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# Singular boundary condition for a degenerated turbulent toy model

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## Abstract

We consider a turbulent toy model with the Prandtl mixing length, that vanishes at the boundary, as eddy viscosity and a Navier-like friction law as boundary condition. We address the paradox of the degeneracy of the boundary condition, by means of an approach by a problem of singular perturbations. We show a convergence theorem for well-prepared source terms, and we illustrate our analysis with a series of analytical examples, showing both blow-up cases for ill-prepared data and convergence cases for well-prepared data.

**Key words** : Fluid mechanics, Turbulence models, singular perturbations.

**2010 MSC**: 76D05, 35Q30, 76F65, 76D03, 35Q30.

## 1 Introduction

The two main characteristics of a turbulence model for the simulation of a fluid in the layer near a solid a wall are the turbulent viscosity and the boundary condition of friction at the wall; the latter is often called the wall law. In many cases, the turbulent viscosity  $\nu_{\text{turb}} \geq 0$  is proportional to a given power of the mixing length  $\varrho$ , which is, in several physical models, of the order of the distance from the wall. If we assume that molecular diffusion is negligible compared to turbulent diffusion, we are led to write as wall law:

$$(1.1) \quad \nu_{\text{turb}} \frac{\partial \mathbf{u}_\tau}{\partial \mathbf{n}} = f(\mathbf{u}_\tau), \quad \text{on } \Gamma,$$

where  $\Gamma$  denotes the boundary of the flow domain  $\Omega$  and  $\mathbf{u}_\tau$  the tangential fluid velocity at  $\Gamma$ .

Obviously written in this way, this condition doesn't make much sense because  $\nu_{\text{turb}}$  is zero over  $\Gamma$  (according to the above discussion), which would mean that  $\frac{\partial \mathbf{u}_\tau}{\partial \mathbf{n}}$  is infinite over  $\Gamma$ , unless  $f(\mathbf{u}_\tau)$  vanishes, and  $f$  satisfies appropriate conditions (see in [2]). This means that no friction occurs in some sense when  $f$  vanishes only at 0 such as  $f(\mathbf{w}) = C_D \mathbf{w}|\mathbf{w}|$ ,

and we are left with a no-slip boundary condition. This is a classical and old debate. The paradox here is that in a classical variational formulation, we will formally have a boundary term of the form

$$\int_{\Gamma} f(\mathbf{u}_{\tau}) \cdot \mathbf{v}_{\tau}$$

which is totally well-defined (for smooth enough  $f, \mathbf{u}, \mathbf{v}$ ), even if  $\mathbf{u}_{\tau} \neq 0$  at  $\Gamma$ .

Following a classical approach introduced by J.-L. Lions [6], it is natural to introduce a problem of *singular perturbations* to understand this paradox. In particular, in this present case, we are led to consider a viscosity of the form  $\nu_{\text{turb}} + \varepsilon$ , with  $\varepsilon > 0$ , which amounts to reconsidering the molecular viscosity, and to asking the question of the limit problem and solution, when  $\varepsilon \rightarrow 0$ .

In this paper, we study a scalar toy model, considering only the tangential velocity denoted by  $u$ , the friction is linear  $f(u) = u$ ,  $\nu_{\text{turb}}(x) = \varrho(x)$ , and only a diffusion term with a source term; the problem is then a sort of singular perturbed Robin problem. Therefore, the singular perturbation problem we consider is the following:

$$(1.2) \quad \begin{cases} -\nabla \cdot ((\varrho + \varepsilon)\nabla u) = f & \text{over } \Omega, \\ -\varepsilon \frac{\partial u}{\partial n} = u & \text{at } \Gamma, \end{cases}$$

where  $\Omega \subset \mathbb{R}^N$  ( $N \leq 3$ ) is a  $\mathcal{C}^2$ -bounded domain and the properties of  $\varrho$  are described by (2.1) below. We show in this paper that when the source term satisfies appropriate compatibility conditions, then it is possible to pass to the limit in this problem, in a certain sense. One main problem is the verification of the boundary condition, which will be *formally*  $u = 0$ , when  $\varepsilon = 0$ . To do so, we carefully define the space function in which we pass to the limit.

Then, we provide several analytical examples of blowing up cases for not well prepared data, as well as examples as for well prepared data, for which convergence to the homogeneous Dirichlet holds true. Note that in all these positive examples, when passing to the limit  $\varepsilon \rightarrow 0$ , we find  $u = 0$  at  $\Gamma$ , a result which we are not able to prove theoretically at present and which seems out of the classical theory of variational problems in weighted spaces.

## 2 The Limit problem

In this section we describe the limit problem and we give the main theoretical results.

### 2.1 Well prepared source term and energy balance

We first need to clarify the hypotheses on the degenerate viscosity, to define what is a well-prepared source term and how it influences the energy balance.

#### 2.1.1 Mixing length

We assume that  $\varrho : \Omega \rightarrow \mathbb{R}^+$  is a  $\mathcal{C}^2(\overline{\Omega}) \cap W^{2,\infty}(\Omega)$  function that satisfies

$$(2.1) \quad \lim_{\substack{d(x,\Gamma) \rightarrow 0 \\ x \in \Omega}} \frac{\varrho(x)}{d(x,\Gamma)} = 1, \quad \text{and} \quad \inf_{\substack{d(x,\Gamma) \geq \frac{1}{n} \\ x \in \Omega}} \varrho(x) > 0,$$

so that it behaves in the same way as the distance function from the boundary. In what follows, we set

$$(2.2) \quad \varrho_\varepsilon := \varrho + \varepsilon,$$

which will be used for the perturbed problem.

### 2.1.2 Well prepared source term

We take as a source term  $f \in \mathcal{C}^0(\overline{\Omega}) \cap L^\infty(\Omega)$  such that

$$(2.3) \quad \int_{\Omega} f \, dx = 0.$$

Let  $\mathbf{g} \in \mathcal{C}^1(\overline{\Omega})$  be the solution to the lifting problem

$$(2.4) \quad \begin{cases} \nabla \cdot \mathbf{g} = f & \text{in } \Omega, \\ \mathbf{g} \cdot \mathbf{n} = 0 & \text{on } \Gamma. \end{cases}$$

Notice that  $\mathbf{g}$  does exist and it is unique up to a constant, by the compatibility condition (2.3) (see the proof in [4]).

We also assume that there exists a constant  $C > 0$  such that for all  $\varepsilon > 0$  one has

$$(2.5) \quad \int_{\Omega} \frac{|\mathbf{g}|^2}{\varrho} \, dx + \int_{\Omega} \frac{|\mathbf{g}|^2}{\varrho_\varepsilon} \, dx \leq C.$$

The reason for this choice will be clear in what follows and note that it is a sort of request that  $g$  has to vanish fast enough when  $x$  approaches the boundary  $\Gamma$ .

### 2.1.3 Energy balance

We write now the standard variational formulation for the perturbed problem: Assume that (2.3), (2.4), and (2.5) hold. We say that  $u \in H^1(\Omega)$  is a weak solution to (1.2) if  $\forall v \in H^1(\Omega)$ , we have

$$(2.6) \quad \int_{\Omega} \varrho_\varepsilon \nabla u \cdot \nabla v + \int_{\Gamma} uv = \int_{\Omega} fv.$$

We now give the main existence theorem.

