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Central limit theorem under the Dedecker-Rio condition in some Banach spaces

Aurélie Bigot*

Abstract

We extend the central limit theorem under the Dedecker-Rio condition to adapted stationary and ergodic sequences of random variables taking values in a class of smooth Banach spaces. This result applies to the case of random variables taking values in $L^p(\mu)$, with $2 \leq p < \infty$ and μ a σ -finite real measure. As an application we give a sufficient condition for empirical processes indexed by Sobolev balls to satisfy the central limit theorem, and discuss about the optimality of these conditions.

Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(X_i)_{i \in \mathbb{Z}}$ be a strictly stationary sequence of centered real-valued random variables in \mathbb{L}^2 , adapted to a stationary filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$. In 2000, Dedecker and Rio proved in [DR00] the central limit theorem (in short CLT) for $(X_n)_n$ under the condition

$$X_0 \mathbb{E}(S_n | \mathcal{F}_0) \text{ converges in } \mathbb{L}^1 \quad (\mathbf{DR})$$

where $S_n = X_1 + \dots + X_n$. Let us look at the particular case where we consider a Markov sequence $(\xi_n)_{n \in \mathbb{Z}}$ with stationary distribution ν and transition operator P . Let $X_n = f(\xi_n)$ be a centered functional of our Markov sequence such that $\nu(f^2) < \infty$, then the CLT holds under the condition of convergence in $\mathbb{L}^1(\nu)$ of the series $\sum_{n=0}^{\infty} f P^n f$.

For α -mixing random variables, condition (\mathbf{DR}) leads to the following condition:

$$\sum_{n=0}^{+\infty} \int_0^{\alpha(\mathcal{F}_0, \sigma(X_n))} Q_{|X_0|}^2(u) du < \infty, \quad (\mathbf{0.1})$$

where $Q_{|X_0|}$ is the upper tail quantile function of $|X_0|$, and $(\alpha(\mathcal{F}_0, \sigma(X_n)))_{n \geq 1}$ is the sequence of strong mixing coefficients associated with $(X_n)_{n \in \mathbb{Z}}$ (see for instance [DR00, (2.1)]).

The CLT under condition (\mathbf{DR}) has been extended in [DM03] to random variables taking values in a separable Hilbert space. In this paper we extend this CLT to the case of r.v.'s taking values

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in a 2-smooth and Schauder decomposable Banach space. As we shall see, the main ingredients to obtain such a result are a martingale blocks decomposition, Theorem 5 in [Ros82] and Theorem 2.1 in [DM15]. Typically, the L^p spaces for $p \geq 2$ fit into our framework of Banach spaces, and this particular case will lead to a CLT for the empirical process as will be shown in Section 2.

Recently, several authors have extended other projective criteria valid in the real spaces context to the case of smooth Banach spaces. For instance, the Hannan condition (see [Han73]) has been extended in [DMP13] for random variables taking values in a 2-smooth Banach space having a Schauder basis. Such a condition can be written in the Banach space setting: $\sum_{n \in \mathbb{Z}} (\mathbb{E} \|P_0(X_n)\|_{\mathbb{B}}^2)^{1/2} < \infty$ where P_0 is the operator defined by $P_0 = \mathbb{E}(\cdot | \mathcal{F}_0) - \mathbb{E}(\cdot | \mathcal{F}_{-1})$, \mathbb{B} is the real and separable Banach space and $\|\cdot\|_{\mathbb{B}}$ its associated norm. Very recently the condition of Maxwell-Woodroffe (see [MW00]) has been extended by [Cun17] in Banach space settings. In the case of 2-smooth Banach spaces, the condition becomes $\sum_{n=1}^{\infty} n^{-3/2} (\mathbb{E} [\|\mathbb{E}(S_n | \mathcal{F}_0)\|_{\mathbb{B}}^2])^{1/2} < \infty$. Note that it has been shown in [DV08] that the conditions by Dedecker-Rio, Hannan and Maxwell-Woodroffe are independent.

The paper is organised as follows. In Section 1, we state an extension of the CLT under the condition **(DR)** to the case of 2-smooth and Schauder decomposable Banach spaces. As a consequence, in Section 2, we derive in Corollary 2.3 a sufficient condition for empirical processes indexed by Sobolev balls to satisfy the CLT, and discuss about the optimality of this condition. The proofs of the main results are postponed to Section 3.

1 A CLT in some smooth Banach spaces

In all the paper, $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ will be a real and separable Banach space. We shall consider the class of Banach spaces that are 2-smooth. This notion introduced by Pisier in [Pis75, Section 3] plays the same role with respect to vector martingales as spaces of type 2 do with respect to the sums of independent random vectors. Let us consider the following definition of 2-smooth Banach spaces.

Definition 1.1. *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable Banach space and define $\psi_2 : x \mapsto \|x\|_{\mathbb{B}}^2$. $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ is said to be 2-smooth if there exists $D > 0$ such that for any $x, u \in \mathbb{B}$,*

$$D^2\psi_2(x)(u, u) \leq D^2 \|u\|_{\mathbb{B}}^2.$$

Here $D^2\psi_2(x)$ denotes the usual second order Fréchet derivative of ψ_2 at point x .

As quoted in [Pin94], this definition implies the $(2, D)$ -smoothness in the sense of Pisier meaning that

$$\|x + y\|_{\mathbb{B}}^2 + \|x - y\|_{\mathbb{B}}^2 \leq 2\|x\|_{\mathbb{B}}^2 + 2D^2\|y\|_{\mathbb{B}}^2, \quad \forall x, y \in \mathbb{B}.$$

We shall also need the notion of Schauder decomposable Banach spaces.

Definition 1.2. *A family $\{x_n : n \in \mathbb{N}\}$ of elements of \mathbb{B} is called a Schauder basis if for any vector $x \in \mathbb{B}$ there exists a unique series $\sum_{n=0}^{\infty} a_n x_n$, with $a_n = a_n(x) \in \mathbb{R}$, which converges to x with respect to $\|\cdot\|_{\mathbb{B}}$.*

One of the properties of Schauder decomposable spaces is the uniform boundedness of the family of operators $(P_n)_n$, where P_n is the projection on the space generated by $\{x_k : k \leq n\}$. More precisely, there exists a constant $c > 0$ such that for any $n \in \mathbb{N}$, $\|P_n\| \leq c$, where $\|\cdot\|$ is the operator norm.

Example 1.3. Let $p \geq 2$ and μ a σ -finite measure on \mathbb{R} . As a consequence of [LPP11, Proposition 5.1], $L^p(\mu)$ is Schauder decomposable. In addition, equipped with its usual norm, it is also $(2, \sqrt{p-1})$ -smooth as shown in [Pin94, Proposition 2.1].

The main result of this paper is Theorem 1.4 below which extends [DR00, Theorem 1] to the Banach space setting.

Theorem 1.4. *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a 2-smooth and Schauder decomposable Banach space. Let $(X_i)_{i \in \mathbb{Z}}$ be an ergodic stationary sequence of centered \mathbb{B} -valued random variables, adapted to a non-decreasing and stationary filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ and such that $\mathbb{E}(\|X_0\|_{\mathbb{B}}^2) < \infty$. Set $S_n := \sum_{k=1}^n X_k$ and assume that*

$$\|X_0\|_{\mathbb{B}} \mathbb{E}(S_n | \mathcal{F}_0) \text{ converges in } \mathbb{L}_{\mathbb{B}}^1. \quad (1.1)$$

Then $(\frac{1}{\sqrt{n}}S_n)_{n \geq 1}$ converges in distribution to G , where G is a Gaussian \mathbb{B} -valued random variable whose covariance operator is given by : $K_G(x^, y^*) = \sum_{k \in \mathbb{Z}} \text{cov}(x^*(X_0), y^*(X_k))$ for any $x^*, y^* \in \mathbb{B}^*$.*

In view of applications, we shall give in Corollary 1.7 sufficient conditions for (1.1) to be verified. With this aim, we first introduce useful notations.

Notations 1.5. Let X and Y be real-valued random variables. Denote by :

- $Q_{|X|}$ the generalized inverse of the upper tail function $t \mapsto \mathbb{P}(|X| > t)$
- $G_{|X|}$ the inverse of $x \in [0, \mathbb{P}(|X| > 0)] \mapsto \int_0^x Q_{|X|}(u) du$
- $H_{X,Y}$ the generalized inverse of $x \mapsto \mathbb{E}(|X| \mathbb{1}_{|Y| > x})$.

Definition 1.6. *Consider a stationary sequence of random variables $(X_i)_{i \in \mathbb{Z}}$ adapted to a non-decreasing and stationary filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$. We define for any nonnegative integer k :*

$$\beta_{1,X}(k) = \left\| \sup_{\|f\|_{\infty} \leq 1} \left| P_{X_k | \mathcal{F}_0}(f) - P_{X_k}(f) \right| \right\|_1.$$

If there is no confusion on the r.v. to which we refer, we shall write $\beta_1(k)$ instead of $\beta_{1,X}(k)$ for the sake of clarity.

Corollary 1.7. *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a 2-smooth and Schauder decomposable Banach space. Let $(X_i)_{i \in \mathbb{Z}}$ be an ergodic stationary sequence of centered \mathbb{B} -valued random variables, adapted to a non-decreasing and stationary filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ and such that $\mathbb{E}(\|X_0\|_{\mathbb{B}}^2) < \infty$. Set $\gamma_i := \mathbb{E}(\|\mathbb{E}[X_i | \mathcal{F}_0]\|_{\mathbb{B}})$. Consider the conditions*

$$(i) \sum_{n \geq 1} \int_0^{\beta_1(n)} Q_{\|X_0\|_{\mathbb{B}}}^2(u) du < \infty,$$

$$(ii) \sum_{n \geq 1} \int_0^{\gamma_n} Q_{\|X_0\|_{\mathbb{B}}} \circ G_{\|X_0\|_{\mathbb{B}}}(u) du < \infty .$$

We have the implications (i) \Rightarrow (ii) \Rightarrow (1.1).

