Hausdorff (resp. metrizable), then so is M' , see (Hirsch, 1976, §2, p. 184), (Kosinski, 1993, VI.4, VI.5, pp. 99–103), or (Wall, 2016, §2.7, p. 63) for technical details. A drastically more general construction is possible in the non-Hausdorff framework: instead of ‘replacing B by C ’ to obtain M' , one can ‘add C without removing B ’ to obtain M'' in a way that we have $b \sim \phi(b)$ for every $b \in \partial B$ with \sim as in Rem. 5.20. The boundary

⁽⁸⁾The fourth dimension is known to be like no other (Scorpan, 2005, p. vii).

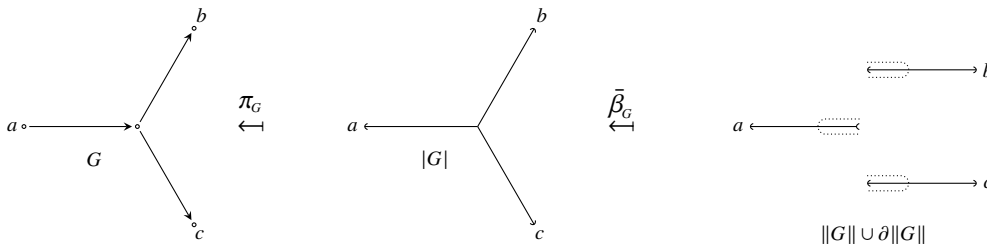


Figure 6. The standard atlas with boundary of a graph.

of B is in some sense duplicated, and if $B = C$ then the whole submanifold B is duplicated. For example, by duplicating the submanifold $]1, 2[$ of the manifold $]0, 2[$, i.e. $B = C =]1, 2[$ and $M =]0, 2[$, we obtain a manifold M'' which is diffeomorphic to the one shown on Fig. 3, see also (Baillif & Gabard, 2008, Fig. 1, p. 1106). In particular we have a canonical embedding of M (resp. M') into M'' which induces an embedding of the branching topology of M (resp. M') into that of M'' (in general, these embeddings are not homeomorphisms).

7) The duplication of submanifolds has further drastic consequences. An atlas (resp. a manifold) is said to be *closed* when it is compact Hausdorff without boundary. A *cobordism* between the closed atlases \mathcal{A}_0 and \mathcal{A}_1 is a compact Hausdorff atlas C such that $\partial C = \mathcal{A}_0 \sqcup \mathcal{A}_1$ with \sqcup denoting the disjoint union of manifolds of the same dimension, see (Hirsch, 1976, §1, pp. 169-172), (Wall, 2016, §5.1, p. 129), (Mukherjee, 2015, Def. 10.1.3, p. 301). If C satisfies all the requirements of a cobordism between \mathcal{A}_0 and \mathcal{A}_1 but the Hausdorff property, then we say that C is a *non-Hausdorff cobordism*, in which case we write $\mathcal{A}_0 \approx \mathcal{A}_1$. Given a closed atlas \mathcal{A} , the cylinder $\mathcal{A} \times [0, 1]$ is known as the *trivial cobordism* from \mathcal{A} to itself. Then $\mathcal{A} \times [\frac{1}{2}, 1]$ is a submanifold of $\mathcal{A} \times [0, 1]$ that can be duplicated to obtain a non-Hausdorff cobordism between \mathcal{A} and $\mathcal{A} \sqcup \mathcal{A}$, i.e. $\mathcal{A} \approx \mathcal{A} \sqcup \mathcal{A}$. We deduce from the equality $\partial((\mathcal{A}_0 \sqcup \mathcal{A}_1) \times [0, 1]) = \mathcal{A}_0 \sqcup \mathcal{A}_0 \sqcup \mathcal{A}_1 \sqcup \mathcal{A}_1$ that $\mathcal{A}_0 \approx \mathcal{A}_0 \sqcup \mathcal{A}_1 \sqcup \mathcal{A}_1$, and therefore $\mathcal{A}_0 \approx \mathcal{A}_0 \sqcup \mathcal{A}_1$ because $\mathcal{A}_1 \approx \mathcal{A}_1 \sqcup \mathcal{A}_1$. Similarly we prove that $\mathcal{A}_1 \approx \mathcal{A}_0 \sqcup \mathcal{A}_1$. Hence there is a non-Hausdorff cobordism between any two closed atlases of the same dimension. By contrast, the only 1-dimensional compact Hausdorff connected manifolds (possibly with boundary) are the circle S^1 ($\partial S^1 = \emptyset$), and the unit segment $[0, 1]$ ($\partial[0, 1] = \{0, 1\}$), see (Milnor, 1965, Appendix) or (Gale (1987)). Consequently, the boundary of a (possibly not connected) 1-dimensional compact Hausdorff manifold is a 0-dimensional compact Hausdorff manifold, i.e. a finite set, whose cardinal is even. It follows that there is a cobordism between two 0-dimensional closed manifolds if, and only if, their cardinals have the same parity.

In the light of the above observations, the gap between Hausdorff and non-Hausdorff manifolds seems to be an abyss. Nevertheless, the standard atlases (Def. 3.3) are so tractable that they are usable by a non-expert. Due to their connection with graphs, which are pervasive in computer science, we believe that they could be relevant to other branches of the discipline. In particular, standard atlases provide a counter-example to the claim «It is exceedingly rare to encounter a space “in nature” that would be a manifold except for the failure of one or the other of these hypotheses.»⁽⁹⁾ (Lee, 2012, p.4).

Beyond manifolds. The notion of a manifold has been generalized in many different ways, often with the aim of dealing with singularities, which can arise when more than two smooth manifolds are glued along their common boundary (Pflaum, 2001, p. 1, 2), or when quotients of (finite dimensional) manifolds are considered, for example in physics (Iglesias-Zemmour, 2013, Preface; pp. xvii-xx). We are interested in two of these generalisations, and more specifically in their relevance comparing to standard atlases (Def. 3.3 and 7.2).

⁽⁹⁾The quotation refers to the Hausdorff and the second-countability properties.

Diffeological spaces (a.k.a. *Souriau spaces*) and their morphisms form a quasitopos (Baez & Hoffnung (2011)) in which the category of smooth manifolds fully embeds; see (Stacey (2011)) for a categorical comparison with similar kind of ‘smootheologies’. This means that diffeological spaces provide a framework in which every standard construction on topological spaces is available for manifolds. We provide details and examine a concrete example related to this paper.

A *parametrization* of dimension $n \in \mathbb{N}$ in a set X is a map $p : U \rightarrow X$ where U is a *domain* of dimension n , i.e. an open subset of \mathbb{R}^n . A collection \mathcal{P} of n -parametrizations in X (with $n \in \mathbb{N}$ fixed) is said to be *glueable* when for all $p, q \in \mathcal{P}$, the maps p and q coincide on $\text{dom } p \cap \text{dom } q$. The *glueing* of such a collection \mathcal{P} is the n -parametrization g defined on $\bigcup \{\text{dom } p \mid p \in \mathcal{P}\}$ by $g(u) = p(u)$ for any $p \in \mathcal{P}$ such that $u \in \text{dom } p$; we denote the glueing of \mathcal{P} by $\cup \mathcal{P}$. A *diffeology* on a set X is a collection \mathcal{D} of parametrization in X containing all constant parametrizations, and which is closed under glueing and precomposition by smooth maps between domains (because of this last property, a diffeology contains parametrizations in every dimension). The collection of constant parametrizations in X is a diffeology on X , and so is the collection of all parametrizations in X . For every smooth manifold \mathcal{M} (with or without boundary, see Def. 7.1), the collection $\{\phi : U \rightarrow \mathcal{M} \mid U \text{ a domain and } \phi \text{ smooth}\}$ is a diffeology; we refer to it as the *standard diffeology* of \mathcal{M} . In particular, the standard diffeology of a domain U is the collection $\{\phi : V \rightarrow U \mid V \text{ domain and } \phi \text{ smooth}\}$. Moreover, the collection of diffeologies on X is closed under intersection. In particular, for every collection \mathcal{P} of parametrizations in X , the diffeology on X generated by \mathcal{P} is (by definition) the smallest one containing \mathcal{P} . This gives an insight of how flexible the notion of a diffeology is: any collection of parametrizations in \mathbb{R}^n (they may not even be continuous) induces a diffeology on \mathbb{R}^n ; in comparison, if $n \neq 4$, there is a unique manifold whose underlying topological space is \mathbb{R}^n (Scorpan, 2005, p. vii). Given a map $f : X \rightarrow X'$ we define $f_*\mathcal{D}$, the *pushforward* of \mathcal{D} by f , as the diffeology on X' generated by $f \circ \mathcal{D}$. Assuming that f is surjective, a parametrization q (in X') belongs to $f_*\mathcal{D}$ if, and only if, for every $u \in \text{dom}(q)$ there exist an open subset U of $\text{dom}(q)$ containing u , and a parametrization $p \in \mathcal{D}$ with $U \subseteq \text{dom}(p)$ such that the restriction of q to U is the composite $f \circ p$ (Iglesias-Zemmour, 2013, 1.68, p. 41); in other words $f_*\mathcal{D} = \{\cup \mathcal{P} \mid \mathcal{P} \subseteq f \circ \mathcal{D} \text{ glueable}\}$. Note that, in the preceding criterion, we can always suppose that U (resp. every domain on which an element of \mathcal{P} is defined) is connected. Given another diffeology \mathcal{D}' on X' , a map $f : X \rightarrow X'$ is said to be *smooth* from \mathcal{D} to \mathcal{D}' when $f_*\mathcal{D} \subseteq \mathcal{D}'$, which is equivalent to have $f \circ \phi \in \mathcal{D}'$ for every $\phi \in \mathcal{D}$, i.e. $f \circ \mathcal{D} \subseteq \mathcal{D}'$. Given an equivalence relation \sim on the underlying set of a diffeology \mathcal{D} , the quotient diffeology \mathcal{D}/\sim is (by definition), the pushforward $q_*\mathcal{D}$ with q the quotient map corresponding to \sim .

Given a graph G and a map $\ell : G^{(1)} \rightarrow [R, \infty[$ with $R > 0$, the following spaces are disconnected unions (i.e. coproducts in the category of topological spaces) of intervals:

$$X_G = \bigcup \{ \{a\} \times]0, \ell(a)[\mid a \in G^{(1)} \} \quad \text{and} \quad \bar{X}_G = \bigcup \{ \{a\} \times [0, \ell(a)] \mid a \in G^{(1)} \}.$$

In particular \bar{X}_G (resp. X_G) is a 1-dimensional smooth manifold with boundary (resp. without boundary), hence it induces a diffeology. Let \sim be the least equivalence relation such that $(a, \ell(a)) \sim (b, 0)$ when $\text{tgt } a = \text{src } b$, $(a, 0) \sim (b, 0)$ when $\text{src } a = \text{src } b$, and $(a, \ell(a)) \sim (b, \ell(b))$ when $\text{tgt } a = \text{tgt } b$. The underlying topological space of the quotient diffeology \bar{X}_G/\sim is $|G|$ equipped with the topology described in Def. 2.10. This topology is the largest one making the quotient map $q : \bar{X}_G \rightarrow |G|$ associated with \sim continuous; it is also the largest one making the map $\bar{\beta}_G$ (Def. 7.2) continuous. So the set $|G|$ admits two ‘natural’ diffeologies with the same underlying topology: the first one is $\mathcal{D}_1 = q_*(\bar{X}_G)$ the pushforward of the (Hausdorff) manifold with boundary \bar{X}_G by q , the second one is $\mathcal{D}_2 = (\bar{\beta}_G)_*(\bar{\mathcal{A}}_G)$ the pushforward of the (non-Hausdorff) standard atlas with boundary $\bar{\mathcal{A}}_G$ by $\bar{\beta}_G$. Let us see which one is relevant.

Given a traversal (a, b) at a vertex v of G we consider the path

$$\gamma : t \in]-\ell(a), \ell(b)[\mapsto \begin{cases} (a, t + \ell(a)) & \text{if } t < 0 \\ v & \text{if } t = 0 \\ (b, t + \ell(b)) & \text{if } t > 0 \end{cases} \in |G|.$$

We note that the map $\tilde{\gamma} :]-\ell(a), \ell(b)[\rightarrow \tilde{\mathcal{A}}_G$ defined by $\tilde{\gamma}(0) = (a, b)$ and $\tilde{\gamma}(t) = \gamma(t)$ for $t \neq 0$ is a smooth map such that $\gamma = \tilde{\beta}_G \circ \tilde{\gamma}$, hence γ is smooth according to \mathcal{D}_2 . By contrast, as we shall see, γ is not smooth according to \mathcal{D}_1 . Formally speaking, the identity map $id_{]-\ell(a), \ell(b)[}$ is an element of the standard diffeology of $]-\ell(a), \ell(b)[$, so if γ were smooth with respect to \mathcal{D}_1 , then it should belong to \mathcal{D}_1 ; we now prove that it is not the case. Let U be an open interval containing 0 and included in $]-\ell(a), \ell(b)[$, and let p be an element of the standard diffeology of the manifold \tilde{X}_G . By connectedness we have $\text{img}(p) \subseteq \{a\} \times [0, \ell(a)]$ for a certain $a \in G^{(1)}$, and therefore $\text{img}(q \circ p) \subseteq \{\text{src}(a), \text{tgt}(a)\} \cup]0, \ell(a)[$. In particular $q \circ p$ is not the restriction of γ to U , so according to the description of pushforwards, $\gamma \notin \mathcal{D}_1$. The vertices are like impassable frontiers for paths on $|G|$ that are smooth according to \mathcal{D}_1 , which is certainly not what we expected.

