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# Singular boundary condition for a degenerated turbulent toy model

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## Abstract

We consider a toy model with the Prandtl mixing length as eddy viscosity, that vanishes at the boundary, and a Navier like friction law as boundary condition. We address the paradox of the degeneracy of the boundary condition, which we approach by a problem of singular perturbations. We show a convergence theorem for well-prepared source terms, and we illustrate our analysis with a series of analytical examples, showing blow up cases and convergence cases for well-prepared data.

**Key words** : Fluid mechanics, Turbulence models, singular perturbations.

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## 1 Introduction

The two main characteristics of a turbulence model for the simulation of a fluid in the vicinity of a wall are the turbulent viscosity and the condition of friction at the wall, often called the wall law. In many cases, the turbulent viscosity  $\nu_{\text{turb}}$  is proportional to a power of the mixing length  $\varrho$ , which is of the order of the distance from the wall in many physical models. If we consider that molecular diffusion is negligible compared to turbulent diffusion, we are led to write as wall law, where  $\Gamma$  denotes the boundary of the flow domain  $\Omega$ ,  $\mathbf{u}_\tau$  the tangential fluid velocity at  $\Gamma$ :

$$(1.1) \quad \nu_{\text{turb}} \frac{\partial \mathbf{u}_\tau}{\partial \mathbf{n}} = f(\mathbf{u}_\tau).$$

Obviously this doesn't make much sense because  $\nu_{\text{turb}}$  is zero over  $\Gamma$  according to the above, which would mean that  $\frac{\partial \mathbf{u}_\tau}{\partial \mathbf{n}}$  is infinite over  $\Gamma$ , unless  $f(\mathbf{u}_\tau)$  vanishes, so far  $f$  satisfies appropriate conditions (see in [3]). This means that no friction occurs in some sense when  $f$  vanishes only at 0 such as  $f(\mathbf{w}) = C_D \mathbf{w}|\mathbf{w}|$ , and we are left with a no

slip boundary condition. This is an old debate. The paradox here is that in a classical variational formulation, we will formally have a term of the form

$$\int_{\Gamma} f(\mathbf{u}_{\tau}) \cdot \mathbf{v}_{\tau}$$

which is totally well defined, even if  $\mathbf{u}_{\tau} \neq 0$  at  $\Gamma$ .

Following J.-L. Lions [5], it is natural to introduce a problem of singular perturbations to understand this paradox. In particular in this present case, we are led to consider a viscosity of the form  $\nu_{\text{turb}} + \varepsilon$ ,  $\varepsilon > 0$ , which amounts to reconsidering the molecular viscosity, and to asking the question of the limit problem when  $\varepsilon \rightarrow 0$ .

We study in this paper a toy model, considering only the tangential velocity denoted  $u$ ,  $f(u) = u$ ,  $\nu_{\text{turb}}(x) = \varrho(x)$ , and only a diffusion term with a source term. Therefore, the singular perturbation problem we consider is the following:

$$(1.2) \quad \begin{cases} -\operatorname{div}((\varrho + \varepsilon)\nabla u) = f & \text{over } \Omega, \\ -\varepsilon \frac{\partial u}{\partial n} = u & \text{at } \Gamma, \end{cases}$$

where  $\Omega \subset \mathbb{R}^N$  ( $N \leq 3$ ) is a  $C^2$ -domain and  $\varrho$  is given by (2.1) below. We show in this paper that when the source term satisfies appropriate compatibility conditions, then it is possible to pass to the limit in this problem in a certain sense. Then we provide several analytical examples of blowing up cases for not well prepared data, as well as examples as for well prepared data, for which convergence holds. In all these positive examples we find  $u = 0$  at  $\Gamma$  when passing to the limit, which we are not able to prove theoretically at the time being.