Proof of Corollary 1.7. By Berbee's coupling lemma (see [Rio17, Lemma 5.1]), there exists a random variable X_n^* distributed as X_n , independent of \mathcal{F}_0 and such that $\mathbb{P}(X_n \neq X_n^*) = \beta_1(n)$. Hence

$$\begin{aligned} \gamma_n &= \mathbb{E}(\|\mathbb{E}(X_n - X_n^* | \mathcal{F}_0)\|_{\mathbb{B}}) \leq \mathbb{E}(\|X_n - X_n^*\|_{\mathbb{B}} \mathbb{1}_{X_n \neq X_n^*}) \\ &\leq \mathbb{E}(\|X_n\|_{\mathbb{B}} \mathbb{1}_{X_n \neq X_n^*}) + \mathbb{E}(\|X_n^*\|_{\mathbb{B}} \mathbb{1}_{X_n \neq X_n^*}). \end{aligned}$$

Using the same argument as in the proof of Proposition 1 in [DD03], we get

$$\mathbb{E}(\|X_n\|_{\mathbb{B}} \mathbb{1}_{X_n \neq X_n^*}) \leq \int_0^{\mathbb{P}(X_n \neq X_n^*)} Q_{\|X_n\|_{\mathbb{B}}}(u) du,$$

implying that

$$\gamma_n \leq 2 \int_0^{\beta_1(n)} Q_{\|X_n\|_{\mathbb{B}}}(u) du \quad \text{and then} \quad G_{\|X_0\|_{\mathbb{B}}}^{-1}(\gamma_n/2) \leq \beta_1(n).$$

Therefore, using a change of variables, it follows that

$$\int_0^{\gamma_n} Q_{\|X_0\|_{\mathbb{B}}} \circ G_{\|X_0\|_{\mathbb{B}}}(u) du \leq 2 \int_0^{\gamma_n/2} Q_{\|X_0\|_{\mathbb{B}}} \circ G_{\|X_0\|_{\mathbb{B}}}(u) du \leq 2 \int_0^{\beta_1(n)} Q_{\|X_0\|_{\mathbb{B}}}^2(u) du.$$

This ends the proof of the first implication. The second implication is an immediate consequence of the proof of Proposition 1 in [DD03] by replacing the absolute values by the norm $\|\cdot\|_{\mathbb{B}}$. \square

In view of applications, let us give the following result which specifies the rates of decrease of $(\beta_1(k))_{k>0}$ and moments of $\|X_0\|_{\mathbb{B}}$ for (1.1) to hold. Its proof follows directly from [Rio17, Annex C with $p = 2$].

Corollary 1.8. *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a 2-smooth and Schauder decomposable Banach space. Let $(X_k)_{k \in \mathbb{Z}}$ be an ergodic stationary sequence of centered \mathbb{B} -valued random variables such that $\mathbb{E}\|X_0\|_{\mathbb{B}}^2 < \infty$, and adapted to a non-decreasing and stationary filtration $(\mathcal{F}_k)_k$. Assume that one of the following conditions holds:*

$$(i) \text{ there exists } r > 2 \text{ such that } \mathbb{E}(\|X_0\|_{\mathbb{B}}^r) < \infty \text{ and } \sum_{n \geq 0} (n+1)^{2/(r-2)} \beta_1(n) < \infty$$

$$(ii) \text{ there exist } r > 2 \text{ and } c > 0 \text{ such that for any } x, \mathbb{P}(\|X_0\|_{\mathbb{B}} > x) \leq \left(\frac{c}{x}\right)^r \text{ and } \sum_{n \geq 0} \beta_1(n)^{1-2/r} < \infty$$

$$(iii) \text{ there exist } a > 0 \text{ and } \tau > 0 \text{ such that } \mathbb{E}\left[\|X_0\|_{\mathbb{B}}^2 (\log(1 + \|X_0\|_{\mathbb{B}}))^a\right] < \infty \text{ and } \beta_1(n) = O\left(e^{-\tau n^{1/a}}\right).$$

Then $\sum_{n \geq 1} \int_0^{\beta_1(n)} Q_{\|X_0\|_{\mathbb{B}}}^2(u) du < \infty$ is verified and the conclusion of Theorem 1.4 holds.

2 Applications to the empirical processes over Sobolev balls for dependent sequences

Let us consider $(Y_i)_{i \in \mathbb{Z}}$ a stationary and ergodic sequence of real random variables, whose cumulative distribution function is denoted F , and define $F_n(t) = \frac{1}{n} \sum_{k=1}^n \mathbb{1}_{Y_k \leq t}$ the empirical distribution function. We are interested in the asymptotic behavior of the centered empirical distribution function in $L^p(\mu)$, where μ is a σ -finite measure on \mathbb{R} .

We suppose that

$$\int_{\mathbb{R}_-} F(t)^p d\mu(t) + \int_{\mathbb{R}_+} (1 - F(t))^p d\mu(t) < \infty \quad (2.1)$$

so that $F_n - F$ is an element of $L^p(\mu)$.

In [DM07] the link between the convergence in distribution of $\sqrt{n}(F_n - F)$ in $L^p(\mu)$ and Donsker classes has been clearly established. More precisely, let us denote

$$W_{1,q}(\mu) := \left\{ f : \mathbb{R} \rightarrow \mathbb{R} : f(x) = f(0) + \mathbb{1}_{x>0} \int_{[0,x[} g d\mu - \mathbb{1}_{x \leq 0} \int_{[x,0[} g d\mu, \|g\|_{q,\mu} \leq 1 \right\},$$

where q is the conjugate exponent of p and $\|\cdot\|_{q,\mu}$ is the usual norm on $L^q(\mu)$. Then according to [DM07, Lemma 1] the following convergences are equivalent:

- (i) $\{\sqrt{n}(F_n - F)(t)\}_t \xrightarrow{\mathcal{L}} \{G(t)\}_t$ in $L^p(\mu)$
- (ii) $\left\{ \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n f(Y_i) - \mathbb{E}f(Y_0) \right) \right\} \xrightarrow{\mathcal{L}} \{G_1(f)\}$ in $\ell^\infty(W_{1,q}(\mu))$

where $\ell^\infty(W_{1,q}(\mu))$ is the space of all functions $\phi : W_{1,q}(\mu) \rightarrow \mathbb{R}$ such that $\sup_{f \in W_{1,q}(\mu)} |\phi(f)|$ is finite, and $G_1(f) = \int g(t)G(t)d\mu(t)$.

Hence, proving that $W_{1,q}(\mu)$ is a Donsker class for $(Y_n)_n$ is equivalent to proving that $\sqrt{n}(F_n - F)$ converges weakly in $L^p(\mu)$ to a Gaussian process.

To study the asymptotic behavior of $\sqrt{n}(F_n - F)$, we then define the random process:

$$\forall k \in \mathbb{Z}, X_k = \{\mathbb{1}_{Y_k \leq t} - F(t) : t \in \mathbb{R}\} \quad (2.2)$$

which takes values in $L^p(\mu)$. With such a notation, the study of the asymptotic behavior of $(\sqrt{n}(F_n - F))_n$ is equivalent to the study of the asymptotic behavior of $(S_n/\sqrt{n})_n$ in $L^p(\mu)$ where $S_n = X_1 + \dots + X_n$. Hence, since $L^p(\mu)$ is a 2-smooth and Schauder decomposable Banach space, the centered empirical distribution behavior in $L^p(\mu)$ will follow from an application of Theorem 1.4 and in particular of Corollary 1.7. To state the condition in terms of dependence conditions on the sequence $(Y_k)_{k \in \mathbb{Z}}$, we introduce the following weak dependence coefficients (see [DP05]).

Definition 2.1. *Let $(Y_i)_{i \in \mathbb{Z}}$ be a stationary sequence of real random variables adapted to a stationary filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$. For any nonnegative integer k , let*

$$b_k = \sup_{t \in \mathbb{R}} |\mathbb{E}(\mathbb{1}_{Y_k \leq t} | \mathcal{M}_0) - \mathbb{E}(\mathbb{1}_{Y_k \leq t})| \quad \text{and} \quad \tilde{\beta}_{1,Y}(k) := \mathbb{E}(b_k).$$

Let us consider the same notations as in [DM07]:

Notations 2.2. Define the function F_μ by: $F_\mu(x) = \mu(]0, x])$ if $x \geq 0$ and $F_\mu(x) = -\mu([x, 0[)$ if $x \leq 0$. Define also the nonnegative random variable $Y_{p,\mu} = |F_\mu(Y_0)|^{1/p}$.

As an application of Corollary 1.7, we derive the following result.

Corollary 2.3. *Let $p \in [2, +\infty[$ and assume that*

$$\sum_{n \geq 0} \int_0^{\tilde{\beta}_{1,Y}(n)} Q_{Y_{p,\mu}}^2(u) du < \infty. \quad (2.3)$$

Then

$$\sqrt{n}(F_n - F) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} G \quad \text{in } L^p(\mu), \quad (2.4)$$

where G is a Gaussian process whose covariance operator is given, for any $x^*, y^* \in L^q(\mu)$, by $K_G(x^*, y^*) = \sum_{k \in \mathbb{Z}} \text{cov}(x^*(X_0), x^*(X_k))$. In particular,

$$n^{p/2} \int |F_n - F|^p d\mu \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \int |G|^p d\mu. \quad (2.5)$$

Remark 2.4. When μ is finite, (2.3) simply reads as $\sum_{n \geq 0} \tilde{\beta}_{1,Y}(n) < \infty$.

