Proving the Convergence to Limit Cycles using Periodically Decreasing Jacobian Matrix Measures

Jawher Jerray, Laurent Fribourg

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Jawher Jerray[1] Laurent Fribourg[2]

***Abstract*— Methods based on "(Jacobian) matrix measure" to show the convergence of a dynamical system to a limit cycle (LC), generally assume that the measure is negative everywhere on the LC. We relax this assumption by assuming that the matrix measure is negative "on average" over one period of LC. Using an approximate Euler trajectory, we thus present a method that guarantees the LC existence, and allows us to construct a basin of attraction. This is illustrated on the example of the Van der Pol system.**

I. Introduction

Consider the nonlinear dynamical system defined by

$$\dot{x}(t) = f(x(t)). \tag{1}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ and $x(0) = x_0 \in \mathbb{R}^n$ as initial condition. Let $\xi_{x_0}(t)$ denote the solution of (1) at time t with initial condition $\xi_{x_0}(0) = x_0$.

For a long time, methods based on the notion of (Jacobian) matrix measure (noted as $\mu_P(\cdot)$) have been used to show the convergence of a solution $\xi_{x_0}(t)$ to a stationary point (see e.g. [1], [2], [3]). These methods essentially assume that a bound c is known for the measure of the (transverse) Jacobian matrix $J(x)$ with respect to x for any x belonging to a Ω forward invariant space. Formally, there exists $c \in \mathbb{R}$ such that, for all $x \in \Omega$:

$$\mu_P(J(x)) \leqslant c. \tag{2}$$

Under these conditions, we have the following information about the distance between two solutions $\xi_{x_0}(t)$ and $\xi_{y_0}(t)$ starting from two different initial conditions x_0 and y_0:

$$\text{if } |y_0 - x_0| \leqslant \varepsilon_0 \text{ then } |\xi_{y_0}(t) - \xi_{x_0}(t)| \leqslant \varepsilon_0 e^{ct},$$

for all $t \geqslant 0$ (see, e.g., [4], [5]).

Similarly, methods based on the transverse component $\mu_P^{\perp}(\cdot)$ of $\mu_p(\cdot)$ are used to show the convergence of $\xi_{x_0}(t)$ towards a limit cycle Γ (see e.g. [6]). Equation (2) becomes, for all $x \in \Omega$:

$$\mu_P^{\perp}(J(x)) \leqslant c. \tag{3}$$

If $c < 0$, the system is said to be *contractive*: The solutions $\xi_{y_0}(t)$ and $\xi_{x_0}(t)$ converge asymptotically to each other. Within the framework of periodic systems, the transverse contractivity leads to the existence and uniqueness of a limit cycle Γ inside Ω. (see, e.g., [6]). These methods of proof of convergence consist essentially in finding a positive definite matrix P (possibly depending on x) ensuring (3) with $c < 0$ for any point x. The discovery of such a matrix can be done by solving a convex optimization problem (via a linear matrix inequality or polynomial sum of squares, see [6]), but such a problem of optimization does not always have a solution.

We propose here to relax the criterion (3) by allowing that $\mu_P^{\perp}(J(x))$ can be locally $\geqslant 0$ on Γ, provided that $\mu_P^{\perp}(J(x))$ is negative on average on Γ, i.e.:

$$\int_0^T \mu_P^{\perp}(J(\xi_{x*}(t)))dt \leqslant c < 0, \tag{4}$$

where T is the period of Γ, and x^* a point of Γ.

We give a set-based criterion, based on the use of an Euler trajectory $\tilde{x}(t)$ of initial condition $\tilde{x}_0$. More precisely, we give an upperbound $\delta(t)$ of the error $e_{y_0}(t) = |\tilde{x}(t) - \xi_{y_0}(t)|$ where the initial point y_0 of the exact solution $\xi_{y_0}(t)$ is located in the vicinity of $\tilde{x}_0(t)$. By showing that $\delta(t)$ decreases at each round of the Euler trajectory, we prove the contraction "in average" of the system, thus highlighting the presence of a limit cycle Γ in the vicinity of $\tilde{x}(t)$ (see Theorem 1). This also allows us to construct an invariant zone $\mathcal{C}$ around Γ.

Thanks to an additional numeric criterion, which is a discrete version of (4) (see (28), Section IV) we then determine a basin of attraction of Γ. See Theorem 2. The method is illustrated on the example of the Van der Pol system.

Note that, although our method can be defined using a norm $|x|_P := \sqrt{x^\top P x}$ for a symmetric positive definite matrix P, we restrict ourselves in the following to the case where P is the identity matrix I, the norm $|\cdot|_I$ (just denoted $|\cdot|$) is the Euclidean norm, and $\mu_I(\cdot)$ (just denoted $\mu(\cdot)$) is the associated matrix measure (see Section II).

In summary, our contribution is to

- give a sufficient set-based condition (see (18)) that guarantees the existence of a limit cycle Γ, and allows us to construct an invariant zone $\mathcal{C}$ around Γ,
- give an additional numeric condition (28) (which is a discrete version of (4)) that allows us to determine a basin of attraction of Γ,
- illustrate these points on the Van der Pol example.

Plan of the paper

After some preliminaries (Section II), we give a criterion that guarantees the existence of a limit cycle Γ (Section III). We then give an additional condition that allows us to determine a basin of attraction of Γ (Section IV). We conclude in Section V.

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II. Preliminaries

We denote by $\mathbb{R}$ and $\mathbb{N}$ the set of real and natural numbers, respectively. These symbols are annotated with subscripts to restrict them in the usual way, e.g., $\mathbb{R}_{\geqslant 0}$ denotes the non-negative real numbers. We denote by $\mathbb{R}^n$ a n-dimensional Euclidean space. For a matrix A, we denote by $A^\top$ the transpose of A. The Euclidean norm is denoted by $|\cdot|$. The ball of center $x \in \mathbb{R}^n$ and radius $\delta \in \mathbb{R}_{\geqslant 0}$ is denoted by $B(x,\delta)$ (i.e., $B(x,\delta) = \{y \in \mathbb{R}^n : |x-y| \leqslant \delta\}$). The distance $d(x,\Gamma)$ of $x \in \mathbb{R}^n$ to $\Gamma \subset \mathbb{R}^n$ is defined as $\inf\{|x-y| : y \in \Gamma\}$. A set of successive integers of the form $\{1,2,\dots,k\}$ is abbreviated as $[k]$. The scalar product of $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ is written $\langle x,y\rangle$. Let $J(x)$ be the Jacobian matrix of the vector field $f(x)$. Among the eigenvectors of matrix $\frac{J(x)+J^\top(x)}{2}$, let $v_0(x)$ (resp. $\mathcal{E}_0(x)$) be the tangent eigenvector (resp. associated eigenvalue) led by $f(x)$, and $v_1(x),\dots,v_{n-1}(x)$ (resp. $\mathcal{E}_1(x),\dots,\mathcal{E}_n(x)$) be the other eigenvectors (resp. associated eigenvalues). Let