We now check that $\mathcal{D}_1 \subseteq \mathcal{D}_2$ (the path γ witnesses that if G has a traversal at a vertex that is not an endpoint, then this inclusion is strict). Let U be a connected domain and $\phi : U \rightarrow \tilde{X}_G$ be a smooth map. There is a (unique) arrow a of G such that $\text{img}(\phi) \subseteq \{a\} \times [0, \ell(a)]$, we denote by $\phi_a : U \rightarrow \{a\} \times [0, \ell(a)]$ the corestriction of ϕ . Given $\tau_1, \tau_2 \in \tilde{\mathcal{A}}_G$ traversals at $\text{src}(a)$ and $\text{tgt}(a)$ (which may be endpoints of G , see Def. 7.2) we have the canonical smooth map

$$\iota_a : (a, t) \in \{a\} \times [0, \ell(a)] \mapsto \begin{cases} (a, t) & \text{if } t \in]0, \ell(a)[\\ \tau_1 & \text{if } t = 0 \\ \tau_2 & \text{if } t = \ell(a) \end{cases} \in \{\tau_1, \tau_2\} \cup]0, \ell(a)[$$

with $\{\tau_1, \tau_2\} \cup]0, \ell(a)[$ submanifold of $\tilde{\mathcal{A}}_G$. The composite of the sequence of smooth maps

$$U \xrightarrow{\phi_a} \{a\} \times [0, \ell(a)] \xrightarrow{\iota_a} \{\tau_1, \tau_2\} \cup]0, \ell(a)[\hookrightarrow \tilde{\mathcal{A}}_G$$

is thus a smooth map on $\tilde{\mathcal{A}}_G$ which we denote by ψ and satisfies $\tilde{\beta}_G \circ \psi = q \circ \phi$. We have proven that the family of parametrizations $\{q \circ \phi \mid \phi \text{ smooth and } \text{dom}(\phi) \text{ connected}\}$ (which generates \mathcal{D}_1) is contained in \mathcal{D}_2 , therefore $\mathcal{D}_1 \subseteq \mathcal{D}_2$.

Hence the ‘right’ diffeology on $|G|$ is the pushforward $(\tilde{\beta}_G)_*(\tilde{\mathcal{A}}_G)$ with $\tilde{\beta}_G$ ‘the best Hausdorff approximation’ under (the topological space of) $\tilde{\mathcal{A}}_G$ (Rem. 7.3). So even if we worked with diffeologies to save Hausdorffness, we would not avoid dealing with non-Hausdorff manifolds. Moreover, in view of the importance of tangent vectors in this article, the following issue is to be taken very seriously: «There are many ways to think about tangent spaces, which are equivalent for manifolds but not when applied to diffeological spaces» (Iglesias-Zemmour, 2013, p. 161).

Diffeological spaces are easy to define, and their category extends that of smooth manifolds in a way that makes standard categorical constructions available. The (probably unavoidable) drawback is that we have no control over the outcomes of such constructions (even if they are applied to manifolds). As a consequence, there are diffeological spaces that have nothing to do with manifolds. The concept of ‘smooth stratification’ overcomes this problem; it was formalized in many different ways, giving rise, for example, to *Whitney stratified spaces* and *conically smooth stratified spaces* (Nocera & Volpe, 2023, Def. 2.5 and 2.23). Roughly speaking, a space of the first kind consists of a topological space X equipped with a partition whose elements are topological submanifolds of X equipped with a smooth structure. Although the elements of the partition may not be of the same dimension, additional requirements ensure coherence between them: as the author understands (Nocera & Volpe, 2023, Def. 2.1 and 2.5) and (Pflaum, 2001, second paragraph,

p. 6), X is a subset of an ‘ambient’ smooth manifold M such that every subpartition \mathcal{P} whose union is a topological submanifold of X inherits a smooth structure from M containing every $S \in \mathcal{P}$ as a smooth submanifold. So providing a topological space with a smooth stratification requires an embedding of it into a smooth manifold. The exact definitions are quite intricate, see in particular (Ayala et al., 2017, Def. 3.2.21), so we just provide a prototypical example related to graphs. For every $\alpha, \beta \in \mathbb{R} \setminus \{0\}$, consider the map

$$\theta_{\alpha\beta} : t \in \mathbb{R} \mapsto \begin{cases} \alpha \exp(\frac{1}{t}) & \text{if } t < 0 \\ 0 & \text{if } t = 0 \\ \beta \exp(-\frac{1}{t}) & \text{if } t > 0 \end{cases} \in \mathbb{R}.$$

The value of the n^{th} derivative of $\theta_{\alpha\beta}$ at $t < 0$ (resp. $t > 0$) is of the form $\alpha A_n(\frac{1}{t}) \exp(\frac{1}{t})$ (resp. $\beta B_n(-\frac{1}{t}) \exp(-\frac{1}{t})$) with A_n (resp. B_n) a polynomial only depending on $n \in \mathbb{N}$. According to the well-known properties of the exponential map, we have

$$\lim_{t \rightarrow 0^-} A_n(\frac{1}{t}) \exp(\frac{1}{t}) = \lim_{t \rightarrow 0^+} B_n(-\frac{1}{t}) \exp(-\frac{1}{t}) = 0.$$

It follows that the maps $\gamma_{\alpha\beta}$ defined by $\gamma_{\alpha\beta}(t) = (t, \theta_{\alpha\beta}(t))$ for every $t \in \mathbb{R}$ are smooth curves on \mathbb{R}^2 (which plays the role of the ambient manifold). Given a finite set $\mathcal{F} \subseteq (\mathbb{R} \setminus \{0\})^2$ the subset

$$X = \bigcup \{ \gamma_{\alpha\beta}([-1, 1]) \mid (\alpha, \beta) \in \mathcal{F} \} \quad (19)$$

of \mathbb{R}^2 is a Whitney stratified space (the partition being given by the sets $\{\gamma_{\alpha\beta}(t)\}$ and $\gamma_{\alpha\beta}(J)$ for $(\alpha, \beta) \in \mathcal{F}$, $t \in \{-1, 0, 1\}$, and $J \in \{-1, 0[,]0, 1[\}$): the crucial point is that all the curves $\gamma_{\alpha\beta}$, as well as their higher derivatives, coincide at $t = 0$.

Now consider the graph G whose set of vertices is $\{-n, \dots, m\}$ for $n, m \in \mathbb{N}$ fixed, and whose arrows are $(\alpha, 0)$ and $(0, \beta)$ for $\alpha \in \{-n, \dots, -1\}$ and $\beta \in \{1, \dots, m\}$ (the source and the target of (x, y) being x and y respectively). The underlying topological space $Sp(\mathcal{X}_G)$ of the standard local order on $|G|$ (Def. 5.10) is (canonically) homeomorphic to the subspace X of \mathbb{R}^2 given by (19) with $\mathcal{F} = \{-n, \dots, -1\} \times \{1, \dots, m\}$. From this example, one conjectures that for every finite graph G the space $Sp(\mathcal{X}_G)$ is homeomorphic to a Whitney stratified subspace X_G of \mathbb{R}^n (actually $n = 3$ should be enough). Nevertheless, such an embedding f is by no means canonical, and the composite $f \circ \bar{\beta}_G$, with $\bar{\beta}_G$ from Def. 7.2, is a local embedding (it is not global because a non-Hausdorff topological space cannot be embedded into a Hausdorff one). One can deduce, through $f \circ \bar{\beta}_G$, the smooth structure of X_G from that of $\bar{\mathcal{A}}_G$, and *vice versa*. Due to the technicalities inherent in dealing with Whitney stratified spaces (for example, see (Pflaum, 2001, §2.1 and §2.2) about tangent bundles and vector fields), it seems to be much easier to work with standard atlases.

In addition to the fact that standard atlases are much more tractable than the corresponding diffeological (resp. Whitney stratified) spaces, we emphasize that, as far as we know, there is no extension of the Lawson correspondence (§5.4) to these kinds of spaces.

Appendix A. Atlases

The notion of an atlas allows us to apply differential calculus beyond the class of maps between open subsets of (finite dimensional) normed spaces⁽¹⁰⁾. We recall some basic definitions without using local coordinates⁽¹¹⁾, and atlases are defined on sets without presupposing any topology (the latter can indeed be recovered from the charts).

⁽¹⁰⁾ « Differentiability is a local phenomenon, so to talk about it, we need only a space which is locally like euclidean space» / «a manifold is locally like the arena of calculus» (Gauld, 1982, p.28, 53). See also (do Carmo, 1992, p.1).

⁽¹¹⁾ This approach is advocated in (Lang, 1999, Foreword) and (Nachbin, 1981, chap.13), in accordance with the principle that local coordinates are only meant to be a tool for computations.

A.1 Differential calculus

We assume that differential calculus for smooth maps $f : U \rightarrow \mathbb{R}^m$ with U an open subset of \mathbb{R}^n is known, see (Lang, 2002, Chapter I, §2-4) or (Nachbin, 1981, §11,13,14,16). The derivative of f at $x \in U$ is a linear map $Df_x : \mathbb{R}^n \rightarrow \mathbb{R}^m$ which we identify with an $m \times n$ matrix with entries in \mathbb{R} , i.e. an element of $\text{Mat}_n^m(\mathbb{R})$. All one really needs here is the derivative of identities and the *chain rule*

$$D(id_U)_x = id_{\mathbb{R}^n} \quad \text{and} \quad D(g \circ f)_x = Dg_{f_x} \circ Df_x \quad (20)$$

for every element x of an open subset U of \mathbb{R}^n . If the map f has a smooth inverse f^{-1} with $fx = y$, then it is called a *smooth diffeomorphism*, and the above relations imply that

$$D(f^{-1})_y = (Df_x)^{-1}. \quad (21)$$

A *translation* is a map f of the form $x \in X \mapsto x + t \in Y$ with X and Y subsets of \mathbb{R}^n and $t \in \mathbb{R}^n$ fixed; if X is open then for every $x \in X$ we have $Df_x = id_{\mathbb{R}^n}$.

A.2 Atlases

A *chart* of dimension $n \in \mathbb{N}$, or *n-chart*, is a *bijection* ϕ whose codomain is an *open* subset of \mathbb{R}^n . The n -dimensional chart $\tilde{\phi}$ is a *subchart* of ϕ , which we denote by $\tilde{\phi} \subseteq \phi$, when $\tilde{\phi}$ and ϕ agree on $\text{dom } \tilde{\phi} \subseteq \text{dom } \phi$ (note that we necessarily have $\text{cod } \tilde{\phi} \subseteq \text{cod } \phi$ since $\tilde{\phi}$ is onto). The subcharts of ϕ are canonically identified with the subsets of $\text{dom } \phi$ whose images under ϕ are open, and therefore with the open subsets of $\text{cod } \phi$. The initial topology on $\text{dom } \phi$ is the only one that makes ϕ a homeomorphism. A subset $U \subseteq \text{dom } \phi$ is open in the initial topology if, and only if, its image under ϕ is open. We denote by ϕ_U the corresponding subchart of ϕ .

Definition A.1. Given the charts ϕ and ψ , a map f is said to be *smooth around* $p \in \text{dom } f$ from ϕ to ψ when there are open subsets U and V of $\text{dom } \phi$ and $\text{dom } \psi$ such that $p \in U \subseteq \text{dom } f$, $f(U) \subseteq V$, and the composite $\psi_V \circ f \circ \phi_U^{-1}$ is smooth as a map from ϕU to ψV . The sets U and V are called *witnesses of smoothness*. The map $\psi_V \circ f \circ \phi_U^{-1}$ is referred to as the *representation* of f in the charts ϕ_U and ψ_V .

Charts are meant to allow differential calculus for mappings whose (co)domains may not be subsets of \mathbb{R}^n . Indeed, with the denotation from Def. A.1, it is natural to think of the linear map $D(\psi \circ f \circ \phi^{-1})_{\phi p}$ as the representation of ‘the’ derivative of f at p (Def. A.14). However, such a representation essentially depends on the charts ϕ and ψ . The concept of an atlas arises from the need to regulate this dependency:

Definition A.2. The n -charts ϕ and ψ are said to be *compatible*, which we denote by $\phi \sim \psi$, when:

- i) the intersection $U = \text{dom } \phi \cap \text{dom } \psi$ is open in both $\text{dom } \phi$ and $\text{dom } \psi$, and
- ii) the composites $\phi_U \circ \psi_U^{-1}$ and $\psi_U \circ \phi_U^{-1}$, which we call *transition maps*, are smooth.

Following the common usage, we ‘forget’ the subscript U and just write $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ instead of $\phi_U \circ \psi_U^{-1}$ and $\psi_U \circ \phi_U^{-1}$.

Remark A.3. With the notation from Def. A.2, for every $X \subseteq U$, if $\phi_U X$ is open in \mathbb{R}^n , then so is $\psi_U X$ because we have $\psi_U X = (\psi_U \circ \phi_U^{-1})(\phi_U X)$ and $\psi_U \circ \phi_U^{-1}$ is a diffeomorphism, so the *inverse mapping theorem* applies (Lang, 2002, Thm. 5.2, p. 13). Consequently, the charts ϕ_U and ψ_U induce the same topology on U , so it suffices to require that U is open in $\text{dom } \phi$ (resp. $\text{dom } \psi$) in the first point of Def. A.2.

