Improved Bounds for Twin-Width Parameter Variants with Algorithmic Applications to Counting Graph Colorings

Ambroise Baril, Miguel Couceiro, Victor Lagerkvist

Linear Bounds between Component Twin-Width and Clique-Width with Algorithmic Applications to Counting Graph Colorings

Ambroise Baril
LORIA, Université de Lorraine, France

Miguel Couceiro
LORIA, Université de Lorraine, France

Victor Lagerkvist
Linköping Universitet, Sweden

Abstract

The H-Coloring problem is a well-known generalization of the classical NP-complete problem k-Coloring where the task is to determine whether an input graph admits a homomorphism to the template graph H. This problem has been the subject of intense theoretical research and in this paper we study the complexity of H-Coloring with respect to the parameters *clique-width* and the more recent *component twin-width*, which describe desirable computational properties of graphs. We give two surprising linear bounds between these parameters, thus improving the previously known exponential and double exponential bounds. These linear bounds entail natural approximations of component twin-width, by making use of the results known for clique-width. On the algorithmic side we target the richer problem of counting the number of homomorphism to H ($\#H$-Coloring). The first algorithm uses a contraction sequence of the input graph G parameterized by the component twin-width of G. This leads to a positive FPT result for the counting version. The second uses a contraction sequence of the template graph H and here we instead measure the complexity with respect to the number of vertices in the input graph. Using our linear bounds we show that the latter *always* beats the previously best algorithm (based on clique-width) and we thus obtain the fastest general purpose algorithm for graph coloring.

2012 ACM Subject Classification Discrete mathematics

Keywords and phrases Component twin-width, Clique-width, Graph coloring, Parameterized complexity, Fine-grained complexity

1 Introduction

Graph coloring is a well-known computational problem where the goal is to color a graph in a consistent way. This problem is one of the most well-studied NP-hard problems and enjoys a wide range of applications *e.g.*, in planning, scheduling, and resource allocation [20]. There are many variants and different formulations of the coloring problem, but the most common formulation is certainly the k-Coloring problem that asks whether the vertices of an input graph can be colored using k available colors in such a way that no two adjacent vertices are assigned the same color. This problem can be extended in many ways and in this paper we are particularly interested in the more general problem where any two adjacent vertices in the input graph G have to be mapped to two adjacent vertices in a fixed template graph H (the H-Coloring problem). It is not difficult to see that k-Coloring is then K_k-Coloring, where K_k is the k-vertex clique.

The basic H-Coloring problem has been extended in many directions, of which one of the most dominant formalisms is the *counting* extension of where the task is not only to decide whether there is at least one solution (coloring) but to return the number of solutions ($\#H$-Coloring). This framework makes it possible to encode phase transition systems

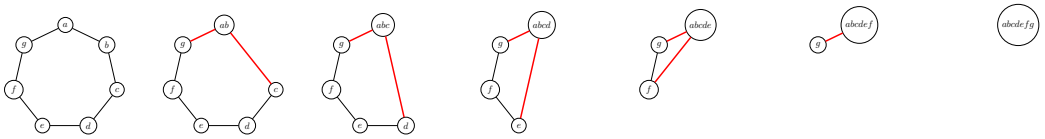


Figure 1 A contraction sequence of the 7-cycle.

modelled by partition functions, modeling problems from statistical physics such as counting q-particle Widom–Rowlinson configurations and counting Beach models, or the classical Ising model (for further examples, see *e.g.* Dyer & Greenhill [17]). The $\#H$-Coloring problem is #P-hard unless every connected component of H is either a single vertex without a loop, a clique or a bipartite complete graph, and it is in P otherwise [17]. The question is then to which degree we can still hope to solve it efficiently, or at least improve upon the naive bound of $|V_H|^{|V_G|}$ (where V_H is the set of vertices in the template graph H and V_G the set of vertices in the input graph G).

In this paper we tackle this question by targeting properties of graphs, so-called *graph parameters*, which give rise to efficiently solvable subproblems. We will see below several concrete examples of graph parameters, but for the moment we simply assume that each graph G is associated with a number $k \in \mathbb{N}$, a *parameter*, which describes a structural property of G. Here, the idea is that small values of k correspond to graphs with a simple structure, while large values correspond to more complicated graphs.

There are then two ways to approach intractable H-Coloring problems: we either restrict the class of *input* graphs G, or the class of *template* graphs H to graphs where the parameter is bounded by some reasonably small constant. The first task is typically studied using tools from *parameterized* complexity where goal is to prove that problems are *fixed-parameter tractable* (FPT), *i.e.*, obtaining running times of the form $f(k) \cdot \|G\|^{O(1)}$ for a computable function $f\colon \mathbb{N} \to \mathbb{N}$ (where $\|G\|$ is the number of bits required to represent the input graph G). The second task is more closely related to *fine-grained* complexity where the goal is to prove upper and lower bounds of the form $2^{f(k)} \cdot \|G\|^{O(1)}$ for a sufficiently "fine-grained" parameter k, which in our case is always going to denote the number of vertices $|V_G|$ in the input graph G. Here, it is worth remarking that H-Coloring is believed to be a very hard problem, and the general Coloring problem, where the template is part of the input, is strongly believed to be unsolvable in $2^{O(|V_G|)} \cdot (\|G\| + \|H\|)^{O(1)}$ time [19]. Hence, regardless whether one studies the problem under the lens of parameterized or fine-grained complexity, one needs to limit the class of considered graphs via a suitable parameter.

There are several graph parameters proposed to address the limitations of tree-width and that generalize the class of co-graphs. We briefly survey two noteworthy graph parameters (see Section 2 for formal definitions).

1. *clique-width* (**cw**). The class of graphs (with labelled vertices) with clique-width $k \geq 1$ is defined as the smallest class of graphs that contains the one vertex graphs $\bullet_i$ with 1 vertex labelled by $i \in [k]$, and that is stable by the following operations for $(i, j) \in [k]^2$ with $i \neq j$: (i) disjoint union of graphs, (ii) relabelling every vertex of label i to label j, and (iii) constructing edges between every vertex labelled by i and every vertex labelled by j. Note that the class of cographs (which contains cliques) is exactly that of graphs with clique-width at most 2.
2. *twin-width* (**tww**). The class of graphs of twin-width $k \geq 1$ is usually formulated via *contraction sequences* where graphs are gradually merged into a single vertex (see Figure 1 for an example). Red edges represent an inconsistency in the merged vertex (see Section

2.3 for a formal definition), and the maximum red degree in the sequence thus represents the largest loss of information. A graph has twin-width $\leq k$ if it admits such a contraction sequence where the maximum red degree does not exceed k.

For clique-width, Ganian et. al [21] identified a structural parameter s of graphs and presented an algorithm for H-Coloring that runs in $O^*(s(H)^{\mathbf{cw}(G)})$ time[1]. It is also optimal in the sense that if there exists an algorithm that solves H-Coloring in time $O^*((s(H)-\varepsilon)^{\mathbf{cw}(G)})$, then the SETH fails [21]. Again, alternative algorithms exist for templates of bounded clique-width, and Wahlström [27] solves #H-Coloring in $O^*((2\mathbf{cw}(H)+1)^{|V_G|})$ time.

Twin-width, on the other hand, is a much more recent parameter, but has in only a few years attracted significant attention [1, 2, 3, 5, 4, 6, 7, 8, 9, 11, 12]. One of its greatest achievement is that checking if a graph is a model of any first order formula can be decided in FPT time parameterized by the twin-width of the input graph. Thus, a very natural research question in light of the above results concerning tree- and clique-width is to study the complexity of (#)H-Coloring via twin-width. Unfortunately, it is easy to see that under standard assumptions, H-Coloring is generally not FPT parameterized by twin-width. Indeed, since twin-width is bounded on planar graphs [23], the existence of an FPT algorithm for 3-Coloring running in $O^*(f(\mathbf{tww}(G)))$ time implies an $O^*(1)$ time (ie. a polynomial time) algorithm for 3-Coloring on planar graphs (since $f(\mathbf{tww}(G))=O(1)$ if G is a planar graph). Since 3-Coloring is NP-hard on planar graphs, this would imply P=NP. Thus, 3-Coloring is *para*-NP-*hard* [16] parameterized by twin-width.

Despite this hardness result it is possible to analyse H-Coloring by a variant of twin-width known as *component twin-width* (**ctww**) [8]. This parameter equals the maximal size of a red-connected component (instead of the maximal red-degree for twin-width). It is then known that component twin-width is functionally equivalent[2] to boolean-width [8], which in turn is functionally equivalent to clique-width [13]. Hence, H-Coloring is FPT parameterized by component twin-width, and the specific problem k-Coloring is additionally known to be solvable in $O^*((2^k-1)^{\mathbf{ctww}(G)})$ time [8]. As remarked Bonnet et al., the theoretical implications of this particular algorithm are limited due to the aformentioned (under the SETH) optimal algorithm parameterized by clique-width [21]. However, this still leaves several gaps in our understanding of component twin-width for H-Coloring and its counting extension #H-Coloring.