$$\mu(J(x)) = \max_{i=0,1,\dots,n-1} \mathcal{E}_i(x). \tag{5}$$

Given a global bound λ on $\mu(J(x))$, we know that all trajectories of (1) with initial conditions in $B(x_0,\delta_0)$ lie in $B(\xi_{x_0}(t),\delta_0 e^{\lambda t})$ (see, e.g., [4], [5]). If $\lambda < 0$ then the system (1) is said to be *contracting* (cf. [7]). Let

$$\mu^\perp(J(x)) = \max_{i=1,\dots,n-1} \mathcal{E}_i(x). \tag{6}$$

Note that, while $i = 0$ belongs to the index domain of (5), the index $i = 0$ is discarded in (6) because it corresponds to the tangent direction.

Let L denote the Lipschitz constant of vector field f, and M_f an upperbound on magnitude of f (i.e: $M_f \geqslant |f(z)|$ for all $z \in \Omega$).

We denote by $\tilde{x}_i$ the Euler discretization of (1) at time $t_i = ih$, for $i \in \mathbb{N}$, where h is the time-step size. Given an initial point $\tilde{x}_0 \in \mathbb{R}^n$, $\tilde{x}_{i+1}$ is defined, for $i \in \mathbb{N}$, by:

$$\tilde{x}_{i+1} = \tilde{x}_i + hf(\tilde{x}_i). \tag{7}$$

For $s \in [0,h]$, $t = ih + s$, let $\tilde{x}_i(s) = \tilde{x}_i + sf(\tilde{x}_i)$ (so $\tilde{x}_i(0) = \tilde{x}_i$, $\tilde{x}_i(h) = \tilde{x}_{i+1}$), and $\tilde{x}(t) = \tilde{x}_i(s)$.

III. Proof of existence of a limit cycle

We now give a sufficient condition to guarantee the existence of a limit cycle Γ solution of (1), and construct a forward invariant zone around Γ.

Consider an Euler trajectory $\tilde{\Gamma}$, i.e. a solution $\tilde{x}(t)$ of (7) with time-step $h \in \mathbb{R}_{>0}$ and initial condition $\tilde{x}_0 \in \mathbb{R}^n$. Let S_0 be the hyperplan through $\tilde{x}_0$ orthogonal to $f(\tilde{x}_0)$. Let $\tilde{x}_i = \tilde{x}(ih)$, for $i \in \mathbb{N}$. We denote by S_i the plan passing through $\tilde{x}_i$ and orthogonal to $f(\tilde{x}_i)$. Likewise, for $s \in [0,h)$, $S_i(s)$ is the plan through $\tilde{x}_i(s)$ orthogonal to $f(\tilde{x}_i(s))$.

We suppose that $\tilde{x}(t)$ returns for the first time (in the good direction) to S_0 at time $t = \tilde{R}_1 \in ((N_1-1)h, N_1 h]$ for some $N_1 \in \mathbb{N}_{>0}$. More generally, we suppose that $\tilde{x}(t)$ returns for the p-th time ($p \in \mathbb{N}_{>0}$) to S_0 at time $t = \tilde{R}_p \in ((N_p-1)h, N_p h]$ for some $N_p \in \mathbb{N}_{>0}$, so we have:

$$\tilde{x}(\tilde{R}_p) \in S_0 \text{ and } f(\tilde{x}_0)^\top f(\tilde{x}(\tilde{R}_p)) > 0.$$

By convention, let $\tilde{R}_0 = 0$. Note that S_0 is "between" S_{N_p-1} and S_{N_p} for all $p \in \mathbb{N}_{>0}$ (see Figure 2, Section III).

Given $\delta_0 \in \mathbb{R}_{>0}$ and $y_0 \in B(\tilde{x}_0,\delta_0) \cap S_0$, we suppose that $\xi_{y_0}(t)$ returns to S_0 for the 1st time at $t = T_{y_0}(1) \in \mathbb{R}_{>0}$. So we have:

$$\xi_{y_0}(T_{y_0}(1)) \in S_0 \text{ and } f(\tilde{x}_0)^\top f(\xi_{y_0}(T_{y_0}(1))) > 0.$$

We will suppose that there exists $\eta \in \mathbb{R}_{>0}$ such that:

$$T_{y_0}(1) \geqslant \eta > 0 \text{ for all } y_0 \in B(\tilde{x}_0,\delta_0) \cap S_0. \tag{8}$$

This will be used in Theorem 1 to show $T_{y_0}(p) \to \infty$ as $p \to \infty$, where $T_{y_0}(p)$ is the p-th return time of $\xi_{y_0}(t)$ to S_0.

Along the lines of [3], we now synchronize the time between $\tilde{x}_i(t)$ and the corresponding solution $\xi_{y_i}(t)$ of (1) with $y_i \in B(\tilde{x}_i,\delta_i) \cap S_i$. More precisely, we consider the time-reparametrisation $\theta_{y_i}(\cdot)$ defined so that $\xi_{y_i}(\theta_{y_i}(t))$ belongs to $S_i(t)$. Hence, given two adjacent points $\tilde{x}_i \in S_i$ and $y_i \in S_i$ with $(\tilde{x}_i - y_i)$ perpendicular to $f(\tilde{x}_i)$, we define $\theta_{y_i}(t)$ for all $s \in [0,h)$ in an "implicit" manner as follows:

$$(\xi_{y_i}(\theta_{y_i}(s)) - \tilde{x}_i(s))^\top f(\tilde{x}_i(s)) = 0. \tag{9}$$

This is possible due to the implicit function theorem if h and $|y_i - x_i|$ are sufficiently small (see [3], Section 2.1). Using (9), we define $\theta_{y_i}(s)$ with $\theta_{y_i}(0) = y_0$, and $y_{i+1} = \xi_{y_i}(\theta_i(h))$. Note that $y_i \in S_i$ for all $i \in \mathbb{N}$. We can now define $\Theta_{y_0}(\cdot)$ so that $\xi_{y_0}(\Theta_{y_0}(t))$ is synchronized with the solution $\tilde{x}(t)$ of (7) (with initial condition $\tilde{x}_0$). Formally, given $y_0 \in \mathbb{R}^n$, $\Theta_{y_0}(0) := y_0$ and $\Theta_{y_0}(t) := \theta_{y_i}(s)$ for $t = ih + s$ with $s \in (0,h]$. Note $\xi_{y_0}(\Theta_{y_0}(ih)) = y_i \in S_i$ for all $i \in \mathbb{N}$. We will ensure $\Theta_{y_0}(t) \to \infty$ as $t \to \infty$ thanks to assumption (8) coupled with assumption (18) (see Theorem 1).