**Lemma 2.1.** *Problem (1.2) has a unique weak solution  $u \in H^1(\Omega)$ . In addition,  $u \in H^2(\Omega) \cap C^1(\overline{\Omega})$  and we have the following estimate*

$$(2.7) \quad \int_{\Omega} \varrho_\varepsilon |\nabla u|^2 + \int_{\Gamma} |u|^2 \leq \int_{\Omega} \frac{|\mathbf{g}|^2}{\varrho_\varepsilon} \leq C.$$

*Proof.* Given  $\varepsilon > 0$ , the existence and the uniqueness of a weak solution to Problem (1.2) is straightforward by Lax-Milgram Theorem. To check the regularity, we note that if  $f$  is in  $L^2(\Omega)$ , then  $u \in H^2(\Omega)$  (see in [1]) and we can write that almost everywhere  $x \in \Omega$

$$(2.8) \quad -\Delta u = \frac{1}{\varrho_\varepsilon} (\nabla \varrho_\varepsilon \cdot \nabla u + f).$$

In a first analysis, we see that when  $N = 2$ , hence  $u \in H^2(\Omega) \subseteq \mathcal{C}^{0,\alpha}(\overline{\Omega}) \cap W^{1,p}(\Omega)$ , for all  $p < \infty$ . Hence,  $\Delta u \in L^p(\Omega)$ , which implies  $u \in W^{2,p}(\Omega) \subset \mathcal{C}^1(\overline{\Omega})$ . When  $N = 3$ , we

get from the Sobolev embedding that  $u \in W^{1,6}(\Omega)$ , then  $\nabla u \in \mathcal{C}^0(\overline{\Omega})$ , and the proof then follows as for  $N = 2$ , since this implies again that  $\Delta u \in L^p(\Omega)$  for all  $p < \infty$ , due to the fact that  $\nabla \varrho_\varepsilon \in \mathcal{C}^0$ .

We now check the energy balance (2.7), which explains the choice for the assumption (2.5). Taking  $u$  as test function, and by the particular choice of the source term, we have

$$(2.9) \quad \left| \int_{\Omega} uf \right| = \left| - \int_{\Omega} \mathbf{g} \cdot \nabla u \right| \leq \left( \int_{\Omega} \frac{|\mathbf{g}|^2}{\varrho_\varepsilon} \right)^{\frac{1}{2}} \left( \int_{\Omega} \varrho_\varepsilon |\nabla u|^2 \right)^{\frac{1}{2}},$$

and estimate (2.7) follows from standard calculus.  $\square$

## 2.2 Functional space

### 2.2.1 Definitions

Beside the classical Sobolev spaces, we define in this section the functional setting, adapted to the problem we will consider, taking care to define the notion of trace that we will use in the following.

Let  $\mathcal{W}$  be the space  $\mathcal{C}^1(\overline{\Omega})$  equipped with the norm

$$(2.10) \quad \|u\|_{1,\varrho,\Gamma} = \left( \int_{\Omega} \varrho |\nabla u|^2 + \int_{\Gamma} |u|^2 \right)^{\frac{1}{2}},$$

which derives from the scalar product

$$(2.11) \quad (u, v)_{1,\varrho,\Gamma} = \int_{\Omega} \varrho \nabla u \cdot \nabla v + \int_{\Gamma} uv.$$

This space is not complete. Indeed, let  $(u_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in this space. Then the sequence of traces on  $\Gamma$ , say  $(\text{tr} u_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^2(\Gamma)$ , which is a complete space. Therefore, it converges to some  $g$  in  $L^2(\Gamma)$ . Furthermore, by (2.1) we can claim that given any  $\omega \subset\subset \Omega$ , the sequence  $(\nabla u_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^2(\omega)^N$ , which is also complete, let  $\mathbf{z}_\omega \in L^2(\omega)^N$  be its limit. From there we can construct  $\mathbf{z} \in L^2_{loc}(\Omega)^N$  such that  $\forall \omega \subset\subset \Omega$ ,  $\mathbf{z}|_\omega = \mathbf{z}_\omega$ . However nothing authorizes us to deduce that  $\mathbf{z} = \nabla u$  for a given  $u$  such that  $\sqrt{\rho} \nabla u \in L^2(\Omega)$ .

From there, let

$$\mathcal{W}_{\mathcal{F}} = \{u = (u_n)_{n \in \mathbb{N}} \in \mathcal{W}^{\mathbb{N}}, u \text{ is a Cauchy sequence}\}$$

with the equivalence relation

$$u = (u_n)_{n \in \mathbb{N}} \mathcal{R} v = (v_n)_{n \in \mathbb{N}} \quad \text{if and only if} \quad \lim_{n \rightarrow \infty} \|u_n - v_n\|_{1,\varrho,\Gamma} = 0.$$

**Remark 2.1.** Let  $u = (u_n)_{n \in \mathbb{N}}$  and  $v = (v_n)_{n \in \mathbb{N}}$  be such that  $u \mathcal{R} v$ . Then we have in  $L^2(\Gamma)$ ,  $\lim_{n \rightarrow \infty} \text{tr} u_n = \lim_{n \rightarrow \infty} \text{tr} v_n$ .

Let  $W = \mathcal{W}_{\mathcal{F}}/\mathcal{R}$  be the quotient space, equipped with the scalar product defined by, for any given  $\bar{u} \in W$  and  $\bar{v} \in W$

$$(2.12) \quad (\bar{u}, \bar{v})_W = \lim_{n \rightarrow \infty} (u_n, v_n)_{1,\varrho,\Gamma}.$$

It is easy checked that the sequence  $((u_n, v_n)_{1,\varrho,\Gamma})_{n \in \mathbb{N}}$  is a Cauchy sequence and that the limit in (2.12) does not depend on the choice of the representatives in the class of  $\bar{u}$  and  $\bar{v}$ . Furthermore,

$$(2.13) \quad \|\bar{u}\|_W = \lim_{n \rightarrow \infty} \|u_n\|_{1,\varrho,\Gamma},$$

where the limit does not depend of the representatives in the class  $\bar{u}$ . We know that  $W$  is an Hilbert space for this scalar product (see [5]). Moreover, the space  $\mathcal{W}$  is embedded in  $W$  and is everywhere dense in  $W$ .

**Definition 2.1.** Let  $\bar{u} = \overline{(u_n)_{n \in \mathbb{N}}} \in W$ . Let  $g \in L^2(\Gamma)$  be the limit in  $L^2(\Gamma)$  of the sequence  $(\text{tr}u_n)_{n \in \mathbb{N}}$ . Then  $g$  does not depend on the representatives in the class of  $\bar{u}$  (see Remark 2.1). We say that  $g$  is the generalized trace of  $\bar{u}$  over  $\Gamma$ , and we still denote  $g = \text{tr}\bar{u}$ , so far no risk of confusion occurs. We also set  $\|\text{tr}\bar{u}\|_{0,2,\Gamma} = \|g\|_{0,2,\Gamma}$ .

From now we note  $u \in W$  instead of  $\bar{u}$ . The space  $W$  is the Cauchy completion of  $\mathcal{W}$  (see also in [3]).

**Remark 2.2.** Let  $f$  such that (2.4), (2.3) and (2.5) are satisfied. Then for  $u \in \mathcal{C}^1(\bar{\Omega})$  and arguing as in (2.9), we obtain

$$\left| S_f(u) = \int_{\Omega} fu \right| \leq C \left( \int_{\Omega} \varrho |\nabla u|^2 \right)^{1/2} \leq C \|u\|_{1,\rho,\Gamma}.$$

Then as  $\mathcal{C}^1(\bar{\Omega})$  is dense in  $W$ ,  $u \rightarrow S_f(u)$  can be extended to  $W$ , so that in this sense we have  $f \in W'$  and we write the extension

$$u \in W \rightarrow_{W'} \langle f, u \rangle_W.$$

### 2.2.2 More about the structure of the space $W$

We aim at a better understanding of the structure of this space. Do do so we introduce the quotient space  $\mathcal{C}^1(\bar{\Omega})/\mathbb{R}$  equipped the quotient norm

$$\|u\|_V = \inf \left\{ \left( \int_{\Omega} \rho |\nabla v|^2 \right)^{\frac{1}{2}}, v - u = Cte \right\}$$

knowing that if  $v = u + Cte$ , then  $\int_{\Omega} \rho |\nabla v|^2 = \int_{\Omega} \rho |\nabla u|^2$ . Therefore,  $\|\bar{u}\|_V = \|\sqrt{\rho} \nabla u\|_{0,2,\Omega}$ , and this does not depend on the representative in the class  $\bar{u}$ . Then let  $V$  be the Cauchy completion of  $\mathcal{C}^1(\bar{\Omega})/\mathbb{R}$  for this norm, following the same construction as in the previous section. In particular, writting  $u = (u_n)_{n \in \mathbb{N}}$  instead of  $\bar{u} = (\bar{u}_n)_{n \in \mathbb{N}}$ , where  $(u_n)_{n \in \mathbb{N}}$  is a Cauchy sequence for the  $V$ -norm, so far no risk of confusion occurs,

$$(2.14) \quad \|u\|_V = \lim_{n \rightarrow \infty} \left( \int_{\Omega} \rho |\nabla u_n|^2 \right)^{\frac{1}{2}},$$

and for  $v = (v_n)_{n \in \mathbb{N}}$ ,

$$(2.15) \quad (u, v)_V = \lim_{n \rightarrow \infty} \int_{\Omega} \rho \nabla u_n \cdot \nabla v_n,$$

which is easily checked that both sequence involed in the r.h.s of (2.14) and (2.15) are Cauchy sequence in  $\mathbb{R}$ . Therefore,  $V$  is an Hilbert space.