Our paper has three major contributions to bridge these gaps. *Firstly,* the best known bounds between clique-width and component twin-width are obtained by following the proof of functional equivalence between component twin-width and boolean-width, and then between boolean-width and clique-width. We thereby obtain

$$\mathbf{ctww} \leq 2^{\mathbf{cw}+1} \quad \text{and} \quad \mathbf{cw} \leq 2^{2^{\mathbf{ctww}}}$$

and H-Coloring is thus solvable in $O^*(s(H)^{2^{2^{\mathbf{ctww}(G)}}})$ time. This proves FPT but is clearly not a practically applicable algorithm and the main question is whether it is possible to improve this to a single-exponential running time $O^*(2^{O(\mathbf{ctww}(G))})$. (This line of research is relatively new but of growing importance and has seen several landmark results [15].) We prove that it is indeed possible by significantly strengthening the bounds between **cw** and

[1] The notation O^* means that we ignore polynomial factors.
[2] *I.e.*, each parameter is bounded by a function of the other.

ctww and obtain the linear bounds

$$\mathbf{cw} \leq \mathbf{ctww} + 1 \leq 2\mathbf{cw}.$$

Our proof is constructive which gives a fast algorithm to derive a contraction-sequence from a clique-width expression and vice versa.

Secondly, we discuss how these bounds can be exploited to *approximate* **ctww** by making use of the results known on **cw**. Thus, an immediate consequence of our linear bounds is that H-Coloring is solvable in $O^*(s(H)^{\mathbf{ctww}(G)+1})$ time, which is a major improvement to the aforementioned triple exponential upper bound.

Thirdly, we consider the generalized problem of counting the number of solutions. It seems unlikely that the optimal algorithm (under SETH) by Ganian et al. [21] can be lifted to $\#H$-Coloring, and while the algorithm by Wahlström [27] successfully solves $\#H$-Coloring, it does so with the significantly worse bound of $O^*((2\mathbf{cw}(H)+1)^{|V_G|})$. We tackle this problem in Section 5 by designing a novel algorithm for $\#H$-Coloring for input graphs with bounded component twin-width and which runs in $O^*((2^{|V_H|}-1)^{\mathbf{ctww}(G)})$ time.

We also consider $\#H$-Coloring when the template graph H has bounded component twin-width. We use an optimal contraction sequence of H in order to obtain a $O^*((\mathbf{ctww}(H)+2)^{|V_G|})$ algorithm for $\#H$-Coloring. Combining this result with our linear bounds $\mathbf{cw} \leq \mathbf{ctww}+1 \leq 2\mathbf{cw}$, we conclude that our algorithm always runs faster (asymptotically) than the $O^*((2\mathbf{cw}(H)+1)^{|V_G|})$ time algorithm by Wahlström [27], which was, to our knowledge, the fastest general $\#H$-Coloring algorithm available in the literature. Moreover, the technique employed in this paper could similarly be used to derive the same results applied to the more general frameworks of counting the solutions of *binary constraint satisfaction problems*, ie. problems of the forms #Binary-Csp(Γ) with Γ a set of binary relations over a finite domain, even though we restrict to $\#H$-Coloring here for the sake of simplicity.

2 Preliminaries

Throughout this paper, a *graph* G is a tuple (V_G, E_G), where V_G is a finite set (the set of vertices of G), and E_G is a binary irreflexive symmetric relation over V_G (the set of edges of G). We will denote the number of vertices of a graph G by $n(G)$ or, simply, by n when there is no danger of ambiguity. The neighborhood of a vertex u of a graph G is the set $N_G(u) = \{v \in V_G \mid (u,v) \in E_G\}$. For a graph H we let H-Coloring be the computational problem of deciding whether there exists an homomorphism from an input graph G to H, *i.e.*, whether there exists a function $f\colon V_G \to V_H$ such that $(x,y) \in E_G$ implies that $(f(x), f(y)) \in E_H$. We write $\#H$-Coloring for the associated *counting* problem where we instead wish to determine the exact number of such homomorphisms. As remarked in Section 1, the template graph H can be chosen with great flexibility to model many different types of problems.

2.1 Parameterized complexity

We assume that the reader is familiar with parameterized complexity and only introduce the strictly necessary concepts (we refer to Flum & Grohe [18] for further background). A *parameterized counting problem* is a pair (F, dom) where $F : \Sigma^* \mapsto \mathbb{N}$ (for an alphabet Σ, *i.e.*, a finite set of symbols) and dom is a subset of $\Sigma^* \times \mathbb{N}$. A parameterized counting problem (F, dom) is said to be *fixed-parameter tractable* (FPT) if there exists a computable function $f\colon \mathbb{N} \to \mathbb{N}$ such that for any instance $(x,k) \in dom$ of F, we can compute $F(x)$ in

$f(k) \times \|x\|^{O(1)}$ time. An algorithm with this complexity is said to be an FPT *algorithm*. Note that even though f might be superpolynomial, which is expected if the problem is NP-hard, instances where k is reasonably small can still be efficiently solved.

In practice, when studying FPT algorithms for an NP-hard counting problem, it is very natural to optimize the superpolynomial function f that appears in the complexity of the algorithm solving it. Typically, when dealing with graphs problems parameterized by the number of vertices n, an algorithm running in $c^n \times \|x\|^{O(1)}$ will be considered efficient in practice if $c > 1$ is small. This field of research is referred to as *fine-grained complexity*.

2.2 Clique-width

For $k \geq 1$, let $[k] = \{1, \dots, k\}$. A *k-labelled graph* G is a tuple (V_G, E_G, l_G), where (V_G, E_G) is a graph and $l_G : V_G \to [k]$. For $i \in [k]$ and a k-labelled graph G, denote by $V_G^i = l_G^{-1}(\{i\})$ the set of vertices of G of label i. A *k-expression* φ of a k-labelled graph G, denoted $[\varphi] = G$, is an expression defined inductively [14] using:

1. $\bullet_i$ with $i \in [k]$: $[\bullet_i]$ is a k-labelled graph with one vertex labelled by i,
2. $\rho_{i\to j}(\varphi)$ with $(i,j) \in [k]^2$ and $i \neq j$: $[\rho_{i\to j}(\varphi)]$ is the same graph as $[\varphi]$, but where all vertices of G with label i now have label j,
3. $\eta_{i,j}(\varphi)$ with $(i,j) \in [k]^2$ and $i \neq j$: $[\eta_{i,j}(\varphi)]$ is the same graph as $[\varphi]$, but where all tuples of the form (u,v) with $\{l_G(u), l_G(v)\} = \{i,j\}$ is now an edge, and
4. $\varphi_1 \oplus \varphi_2$: $[\varphi_1 \oplus \varphi_2]$ is the disjoint union of the graphs $[\varphi_1]$ and $[\varphi_2]$.

A graph G has a k-expression φ if there exists $l : V_G \mapsto [k]$ such that $[\varphi] = (V_G, E_G, l)$. The *clique-width* of a graph G (denoted by $\mathbf{cw}(G)$) is the minimum $k \geq 1$ such that G has a k-expression. An *optimal expression* of a graph G is a $\mathbf{cw}(G)$-expression of G.

The *subexpressions* of an expression φ are defined similarly: the only subexpression of $\bullet_i$ is $\bullet_i$, the subexpressions of $\varphi = \varphi_1 \oplus \varphi_2$ are φ and the subexpressions of φ_1 and φ_2, the subexpressions of $\varphi = \rho_{i\to j}(\varphi')$ and $\varphi = \eta_{i,j}(\varphi')$ are φ and the subexpressions of φ'.

A *linear* k-expression is a k-expression φ where for every subexpression of φ of the form $\varphi_1 \oplus \varphi_2$, φ_2 is of the form $\bullet_i$ with $i \in [k]$. The *linear clique-width* (denoted by $\mathbf{linearcw}(G)$) of a graph G is the minimum $k \geq 1$ such that G has a linear k-expression.

2.3 Component twin-width

A *trigraph* is a triplet $G = (V_G, E_G, R_G)$ where (V_G, E_G) and (V_G, R_G) are graphs, and $E_G \cap R_G \neq \emptyset$. The set V_G is said to be the set of vertices of G, E_G is the set of (black) edges of G, and R_G the set of *red edges* of G. A red-connected component of a trigraph G is a connected component of the graph (V_G, R_G).