We now define $a_{i+1}(y_0)$ and $b_{i+1}(y_0)$ ($i \in \mathbb{N}$), or for the sake of notation simplicity, just a_{i+1} and b_{i+1}, as follows:

Definition 1. For $i \in \mathbb{N}$, let:

$$0 < a_{i+1} \leqslant \inf_{s\in[0,h]} \dot{\theta}_{y_i}(s), \qquad b_{i+1} \geqslant \sup_{s\in[0,h]} \dot{\theta}_{y_i}(s).$$

Remark 1. The existence of a_{i+1}, b_{i+1} for all $i \in \mathbb{N}$ follows from the assumptions that the 1st return times of $\tilde{x}_0(t)$ and $\xi_{y_0}(t)$ to S_0 are *finite* (i.e., $\tilde{R}_1 \in \mathbb{R}_{>0}$ and $T_{y_0}(1) \in \mathbb{R}_{>0}$ for all $y_0 \in B(\tilde{x}_0,\delta_0) \cap S_0$) together with conditions (8)-(18) (see Theorem 1). The bounds a_i and b_i can be computed using the implicit function theorem by time derivation of (9) (see [3], Section 2.1). The assumption that $a_{i+1} > 0$ (i.e., $\inf_{s\in[0,h]} \dot{\theta}_{y_i}(s) > 0$) is true when h is sufficiently small and a lowerbound $m > 0$ exists on the magnitude of vector field f (i.e., $|f(x)| > m$ for all $x \in \Omega$).

We now define α_i, Y_i and Z_{i+1} along the lines of [4] (Algorithm 1), as follows:

Definition 2.
- Let
  $\alpha_0 = \delta_0$,
  $\alpha_{i+1} = \alpha_i + b_{i+1} M_f h$ for $i \in \mathbb{N}$,
  More generally let
  $\alpha_i(s) = \alpha_i + b_{i+1} M_f s$ for $i \in \mathbb{N}, s \in [0,h]$.

- Let $Y_i = B(\tilde{x}_i, \alpha_i) \cap S_i$, for all $i \in \mathbb{N}$.
  More generally let:
  $Y_i(s) = B(\tilde{x}_i(s), \alpha_i(s)) \cap S_i(s)$ for all $i \in \mathbb{N}, s \in [0,h]$.
- Let $Z_{i+1} = \bigcup_{s\in[0,h]} Y_i(s)$ for all $i \in \mathbb{N}$.

Remark 2. Note that $Y_0 = B(\tilde{x}_0, \delta_0) \cap S_0$ and $Y_{i+1} = Z_{i+1} \cap S_{i+1}$. Note also that, given $y_0 \in Y_0$, we have: $\xi_{y_i}(\theta_{y_i}(s)) \in Z_{i+1}$ for all $i \in \mathbb{N}, s \in [0,h]$. We have also: $\xi_{y_0}(\Theta_{y_0}(t)) \in Z_{i+1}$ for all $t = ih + s$ with $s \in [0,h]$. This implies that Z_{i+1} is an overapproximation of the "reachability" set $\{z \in \mathbb{R}^n |\ z = \xi_{y_0}(\Theta_{y_0}(ih+s))$ for some $y_0 \in Y_0$ and $s \in [0,h]\}$.

Let us now define $\Lambda_{i+1} \in \mathbb{R}$ ($i \in \mathbb{N}$), as follows:

$$\Lambda_{i+1} \geqslant \sup_{z\in Z_{i+1}} \mu^{\perp}[J(z)],$$

Note that Λ_{i+1} satisfies for all $z_1, z_2 \in Z_{i+1}$ (see e.g. [1]):

$$\langle f(z_1) - f(z_2), z_1 - z_2\rangle \leqslant \Lambda_{i+1}|z_1 - z_2|^2. \qquad (10)$$

Let $\gamma > 0$ be a positive real (arbitrarily chosen).

Definition 3. For all $i \in \mathbb{N}$, let

$\sigma_{i+1} = \frac{1}{2}a_{i+1}\Lambda_{i+1}$ if $\Lambda_{i+1} < -\gamma$,
$\sigma_{i+1} = \frac{3}{2}b_{i+1}\max(|\Lambda_{i+1}|, \gamma)$ if $\Lambda_{i+1} \geqslant -\gamma$.

Remark 3. Note that $a_i, b_i, Z_i, \Lambda_i, \sigma_i$ are *not* defined for $i = 0$. The constant σ_{i+1} is a conservative approximation of the matrix measure Λ_{i+1} on Z_{i+1}. The constant γ is used to isolate the problematic neighborhood of $\mu^{\perp}[J(z)]$ around 0, (and get a *positive* lowerbound of $|\sigma_{i+1}|$): If z is such that $|\mu^{\perp}[J(z)]| < \gamma$, then $|\mu^{\perp}[J(z)]|$ is replaced by γ. It follows from the definition that for all $i \in \mathbb{N}$:

$$|\sigma_{i+1}| \geqslant \frac{\gamma}{2}a_{i+1} > 0 \qquad (11)$$

Given δ_0, let us define δ_i, $\delta(t)$ and $\tilde{e}_{y_i}, \tilde{e}_{y_0}(t)$ as follows:

Definition 4. Let $\tilde{e}_{y_0} = |\tilde{x}_0 - y_0|$, and for $i \in \mathbb{N}$ let:
$\delta_{i+1} = \delta_i e^{\sigma_{i+1}h}$, $\tilde{e}_{y_{i+1}} = |\tilde{x}_{i+1} - y_{i+1}|$.
More generally for $i \in \mathbb{N}, s \in [0,h], t = ih + s$, let:
$\delta_i(s) = \delta_i e^{\sigma_{i+1}s}$ $\tilde{e}_{y_i}(s) = |\tilde{x}_i(s) - y_i(\theta_{y_i}(s))|$,
$\delta(t) = \delta_i(s)$, $\tilde{e}_{y_0}(t) = |\tilde{x}(t) - \xi_{y_0}(\Theta_{y_0}(t))|$.

Remark 4. For $i \in \mathbb{N}$ and $s \in [0,h]$, we have:
$\delta(ih + s) = \delta_i(s) = \delta_0 e^{h\Sigma_{k=1}^{i}\sigma_k + s\sigma_{i+1}}$
if we adopt the convention $\Sigma_{k=1}^{i}\sigma_k = 0$ for $i = 0$.