We follow (Hicks, 1965, p.2), (Brickell & Clark, 1970, p. 15), (Ellis & Hawking, 1973, p. 11-12), (Marsden et al., 1988, p.142), (Lang, 1999, p.22), and (Nestruev, 2020, 5.5, p. 56):

Definition A.4. An *atlas* is a collection \mathcal{A} of pairwise compatible charts, its *underlying set* $|\mathcal{A}|$ is the union $\bigcup \{ \text{dom } \phi \mid \phi \in \mathcal{A} \}$. All the charts of \mathcal{A} have the same dimension (Def. A.2) which is the *dimension* of \mathcal{A} .

Definition A.5. Given the atlases \mathcal{A} and \mathcal{B} , a map $f : |\mathcal{A}| \rightarrow |\mathcal{B}|$ is said to be *smooth around* $p \in |\mathcal{A}|$ when there exist two charts $\phi \in \mathcal{A}$ and $\psi \in \mathcal{B}$ such that f is smooth around p from ϕ to ψ (Def. A.1). It is said to be *smooth* when it is so around every point of \mathcal{A} . Smooth maps are the morphisms of the category of atlases, which we denote by **Atl**. An isomorphism of **Atl** is called a *smooth diffeomorphism*. The cartesian product of \mathcal{A} and \mathcal{B} in **Atl** is the atlas

$$\mathcal{A} \times \mathcal{B} = \{ \phi \times \psi \mid \phi \in \mathcal{A}; \psi \in \mathcal{B} \}.$$

The *transition map* between the charts $\phi \times \psi$ and $\phi' \times \psi'$ of $\mathcal{A} \times \mathcal{B}$ is the product $\tau \times \tau'$ of the transition map τ between ϕ and ϕ' and the transition map τ' between ψ and ψ' .

Smoothness around p does not depend on the choice of the charts ϕ and ψ :

Lemma A.6. Suppose that $\phi_0 \sim \phi_1$ and $\psi_0 \sim \psi_1$. Let f be a map such that $\text{dom } \phi_0 \cap \text{dom } \phi_1 \subseteq \text{dom } f$ and $f(\text{dom } \phi_0 \cap \text{dom } \phi_1) \subseteq \text{dom } \psi_0 \cap \text{dom } \psi_1$. Given any $p \in \text{dom } \phi_0 \cap \text{dom } \phi_1$, the map f is smooth around p from ϕ_0 to ψ_0 if, and only if, so it is from ϕ_1 to ψ_1 . In particular we have

$$\psi_1 \circ f \circ \phi_1^{-1} = (\psi_1 \circ \psi_0^{-1}) \circ (\psi_0 \circ f \circ \phi_0^{-1}) \circ (\phi_1 \circ \phi_0^{-1})^{-1}$$

being understood that the ϕ_i 's and the ψ_i 's in the above equation are subcharts of ϕ_i and ψ_i whose domains respectively contain p and $f(p)$.

Example A.7. Any open subset U of \mathbb{R}^n is equipped with the atlas $\{id_U\}$. For any chart ϕ , the one element set $\{\phi\}$ is an atlas, and ϕ induces a smooth map from $\{\phi\}$ to $\{id_{\text{cod } \phi}\}$. A *curve* is a smooth map defined on an open interval of \mathbb{R} . A path on an atlas \mathcal{A} is said to be *smooth* when it is the restriction of a curve on \mathcal{A} . A *piecewise smooth* path on \mathcal{A} is a concatenation of smooth paths.

Two atlases with the same underlying set are said to be *equivalent* when their union is still an atlas. A *manifold* is an equivalence class of atlases.

Definition A.8. The topology of the atlas \mathcal{A} is the coarsest one in which $\text{dom}(\phi)$ (with the initial topology) is an *open* subspace for every $\phi \in \mathcal{A}$.

Thus any chart of \mathcal{A} is a homeomorphism. As in (Hicks, 1965, p.3) and (Lang, 1999, Chap.II), we *do not require* the topology of an atlas to be Hausdorff; yet we have:

Lemma A.9. (Gauld, 2014, Prop. 9.2, p.153). The topology of any atlas is T_1 .

A.3 Tangent bundles and vector fields

Tangent vectors. Let \mathcal{A} be an atlas of dimension n . Given $\phi, \psi \in \mathcal{A}$, $p \in \text{dom}(\phi)$, $q \in \text{dom}(\psi)$, and $a, b \in \mathbb{R}^n$, we write $(p, \phi, a) \sim (q, \psi, b)$ when $p = q$ and $D(\psi \circ \phi^{-1})_{\phi p} a = b$. This makes sense because the elements of \mathcal{A} are pairwise compatible. The relation \sim is reflexive, transitive, and symmetric as a consequence of (20) and (21), p. 43. We follow (Hirsch, 1976, pp.16-17):

Definition A.10. The *tangent vectors* of \mathcal{A} are the elements of the quotient

$$T\mathcal{A} = \{(p, \phi, a) \mid \phi \in \mathcal{A}, p \in \text{dom } \phi, \text{ and } a \in \mathbb{R}^n\} / \sim$$

All the tuples contained in $v \in T\mathcal{A}$ have the same first component p , we say that v is a *tangent vector at p* , or that p is the *attachment point* of v . We denote by $T_p\mathcal{A}$ the set of all such vectors.

Example A.11. Following Ex. A.7, we can suppose that the set of tangent vectors of a single element atlas $\{\phi\}$ is $\text{dom } \phi \times \mathbb{R}^n$, with n the dimension of the chart ϕ .

For every $\phi \in \mathcal{A}$, $p \in \text{dom } (\phi)$, and $v \in T_p\mathcal{A}$, there is a unique $[[v]]_p^\phi \in \mathbb{R}^n$ such that v is the equivalence class of the tuple $(p, \phi, [[v]]_p^\phi)$ (because every mapping $D(\psi \circ \phi^{-1})_{\phi p}$ with $\phi, \psi \in \mathcal{A}$ and $p \in \text{dom } \phi \cap \text{dom } \psi$, is a bijection). By definition of the relation \sim we have

$$[[v]]_p^\psi = (D(\psi \circ \phi^{-1})_{\phi p}) [[v]]_p^\phi \quad (22)$$

for every $\psi \in \mathcal{A}$ with $p \in \text{dom } \psi$.

Tangent spaces. We transfer the (topological) real vector space structure of \mathbb{R}^n to $T_p\mathcal{A}$ in the obvious way: the linear combination $u + \lambda v$ in $T_p\mathcal{A}$ is characterized by the equality

$$[[u + \lambda v]]_p^\phi = [[u]]_p^\phi + \lambda [[v]]_p^\phi.$$

It does not depend on ϕ because we have the relation (22) and the map $D(\psi \circ \phi^{-1})_{\phi p}$ is linear. Given $n \in \mathbb{N}$ there is a unique, up to isomorphism, structure of n -dimensional Hausdorff *topological vector spaces* (tvs for short) (Schaefer & Wolff, 1999, Chap.I, 3.2, 3.4). Hence the following is sound:

Definition A.12. For any chart ϕ of \mathcal{A} with $p \in \text{dom } \phi$, the *tangent space* of \mathcal{A} at p is $T_p\mathcal{A}$ with the only topological vector space structure that makes the map $[[\cdot]]_p^\phi : T_p\mathcal{A} \rightarrow \mathbb{R}^n$ an isomorphism.

Tangent bundle. We denote by $GL_n(\mathbb{R})$ the group of $n \times n$ invertible matrices with real entries; its underlying set is open in $\text{Mat}_n(\mathbb{R})$ equipped with the standard topology.

The set $T\mathcal{A}$ (Def. A.10) is the underlying set of an atlas: for every $\phi \in \mathcal{A}$ one defines the chart

$$T\phi : \bigsqcup_{p \in \text{dom } \phi} T_p\mathcal{A} \rightarrow \text{cod } \phi \times \mathbb{R}^n \quad (23)$$

by $T\phi(v) = (\phi p, [[v]]_p^\phi)$ for every $v \in T_p\mathcal{A}$. Given $\phi, \psi \in \mathcal{A}$, and $(x, t) \in \text{cod } \phi \times \mathbb{R}^n$, we have $T\psi \circ (T\phi)^{-1}(x, t) = ((\psi \circ \phi^{-1})x, D(\psi \circ \phi^{-1})_x t)$ so the map $T\psi \circ (T\phi)^{-1} : \text{cod } \phi \times \mathbb{R}^n \rightarrow \text{cod } \psi \times \mathbb{R}^n$ is smooth because so are the maps $\psi \circ \phi^{-1} : \text{cod } \phi \rightarrow \text{cod } \psi$ and $D(\psi \circ \phi^{-1}) : \text{cod } \phi \rightarrow GL_n(\mathbb{R})$. This atlas is called the *tangent bundle* of \mathcal{A} ; it is also denoted by $T\mathcal{A}$. The construction extends to a functor $T : \mathbf{Atl} \rightarrow \mathbf{Atl}$. More precisely, if the map $f : \mathcal{A} \rightarrow \mathcal{B}$ is smooth, then so is the map $Tf : T\mathcal{A} \rightarrow T\mathcal{B}$ defined by $Tf(v) = Df_p v$ for every $v \in T_p\mathcal{A}$. Indeed, if $T\phi(v) = (x, t)$ then we have

$$(T\psi \circ Tf \circ (T\phi)^{-1})(x, t) = (\psi \circ f \circ \phi^{-1}x, D(\psi \circ f \circ \phi^{-1})_x \cdot t),$$

and $D(\psi \circ f \circ \phi^{-1})$ is smooth as a map from $\text{cod } \phi$ to $\text{Mat}_n^m(\mathbb{R})$. The tangent bundle functor preserves binary products in the sense that the bijection

$$((p, q), \phi \times \psi, (a, b)) \leftrightarrow ((p, \phi, a), (q, \psi, b))$$

induces a smooth diffeomorphism between $T(\mathcal{A} \times \mathcal{B})$ and $T\mathcal{A} \times T\mathcal{B}$.

Example A.13. Following Ex. A.11, we can suppose that the tangent bundle of a single element atlas $\{\phi\}$ is $\{T\phi\}$ with $T\phi : (p, v) \in \text{dom } (\phi) \times \mathbb{R}^n \mapsto (\phi(p), v) \in \text{cod } (\phi) \times \mathbb{R}^n$.

Derivative of a smooth map. Given a smooth map $f : \mathcal{A} \rightarrow \mathcal{B}$ (Def. A.5) and the charts $\phi \in \mathcal{A}$ and $\psi \in \mathcal{B}$ with $p \in \text{dom } \phi$ and $f(p) \in \text{dom } \psi$, neither the smoothness of f around p (Def. A.1) nor the validity of the following equality

$$\llbracket Df_p v \rrbracket_{f_p}^\psi = D(\psi \circ f \circ \phi^{-1})_{\phi p} \llbracket v \rrbracket_p^\phi \quad (24)$$

depend on ϕ and ψ : it suffices to apply i) the chain rule (§A.1(20)) to the formula from Lem. A.6, and ii) the relation (22). Hence the following definition is sound:

Definition A.14. The *derivative* of f at p is the only map $Df_p : T_p \mathcal{A} \rightarrow T_{f_p} \mathcal{B}$ satisfying the equality (24) for every $v \in T_p \mathcal{A}$. The map Df_p is linear because so are the mappings $\llbracket - \rrbracket_p^\phi$, $\llbracket - \rrbracket_{f_p}^\psi$, and $D(\psi \circ f \circ \phi^{-1})_{\phi p}$, see §A.1 and Def. A.12. The map $D(\psi \circ f \circ \phi^{-1})_{\phi p}$ is called the *representation* of Df_p in the charts ϕ and ψ , it is denoted by $\llbracket Df_p \rrbracket_\phi^\psi$. (We switch from a representation of Df_p to another by applying the chain rule (§A.1(20)) to the formula from Lem. A.6.)

Vector fields and parallelizations. The projection $\pi : T\mathcal{A} \rightarrow \mathcal{A}$ sends every tangent vector to its attachment point. It is smooth because its local representation in the charts $T\phi$ and ϕ , see (23), is the projection $\text{cod } \phi \times \mathbb{R}^n \rightarrow \text{cod } \phi$. A *vector field* over \mathcal{A} is a *section* of π , i.e. a smooth map $f : \mathcal{A} \rightarrow T\mathcal{A}$ such that $\pi \circ f = \text{id}$. A *parallelization* of \mathcal{A} is a tuple (f_1, \dots, f_n) of vector fields over \mathcal{A} such that $(f_1(p), \dots, f_n(p))$ is a vector basis of $T_p \mathcal{A}$ for every $p \in \mathcal{A}$.

Remark A.15. For every parallelization (f_1, \dots, f_n) of \mathcal{A} we have the smooth diffeomorphism

$$(p, (a_1, \dots, a_n)) \in \mathcal{A} \times \mathbb{R}^n \quad \mapsto \quad \sum_{i=1}^n a_i f_i(p) \in T\mathcal{A}.$$

Conversely, from any smooth diffeomorphism $\Pi : \mathcal{A} \times \mathbb{R}^n \rightarrow T\mathcal{A}$ one recovers a parallelization of \mathcal{A} by putting $f_i = \Pi \circ (p \mapsto (p, e_i))$ with $p \in \mathcal{A}$ and e_i denoting the i^{th} canonical vector of \mathbb{R}^n .