Let G be a trigraph and $(u,v) \in (V_G)^2$ with $u \neq v$. The trigraph $G/(u,v)$ is defined as the graph G where u and v have been removed and replaced by a new vertex uv, and where for all $z \in V_G \setminus \{u,v\}$:

- $(uv,z) \in E_{G/(u,v)}$ if $(u,z) \in E_G$ and $(v,z) \in E_G$,
- $(uv,z) \notin (E_{G/(u,v)} \cup R_{G/(u,v)})$ if $(u,z) \notin (E_G \cup R_G)$ and $(v,z) \notin (E_G \cup R_G)$, and
- $(uv,z) \in R_{G/(u,v)}$ otherwise, *i.e.* when (u,z) or (v,z) is already a red-edge, or when among (u,z) and (v,z), one is a black edge and one is a non-edge.

A *contraction* of a trigraph G is a graph of the form $G/(u,v)$ with $(u,v) \in (V_G)^2$ and $u \neq v$. A contraction sequence of a graph G is a sequence of trigraphs of the form $(G_n, \dots, G_1)$ with $n = |V_G|$, $G_n = (G, \emptyset)$, and for all $k \in [n-1]$, G_k is a contraction of G_{k+1}. Note that

G_k has k vertices and, in particular, that G_1 has only one vertex, which is necessarily the trigraph[3] $G_1 = (\{V_G\}, \emptyset, \emptyset)$. A contraction sequence is said to be *linear* if any of its trigraph contains at most one red-connected component with at least 2 vertices of the original graph.

The *component twin-width* of a contraction sequence $(G_n, \dots, G_1)$ is the maximal size of a red-connected component among the trigraphs G_k for $k \in [n]$. The (*linear*) *component twin-width* of a graph G denoted by $\mathbf{linearctww}(G)/\mathbf{ctww}(G)$ is the minimum (linear) component twin-width of all its contraction sequences. The soundness proof of our algorithms that make use of contraction sequences rely on the following fundamental property of contraction sequences. This property can be easily proven by induction on k, knowing that it is indeed true for $G_n = (V_G, E_G, \emptyset)$, and using the definition of a contraction of a trigraph.

▶ **Property 1.** *Let $(G_n, \dots, G_1)$ be a contraction sequence of a graph G, $k \in [n]$, U and V two different vertices of G_k, $u \in U$ and $v \in V$.*[3] *Then*

- $(u, v) \in E_G$, *whenever* $(U, V) \in E_{G_k}$, *and*
- $(u, v) \notin E_G$, *whenever* $(U, V) \notin E_{G_k} \cup R_{G_k}$.

A *cograph* is a graph that has a contraction sequence with no red-edges, *i.e.,* graphs with component twin-width 1 [10].

2.4 Rank-width

A *branch decomposition* of a graph G is a ternary tree T (a tree where each non-leaf vertex has degree 3) whose set of leaves is exactly V_G. Let G be a graph and T a branch decomposition of G. Every edge e of T corresponds to a bipartition (X_e, Y_e) of V_G by considering the bipartition of the leaves of T into their connected components of $T - e$ (the tree T but where the edge e have been removed). For every edge e of T, let A_e be the $(\mathbb{Z}/2\mathbb{Z})$-matrix whose set of rows is X_e and whose set of columns is Y_e, and whose coefficient of index $(u, v) \in X_e \times Y_e$ is 1 if $(u, v) \in E_G$, and 0 otherwise.

Finally, let $\rho_G(T) = \max_{e \in E_T} \mathbf{rank}(A_e)$. The *rank-width* of G denoted by $\mathbf{rw}(G)$ is the minimum of $\rho_G(T)$ for T a branch-decomposition of G. A branch decomposition T realising this minimum is called an *optimal* branch-decomposition of G. The main interest of rank-width is made clear in the following remark.

▶ Remark 2. Let T be an optimal branch-decomposition of a graph G, and $e \in E_T$. If $|X_e| > 2^{\mathbf{rw}(G)}$, then there exists $(u, u') \in (X_e)^2$ with $u \neq u'$ such that $N_G(u) \cap Y_e = N_G(u') \cap Y_e$.

Proof. Since the rank of the matrix A_e is lower than $\mathbf{rw}(G)$, the rows of G all belong to a $(\mathbb{Z}/2\mathbb{Z})$-vector space of dimension at most $\mathbf{rw}(G)$. The latter has a cardinality of at most $2^{\mathbf{rw}(G)}$, and therefore, X_e has 2 identical rows, which proves the result. ◀

3 Comparing Clique-Width and Component Twin-Width

In this section, we prove the linear bounds between clique-width $\mathbf{cw}$ and component twin-width $\mathbf{ctww}$:

$$\mathbf{cw} \leq \mathbf{ctww} + 1 \leq 2\mathbf{cw}$$

and that the similar bounds hold for their linear versions.

Firstly, we prove the leftmost inequality.

[3] Each vertex of G_k is a set of vertices of G that have been contracted.

▶ **Theorem 3.** *For every graph G,* $\mathbf{cw}(G) \leq \mathbf{ctww}(G) + 1$.

Proof. Let $(G_n, \dots, G_1)$ be an optimal contraction sequence of G, and let $\kappa = \mathbf{ctww}(G)$. Note that, for all $k \in [n]$, every red-connected component of G_k has size $\leq \kappa$. We explain how to construct a $(\kappa+1)$-expression of G.

We show the following invariant for all $k \in [n]$:
$\mathcal{P}(k)$: "Let $C = \{S_1, \dots, S_p\}$ be a red-connected component of G_k and[3] $\bigcup C = S_1 \cup \dots \cup S_p$. There exists a $(\kappa+1)$-expression φ_C of the p-labelled graph $G_C = G[\bigcup C]$ with $\forall i \in [p], V^i_{G_C} = S_i$."

We first prove $\mathcal{P}(n)$. In $G_n = (V_G, E_G, \emptyset)$, there are no red edges: the red-connected component are the singletons $\{u\}$ for $u \in V_G$. Thus $\bullet_1$ is a $(\kappa+1)$-expression of $(G[\{u\}], l_u)$ (with $l_u : u \mapsto 1$), which proves $\mathcal{P}(n)$.

Now, take $k \in [n-1]$ and assume $\mathcal{P}(k+1)$. We will prove $\mathcal{P}(k)$. By definition of a contraction sequence, G_k is of the form $G_k = G_{k+1}/(u,v)$ for two different vertices u and v of G_{k+1}. Observe that each red-connected component of G_k is also a red-connected component of G_{k+1}, except the red-connected component C containing uv. Hence, it suffices to prove $\mathcal{P}(k)$ for the red-connected component C. Notice also that $(C \setminus \{uv\}) \cup \{u, v\}$ is a union of red-connected component $C_1, \dots, C_q$ of G_{k+1} (every pair of red-connected vertices in G_{k+1} that does not contain u or v is also red-connected in G_k). We thus have that $C =: (C_1 \cup \dots \cup C_q \cup \{uv\}) \setminus \{u, v\}$.

Denote by $\{S_1, \dots, S_{p-1}, S'_p\}$ the set of vertices of C, with $p = |C|$, and $S'_p = uv$. We have seen that $C_1 \cup \dots \cup C_q = \{S_1, \dots, S_{p-1}, S_p, S_{p+1}\}$, with $S_p := u$ and $S_{p+1} := v$. For each $i \in [p+1]$, S_i belongs to a unique C_j with $j \in [q]$: let $j(i) \in [q]$ be such that $S_i \in C_{j(i)}$.

By $\mathcal{P}(k+1)$ and up to interchanging labels, for every $j \in [q]$ there exists a $(\kappa+1)$-expression φ_{C_j} of the p-labelled graph $G_{C_j} = G[\bigcup C_j]$ with for all $i \in [p]$ with $j(i) = j$, $V^i_{G_{C_j}} = S_i$. Therefore, $\varphi' := \varphi_{C_1} \oplus \dots \oplus \varphi_{C_q}$ expresses the disjoint union of the graphs $G_{C_1}, ..., G_{C_q}$. Furthermore, φ' is an expression of a graph over the same vertices as $G[\bigcup C]$, however, we still need to construct the black edges crossing these red-connected components.

We thus apply $\eta_{i,i'}$ to φ' for every black edge of the form $(S_i, S_{i'})$ in G_{k+1}, to obtain an expression φ''. Since the vertices with labels i and i' are exactly the vertices of S_i and $S_{i'}$, we create exactly the edges between vertices of S_i and of $S_{i'}$. By Property 1, we construct the correct black edges in $G[\bigcup C]$, and thus φ'' is an expression of $G[\bigcup C]$.

Moreover, we need to make sure that the labels in φ'' match the expectations of $\mathcal{P}(k)$. For that, we apply $\rho_{p+1 \to p}$ to φ'' to get an expression φ_{G_C}. By doing so, S_p (say, u) and S_{p+1} (say, v) have the same label in φ_{G_C}. Hence, φ_{G_C} witnesses $\mathcal{P}(k)$ (since $S_p = u$ and $S_{p+1} = v$ are now contracted together as the vertex $S'_p = uv$ in G_k) for the red-connected component C. Indeed, we have used $p + 1 = |C| + 1 \leq \kappa + 1$ different labels to construct φ_{G_C} from $\varphi_{C_1}, \dots, \varphi_{C_q}$.