## 2 Limit problem

### 2.1 Well prepared source term and energy balance

#### 2.1.1 Mixing length

We assume that  $\varrho : \Omega \rightarrow \mathbb{R}$  a  $C^2 \cap W^{2,\infty}$  function that satisfies

$$(2.1) \quad \lim_{\substack{d(x,\Gamma) \rightarrow 0 \\ x \in \Omega}} \frac{\varrho(x)}{d(x,\Gamma)} = 1, \quad \inf_{\substack{d(x,\Gamma) \geq \frac{1}{n} \\ x \in \Omega}} \varrho(x) > 0,$$

In what follows, we set

$$(2.2) \quad \varrho_{\varepsilon} = \varrho + \varepsilon$$

#### 2.1.2 Well prepared source term

We take as a source term  $f \in C^0(\overline{\Omega}) \cap L^{\infty}(\Omega)$  such that

$$(2.3) \quad \int_{\Omega} f = 0.$$

Let  $\mathbf{g} \in C^1(\overline{\Omega})$  be the unique solution to

$$(2.4) \quad \begin{cases} \operatorname{div} \mathbf{g} = f & \text{in } \Omega, \\ \mathbf{g} \cdot \mathbf{n} = 0 & \text{on } \Gamma. \end{cases}$$

Notice that  $\mathbf{g}$  does exist and is unique up to the constants, by the compatibility condition (2.3) (see in [4]).

We assume that there exists a constant  $C$  such that for all  $\varepsilon > 0$  one has

$$(2.5) \quad \int_{\Omega} \frac{|\mathbf{g}|^2}{\varrho_\varepsilon} \leq C.$$

The reason for this choice will be clear in what follows.

### 2.1.3 Energy balance

Assume that (2.3), (2.4) and (2.5) hold. We say that  $u \in H^1(\Omega)$  is a weak solution to (1.2) if  $\forall v \in H^1(\Omega)$ , we have

$$(2.6) \quad \int_{\Omega} \varrho_\varepsilon \nabla u \cdot \nabla v + \int_{\Gamma} uv = \int_{\Omega} fv.$$

**Lemma 2.1.** *Problem (1.2) has a unique weak solution  $u \in H^1(\Omega)$ . In addition,  $u \in C^2(\overline{\Omega})$  and we have*

$$(2.7) \quad \int_{\Omega} \varrho_\varepsilon |\nabla u|^2 + \int_{\Gamma} |u|^2 \leq \int_{\Omega} \frac{|\mathbf{g}|^2}{\varrho_\varepsilon} \leq C.$$

*Proof.* Given  $\varepsilon > 0$ , the existence and the uniqueness of a weak solution to Problem (1.2) is straightforward by Lax-Milgram Theorem. To check the regularity, write

$$(2.8) \quad -\Delta u = \frac{1}{\varrho_\varepsilon} (\nabla \varrho_\varepsilon \cdot \nabla u + f).$$

In a first analysis, we see that the r.h.s is in  $L^2$ , then  $u \in H^2(\Omega)$  (see in [1]). Therefore, the r.h.s is in  $C^0(\overline{\Omega})$  when  $N = 2$ , hence  $u \in C^2(\Omega)$ . When  $N = 3$ , we get  $u \in W^{2,6}$ , then  $\nabla u \in C^0$ , hence  $u \in C^2(\overline{\Omega})$  since  $\nabla \varrho_\varepsilon \in C^0$ .

We now check the energy balance (2.7), which explains why (2.5). Take  $u$  as test function. By the particular choice of the source term, we have

$$(2.9) \quad \left| \int_{\Omega} uf \right| = \left| - \int_{\Omega} \mathbf{g} \cdot \nabla u \right| \leq \left( \int_{\Omega} \frac{|\mathbf{g}|^2}{\varrho_\varepsilon} \right)^{\frac{1}{2}} \left( \int_{\Omega} \varrho_\varepsilon |\nabla u|^2 \right)^{\frac{1}{2}}.$$

Estimate (2.7) follows from standard calculus. □

## 2.2 Functional space

We equip  $C^1(\overline{\Omega})$  with the norm

$$(2.10) \quad \|u\|_{1,\varrho} = \left( \int_{\Omega} \varrho |\nabla u|^2 + \int_{\Gamma} |u|^2 \right)^{\frac{1}{2}}.$$

Let  $W$  be the completion of  $C^1(\overline{\Omega})$  for the norm  $\|\cdot\|_{1,\varrho}$ .