With the equivalence with Donsker classes in mind, it is interesting to study the envelope function of our class of functions. Note that $x \mapsto |F_\mu(x)|^{1/p}$ is the smallest envelope function of $W_{1,q}(\mu)$ which means that

$$|F_\mu(x)|^{1/p} = \sup\{|f(x) - f(0)| : f \in W_{1,q}(\mu)\}. \quad (2.6)$$

Indeed, for any $f \in W_{1,q}(\mu)$ and any real x , by Hölder's inequality,

$$|f(x) - f(0)| \leq |F_\mu(x)|^{1/p} \|g\|_{q,\mu} \leq |F_\mu(x)|^{1/p}.$$

On the other hand, define for any $x \in \mathbb{R}$:

$$f_x : y \mapsto \mathbb{1}_{y>0} \int_{]0,y[} g_x d\mu - \mathbb{1}_{y \leq 0} \int_{]y,0[} g_x d\mu$$

$$\text{with } g_x : t \mapsto \begin{cases} \mathbb{1}_{]0,x]}(t) |F_\mu(x)|^{-1/q} & \text{if } F_\mu(x) \neq 0 \text{ and } x > 0 \\ -\mathbb{1}_{]x,0]}(t) |F_\mu(x)|^{-1/q} & \text{if } F_\mu(x) \neq 0 \text{ and } x < 0 \\ 0 & \text{either} \end{cases}.$$

One has that f_x belongs to $W_{1,q}(\mu)$ for any real x and $f_x(x) = |F_\mu(x)|^{1/p}$. This ends the proof of (2.6).

Note that when $(Y_i)_{i \in \mathbb{Z}}$ is a sequence of i.i.d. random variables, (2.3) reads as

$$\int_0^1 Q_{Y_{p,\mu}}^2(u) du < \infty \quad \text{i.e.} \quad \mathbb{E}[|F_\mu(Y_0)|^{2/p}] < \infty.$$

It means that the smallest envelope function of $W_{1,q}(\mu)$ is square integrable. This condition together with an entropy condition ensure the CLT (see Theorem 2.5.2 in [VW96]).

Remark 2.5. It is worth noting that, in the particular case where μ is the Lebesgue measure on \mathbb{R} denoted by λ , $W_{1,q}(\lambda)$ can be rewritten

$$W_{1,q}(\lambda) = \left\{ f : \mathbb{R} \rightarrow \mathbb{R} : f(x) = f(0) + \mathbb{1}_{x>0} \int_{[0,x[} g(t) dt - \mathbb{1}_{x \leq 0} \int_{[x,0[} g(t) dt, \|g\|_q \leq 1 \right\}$$

which is the space of absolutely continuous functions f such that $\lambda(|f'|^q) \leq 1$. In particular, it contains the unit ball of the Sobolev space of order 1 with respect to $L^q(\lambda)$. Moreover, since in this case F_μ is the identity function, condition (2.3) can be rewritten

$$\sum_{n \geq 0} \int_0^{\tilde{\beta}_{1,Y}^{(n)}} Q_{|Y_0|}^{2/p}(u) du < \infty.$$

Comment on the optimality of condition (2.3).

In the dependent case, condition (2.3) of Corollary 2.3 implies that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n (|F_\mu(Y_i)|^{1/p} - \mathbb{E}(|F_\mu(Y_i)|^{1/p})) \quad (2.7)$$

converges in distribution to a Gaussian r.v. The proposition below which is essentially due to [DMR94], shows that condition (2.3) is essentially optimal for the convergence in distribution of (2.7) to a Gaussian r.v. to hold.

Proposition 2.6. [DMR94] *Let $a > 1$. Suppose that Y_0 is a real-valued r.v. whose distribution function F is continuous and such that $\int_0^1 u^{-1/a} Q_{Y_{p,\mu}}^2(u) du = +\infty$. Then, there exists a stationary Markov chain $(Z_i)_{i \in \mathbb{Z}}$ with marginal distribution function F and such that*

(i) $0 < \liminf_{n \rightarrow \infty} n^\alpha \beta_1(n) \leq \limsup_{n \rightarrow \infty} n^\alpha \beta_1(n) < \infty$, here $(\beta_1(n))_n$ denotes the sequence of strong β_1 -mixing coefficients of $(Z_i)_{i \in \mathbb{Z}}$

(ii) $\frac{1}{\sqrt{n}} \sum_{i=1}^n (|F_\mu(Z_i)|^{1/p} - \mathbb{E}(|F_\mu(Z_i)|^{1/p}))$ does not converge in distribution to a Gaussian law.

Let us see how Proposition 2.6 can be deduced from Theorem 5 in [DMR94]. Let $\tilde{F} := |F_\mu \circ F^{-1}(\cdot)|^{1/p}$. Note that $\int_0^1 u^{-1/a} Q_{Y_{p,\mu}}^2(u) du$ is convergent if and only if, with \tilde{F} defined above, $\int_0^{1/2} u^{-1/a} \tilde{F}^2(u) du$ and $\int_0^{1/2} u^{-1/a} \tilde{F}^2(1-u) du$ are convergent. In other words, in our setting one of

the two previous integrals is infinite. Let us assume for instance that $\int_0^{1/2} u^{-1/a} \tilde{F}^2(1-u) du = +\infty$. Theorem 5 in [DMR94], applied to the function $f : x \mapsto \tilde{F}(1-x)$, asserts that there exists a stationary Markov chain $(U_i)_{i \in \mathbb{Z}}$ with uniform marginal distributions on $[0, 1]$ such that $(\beta_{1,U}(n))_{n \in \mathbb{Z}}$ satisfies (i) and

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n (|F_\mu \circ F^{-1}(1-U_i)|^{1/p} - \mathbb{E}(|F_\mu \circ F^{-1}(1-U_i)|^{1/p}))$$

does not converge in distribution to a Gaussian distribution. Furthermore, setting $Z_i = F^{-1}(1-U_i)$, the Markov chain $(Z_i)_{i \in \mathbb{Z}}$ admits F as marginal distribution function and the same mixing coefficients as $(U_i)_{i \in \mathbb{Z}}$ so that (i) is verified.

Furthermore, analogously to the first section, we get sufficient conditions for (2.3) to be satisfied.

Corollary 2.7. *Let $(X_k)_{k \in \mathbb{Z}}$ and $(Y_k)_{k \in \mathbb{Z}}$ be as defined in (2.1) and (2.2), and $(\mathcal{F}_k)_k$ be a filtration to which $(X_k)_k$ is adapted. Then under one of these conditions:*

(i) *there exists $r > 2$ such that $\mathbb{E}(Y_{p,\mu}^r) < \infty$ and $\sum_{n \geq 0} (n+1)^{2/(r-2)} \tilde{\beta}_{1,Y}(n) < \infty$*

(ii) *there exists $r > 2$ and $c > 0$ such that for any x , $\mathbb{P}(Y_{p,\mu} > x) \leq (\frac{c}{x})^r$ and $\sum_{n \geq 0} \tilde{\beta}_{1,Y}(n)^{1-2/r} < \infty$*

(iii) *there exists $a > 0$ and $\tau > 0$ such that $\mathbb{E} \left[Y_{p,\mu}^2 (\log(1 + Y_{p,\mu}))^a \right] < \infty$ and $\tilde{\beta}_{1,Y}(n) = O(e^{-\tau n^{1/a}})$*

the condition (2.3) is verified, so Corollary 2.3 applies.

Application to the empirical process in L^p for intermittent maps.

For $\gamma \in]0, 1[$, let $T_\gamma : [0, 1] \rightarrow [0, 1]$ be the intermittent map defined by [LSV99] as follows:

$$T_\gamma(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1] \end{cases} \quad (2.8)$$

If there is no confusion we write T for the sake of clarity. As shown in [LSV99], for all $\gamma \in]0, 1[$, there exists a unique absolutely continuous T_γ -invariant probability measure ν_γ (or simply ν) on $[0, 1]$ whose density h_γ satisfies: there exist two finite constants $c_1, c_2 > 0$ such that for all $x \in [0, 1]$, $c_1 \leq x^\gamma h_\gamma(x) \leq c_2$. Let us fix γ and consider K the Perron-Frobenius operator of T with respect to ν defined by

$$\nu(f \circ T.g) = \nu(f.Kg), \quad \text{for any } f, g \in \mathbb{L}^2(\nu). \quad (2.9)$$

Then, by considering $(X_i)_{i \in \mathbb{Z}}$ a stationary Markov chain with invariant measure ν and transition kernel K , for any positive integer n , on the probability space $([0, 1], \nu)$, (T, T^2, \dots, T^n) is distributed as $(X_n, X_{n-1}, \dots, X_1)$ (see for instance Lemma XI.3 in [HH01]). Consequently, the two following empirical processes have the same distribution

-
- $\left\{ G_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^n [\mathbb{1}_{T^k \leq t} - F(t)]; t \in [0, 1] \right\}$
 - $\left\{ L_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^n [\mathbb{1}_{X_k \leq t} - F(t)]; t \in [0, 1] \right\}$

where $F(t) = \nu([0, t])$. Since ν is supported on $[0, 1]$, condition (2.3) reads as $\sum_{n \geq 0} \tilde{\beta}_{1,X}(n) < \infty$. Now, from [DDT15, Proposition 6.2], we have the upper bound

$$\tilde{\beta}_{1,X}(n) \leq \frac{C}{(n+1)^{(1-\gamma)/\gamma}}. \quad (2.10)$$

Hence, applying Corollary 2.3, we derive that for any $\gamma \in]0, 1/2[$ and any $p \geq 2$

$$\{G_n(t) : t \in [0, 1]\} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \{G(t) : t \in [0, 1]\} \text{ in } \mathbb{L}^p([0, 1], \lambda) \quad (2.11)$$

where G is a Gaussian process.