Example A.16. Following Ex. A.13, a parallelization of a single element n -dimensional atlas $\{\phi\}$ is given by the tuple (f_1, \dots, f_n) with $f_i(p) = (p, e_i)$ and e_i the i^{th} canonical vector of \mathbb{R}^n .

Remark A.17. If (f_1, \dots, f_n) and (g_1, \dots, g_m) are parallelizations of \mathcal{A} and \mathcal{B} , then $(f_1, \dots, f_n, g_1, \dots, g_m)$ is a parallelization of $\mathcal{A} \times \mathcal{B}$.

Appendix B. Length metrics

A continuous map $\gamma : [a, b] \rightarrow X$ with $a \leq b$ is called a *path* on X . The *opposite* of γ is the path $\bar{\gamma} : [a, b] \rightarrow X$ with $\bar{\gamma}(t) = \gamma(a + b - t)$. The *endpoints* of γ are $\gamma(a)$ and $\gamma(b)$, more precisely its *starting* and *finishing* points. A *subdivision* of $[a, b]$ is a finite increasing sequence of elements of $[a, b]$ whose first and last elements are a and b . We say that γ is an *arc* when it is one-to-one. We say that γ is a *pseudo-arc* when $\gamma^{-1}\{p\}$ is connected for every $p \in X$ (the empty set is connected). A *weak reparametrization* is a non-decreasing path onto an interval (equivalently, a non-decreasing surjection between two intervals). A *reparametrization* is an increasing path onto an interval (equivalently, an increasing surjection between two intervals).

Remark B.1. Two arcs α and β with the same image and the same starting point are equal up to reparametrization. Indeed, denote by $\tilde{\alpha}$ and $\tilde{\beta}$ the homeomorphisms induced by α and β on their common image. Putting $\theta = \tilde{\beta}^{-1} \circ \tilde{\alpha}$, we have $\alpha = \beta \circ \theta$ and the homeomorphism θ is increasing. The case where α and β are mere paths is studied in (Fahrenberg & Raussen (2007)).

Definition B.2. Given paths $\gamma : [a, b] \rightarrow X$ and $\delta : [b, c] \rightarrow X$ with $\gamma(b) = \delta(b)$, the *concatenation* $\gamma \cdot \delta$ is the path defined on $[a, c]$ by $\gamma \cdot \delta(t) = \gamma(t)$ if $t \leq b$, and by $\gamma \cdot \delta(t) = \delta(t)$ if $b \leq t$. If $\text{dom}(\delta) = [b', c]$ then we still write $\gamma \cdot \delta$ to mean $\gamma \cdot (\delta \circ \tau)$ with τ denoting the translation $t \mapsto t + b' - b$ from $[b, c - b' + b]$ to $[b', c]$.

For every path γ on a Hausdorff space X , there exists an *arc* α on X whose endpoints are those of γ , see (Engelking, 1989, 6.3.12(a), p.376). Since any subspace of a Hausdorff space is Hausdorff, we can suppose that $\text{img}(\alpha) \subseteq \text{img}(\gamma)$. The arc α can be chosen so as to satisfy extra properties, as stated in the next result. Its proof, given in Appendix C, follows (Douady & Douady, 2020, Exercise 1.4, p. 12).

Lemma B.3. For every path γ from p to q (with $p \neq q$) on a Hausdorff space X , there exists an increasing map $j : [0, 1] \rightarrow \text{dom}(\gamma)$ such that $\gamma \circ j$ is an arc from p to q , and j is continuous if, and only if, γ is an arc. Moreover we can require that j be left-continuous with $j(0) = \min(\text{dom}(\gamma))$, or that j be right-continuous with $j(1) = \max(\text{dom}(\gamma))$.

The Hausdorff property cannot be omitted: suppose that G is the graph from Ex. 2.21 and that $\|G\|$ is equipped with the topology of the standard atlas of G (Def. 3.3). Let γ be a path on $\|G\|$ from (b, t) to (c, t') . By a connectedness argument γ visits some point (a, t'') , so we can split γ as a concatenation $\gamma_1 \cdot \gamma_2$ with γ_1 ending at (a, t'') . Still by connectedness, both γ_1 and γ_2 cover the subset $\{a\} \times]t'', 1[$ of $\|G\|$, so γ is not one-to-one.

Remark B.4. A path on a Hausdorff space preserves closures. An arc on a Hausdorff space induces a homeomorphism on its image (Gauld, 1982, Thm. 6, p.32).

B.1 Metrics and pseudometrics

A *pseudometric* on a set M is a map $d : M \times M \rightarrow \mathbb{R}_+ \cup \{\infty\}$ such that $d(x, x) = 0$, $d(x, y) = d(y, x)$, and $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y , and $z \in M$. It is a *metric* when $d(x, y) = 0$ implies that $x = y$. A *(pseudo)metric space* is a set equipped with a (pseudo)metric; it comes with a topology whose open sets are unions of *open balls*, namely the sets of the form $\{p \in M \mid d(p, c) < r\}$ with $r > 0$ and $c \in M$. The *open r -neighborhood* of $X \subseteq M$ is the union of the open balls of radius r centered at some point of X .

Remark B.5. Any path γ on a pseudometric space (M, d) is *uniformly continuous*, i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|t - t'| < \delta$ implies $d(\gamma(t), \gamma(t')) < \varepsilon$ (Rudin, 1976, 4.19, p.91).

Definition B.6. The *length* of a path γ on the (pseudo)metric space (M, d) , which we denote by $L(\gamma)$, is defined as the least upper bound of the collection of sums

$$\sum_{k=1}^n d(\gamma(t_{k-1}), \gamma(t_k))$$

with $t_0 \leq \dots \leq t_n$ a subdivision of $[a, b]$. We say that γ is *rectifiable* when $L(\gamma)$ is finite.

Remark B.7 (Bridson & Haefliger (1999), Prop. 1.20(3), p. 12). Given a concatenation of paths $\gamma_1 \cdots \gamma_n$ (Def. B.2), we have $L(\gamma_1 \cdots \gamma_n) = L(\gamma_1) + \dots + L(\gamma_n)$.

Remark B.8. The length of a path is invariant under weak reparametrization: for every path $\gamma : [a, b] \rightarrow (M, d)$ and every weak reparametrization $\theta : [x, y] \rightarrow [a, b]$, both γ and $\gamma \circ \theta$ have the same length (Papadopoulos, 2013, 1.1.8). For this reason, when γ is rectifiable, it is often convenient to suppose that γ is *arclength parametrized*, i.e. for all $a \leq t \leq t' \leq b$, the length of the subpath $\gamma|_{[t, t']}$ is $t' - t$. Indeed, for any path γ there is a unique reparametrization $\theta : [0, L(\gamma)] \rightarrow \text{dom}(\gamma)$

such that $\gamma \circ \theta$ is arclength parametrized, see (Papadopoulos, 2013, §1.2) or (Bridson & Haefliger, 1999, Prop. 1.20(5), p.13).

Remark B.9. Suppose that $t \leq t'$ and $\alpha(t) = \alpha(t')$ with α an arclength parametrized pseudo-arc. We have $t' - t = L(\alpha|_{[t, t']})$ and α constant on $[t, t']$. Therefore $t = t'$ and α is an arc.

Definition B.10. Let (M, d) be a metric space. The path $\gamma : [a, b] \rightarrow M$ is a *geodesic* when $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [a, b]$. It is a *local geodesic* when for all $t \in [a, b]$ there exists $\varepsilon > 0$ such that the restriction of γ to $[a, b] \cap [t - \varepsilon, t + \varepsilon]$ is a geodesic. A subset $X \subseteq M$ is said to be *geodesically convex* when for all $p, q \in X$ there exists a geodesic from p to q and they are all contained in X .

Definition B.11. The *length (pseudo)metric* d_L associated with d is given by

$$d_L(p, q) = \inf \{L(\gamma) \mid \gamma \text{ path on } M \text{ from } p \text{ to } q\}.$$

We have $d_L(p, q) = \infty$ when there is no path from p to q .

The inequality $d \leq d_L$ is always satisfied: the (pseudo)metric d is said to be *intrinsic* when $d = d_L$. Note that if d is a metric, then so is d_L because $d_L(p, q) = 0$ implies $d(p, q) = 0$, and therefore $p = q$. The converse is also true: assume that d_L is a metric and $d(p, q) = 0$. Any map $\gamma : [a, b] \rightarrow M$ whose image is contained in $\{p, q\}$ is continuous with respect to the topology induced on M by d . Moreover we have $L(\gamma) = 0$, therefore $d_L(p, q) = 0$ and $p = q$.

Given the metric spaces (M_i, d_i) for $i \in \{1, \dots, n\}$, there is no preferred metric d on the product $M_1 \times \dots \times M_n$ among those making the projection maps continuous. The 1-Lipschitz maps are often taken as metric space morphisms, so we require that $d_i(p_i, q_i) \leq d(p, q)$ for all points $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$, and all $i \in \{1, \dots, n\}$. In the context of this article, it is also desirable for each slice

$$\sigma_i : M_i \hookrightarrow \{p_1\} \times \dots \times \{p_{i-1}\} \times M_i \times \{p_{i+1}\} \times \dots \times \{p_n\}$$

to be an isometry, i.e. $d_i(x, y) = d(\sigma_i(x), \sigma_i(y))$ for all $i \in \{1, \dots, n\}$ and $x, y \in M_i$. The most well-known example of product metric is the euclidean one, i.e. $\sqrt{d_1^2 + \dots + d_n^2}$, which is a special instance of an α -product metric:

Definition B.12. For $\alpha \in [1, \infty]$ the α -product metric is given by $(d_1^\alpha + \dots + d_n^\alpha)^{\frac{1}{\alpha}}$ with the usual convention that the preceding formula is replaced by $\max \{d_i \mid i \in \{1, \dots, n\}\}$ when $\alpha = \infty$. The α -product metric is also given by

$$(p, q) \mapsto |d_1(p_1, q_1), \dots, d_n(p_n, q_n)|_\alpha$$

with $|\cdot|_\alpha$ denoting the α -norm on \mathbb{R}^n , i.e. $|t|_\alpha = (|t_1|^\alpha + \dots + |t_n|^\alpha)^{\frac{1}{\alpha}}$ when $\alpha \neq \infty$ and $|t|_\infty = \max \{|t_i| \mid i \in \{1, \dots, n\}\}$ for every $t \in \mathbb{R}^n$.

Proposition B.13. Any (finite) α -product of intrinsic (pseudo)metrics is an intrinsic (pseudo)metric.

Proof. See the proof of (Bridson & Haefliger, 1999, 5.3(1), p.56) for the case $\alpha = 2$. One readily checks that it remains valid for any $\alpha \in [1, \infty[$ instead of 2; see also (Bridson & Haefliger, 1999, 5.5(1), p.58) for $\alpha < \infty$; the case $\alpha = \infty$ just requires to drop the exponent α and replace some occurrences of '+' by 'max'.

B.2 Length of piecewise smooth paths

For every $t \in]a, b[$ the derivative of a smooth path $\gamma : [a, b] \rightarrow \mathcal{A}$ at t is a linear map $D\gamma_t : T_t]a, b[\rightarrow T_{\gamma(t)}\mathcal{A}$. Hence $D\gamma_t$ can be identified with the tangent vector $D\gamma_t(1)$, which we denote by $\gamma'(t)$. We obtain a continuous map $\gamma' : [a, b] \rightarrow T\mathcal{A}$ setting $\gamma'(a) = c'(a)$ and $\gamma'(b) = c'(b)$ with c being any curve extending γ . Assuming that every tangent space $T_p\mathcal{A}$ of the atlas \mathcal{A} is equipped with a norm $|\cdot|_p$ in such a way that the map $t \rightarrow |\gamma'(t)|_{\gamma(t)}$ is continuous, one defines the length of γ as the sum of the norms of the speed vectors along γ : this is the infinitesimal version of Def. B.6. *Riemannian metrics* (do Carmo, 1992, 2.1) are such structures, nevertheless, they do not fit our needs (§7, *Finsler geometry*, p. 35).

Definition B.14. Given a parallelization (f_1, \dots, f_n) of the atlas \mathcal{A} , and a norm $|\cdot|_*$ on \mathbb{R}^n ; the mapping

$$v \in T_p\mathcal{A} \mapsto |(a_1, \dots, a_n)|_* \in \mathbb{R}_+ \quad (25)$$

with $v = a_1 f_1(p) + \dots + a_n f_n(p)$ is a norm on $T_p\mathcal{A}$ which we still denote by $|\cdot|_*$. The *length* of a smooth path $\gamma : [x, y] \rightarrow \mathcal{A}$ is defined as

$$\mathcal{L}(\gamma) = \int_x^y |\gamma'(t)|_* dt .$$

The length of a piecewise smooth path $\gamma_1 \cdots \gamma_k$ is $\mathcal{L}(\gamma_1) + \dots + \mathcal{L}(\gamma_k)$. The *distance* between two points p and q of \mathcal{A} is then defined as

$$d_{\mathcal{A}} = \inf \{ \mathcal{L}(\gamma) \mid \gamma : (\text{piecewise}) \text{ smooth path from } p \text{ to } q \} .$$

One can indifferently consider smooth or piecewise smooth paths in the above definition. Indeed, for every piecewise smooth path γ one has a finite set $F \subseteq \text{dom } \gamma$ such that $\gamma'(t)$ exist for all $t \notin F$. If $\theta : [a, b] \rightarrow [a, b]$ is a non-decreasing surjective smooth path whose derivatives $D^{(k)}\theta$ vanish at each point of F and at every order $k \geq 1$, then the composite $\gamma \circ \theta$ is a smooth path. By a mere change of variable we have $L(\gamma) = L(\gamma \circ \theta)$. The map θ is obtained, for example, as

$$t \mapsto a + M \int_a^t \left(1 - \sum_{x \in F} \beta_x \right)$$

with $\{\beta_x : \mathbb{R} \rightarrow [0, 1] \mid x \in F\}$ a family of *bump functions* (Hirsch, 1976, p. 41,42) such that $\beta_x(x) = 1$, $D^{(k)}\beta_x(x) = 0$ at every order $k \geq 1$, and $\beta_x \beta_y = 0$ when $x \neq y$. The constant M has to be chosen in a way that $\theta(b) = b$, in other words

$$M = \frac{b - 2a}{S} \quad \text{with} \quad S = \int_a^b \left(1 - \sum_{x \in F} \beta_x \right) .$$

B.3 Technical facts about metric graphs

We are given a graph G together with a map $\ell : G^{(1)} \rightarrow [R, \infty[$ with $R > 0$.