Since $\{V_G\}$ is a red-connected component of $G_1 = (\{V_G\}, \emptyset, \emptyset)$, it follows from $\mathcal{P}(1)$ that $G[V_G] = G$ has a $(\kappa+1)$-expression, and thus $\mathbf{cw}(G) \leq \kappa + 1$. Recall that $\kappa = \mathbf{ctww}(G)$, and thus $\mathbf{cw}(G) \leq \mathbf{ctww}(G) + 1$. ◀

Moreover, if the contraction sequence given is linear, we obtain a linear $(\kappa+1)$-expression. We thus get for all graph G that $\mathbf{linearcw}(G) \leq \mathbf{linearctww}(G) + 1$.

Notice that **linearcw** can not be bounded by a function of **ctww**. For instance, the class of cographs (and even of trees) have unbounded linear clique-width [22], despite having a bounded component twin-width of 1. We now prove the rightmost bound.

▶ **Theorem 4.** *For every graph G, we have:*
(i) $\mathbf{ctww}(G) \leq 2\mathbf{cw}(G) - 1$, *and*

(*ii*) $\mathbf{ctww}(G) \leq \mathbf{linearcw}(G)$.

Proof. We first prove (*i*) and then adapt it to prove (*ii*). Let $k := \mathbf{cw}(G)$ and take a k-expression of G. We will explain how to construct a contraction sequence of G in which every red-connected component has size $\leq 2k-1$. The following remark will be implicitly used throughout this proof.

▶ Remark 5. Two vertices that have the same label in an expression φ' also have the same label in any expression of φ that has φ' as a sub-expression.

We prove the following property of k-expressions of φ by structural induction:
$\mathcal{H}(\varphi)$: "Let $(G, l_G) := [\varphi]$. There exists a (partial) contraction sequence $(G_n, \ldots, G_{k'})$ with $k' \leq k$ of G such that:

- every red-connected component in the trigraphs $G_n, \ldots, G_{k'}$ has size $\leq 2k-1$,
- the vertices of $G_{k'}$ are exactly the non-empty V_G^i for $i \in [k]$, and
- every pair of vertices contracted have the same labels in (G, l_G)[4].

If $\varphi = \bullet_i$ with $i \in [k]$, there is nothing to do since G has only one vertex. If φ is of the form $\rho_{i\to j}(\varphi')$ (with $(i,j) \in [k]^2$ and $i \neq j$), consider for G the partial contraction sequence of $(G', l_{G'}) := [\varphi']$ given by $\mathcal{H}(\varphi')$, and then contract $V_{G'}^i$ and $V_{G'}^j$ to obtain $V_G^j = V_{G'}^i \cup V_{G'}^j$. Since φ' is also a k-expression of G, this partial contraction sequence of G satisfies $\mathcal{H}(\varphi)$.

If φ is of the form $\eta_{i,j}(\varphi')$ (with $(i,j) \in [k]^2$ and $i \neq j$), consider for G the partial contraction sequence of $(G', l_{G'}) := [\varphi']$ given by $\mathcal{H}(\varphi')$. To prove that it is sufficient to prove $\mathcal{H}(\varphi)$, it is sufficient to justify that it does not create any red edge in the contraction of G that was not present in the contraction of G'. The first red-edge (x, y) that would appear in the contraction of $G = [\eta_{i,j}(\varphi')]$ that does not appear in the same contraction of $G' = [\varphi']$, results necessarily of the contraction of two vertices u and v with $x = uv$ and y being in the symmetric difference of the neighborhoods of u and v in $G = [\eta_{i,j}(\varphi')]$ but not in $G' = [\varphi']$. Such a red-edge can not exist because we contract only vertices with the same label in φ' (or, equivalently, in φ), and that $\eta_{i,j}$ can only decrease (w.r.t. $\subseteq$) the symmetric difference between the neighborhood of vertices with the same label in φ. By Remark 5, this implies that it is also true for vertices having the same label in any subexpression of φ.

If φ is of the form $\varphi = \varphi' \oplus \varphi''$: denote $(G', l') := [\varphi']$ and $(G'', l'') := [\varphi'']$, thereby, $V_G = V_{G'} \cup V_{G''}$. Consider the partial contraction sequence of G given by:

1. contract the vertices in $V_{G'}$ in accordance to the contraction sequence given by $\mathcal{H}(\varphi')$,
2. contract the vertices in $V_{G''}$ in accordance to the contraction sequence given by $\mathcal{H}(\varphi'')$
3. for all $i \in [k]$, contract $V_{G'}^i$ with $V_{G''}^i$ (if both are nonempty) to get $V_G^i = V_{G'}^i \cup V_{G''}^i$.

Steps 1 and 2 do not create a red-edge adjacent to both $V_{G'}$ and $V_{G''}$ (since these are two distinct connected components of G).

Thus, before step 3, we have a trigraph with $\leq 2k$ vertices (because both trigraphs obtained after $\mathcal{H}(\varphi')$ and $\mathcal{H}(\varphi'')$ have less than k vertices), and every red-component that have appeared so far has size $\leq 2k-1$. After the first contraction of step 3, the resulting trigraph has $\leq 2k-1$ vertices, and thus no red-connected component of size $> 2k-1$ can emerge. Such a contraction satisfies every requirement of $\mathcal{H}(\varphi)$. We have thus proven $\mathcal{H}(\varphi)$ for every k-expression.

[4] Inductively, we say that the label of a vertex $S \in V_{G_l}$ $(k' \leq l \leq n)$ is then the common label of the vertices that have been contracted together to produce S.

Now, take a k-expression φ of G. Up to applying $\rho_{i\to 1}$ for all $i \in [k]$ to φ, we can assume that $(G, l_G) := [\varphi]$ with l_G being constant equal to 1. The partial contraction sequence of G given by $\mathcal{H}(\varphi)$ is a total contraction sequence of G of component twin-width $\leq 2k-1$. Since $k = \mathbf{cw}(G)$, we have proven that $\mathbf{ctww}(G) \leq 2\mathbf{cw}(G) - 1$.

To prove (ii), we show a similar property $\mathcal{H}_{\text{lin}}(\varphi)$ for every linear k-expression. The only difference between $\mathcal{H}_{\text{lin}}$ and $\mathcal{H}$ is that we replace the condition $\leq 2k-1$ by $\leq k$. The proof then follows exactly the same steps, except for the case $\varphi = \varphi' \oplus \varphi''$, where step 2 (the contraction according to $\mathcal{H}_{\text{lin}}(\varphi'')$) is not necessary anymore, since φ'' is of the form $\bullet_i$ ($i \in [k]$), and we obtain a trigraph of size $k+1$ instead of $2k$, since φ'' has 1 vertex instead of k. This ensures that every red-connected component has size $\leq (k+1) - 1 = k$ instead of $2k-1$ in the non-linear case.

For step 3, *i.e.,* contracting vertices of the same color in φ' and in φ'', just note that it consists of at most 1 contraction instead of k in the linear case. ◀

Moreover, in the proof of (i), if the k-expression given is linear, we obtain a linear contraction sequence. We get that for every graph G, $\mathbf{linearctww}(G) \leq 2\mathbf{linearcw}(G) - 1$.

4 Approximating Component Twin-Width

The linear bounds established in Section 3 entail reasonable approximation results for component twin-width by making use of known approximations of clique-width [26]. The best currently known approximation algorithm for clique-width is given by Theorem 6.

▶ **Theorem 6.** *[26] Let k be a fixed positive integer. There is an $O(|V_G|^3)$-time algorithm that either outputs an (8^k-1)-expression of an input graph G or confirms that the clique-width of G is larger than k.*

From Theorem 6 and the linear bounds established in Theorem 3 and Theorem 4, we immediately obtain an approximation algorithm for component twin-width.

▶ **Theorem 7.** *Let p be a fixed positive integer. There is an $O(|V_G|^3)$-time algorithm that either outputs a contraction sequence of component twin-width $\leq 2^{3p+4} - 3$ of an input graph G or confirms that the component twin-width of G is larger than p.*

Proof. The algorithm consists of applying the algorithm of Theorem 6 to G with $k := p+1$. If the algorithm confirms that $\mathbf{cw}(G) > p+1$, then we know that $\mathbf{ctww}(G) > p$ by Theorem 3. Otherwise, it outputs a $(2^{3(p+1)}-1)$-expression of G, which we transform into a contraction sequence of G of component twin-width $\leq 2 \times (2^{3(p+1)} - 1) - 1 = 2^{3p+4} - 3$ through the constructive proof of Theorem 4, which can be performed in linear time in the size of the $(2^{3(p+1)} - 1)$-expression of G. ◀

It is still interesting to see that a direct comparison between component twin-width and rank-width yields to a better approximation ratio, thanks to Theorem 8. In fact, Theorem 6 was also obtained by this method. By avoiding using clique-width as an intermediate parameter, it is not surprising that we obtain a better ratio.