We could also consider unbounded but monotonic observables φ of the iterates such as $\varphi(x) = 1/x^\alpha$ or $\varphi(x) = 1/(1-x)^\alpha$ since for such functions, the $\tilde{\beta}_1$ -coefficients of $(\varphi(X_k))_{k \geq 0}$ are of the same order than the initial ones. More precisely taking into account the behavior of the density h_γ of ν , one can prove that condition (2.3) reads as $\alpha < \frac{p}{2}(1-2\gamma)$ when $\varphi(x) = 1/x^\alpha$ and $\alpha < \frac{p}{2} \frac{1-2\gamma}{1-\gamma}$ when $\varphi(x) = 1/(1-x)^\alpha$.

3 Proofs

We start this section by a general CLT for random variables taking values in 2-smooth Banach spaces. It will be a building block in the proof of Theorem 1.4 and has interest in itself.

3.1 A general result

Theorem 3.1. *Let \mathbb{B} be a 2-smooth Banach space and let $(X_i)_{i \in \mathbb{Z}}$ be a stationary sequence of \mathbb{B} -valued centered random variables, adapted to a non-decreasing and stationary filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$, and such that $\mathbb{E}(\|X_0\|_{\mathbb{B}}^2) < \infty$. Set $S_n := \sum_{k=1}^n X_k$ and assume that*

$$\text{for any } x^* \in \mathbb{B}^*, \left(\frac{[x^*(S_n)]^2}{n} \right)_n \text{ is an uniformly integrable family,} \quad (3.1)$$

$$\frac{1}{n} \mathbb{E} \left[\|\mathbb{E}(S_n | \mathcal{F}_0)\|_{\mathbb{B}}^2 \right] \xrightarrow[n \rightarrow +\infty]{} 0, \quad (3.2)$$

$$\text{for any } x^* \in \mathbb{B}^*, \text{ there exists } \sigma^2(x^*) \text{ such that } \mathbb{E} \left| \mathbb{E} \left(\frac{[x^*(S_n)]^2}{n} \middle| \mathcal{F}_0 \right) - \sigma^2(x^*) \right| \xrightarrow[n \rightarrow +\infty]{} 0, \quad (3.3)$$

and there exists $(F_l)_l$ a sequence of finite dimensional subspaces of \mathbb{B} such that

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E}[q_{F_l}^2(S_n)] \xrightarrow[l \rightarrow +\infty]{} 0, \quad (3.4)$$

where $q_F(x) := \inf\{\|x - y\|_{\mathbb{B}} : y \in F\}$ for any $x \in \mathbb{B}$.

Then $\left(\frac{1}{\sqrt{n}}S_n\right)_{n \geq 1}$ converges in distribution to G , where G is a \mathbb{B} -valued Gaussian random variable whose law γ is such that for any $x^* \in \mathbb{B}^*$, $\hat{\gamma}(x^*) = e^{-\sigma^2(x^*)/2}$.

Remark 3.2. Conditions (3.1) and (3.2) can be replaced by the two following conditions:

$$\left(\frac{\|S_n\|_{\mathbb{B}}^2}{n}\right)_n \text{ is an uniformly integrable family,} \quad (3.5)$$

$$\frac{1}{\sqrt{n}}\mathbb{E}\|\mathbb{E}(S_n|\mathcal{F}_0)\|_{\mathbb{B}} \xrightarrow{n \rightarrow +\infty} 0. \quad (3.6)$$

Proof of Remark 3.2. Let x^* be an element of \mathbb{B}^* . Then, there exists $c(x^*) > 0$ such that for any $x \in \mathbb{B}$, $|x^*(x)| \leq c(x^*)\|x\|_{\mathbb{B}}$. Hence,

$$\forall n \in \mathbb{N}^*, \left| \frac{[x^*(S_n)]^2}{n} \right| \leq c(x^*)^2 \frac{\|S_n\|_{\mathbb{B}}^2}{n}.$$

So, condition (3.5) implies that $\left(\frac{[x^*(S_n)]^2}{n}\right)_n$ is an uniformly integrable family.

On another hand, for any $A > 0$,

$$\frac{1}{n} \|\mathbb{E}(S_n|\mathcal{F}_0)\|_{\mathbb{B}}^2 \leq \left(\frac{1}{\sqrt{n}} \|\mathbb{E}(S_n|\mathcal{F}_0)\|_{\mathbb{B}} \mathbb{1}_{\left\| \mathbb{E}\left(\frac{S_n}{\sqrt{n}}|\mathcal{F}_0\right)\right\|_{\mathbb{B}} \leq A} \right)^2 + \left(\frac{1}{\sqrt{n}} \|\mathbb{E}(S_n|\mathcal{F}_0)\|_{\mathbb{B}} \mathbb{1}_{\left\| \mathbb{E}\left(\frac{S_n}{\sqrt{n}}|\mathcal{F}_0\right)\right\|_{\mathbb{B}} > A} \right)^2.$$

Clearly,

$$\left(\frac{1}{\sqrt{n}} \|\mathbb{E}(S_n|\mathcal{F}_0)\|_{\mathbb{B}} \mathbb{1}_{\left\| \mathbb{E}\left(\frac{S_n}{\sqrt{n}}|\mathcal{F}_0\right)\right\|_{\mathbb{B}} \leq A} \right)^2 \leq \frac{A}{\sqrt{n}} \|\mathbb{E}(S_n|\mathcal{F}_0)\|_{\mathbb{B}},$$

which converges to 0 in \mathbb{L}^1 by condition (3.6). On another hand, by Lemma 6.3 in [DMP14], we have for any \mathbb{B} -valued r.v. Y , any $\varepsilon > 0$ and any σ -algebra \mathcal{B} ,

$$\mathbb{E}\left(\|Y\|_{\mathbb{B}}^2 \mathbb{1}_{\|\mathbb{E}(Y|\mathcal{B})\|_{\mathbb{B}} > 2\varepsilon}\right) \leq \mathbb{E}\left(\|Y\|_{\mathbb{B}}^2 \mathbb{1}_{\|Y\|_{\mathbb{B}} > \varepsilon}\right). \quad (3.7)$$

Hence, applying Jensen's inequality and taking into account (3.7), we get

$$\begin{aligned} \mathbb{E}\left[\left(\frac{1}{\sqrt{n}} \|\mathbb{E}(S_n|\mathcal{F}_0)\|_{\mathbb{B}} \mathbb{1}_{\left\| \mathbb{E}\left(\frac{S_n}{\sqrt{n}}|\mathcal{F}_0\right)\right\|_{\mathbb{B}} > A}\right)^2\right] &\leq \mathbb{E}\left(\frac{\|S_n\|_{\mathbb{B}}^2}{n} \mathbb{1}_{\left\| \mathbb{E}\left(\frac{S_n}{\sqrt{n}}|\mathcal{F}_0\right)\right\|_{\mathbb{B}} > A}\right) \\ &\leq \mathbb{E}\left(\frac{\|S_n\|_{\mathbb{B}}^2}{n} \mathbb{1}_{\|S_n\|_{\mathbb{B}}/\sqrt{n} > A/2}\right), \end{aligned}$$

which converges to zero by (3.5) by first letting n tend to infinity and after A . So, overall, (3.5) together with (3.6) imply (3.2). \square

3.2 Proof of Theorem 3.1

To prove it, we follow the method of proof given in the proof of [MPU19, Theorem 4.31]. This method consists in constructing super big blocks of random variables and then in approximating them by a triangular array of martingales to which a CLT is applied. In the Banach space setting, to prove Theorem 3.1 we shall rather use [Ros82, Theorem 5] instead of [MPU19, Theorem 2.4]. For any fixed positive integer m , set $p = \lfloor n/m \rfloor$ and let

$$X_{n,j} := \sum_{k=p(j-1)+1}^{pj} \frac{1}{\sqrt{n}} X_k \quad \text{and} \quad \mathcal{F}_{n,j} := \mathcal{F}_{jp}.$$

Let us consider the conditionally centered random variables

$$\widetilde{X}_{n,j} := X_{n,j} - \mathbb{E}(X_{n,j} | \mathcal{F}_{n,j-1}),$$

so that $(\widetilde{X}_{n,j})_{1 \leq j \leq n}$ is a triangular array of martingale differences adapted to the array of filtrations $(\mathcal{F}_{n,j})_{1 \leq j \leq n}$.

As done in [MPU19, proof of Theorem 4.31, p. 133], let consider $(m_n)_n$ a sequence sufficiently slow growing such that, as n tends to infinity

$$m = m_n \rightarrow +\infty, \quad \frac{m}{\sqrt{n}} \rightarrow 0 \tag{3.8}$$

and,

$$\sqrt{m} \mathbb{E} \left(\frac{1}{\sqrt{p}} S_p \middle| \mathcal{F}_0 \right) \xrightarrow{n \rightarrow +\infty} 0 \quad \text{in } \mathbb{L}^1(\mathbb{B}). \tag{3.9}$$

We have in particular, from (3.8), that $\frac{mp}{n} \rightarrow 1$.

Note that

$$\begin{aligned} \sum_{j=1}^m \widetilde{X}_{n,j} &= \sum_{j=1}^m X_{n,j} - \sum_{j=1}^m \mathbb{E}(X_{n,j} | \mathcal{F}_{n,j-1}) \\ &= \frac{1}{\sqrt{n}} S_n + \frac{1}{\sqrt{n}} (S_{pm} - S_n) - \sum_{j=1}^m \mathbb{E}(X_{n,j} | \mathcal{F}_{n,j-1}). \end{aligned}$$

On the one hand,

$$\mathbb{E} \left\| \frac{1}{\sqrt{n}} (S_{pm} - S_n) \right\|_{\mathbb{B}} \leq \frac{m}{\sqrt{n}} \mathbb{E} \|X_0\|_{\mathbb{B}},$$

which converges to 0 as n tends to $+\infty$ from (3.8). On another hand, bearing in mind stationarity,

$$\mathbb{E} \left\| \sum_{j=1}^m \mathbb{E}(X_{n,j} | \mathcal{F}_{n,j-1}) \right\|_{\mathbb{B}} \leq \sqrt{\frac{mp}{n}} \sqrt{m} \mathbb{E} \left\| \mathbb{E} \left(\frac{S_p}{\sqrt{p}} \middle| \mathcal{F}_0 \right) \right\|_{\mathbb{B}},$$

which converges to 0 as n tends to $+\infty$ combining conditions (3.8) and (3.9).