Steps. Each arrow a comes with the canonical map (see Def. 2.6 for $|G|$)

$$\chi_a : t \in [0, \ell(a)] \mapsto \begin{cases} \text{src } a & \text{if } t = 0 \\ \text{tgt } a & \text{if } t = \ell(a) \in |G| \\ (a, t) & \text{otherwise} \end{cases} \quad (26)$$

Remark B.15. The *finest* topology making all the maps χ_a continuous is the topology of $|G|$ (Def. 2.10). It suffices to note that if V is an open set of the latter topology containing the vertex v ,

and if a is an arrow of G such that $\text{src}(a) = v$ (resp. $\text{tgt}(a) = v$), then $\chi_a^{-1}(V)$ contains $]0, \varepsilon[$ (resp. $] \ell(a) - \varepsilon, \ell(a)[$).

Lemma B.16. If $I =]0, \ell(a)[$ or if $I \subseteq [0, \ell(a)]$ is closed, then χ_a induces a homeomorphism from I to $\chi_a(I)$. If $\text{src}(a) \neq \text{tgt}(a)$, then χ_a induces a homeomorphism on its image.

Proof. Denote by $\tilde{\chi}_a :]0, \ell(a)[\rightarrow \chi_a(]0, \ell(a)[$ the bijection induced by χ_a . The direct image map $\tilde{\chi}_a^*$ induces a bijection from the collection of open subintervals of $]0, \ell(a)[$ to the collection of segments of G (Def. 2.8) contained in $\chi_a(]0, \ell(a)[$. The former (resp. the latter) is a base of the topology of $]0, \ell(a)[$ (resp. $\chi_a(]0, \ell(a)[$) by Rem. 2.9). Let I be a proper closed subinterval of $[0, \ell(a)]$. If $I \subseteq]0, \ell(a)[$ the preceding case applies. If $0 \in I$ then $I = [0, r]$ with $r < \ell(a)$ because I is a proper subset of $[0, \ell(a)]$. So $\tilde{\chi}_a : [0, r] \rightarrow \chi_a([0, r])$, the restriction of χ_a , is a bijection. Moreover $\tilde{\chi}_a$ is continuous because so is χ_a (Rem. 2.9). We conclude that $\tilde{\chi}_a$ is a homeomorphism by Rem. B.4, which applies because $|G|$ is Hausdorff (Rem. 2.9).

Remark B.17. Given an arrow a and a traversal (a, b) of G (Def. 2.14), the standard charts ϕ_a and ϕ_{ab} (Def. 3.1) are related to the mappings χ_a and χ_b : for all $t \in \text{cod } \phi_a$ we have $\beta_G \circ \phi_a^{-1}(t) = \chi_a(t) = (a, t)$, and for all $t \in \text{cod } \phi_{ab}$ we have

$$\beta_G \circ \phi_{ab}^{-1}(t) = \begin{cases} \chi_a(\ell(a) + t) = (a, \ell(a) + t) & \text{if } t < 0 \\ \chi_a(\ell(a)) = \text{tgt } a = \text{src } b = \chi_b(0) & \text{if } t = 0 \\ \chi_b(t) = (b, t) & \text{if } t > 0 \end{cases}$$

with β_G denoting the blowup of G (Def. 2.15).

We define the standard metric and pseudometric structures on $|G|$ and $\|G\|$.

Lemma B.18. Any path γ on $|G|$ visits finitely many vertices.

Proof. We recall that $\ell \geq R$ with $R > 0$. The union V of stars of radius $\frac{R}{3}$ centered at some vertex of G (Def. 2.7) does not contain any point of the form $(a, \ell(a)/2)$ with $a \in G^{(1)}$; such a point is called a *middle point*. The set $U = |G| \setminus G^{(0)}$ is an open neighborhood of $|G| \setminus V$ that contains all the middle points but no vertex of G . The family \mathcal{C} made of U and the stars of radius $\frac{R}{3}$ centered at some vertex of G visited by γ is an open covering of the image of γ . Suppose that γ visits at least two vertices. By a connectedness argument, γ visits at least one middle point. Moreover, the stars of radius $\frac{R}{3}$ are pairwise disjoint. It follows that no *strict* subfamily of \mathcal{C} covers the image of γ which is compact, so the family \mathcal{C} is finite, and γ visits finitely many vertices of G .

Definition B.19. A *step* s is a path of the form $\chi_a \circ \theta$ with θ an arc on $[0, \ell(a)]$; it is said to be *affine* if so is θ ; *directed* or *antidirected* according to whether θ is increasing or decreasing; *standard* if θ is an inclusion $[x, y] \hookrightarrow [0, \ell(a)]$ or its opposite, in which case the step s is denoted by (a, x, y) or (a, y, x) accordingly. We define the *length* of s , denoted by $\ell(s)$, as the length of the interval $\text{img } \theta$. The *support* of s is the arrow a . The steps $\chi_a \circ \theta$ and $\chi_b \circ \theta'$ are said to be *overlapping* when $a = b$ and the interior of $\text{img } \theta \cap \text{img } \theta'$ is nonempty. In higher dimensions, a step of $|G_1| \times \cdots \times |G_n|$ is a product $s_1 \times \cdots \times s_n$ of steps s_i of $|G_i|$.

Remark B.20. If the step $s = \chi_a \circ \theta$ is written as a concatenation $s_1 \cdots s_n$ (Def. B.2), then we have (up to translation of the domains of definitions of the paths s_i) a subdivision $t_0 < \cdots < t_n$ of $\text{dom}(s)$ such that $\text{dom}(s_i) = [t_{i-1}, t_i]$ and the restriction of s to $[t_{i-1}, t_i]$ coincide with s_i for every $i \in \{1, \dots, n\}$. If $n = 1$ then $s = s_1$; otherwise each interval $[t_{i-1}, t_i]$ is proper in $[0, \ell(a)]$, so Lem. B.16 applies and $s_i = \chi_a \circ \theta|_{[t_{i-1}, t_i]}$. Since θ is an arc on $[0, \ell(a)]$, it induces an homeomorphism

on its image; in particular it is either increasing or decreasing. It follows that $\theta(t_{i-1}) \leq \theta(t_i)$ for every $i \in \{1, \dots, n\}$ or $\theta(t_{i-1}) \geq \theta(t_i)$ for every $i \in \{1, \dots, n\}$. In the first case we have

$$|\theta(t_n) - \theta(t_0)| = \theta(t_n) - \theta(t_0) = \sum_{i=1}^n \theta(t_i) - \theta(t_{i-1}) = \sum_{i=1}^n |\theta(t_i) - \theta(t_{i-1})|$$

and in the second one we have

$$|\theta(t_n) - \theta(t_0)| = \theta(t_0) - \theta(t_n) = \sum_{i=1}^n \theta(t_{i-1}) - \theta(t_i) = \sum_{i=1}^n |\theta(t_i) - \theta(t_{i-1})|.$$

We conclude that $\ell(s) = \ell(s_1) + \dots + \ell(s_n)$.

The next result justifies the notion of a step:

Lemma B.21. If the concatenations of steps $s_1 \dots s_n$ and $\tilde{s}_1 \dots \tilde{s}_m$ are equal, then $\ell(s_1) + \dots + \ell(s_n) = \ell(\tilde{s}_1) + \dots + \ell(\tilde{s}_m)$.

Proof. Let $t_0 < \dots < t_n$ and $\tilde{t}_0 < \dots < \tilde{t}_m$ with $\text{dom}(s_i) = [t_{i-1}, t_i]$ and $\text{dom}(\tilde{s}_i) = [\tilde{t}_{i-1}, \tilde{t}_i]$. Following our notion of concatenation we have $[t_0, t_n] = [\tilde{t}_0, \tilde{t}_m]$. Suppose that $t_0 < \dots < t_n$ is finer than $\tilde{t}_0 < \dots < \tilde{t}_m$ (otherwise consider a third subdivision that is finer than both of them) and conclude by Rem. B.20.

We recall from Def. 4.1 that a path γ on $|G|$ is said to be *admissible* when it can be written as a (finite) concatenation of steps (Def. B.19).

Lemma B.22. Every arc α on $|G|$ is admissible.

Proof. The set of vertices visited by α is finite (Lem. B.18) and α is one-to-one, so the set $\gamma^{-1}(G^{(0)})$ is finite: let $t_0 < \dots < t_k$ be its enumeration. Given $i \in \{1, \dots, k\}$, the map $\theta : \xi \in [t_{i-1}, t_i] \rightarrow \chi_a^{-1}(\alpha(\xi)) \in [0, \ell(a)]$ is an arc (Rem. B.4 and Lem. B.16) such that the restriction of α to $[t_{i-1}, t_i]$ is $\chi_a \circ \theta$.

Lemma B.23. If the image of an arc α is contained in the image of an admissible path γ , then $\ell(\alpha) \leq \ell(\gamma)$. Equality holds if, and only if, γ is an arc and $\text{img}(\alpha) = \text{img}(\gamma)$.

Proof. We write γ as a concatenation of steps $\gamma_1 \dots \gamma_n$ (Def. 4.1). If α is reduced to a single step, then we can write it as $\alpha_1 \dots \alpha_m$ so that for every $j \in \{1, \dots, m\}$ and every $i \in \{1, \dots, n\}$, if α_j and γ_i are overlapping (Def. B.19), then $\text{img}(\alpha_j) \subseteq \text{img}(\gamma_i)$; in particular there is some $i \in \{1, \dots, n\}$ such that $\text{img}(\alpha_j) \subseteq \text{img}(\gamma_i)$. As a consequence of Lem. B.22, the same holds for any arc α . Hence we have a map $f : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ defined by

$$f(j) = \min \{ i \in \{1, \dots, n\} \mid \text{img}(\alpha_j) \subseteq \text{img}(\gamma_i) \}$$

and we observe that

$$\ell(\alpha) = \sum_{i=1}^n \left(\sum_{j \in f^{-1}\{i\}} \ell(\alpha_j) \right). \quad (27)$$

For any $i \in \{1, \dots, n\}$ and any distinct $j, j' \in f^{-1}\{i\}$, the steps α_j and $\alpha_{j'}$ do not overlap, so we have

$$\sum_{j \in f^{-1}\{i\}} \ell(\alpha_j) \leq \ell(\gamma_i). \quad (28)$$

Since $\ell(\gamma) = \ell(\gamma_1) + \dots + \ell(\gamma_n)$ (Def. 4.1), we deduce from (27) and (28) that $\ell(\alpha) \leq \ell(\gamma)$. Under the assumption that γ is an arc such that $\text{img}(\alpha) = \text{img}(\gamma)$, the roles of α and γ can be swapped, so we have $\ell(\alpha) = \ell(\gamma)$.

If the image of γ strictly contains that of α , then some step γ_i is not covered by α and consequently, we have some index i for which the inequality (28) is strict.

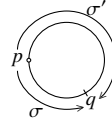
Suppose that γ is not an arc and $\text{img}(\alpha) = \text{img}(\gamma)$. We have a point $p \in \text{img}(\gamma)$ such that $\gamma^{-1}\{p\}$ is not a singleton, and (a unique) $\tau \in \text{dom}(\alpha)$ such that $\alpha(\tau) = p$. Let $j \in \{1, \dots, m\}$ be such that $\tau \in \text{dom}(\alpha_j)$ (there are at most two such indices, and they are consecutive). In particular $\text{img}(\alpha_j) \subseteq \text{img}(\gamma_k)$ with $k = f(j)$, and $t \in \text{dom}(\gamma_k)$ with $t = \min(\gamma^{-1}\{p\})$. Let $t' = \max(\gamma^{-1}\{p\})$, we have $t < t'$ and $\gamma(t') = \gamma(t)$. Denote by a the support of the step γ_k . If the image of χ_a is covered by γ , then it is also covered by the arc α , so $\text{src}(a) \neq \text{tgt}(a)$. In any case the step γ_k is an arc (Lem. B.16), so $t' \notin \text{dom}(\gamma_k)$. Let $k' \in \{1, \dots, n\}$ such that $t' \in \text{dom}(\gamma_{k'})$; one of the steps $\gamma_{k'-1}$, $\gamma_{k'}$, and $\gamma_{k'+1}$ is overlapping α_j . Up to renaming, suppose that it is $\gamma_{k'}$. We have $\text{img}(\alpha_j) \subseteq \text{img}(\gamma_{k'})$ and $k < k'$. It follows that the inequality (28) is strict for $i = k'$. In both cases ($\text{img}(\alpha) \neq \text{img}(\gamma)$ or γ not one-to-one), we have $\ell(\alpha) < \ell(\gamma)$.