We now show that, under certain condition, $\tilde{e}_{y_0}(t) \leqslant \delta(t)$.

Proposition 1. *Let $i \in \mathbb{N}, y_i \in Y_i$.*
If $\tilde{e}_{y_i} \leqslant \delta_i$, we have:

$$\tilde{e}_{y_i}(s) \leqslant \max(\delta_i(s), h(\tilde{M}_{i+1}(\frac{2L}{\gamma a_{i+1}} + 1) + b_{i+1}M_f)) \quad (12)$$

for all $s \in [0,h]$, where $\tilde{M}_{i+1}$ an upperbound of $|f(\tilde{x}_i(s))|$ on $s \in [0,h]$.

Furthermore, for all $i \in \mathbb{N}_{>0}, s \in [0,h], y_0 \in Y_0$:

$$\tilde{e}_{y_{i-1}}(s) \leqslant \max(\delta_{i-1}(s), h(\tilde{M}_i(\frac{2L}{\gamma a_i} + 1) + b_i M_f)). \quad (13)$$

Proof. The proof of (12) is an adaptation of the proof of Theorem 1 of [8] to the context of *transverse* contraction.

Consider $y_i \in Y_i$. Let: $\tilde{e}_{y_i}(t) = |\xi_{y_i}(\theta_{y_i}(t)) - \tilde{x}_i(t)|$ for $t \in [0,h]$. Hence $\tilde{e}_{y_i}(0) = |y_i - \tilde{x}_i| \leqslant \delta_i$. For the sake of simplicity, we will write $\tilde{x}_t$ instead of $\tilde{x}_i(t)$, $y_{\theta t}$ instead of $\xi_{y_i}(\theta_{y_i}(t))$, $\dot{\theta}_t$ instead of $\dot{\theta}_{y_i}(t)$.

For all $t \in [0,h]$, we have by (10):

$$\langle f(y_{\theta t}) - f(\tilde{x}_t), y_{\theta t} - \tilde{x}_t\rangle \leqslant \Lambda_{i+1}|y_{\theta t} - \tilde{x}_t|^2 \qquad (14)$$

since $y_{\theta t}, \tilde{x}_t \in Z_{i+1}$. So for all $t \in [0,h]$, $\tilde{e}_{y_i}(t) = |y_{\theta t} - \tilde{x}_t|$ satisfies:

$$\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}(\tilde{e}_{y_i}^2(t)) &= \langle \dot{\theta}_t f(y_{\theta t}) - f(\tilde{x}_i), y_{\theta t} - \tilde{x}_t\rangle \\
&= \dot{\theta}_t\langle f(y_{\theta t}) - f(\tilde{x}_t), y_{\theta t} - \tilde{x}_t\rangle \\
&\qquad + \langle \dot{\theta}_t f(\tilde{x}_t) - f(\tilde{x}_i), y_{\theta t} - \tilde{x}_t\rangle \\
&= \dot{\theta}_t\langle f(y_{\theta t}) - f(\tilde{x}_t), y_{\theta t} - \tilde{x}_t\rangle + \langle -f(\tilde{x}_i), y_{\theta t} - \tilde{x}_t\rangle \\
&\qquad \text{(using the fact } f(\tilde{x}_t) \perp (y_{\theta t} - \tilde{x}_t)) \\
&\leqslant \dot{\theta}_t\Lambda_{i+1}\tilde{e}_{y_i}^2(t) + \langle -f(\tilde{x}_i), y_{\theta t} - \tilde{x}_t\rangle \qquad \text{(using (14))} \\
&\leqslant \dot{\theta}_t\Lambda_{i+1}\tilde{e}_{y_i}^2(t) + L|f(\tilde{x}_i)|t\tilde{e}_{y_i}(t) \\
&\qquad \text{(because, using again } f(\tilde{x}_t) \perp (y_{\theta t} - \tilde{x}_t)\text{:} \\
&\qquad |\langle -f(\tilde{x}_i), y_{\theta t} - \tilde{x}_t\rangle| = |\langle f(\tilde{x}_t) - f(\tilde{x}_i), y_{\theta t} - \tilde{x}_t\rangle| \\
&\qquad \leqslant L|\tilde{x}_t - \tilde{x}_i||y_{\theta t} - \tilde{x}_t| \leqslant Lt|f(\tilde{x}_i)|\tilde{e}_{y_i}(t)
\end{aligned}$$

Hence for all $t \in [0,h]$:

$$\frac{1}{2}\frac{d}{dt}(\tilde{e}_{y_i}^2(t)) \leqslant \rho_{i+1}\Lambda_{i+1}\tilde{e}_{y_i}^2(t) + L|f(\tilde{x}_i)|t\tilde{e}_{y_i}(t) \qquad (15)$$

(with $\rho_{i+1} = a_{i+1}$ if $\Lambda_{i+1} < -\gamma$, $\rho_{i+1} = b_{i+1}$ otherwise).
By subtracting $L|f(\tilde{x}_i)|t\tilde{e}_{y_i}(t)$ from $\rho_{i+1}\Lambda_{i+1}\tilde{e}_{y_i}^2(t)$ in the right-hand side of (15), we get using (11)

If $h \leqslant \frac{\gamma a_{i+1}}{2L|f(\tilde{x}_i)|}\tilde{e}_{y_i}(t)$ for all $t \in (0,h]$, then
$\frac{1}{2}\frac{d}{dt}(\tilde{e}_{y_i}^2(t)) \leqslant \sigma_{i+1}\tilde{e}_{y_i}^2(t)$.

It then follows by integration that, for all $i \in \mathbb{N}, y_i \in Y_i$:
If $h \leqslant \frac{\gamma a_{i+1}}{2L|f(\tilde{x}_i)|}\tilde{e}_{y_i}(s)$ for all $s \in [0,h]$, then, assuming $\tilde{e}_{y_i} \leqslant \delta_i$:

$$\tilde{e}_{y_i}(s) \leqslant \delta_i e^{\sigma_{i+1}s} = \delta_i(s) \text{ for all } s \in [0,h]. \qquad (16)$$

Suppose that the hypothesis $h \leqslant \frac{\gamma a_{i+1}}{2L|f(\tilde{x}_i)|}\tilde{e}_{y_i}(s)$ is not satisfied for some $s_0 \in [0,h]$, i.e: $\tilde{e}_{y_i}(s_0) < Ch$ where $C = \frac{2L|f(\tilde{x}_i)|}{\gamma a_{i+1}}$. It then follows for all $s \in [0,h]$:

$$\begin{aligned}
&\tilde{e}_{y_i}(s) < Ch + |\tilde{e}_{y_i}(s_0) - e_{y_i}(s)| \\
&\quad \leqslant Ch + |\xi_{y_i}(\theta_{y_i}(s)) - \xi_{y_i}(\theta_{y_i}(s_0))| + |\tilde{x}_i(s) - \tilde{x}_i(s_0)| \\
&\quad \leqslant Ch + hb_{i+1}M_f + h\tilde{M}_{i+1}.
\end{aligned}$$

Using (16), we have then for all $s \in [0,h]$:
$\tilde{e}_{y_i}(s) \leqslant \max(\delta_i(s), h\tilde{M}_{i+1}(\frac{2L}{\gamma a_{i+1}} + 1) + b_{i+1}M_f)$, i.e. (12).