**Theorem 2.1.** *The space  $W$  is isomorphic to a close subspace of*

$$(2.11) \quad Z = (H^{1/2}(\Omega)/\mathbb{R}) \times L^2(\Gamma).$$

*Proof.* We equip the space  $Z$  with the natural product norm. We know that if  $\sqrt{\varrho}\nabla u \in L^2(\Omega)^N$ , then  $u \in H^{1/2}(\Omega)/\mathbb{R}$  (see in [2] for instance). Therefore

$$(2.12) \quad \psi : \begin{cases} W & \rightarrow & Z \\ u & \rightarrow & (u, \text{tru}), \end{cases}$$

is an isomorphism from  $W$  to  $\text{Im}\psi$  which is closed in  $Z$ . □

### 2.3 Passing to the limit: main result

We still assume that (2.3), (2.4) and (2.5) hold. We study in this section how to pass to the limit when  $\varepsilon \rightarrow 0$ . We still denote by  $\psi$  the isomorphism defined by (2.12). We assume in addition that  $\Omega$  is bounded. Our main result is the following.

**Theorem 2.2.** *Let  $u = u_\varepsilon$  be the solution of Problem (1.2) for a given  $\varepsilon > 0$ . Then there exists  $(\varepsilon_n)_{n \in \mathbb{N}}$  that converges to 0,  $u \in W$ , such that  $(u_{\varepsilon_n})_{n \in \mathbb{N}}$  weakly converges in  $W$  to  $u$ , that satisfies  $\forall v \in W$ ,*

$$(2.13) \quad \int_{\Omega} \varrho \nabla u \cdot \nabla v + \int_{\Gamma} uv = \int_{\Omega} fv$$

*Proof.* Let  $u_\varepsilon$  be the solution of Problem (1.2) for a given  $\varepsilon > 0$ . It verifies the estimate (2.7), i.e.<sup>1</sup>

$$\int_{\Omega} \varrho_\varepsilon |\nabla u_\varepsilon|^2 + \int_{\Gamma} |u_\varepsilon|^2 \leq C,$$

which yields

$$(2.14) \quad \varepsilon \int_{\Omega} |\nabla u_\varepsilon|^2 \leq C$$

and

$$\|u_\varepsilon\|_W \leq C.$$

Therefore, there exists  $(\varepsilon_n)_{n \in \mathbb{N}}$  that converges to 0,  $u \in W$ , such that  $(u_{\varepsilon_n})_{n \in \mathbb{N}}$  weakly converges to  $u$  in  $W$ . It remains to check what variational problem is satisfied by  $u$ . From the weak convergence in  $W$  we deduce that  $\forall v \in W$ ,

$$(2.15) \quad \int_{\Omega} \varrho \nabla u_{\varepsilon_n} \cdot \nabla v + \int_{\Gamma} u_{\varepsilon_n} v \xrightarrow{n \rightarrow \infty} \int_{\Omega} \varrho \nabla u \cdot \nabla v + \int_{\Gamma} hv.$$

Since  $u_{\varepsilon_n} \in C^2(\overline{\Omega})$  is a weak solution to (1.2), we have  $\forall v \in C^1(\overline{\Omega})$ ,

$$(2.16) \quad \int_{\Omega} \varrho_{\varepsilon_n} \nabla u_{\varepsilon_n} \cdot \nabla v + \int_{\Gamma} u_{\varepsilon_n} v = \int_{\Omega} fv.$$

In particular,

$$(2.17) \quad \int_{\Omega} \varrho_{\varepsilon_n} \nabla u_{\varepsilon_n} \cdot \nabla v = \varepsilon_n \int_{\Omega} \nabla u_{\varepsilon_n} \cdot \nabla v + \int_{\Omega} \varrho \nabla u_{\varepsilon_n} \cdot \nabla v.$$

We deduce from (2.14)

$$(2.18) \quad \varepsilon_n \left| \int_{\Omega} \nabla u_{\varepsilon_n} \cdot \nabla v \right| \leq \varepsilon_n \|\nabla u_{\varepsilon_n}\|_{L^2} \|\nabla v\|_{L^2} \leq \sqrt{\varepsilon_n} \sqrt{C} \|\nabla v\|_{L^2} \xrightarrow{n \rightarrow \infty} 0,$$

hence (2.13) by (2.15), (2.18) and the density of  $C^1(\overline{\Omega})$  in  $W$ . □

<sup>1</sup> $C$  denotes any generic constant

### 3 1D case example

#### 3.1 Example of convergence

We seek for  $u = u(z) : [0, 1] \rightarrow \mathbb{R}$ , solution of

$$(3.1) \quad \begin{cases} -((z + \varepsilon)u')' = 1 & \text{over } ]0, 1[, \\ \varepsilon u'(0) = u(0) = u_0, \\ -(\varepsilon + 1)u'(1) = u(1) = u_1 \end{cases}$$