▶ **Theorem 8.** *[26] Let k be a fixed positive integer. There is an $O(|V(G)|^3)$-time algorithm that either outputs a rank-decomposition (of an input graph G) of width at most $3k-1$ or confirms that the rank-width of G is larger than k.*

We can indeed make use of the bounds given by Theorem 9. The proof is very similar to the proof of functional equivalence between boolean-width and component twin-width [8], which is not surprising, since Remark 2 applies both to rank-with and boolean-width.

▶ **Theorem 9.** *For every graph G,* $\mathbf{rw}(G) - 1 \leq \mathbf{ctww}(G) \leq 2^{\mathbf{rw}(G)+1}$

Proof. The first inequality $\mathbf{rw}(G) - 1 \leq \mathbf{ctww}(G)$ follows from Theorem 3, stating that $\mathbf{cw}(G) - 1 \leq \mathbf{ctww}(G)$ and the fact that $\mathbf{rw}(G) \leq \mathbf{cw}(G)$ [25].

We now focus on proving the second bound of $\mathbf{ctww}(G) \leq 2^{\mathbf{rw}(G)+1}$. This proof follows the same scheme as the proof of the functional equivalence between boolean-width and component twin-width [8], primarily since Remark 2 also applies to both boolean-width.

Similarly to a branch-decomposition of graphs, a branch-decomposition of a trigraph G' is a ternary tree whose set of leaves is $V_{G'}$. It is said to be rooted if a non-leaf vertex has been chosen to be the root, which leads to the usual definition of children and descendants in a rooted tree. The set of leaves descending from a vertex v of a tree T is denoted by D_v. Now, let G be a graph and T be an optimal branch decomposition of G, and let $r := \mathbf{rw}(G)$. We prove by induction the following property for $k \in [n]$.

$\mathcal{P}(k)$: "There exists a (partial) contraction sequence $(G_n, \ldots, G_k)$ of G of component twin-width $\leq 2^{r+1}$. Moreover, there exists a branch-decomposition T_k of G_k such that for every $t \in V_{T_k}$ with $|D_t| > 2^r$, there is no red-edge crossing the bipartition $(D_t, V_{G_k} \setminus D_t)$."

Note that $\mathcal{P}(n)$ is indeed true since $G = G_n$ has no red-edge. Now assume $\mathcal{P}(k+1)$ with $k \in [n-1]$. We will prove $\mathcal{P}(k)$. First, note that if $k \leq 2^r$, contracting any two arbitrary vertices and giving any branch decomposition of G_k proves $\mathcal{P}(k)$. We may thus assume that $k > 2^r$. The root ρ satisfies $|D_\rho| \geq 2^r + 1$. Observe that there exists a node v of T_{k+1} such that $2^r + 1 \leq |D_v| \leq 2^{r+1}$: a node v such that D_v has size at least $2^r + 1$ and which is furthest from the root meets the condition. Moreover, any child w of v verifies $|D_w| \leq 2^r$.

If we use Remark 2 with respect to an edge adjacent to v, then there are two vertices u and u' that satisfy $N_G(u) \cap (V_{G_{k+1}} \setminus D_v) = N_G(u') \cap (V_{G_{k+1}} \setminus D_v)$.

To prove $\mathcal{P}(k)$, we will prove that it is sufficient to contract the vertices u and u' of G_{k+1} to obtain G_k, and to identify the leaves u and u' of T_{k+1} to obtain T_k (ie. we remove u' and shortcut every degree 2 vertex that appear, and we then rename u as uu'). Note that all the red-edges created by the contraction of u and u' are adjacent to the new vertex uu'.

First, by our choice of u and u', we do not create any red-edge crossing $(D_v, V_{G_k} \setminus D_v)$. Due to the property of T_{k+1} ensured by $\mathcal{P}(k+1)$ (recall that $|D_v| > 2^r$), there is no red-edge crossing $(D_v, V_{G_k} \setminus D_v)$ in T_k. The red-connected component C of the new vertex uu' is thus contained in D_v, and thus has size at most 2^{r+1}. Since C is the only red-connected component of G_k that was not a red-connected component of G_{k+1}, G_k indeed meets the requirements of $\mathcal{P}(k)$.

Second, due to the choice of v, any node t of T_k with $|D_t| > 2^r$ containing the new vertex uu' is an ancestor of v. Since $D_v \subseteq D_t$, by the above argument, there is no red-edge crossing $(D_t, V_{G_k} \setminus D_t)$.

The proof of $\mathcal{P}(k)$ is now complete: $\mathcal{P}(1)$ justifies that $\mathbf{ctww}(G) \leq 2^{r+1}$. ◀

This bounds naturally leads to the approximation given in Theorem 10.

▶ **Theorem 10.** *Let p be a fixed positive integer. There is an $O(|V_G|^3)$-time algorithm that either outputs a contraction sequence of component twin-width $\leq 8^{p+1}$ of an input graph G or confirms that the component twin-width of G is larger than p.*

Proof. This result can be obtained similarly to Theorem 7, by using Theorems 8 and 9 instead of Theorems 3 and 4 and 6. ◀

5 Complexity Results

We show two algorithmic applications of dynamic programming over component twin-width to #H-Coloring. The first assumes that an optimal contraction sequence of the input graph G is given, and results in a FPT algorithm parameterized by **ctww**, running in time $O^*((2^{|V_H|}-1)^{\mathbf{ctww}(G)})$. The second approach uses an optimal contraction sequence of the template H (whose computation can be seen as a pre-computation, since it does not involve the input graph G): we obtain a fine-grained algorithm running in time $O^*((\mathbf{ctww}(H)+2)^{|V_G|})$, whose complexity beats the best algorithms of the literature and that runs in time $O^*((2\mathbf{cw}(H)+1)^{|V_G|})$ and $O^*((\mathbf{linearcw}(H)+2)^{|V_G|})$ [27] through the linear bounds of Section 3. Note that the technique employed in this paper could similarly be used to derive the same complexity results applied to the more general frameworks of counting the solutions of *binary constraint satisfaction problems*, ie. problems of the forms #Binary-Csp(Γ) with Γ a set of binary relations over a finite domain, even though we restrict to the simpler case of #H-Coloring here to avoid having to define contraction sequences of instances and template of binary constraint satisfaction problems.

5.1 Parameterized complexity

We present an algorithm solving #H-Coloring in FPT time parameterized by component twin-width. It is inspired by the algorithm solving k-Coloring [8], thus proving that #H-Coloring is FPT parameterized by component twin-width and thus also by clique-width (by functional equivalence). The proof can be found in Appendix A.

▶ **Theorem 11.** *For any graph H, there exists an algorithm running in time $O^*((2^{|V_H|}-1)^{\mathbf{ctww}(G)})$ that solves #H-Coloring on any input graph G (assuming that an optimal contraction sequence $(G_n,\dots,G_1)$ of G is given).*

If one only wishes to solve H-Coloring rather than the counting problem, the algorithm by Ganian et al. [21] which runs in $O^*(s(H)^{\mathbf{cw}(G)})$ for a graph parameter s, is strictly more efficient. Indeed, for any graph H, its structural parameter $s(H)$ is bounded by $2^{|V_H|}-2$ [21] (the equality happens if and only if H is a clique), and as we have proven in Theorem 3, for any graph G, $\mathbf{cw}(G) \leq \mathbf{ctww}(G)+1$. However, it is difficult to extend this algorithm to the counting problem since the sets stored as invariants in the algorithm do not necessarily represent disjoint subsets of partial coloring. This is acceptable if one only wants to determine the existence of a total coloring (as long as every coloring is represented at least once), but which causes issues when counting the number of colorings.

More precisely, for the #k-Coloring problem, one may remark that the $O^*((2^k-2)^{\mathbf{cw}(G)})$ algorithm by [24] also solves the counting problem, and thus runs faster than the algorithm given in the proof of Theorem 11. However, our algorithm is strictly more general since it is applicable to *any* #H-Coloring problem.

5.2 Fine-grained complexity

We now consider the dual problem of solving #H-Coloring when H has bounded component twin-width. We therefore use an optimal contraction sequence of the template H instead of the input G, and obtain a fine-grained algorithm for #H-Coloring which runs in $O^*((\mathbf{ctww}(H)+2)^n)$ time. The algorithm is inspired by the algorithms running in $O^*((2\mathbf{cw}(H)+1)^n)$ and $O^*((\mathbf{linearcw}(H)+2)^n)$ time by Wahlström [27], but it is *always* at least as fast as both.