As the  $V$ -norm does not depend on the representative in the quotient by the constants, and because a Cauchy sequence of  $\mathcal{C}^1(\overline{\Omega})$  for the  $W$ -norm is also a Cauchy sequence for the  $V$ -norm, we obviously have  $W \hookrightarrow V$ , and we see that

$$(2.16) \quad \forall u \in W, \quad \|u\|_V + \|\text{tru}\|_{0,2,\Gamma} = \|u\|_W,$$

where  $\text{tru} \in L^2(\Gamma)$  denotes the generalized trace given in Definition (2.1). Moreover, for  $u, v \in W$ ,

$$(2.17) \quad \forall u, v \in W, \quad (u, v)_V + (\text{tru}, \text{tr}v)_{0,2,\Gamma} = (u, v)_W,$$

which allows to define the projection of  $u \in W$  on  $V$ , which amounts to "forgetting" the trace of  $u$ , and see  $W$  as the product of a closed subspace of  $W$  by  $L^2(\Gamma)$ . Unfortunately it is a little more complicated than that. The following construction aims to try to explain this complexity.

The issue is due to the orthogonal of  $\overline{\mathcal{D}(\Omega)}$ , where  $\overline{\mathcal{D}(\Omega)}$  is the adherence of  $\mathcal{D}(\Omega)$  in  $V$ . Let  $T$  denotes this orthogonal, which may be not reduced to 0. Consider indeed the two dimensional torus  $\Omega = \{x \in \mathbb{R}^2, 0 < R_m < |x| < R_M\}$ ,

$$\rho(r) = \frac{(r - R_m)(R_M - r)}{R_M - R_m}.$$

Then it is easy checked that there is a non trivial radial solution  $u(r) \in \mathcal{C}^1(\overline{\Omega})$  to the equation  $\text{div}(\rho \nabla u) = 0$  (see section 4 for such calculations). Therefore, for all  $v \in \mathcal{D}(\Omega)$ ,

$$(u, v)_V = \int_{\Omega} \rho \nabla u \cdot \nabla v = 0,$$

hence  $u \in T$ .

We are led to introduce the quotient space  $V/T$  equipped with the quotient norm

$$\|\bar{u}\|_{V/T} = \inf\{\|v\|_V, u - v \in T\}.$$

We are able to state the following structure result.

**Lemma 2.2.** *The space  $W$  is continuously embedded in the space*

$$(2.18) \quad Z = V/T \times L^2(\Gamma).$$

*Proof.* We equip the space  $V/\text{Ker}T$  with the standard quotient norm, the space  $Z$  with the product norm. Let

$$(2.19) \quad \psi : \begin{cases} W & \rightarrow & Z \\ u & \rightarrow & (\bar{u}, \text{tru}), \end{cases}$$

From the definition of the quotient norm,  $\|\bar{u}\|_{V/T} \leq \|u\|_V$ . Therefore by (2.16)

$$\|\psi(u)\|_Z \leq \|u\|_W.$$

□

Unfortunately, we are not able to prove that  $\text{Im}(\psi)$  is closed in  $Z$ . At this stage, we are able to prove the following result.

**Proposition 2.1.** *Let  $(\bar{u}_n, h_n)_{n \in \mathbb{N}}$  be a sequence in  $\text{Im}(\psi)$  that converges to some  $(\bar{u}, h) \in Z$ . Then there exists  $v \in W$  such that  $\bar{u} = \bar{v}$ , and a function  $\tilde{h} \in L^2(\Gamma)$  such that  $\text{tr}v = h + \tilde{h}$ .*

*Proof.* Because  $(\bar{u}_n, h_n) \in \text{Im}(\psi)$ , there exists  $v_n \in W$  such that  $(\bar{u}_n, h_n) = (\bar{v}_n, \text{tr}v_n)$ . Therefore,  $u_n - v_n \in T$ , which means

$$(2.20) \quad \forall \zeta \in \mathcal{D}(\Omega), \quad (v_n, \zeta)_V = (u_n, \zeta)_V.$$

Moreover, by the convergence of  $(\bar{u}_n)_{n \in \mathbb{N}}$ , the sequence  $(v_n)_{n \in \mathbb{N}}$  is bounded in  $W$ . Therefore we can extract a subsequence, still denoted  $(v_n)_{n \in \mathbb{N}}$  that weakly converges to some  $v$  in  $W$ , which means that for all  $\varphi \in C^1(\bar{\Omega})$ , we have by (2.17)

$$(2.21) \quad (v_n, \varphi)_V + \int_{\Gamma} v_n \varphi \rightarrow (v, \varphi)_V + \int_{\Gamma} v \varphi.$$

Taking  $\zeta \in \mathcal{D}(\Omega)$  in (2.21) combined with (2.20) and the convergence of  $(\bar{u}_n)_{n \in \mathbb{N}}$  to  $\bar{u}$  in  $V$ , yields

$$(v_n, \zeta)_V = (u_n, \zeta)_V \rightarrow (v, \zeta)_V = (u, \zeta)_V.$$

Therefore,  $\bar{u} = \bar{v}$ , which is the first part of the claim.

Moreover, for all  $g \in L^2(\Gamma)$ ,

$$(2.22) \quad \int_{\Gamma} \text{tr}v_n g = \int_{\Gamma} h_n g \rightarrow \int_{\Gamma} h g$$

Let  $s > 5/2$  such that  $s - 3/2 \notin \mathbb{N}$ . As  $\Omega$  is of class  $\mathcal{C}^2$ , then  $H^{s-1/2}(\Gamma)$  is everywhere dense in  $L^2(\Gamma)$ , and  $H^s(\Omega)$  is everywhere dense in  $\mathcal{C}^1(\Omega)$  (see [7, chapter 4]). Let  $g \in H^{s-1/2}(\Gamma)$ , and let  $\varphi \in H^s(\Omega)$  be such that  $\text{tr}\varphi = g$ . Then let  $\zeta_\varepsilon \in C^\infty(\bar{\Omega})$  be a cutoff function, be such that  $\text{supp } \zeta_\varepsilon \subset \{x \in \Omega, \rho(x) \leq \varepsilon\} = C_\varepsilon$ ,  $\zeta_\varepsilon = 1$  over  $\Gamma$ , with  $|\nabla \zeta_\varepsilon| = O(1/\varepsilon)$ . From (2.21) and (2.22), we get

$$(2.23) \quad \int_{\Gamma} v g = \int_{\Gamma} h_n g + (v_n - v, \zeta_\varepsilon \varphi)_V$$

Let us focus on the term

$$F_{\varepsilon, n} = (v_n - v, \zeta_\varepsilon \varphi)_V.$$

Of course, if we would be able to prove the convergence of  $(v_n)_{n \in \mathbb{N}}$  to  $v$  in  $V$ , then the term  $F_{\varepsilon, n}$  would go to zero when  $n \rightarrow \infty$ , and from (2.23) we deduce that for all  $g \in H^{s-1/2}(\Gamma)$

$$\int_{\Gamma} h g = \int_{\Gamma} h \text{tr}v,$$

then  $h = \text{tr}v$  by the density of  $H^{s-1/2}(\Gamma)$  in  $L^2(\Gamma)$ , hence  $(\bar{u}, h) = \psi(v)$ . But things does not work that easily. Indeed from  $u_n - v_n \in T$ , we only can decompose  $v_n$  as  $v_n = u_n + r_n$ . Here so far we have no more control about  $r_n$ , we cannot take advantage from the convergence of  $(u_n)_{n \in \mathbb{N}}$  to  $u$  in  $V$ . On the other hand, we observe that because  $\zeta_\varepsilon \varphi \in \mathcal{C}^1(\bar{\Omega})$ , the support of which is in  $C_\varepsilon$ ,

$$\left( \int_{C_\varepsilon} \rho |\nabla(\zeta_\varepsilon \varphi)|^2 \right)^{1/2} = \|\zeta_\varepsilon \varphi\|_V.$$