In this 1D case,  $\Omega = [0, 1]$ ,  $\Gamma = \{0, 1\}$ ,  $\varrho(z) = z$ ,  $f = \text{div}\varrho$ . Notice that  $\varrho(z) = d(z, 0)$ , and is not of order of  $d(z, \Gamma)$ . It is just the distance to the ground, which is a case slightly different from the general case studied in section 2.3, which does not change much in our study and our conclusions. In particular (2.5) is satisfied. We observe that one can pass to the limit, and that the limit  $u$  given by (3.6) below satisfies  $u(0) = 0$ . However,  $u'$  is singular since it is a Dirac mass at 0. In what follows we develop the analytical calculations. By integrating the equation over  $[0, z]$ , we get

$$-(z + \varepsilon)u'(z) + \varepsilon u'(0) = z,$$

which yields by using the boundary condition:

$$(3.2) \quad u'(z) = \frac{u_0 - z}{z + \varepsilon} = \frac{u_0 + \varepsilon}{z + \varepsilon} - 1,$$

that we integrate once again over  $[0, z]$  to get,

$$(3.3) \quad u(z) = (u_0 + \varepsilon) \ln\left(\frac{z + \varepsilon}{\varepsilon}\right) - z + u_0.$$

In order to close the formula, we use the second boundary condition at  $z = 1$ . By (3.3) we have on one hand:

$$(3.4) \quad u_1 = (u_0 + \varepsilon) \ln\left(\frac{1 + \varepsilon}{\varepsilon}\right) - 1 + u_0,$$

and on the other hand by (3.2),

$$u'(1) = \frac{u_0 - 1}{1 + \varepsilon} = -\frac{u_1}{1 + \varepsilon},$$

giving

$$u_1 = 1 - u_0,$$

which, combined with (3.4) yields

$$u_0 = \frac{2 + \varepsilon \ln\left(\frac{\varepsilon}{1 + \varepsilon}\right)}{2 - \ln\left(\frac{\varepsilon}{1 + \varepsilon}\right)} \underset{\varepsilon \rightarrow 0}{\sim} -\frac{2}{\ln \varepsilon} \underset{\varepsilon \rightarrow 0}{\rightarrow} 0,$$

and for  $z > 0$ ,

$$(3.5) \quad u(z) \underset{\varepsilon \rightarrow 0}{\sim} -\frac{2 \ln(z + \varepsilon)}{\ln \varepsilon} + 2 - z,$$

and therefore the limit as  $\varepsilon \rightarrow 0$  is the discontinuous function

$$(3.6) \quad \begin{cases} u(z) = 2 - z & z > 0, \\ u(0) = 0, \end{cases}$$

which solves the boundary problem

$$(3.7) \quad \begin{cases} -(z u')' = 1 & \text{over } ]0, 1[, \\ u(0) = 0 \\ u(1) = 1. \end{cases}$$

## 4 Case of the Disk

We assume in this section that the domain  $\Omega$  is the unit ball  $\mathbb{B}(0, 1) \subset \mathbb{R}^2$ . What is convenient in this geometry is the fact that one can look for radial solutions as long as the source  $f$  is radial. In particular, the function  $\varrho$  is given by

$$(4.1) \quad \varrho(r) = 1 - r.$$

The equation gets:

$$(4.2) \quad -\frac{\partial u_\varepsilon}{\partial r} + (1 - r + \varepsilon) \left( \frac{\partial^2 u_\varepsilon}{\partial r^2} + \frac{1}{r} \frac{\partial u_\varepsilon}{\partial r} \right) = f(r).$$

### 4.1 Homogeneous equation

We start with the corresponding homogeneous equation is

$$(4.3) \quad \frac{\partial^2 u_\varepsilon}{\partial r^2} + \left( \frac{1 + \varepsilon - 2r}{1 + \varepsilon - r} \right) \frac{1}{r} \frac{\partial u_\varepsilon}{\partial r} = 0,$$

for every  $r > 0$ . As

$$\left( \frac{1 + \varepsilon - 2r}{1 + \varepsilon - r} \right) \frac{1}{r} = \frac{1}{r} - \frac{1}{1 + \varepsilon - r},$$

we find that the fundamental solution of (4.3) is the function

$$r \mapsto \frac{1}{r(1 + \varepsilon - r)},$$

### 4.2 Blow up example

Assume that the source  $f$  is constant equal to 1, that does not longer satisfies (2.3), (2.4) and (2.5). The equation (4.2) becomes

$$(4.4) \quad \frac{\partial^2 u_\varepsilon}{\partial r^2} + \left( \frac{1 + \varepsilon - 2r}{1 + \varepsilon - r} \right) \frac{1}{r} \frac{\partial u_\varepsilon}{\partial r} = \frac{1}{1 - r + \varepsilon}.$$