So, overall, Theorem 1.4 will be proved if one can show that $\sum_{j=1}^m \widetilde{X}_{n,j}$ converges in distribution to G . To this end, we shall use Theorem 5 in [Ros82]. Hence, we have to show that the following conditions are verified:

(i) there exists $\psi : \mathbb{B}^* \rightarrow \mathbb{R}_+$ such that for any $x^* \in \mathbb{B}^*$, $\sigma_n^2(x^*) \xrightarrow{\mathbb{P}} \psi(x^*)$ as $n \rightarrow +\infty$, where

$$\sigma_n^2(x^*) = \sum_{j=1}^m \mathbb{E} \left([x^*(\widetilde{X}_{n,j})]^2 \middle| \mathcal{F}_{n,j-1} \right)$$

(ii) for any $x^* \in \mathbb{B}^*$ and any $\varepsilon > 0$, $\sum_{j=1}^m \mathbb{E} \left[(x^*(\widetilde{X}_{n,j})^2 \mathbb{1}_{|x^*(\widetilde{X}_{n,j})| > \varepsilon} \middle| \mathcal{F}_{n,j-1} \right] \xrightarrow{\mathbb{P}} 0$ as $n \rightarrow +\infty$

(iii) there exists a sequence $(F_l)_l$ of finite subspaces of \mathbb{B} such that

$$\limsup_{n \rightarrow +\infty} \sum_{j=1}^m \mathbb{E} \left[q_{F_l}^2(\widetilde{X}_{n,j}) \middle| \mathcal{F}_{n,j-1} \right] \xrightarrow{\mathbb{P}} 0 \text{ as } l \rightarrow +\infty.$$

First, since for any integer k , $\mathbb{E}(\|X_k\|_{\mathbb{B}}^2)$ is finite, it appears immediately that for any $x^* \in \mathbb{B}^*$ and any n, j , $\mathbb{E}[x^*(\widetilde{X}_{n,j})]^2 < \infty$.

Let us begin with the proof of condition (i). Let x^* be a linear form defined on \mathbb{B} . Using the same notations as in Rosinski's paper, we write

$$\sigma_n^2(x^*) := \sum_{j=1}^m \mathbb{E} \left[[x^*(\widetilde{X}_{n,j})]^2 \middle| \mathcal{F}_{n,j-1} \right] = \sum_{j=1}^m \left(\mathbb{E} \left[[x^*(X_{n,j})]^2 \middle| \mathcal{F}_{n,j-1} \right] - \mathbb{E} \left[x^*(X_{n,j}) \middle| \mathcal{F}_{n,j-1} \right]^2 \right).$$

Hence, by stationarity and condition (3.8), we derive

$$\limsup_{n \rightarrow +\infty} \mathbb{E} \left| \sigma_n^2(x^*) - \sigma^2(x^*) \right| \leq \limsup_{p \rightarrow +\infty} \mathbb{E} \left| \mathbb{E} \left(\frac{[x^*(S_p)]^2}{p} \middle| \mathcal{F}_0 \right) - \sigma^2(x^*) \right| + \limsup_{p \rightarrow +\infty} \mathbb{E} \left(\left| \mathbb{E} \left(\frac{x^*(S_p)}{\sqrt{p}} \middle| \mathcal{F}_0 \right) \right|^2 \right). \quad (3.10)$$

Then, combining (3.10), (3.2) and (3.3), we get

$$\sigma_n^2(x^*) \xrightarrow[n \rightarrow +\infty]{\mathbb{L}^1} \sigma^2(x^*).$$

This ends the proof of (i).

Let us now check condition (ii). Let ε be a positive real number. For any $n \in \mathbb{N}^*$, in accordance with [MPU19, Lemma 2.28] and by stationarity

$$\begin{aligned} & \mathbb{E} \left| \sum_{j=1}^m \mathbb{E} \left([x^*(\widetilde{X}_{n,j})]^2 \mathbb{1}_{|x^*(\widetilde{X}_{n,j})| > \varepsilon} \middle| \mathcal{F}_{n,j-1} \right) \right| \\ & \leq 12 \mathbb{E} \sum_{j=1}^m \mathbb{E} \left([x^*(X_{n,j})]^2 \mathbb{1}_{|x^*(X_{n,j})| > \varepsilon/4} \middle| \mathcal{F}_{n,j-1} \right) = 12 \frac{mp}{n} \mathbb{E} \left[\frac{[x^*(S_p)]^2}{p} \mathbb{1}_{\left| \frac{x^*(S_p)}{\sqrt{p}} \right| > \frac{\sqrt{n}}{4\sqrt{p}} \varepsilon} \right]. \end{aligned}$$

Then, from (3.1) and since $\frac{mp}{n} \rightarrow 1$ as n tends to $+\infty$,

$$\sum_{j=1}^m \mathbb{E} \left([x^*(\widetilde{X}_{n,j})]^2 \mathbb{1}_{|x^*(\widetilde{X}_{n,j})| > \varepsilon} \middle| \mathcal{F}_{n,j-1} \right) \xrightarrow{n \rightarrow +\infty} 0 \quad \text{in } \mathbb{L}^1.$$

It remains to verify condition (iii). It is easy to see that for any $x, y \in \mathbb{B}$, $q_{F_l}(x + y) \leq q_{F_l}(x) + \|y\|_{\mathbb{B}}$. Therefore

$$q_{F_l}(\widetilde{X}_{n,j}) \leq q_{F_l}(X_{n,j}) + \|\mathbb{E}(X_{n,j} | \mathcal{F}_{n,j-1})\|_{\mathbb{B}},$$

implying that

$$q_{F_l}^2(\widetilde{X}_{n,j}) \leq 2q_{F_l}^2(X_{n,j}) + 2\|\mathbb{E}(X_{n,j} | \mathcal{F}_{n,j-1})\|_{\mathbb{B}}^2.$$

On the one hand, taking into account stationarity, we derive

$$\mathbb{E} \sum_{j=1}^m \|\mathbb{E}[X_{n,j} | \mathcal{F}_{n,j-1}]\|_{\mathbb{B}}^2 = \frac{mp}{n} \mathbb{E} \left\| \mathbb{E} \left(\frac{S_p}{\sqrt{p}} \middle| \mathcal{F}_0 \right) \right\|_{\mathbb{B}}^2,$$

which converges to 0 as n tends to infinity by conditions (3.2) and (3.8).

On the other hand, by stationarity

$$\sum_{j=1}^m \mathbb{E} [q_{F_l}^2(X_{n,j})] = m \mathbb{E} [q_{F_l}^2(X_{n,1})] = \frac{mp}{n} \frac{1}{p} \mathbb{E}[q_{F_l}^2(S_p)].$$

Therefore, combining (3.4) and (3.8), we derive

$$\lim_{l \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{j=1}^m \mathbb{E} [q_{F_l}^2(X_{n,j})] = 0.$$

Hence, $(\widetilde{X}_{n,j})_{n,j}$ satisfies the conditions of [Ros82, Theorem 5]. This ends the proof of Theorem 3.1.

3.3 Proof of Theorem 1.4

We shall apply Theorem 3.1.

Let start by showing that (1.1) implies (3.1). Denote for any nonnegative integer k ,

$$Y_k = x^*(X_k) \quad \text{and} \quad T_k = \sum_{i=1}^k Y_i. \tag{3.11}$$

Condition (1.1) implies that for any $x^* \in \mathbb{B}^*$, $(x^*(X_0) \mathbb{E}(x^*(S_n) | \mathcal{F}_0))_n$ converges in \mathbb{L}^1 . Hence, applying [DR00, Proposition 1(b)], it follows that $\left(\frac{T_n^2}{n}\right)_n$ is a uniformly integrable family, which

proves (3.1).

We prove now that under (1.1), (3.4) is verified. Let $(e_n)_{n \in \mathbb{N}}$ be a Schauder basis of \mathbb{B} . For any l , denote by P_l the projection on the subspace F_l generated by the l first vectors of the basis. For any integers n and l , since $P_l(S_n) \in F_l$,

$$\frac{1}{n} \mathbb{E} q_{F_l}^2(S_n) \leq \frac{1}{n} \mathbb{E} \|(\text{id} - P_l)S_n\|_{\mathbb{B}}^2.$$

From Theorem 2.1 together with Lemma 1.1 and Remark 2.1 in [DM15] with $p = 2$, we get that there exists $c > 0$ such that for any positive integer n

$$\mathbb{E} \|(\text{id} - P_l)S_n\|_{\mathbb{B}}^2 \leq c \sum_{i=1}^n \max_{i \leq j \leq n} \mathbb{E} \left(\|(\text{id} - P_l)X_i\|_{\mathbb{B}} \left\| \sum_{k=i}^j \mathbb{E} [(\text{id} - P_l)X_k | \mathcal{F}_i] \right\|_{\mathbb{B}} \right).$$