▶ **Theorem 12.** *$\#H$-Coloring is solvable in time*

$$O^*((\mathbf{ctww}(H)+2)^{|V_G|}).$$

The proof can be found in Appendix B. By Theorem 4, $\mathbf{ctww}(H)+2 \leq \mathbf{linearcw}(H)+2$ and $\mathbf{ctww}(H)+2 \leq 2\mathbf{cw}(H)+1$ for any graph H. Therefore, the algorithm in the proof of Theorem 12 is always asymptotically faster than the clique-width approach by Wahlström [27] that solves the problem in $O^*((\mathbf{linearcw}(H)+2)^{|V_G|})$ and $O^*((2\mathbf{cw}(H)+1)^{|V_G|})$ time.

6 Conclusion and Future Research

In this paper we explored component twin-width in the context of $\#H$-Coloring problems. We improved the bounds of the functional equivalence between component twin-width and clique-width from the (doubly) exponential $\mathbf{cw} \leq 2^{2^{\mathbf{ctww}}}$ and $\mathbf{ctww} \leq 2^{\mathbf{cw}+1}$ to the linear

$$\mathbf{cw} \leq \mathbf{ctww}+1 \leq 2\mathbf{cw}.$$

In particular, this entails a practical FPT algorithm for H-Coloring parameterized by component twin-width. From these linear bounds derives an approximation algorithm with exponential ratio, that can even be improved by a direct comparison with rank-width. Finally, we constructed two algorithms for solving $\#H$-Coloring. The first uses a given optimal contraction sequence of the input graph G to solve $\#H$-Coloring in FPT time parameterized by component twin-width. The second uses a contraction sequence of the template graph H and beats the clique-width approach for solving $\#H$-Coloring (with respect to $|V_G|$).

We now discuss some topics for future research.

6.1 Tightness of the bounds

Even though the bound $\mathbf{cw} \leq \mathbf{ctww}+1$ given by Theorem 3 is tight for any cograph with at least 1 edge, we do not currently know if this bound can be improved for graphs with greater clique-width or component twin-width. Moreover, it would be interesting to determine whether the bound $\mathbf{ctww} \leq 2\mathbf{cw}-1$ given by Theorem 4 is tight. The same remark holds for the bounds between component twin-width and rank-width given by Theorem 9. It would be interesting to study the tightness of the bound $\mathbf{tww} \leq 2\mathbf{cw}-2$ (where $\mathbf{tww}$ designs the twin-width), which is a direct consequence of Theorem 4.

6.2 Lower bounds on complexity

The algorithms relying on clique-width to solve H-Coloring by [21] in $O^*(s(H)^{\mathbf{cw}(G)})$ time are known to be optimal under the SETH. We have a similar optimality result for treewidth ($\mathbf{tw}$), with an algorithm solving H-Coloring in time $|V_H|^{\mathbf{tw}(G)}$, despite the existence of an algorithm in $(|V_H|-\varepsilon)^{\mathbf{tw}(G)}$ with $\varepsilon > 0$ being ruled out under SETH. A natural research direction is then to optimize the running time of the algorithm of Theorem 11, possibly by making use of $s(H)$, and prove a similar lower bound.

6.3 Extensions

Instead of solving $\#H$-Coloring the results of Section 5 can be extended to arbitrary binary constraints (*binary constraint satisfaction problems*, Bcsps). The notion of component twin-width indeed generalize naturally to both instances and a templates of a Bcsp.

Additionally, one may note that the algorithms detailed in Section 5 can be adapted to solve a "cost" version of $\#H$-Coloring: given a weight matrix C, the cost of an homomorphism f is $\sum_{u \in V_G} C(u, f(u))$, and we want to find an homomorphism of minimal cost.

A Proof of Theorem 11

▶ **Theorem 11.** *For any graph H there exists an algorithm running in time $O^*((2^{|V_H|}-1)^{\mathbf{ctww}(G)})$ which solves #H-COLORING on any input graph G (assuming that an optimal contraction sequence $(G_n,\dots,G_1)$ of G is given).*

Proof. For $k\in[n]$, $C=\{S_1,\dots,S_p\}\subseteq V_{G_k}$ a red-connected component of vertices of G_k, and for $\gamma: C\mapsto(2^{V_H}\setminus\emptyset)$, a *$H$-coloring of $G[\cup C]$ with profile γ* is a H-coloring f of $G[\cup C]$ such that for all $i\in[p]$, $f(S_i)=\gamma(S_i)$. *I.e.* the vertices of H used to color S_i are exactly the colors of the set $\gamma(S_i)$.

Then, define the set $COL(C,\gamma)$ as the set of H-colorings of $G[\cup C]$ with profile γ. We see that for every red-connected component C of G_k, the sets $COL(C,\gamma)$ for $\gamma: C\mapsto(2^{V_H}\setminus\emptyset)$ form a partition of the set of the H-colorings of $G[\cup C]$.

The principle of the algorithm is, similarly to the proof of Theorem 2 in the main text, to inductively maintain (from $k=n$ to 1) the knowledge of every $|COL(C,\gamma)|$ (stored in a tabular $\#col(C,\gamma)$) for a red-connected component C of G_k and $\gamma: C\mapsto(2^{V_H}\setminus\emptyset)$. In this way, since $\{V_G\}$ is a red-connected component of $G_1=(\{V_G\},\emptyset,\emptyset)$, we can obtain the number of H-colorings of $G[V_G]=G$ by computing

$$\sum_{T\in(2^{V_H}\setminus\emptyset)}\#col(\{V_G\},V_G\mapsto T).$$

First, note that the red-connected components of G_n are the $\{u\}$ for $u\in V_G$ (since $G_n=(V_G,E_G,\emptyset)$ has no red edge). For every $\gamma: u\mapsto\gamma(u)\in(2^{|V_H|}\setminus\emptyset)$ we let $\#col(\{u\},\gamma)\leftarrow 0$ if $|\gamma(u)|\neq 1$ and $\#col(\{u\},\gamma)\leftarrow 1$ if $|\gamma(u)|=1$. Hence, we correctly store the value of $|COL(\{u\},\gamma)|$ in the tabular $\#col(\{u\},\gamma)$.

We explain how to maintain this invariant after the contraction from G_{k+1} to G_k (with $k\in[n-1]$). By definition of a contraction sequence, G_k is of the form $G_k=:G_{k+1}/(u,v)$ with u and v two different vertices of G_{k+1}.

Note that every red-connected component of G_k is also a red-connected component of G_{k+1}, except the red-connected component C containing uv. We only have to compute $|COL(C,\gamma)|$ for any $\gamma: C\mapsto 2^{V_H}\setminus\emptyset$, and to store it in the tabular $\#col(C,\gamma)$. Initialize the value of $\#col(C,\gamma)$ with 0.

Let $C=:\{S_1\dots,S_{p-1},S'_p\}$, with $S'_p:=uv$, and $p:=|C|\leq\mathbf{ctww}(G)$. Since every pair of red-connected vertices in G_{k+1} (that contains neither u nor v) are red-connected in G_k (except u and v), C must be of the form

$$C:=(C_1\cup\dots\cup C_q\cup\{S'_p\})\setminus\{S_p,S_{p+1}\},$$

with $S_p:=u$ and $S_{p+1}:=v$ and $C_1\cup\dots\cup C_q=\{S_1,\dots,S_{p-1},S_p,S_{p+1}\}$,[5] and where $C_1,\dots,C_q$ (with $q>0$) are red-connected components of G_{k+1} whose union contains both $S_p=u$ and $S_{p+1}=v$. Notice that each S_i (for $i\in[p+1]$) belongs to a unique $C_{j(i)}$ with $j(i)\in[q]$.

An illustration of these notations are given in Figure 2.

The algorithm iterates over every family $(\gamma_j: C_j\mapsto(2^{V_H}\setminus\emptyset))_{1\leq j\leq q}$. Let $\gamma=\gamma_1\cup\dots\cup\gamma_q$ that maps every S_i (with $i\in[p-1]$) to $\gamma_{j(i)}(S_i)$, and that maps $S'_p=uv=S_p\cup S_{p+1}=u\cup v$ to $\gamma_{j(p)}(S_p)\cup\gamma_{j(p+1)}(S_{p+1})$. We check if there exists a $(i,i')\in[p]^2$ with $i\neq i'$, a black edge

[5] Note that $uv=S'_p=S_p\cup S_{p+1}=u\cup v$.

between S_i and $S_{i'}$ in G_{k+1}, and $(\gamma(S_i) \times \gamma(S_{i'})) \setminus E_H \neq \emptyset$, in time $O(p^2)$. If so, we move to the next family $(\gamma_j)_{1\leq j\leq q}$. Otherwise, we increment $\#col(C,\gamma)$ by $\prod_{j=1}^{q} \#col(C_j,\gamma_j)$.