Hence, we deduce from the Cauchy-Schwarz inequality,

$$(2.24) \quad |F_{\varepsilon,n}| \leq \|v_n - v\|_V \left( \int_{C_\varepsilon} \rho |\nabla(\zeta_\varepsilon \varphi)|^2 \right)^{1/2} \leq C \left( \int_{C_\varepsilon} \rho |\nabla(\zeta_\varepsilon \varphi)|^2 \right)^{1/2}.$$

We have

$$\int_{C_\varepsilon} \rho |\nabla(\zeta_\varepsilon \varphi)|^2 = O \left( \int_{C_\varepsilon} \rho \zeta_\varepsilon^2 |\nabla \varphi|^2 + \frac{2}{\varepsilon} \int_{C_\varepsilon} \zeta_\varepsilon \rho \varphi \nabla \varphi + \frac{1}{\varepsilon^2} \int_{C_\varepsilon} \rho \varphi^2 \right) \\ I_1 + I_2 + I_3$$

Because  $\rho \leq \varepsilon$  in  $C_\varepsilon$ , we get  $I_1 = O(\varepsilon^2)$ ,  $I_2 = O(\varepsilon)$ . The third integral is more problematic. Indeed we have

$$I_3 \leq \frac{1}{\varepsilon} \int_{C_\varepsilon} \varphi^2 \rightarrow \int_\Gamma g^2,$$

by the mean value theorem, and also because  $\text{tr} \varphi = g$  on  $\Gamma$ . Therefore,  $F_{n,\varepsilon}$  is uniformly bounded in  $n$  and  $\varepsilon$ , and at fixed  $n$ , eventually up to a subsequence,  $F_{n,\varepsilon} \rightarrow \mu_n(g)$  as  $\varepsilon$  goes to 0, and  $\mu_n$  satisfies  $|\mu_n(g)| \leq C \|g\|_{L^2(\Gamma)}$ . Then, up to a subsequence, when  $n \rightarrow \infty$ , we deduce from (2.22) and (2.23) that there exists a continuous linear operator on  $L^2(\Gamma)$  such that for all  $g \in L^2(\Gamma)$ ,

$$\int_\Gamma \text{tr} v g = \int_\Gamma h g + \mu(g).$$

But as  $\mu$  is linear continuous on  $L^2(\Gamma)$ , there exists  $\tilde{h} \in L^2(\Gamma)$  such that  $\mu(g) = (\tilde{h}, g)$ , hence  $\text{tr} v = h + \tilde{h}$ , which concludes the proof.  $\square$

### 2.3 Passing to the limit: main result

We still assume that (2.3), (2.4) and (2.5) hold. We study in this section how to pass to the limit when  $\varepsilon \rightarrow 0$ . We still denote by  $\psi$  the isomorphism defined by (2.19). We assume in addition that  $\Omega$  is bounded. Our main result is the following.

**Theorem 2.1.** *Let  $u = u_\varepsilon \in \mathcal{C}^1(\overline{\Omega}) \cap H^2(\Omega)$  be the solution of Problem (1.2) for a given  $\varepsilon > 0$ . Then, there exists a sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  that converges to 0, and a function  $u \in W$ , such that  $(u_{\varepsilon_n})_{n \in \mathbb{N}}$  weakly converges in  $W$  to  $u$ , which satisfies  $\forall v \in W$ ,*

$$(2.25) \quad (u, v)_W =_{W'} \langle f, u \rangle_W$$

*Proof.* The function  $u_\varepsilon$  verifies the estimate (2.7), i.e.<sup>1</sup>

$$(2.26) \quad \int_\Omega \varrho_\varepsilon |\nabla u_\varepsilon|^2 + \int_\Gamma |u_\varepsilon|^2 \leq C,$$

which yields, since  $\varrho \geq 0$

$$(2.27) \quad \varepsilon \int_\Omega |\nabla u_\varepsilon|^2 \leq C.$$

In particular as  $u_\varepsilon \in \mathcal{C}^1(\overline{\Omega})$ ,  $u_\varepsilon \in W$ , then we can rewrite (2.26) as

$$(2.28) \quad \|u_\varepsilon\|_W \leq C.$$

---

<sup>1</sup> $C$  denotes any generic constant

Therefore, there exists  $(\varepsilon_n)_{n \in \mathbb{N}}$  that converges to 0,  $u \in W$ , such that  $(u_{\varepsilon_n})_{n \in \mathbb{N}}$  weakly converges to  $u$  in  $W$ . It remains to check what variational problem is satisfied by  $u$ . From the weak convergence in  $W$  we deduce that  $\forall v \in \mathcal{C}^1(\bar{\Omega})$ ,

$$(2.29) \quad \int_{\Omega} \varrho \nabla u_{\varepsilon_n} \cdot \nabla v + \int_{\Gamma} u_{\varepsilon_n} v \xrightarrow{n \rightarrow \infty} (u, v)_W.$$

Since  $u_{\varepsilon_n} \in \mathcal{C}^1(\bar{\Omega}) \cap H^2(\Omega)$  is a weak solution to (1.2), we have  $\forall v \in \mathcal{C}^1(\bar{\Omega})$ ,

$$(2.30) \quad \int_{\Omega} \varrho_{\varepsilon_n} \nabla u_{\varepsilon_n} \cdot \nabla v + \int_{\Gamma} u_{\varepsilon_n} v = \int_{\Omega} f v.$$

In particular,

$$(2.31) \quad \int_{\Omega} \varrho_{\varepsilon_n} \nabla u_{\varepsilon_n} \cdot \nabla v = \varepsilon_n \int_{\Omega} \nabla u_{\varepsilon_n} \cdot \nabla v + \int_{\Omega} \varrho \nabla u_{\varepsilon_n} \cdot \nabla v.$$

We deduce from (2.27)

$$(2.32) \quad \varepsilon_n \left| \int_{\Omega} \nabla u_{\varepsilon_n} \cdot \nabla v \right| \leq \varepsilon_n \|\nabla u_{\varepsilon_n}\|_{L^2} \|\nabla v\|_{L^2} \\ \sqrt{\varepsilon_n} \sqrt{C} \|\nabla v\|_{L^2} \xrightarrow{n \rightarrow \infty} 0,$$

hence (2.25) follows by (2.29), (2.32) and the density of  $\mathcal{C}^1(\bar{\Omega})$  in  $W$ .  $\square$

**Remark 2.3.** *We could have first remarked that Problem (2.25) has a unique solution by the Lax-Milgram Theorem. Theorem 2.1 states that this solution is a limit of solutions of partial differential equations. But nothing indicates that it is a solution to a PDE. At this stage it is only a solution to a variational problem.*

### 3 Some explicit 1D Examples

In this section we provide some explicit examples which illustrate the different behavior of the solutions, even of a further simplification of the toy problem, restricting to a 1D case in order to be able to perform the explicit and exact computation of the solution, obtaining results which are not provable by the classical variational methods for weak solutions and weak limits, as those constructed in the previous section.