Hence, by using stationarity, we derive

$$\begin{aligned} \mathbb{E} \|(\text{id} - P_l)S_n\|_{\mathbb{B}}^2 &\leq c \sum_{i=1}^n \max_{i \leq j \leq n} \mathbb{E} \left(\|(\text{id} - P_l)X_0\|_{\mathbb{B}} \left\| \sum_{k=0}^{j-i} \mathbb{E} [(\text{id} - P_l)X_k | \mathcal{F}_0] \right\|_{\mathbb{B}} \right) \\ &\leq nc \max_{1 \leq j \leq n} \mathbb{E} \left(\|(\text{id} - P_l)X_0\|_{\mathbb{B}} \left\| \sum_{k=0}^{j-1} \mathbb{E} [(\text{id} - P_l)X_k | \mathcal{F}_0] \right\|_{\mathbb{B}} \right). \end{aligned}$$

Furthermore, $(P_l)_l$ is uniformly bounded for the operator norm, then so is $(\text{id} - P_l)_l$. Hence there exists $C > 0$ such that for any l and any $n > N$,

$$\begin{aligned} &\frac{1}{n} \mathbb{E} \|(\text{id} - P_l)S_n\|_{\mathbb{B}}^2 \\ &\leq C \mathbb{E} \left(\|(\text{id} - P_l)X_0\|_{\mathbb{B}} \left\| \sum_{k=0}^N \mathbb{E} (X_k | \mathcal{F}_0) \right\|_{\mathbb{B}} \right) + C \max_{N+1 \leq j \leq n} \mathbb{E} \left(\|X_0\|_{\mathbb{B}} \left\| \sum_{k=N+1}^{j-1} \mathbb{E} (X_k | \mathcal{F}_0) \right\|_{\mathbb{B}} \right). \end{aligned} \tag{3.12}$$

The second term in the right hand side converges to zero by condition (1.1) by letting first n tend to $+\infty$ and after N .

Since $\mathbb{E}(\|X_0\|_{\mathbb{B}}^2) < \infty$ and $\|(\text{id} - P_l)X_0\|_{\mathbb{B}}$ converges a.s. to zero as l tends to $+\infty$, the first term in the right hand side of (3.12) converges to zero by letting first l tend to $+\infty$. So, overall,

$$\lim_{l \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E} \|(\text{id} - P_l)S_n\|_{\mathbb{B}}^2 = 0, \tag{3.13}$$

and (3.4) follows.

We prove now that (1.1) implies (3.2). For any positive integers n and l , we have

$$\frac{1}{n} \mathbb{E} \left[\|\mathbb{E}(S_n | \mathcal{F}_0)\|_{\mathbb{B}}^2 \right] \leq \frac{2}{n} \mathbb{E} \left[\|\mathbb{E}(P_l S_n | \mathcal{F}_0)\|_{\mathbb{B}}^2 \right] + \frac{2}{n} \mathbb{E} \left[\|\mathbb{E}[(\text{id} - P_l)S_n | \mathcal{F}_0]\|_{\mathbb{B}}^2 \right].$$

The second term in the right hand side is going to zero as n tends to infinity by using (3.13) and Jensen's inequality.

Furthermore, to prove the convergence of $\frac{2}{n} \mathbb{E} \left[\|\mathbb{E}(P_t S_n | \mathcal{F}_0)\|_{\mathbb{B}}^2 \right]$ to 0 as n tends to $+\infty$, it is sufficient to prove that for any $x^* \in \mathbb{B}^*$ we have $\frac{1}{n} \mathbb{E} \left[\mathbb{E}(x^* S_n | \mathcal{F}_0)^2 \right] \rightarrow 0$. Since $\left(\frac{[x^*(S_n)]^2}{n} \right)_n$ is uniformly integrable, it is then enough to prove that

$$\frac{1}{\sqrt{n}} \mathbb{E} |\mathbb{E}(x^*(S_n) | \mathcal{F}_0)| \xrightarrow{n \rightarrow +\infty} 0.$$

This holds under the condition $(x^*(X_0) x^*(\mathbb{E}(S_n | \mathcal{F}_0)))_n$ converges in \mathbb{L}^1 by the arguments developed in Step 3 of the proof of Theorem 4.18 in [MPU19].

It remains to check Condition (3.3). Applying x^* , we place ourselves in the well-known context of real random variables. From Condition (1.1) and using the same notations as in (3.11), $(Y_0 \mathbb{E}(T_n | \mathcal{F}_0))_n$ converges in \mathbb{L}^1 then the series of covariance associated to $(Y_k)_k$ converges. Let us denote it $\sigma^2(x^*)$.

From [MPU19, Proof of Theorem 4.18, Step 4], since the sequence $(X_n)_{n \in \mathbb{Z}}$ is ergodic, we get

$$\frac{1}{n} \mathbb{E} (T_n^2 | \mathcal{F}_0) \xrightarrow[n \rightarrow +\infty]{\mathbb{L}^1} \sigma^2(x^*).$$

That is

$$\mathbb{E} \left(\frac{[x^*(S_n)]^2}{n} \middle| \mathcal{F}_0 \right) \xrightarrow[n \rightarrow +\infty]{\mathbb{L}^1} \sigma^2(x^*) := \sum_{k \in \mathbb{Z}} \text{Cov}(x^*(X_0), x^*(X_k)).$$

This ends the proof of Theorem 1.4.

3.4 Proof of Corollary 2.3

Note that when μ is finite, the result follows immediately. Assume from now that μ is not finite.

Let start by noting that $\|X_0\|_{p,\mu} \leq Y_{p,\mu} + \mathbb{E}Y_{p,\mu}$, so that

$$Q_{\|X_0\|_{p,\mu}} \leq Q_{Y_{p,\mu}} + \mathbb{E}Y_{p,\mu} \quad \text{and} \quad G_{\|X_0\|_{p,\mu}}(\cdot) \geq G_{Y_{p,\mu}}(\cdot/2), \quad (3.14)$$

where $\|X_0\|_{p,\mu} = \left(\int_{\mathbb{R}} |X_0(t)|^p d\mu(t) \right)^{1/p}$. Indeed, by definition of X_0 and as an application of Minkowski's inequality,

$$\begin{aligned} \|X_0\|_{p,\mu} &= \left(\int_{-\infty}^0 |\mathbb{1}_{Y_0 \leq t} - F(t)|^p d\mu(t) + \int_0^{+\infty} |\mathbb{1}_{Y_0 > t} + 1 - F(t)|^p d\mu(t) \right)^{1/p} \\ &\leq \left(\int_{-\infty}^0 \mathbb{1}_{Y_0 \leq t} d\mu(t) + \int_0^{+\infty} \mathbb{1}_{Y_0 > t} d\mu(t) \right)^{1/p} + \left(\int_{-\infty}^0 |F(t)|^p d\mu(t) + \int_0^{+\infty} |1 - F(t)|^p d\mu(t) \right)^{1/p} \\ &\leq Y_{p,\mu} + \mathbb{E}Y_{p,\mu}. \end{aligned}$$

The inequality for the quantile function follows immediately. Let us prove the last inequality. For any $x \in [0, 1]$,

$$\int_0^x Q_{\|X_0\|_{p,\mu}}(u) du \leq \int_0^x Q_{Y_{p,\mu}}(u) du + x \mathbb{E}Y_{p,\mu} \leq \int_0^x Q_{Y_{p,\mu}}(u) du + x \int_0^1 Q_{Y_{p,\mu}}(u) du.$$

Hence, as $Q_{Y_{p,\mu}}$ is non-increasing, $\int_0^x Q_{\|X_0\|_{p,\mu}}(u) du \leq 2 \int_0^{x/2} Q_{Y_{p,\mu}}(u) du$. Thus, $G_{\|X_0\|_{p,\mu}}(\cdot) \geq G_{Y_{p,\mu}}(\cdot/2)$ and (3.14) is established.

From (3.14) and after a change of variables, for any $a > 0$,

$$\begin{aligned} \int_0^{\mathbb{E}\|\mathbb{E}(X_n|\mathcal{F}_0)\|_{p,\mu}} Q_{\|X_0\|_{p,\mu}} \circ G_{\|X_0\|_{p,\mu}}(u) du & \\ & \leq 2 \int_0^{\mathbb{E}\|\mathbb{E}(X_n|\mathcal{F}_0)\|_{p,\mu}/2} Q_{Y_{p,\mu}} \circ G_{Y_{p,\mu}}(u) du + \mathbb{E}\|\mathbb{E}(X_n|\mathcal{F}_0)\|_{p,\mu} \cdot \mathbb{E}Y_{p,\mu} \\ & \leq 2a \int_0^{G_{Y_{p,\mu}}(\mathbb{E}\|\mathbb{E}(X_n|\mathcal{F}_0)\|_{p,\mu}/a)} Q_{Y_{p,\mu}}^2(u) du + \mathbb{E}\|\mathbb{E}(X_n|\mathcal{F}_0)\|_{p,\mu} \cdot \mathbb{E}Y_{p,\mu}. \end{aligned} \quad (3.15)$$

Let U and V be real r.v.'s such that U is distributed as $\|\mathbb{E}(X_n|\mathcal{F}_0)\|_{p,\mu}$, V is distributed as $Y_{p,\mu}$ and U and V are independent. It follows that $\mathbb{E}\|\mathbb{E}(X_n|\mathcal{F}_0)\|_{p,\mu} \cdot \mathbb{E}Y_{p,\mu} = \mathbb{E}(UV)$.

Following the proof of Proposition 1, (4.1) in [DD03] and after taking into account (3.14), we derive

$$\mathbb{E}(UV) \leq \int_0^{\mathbb{E}\|\mathbb{E}(X_n|\mathcal{F}_0)\|_{p,\mu}} Q_{Y_{p,\mu}} \circ G_{\|X_0\|_{\mathbb{B}}}(u) du \leq 2a \int_0^{G_{Y_{p,\mu}}(\mathbb{E}\|\mathbb{E}(X_n|\mathcal{F}_0)\|_{p,\mu}/a)} Q_{Y_{p,\mu}}^2(u) du. \quad (3.16)$$

Finally, (3.15) together with (3.16) imply that

$$\int_0^{\mathbb{E}\|\mathbb{E}(X_n|\mathcal{F}_0)\|_{p,\mu}} Q_{\|X_0\|_{p,\mu}} \circ G_{\|X_0\|_{p,\mu}}(u) du \leq 4a \int_0^{G_{Y_{p,\mu}}(\mathbb{E}\|\mathbb{E}(X_n|\mathcal{F}_0)\|_{p,\mu}/a)} Q_{Y_{p,\mu}}^2(u) du. \quad (3.17)$$

In the following, we will assume without loss of generality that $\mathbb{P}(F_\mu(Y_0) > 0) > 0$ and $\mathbb{P}(F_\mu(Y_0) < 0) > 0$, as if it is not the case, in the following calculations some terms disappear making it simpler.