By induction on i, using the fact that $\tilde{e}_{y_0} = |y_0 - \tilde{x}_0| \leqslant \delta_0$ (since $y_0 \in Y_0 \subseteq B(\tilde{x}_0, \delta_0)$), we prove (13). □

Definition 5. For all $i \in \mathbb{N}$, let

$$\mathcal{C}_{i+1} = \bigcup_{s\in[0,h]} B(\tilde{x}_i(s), \delta_i(s)) \cap S_i(s).$$

Besides

$$\mathcal{C} = \bigcup_{i=1}^{N_1} \mathcal{C}_i,$$

where $N_1 = \lceil \frac{\tilde{R}_1}{h} \rceil$ and $\tilde{R}_1 \in \mathbb{R}_{>0}$ is the 1st return time of $\tilde{x}(t)$ to S_0.

Using Proposition 1, we now show that a simple test of set inclusion (see (18)) guarantees the existence of a limit cycle.

Theorem 1. *Consider an Euler trajectory $\tilde{\Gamma}$ and $\delta_0 \in \mathbb{R}_{>0}$. Suppose that, for all $i \in [N_1], s \in [0,h]$:*

$$\delta_{i-1}(s) \geqslant h(\tilde{M}_i(\frac{2L}{\gamma a_i}+1)+b_i M_f), \tag{17}$$

and

$$\mathcal{C}_{N_1} \cap S_0 \subseteq Y_0. \tag{18}$$

Suppose furthermore that there exists $\eta \in \mathbb{R}_{>0}$ such that (8) holds. Then

- *$\mathcal{C}$ is forward invariant w.r.t. Y_0, i.e.: For all $t \in [0,\infty), y \in Y_0$, we have $\xi_y(t) \in \mathcal{C}$, and*
- *$\mathcal{C}$ contains a limit cycle Γ.*

Proof. From (13) and (17), we deduce: $e_{y_0}(t) \leqslant \delta(t)$ for all $y_0 \in Y_0, t \in [0, N_1 h]$. In particular, at the time of the 1st return of $\tilde{x}(t)$ to S_0 at $t = \tilde{R}_1 \in ((N_1-1)h, N_1 h]$, we have $e_{y_0}(\tilde{R}_1) \leqslant \delta(\tilde{R}_1)$. So: $e_{y_0}(\tilde{R}_1) = |y_1 - \tilde{x}_1| \leqslant \delta(\tilde{R}_1)$, where $\tilde{x}_1 = \tilde{x}(\tilde{R}_1)$ and $y_1 = \xi_{y_0}(T_{y_0}(1))$ are the 1st return points of $\tilde{x}(t)$ and $\xi_{y_0}(t)$ to S_0 respectively.

From the definition of $\mathcal{C}_{N_1}$ we can deduce that $y_1 \in \mathcal{C}_{N_1} \cap S_0$. Therefore, using (18), we have: $y_1 \in Y_0$. Thus for all $y \in Y_0$, we have $y' = \xi_{y_0}(T_y(1)) \in Y_0$. This shows by the Brouwer fixed-point theorem applied to the continuous function: $y \in Y_0 \mapsto \xi_y(T_y(1))$, that there exists $y^* \in \mathcal{C}_{N_1} \cap S_0 \subseteq Y_0$ such that $\xi_{y^*}(T_{y^*}(1)) = y^*$. We deduce that there exists a limit cycle Γ (closed solution de (1)) of period $T = T_{y^*}(1)$ which passes through $y^* \in \mathcal{C}_{N_1} \cap S_0 \subseteq Y_0$.

Observe that for all $y_0 \in Y_0$ and $t \in [0, T_{y_0}(1)]$, we have, by (13) of Proposition 1, $e_{y_k}(s) = |\tilde{x_k}(s) - \xi(\theta_{y_k}(s))| \leqslant \delta_k(s)$, hence $\xi(\theta_{y_k}(s)) \in B(\tilde{x}_k(s), \delta_k(s)) \cap S_k(s) \in \mathcal{C}_k$, and $\xi_{y_0}(t) \in \bigcup_{k=1}^{N_1} \mathcal{C}_k = \mathcal{C}$ for all $t \in [0, T_{y_0}(1)]$.
Since $y_1 = \xi_{y_0}(T_{y_0}(1)) \in \mathcal{C}_{N_1} \cap S_0 \subseteq Y_0$, we can repeat the reasoning starting from y_1, which gives $\xi_{y_1}(t) \in \mathcal{C}$ for $t \in [0, T_{y_1}(1)]$, hence $\xi_{y_0}(t) \in \mathcal{C}$ for $t \in [0, T_{y_0}(2)]$. By induction on $p \in \mathbb{N}_{>0}$, we have more generally

$$\xi_{y_0}(t) \in \mathcal{C} \text{ for all } t \in [0, T_{y_0}(p)]. \tag{19}$$

Now, using (8) and (18), we have: $T_{y_0}(p) \geqslant p\eta$ for all $p \in \mathbb{N}_{>0}$, hence $T_{y_0}(p) \to \infty$ as $p \to \infty$. So we deduce from (19): $\xi_{y_0}(t) \in \mathcal{C}$ for all $t \in [0,\infty)$. In particular, the limit cycle Γ which is the trajectory $\xi_{y^*}(t)$ for $t \in [0,\infty)$, is included in $\mathcal{C}$. □

Note that a simple numerical sufficient condition for (18) is: $|\tilde{x}(\tilde{R}_1) - \tilde{x}_0| + \delta(\tilde{R}_1) < \delta_0$.