Dashes. Let $p = \chi_a(s)$ and $q = \chi_a(t)$ with $\{s, t\} \not\subseteq \{0, \ell(a)\}$, which amounts to say that at least one of the points p and q is not a vertex.

i) If $\{s, t\} \subseteq]0, \ell(a)[$ or $\text{src } a \neq \text{tgt } a$, then (a, s, t) is the unique standard step from p to q (Def. B.19),

ii) If $\{s, t\} \not\subseteq]0, \ell(a)[$ and $\text{src } a = \text{tgt } a$, then we have exactly two standard steps σ and σ' from p to q , namely:

- $(a, s, 0)$ and $(a, s, \ell(a))$ if $t \in \{0, \ell(a)\}$, or
- $(a, 0, t)$ and $(a, \ell(a), t)$ if $s \in \{0, \ell(a)\}$.



In both cases the sum of their lengths is $\ell(a)$, and only one of them is directed. From the extra assumption that $\frac{\ell(a)}{2} \notin \{s, t\}$ (i.e. neither p nor q stands on the midpoint of $\{a\} \times]0, a[$) we deduce that one of the steps σ and σ' is strictly shorter than the other.

Definition B.24. Let $p, q \in \text{img}(\chi_a)$, one of them not being a vertex. The *dash* from p to q , denoted by $p \rightarrow q$, is the shortest standard step from p to q (Def. B.19) with the convention that if there are two such steps, then $p \rightarrow q$ is the directed one. When it is not the only step from p to q , the other one is called the *complement* of $p \rightarrow q$. In higher dimensions, for $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ in $|G_1| \times \dots \times |G_n|$, if the dash $p_k \rightarrow q_k$ exists for every $k \in \{1, \dots, n\}$, then we write $p \rightarrow q$ to denote the product $(p_1 \rightarrow q_1) \times \dots \times (p_n \rightarrow q_n)$.

Remark B.25. Let γ be an admissible path (Def. 4.1) from (a, t) to some vertex. By a connectedness argument, we have $\chi_a[0, t] \subseteq \text{img}(\gamma)$ or $\chi_a[t, \ell(a)] \subseteq \text{img}(\gamma)$. In particular the restrictions of χ_a to $[0, t]$ and $[t, \ell(a)]$ are arcs (Lem. B.16). It follows from Lem. B.23 that $\ell(\gamma) \geq \min\{t, \ell(a) - t\}$. If γ visits two *distinct* vertices, then by a connectedness argument we have $\text{img}(\chi_a) \subseteq \text{img}(\gamma)$ for some arrow a , so $\ell(\gamma) \geq \ell(a) \geq R$.

Remark B.26. The step $\chi_a \circ \theta$ is an arc if, and only if, $\text{src}(a) \neq \text{tgt}(a)$ or $\text{img } \theta \neq \text{dom}(\chi_a)$. It follows that every dash is an arc.

Any two neighbors (Def. 4.3) are related by a unique geodesic:

Lemma B.27. Suppose that $p = \chi_a(s)$ and $q = \chi_b(t)$ with $d_{|G|}(p, q) < \frac{R}{2}$ and $p \neq q$.

If $a = b$ and ($\text{src } a \neq \text{tgt } a$ or $|t - s| < \frac{1}{2}\ell(a)$), then $\ell(p \rightarrow q) = d_{|G|}(p, q) = |t - s|$.

If $a = b$ and $\text{src } a = \text{tgt } a$ and $|t - s| \geq \frac{1}{2}\ell(a)$, or if $a \neq b$, then there is a unique vertex v (which is an endpoint shared by a and b) such that $\ell(p \rightarrow v \rightarrow q) = d_{|G|}(p, q)$, i.e. $\ell(a) - |t - s|$ in the case where $a = b$.

Moreover, the only admissible path of length $d_{|G|}(p, q)$ from p to q is, up to reparametrization, the arc $p \rightarrow q$ or the arc $p \rightarrow v \rightarrow q$ accordingly.

Proof. The arrows a and b share at least one endpoint, and one (at least) of the points p and q is not a vertex (Rem. B.25). We have an admissible path γ from p to q with $\ell(\gamma) < R/2$ because $d_{|G|}(p, q) < R/2$ (Def. 4.3). There are three cases to examine:

Case 1. Assume that $a \neq b$.

By Rem. B.25 and because $\ell(\gamma) < R/2$ we have $\text{img}(p \rightarrow v) \subseteq \text{img}(\gamma)$ for some $v \in \{\text{src } a, \text{tgt } a\}$ (Def. B.24). Similarly we have $\text{img}(v' \rightarrow q) \subseteq \text{img}(\gamma)$ for some $v' \in \{\text{src } b, \text{tgt } b\}$. We deduce that $v = v'$ from Rem. B.25. The admissible path $\alpha = (p \rightarrow v \rightarrow q)$ is an arc because so are the dashes $p \rightarrow v$ and $v \rightarrow q$ (Rem. B.26) and $\text{img}(p \rightarrow v) \cap \text{img}(v \rightarrow q) = \{v\}$. We apply Lem. B.23 to conclude that $\ell(\alpha) \leq \ell(\gamma)$ with equality if, and only if, γ is an arc having the same image as α , in which case γ and α are equal up to reparametrization (Rem. B.1). By Def. 4.3 we have $\ell(\alpha) = d_{|G|}(p, q)$.

The two remaining cases share the assumption that $a = b$. Up to exchanging the roles of s and t we can suppose that $s \leq t$. By Rem. B.25, the image of γ either contains the set $B = \{a\} \times [s, t]$ or the set $C = \{a\} \times ([0, s] \cup [t, \ell(a)])$.

Case 2. Assume that $\text{src } a = \text{tgt } a = v$, and $t - s \geq \frac{1}{2}\ell(a)$.

The standard steps $(a, s, 0)$ and $(a, \ell(a), t)$ are $p \rightarrow v$ and $v \rightarrow q$ (Def. B.24). The path γ does not cover B , otherwise we would have $\ell(\gamma) \geq R/2$ (Lem. B.23). Therefore $\text{img}(\gamma)$ contains C which is the image of $p \rightarrow v \rightarrow q$. We have $\text{img}(p \rightarrow v) \cap \text{img}(v \rightarrow q) = \{v\}$, from which we deduce, as in the first case, that $\ell(p \rightarrow v \rightarrow q) \leq \ell(\gamma)$ with equality if, and only if, γ and $p \rightarrow v \rightarrow q$ are equal up to reparametrization.

Case 3. Assume that $\text{src } a \neq \text{tgt } a$ or $t - s < \frac{1}{2}\ell(a)$.

The dash $p \rightarrow q$ is well-defined (Def. B.24), and its length is $t - s$. If $C \subseteq \text{img}(\gamma)$ then one of the following situations occurs:

- γ visits $\text{src } a$ and $\text{tgt } a$ (which are distinct), so $\ell(\gamma) > R$ (Rem. B.25), or
- $\ell(\gamma) \geq \ell(a) - (t - s) > \frac{1}{2}\ell(a)$ (Lem. B.23).

In both situations the inequality $\ell(\gamma) < R/2$ is not satisfied, therefore $\text{img}(\gamma)$ contains B which is the trace of the arc $p \rightarrow q$ (Rem. B.26). With the same arguments as in the two first cases, we deduce that $\ell(p \rightarrow q) \leq \ell(\gamma)$ with equality if, and only if, γ and $p \rightarrow q$ are equal up to reparametrization.

Lemma B.28. For every admissible path γ on $|G|$ we have $L(\gamma) = \ell(\gamma)$, i.e. the length in the sense of Def. B.6 is equal to the length in the sense of Def. 4.1.

Proof. Given a step s , if the subdivision $t_0 < \dots < t_n$ of $\text{dom}(s)$ is fine enough, we have $\ell(s_i) < \frac{R}{2}$ with s_i denoting the restriction of s to $[t_{i-1}, t_i]$. It follows that $d_{|G|}(s(t_i), s(t_{i-1})) < \frac{R}{2}$ (Def. 4.3). From Lem. B.27 we deduce that $\ell(s_i) = d_{|G|}(s(t_i), s(t_{i-1}))$, and therefore

$$\sum_{i=1}^n d_{|G|}(s(t_i), s(t_{i-1})) = \sum_{i=1}^n \ell(s_i) = \ell(s),$$

the second equality being given by Rem. B.20. Hence $L(s) = \ell(s)$. Given a concatenation of steps $\gamma_1 \cdots \gamma_n$ we have (the last equality being given by Def. 4.1)

$$L(\gamma_1 \cdots \gamma_n) = \sum_{i=1}^n L(\gamma_i) = \sum_{i=1}^n \ell(\gamma_i) = \ell(\gamma_1 \cdots \gamma_n).$$

The length metric is approximated by arcs.

Lemma B.29. For every path γ from p to q on the metric graph $|G|$, there exists an arc α from p to q such that $\text{img}(\alpha) \subseteq \text{img}(\gamma)$, $L(\alpha) \leq L(\gamma)$, and γ is a pseudo-arc if, and only if $L(\alpha) = L(\gamma)$.

Proof. Let γ be a path on $|G|$ from p to q . If γ is not rectifiable, then consider any arc α from p to q such that $\text{img}(\alpha) \subseteq \text{img}(\gamma)$. By Lem. B.22 we know that α is admissible, hence it is rectifiable (Lem. B.28).

If γ is rectifiable, then we can suppose that it is arclength parametrized (Rem. B.8). Let $f: \text{dom}(\gamma) \rightarrow \text{dom}(\gamma)$ be an increasing map such that $\alpha = \gamma \circ f$ is an arc from p to q (Lem. B.3). We have $\text{img}(\alpha) \subseteq \text{img}(\gamma)$ and given a subdivision $\tau_0 < \cdots < \tau_n$ of $\text{dom}(\alpha) = \text{dom}(f)$, the sequence $f(\tau_0) < \cdots < f(\tau_n)$ of elements of $\text{dom}(\gamma)$ satisfies $\gamma(f(\tau_i)) = \alpha(\tau_i)$. It follows that $L(\alpha) \leq L(\gamma)$. If γ is an arc, then f is a reparametrisation, and therefore $L(\alpha) = L(\gamma)$; otherwise f is not continuous, which means here that we have $\tau \in \text{dom}(f)$ such that

$$r = \inf \{ f(\tau + \varepsilon) - f(\tau) \mid \varepsilon > 0; \tau + \varepsilon \in \text{dom}(\gamma) \} > 0$$

Let $t = f(\tau)$ and $t' = t + r$, the subpath $\gamma|_{[t, t']}$ is a loop (it has to be so because $\gamma \circ f$ is continuous), and $\text{img}(\alpha) = \text{img}(\gamma \circ f) \subseteq \text{img}(\gamma|_{[a, t]} \cdot \gamma|_{[t', b]})$ with $[a, b] = \text{dom}(\gamma)$. By applying the first part of the proof we obtain that $L(\alpha) \leq L(\gamma|_{[a, t]} \cdot \gamma|_{[t', b]})$. We have $L(\gamma) = L(\gamma|_{[a, t]} \cdot \gamma|_{[t', b]}) + L(\gamma|_{[t, t']})$ by Rem. B.7, and $L(\gamma|_{[t, t']}) > 0$ because γ is arclength parametrized. Hence $L(\alpha) < L(\gamma)$.

Geodesically convex balls. Open balls with ‘small’ radii are geodesically convex (Def. B.10):

Proposition B.30. Every open ball B of radius $r \leq \frac{R}{4}$ is geodesically convex. If the center of B is an isolated vertex then B is a singleton; otherwise the center is $\chi_c(\rho)$ for some arrow c and $\rho \in [0, \ell(c)]$, and

$$B = \begin{cases} \{c\} \times]\rho - r, \rho + r[& \text{if } r \leq \rho \leq \ell(c) - r, \\ S(\text{src } c, r - \rho) \cup \{c\} \times]0, \rho + r[& \text{if } r > \rho, \text{ and} \\ \{c\} \times]\rho - r, \ell(c)[\cup S(\text{tgt } c, \rho + r - \ell(c)) & \text{if } \rho + r > \ell(c). \end{cases} \quad (29)$$

with $S(v, x)$ denoting the star centered at the vertex v with radius x (Def. 2.7).

Proof. The case where the center of B is an isolated vertex is obvious, so we assume it is not. First we prove that the right hand part of each equality appearing in (29) is a geodesically convex set. Let p and q be two distinct points of such a set, which we denote by X in the sequel.

If X is described by the 1st case of (29), then given $p = (c, s)$ and $q = (c, t)$ with $s, t \in]\rho - r, \rho + r[$, we have $|t - s| < 2r \leq R/2 \leq \ell(c)/2$; so we are in the first case of Lem. B.27, the step $p \rightarrow q$ is the unique geodesic from p to q , and $\text{img}(p \rightarrow q) \subseteq \{c\} \times]\rho - r, \rho + r[$.