#### 3.1 General solutions on an interval

On the interval  $\Omega = ]0, 1[$ , the problem (1.2) becomes

$$(3.1) \quad \forall z \in ]0, 1[, \quad -\frac{d}{dz} ((\varrho(z) + \varepsilon) u'_\varepsilon(z)) = f(z), \\ \varepsilon u'_\varepsilon(0) = u_\varepsilon(0), \\ -\varepsilon u'_\varepsilon(1) = u_\varepsilon(1),$$

where  $\varrho \in \mathcal{C}^1([0, 1])$  satisfies

$$(3.2) \quad \varrho(z) \underset{z \rightarrow 0^+}{\sim} z \quad \text{and} \quad \varrho(z) \underset{z \rightarrow 1^-}{\sim} 1 - z.$$

In this case, the function  $g$  of (2.4) is given by simply by

$$(3.3) \quad \forall z \in [0, 1], \quad g(z) = \int_0^z f(s) ds.$$

And (2.3) yields

$$(3.4) \quad g(1) = \int_0^1 f(s) ds = 0.$$

**Lemma 3.1.** *The problem (3.1) has a unique solution on  $[0, 1]$ , which is given by*

$$(3.5) \quad u_\varepsilon : z \mapsto R_\varepsilon(z) \int_0^1 \frac{g(t)}{\varrho(t) + \varepsilon} dt - \int_0^z \frac{g(t)}{\varrho(t) + \varepsilon} dt,$$

Where  $R_\varepsilon$  is given by

$$(3.6) \quad \forall z \in [0, 1], \quad R_\varepsilon(z) = \frac{1 + \int_0^z \frac{dt}{\varrho(t) + \varepsilon}}{2 + \int_0^1 \frac{dt}{\varrho(t) + \varepsilon}}.$$

*Proof.* Let  $u_\varepsilon$  satisfy (3.1). The solution of first equation is

$$(3.7) \quad \forall z \in [0, 1], \quad u_\varepsilon(z) = d_0 + c_0 \int_0^z \frac{dt}{\varrho(t) + \varepsilon} - \int_0^z \frac{g(t)}{\varrho(t) + \varepsilon} dt.$$

The first boundary condition of (3.1) directly yields  $d_0 = c_0$ , which means

$$(3.8) \quad u_\varepsilon(z) = c_0 \left( 1 + \int_0^z \frac{dt}{\varrho(t) + \varepsilon} \right) - \int_0^z \frac{g(t)}{\varrho(t) + \varepsilon} dt.$$

In (3.8), the constant  $c_0$  cannot be chosen arbitrarily. Indeed, it comes from the second boundary condition of (3.1).

$$(3.9) \quad \begin{cases} u_\varepsilon(1) = c_0 \left( 1 + \int_0^1 \frac{dt}{\varrho(t) + \varepsilon} \right) - \int_0^1 \frac{g(t)}{\varrho(t) + \varepsilon} dt, \\ -\varepsilon u'_\varepsilon(1) = -c_0. \end{cases}$$

Therefore,  $c_0$  satisfies

$$(3.10) \quad c_0 \left( 2 + \int_0^1 \frac{dt}{\varrho(t) + \varepsilon} \right) = \int_0^1 \frac{g(t)}{\varrho(t) + \varepsilon} dt.$$

This expression of  $c_0$  injected in (3.8) yields (3.5). □

Before considering explicit examples, there is an interesting convergence result.

**Lemma 3.2.** *Let  $\varrho$  satisfy (3.2) and  $g \in \mathcal{C}^1([0, 1])$  such that  $g(0) = g(1) = 0$ . Then the function  $\left[ t \mapsto \frac{g(t)}{\varrho(t)} \right]$  belongs to  $L^1(0, z)$  for  $0 \leq z \leq 1$ , and we have*

$$(3.11) \quad \int_0^z \frac{g(t)}{\varrho(t) + \varepsilon} dt \xrightarrow{\varepsilon \rightarrow 0} \int_0^z \frac{g(t)}{\varrho(t)} dt.$$

*Proof.* Let  $z \in [0, 1[$ . It is clear that  $\left[ t \mapsto \frac{|g(t)|}{\varrho(t)} \right]$  is continuous on  $]0, z]$  since  $\varrho$  is strictly positive in the interior. Moreover, we have

$$(3.12) \quad \left| \frac{g(t)}{\varrho(t)} \right| = \frac{|g(t)|}{\varrho(t)} \quad \text{and also} \quad \lim_{t \rightarrow 0^+} \frac{|g(t)|}{\varrho(t)} = \frac{|g'(0)|}{|\varrho'(0)|} = |g'(0)|,$$

and the limit vanishes if  $g'(0) = 0$ .

Consequently, the function  $g(t)/\varrho(t)$  extended by 0 at  $z = 0$  is continuous on  $[0, z]$ , then  $\left[ t \mapsto \frac{g(t)}{\varrho(t)} \right]$  is in  $L^1(0, z)$  for every  $z$  such that  $0 \leq z < 1$ .

The Lebesgue-Levi theorem on monotone convergence then yields

$$(3.13) \quad \int_0^z \frac{g(t)}{\varrho(t) + \varepsilon} dt \xrightarrow{\varepsilon \rightarrow 0} \int_0^z \frac{g(t)}{\varrho(t)} dt.$$

If  $z = 1$ , we use a similar argument to prove

$$(3.14) \quad \left| \frac{g(t)}{\varrho(t)} \right| = \frac{|g(t)|}{\varrho(t)} \quad \text{and} \quad \lim_{t \rightarrow 1^-} \frac{|g(t)|}{\varrho(t)} = \frac{|g'(1)|}{|\varrho'(1)|} = |g'(1)|.$$

If  $g'(1) = 0$ , the limit vanishes and again

$$(3.15) \quad \int_0^1 \frac{g(t)}{\varrho(t) + \varepsilon} dt \xrightarrow{\varepsilon \rightarrow 0} \int_0^1 \frac{g(t)}{\varrho(t)} dt.$$

□

**Corollary 3.1.** *With the same assumptions as before, we have*

$$(3.16) \quad \int_0^z \frac{g(t)}{\varrho(t) + \varepsilon} dt \xrightarrow[\varepsilon \rightarrow 0]{L^p} \int_0^z \frac{g(t)}{\varrho(t)} dt,$$

for  $1 \leq p < +\infty$ .

*Proof.* The previous Lemma has shown the pointwise convergence, then only the domination has to be proved. Let  $\varepsilon > 0$ , we have

$$\left| \int_0^z \frac{g(t)}{\varrho(t) + \varepsilon} dt - \int_0^z \frac{g(t)}{\varrho(t)} dt \right| = \left| \int_0^z \frac{\varepsilon g(t)}{\varrho(t)(\varrho(t) + \varepsilon)} dt \right| \leq \int_0^z \frac{|g(t)|}{\varrho(t)} dt \leq \int_0^1 \frac{|g(t)|}{\varrho(t)} dt.$$

By the previous lemma, this constant is finite, then the  $L^p$  Lebesgue Theorem yields the result.

□

### 3.2 Examples with $\varrho(z) = z(1 - z)$

We now give some explicit examples when the function  $\varrho$  vanishes linearly at both  $z = 0$  and  $z = 1$  and we use the explicit form to study the regularity of the solution to (3.1).

**Lemma 3.3.** *Let us suppose that  $\varrho(z) = z(1 - z)$ . Then, the function  $R_\varepsilon$  is continuous on  $[0, 1]$  and  $\lim_{\varepsilon \rightarrow 0^+} R_\varepsilon = \frac{1}{2}$  in  $L^p(0, 1)$ , for  $1 \leq p < +\infty$ .*

*Proof.* We calculate for  $0 \leq z \leq 1$  :

$$(3.17) \quad \int_0^z \frac{dt}{\varrho(t) + \varepsilon} = \int_0^z \frac{dt}{\varepsilon + t - t^2}.$$

The two zeros of the polynomial are

$$(3.18) \quad r_1(\varepsilon) = \frac{1 + \sqrt{1 + 4\varepsilon}}{2} > 1 \quad \text{and} \quad r_2(\varepsilon) = \frac{1 - \sqrt{1 + 4\varepsilon}}{2} < 0.$$

that yields

$$(3.19) \quad \begin{aligned} \int_0^z \frac{dt}{\varrho(t) + \varepsilon} &= \int_0^z \frac{dt}{(r_1(\varepsilon) - t)(t - r_2(\varepsilon))} \\ &= \frac{1}{r_1(\varepsilon) - r_2(\varepsilon)} \left( \int_0^z \frac{dt}{r_1(\varepsilon) - t} + \int_0^z \frac{dt}{t - r_2(\varepsilon)} \right) \\ &= \frac{1}{\sqrt{1 + 4\varepsilon}} \left( \ln \left( \frac{r_1(\varepsilon)}{r_1(\varepsilon) - z} \right) + \ln \left( \frac{z - r_2(\varepsilon)}{-r_2(\varepsilon)} \right) \right). \end{aligned}$$