Soundness: The soundness of this algorithm follows from the fact that for each $\gamma : C \mapsto 2^{V_H} \setminus \emptyset$, $COL(C,\gamma)$ is the disjointed union, for $(\gamma_1,\dots,\gamma_q)$ such that $\gamma = \gamma_1 \cup \dots \cup \gamma_q$, of the sets of H-colorings f such that for all $j \in [q]$ the profile of $f|_{C_j}$ is γ_j, that we denote by $COL(C,\gamma_1,\dots,\gamma_q)$. We only need to compute $|COL(C,\gamma_1,\dots,\gamma_q)|$, which can be derived by Claim 13. We then store the sum over $(\gamma_1,\dots,\gamma_q)$ such that $\gamma = \gamma_1 \cup \dots \cup \gamma_q$ in $\#col(C,\gamma)$.

▷ Claim 13. There are two distinct cases:

1. If there exists $(i,i') \in [p]^2$ such that $(S_i,S_{i'})$ is a black edge of G_{k+1} and $(\gamma_{j(i)}(S_i) \times \gamma_{j(i')}(S_{i'})) \setminus E_H \neq \emptyset$, then $COL(C,\gamma_1,\dots,\gamma_q) = \emptyset$.
2. If for all $(i,i') \in [p]^2$ such that $(S_i,S_{i'})$ is a black edge of $G_{k+1}, (\gamma_{j(i)}(S_i) \times \gamma_{j(i')}(S_{i'})) \subseteq E_H$, then a function $f : \cup C \mapsto V_H$ belongs to $COL(C,\gamma_1,\dots,\gamma_q)$ iff, for all $j \in [q]$, f restricted to C_j (denoted by f_j) belongs to $COL(C_j,\gamma_j)$.

Proof. We treat the two cases separately. In the first case, assume that there exists $(i,i') \in [p]^2$ such that $(S_i,S_{i'})$ is a black edge of G_{k+1} and $(\gamma_{j(i)}(S_i) \times \gamma_{j(i')}(S_{i'})) \setminus E_H \neq \emptyset$ and, for the sake of contradiction, suppose that there is $f \in COL(C,\gamma_1,\dots,\gamma_q)$. Take $(v_i,v_{i'}) \in (\gamma_{j(i)}(S_i) \times \gamma_{j(i')}(S_{i'})) \setminus E_H$. By definition of a profile, there exists $(u_i,u_{i'}) \in S_i \times S_{i'}$ with $f(u_i) = v_i$ and $f(u_{i'}) = v_{i'}$. Then, since there exists a black edge between S_i and $S_{i'}$ in G_{k+1}, this means by Property 1 that $(u_i,u_{i'}) \in E_G$. But $(f(u_i),f(u_{i'})) = (v_i,v_{i'}) \notin E_H$, so f is not a H-coloring, which contradicts the definition of f.

In the second case, assume that for all $(i,i') \in [p]^2$ such that $(S_i,S_{i'})$ is a black edge of $G_{k+1}, (\gamma_{j(i)}(S_i) \times \gamma_{j(i')}(S_{i'})) \subseteq E_H$. To prove necessity, notice that the restriction of a partial H-coloring is also a partial H-coloring, and by definition of $COL(C,\gamma_1,\dots,\gamma_q)$, if $f \in COL(C,\gamma_1,\dots,\gamma_q)$, then $f_j \in COL(C_j,\gamma_j)$.

To prove sufficiency, assume that $f : \cup C \mapsto V_H$ is such that for all $j \in [q], f_j \in COL(C_j,\gamma_j)$. Then, provided that f is a H-coloring of $G[\cup C]$, $f \in COL(C,\gamma_1,\dots,\gamma_q)$. Hence, we only have to prove that f is a H-coloring. So let $(u,u') \in E_G$. We prove that $(f(u),f(u')) \in E_H$. Observe that there exist S_i and $S_{i'}$ (with $(i,i') \in [p]^2$) such that $u \in S_i$ and $v \in S_{i'}$. If S_i and $S_{i'}$ are in the same red-connected component C_j (with $j \in [q]$) of G_{k+1}, then $(f(u),f(u')) = (f_j(u),f_j(u')) \in E_H$ because f_j is a H-coloring. Otherwise, $(S_i,S_{i'})$ is not a red edge of G_{k+1}, so $(S_i,S_{i'})$ is a black edge of G_{k+1}, since $(u,u') \in E_G$ and $(u,u') \in S_i \times S_{i'}$, by Property 1. By assumption, $(\gamma_{j(i)}(S_i) \times \gamma_{j(i')}(S_{i'})) \subseteq E_H$ and, by definition of a profile, $(f(u),f(u')) = (f_{j(i)}(u),f_{j(i')}(u')) \in \gamma_{j(i)}(S_i) \times \gamma_{j(i')}(S_{i'}) \subseteq E_H$. The latter shows that f is indeed a H-coloring. ◀

From Claim 13 it follows that choosing an f in $COL(C,\gamma_1,\dots,\gamma_q)$ is either impossible, or equivalent to choosing $f_j \in COL(C_j,\gamma_j)$ for all $j \in [q]$, which is why we add either 0 or $\prod_{j=1}^{q} \#col(C_j,\gamma_j)$ when treating the part of $\#col(C,\gamma)$ relative to the family $(\gamma_1,\dots,\gamma_q)$.

Complexity: To treat the red-connected component C, the only non-polynomial part is to iterate over every family $(\gamma_1,\dots,\gamma_q)$, which represents

$$\prod_{j=1}^{q} (2^{|V_H|}-1)^{|C_j|} = (2^{|V_H|}-1)^{|C|+1} \leq (2^{|V_H|}-1)^{\mathbf{ctww}(G)+1}$$

families to treat (recall that for all $j \in [q]$, γ_j is a non-empty subset of C_j). ◀

B Proof of Theorem 12

▶ **Theorem 12.** *#H-*Coloring *is solvable in time*

$$O^*((\mathbf{ctww}(H)+2)^{|V_G|}).$$

Proof. Consider an optimal contraction sequence $(H_m,\dots,H_1)$ of H, with $m := |V_H|$. We give an algorithm similar to that described in the proof of Theorem 11, except that we define profiles for red-connected component of each H_k, $k \in [m]$.

Let $C = \{T_1,\dots,T_p\}$ be a red connected component of H_k and let $\gamma = (S_1,\dots,S_p)$ be a p-tuple of pairwise disjoint subsets of V_G. An H-coloring f of $G[S_1 \cup \dots, \cup S_p]$ is said to have *C-profile* γ if for each $i \in [p]$, $f(S_i) \subseteq T_i$. Denote by $COL(\gamma, C)$ the set of partial H-colorings of G (*i.e.,* a H-Coloring of an induced subgraph) with C-profile γ. It is easy to compute the $|COL(\gamma, C)|$ for a red-connected component C of $H_m = (V_H, E_H, \emptyset)$ and $\gamma = (S)$ with $S \subseteq V_G$, since C is of the form $C = \{v\}$ with $v \in E_H$. We have $|COL((S), \{v\})| = 1$ if $S^2 \cap E_G = \emptyset$, and $|COL((S), \{v\})| = 0$, otherwise.

As in the proof of Theorem 11, for $k \in [m-1]$ the only red-connected component of H_k that is not a red-connected component of H_{k+1}, is the red-connected component $C = \{T_1,\dots,T_{p-1},T'_p\}$ that contains $T'_p = uv$ (the vertex obtained by contraction of $T_p = u$ and $T_{p+1} = v$ in H_{k+1}). Hence, C is of the form

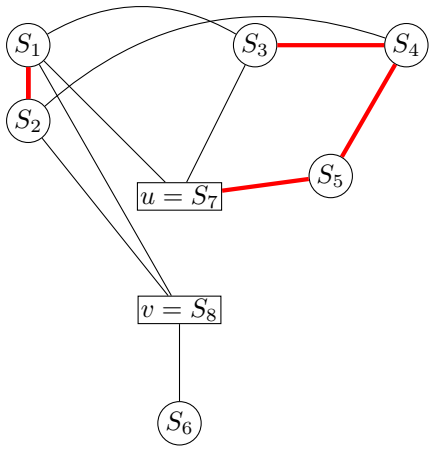
$$C = (C_1 \cup \dots \cup C_q \cup \{T'_p\}) \setminus \{T_p, T_{p+1}\},$$

with $C_1 \cup \dots \cup C_q = \{T_1,\dots,T_{p-1},T_p,T_{p+1}\}$, where $C_1,\dots,C_q$ are the red-connected components of H_{k+1} whose union contains $T_p = u$ and $T_{p+1} = v$. Again, each T_i belongs to a unique $C_{j(i)}$ with $j(i) \in [q]$.