If X is described by the 2nd (resp. 3rd) case of (29), then it contains a unique vertex v , which is the center of the star appearing in this description, i.e. $v = \text{src } c$ (resp. $v = \text{tgt } c$). In particular, the connected components of $X \setminus \{v\}$ are of the form $\{x\} \times I$ with I an open interval of $]0, \ell(x)[$ of length at most $2r$ which is either initial (in which case $\text{src } x = v$) or final (in which case $\text{tgt } x = v$). Since p and q are neighbors ($d_{|G|}(p, q) < R/2$), the shortest path α from $p = \chi_a(s)$ to $q = \chi_b(t)$ is

the only admissible path of length $d_{|G|}(p, q)$ up to reparametrization (Lem. B.27). We have one of the following situations:

- There is a connected component $\{x\} \times I$ of $X \setminus \{v\}$ such that $p, q \in \{v\} \cup \{x\} \times I$ (therefore $x \in \{a, b\}$). Since the length of I is at most $2r$, the arc α is reduced to the dash $p \rightarrow q$ (Def. B.24) and its image is included in $\{v\} \cup \{x\} \times I$.
- The points p and q respectively belong to the connected components $\{a\} \times I$ and $\{b\} \times J$ of $X \setminus \{v\}$. We observe that $\text{length}(I) + \text{length}(J) \leq 2r$. If $a = b$, then $I \cap J = \emptyset$ and

$$|t - s| \geq \ell(a) - (\ell(I) + \ell(J)) \geq \ell(a) - 2r \geq \ell(a) - \frac{R}{2} \geq \frac{1}{2}\ell(a).$$

Hence no matter that $a \neq b$ or $a = b$, the arc α is $p \rightarrow v \rightarrow q$ and its image is contained in $(\{a\} \times I) \cup \{v\} \cup (\{b\} \times J)$.

By Lem. B.29 any geodesic γ from p to q is an arc up to reparametrization. Therefore α and γ are equal up to weak reparametrization, and we have in particular $\text{img}(\gamma) = \text{img}(\alpha)$. Hence X is geodesically convex.

The open ball B obviously contains X , so it remains to check the converse inclusion. Any point p of B is a neighbor of the center q of B . Let α be the shortest path from p to q , which is actually an arc (Lem. B.27), we have $L(\alpha) < r$. According to the description of α given in Lem. B.27 we have $p = \chi_a(t)$ and $q = \chi_c(\rho)$ with $t \in [0, \ell(a)]$ and $\rho \in [0, \ell(c)]$. We have three cases to deal with according to the inequalities relating r and ρ .

Suppose that $r \leq \rho \leq \ell(c) - r$. The length of any path whose image covers $\{c\} \times]0, r]$ or $\{c\} \times [\ell(c) - r, \ell(c)[$ is at least r . Hence the image of α is included in $\{c\} \times]\rho - r, \rho + r[$.

Now assume that $r > \rho$. If p and q are on the same arrow of $|G|$ (i.e. $a = c$) then α is reduced to the dash $p \rightarrow q$ whose image is $\chi_c[\min(t, \rho), \max(t, \rho)]$. This latter is included in $\{\text{src } c\} \cup (\{c\} \times]0, \rho + r[)$. If p and q are not on the same arrow of $|G|$ (i.e. $a \neq c$), then α is $p \rightarrow v \rightarrow q$ with v the only vertex of B . The length of $v \rightarrow q$ is ρ so the length of $p \rightarrow v$ is $d_{|G|}(p, q) - \rho < r - \rho$. Moreover $v = \text{src } c$ because $r < R \leq \ell(c)$. Hence $p \in S(v, r - \rho)$. The last case is dealt with the same way.

A similar result holds for closed balls with the constraint $r < \frac{R}{4}$ and the boundaries in the description (29) adapted accordingly.

Uniqueness of geodesics and geodesical convexity are not preserved under the ∞ -product metric: for example, the length of the broken line γ (on the following picture) is equal to the length of the vertical segment from p to q . In particular γ is a geodesic joining two points of the ball B which gets out of B .



Let $p, q \in |G_1| \times \cdots \times |G_n|$ such that $d^{(\infty)}(p, q) < \frac{R}{2}$. For every $k \in \{1, \dots, n\}$, the unique geodesic γ_k on $|G_k|$ from p_k to q_k is given by $p_k \rightarrow q_k$ or $p_k \rightarrow v_k \rightarrow q_k$ depending on whether we are in the first or the second case of Lem. B.27.

We recall (Def. 6.2, p. 34) that an open ball B of radius r in the metric space $(|G_1| \times \cdots \times |G_n|, d^{(\infty)})$ is said to be admissible when $r \leq \frac{R}{4}$ and for every $k \in \{1, \dots, n\}$, if the k^{th} projection of the center of B is not a vertex, then the k^{th} projection of B contains no vertex.

Proposition B.31. Denote by p the center of an admissible open ball B . For every $q \in B$ and every $k \in \{1, \dots, n\}$, the shortest path from p_k to q_k is the dash $p_k \rightarrow q_k$ (Def. B.24). The higher dimensional dash $p \rightarrow q = (p_1 \rightarrow q_1) \times \cdots \times (p_n \rightarrow q_n)$ (with affine parametrization) is of length

$d(p, q)$ and its image is contained in the union of the tiles containing p and q respectively. Moreover, if $p \leq_U q$ (resp. $q \leq_U p$) for some pospace U of the standard local order $\mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$ (Def. 5.10), then $p \rightarrow q$ (resp. $q \rightarrow p$) induces (as a map) a pospace morphism with values in U , and therefore a directed path on $\mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$.

Proof. Let $q \in B$ and $k \in \{1, \dots, n\}$. The (image of the) shortest path α from p_k to q_k (Lem. B.27) is contained in B (the distance from any point along it to p is less than that $d_{|G_k}(p_k, q_k) < r$). The arc α is either of the form $p_k \rightarrow q_k$ or $p_k \rightarrow v_k \rightarrow q_k$ for a unique vertex v_k (Lem. B.27); however, the second form is excluded by hypothesis on B (if p_k is not a vertex then B_k contains no vertex). We have $\text{img}(\gamma) \setminus \{p\} \subseteq \tau$ with τ the tile containing q (Prop. B.30). The last point is an immediate consequence of Def. 5.10 and B.24.

Appendix C. Proof of Lemma B.3

We are given a Hausdorff space X and a path $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) \neq \gamma(1)$. Suppose that $\gamma^{-1}\{p\}$ contains more than one element. Let a and b be the least and the greatest elements of $\gamma^{-1}\{p\}$. The subpath $\gamma|_{[a,b]}$ is a nontrivial loop that we can shunt by considering $\gamma|_{[0,a]} \cdot \gamma|_{[b,1]}$. In order to obtain an arc, we need to shunt all the loops; the problem is that there might be infinitely many of them, and they might be nested (e.g. space filling curves). In spite of these difficulties, we will prove that there exists an increasing map $j : [0, 1] \rightarrow [0, 1]$ such that $\gamma \circ j$ is an arc from $\gamma(0)$ to $\gamma(1)$. The map j can be defined so as to be left-continuous with $j(0) = 0$ (resp. right-continuous with $j(1) = 1$); it is continuous if, and only if, γ is already an arc. Intuitively, the map j ‘jumps’ from a to b when $\gamma(a) = \gamma(b)$. We start with some technical results.

Lemma C.1. Let W be an open subset of $]0, 1[$ whose complement in $[0, 1]$, which we denote by W^c , has no isolated points in $[0, 1]$ but a subset of $\{0, 1\}$. There exists a non-decreasing continuous map $h : [0, 1] \rightarrow [0, 1]$ that is constant on every connected component of W , and such that for all $t, t' \in [0, 1]$, if $h(t) = h(t')$ with $t \neq t'$ then t and t' belong to the closure of a (necessarily unique) connected component of W ; for short, we say that such a map h is compatible with W .

Proof. We denote by λ the Lebesgue measure on \mathbb{R} (we just need to measure the compact subsets of \mathbb{R} actually), and consider the map $f : [0, 1] \rightarrow [0, 1]$ defined by $f(t) = \lambda(W^c \cap [0, t])$. For all $t, t' \in [0, 1]$ with $t < t'$ we have

$$f(t') - f(t) = \lambda(W^c \cap [0, t']) - \lambda(W^c \cap [0, t]) = \lambda(W^c \cap]t, t']) \leq t' - t \quad (30)$$

so f is non-decreasing and continuous. We denote by \mathcal{J} the set of connected components of W . Given $J, J' \in \mathcal{J}$, we write $J \leq J'$ to mean that $J = J'$ or $t < t'$ holds for every $t \in J$ and every $t' \in J'$; (\mathcal{J}, \leq) is a total order and we have

$$J < J' \implies \sup J < \inf J' \quad (31)$$

because $\sup J$ is not isolated in W^c . We denote by \bar{J} the closure of $J \in \mathcal{J}$. Given $t, t' \in \bar{J}$ with $t < t'$ we deduce from (30) that $f(t') - f(t) \leq \lambda(\{t'\}) = 0$. We have proven that $\bar{J} \subseteq f^{-1}(\{f(t)\})$ for every $t \in \bar{J}$. We say that there is a *gap* on the left (resp. right) of $t \in [0, 1]$ when there exists $\varepsilon > 0$ such that $]t - \varepsilon, t[\cap W = \emptyset$ (resp. $]t, t + \varepsilon[\cap W = \emptyset$). Suppose that $t' < t$ with a gap on the left of t ; by taking $t'' \in [0, 1]$ such that $\max(t', t - \varepsilon) < t'' < t$, we have $f(t') \leq f(t'') < f(t)$. Dually, if $t < t'$ with a gap on right of t , then we have $f(t) < f(t')$. If \mathcal{J} is finite, then for every $J \in \mathcal{J}$ there is a gap on the left of $\inf J$, and a gap on the right of $\sup J$ (because we have (31)), and therefore $\bar{J} = f^{-1}(\{f(t)\})$ for every $t \in \bar{J}$. Consequently, the map $h = f$ satisfies the expected properties when \mathcal{J} is finite.

If \mathcal{J} is infinite, then we may have $J \in \mathcal{J}$ with $t \in \bar{J}$ and $t' \notin \bar{J}$ such that $f(t) = f(t')$, which amounts to have $\lambda(W^c \cap [\min(t, t'), \max(t, t')]) = 0$; e.g. it happens when W^c is the Cantor set

and $J =]\frac{1}{3}, \frac{2}{3}[$ (Brokate & Kersting, 2015, p. 92-95). In order to address this issue, suppose that we have a non-decreasing continuous map g that is constant on every \bar{J} with $J \in \mathcal{J}$, and such that for every $t \in [0, 1]$ with no gap on the left (resp. right), we have $t' < t \Rightarrow g(t') < g(t)$ (resp. $t' > t \Rightarrow g(t') > g(t)$). Then $h = \frac{1}{2}(f + g)$ has the expected properties. The map g is a kind of *Cantor function*⁽¹²⁾, not surprisingly it is defined in a similar way, see (Brokate & Kersting, 2015, p. 99) or (Dovgoshey et. al. (2006)) for a very detailed study.

We fix a bijection $\phi : \mathcal{J} \rightarrow \mathbb{N}$ and write $J \sqsubseteq J'$ when $\phi(J) \leq \phi(J')$. Every nonempty subset \mathcal{S} of \mathcal{J} has a least element w.r.t. \sqsubseteq which we call the *distinguished* element of \mathcal{S} . Given a finite subset \mathcal{F} of \mathcal{J} , and $F, F' \in \mathcal{F}$, we say that F is the *predecessor* of F' in \mathcal{F} when $F < F'$ and for every $X \in \mathcal{F}$ we have $X \leq F$ or $F' \leq X$. A *segment* of \mathcal{J} bounded by \mathcal{F} is a \sqsubseteq -maximal element of the collection of order-convex subsets of (\mathcal{J}, \leq) that are included in $\mathcal{J} \setminus \mathcal{F}$; every such segment is:

- the whole set \mathcal{J} (when $\mathcal{F} = \emptyset$), or
- the initial segment $\{J \in \mathcal{J} \mid J < F\}$ with $F = \min(\mathcal{F}, \leq)$, or
- the final segment $\{J \in \mathcal{J} \mid F < J\}$ with $F = \max(\mathcal{F}, \leq)$, or
- $\{J \in \mathcal{J} \mid F < J < F'\}$ with F predecessor of F' in \mathcal{F} (uniquely defined).