This function is continuous on  $[0, 1]$ , which implies  $R_\varepsilon \in \mathcal{C}^0([0, 1])$ . By observing that  $1 - r_2(\varepsilon) = r_1(\varepsilon)$ , we choose to write  $r(\varepsilon) := r_1(\varepsilon)$ , and after some manipulations the expression of  $R_\varepsilon(z)$  for  $0 < z < 1$  becomes the following

$$(3.20) \quad R_\varepsilon(z) = \frac{1 + \frac{1}{\sqrt{1+4\varepsilon}} \left( \ln \left( \frac{r(\varepsilon)}{r(\varepsilon)-z} \right) + \ln \left( \frac{r(\varepsilon)-(1-z)}{r(\varepsilon)-1} \right) \right)}{2 + \frac{2}{\sqrt{1+4\varepsilon}} \ln \left( \frac{r(\varepsilon)}{r(\varepsilon)-1} \right)}.$$

Then, we can calculate the following limit

$$(3.21) \quad R_\varepsilon(z) \underset{\varepsilon \rightarrow 0}{\sim} \frac{-\ln(r(\varepsilon) - 1)}{-2\ln(r(\varepsilon) - 1)} = \frac{1}{2}.$$

We deduce that  $R_\varepsilon$  converges to  $1/2$  almost everywhere on  $[0, 1]$ .

Then (3.20) yields the following estimate

$$(3.22) \quad \left| R_\varepsilon(z) - \frac{1}{2} \right| = \frac{1}{2\sqrt{1+4\varepsilon}} \times \frac{\left| \ln \left( \frac{r(\varepsilon)-(1-z)}{r(\varepsilon)-z} \right) \right|}{1 + \frac{1}{\sqrt{1+4\varepsilon}} \ln \left( \frac{r(\varepsilon)}{r(\varepsilon)-1} \right)}.$$

Which allows to obtain the simpler upper bound

$$(3.23) \quad \left| R_\varepsilon(z) - \frac{1}{2} \right| \leq \frac{1}{2},$$

which shows, by applying the Lebesgue dominated convergence theorem the result.  $\square$

The above result has the following consequence on the characterization of the limit of  $u_\varepsilon$ .

**Corollary 3.2** (Condition for the boundary conditions of the limit). *If  $\varrho(z) = z(1 - z)$ , the sequence of solutions  $(u_\varepsilon)_{\varepsilon > 0}$  converges to  $u \in L^p(0, 1)$ , for all  $1 \leq p < +\infty$ .*

*Moreover, the limit  $u$  satisfies  $u(0) = u(1) = 0$  if and only if  $\int_0^1 \frac{g(t)}{t(1-t)} dt = 0$ .*

*Proof.* Lemmas 3.2 and 3.3 yield the expression for  $u$ :

$$(3.24) \quad \forall z \in ]0, 1[, \quad u(z) = \frac{1}{2} \int_0^1 \frac{g(t)}{t(1-t)} dt - \int_0^z \frac{g(t)}{t(1-t)} dt.$$

In particular

$$(3.25) \quad u(0) = \frac{1}{2} \int_0^1 \frac{g(t)}{t(1-t)} dt \quad \text{and} \quad u(1) = -\frac{1}{2} \int_0^1 \frac{g(t)}{t(1-t)} dt,$$

from which the result follows.  $\square$

### 3.3 An example with less regularity

We give now an example when the function  $g$  is less regular at  $z = 0$ . In particular, we assume that it vanishes logarithmically, instead than linearly. To be more precise let us assume that  $g : [0, 1] \rightarrow \mathbb{R}$  satisfies the following assumptions

- $g(0) = 0$
- $g(t) = 1 - t$  if  $\frac{1}{2} < t \leq 1$
- $g(t) = \frac{1}{\ln(t)}$  if  $0 < t < e^{-1}$
- $g \in \mathcal{C}^1(]0, 1])$ .

It is clear that the class above is non empty, since it is enough to extend  $g$  in the interval  $[e^{-1}, 1/2]$  in a differentiable way matching values of function and derivative.

**Theorem 3.1.** *Let  $g$  satisfy the previous assumptions, and  $s \in [0, 1]$ . Then, it follows that  $g \in H^s(0, 1)$  if and only if  $0 \leq s \leq \frac{1}{2}$ .*

*Proof.* First observe that in the interval  $[e^{-1}, 1]$  the function is smooth so it belongs to  $H^1(e^{-1}, 1)$  and the only problem is the behavior at zero. The derivative of  $g$  on  $]0, e^{-1}[$  is

$$(3.26) \quad \forall t \in ]0; e^{-1}[, \quad g'(t) = -\frac{1}{t \ln(t)^2}.$$

This function is not in  $L^2(0, e^{-1})$ , which means  $g \notin H^1(0; e^{-1})$ , then  $g \notin H^1(0, 1)$ . Let  $s \in ]0, 1[$ , since  $g$  is of class  $\mathcal{C}^1$  on  $]0, 1]$ , it is sufficient to prove the convergence of the double integral

$$(3.27) \quad \int_{x=0}^{e^{-1}} \int_{y=0}^{e^{-1}} \frac{|f(x) - f(y)|^2}{|x - y|^{2s+1}} dx dy,$$

which will give, if finite, the Sobolev semi-norm in  $H^s(0, e^{-1})$ . This integral can be explicitly rewritten as follows

$$(3.28) \quad \int_{x=0}^{e^{-1}} \int_{y=0}^{e^{-1}} \frac{\ln(y/x)^2}{\ln(x)^2 \ln(y)^2 |x - y|^{2s+1}} dx dy.$$

Applying the change of variables

$$(3.29) \quad \begin{cases} u = \ln(y/x) \\ v = \ln(x) \end{cases} \quad \text{hence} \quad \begin{cases} x = e^v \\ y = e^{u+v}, \end{cases}$$

which sends the domain  $(0, e^{-1})^2 \in \mathbb{R}_x \times \mathbb{R}_y$  into  $\mathbb{R} \times (-\infty, 1) \in \mathbb{R}_u \times \mathbb{R}_v$ , the integral (3.28) becomes

$$(3.30) \quad \begin{aligned} & \int_{u=-\infty}^{+\infty} \int_{v=-\infty}^{-1} \frac{u^2 e^{u+2v} \mathbb{1}_{\{u+v < -1\}}}{v^2 (u+v)^2 e^{(2s+1)v} |e^u - 1|^{2s+1}} du dv \\ &= \int_{-\infty}^{+\infty} \frac{u^2 e^u}{|e^u - 1|^{2s+1}} \left( \int_{-\infty}^{-1} \frac{e^{(1-2s)v} \mathbb{1}_{\{u+v < -1\}}}{v^2 (u+v)^2} dv \right) du. \end{aligned}$$

If  $1 - 2s < 0$ , the middle term of (3.30) does not converge. Therefore, the integral is not defined if  $1 - 2s < 0$ , which means  $s > \frac{1}{2}$ . On the other hand, if  $s < \frac{1}{2}$  the integral converges as it easily follows.

Let us now focus on the limiting case  $s = \frac{1}{2}$ . In this case the integral (3.30) then becomes

$$(3.31) \quad \int_{-\infty}^{+\infty} \frac{u^2 e^u}{|e^u - 1|^2} \left( \int_{-\infty}^{-1} \frac{\mathbb{1}_{\{u+v < -1\}}}{v^2 (u+v)^2} dv \right) du.$$

We calculate the middle integral, by using the following decomposition by simple rational functions. Let  $u, v \in \mathbb{R}$  satisfy  $v < -1$  and  $u + v < -1$ , then

$$(3.32) \quad \frac{1}{v^2(u+v)^2} = \frac{1}{u^2} \left( \frac{1}{v^2} + \frac{1}{(u+v)^2} \right) + \frac{2}{u^3} \left( \frac{1}{u+v} - \frac{1}{v} \right).$$