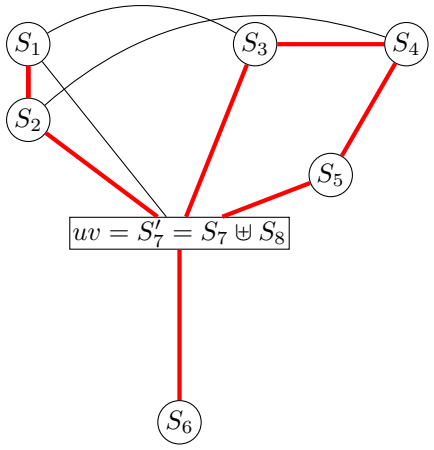
Then, as in the proof of Theorem 11, for all families of disjoint subsets of V_G and $\gamma = (S_1,\dots,S_{p-1},S'_p)$, we can compute the value of $|COL(\gamma, C)|$. Indeed, as in the proof of Theorem 11, it is the sum for every family $(\gamma_j)_{1\le j\le q}$ that defines the profile γ (*i.e.*, every γ_j is a family of pairwise disjoint subsets of V_G, and S'_p is of the form $S'_p = S_p \cup S_{p+1}$ with $S_p \cap S_{p+1} = \emptyset$ and $\forall j \in [q]$, $\gamma_j = (S_i)_{i\in j^{-1}(\{j\})}$)

of the value (1) $\prod_{j=1}^{q} |COL(\gamma_j, C_j)|$ if for every $(i,i') \in [p]^2$ with $j(i) \neq j(i')$ and for every edge $(u_i, u_{i'})$ of G with $u_i \in S_i$ and $u_{i'} \in S_{i'}$, there is a black edge between T_i and $T_{i'}$ in H_{k+1}, and (2) 0, otherwise.

The complexity of computing $|COL(\gamma, C)|$ for every γ is $(\mathbf{ctww}(H)+2)^{|V_G|}$, since exploring every family $(\gamma_j)_{1\le j\le q}$ containing only pairwise disjoint subsets of $|V_G|$ requires to explore $(\sum_{j=1}^{q} |C_j| + 1)^{|V_G|}$ families (any vertex of G can be mapped to a unique element in $\{T_1, T_2, \dots, T_{p+1}\}$ or none of them), which makes $(|C|+2)^n \le (\mathbf{ctww}(H)+2)^n$ possibilities. Since $H_1 = (\{V_H\}, \emptyset, \emptyset)$, we obtain the number of such H-colorings of G in time $O^*((\mathbf{ctww}(H)+2)^{|V_G|})$, and it is equal to $|COL(\{V_G\}, \{V_H\})|$. ◀



G_{k+1}



G_k

Figure 2 An example where merging $u = S_7$ and $v = S_8$ causes $j = 4$ different red-connected components to merge into a red-connected component of size $p = 7$. With the notations of this proof, we could have $C_1 = \{S_1, S_2\}, C_2 = \{S_3, S_4, S_5, S_7\}, C_3 = \{S_6\}$ and $C_4 = \{S_8\}$. For instance, $j(1) = j(2) = 1, j(3) = j(4) = j(5) = j(7) = 2, j(6) = 3$ and $j(8) = 4$

References

1 Pierre Bergé, Édouard Bonnet, and Hugues Déprés. Deciding twin-width at most 4 is NP-complete. In Mikolaj Bojanczyk, Emanuela Merelli, and David P. Woodruff, editors, *Proceedings of the 49th International Colloquium on Automata, Languages, and Programming (ICALP-2022)*, volume 229 of *LIPIcs*, pages 18:1–18:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPIcs.ICALP.2022.18.

2 Édouard Bonnet, Dibyayan Chakraborty, Eun Jung Kim, Noleen Köhler, Raul Lopes, and Stéphan Thomassé. Twin-width VIII: delineation and win-wins. *CoRR*, abs/2204.00722, 2022.

609 **3** Édouard Bonnet and Hugues Déprés. Twin-width can be exponential in treewidth. *CoRR*, abs/2204.07670, 2022. URL: `https://arxiv.org/abs/2204.07670`.

611 **4** Édouard Bonnet, Colin Geniet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width II: small classes. In *Proceedings of the 32nd ACM-SIAM Symposium on Discrete Algorithms (SODA-2021)*, pages 1977–1996. SIAM, 2021.

614 **5** Édouard Bonnet, Colin Geniet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width III: Max Independent Set, Min Dominating Set, and Coloring. In *Proceedings of the 48th International Colloquium on Automata, Languages, and Programming (ICALP-2021)*, volume 198 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 35:1–35:20, Dagstuhl, Germany, 2021. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. URL: `https://drops.dagstuhl.de/opus/volltexte/2021/14104`, `doi:10.4230/LIPIcs.ICALP.2021.35`.

620 **6** Édouard Bonnet, Colin Geniet, Romain Tessera, and Stéphan Thomassé. Twin-width VII: groups. *CoRR*, abs/2204.12330, 2022.

622 **7** Édouard Bonnet, Ugo Giocanti, Patrice Ossona de Mendez, Pierre Simon, Stéphan Thomassé, and Szymon Toruńczyk. Twin-width IV: ordered graphs and matrices. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing (STOC-2022)*, pages 924–937, 2022.

626 **8** Édouard Bonnet, Eun Jung Kim, Amadeus Reinald, and Stéphan Thomassé. Twin-width VI: the lens of contraction sequences. In *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA-2022)*, pages 1036–1056. SIAM, 2022.

629 **9** Édouard Bonnet, Eun Jung Kim, Amadeus Reinald, Stéphan Thomassé, and Rémi Watrigant. Twin-width and polynomial kernels. *Algorithmica*, 84:1–38, 2022.

631 **10** Édouard Bonnet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width I: tractable FO model checking. *ACM Journal of the ACM (JACM)*, 69(1):1–46, 2021.

633 **11** Édouard Bonnet, O-joung Kwon, David R Wood, et al. Reduced bandwidth: a qualitative strengthening of twin-width in minor-closed classes (and beyond). *arXiv preprint arXiv:2202.11858*, "", 2022.

636 **12** Édouard Bonnet, Jaroslav Nešetřil, Patrice Ossona de Mendez, Sebastian Siebertz, and Stéphan Thomassé. Twin-width and permutations. *CoRR*, abs/2102.06880, 2021.

638 **13** Binh-Minh Bui-Xuan, Jan Arne Telle, and Martin Vatshelle. Boolean-width of graphs. *Theoretical Computer Science*, 412(39):5187–5204, 2011.

640 **14** Bruno Courcelle and Stephan Olariu. Upper bounds to the clique width of graphs. *Discrete Applied Mathematics*, 101(1-3):77–114, 2000.

642 **15** Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer Publishing Company, Incorporated, 1st edition, 2015.

645 **16** Ronald de Haan and Stefan Szeider. Parameterized complexity classes beyond para-np. *Journal of Computer and System Sciences*, 87:16–57, 2017.

647 **17** Martin Dyer and Catherine Greenhill. The complexity of counting graph homomorphisms. *Random Structures & Algorithms*, 17(3-4):260–289, 2000.

649 **18** Jörg Flum and Martin Grohe. *Parameterized Complexity Theory*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2006. `doi:10.1007/3-540-29953-X`.

651 **19** Fedor V. Fomin, Alexander Golovnev, Alexander S. Kulikov, and Ivan Mihajlin. Lower bounds for the graph homomorphism problem. In *Proceedings of the 42nd International Colloquium on Automata, Languages, and Programming (ICALP-2015), 2015, Kyoto, Japan, July 6-10, 2015, Proceedings, Part I*, volume 9134 of *Lecture Notes in Computer Science*, pages 481–493. Springer, 2015. `doi:10.1007/978-3-662-47672-7\_39`.

656 **20** Piotr Formanowicz and Krzysztof Tanaś. A survey of graph coloring-its types, methods and applications. *Foundations of Computing and Decision Sciences*, 37(3):223–238, 2012.

658 **21** Robert Ganian, Thekla Hamm, Viktoriia Korchemna, Karolina Okrasa, and Kirill Simonov. The fine-grained complexity of graph homomorphism parameterized by clique-width. *CoRR*, abs/2210.06845, 2022.

22 Frank Gurski and Egon Wanke. On the relationship between nlc-width and linear nlc-width. *Theoretical Computer Science*, 347(1-2):76–89, 2005.

23 Petr Hliněnỳ. Twin-width of planar graphs is at most 11. *CoRR*, abs/2205.05378, 2022.

24 Michael Lampis. Finer tight bounds for coloring on clique-width. *SIAM Journal on Discrete Mathematics*, 34(3):1538–1558, 2020.

25 Sang-Il Oum. *Graphs of bounded rank-width*. Princeton University, 2005.

26 Sang-il Oum. Approximating rank-width and clique-width quickly. *ACM Transactions on Algorithms (TALG)*, 5(1):1–20, 2008.

27 Magnus Wahlström. New plain-exponential time classes for graph homomorphism. *Theory of Computing Systems*, 49(2):273–282, 2011.