Thanks to the variation of constants method, the solutions of (4.4) are

$$(4.5) \quad r \mapsto \frac{K_1}{r(1 + \varepsilon - r)} + \frac{1}{2} \frac{r}{(1 + \varepsilon - r)}.$$

Since

$$\frac{K_1}{r(1 + \varepsilon - r)} = \frac{K_1}{1 + \varepsilon} \left( \frac{1}{r} + \frac{1}{1 + \varepsilon - r} \right),$$

and

$$\frac{r}{1 + \varepsilon - r} = \frac{1 + \varepsilon}{1 + \varepsilon - r} - 1,$$

the radial solution  $u_\varepsilon$  is given by

$$(4.6) \quad u_\varepsilon(r) = \frac{K_1}{1 + \varepsilon} \log\left(\frac{r}{1 + \varepsilon - r}\right) - \frac{r}{2} - \frac{(1 + \varepsilon)}{2} \log(1 + \varepsilon - r) + K_2,$$

where the constants  $K_1$  and  $K_2$  can be retrieved from the boundary conditions. From the condition

$$\varepsilon \frac{\partial u_\varepsilon}{\partial r}(1) = u_\varepsilon(1),$$

we get

$$K_1 + \frac{1}{2} = \frac{K_1}{1 + \varepsilon} \log\left(\frac{1}{\varepsilon}\right) - \frac{1}{2} - \frac{(1 + \varepsilon)}{2} \log(\varepsilon) + K_2,$$

and the relation

$$K_1 \left(1 + \frac{\log(\varepsilon)}{1 + \varepsilon}\right) + 1 = K_2 - \frac{(1 + \varepsilon)}{2} \log(\varepsilon).$$

We want  $u_\varepsilon \in H^1(\Omega)$ . In particular, as  $u'_\varepsilon(r) \sim \frac{K_1}{(1 + \varepsilon)r}$ , we have to take  $K_1 = 0$  to assure that  $u'_\varepsilon \in L^2(\Omega)$ . It gives :

$$K_2 = 1 + \frac{1 + \varepsilon}{2} \log(\varepsilon),$$

and

$$(4.7) \quad u_\varepsilon(r) = 1 - \frac{r}{2} + \frac{(1 + \varepsilon)}{2} \log\left(\frac{\varepsilon}{1 + \varepsilon - r}\right) = 1 - \frac{r}{2} + \frac{(1 + \varepsilon)}{2} \log\left(1 + \frac{r - 1}{1 + \varepsilon - r}\right).$$

We observe that in this case, the solution blows up everywhere when  $\varepsilon \rightarrow 0$ .

### 4.3 Example of a case with a limit

We show how to choose the source term by directly finding an appropriate  $\mathbf{g}$  to satisfy (2.3), (2.4) and (2.5) in order to pass to the limit. The polar basis  $(\mathbf{e}_r, \mathbf{e}_\theta)$  is defined by

$$(4.8) \quad \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{pmatrix}.$$

We are looking for  $\mathbf{g}$  such that

$$\mathbf{g} \cdot \mathbf{n}_\Gamma = 0,$$

To avoid any singularities at 0, we take

$$\mathbf{g} = r \varrho(r) \mathbf{e}_r,$$

Therefore,  $f = \nabla \cdot \mathbf{g}$  is given by:

$$\begin{aligned} f(r) &= \frac{1}{r} \left( \frac{\partial}{\partial r} (r^2 \varrho(r)) \right) \\ &= 2\varrho(r) - r. \end{aligned}$$