This implies that we obtain for  $u < 0$

$$(3.33) \quad \int_{-\infty}^{-1} \frac{\mathbb{1}_{\{u+v < -1\}}}{v^2(u+v)^2} dv = \int_{-\infty}^{-1} \frac{dv}{v^2(u+v)^2} = \frac{1}{u^2} \left( 1 + \frac{1}{1-u} \right) + \frac{2}{u^3} \ln(1-u),$$

and for  $u > 0$

$$(3.34) \quad \int_{-\infty}^{-1} \frac{\mathbb{1}_{\{u+v < -1\}}}{v^2(u+v)^2} dv = \int_{-\infty}^{-(1+u)} \frac{dv}{v^2(u+v)^2} = \frac{1}{u^2} \left( 1 + \frac{1}{1+u} \right) - \frac{2}{u^3} \ln(1+u).$$

By a Taylor series expansion, we can evaluate the two limits as follows

$$\begin{aligned} & \frac{1}{u^2} \left( 1 + \frac{1}{1-u} \right) + \frac{2}{u^3} \ln(1-u) \xrightarrow{u \rightarrow 0^-} \frac{1}{3} \\ & \text{and} \\ & \frac{1}{u^2} \left( 1 + \frac{1}{1+u} \right) - \frac{2}{u^3} \ln(1+u) \xrightarrow{u \rightarrow 0^+} \frac{1}{3}. \end{aligned}$$

Then it is easy to show the convergence of the integral (3.31). □

Once proved the preliminary regularity of  $g$ , let  $f$  be defined as the derivative of  $g$  on  $]0, 1[$ . This nonsmooth function satisfies (in particular)

- $f \in \mathcal{C}^0(]0, 1[)$
- $f(t) = -1$  if  $\frac{1}{2} < t \leq 1$ .
- $f(t) = \frac{-1}{t \ln(t)^2}$  if  $0 < t \leq e^{-1}$

We now try to solve the problem (3.1) with  $f$  defined this way, and  $\varrho(z) = z(1-z)$ . Since  $g$  is absolutely continuous in  $[0, 1]$  (note that it is differentiable in  $]0, 1[$  and  $\int_0^{e^{-1}} |t \log^2(t)|^{-1} dt$  is finite and  $g(0) = 0$ ), we can write

$$\forall t \in ]0, 1[, \quad g(t) = \int_0^t f(s) ds.$$

Therefore, a calculation as in Lemma 3.1 can be applied and yields for  $0 < z < 1$

$$u_\varepsilon(z) = R_\varepsilon(z) \int_0^1 \frac{g(t)}{t(1-t) + \varepsilon} dt - \int_0^z \frac{g(t)}{t(1-t) + \varepsilon} dt,$$

which can be rewritten as follows

$$u_\varepsilon(z) = (R_\varepsilon(z) - 1) \int_0^{e^{-1}} \frac{g(t)}{t(1-t) + \varepsilon} dt + R_\varepsilon(z) \int_{e^{-1}}^1 \frac{g(t)}{t(1-t) + \varepsilon} dt - \int_{e^{-1}}^z \frac{g(t)}{t(1-t) + \varepsilon} dt.$$

One one hand, the Lebesgue theorem yields for  $0 < z < 1$ , the following convergence

$$R_\varepsilon(z) \int_{e^{-1}}^1 \frac{g(t)}{t(1-t) + \varepsilon} dt - \int_{e^{-1}}^z \frac{g(t)}{t(1-t) + \varepsilon} dt \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{2} \int_{e^{-1}}^1 \frac{g(t)}{t(1-t)} dt - \int_{e^{-1}}^z \frac{g(t)}{t(1-t)} dt.$$

On the other hand, the Beppo Levi theorem yields for  $0 < z < 1$

$$-\int_0^{e^{-1}} \frac{g(t)}{t(1-t) + \varepsilon} dt \xrightarrow{\varepsilon \rightarrow 0} \int_0^{e^{-1}} \frac{dt}{t|\ln(t)|(1-t)} = +\infty.$$

Hence, since  $R_\varepsilon(z) \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{2}$ :

$$(3.35) \quad (R_\varepsilon(z) - 1) \int_0^{e^{-1}} \frac{g(t)}{t(1-t) + \varepsilon} dt \xrightarrow{\varepsilon \rightarrow 0} +\infty.$$

This shows that the sequence of the approximated solutions does not converge almost everywhere.

## 4 A 2D problem: The Case of the Disk

In this section we consider a 2D case with circular symmetry (which in turn reduces again to a 1D problem) and we illustrate other interesting phenomena.

In this section we assume that the domain  $\Omega$  is the unit ball  $\mathbb{B}(0,1) \subset \mathbb{R}^2$ . What is convenient in this geometry is the fact that one can look for radial solutions as long as the source  $f$  is radial. In particular, choosing  $\varrho(x) = d(x, \partial\mathbb{B}(0,1))$  the function  $\varrho$  will be given by

$$(4.1) \quad \text{if } |x| = r \quad \text{then} \quad \varrho(r) = 1 - r.$$

The equation (1.2) becomes in polar variables  $(r, \theta)$ , when the data are independent of  $\theta$  by circular symmetry:

$$(4.2) \quad -\frac{\partial u_\varepsilon}{\partial r} + (1 - r + \varepsilon) \left( \frac{\partial^2 u_\varepsilon}{\partial r^2} + \frac{1}{r} \frac{\partial u_\varepsilon}{\partial r} \right) = f(r),$$

and the boundary condition is  $-\varepsilon \frac{\partial u}{\partial r} = u$  on  $r = 1$ .

### 4.1 The associated Homogeneous equation

We start by studying the the corresponding homogeneous equation

$$(4.3) \quad \frac{\partial^2 u_\varepsilon}{\partial r^2} + \left( \frac{1 + \varepsilon - 2r}{1 + \varepsilon - r} \right) \frac{1}{r} \frac{\partial u_\varepsilon}{\partial r} = 0,$$

for every  $r > 0$ . As

$$\left( \frac{1 + \varepsilon - 2r}{1 + \varepsilon - r} \right) \frac{1}{r} = \frac{1}{r} - \frac{1}{1 + \varepsilon - r},$$

we find that the solution of (4.3) is the function  $\frac{\partial u_\varepsilon}{\partial r}$

$$r \mapsto \frac{1}{r(1 + \varepsilon - r)},$$



## 4.2 Blow up example

Assume now that the source  $f$  is constant and it is equal to 1. Note that with these assumptions  $f$  does not longer satisfies (2.3), (2.4), and (2.5). The equation (4.2) becomes

$$(4.4) \quad \frac{\partial^2 u_\varepsilon}{\partial r^2} + \left( \frac{1 + \varepsilon - 2r}{1 + \varepsilon - r} \right) \frac{1}{r} \frac{\partial u_\varepsilon}{\partial r} = \frac{1}{1 - r + \varepsilon}.$$

Thanks to the variation of constants method, the solutions of (4.4) are

$$(4.5) \quad r \mapsto \frac{K_1}{r(1 + \varepsilon - r)} + \frac{1}{2} \frac{r}{1 + \varepsilon - r}.$$

Since

$$\frac{K_1}{r(1 + \varepsilon - r)} = \frac{K_1}{1 + \varepsilon} \left( \frac{1}{r} + \frac{1}{1 + \varepsilon - r} \right),$$

and

$$\frac{r}{1 + \varepsilon - r} = \frac{1 + \varepsilon}{1 + \varepsilon - r} - 1,$$

the radial solution  $u_\varepsilon$  is given by

$$(4.6) \quad u_\varepsilon(r) = \frac{K_1}{1 + \varepsilon} \log \left( \frac{r}{1 + \varepsilon - r} \right) - \frac{r}{2} - \frac{(1 + \varepsilon)}{2} \log(1 + \varepsilon - r) + K_2,$$

where the constants  $K_1$  and  $K_2$  can be retrieved from the boundary conditions. From the condition

$$-\varepsilon \frac{\partial u_\varepsilon}{\partial r}(1) = u_\varepsilon(1),$$

we get

$$-\left( K_1 + \frac{1}{2} \right) = \frac{K_1}{1 + \varepsilon} \log \left( \frac{1}{\varepsilon} \right) - \frac{1}{2} - \frac{(1 + \varepsilon)}{2} \log(\varepsilon) + K_2,$$

and the relation

$$K_1 \left( 1 + \frac{\log(\varepsilon)}{1 + \varepsilon} \right) + K_2 = \frac{(1 + \varepsilon)}{2} \log(\varepsilon).$$

We want  $u_\varepsilon \in H^1(\Omega)$ . In particular, as

$$u'_\varepsilon(r) \sim \frac{K_1}{(1 + \varepsilon)r}, \quad \text{for } r \rightarrow 0^+,$$

we have to fix  $K_1 = 0$  to have  $u'_\varepsilon \in L^2(\Omega)$ .