Example 1. Consider the Van der Pol system with $p = 0.3$.

$$\begin{cases} du_1/dt = u_2 \\ du_2/dt = pu_2 - pu_1^2 u_2 - u_1 \end{cases}$$

We use the software ORBITADOR to perform the simulations (see https://perso.telecom-paristech.fr/jjerray/orbitador/). Let $h = 10^{-4}, \delta_0 = 0.1$, $\tilde{x}_0 = (1.8929; -0.5383), \gamma = 0.015$. We have $L = 1.516, M_f = 2.22, \max_{i\in[N_1]} \tilde{M}_i = 2.22$, $\tilde{R}_1 = 6.314$, $N_1 = \lceil \frac{\tilde{R}_1}{h} \rceil = 63140$, and $\min_{k\in[N_1]} a_k \geqslant 0.9$, $\max_{k\in[N_1]} b_k \leqslant 1.1$.
Besides for all $t \in [0, N_1 h]$: $\delta(t) \geqslant \delta(N_1 h) = 0.0642$.
Note that $\mu^{\perp}[J(\tilde{x}(t))] \in [-4.2, +1.2]$ for $t \in [0, \tilde{R}_1]$, so the system is *not* uniformly contractive.
We check that (17) holds, i.e, for all $i \in [N_1], s \in [0,h]$:
$\delta_{i-1}(s) \geqslant 0.0642 \geqslant h \max_{i\in[N_1]}(\tilde{M}_i(\frac{2L}{\gamma a_i}+1)+b_i M_f) \approx 0.05$.
Besides, as checked by ORBITADOR, the inclusion (18) also holds. Hence by Theorem 1, there is a limit cycle Γ contained in an invariant set $\mathcal{C}$. The inclusion (18) is highlighted in the following figures, using the time-step $h = 0.002$ (instead of $h = 10^{-4}$) in order to make the inclusion more visible. On Figure 1, the initial ball $B(\tilde{x}_0, \delta_0)$ is depicted in orange, the invariant zone $\mathcal{C}$ in red, and the set $\bigcup_{s\in[0,h]} B(\tilde{x}_{N_1-1}(s), \delta_{N_1-1}(s))$ in green. The sets $\mathcal{C}_{N_1} \cap S_{N_1-1}$, Y_0 and $\mathcal{C}_{N_1} \cap S_{N_1}$ have the form of straight line segments which are transverse to the Euler trajectory $\tilde{\Gamma}$. On Figure 1, $\tilde{\Gamma}$ is depicted in black: it starts at the middle $\tilde{x}_0$ of segment Y_0 (in blue), and turns clockwise. The segments $\mathcal{C}_{N_1} \cap S_{N_1-1}$, Y_0 and $\mathcal{C}_{N_1} \cap S_{N_1}$ appear successively from top to bottom in Figures 1 and 2. (The bottom of Figure 1 and Figure 2 gives zoom views on $\mathcal{C}_{N_1} \cap S_{N_1-1}, Y_0, \mathcal{C}_{N_1} \cap S_{N_1}$.)
We see (18): $\mathcal{C}_{N_1} \cap S_0 \subseteq B(\tilde{x}_0, \delta_0)$, where $\mathcal{C}_{N_1}$ is the part of the green zone delimited by $\mathcal{C}_{N_1} \cap S_{N_1-1}$ (fuschia) and $\mathcal{C}_{N_1} \cap S_{N_1}$ (cyan), $B(\tilde{x}_0, \delta_0)$ is the orange ball, and S_0 is the straight line extending Y_0 (blue).

IV. Attractivity of the limit cycle

We know assume that the existence of the limit cycle Γ is known. Let $y^* = \Gamma \cap S_0$ be the intersection of Γ with S_0.

So far, the Euler approximation function $\tilde{x}(t)$ has been defined implicitly with $\tilde{x}_0$ as initial condition. Let us now write $\tilde{x}(t; z)$ to denote the Euler approximation function with z as initial condition, and let us write $\sigma_k(z)$ the corresponding coefficient σ_k. For $p \in \mathbb{N}_{>0}$ and $z \in Y_0$, let also $\tilde{R}_p(z)$ be the time t taken by $\tilde{x}(t; z)$ to make p rounds, and $N_p(z) = \lceil \frac{R_p(z)}{h} \rceil$. Note that under the hypotheses of Theorem 1: $\tilde{x}_p = \tilde{x}(\tilde{R}_p(\tilde{x}_0); \tilde{x}_0) = \tilde{x}(\tilde{R}_1(\tilde{x}_{p-1}); \tilde{x}_{p-1}) \in Y_0$ for all $p \in \mathbb{N}_{>0}$.

Let us suppose that there exists $a, b \in \mathbb{R}_{>0}$ such that

$$0 < a \leqslant a_i \leqslant b_i \leqslant b \text{ for all } i \in \mathbb{N}. \tag{20}$$

This property is guaranteed if we know that all round performed by the solution of (1) is completed in a known interval of time, i.e. if there exists $T', T'' \in \mathbb{R}_{>0}$ such that

$$T' \leqslant T_1(y_0) \leqslant T'' \text{ for all } y_0 \in Y_0. \tag{21}$$

We will also use the assumption: there exists $R' \in \mathbb{R}_{>0}$ such that

$$\tilde{R}_1(z) \leqslant R' \text{ for all } z \in Y_0. \tag{22}$$

Theorem 2. *Consider an Euler trajectory $\tilde{\Gamma}$ and $\delta_0 \in \mathbb{R}_{>0}$. Suppose that the system satisfies (17) and (18). Suppose*

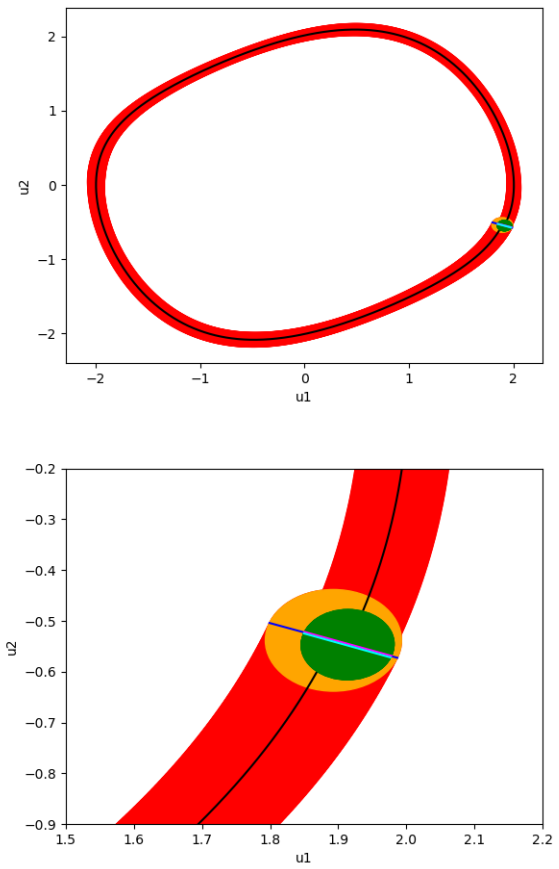

Fig. 1. Top: Euler trajectory $\tilde{\Gamma}$ (black); invariant zone $\mathcal{C}$ (red); $\bigcup_{s\in[0,h]} B(\tilde{x}_{N_1-1}(s), \delta_{N_1-1}(s))$ (green); segments $\mathcal{C}_{N_1} \cap S_{N_1-1}$ (fuschia), Y_0 (blue), and $\mathcal{C}_{N_1} \cap S_{N_1}$ (cyan). Bottom: zoom showing (18): $\mathcal{C}_{N_1} \cap S_0 \subseteq B(\tilde{x}_0, \delta_0)$, where $\mathcal{C}_{N_1}$ is the green part delimited by $\mathcal{C}_{N_1} \cap S_{N_1-1}$ and $\mathcal{C}_{N_1} \cap S_{N_1}$, and $B(\tilde{x}_0, \delta_0)$ is the orange ball.