We take this function as a source term. Using the same notation as before, this gives

$$\frac{K'_1(r)}{r \varrho_\varepsilon(r)} = \frac{f(r)}{\varrho_\varepsilon(r)},$$



and

$$K_1'(r) = 2r - 3r^2.$$

Thus,  $K_1(r) = r^2 - r^3 = r^2 \varrho(r)$ , which gives as a solution:

$$(4.9) \quad u_\varepsilon'(r) = r \frac{\varrho(r)}{\varrho_\varepsilon(r)}.$$

In particular,  $u_\varepsilon'(1) = 0$ . The boundary condition implies that  $u_\varepsilon(1) = 0$ . We finally get

$$(4.10) \quad u_\varepsilon(r) = - \int_1^r r \frac{\varrho(r)}{\varrho_\varepsilon(r)} dr,$$

and by the Lebesgue dominated convergence theorem,

$$(4.11) \quad u_\varepsilon(r) \xrightarrow{\varepsilon \rightarrow 0} \frac{1 - r^2}{2},$$

which is a  $C^\infty$  function that satisfies  $u = 0$  at  $\Gamma = \{r = 1\}$ .

#### 4.4 A general class of solutions

We consider now that the source  $f$  depends on  $\varepsilon$ . Let  $\alpha$  and  $\beta$  be two real numbers, with  $\alpha \geq 0$  and  $\beta \geq 1$ . The idea is to take the source  $f_\varepsilon$  such that the product  $r f_\varepsilon(r)$  gives a derivative with "good" properties. We define the source  $f = f_{\varepsilon, \alpha, \beta}$  for every  $r$  in  $(0, 1)$  as

$$f_\varepsilon(r) = (\alpha + 1)r^{\alpha-1}(1 + \varepsilon - r)^\beta - \beta(1 + \varepsilon - r)^{\beta-1}r^\alpha.$$

In particular, when doing the variation of constants method, we obtain

$$K_1'(r) = f_\varepsilon(r)r = (\alpha + 1)r^\alpha(1 + \varepsilon - r)^\beta - \beta(1 + \varepsilon - r)^{\beta-1}r^{\alpha+1},$$

which gives

$$K_1(r) = r^{\alpha+1}(1 + \varepsilon - r)^\beta,$$

and the solutions of (4.2) are the functions

$$r \mapsto \frac{K_1}{r(1 + \varepsilon - r)} + r^\alpha(1 + \varepsilon - r)^{\beta-1}.$$

The integrability of  $|u_\varepsilon'|^2$  implies that  $K_1 = 0$ , so  $u_\varepsilon'$  is given by

$$(4.12) \quad u_\varepsilon'(r) = r^\alpha(1 + \varepsilon - r)^{\beta-1}.$$

If  $\beta$  is an integer, we can give an explicit expression of  $u$ . We will make this assumption in what follows and rename  $\beta = k$  to make it more clear. We can write

$$u_\varepsilon'(r) = \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i r^{\alpha+i} (1 + \varepsilon)^{k-1-i},$$

which gives by integration

$$(4.13) \quad u_\varepsilon(r) = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^i}{\alpha + i + 1} r^{\alpha+i+1} (1 + \varepsilon)^{k-1-i} + K_2,$$

where  $K_2 \in \mathbb{R}$  is a constant we will determine thanks to the boundary condition  $\varepsilon u'_\varepsilon(1) = u_\varepsilon(1)$ . On the one hand,

$$\varepsilon u'_\varepsilon(1) = \varepsilon^k,$$

on the other hand,

$$u_\varepsilon(1) = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^i}{\alpha+i+1} (1+\varepsilon)^{k-i-1} + K_2.$$

Thus,  $K_2$  is a polynomial on  $\varepsilon$  of degree  $k$ :

$$K_2 = \varepsilon^k + \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^{i+1}}{\alpha+i+1} (1+\varepsilon)^{k-i-1} = \frac{(-1)^k}{\alpha+k} + O(\varepsilon).$$

For instance, if we take  $k = 1$ , we have:

$$u_\varepsilon(r) = \frac{r^{\alpha+1}}{\alpha+1} + \varepsilon - \frac{1}{\alpha+1}$$

and for  $k = 2$ :

$$u_\varepsilon(r) = -\frac{r^{\alpha+2}}{\alpha+2} + \left(\frac{1+\varepsilon}{1+\alpha}\right) r^{\alpha+1} + \varepsilon^2 - \frac{1+\varepsilon}{1+\alpha} + \frac{1}{\alpha+2}.$$

We can show that when  $\varepsilon$  goes to 0,  $u_\varepsilon(r)$  goes to the limit function  $r \mapsto \frac{(-1)^k}{\alpha+k} (1 - r^{\alpha+k})$ . In all cases, it vanishes at the boundary.

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