Let $x > 0$ and $y < 0$, we can write

$$\begin{aligned} \|\mathbb{E}(X_n|\mathcal{F}_0)\|_{p,\mu} &= \left(\int_{-\infty}^y |\mathbb{E}(\mathbb{1}_{Y_n \leq t}|\mathcal{F}_0) - F(t)|^p d\mu(t) + \int_y^x |\mathbb{E}(\mathbb{1}_{Y_n \leq t}|\mathcal{F}_0) - F(t)|^p d\mu(t) \right. \\ & \quad \left. + \int_x^{+\infty} |\mathbb{E}(\mathbb{1}_{Y_n \leq t}|\mathcal{F}_0) - 1 + 1 - F(t)|^p d\mu(t) \right)^{1/p} \\ & \leq \left(\int_{-\infty}^y |\mathbb{E}(\mathbb{1}_{Y_n \leq t}|\mathcal{F}_0)|^p d\mu(t) \right)^{1/p} + \left(\int_{-\infty}^y |\mathbb{E}(\mathbb{1}_{Y_0 \leq t})|^p d\mu(t) \right)^{1/p} + F_\mu(x)^{1/p} b_n \\ & \quad + (-F_\mu(y))^{1/p} b_n + \left(\int_x^{+\infty} |\mathbb{E}(\mathbb{1}_{Y_n > t}|\mathcal{F}_0)|^p d\mu(t) \right)^{1/p} + \left(\int_x^{+\infty} |\mathbb{E}(\mathbb{1}_{Y_0 > t})|^p d\mu(t) \right)^{1/p}, \end{aligned} \quad (3.18)$$

where b_n is defined in Definition 2.1. Considering for any f , $\|f\|_{p,I,\mu} = (\int_I |f(t)|^p d\mu(t))^{1/p}$ and using Jensen's inequality for $\|\cdot\|_{p,[x,+\infty[, \mu}$, we get

$$\begin{aligned} \left(\int_x^{+\infty} |\mathbb{E}(\mathbb{1}_{Y_n > t} | \mathcal{F}_0)|^p d\mu(t) \right)^{1/p} &= \|\mathbb{E}(\mathbb{1}_{Y_n > \cdot} | \mathcal{F}_0)\|_{p,[x,+\infty[, \mu} \\ &\leq \mathbb{E} \left(\|\mathbb{1}_{Y_n > \cdot}\|_{p,[x,+\infty[, \mu} \middle| \mathcal{F}_0 \right) = \mathbb{E} \left[\left(\int_x^{+\infty} \mathbb{1}_{Y_n > t} d\mu(t) \right)^{1/p} \middle| \mathcal{F}_0 \right] \end{aligned}$$

and

$$\begin{aligned} \left(\int_x^{+\infty} |\mathbb{E}(\mathbb{1}_{Y_0 > t})|^p d\mu(t) \right)^{1/p} &= \|\mathbb{E}(\mathbb{1}_{Y_0 > \cdot})\|_{p,[x,+\infty[, \mu} \\ &\leq \mathbb{E} \|\mathbb{1}_{Y_0 > \cdot}\|_{p,[x,+\infty[, \mu} \leq \mathbb{E} \left[\left(\int_x^{+\infty} \mathbb{1}_{Y_0 > t} d\mu(t) \right)^{1/p} \right]. \end{aligned}$$

We proceed in a similar way with $\|\cdot\|_{p,]-\infty, y], \mu}$, so that taking the expectation in (3.18), we finally get

$$\begin{aligned} \mathbb{E} \|\mathbb{E}(X_n | \mathcal{F}_0)\|_{p, \mu} & \tag{3.19} \\ & \leq 2\mathbb{E} \left[\left(\int_x^{+\infty} \mathbb{1}_{Y_0 > t} d\mu(t) \right)^{1/p} \right] + \tilde{\beta}_{1,Y}(n) [F_\mu(x)^{1/p} + (-F_\mu(y))^{1/p}] + 2\mathbb{E} \left[\left(\int_{-\infty}^y \mathbb{1}_{Y_0 \leq t} d\mu(t) \right)^{1/p} \right]. \end{aligned}$$

Note now that

$$\begin{aligned} \mathbb{E} \left[\left(\int_x^{+\infty} \mathbb{1}_{Y_0 > t} d\mu(t) \right)^{1/p} \right] &= \mathbb{E} \left[((F_\mu(Y_0))_+ - F_\mu(x))_+^{1/p} \right] \\ &\leq \mathbb{E} \left[(F_\mu(Y_0))_+^{1/p} \mathbb{1}_{(F_\mu(Y_0))_+^{1/p} > F_\mu(x)^{1/p}} \right] \leq \int_0^{Q_{(F_\mu(Y_0))_+^{1/p}}^{-1}(F_\mu(x)^{1/p})} Q_{(F_\mu(Y_0))_+^{1/p}}(u) du, \end{aligned} \tag{3.20}$$

and similarly,

$$\mathbb{E} \left[\left(\int_{-\infty}^y \mathbb{1}_{Y_0 \leq t} d\mu(t) \right)^{1/p} \right] \leq \int_0^{Q_{(-F_\mu(Y_0))_+^{1/p}}^{-1}((-F_\mu(y))^{1/p})} Q_{(-F_\mu(Y_0))_+^{1/p}}(u) du. \tag{3.21}$$

Furthermore, we can select $x > 0$ such that $F_\mu(x)^{1/p} = Q_{(F_\mu(Y_0))_+^{1/p}}(\tilde{\beta}_{1,Y}(n))$. Indeed, assume that such an x doesn't exist. Then, there exists $x_1 > 0$ such that $F_\mu(x_1^-) \neq F_\mu(x_1)$ and $Q_{(F_\mu(Y_0))_+^{1/p}}(\tilde{\beta}_{1,Y}(n))$ belongs to $]F_\mu(x_1^-)^{1/p}, F_\mu(x_1)^{1/p}[$. Note that $(F_\mu(Y_0))_+^{1/p}$ doesn't take values in $]F_\mu(x_1^-)^{1/p}, F_\mu(x_1)^{1/p}[$. Since $(F_\mu(Y_0))_+^{1/p}$ and $Q_{(F_\mu(Y_0))_+^{1/p}}(U)$ have the same distribution, it implies that $Q_{(F_\mu(Y_0))_+^{1/p}}$ doesn't take values in $]F_\mu(x_1^-)^{1/p}, F_\mu(x_1)^{1/p}[$. There is a contradiction.

In the same manner, reasoning on $(-F_\mu(Y_0))_+^{1/p}$, we can select $y < 0$ such that $(-F_\mu(y))^{1/p} = Q_{(-F_\mu(Y_0))_+^{1/p}}(\tilde{\beta}_{1,Y}(n))$. Hence, selecting $x > 0$ such that $F_\mu(x)^{1/p} = Q_{(F_\mu(Y_0))_+^{1/p}}(\tilde{\beta}_{1,Y}(n))$ and $y < 0$

such that $(-F_\mu(y))^{1/p} = Q_{(-F_\mu(Y_0))_+^{1/p}}(\tilde{\beta}_{1,Y}(n))$, by combining (3.20) and (3.21) we get

$$\mathbb{E} \left[\left(\int_x^{+\infty} \mathbb{1}_{Y_0 > t} d\mu(t) \right)^{1/p} \right] \leq \int_0^{\tilde{\beta}_{1,Y}(n)} Q_{(F_\mu(Y_0))_+^{1/p}}(u) du \leq \int_0^{\tilde{\beta}_{1,Y}(n)} Q_{Y_{p,\mu}}(u) du$$

(3.22)

and

$$\mathbb{E} \left[\left(\int_{-\infty}^y \mathbb{1}_{Y_0 \leq t} d\mu(t) \right)^{1/p} \right] \leq \int_0^{\tilde{\beta}_{1,Y}(n)} Q_{(-F_\mu(Y_0))_+^{1/p}}(u) du \leq \int_0^{\tilde{\beta}_{1,Y}(n)} Q_{Y_{p,\mu}}(u) du.$$

On another hand, as $Q_{Y_{p,\mu}}$ is a nonincreasing function, we have

$$\tilde{\beta}_{1,Y}(n) Q_{Y_{p,\mu}}(\tilde{\beta}_{1,Y}(n)) \leq \int_0^{\tilde{\beta}_{1,Y}(n)} Q_{Y_{p,\mu}}(u) du. \quad (3.23)$$

Starting from (3.19) and taking into account (3.22) and (3.23), we get

$$\mathbb{E} \| \mathbb{E}(X_n | \mathcal{F}_0) \|_{p,\mu} \leq 6 \int_0^{\tilde{\beta}_{1,Y}(n)} Q_{Y_{p,\mu}}(u) du. \quad (3.24)$$

Using the upper bound of (3.24) in (3.17) with $a = 6$, we derive

$$\int_0^{\mathbb{E} \| \mathbb{E}(X_n | \mathcal{F}_0) \|_{p,\mu}} Q_{\|X_0\|_{p,\mu}} \circ G_{\|X_0\|_{p,\mu}}(u) du \leq 24 \int_0^{\tilde{\beta}_{1,Y}(n)} Q_{Y_{p,\mu}}^2(u) du.$$

Thus, as soon as (2.3) is verified, the condition (ii) in Corollary 1.7 holds and Theorem 1.4 applies.