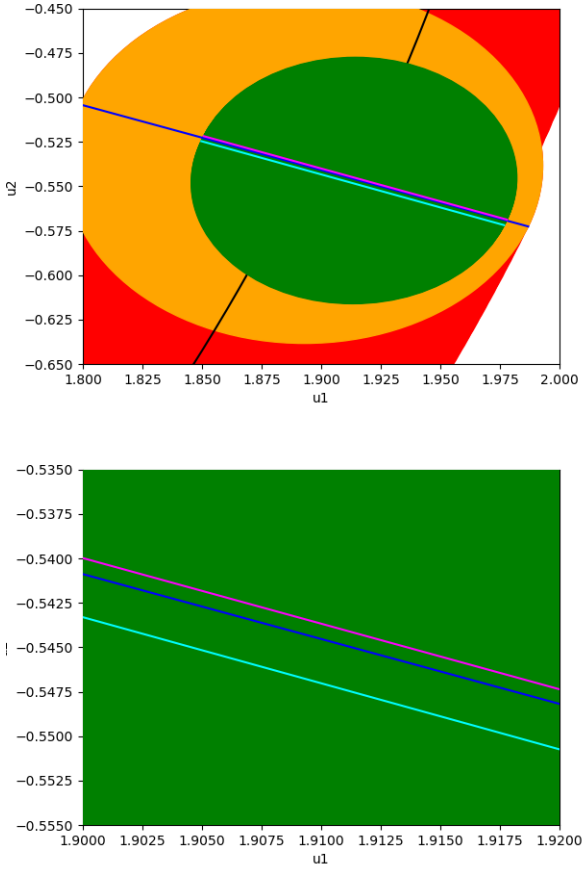

Fig. 2. Further zooms on $\mathcal{C}_{N_1}$ showing from top to bottom $\mathcal{C}_{N_1} \cap S_{N_1-1}$ (fuschia), Y_0 (blue) and $\mathcal{C}_{N_1} \cap S_{N_1}$ (cyan). We see (18): $\mathcal{C}_{N_1} \cap S_0 \subseteq B(\tilde{x}_0, \delta_0)$.

furthermore that there exists $\eta, T', T'', R' \in \mathbb{R}_{>0}$ *such that* (8), (21) *and* (22) *hold. Then we have for all* $t \in [0,\infty), y \in Y_0$*:*

$$\tilde{e}_y(t) \leqslant \max(\delta(t), Dh), \tag{23}$$

where $D = M_{\mathcal{C}}(\frac{2L}{\gamma a} + b + 1)$ *and* $M_{\mathcal{C}}$ *is an upper bound on magnitude of* f *over the elements of* $\mathcal{C}$*. Besides if*

$$\lim_{t\to\infty} \delta(t) = 0, \tag{24}$$

then

$$\limsup_{t\to\infty} \tilde{e}_y(t) \leqslant Dh, \text{ for all } y \in Y_0, \tag{25}$$

$$\limsup_{t\to\infty} d(\tilde{x}(t), \Gamma) \leqslant Dh \tag{26}$$

(hence, the Euler trajectory $\tilde{x}(t;\tilde{x}_0)$ *converges asymptotically to* Γ *as* $h \to 0$*) and:*

$$\lim_{t\to\infty} d(\xi_y(t), \Gamma) = 0 \text{ for all } y \in Y_0, \tag{27}$$

i.e., Y_0 *is a basin of attraction of* Γ*.*
A sufficient condition for (24) *to hold is:*

There exists $d \in \mathbb{R}_{<0}$ *such that*

$$h\Sigma_{k=1}^{N_1(z)-1}\sigma_k(z) + s\sigma_{N_1(z)}(z) \leqslant d < 0 \tag{28}$$

for all $z \in Y_0, s \in [0,h]$*.*

Proof. By Theorem 1, $\mathcal{C}$ is an invariant, and $M_{\mathcal{C}}$ can be taken in place of M_f and $\tilde{M}_i$ for all $i \in \mathbb{N}$. From (13) of Proposition 1 and (20) (which is implied by (21)), we then deduce: $\tilde{e}_y(t) \leqslant \max(\delta(t), M_{\mathcal{C}}h(\frac{2L}{\gamma a} + b + 1)) = \max(\delta(t), Dh)$, i.e. (23). So if $\lim_{t\to\infty} \delta(t) = 0$, $\delta(t)$ becomes after a sufficient long time $t \geqslant t_0$ less than Dh, and by (23), we have: $\tilde{e}_y(t) \leqslant Dh$ for $t \geqslant t_0$, i.e. (25). In particular, since $y^* \in Y_0$, for $y = y^*$, we have $\limsup_{t\to\infty} \tilde{e}_{y^*}(t) \leqslant Dh$, i.e. $\limsup_{t\to\infty} |\tilde{x}(t) - \xi_{y^*}(\Theta_{y^*}(t))| \leqslant Dh$, hence $\limsup_{t\to\infty} d(\tilde{x}(t), \Gamma) \leqslant D$, i.e. (26), since $\xi_{y^*}(t) \in \Gamma$ for all t. Besides we have:

$d(\xi_y(\Theta_y(t)), \Gamma)$
$\leqslant d(\xi_y(\Theta_y(t)), \tilde{x}(t)) + d(\tilde{x}(t), \Gamma)$
$\leqslant e_y(t) + d(\tilde{x}(t), \Gamma)$.