We define the \sqsubseteq -increasing sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ of finite subsets of \mathcal{J} as follows: $\mathcal{F}_0 = \emptyset$ and $\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{\text{distinguished elements of the segments of } \mathcal{J} \text{ bounded by } \mathcal{F}_n\}$. Given $n \in \mathbb{N}$, if $J_n \notin \mathcal{F}_n$, then it belongs to some segment (actually a single one) of \mathcal{J} bounded by \mathcal{F}_n , therefore it belongs to \mathcal{F}_{n+1} . For every increasing map $d : (\mathcal{F}, \leq) \rightarrow (]\alpha, \beta[, \leq)$, with $\alpha = \inf W$ and $\beta = \sup W$, and every \mathcal{S} segment of \mathcal{J} bounded by \mathcal{F} , we set:

$$d^-(\mathcal{S}) = \begin{cases} \alpha & \text{if } \mathcal{S} \text{ is initial,} \\ d(F) & \text{if } \mathcal{S} = \{J \in \mathcal{J} \mid F < J < F'\} \text{ or } \mathcal{S} = \{J \in \mathcal{J} \mid F < J\} \end{cases}$$

$$d^+(\mathcal{S}) = \begin{cases} \beta & \text{if } \mathcal{S} \text{ is final,} \\ d(F') & \text{if } \mathcal{S} = \{J \in \mathcal{J} \mid F < J < F'\} \text{ or } \mathcal{S} = \{J \in \mathcal{J} \mid J < F'\} \end{cases}$$

We inductively define a sequence of increasing maps $d_n : (\mathcal{F}_n, \leq) \rightarrow (]\alpha, \beta[, \leq)$ as follows: the map d_0 is the empty one, and for every $n \in \mathbb{N}$, the map $d_{n+1} : \mathcal{F}_{n+1} \rightarrow]\alpha, \beta[$ is

$$d_{n+1}(X) = \begin{cases} d_n(X) & \text{if } X \in \mathcal{F}_n, \\ \frac{1}{2}(d_n^+(\mathcal{S}) + d_n^-(\mathcal{S})) & \text{if } \mathcal{S} \text{ is the segment of } \mathcal{J} \text{ bounded by } \mathcal{F}_n \text{ that contains } X. \end{cases}$$

For every $n \in \mathbb{N}$ we have:

- i) $d_n(F') - d_n(F) \leq \frac{1}{2^n}$ when F predecessor of F' in (\mathcal{F}_n, \leq) and $\{J \in \mathcal{J} \mid F < J < F'\} \neq \emptyset$,
- ii) $d_n(F) - \alpha \leq \frac{1}{2^n}$ when $F = \min(\mathcal{F}_n, \leq)$ and $\{J \in \mathcal{J} \mid J < F\} \neq \emptyset$, and
- iii) $\beta - d_n(F) \leq \frac{1}{2^n}$ when $F = \max(\mathcal{F}_n, \leq)$ and $\{J \in \mathcal{J} \mid F < J\} \neq \emptyset$.

The preceding property is satisfied by d_0 ; assume that it is satisfied by d_n for some $n \in \mathbb{N}$. Let $F = \min(\mathcal{F}_{n+1}, \leq)$ and assume that there exists $J \in \mathcal{J}$ such that $J < F$. Then F is the distinguished element of the initial segment bounded by \mathcal{F}_n (by definition of \mathcal{F}_{n+1}) which is of the form $\{J \in \mathcal{J} \mid J < X\}$ for a unique $X \in \mathcal{F}_n$. We have $d_{n+1}(F) = \frac{1}{2}(\alpha + d_n(X))$ (by definition of d_{n+1}) and $d_n(X) - \alpha \leq \frac{1}{2^n}$ (by hypothesis), it follows that $d_{n+1}(F) - \alpha \leq \frac{1}{2^{n+1}}$. We deal with the case where $F = \max(\mathcal{F}_{n+1}, \leq)$

⁽¹²⁾The Cantor function is sometimes referred to as the *Devil's staircase*, or as any combination of the names *Lebesgue*, *Cantor*, and *Vitali* followed by either the word *function* or *staircase*.

in a similar way. Suppose that F is the predecessor of F' in $(\mathcal{F}_{n+1}, \leq)$ and that there exists $J \in \mathcal{J}$ such that $F < J < F'$. If $F \in \mathcal{F}_n$, then $F' \notin \mathcal{F}_n$ otherwise F would not be the predecessor of F' in $(\mathcal{F}_{n+1}, \leq)$. It follows that F' is the distinguished element of a segment \mathcal{S} of \mathcal{J} bounded by \mathcal{F}_n (by definition of \mathcal{F}_{n+1}), which is of the form $\{J \in \mathcal{J} \mid F < J\}$ or $\{J \in \mathcal{J} \mid F < J < X\}$ for a unique $X \in \mathcal{F}_n$. In the first case we have $d_{n+1}(F') = \frac{1}{2}(\beta + d_n(F))$ (by definition of d_{n+1}) and $\beta - d_n(F) \leq \frac{1}{2^n}$ (by hypothesis). In the second case we have $d_{n+1}(F') = \frac{1}{2}(d_n(X) + d_n(F))$ (by definition of d_{n+1}) and $d_n(X) - d_n(F) \leq \frac{1}{2^n}$ (by hypothesis). In both cases we deduce that $d_{n+1}(F') - d_{n+1}(F) \leq \frac{1}{2^{n+1}}$. If $F \notin \mathcal{F}_n$, then $F' \in \mathcal{F}_n$ otherwise F would not be the predecessor of F' in $(\mathcal{F}_{n+1}, \leq)$, the rest of the proof is similar to the case where $F \in \mathcal{F}_n$.

For every $n \in \mathbb{N}$, we define $g_n : [0, 1] \rightarrow [0, 1]$ as the only piecewise linear map such that $g_n(0) = 0$, $g_n(1) = 1$, and $g_n(t) = d_n(J)$ for every $J \in \mathcal{F}_n$ and every $t \in J$; the map g_n is well-defined by (31). The map g_n is non-decreasing because d_n is increasing. As a consequence of i), ii), and iii) we have $\max(|g_n - g_{n+k}|) \leq \frac{1}{2^n}$ for all $n, k \in \mathbb{N}$. Hence the sequence $(g_n)_{n \in \mathbb{N}}$ is uniformly convergent; its limit, which we denote by g , is continuous and non-decreasing. Every $J \in \mathcal{J}$ belongs to some \mathcal{F}_n , so the maps g_n and g_{n+k} coincide on \bar{J} for every $k \in \mathbb{N}$. It follows that the maps g_n and g also coincide on \bar{J} ; in particular g is constant on \bar{J} .

Suppose that $t = \inf J$ for some $J \in \mathcal{J}$ with no gap on the left (i.e. $W \cap]t - \varepsilon, t[\neq \emptyset$ for every $\varepsilon \in]0, t[$). If $0 \leq t' < t$ then we have $J'' \in \mathcal{J}$ such that $J'' \cap]t', t[\neq \emptyset$. We know that $\sup J'' \neq t$, so $W \cap]\sup J'', t[\neq \emptyset$. Consequently, we have $J' \in \mathcal{J}$ such that $t' < \sup J' < J' < t$. Repeating the above argument, we have $J \in \mathcal{J}$ such that $J' < J < t$. There exists $n \in \mathbb{N}$ such that $J, J' \in \mathcal{F}_n$; it follows that $g(t') \leq d_n(J) < d_n(J') \leq g(t)$. By a similar reasoning, we show that if $t = \sup J$ has no gap on the right, then $g(t) < g(t')$ for every $t' > t$.

Observe that both f and g are necessary: if $\mathcal{J} = \{J_n \mid n \in \mathbb{N} \cup \{\infty\}\}$ with $J_n < J_{n+1}$ for every $n \in \mathbb{N}$, $\sup(\bigcup_{n \in \mathbb{N}} J_n) = \frac{1}{3}$, and $J_\infty =]\frac{2}{3}, 1[$, then g is constant and f is increasing on $[\frac{1}{3}, \frac{2}{3}]$.

Lemma C.2. Given a map h as in the statement of Lem. C.1, the maps a and b defined by $a(x) = \min(h^{-1}(\{x\}))$ and $b(x) = \max(h^{-1}(\{x\}))$ are increasing and satisfy $\sup(a([0, x[)) = \sup(b([0, x[)) = a(x)$ and $\inf(a(]x, 1])) = \inf(b(]x, 1])) = b(x)$ for every $x \in [0, 1]$ (with the convention that $[0, 0[=]1, 1] = \emptyset$).

Proof. Given $x < y$ we have $h^{-1}(\{x\}) < h^{-1}(\{y\})$ because h is nondecreasing, therefore $a(x) \leq b(x) < a(y) \leq b(y)$. Let $x \in [0, 1]$. Given $\varepsilon \in]0, f(x)[$ we have $h(a(x) - \varepsilon) < x$ by definition of $a(x)$. Since h is continuous non-decreasing, we have $t' \in]a(x) - \varepsilon, a(x)[$ such that $h(a(x) - \varepsilon) < h(t') < x$. We deduce that $a(h(a(x) - \varepsilon)) < a(h(t')) < a(x)$, and from the definition of a we obtain that $a(x) - \varepsilon < a(h(t')) < a(x)$. By definition of b we have $x < h(b(x) + \varepsilon)$ for every $\varepsilon \in]0, 1 - b(x)[$. Since h is continuous non-decreasing, we have $t' \in]b(x), b(x) + \varepsilon[$ such that $x < h(t') < h(b(x) + \varepsilon)$. We deduce that $b(x) < a(h(t'))$ from the definitions of a and b ; it follows that $b(x) < a(h(t')) < a(h(b(x) + \varepsilon))$. We also have $a(h(b(x) + \varepsilon)) \leq b(x) + \varepsilon$ by definition of a , therefore $b(x) < a(h(t')) < b(x) + \varepsilon$. We have proven that $\sup(a([0, x[)) = a(x)$ and $\inf(a(]x, 1])) = b(x)$, the remaining equalities are proven dually.

The equalities $\sup(a([0, x[)) = a(x)$ and $\inf(b(]x, 1])) = b(x)$ (Lem. C.2) are referred to by writing that a is *left-continuous* and that b is *right-continuous* at x . We readily deduce from Lem. C.2 that a (resp. b) is continuous at x if, and only if, $a(x) = b(x)$, that is to say $h^{-1}(\{x\})$ is a one-element set. We conclude:

Proof of Lem. B.3. We consider the set Ω of all the open subsets of $[0, 1]$ neither containing 0 nor 1, and whose connected components $]a, b[$ satisfy $\gamma(a) = \gamma(b)$. Every such connected component corresponds to a loop we wish to shunt. In particular $]0, 1[\notin \Omega$ because $\gamma(0) \neq \gamma(1)$. The set Ω is ordered by inclusion. Let \mathcal{W} be a \subseteq -chain of elements of Ω , and let $]a, b[$ be a connected component of $\bigcup \mathcal{W}$. Let U_a and U_b open subsets of X containing $\gamma(a)$ and $\gamma(b)$ respectively.

Since γ is continuous, we have open intervals J_a and J_b containing a and b respectively, and such that $\gamma(J_a) \subseteq U_a$ and $\gamma(J_b) \subseteq U_b$. There is some connected component J of some $W \in \mathcal{W}$ intersecting both J_a and J_b . Otherwise, the collection of sets $[\sup(J_a), \inf(J_b)] \cap W^c$ with $W \in \mathcal{W}$ would be a \subseteq -chain of *nonempty* compact subsets of $[\sup(J_a), \inf(J_b)]$, so the intersection of the elements of this collection would be nonempty, and we would have $a, b \notin \bigcup \mathcal{W}$. Since $W \in \Omega$ we have $\gamma(\inf J) = \gamma(\sup J)$, so $U_a \times U_b$ intersects the diagonal of $X \times X$. We have proven that every neighborhood of $(\gamma(a), \gamma(b))$ intersects the diagonal of $X \times X$, which is closed because X is Hausdorff⁽¹³⁾, therefore $\gamma(a) = \gamma(b)$, and consequently $\bigcup \mathcal{W}$ belongs to Ω . Hence the poset (Ω, \subseteq) is inductive, so it contains a maximal element W (by Zorn's lemma, see Douady & Douady (2020), p. 12, or Goubault-Larrecq (2013), Thm. 2.4.2, p. 16). The only possible isolated points of $W^c = ([0, 1] \setminus W)$ are 0 and 1 (if $t \notin \{0, 1\}$ were isolated in W^c , then $W \cup \{t\}$ would belong to Ω , thus contradicting the fact that W is maximal).

Let h be compatible with W (in the sense of Lem. C.1) and the maps a and b be as in Lem. C.2.

Given $x \in [0, 1]$ we have $h^{-1}(\{x\}) = [a(x), b(x)]$ (by definition of a and b). If $a(x) \neq b(x)$ then $[a(x), b(x)]$ is the closure of a connected components of W (Lem. C.1), so $\gamma(a(x)) = \gamma(b(x))$ by construction of W . Suppose that $\gamma(a(x)) = \gamma(a(y))$ for some $y \in [0, 1]$. Then W contains $]a(x), a(y)[$ otherwise W would not be maximal in (Ω, \subseteq) . So there exists a connected component J of W such that $[a(x), a(y)] \subseteq \bar{J}$. According to Lem. C.1 we have $h(a(x)) = h(a(y))$, that is to say $x = y$. Let V be a neighborhood of $\gamma(a(x))$. There exists $\varepsilon > 0$ such that $\gamma(]a(x) - \varepsilon, a(x)[\cup]b(x), b(x) - \varepsilon]) \subseteq V$ because h is continuous at points $a(x)$ and $b(x)$. Take $\eta > 0$ such that $a(]x - \eta, x]) \subseteq]a(x) - \varepsilon, a(x)[$ and $b(]x, x + \eta]) \subseteq]b(x), b(x) + \varepsilon]$, which is possible because a is left-continuous and b is right-continuous (Lem. C.2). Let $y \in]x - \eta, x + \eta[$. If $y < x$ then we have $a(x) - \varepsilon < a(y) < a(x)$, so $\gamma(a(y)) \in V$. If $x < y$ then we have $b(x) < b(y) < b(x) + \varepsilon$, so $\gamma(b(y)) \in V$, therefore $\gamma(a(y)) \in V$ because $a(y) = b(y)$. Hence $\gamma \circ a$ is continuous at x .

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⁽¹³⁾The hypothesis that X is Hausdorff does not intervene anywhere else in the proof.

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