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References

- [BGM99] E. del Barrio, E. Giné, and C. Matrán. “Central limit theorems for the Wasserstein distance between the empirical and the true distributions”. In: *Ann. Probab.* 27.2 (1999), pp. 1009–1071. ISSN: 0091-1798. DOI: [10.1214/aop/1022677394](https://doi.org/10.1214/aop/1022677394).
- [Cun17] C. Cuny. “Invariance principles under the Maxwell-Woodroffe condition in Banach spaces”. In: *Ann. Probab.* 45.3 (2017), pp. 1578–1611. ISSN: 0091-1798. DOI: [10.1214/16-AOP1095](https://doi.org/10.1214/16-AOP1095).
- [DM15] J. Dedecker and F. Merlevède. “Moment bounds for dependent sequences in smooth Banach spaces”. In: *Stochastic Process. Appl.* 125.9 (2015), pp. 3401–3429. ISSN: 0304-4149. DOI: [10.1016/j.spa.2015.05.002](https://doi.org/10.1016/j.spa.2015.05.002).
- [DDT15] J. Dedecker, H. Dehling, and M. S. Taqqu. “Weak convergence of the empirical process of intermittent maps in \mathbb{L}^2 under long-range dependence”. In: *Stoch. Dyn.* 15.2 (2015), pp. 1550008, 29. ISSN: 0219-4937. DOI: [10.1142/S0219493715500082](https://doi.org/10.1142/S0219493715500082).
- [DD03] J. Dedecker and P. Doukhan. “A new covariance inequality and applications”. In: *Stochastic Process. Appl.* 106.1 (2003), pp. 63–80. ISSN: 0304-4149. DOI: [10.1016/S0304-4149\(03\)00040-1](https://doi.org/10.1016/S0304-4149(03)00040-1).

-
- [DM03] J. Dedecker and F. Merlevède. “The conditional central limit theorem in Hilbert spaces”. In: *Stochastic Process. Appl.* 108.2 (2003), pp. 229–262. ISSN: 0304-4149. DOI: [10.1016/j.spa.2003.07.004](https://doi.org/10.1016/j.spa.2003.07.004).
- [DM07] J. Dedecker and F. Merlevède. “The empirical distribution function for dependent variables: asymptotic and nonasymptotic results in \mathbb{L}^p ”. In: *ESAIM Probab. Stat.* 11 (2007), pp. 102–114. ISSN: 1292-8100. DOI: [10.1051/ps:2007009](https://doi.org/10.1051/ps:2007009).
- [DMP14] J. Dedecker, F. Merlevède, and M. Peligrad. “A quenched weak invariance principle”. In: *Ann. Inst. Henri Poincaré Probab. Stat.* 50.3 (2014), pp. 872–898. ISSN: 0246-0203. DOI: [10.1214/13-AIHP553](https://doi.org/10.1214/13-AIHP553).
- [DMP13] J. Dedecker, F. Merlevède, and F. Pène. “Empirical central limit theorems for ergodic automorphisms of the torus”. In: *ALEA Lat. Am. J. Probab. Math. Stat.* 10.2 (2013), pp. 731–766.
- [DP05] J. Dedecker and C. Prieur. “New dependence coefficients. Examples and applications to statistics”. In: *Probab. Theory Related Fields* 132.2 (2005), pp. 203–236. ISSN: 0178-8051. DOI: [10.1007/s00440-004-0394-3](https://doi.org/10.1007/s00440-004-0394-3).
- [DR00] J. Dedecker and E. Rio. “On the functional central limit theorem for stationary processes”. In: *Ann. Inst. H. Poincaré Probab. Statist.* 36.1 (2000), pp. 1–34. ISSN: 0246-0203. DOI: [10.1016/S0246-0203\(00\)00111-4](https://doi.org/10.1016/S0246-0203(00)00111-4).
- [Deh83] H. Dehling. “Limit theorems for sums of weakly dependent Banach space valued random variables”. In: *Z. Wahrsch. Verw. Gebiete* 63.3 (1983), pp. 393–432. ISSN: 0044-3719. DOI: [10.1007/BF00542537](https://doi.org/10.1007/BF00542537).
- [DMR94] P. Doukhan, P. Massart, and E. Rio. “The functional central limit theorem for strongly mixing processes”. In: *Ann. Inst. H. Poincaré Probab. Statist.* 30.1 (1994), pp. 63–82. ISSN: 0246-0203.
- [Dur09] O. Durieu. “Independence of four projective criteria for the weak invariance principle”. In: *ALEA Lat. Am. J. Probab. Math. Stat.* 5 (2009), pp. 21–26.
- [DV08] O. Durieu and D. Volný. “Comparison between criteria leading to the weak invariance principle”. In: *Ann. Inst. Henri Poincaré Probab. Stat.* 44.2 (2008), pp. 324–340. ISSN: 0246-0203. DOI: [10.1214/07-AIHP123](https://doi.org/10.1214/07-AIHP123).
- [Han73] E. J. Hannan. “Central limit theorems for time series regression”. In: *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 26 (1973), pp. 157–170. DOI: [10.1007/BF00533484](https://doi.org/10.1007/BF00533484).
- [HH01] H. Hennion and L. Hervé. *Limit theorems for Markov chains and stochastic of dynamical systems by quasi-compactness*. Vol. 1766. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2001, pp. viii+145. ISBN: 3-540-42415-6. DOI: [10.1007/b87874](https://doi.org/10.1007/b87874).
- [HP76] J. Hoffmann-Jørgensen and G. Pisier. “The law of large numbers and the central limit theorem in Banach spaces”. In: *Ann. Probability* 4.4 (1976), pp. 587–599. ISSN: 0091-1798. DOI: [10.1214/aop/1176996029](https://doi.org/10.1214/aop/1176996029).
- [LSV99] C. Liverani, B. Saussol, and S. Vaienti. “A probabilistic approach to intermittency”. In: *Ergodic Theory Dynam. Systems* 19.3 (1999), pp. 671–685. ISSN: 0143-3857. DOI: [10.1017/S0143385799133856](https://doi.org/10.1017/S0143385799133856).
- [LPP11] J. Lukeš, L. Pick, and D. Pokorný. “On geometric properties of the spaces $L^{p(x)}$ ”. In: *Rev. Mat. Complut.* 24.1 (2011), pp. 115–130. ISSN: 1139-1138. DOI: [10.1007/s13163-010-0032-9](https://doi.org/10.1007/s13163-010-0032-9).
- [MW00] M. Maxwell and M. Woodroffe. “Central limit theorems for additive functionals of Markov chains”. In: *Ann. Probab.* 28.2 (2000), pp. 713–724. ISSN: 0091-1798,2168-894X. DOI: [10.1214/aop/1019160258](https://doi.org/10.1214/aop/1019160258).
-

-
- [MPU97] F. Merlevède, M. Peligrad, and S. Utev. “Sharp conditions for the CLT of linear processes in a Hilbert space”. In: *J. Theoret. Probab.* 10.3 (1997), pp. 681–693. ISSN: 0894-9840. DOI: [10.1023/A:1022653728014](https://doi.org/10.1023/A:1022653728014).
- [MPU19] F. Merlevède, M. Peligrad, and S. Utev. *Functional Gaussian approximation for dependent structures*. Vol. 6. Oxford Studies in Probability. Oxford University Press, Oxford, 2019, pp. xv+478. ISBN: 978-0-19-882694-1. DOI: [10.1093/oso/9780198826941.001.0001](https://doi.org/10.1093/oso/9780198826941.001.0001).
- [Pin94] I. Pinelis. “Optimum bounds for the distributions of martingales in Banach spaces”. In: *Ann. Probab.* 22.4 (1994), pp. 1679–1706. ISSN: 0091-1798.
- [Pis75] G. Pisier. “Martingales with values in uniformly convex spaces”. In: *Israel J. Math.* 20.3-4 (1975), pp. 326–350. ISSN: 0021-2172. DOI: [10.1007/BF02760337](https://doi.org/10.1007/BF02760337).
- [Rio17] E. Rio. *Asymptotic theory of weakly dependent random processes*. Vol. 80. Probability Theory and Stochastic Modelling. Translated from the 2000 French edition [MR2117923]. Springer, Berlin, 2017, pp. xviii+204. ISBN: 978-3-662-54322-1. DOI: [10.1007/978-3-662-54323-8](https://doi.org/10.1007/978-3-662-54323-8).
- [Ros82] J. Rosiński. “Central limit theorems for dependent random vectors in Banach spaces”. In: *Martingale theory in harmonic analysis and Banach spaces (Cleveland, Ohio, 1981)*. Vol. 939. Lecture Notes in Math. Springer, Berlin-New York, 1982, pp. 157–180.
- [VW96] A. W. van der Vaart and J. A. Wellner. *Weak convergence and empirical processes*. Springer Series in Statistics. With applications to statistics. Springer-Verlag, New York, 1996, pp. xvi+508. ISBN: 0-387-94640-3. DOI: [10.1007/978-1-4757-2545-2](https://doi.org/10.1007/978-1-4757-2545-2).
- [VR59] V. A. Volkonskii and Y. A. Rozanov. “Some limit theorems for random functions. I”. In: *Theor. Probability Appl.* 4 (1959), pp. 178–197. ISSN: 0040-585X. DOI: [10.1137/1104015](https://doi.org/10.1137/1104015).
- [Woj97] P. Wojtaszczyk. *A mathematical introduction to wavelets*. Vol. 37. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1997, pp. xii+261. ISBN: 0-521-57020-4. DOI: [10.1017/CB09780511623790](https://doi.org/10.1017/CB09780511623790).