By taking the $\limsup$ of both sides of the inequality, we have:

$\limsup_{t\to\infty} d(\xi_y(\Theta_y(t)), \Gamma)$
$\leqslant \limsup_{t\to\infty} \tilde{e}_y(t) + \limsup_{t\to\infty} d(\tilde{x}(t), \Gamma)$,

hence, using (25) and (26):

$\limsup_{t\to\infty} d(\xi_y(\Theta_y(t)),\Gamma) \leqslant 2Dh$,
and since $\Theta_y(t) \to \infty$ as $t \to \infty$:
$\limsup_{t\to\infty} d(\xi_y(t),\Gamma) \leqslant 2Dh$.
Now since $d(\xi_y(t),\Gamma)$ does not depend on h, we have: $\lim_{t\to\infty} d(\xi_y(t),\Gamma) = 0$, i.e. (27).
Let us now prove that (28) implies (24). We first show by induction on p that for all $\tilde{x}_0 \in Y_0$

$$\delta(\tilde{R}_1(\tilde{x}_{p-1})) \leqslant \delta_0 e^{pd} \text{ for all } p \in \mathbb{N}_{>0}. \tag{29}$$

For the base case $p = 1$, we have for some $s \in (0,h]$
$\delta(\tilde{R}_1) \leqslant \delta_0 e^{h\Sigma_{k=1}^{N_1-1}\sigma_k + s\sigma_{N_1}} \leqslant \delta_0 e^d$ (using (28)).
For $p \geqslant 2$, we have for some $s \in (0,h]$
$\delta(\tilde{R}_p) \leqslant \delta(R_{p-1}) e^{h\Sigma_{k=1}^{N_1(\tilde{x}_{p-1})-1}\sigma_k(\tilde{x}_{p-1}) + s\sigma_{N_1(\tilde{x}_{p-1})}(\tilde{x}_{p-1})}$
$\leqslant \delta(R_{p-1}) e^d$ (using (28) and the fact that $\tilde{x}_{p-1} \in Y_0$)
$\leqslant e^{(p-1)d} e^d = e^{pd}$ (by induction hypothesis).
From (29) it then follows that $\delta(t) \leqslant \delta_0 e^{pd+B}$ for $t \in [\tilde{R}_p, \tilde{R}_{p+1}]$, where
$B = h \max_{i\in[N_1(\tilde{x}_p)-1], s\in\{0,s\}} \Sigma_{k=1}^{i} h\sigma_k(\tilde{x}_p) + s\sigma_{i+1}(\tilde{x}_p)$.
Using the fact that $d < 0$ and that under (22), $p \to \infty$ as $t \to \infty$, we have: $\lim_{t\to\infty} \delta(t) = 0$, i.e. (24). □

Remark 5. Using a continuity argument, it can be seen that (28) holds for all $z \in Y_0, s \in [0,h]$, if

$$\int_0^T \rho(t)\mu^{\perp}[J(\xi_{y*}(t)]dt \leqslant 4d \tag{30}$$

holds (with $\rho(t) = \frac{1}{2}$ if $\mu^{\perp}[J(\xi_{y*}(t)] < \gamma$, and $\rho(t) = \frac{3}{2}$ otherwise), and if h is sufficiently small, and δ_0 (hence $|\tilde{x}_0 - y^*|$) is sufficiently small. Note that (30) is a strong form of (4): $\int_0^T \mu^{\perp}[J(\xi_{y*}(t))]dt \leqslant d < 0$ (i.e., (30) implies (4)).

As a recapitulation, it follows from Theorems 1 and 2:

- (17)-(18) coupled with (8) forms a sufficient condition that guarantees the existence of a limit cycle Γ, and allows us to construct an invariant zone $\mathcal{C}$ around Γ;
- (28) (coupled with (21)-(22)) is an additional numeric condition that allows us to determine a basin of attraction of Γ.

Note that although (28) is a criterion that has to be *a priori* checked numerically, it can be formally established using, e.g., interval arithmetic [9].

Example 2. Let us continue the Van der Pol of example 1 (with $h = 10^{-4}$). We have $a = 0.9, b = 1.1$ and $D = M_{\mathcal{C}} h(\frac{2L}{\gamma a} + b + 1) \approx 500$. The numeric computation of

$$\mathcal{K}(z,s) := h\Sigma_{k=1}^{N_1(z)-1}\sigma_k(z) + s\sigma_{N_1(z)}(z)$$

for $z \in Y_0$, and $s \in [0,h]$ gives: $\mathcal{K}(z,s) \in (-0.34, -0.36)$, hence $\mathcal{K}(z,s) < d$ with $d = -0.34$. Therefore (28) holds, hence (24). It then follows from Theorem 2 that (25) holds and $Y_0 = B(\tilde{x}_0, \delta_0) \cap S_0$ is a basin of attraction of Γ.

Finally, we give a numerical evidence of (25). Figure 3 gives the evolution of $\tilde{e}_{y_0}(t) = |\xi_{y_0}(\Theta_{y_0}(t)) - \tilde{x}(t)|$ for $y_0 = (1.8037; -0.5057) \in Y_0$ and different values of time-step h (The exact solution $\xi_{y_0}(t)$ is here approximated as $\tilde{x}(t; y_0)$ with time-step 10^{-6}). The different curves $\tilde{e}_{y_0}(t)$ are given from top to bottom for $h_1 = 5\ 10^{-4}$ (red), $h_2 = 2.5\ 10^{-4}$ (green), $h_3 = 1.25\ 10^{-4}$ (blue). The curves are in agreement with (25): $\limsup_{t\to\infty} \tilde{e}_{y_0}(t) \leqslant Dh$.

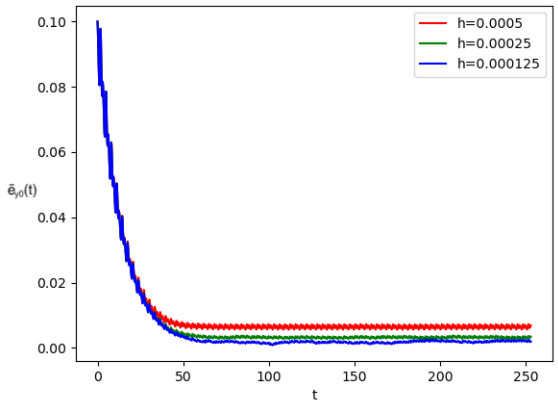

Fig. 3. Evolution of $\tilde{e}_{y_0}(t)$ with $h_1 = 5 \cdot 10^{-4}$ (red), $h_2 = 2.5 \cdot 10^{-4}$ (green) and $h_3 = 1.25 \cdot 10^{-4}$ (blue), showing agreement with (25): $\limsup_{t\to\infty} \tilde{e}_{y_0}(t) \leqslant Dh$.

V. Final Remarks

We have given an original set of conditions involving the Euler approximation of (1) that allows us to prove the existence of a limit cycle Γ, and determine an invariant set $\mathcal{C}$ around Γ as well as a basin of attraction of Γ.

In the future, it would be interesting to apply this method together with non-Euclidean norms such as the weighted norms of [4].

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