It gives :

$$K_2 = \frac{1 + \varepsilon}{2} \log(\varepsilon),$$

and consequently

$$(4.7) \quad u_\varepsilon(r) = -\frac{r}{2} + \frac{(1 + \varepsilon)}{2} \log \left( \frac{\varepsilon}{1 + \varepsilon - r} \right) = -\frac{r}{2} + \frac{(1 + \varepsilon)}{2} \log \left( 1 + \frac{r - 1}{1 + \varepsilon - r} \right).$$

We observe that in this case, the solution blows up everywhere when  $\varepsilon \rightarrow 0$ . Here the problem is not that  $u$  is bounded near the boundary, but that  $u$  itself, as limit  $\varepsilon \rightarrow 0$ .

### 4.3 Example of a case with a limit

We now consider a source term which is compatible with the weak formulation and in particular, we show how to choose  $f$  by directly finding an appropriate  $\mathbf{g}$  to satisfy (2.3), (2.4) and (2.5) in order to pass to the limit. The polar basis  $(\mathbf{e}_r, \mathbf{e}_\theta)$  is defined by

$$(4.8) \quad \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{pmatrix}.$$

We are looking for  $\mathbf{g}$  such that

$$\mathbf{g} \cdot \mathbf{n}|_\Gamma = 0,$$

To avoid any singularities at 0, we take  $\mathbf{g}$  of the following form

$$\mathbf{g} = r\varrho(r)\mathbf{e}_r.$$

Therefore,  $f = \nabla \cdot \mathbf{g}$  is given by:

$$\begin{aligned} f(r) &= \frac{1}{r} \left( \frac{\partial}{\partial r} (r^2 \varrho(r)) \right) \\ &= 2\varrho(r) - r. \end{aligned}$$

We take this function as a source term. Using the same notation as before, this gives

$$\frac{K'_1(r)}{r\varrho_\varepsilon(r)} = \frac{f(r)}{\varrho_\varepsilon(r)},$$

and

$$K'_1(r) = 2r - 3r^2.$$

Thus,  $K_1(r) = r^2 - r^3 = r^2\varrho(r)$ , which gives as a solution:

$$(4.9) \quad u'_\varepsilon(r) = r \frac{\varrho(r)}{\varrho_\varepsilon(r)}.$$

In particular,  $u'_\varepsilon(1) = 0$ . The boundary condition implies that  $u_\varepsilon(1) = 0$ . We finally get

$$(4.10) \quad u_\varepsilon(r) = - \int_1^r r \frac{\varrho(r)}{\varrho_\varepsilon(r)} dr,$$

and by the Lebesgue dominated convergence theorem,

$$(4.11) \quad u_\varepsilon(r) \xrightarrow{\varepsilon \rightarrow 0} \frac{1-r^2}{2},$$

which is a  $\mathcal{C}^\infty$  function that satisfies  $u = 0$  at  $\Gamma = \{r = 1\}$ .

### 4.4 A general class of solutions

We consider now a family of source terms  $f_\varepsilon$  depending on  $\varepsilon$ . Let  $\alpha$  and  $\beta$  be two real numbers, with  $\alpha \geq 0$  and  $\beta \geq 1$ . The idea is to take the source  $f_\varepsilon$  such that the product  $rf_\varepsilon(r)$  gives a derivative with “good” properties. We define the source  $f_\varepsilon = f_{\varepsilon, \alpha, \beta}$  as

$$\forall r \in (0, 1) \quad f_\varepsilon(r) := (\alpha + 1)r^{\alpha-1}(1 + \varepsilon - r)^\beta - \beta(1 + \varepsilon - r)^{\beta-1}r^\alpha.$$

In particular, when applying the variation of constants method, we obtain the following relation

$$K_1'(r) = f_\varepsilon(r)r = (\alpha + 1)r^\alpha(1 + \varepsilon - r)^\beta - \beta(1 + \varepsilon - r)^{\beta-1}r^{\alpha+1},$$

which gives

$$K_1(r) = r^{\alpha+1}(1 + \varepsilon - r)^\beta,$$

and the solutions of (4.2) are the functions

$$r \mapsto \frac{K_1}{r(1 + \varepsilon - r)} + r^\alpha(1 + \varepsilon - r)^{\beta-1}.$$

The integrability of  $|u_\varepsilon'|^2$  implies that  $K_1 = 0$ , so  $u_\varepsilon'$  is given by

$$(4.12) \quad u_\varepsilon'(r) = r^\alpha(1 + \varepsilon - r)^{\beta-1}.$$

If  $\beta$  is an integer, we can give an expression of  $u$  as a sum with a *finite* number of terms). In order to find relevant properties and avoid infinite summation of singular terms, from now on we will make this assumption; we rename  $\beta = k \in \mathbb{N}$  to make it more clear.

We can write

$$u_\varepsilon'(r) = \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i r^{\alpha+i} (1 + \varepsilon)^{k-1-i},$$

which gives by integration

$$(4.13) \quad u_\varepsilon(r) = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^i}{\alpha + i + 1} r^{\alpha+i+1} (1 + \varepsilon)^{k-1-i} + K_2,$$

where  $K_2 \in \mathbb{R}$  is a constant we will determine thanks to the boundary condition  $-\varepsilon u_\varepsilon'(1) = u_\varepsilon(1)$ . On the one hand,

$$\varepsilon u_\varepsilon'(1) = \varepsilon^k,$$

on the other hand,

$$u_\varepsilon(1) = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^i}{\alpha + i + 1} (1 + \varepsilon)^{k-i-1} + K_2.$$

Thus,  $K_2$  is a polynomial function in  $\varepsilon$  of degree  $k$ :

$$K_2 = \varepsilon^k + \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^{i+1}}{\alpha + i + 1} (1 + \varepsilon)^{k-i-1} = \frac{(-1)^k}{\alpha + k} + O(\varepsilon).$$

For instance, if we take  $k = 1$ , we have:

$$u_\varepsilon(r) = \frac{r^{\alpha+1}}{\alpha + 1} + \varepsilon - \frac{1}{\alpha + 1}$$

and for  $k = 2$ :

$$u_\varepsilon(r) = -\frac{r^{\alpha+2}}{\alpha + 2} + \left(\frac{1 + \varepsilon}{1 + \alpha}\right) r^{\alpha+1} + \varepsilon^2 - \frac{1 + \varepsilon}{1 + \alpha} + \frac{1}{\alpha + 2}.$$

By direct computations we can show that when  $\varepsilon \rightarrow 0^+$

$$u_\varepsilon(r) \rightarrow \frac{(-1)^k}{\alpha + k} (1 - r^{\alpha+k})$$

and in all cases, it vanishes at the boundary.

## References

- [1] C. Amrouche, C. Conca, A. Ghosh, and T. Ghosh. Uniform  $W^{1,p}$  estimates for an elliptic operator with Robin boundary condition in a  $C^1$  domain. *Calc. Var. Partial Differential Equations*, 59(2):Paper No. 71, 25, 2020.
- [2] T. Chacòn-Rebollo and R. Lewandowski. *Mathematical and Numerical Foundations of Turbulence Models and Applications*. Modeling and Simulation in Science, Engineering and Technology. Springer New York, 2014.
- [3] Jacques Dixmier. *Topologie générale*. Presses universitaires de France, Paris, 1981.
- [4] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms.
- [5] Bill Lawvere. *Metric spaces, generalized logic, and closed categories*. Rendiconti Seminari Matematici della Facoltà di Scienze della Matematica e delle Sue Pianificazioni "La Sapienza", Milano, 1973.
- [6] J.-L. Lions. *Perturbations singulières dans les problèmes aux limites et en contrôle optimal*. Springer-Verlag, Berlin-New York, 1973.
- [7] Vladimir Maz'ya. *Sobolev Spaces on Domains*, volume 137 of *Teubner-Texte zur Mathematik*. Vieweg+Teubner Verlag, 1998.