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# Hierarchical competition and heterogeneous behavior in noncooperative oligopoly markets

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In this paper, we consider a sequential bilateral oligopoly market which embodies a finite number of leaders and followers who compete on quantities. We define a non-cooperative equilibrium concept for this two-stage market game with complete and perfect information, namely the Stackelberg-Nash equilibrium (SNE). Then, we study the existence of a SNE with trade. The existence proof requires some steps as this market game displays a rich set of strategic interactions. In particular, to show the existence of a pure strategy subgame perfect Nash equilibrium, we have to determine the conditions under which there exist well defined continuously differentiable best responses. Some examples buttress the approach and discuss the assumptions made on the primitives.

*Key Words:* Best responses, Stackelberg-Nash equilibrium, trade, autarky

*Subject Classification:* C72, D51

## 1. INTRODUCTION

The existence of noncooperative oligopoly equilibria in finite strategic market games has been widely studied under Cournot competition (Dubey and Shubik, 1978, Dubey, 1982, Sahi and Yao, 1989, Amir et al., 1990, and Peck and Shell, 1992). An essential issue concerns the existence of a Cournot-Nash equilibrium with trade (Cordella and Gabszewicz, 1998, Bloch and Ferrer, 2001, Giraud, 2003, Busetto and Codognato, 2006, Dickson and Hartley, 2008). In this paper, we consider a two-stage finite market game with observable delays in which several leaders and followers compete on quantities. We define a noncooperative oligopoly equilibrium concept, namely the Stackelberg-Nash equilibrium (SNE thereafter). The main objective is to study the existence of a SNE. Some Stackelberg general equilibrium concepts have already been defined and computed in simple finite exchange economies (Julien and Tricou, 2010, 2012, and Julien 2013). Nevertheless, these contributions provide no existence proof. Thus, this paper is devoted to the existence of a non-autarkic pure strategy subgame perfect-Nash equilibrium (SPNE), that is, a SNE with trade, within the research program for strategic market games founded by Shubik (1973), Shapley (1976), and Shapley and Shubik (1977).

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To study the existence of a SNE with trade, we extend the exchange bilateral oligopoly model introduced by Gabszewicz and Michel (1997), and explored notably by Bloch and Ghosal (1997), Bloch and Ferrer (2001), Dickson (2005), Dickson and Hartley (2008), and Amir and Bloch (2009). The bilateral oligopoly model is a two commodity version of the strategic market game models (Shapley and Shubik, 1977, Sahi and Yao, 1989, and Amir et al., 1990). In the bilateral oligopoly model each trader has corner endowment but wants to consume both commodities. There is a market price which aggregates the strategic supplies of all traders and allocates the amounts traded to each market participant. Therefore, to build a sequential market game with hierarchical competition, we consider a market with a finite number of heterogeneous traders. Heterogeneity does not stem only from the primitives, i.e., from endowments and preferences, it merely relies on asymmetric behavior which is peculiar to hierarchical competition. Thus, this hierarchical competition is modeled within a two-stage game which embodies two simultaneous move subgames. There are strategic interactions on each side and between both sides of the market: the leaders interact with the followers in the two-stage game, but the leaders (followers) interact with each other in a simultaneous move game. Therefore, there are two multiple leader-follower industries which are connected through trade. Thus, this contribution constitutes a first endeavour to cast within a pure exchange two-commodity framework the multiple leader-follower one industry game, which is studied in particular by Sherali (1984), Yu and Wang (2008), and Julien (2017). We assume that the timing of moves is given. In addition, information is assumed to be complete and perfect. Thus, we look for pure strategy SPNE.

There are two main problems involved with the existence of a SNE with trade. The first problem is related to the possibility of autarky. It is well known that autarky is always a Nash equilibrium in finite strategic market games with simultaneous moves (see, in particular, Cordella and Gabszewicz, 1998, Giraud, 2003, and Busetto and Codognato, 2006). Therefore, autarky also holds in the simultaneous move bilateral oligopoly game. Thus, we wonder whether autarky is a plausible outcome in the sequential bilateral oligopoly model. It seems plausible to conjecture that the no trade equilibrium is a possible outcome for the entire sequential game, in which case neither leaders nor followers participate in exchange. But, is it possible that exchange takes place in one subgame whereas autarky prevails in the other subgame, in which case only the leaders or only the followers participate in exchange on the market? An example given in Section 5 provides a positive answer to this question. The second difficulty, which is specific to sequential non-cooperative oligopoly games, concerns the existence of well defined best responses (Julien, 2017). Indeed, in the basic one leader-one follower game, the best response is determined, for any given strategy profile of the leader, as the solution to the maximization of the follower's payoff. But in a game with at least two followers the best responses could not be well-defined. Indeed, in the subgame between followers, any follower's optimal behavior consists in a decision mapping which depends upon two kinds of arguments: the strategy profile of all leaders and the strategy profile of all other followers. Thus, if these mappings were not mutually consistent, the best responses could not be well-defined. This problem is illustrated with an example in Section 5. In this paper, we provide a consistency condition to show the existence of smooth best responses as well as the existence of a SNE with trade.

To the best of our current knowledge, Stackelberg competition has not yet been studied in this way in noncooperative oligopoly. Groh (1999) proposes a bilateral oligopoly market in which the leaders are sellers and the followers are buyers. The

existence of a SPNE with trade is based on three restrictions: the utility function is specific; each side of the market embodies only leaders or followers; with identical traders within each side. Therefore, our contribution to the literature is threefold. First, we consider a SNE concept for a market game in which utility functions are not specific, and in which heterogeneous leaders compete with heterogeneous followers within each side and between both sides of the market. The existence of a SNE with trade requires notably that the utility functions be twice-continuously differentiable, strictly increasing and strictly quasi-concave, with, for some traders, indifference curves which are contained in the strict interior of the commodity space. Second, the study of optimal behavior in subgames leads us to provide a characterization of the strategic equilibrium which brings into light a consistency criterion. This criterion concerns the internal consistency of the system of equations which determines the best responses. Our criterion gives some nondegeneracy sufficient condition on the Jacobian determinant of the set of equations that defines implicitly the followers' best responses. If our criterion holds, then the reduced form payoffs of leaders are well-defined. Therefore, the subgame between leaders may be studied. Thus, the set of pure strategy subgame perfect equilibria in the finite extensive form game with observable delays is not empty. Third, our approach puts forward the beliefs of leaders: by definition, in a SNE, the leaders know perfectly the reactions of the followers. Indeed, the optimal behavior of leaders deserves careful study.

The main objective of the paper is to prove the existence of a SNE with trade. To this end, we consider a slight perturbation of the market game as in Dubey and Shubik (1978) when they show existence of non-autarkic Cournot-Nash equilibria. The proof requires five steps. The first step is devoted to the study of optimal behavior in each perturbed subgame. Thus, we study the optimal decision mappings of followers (Propositions 1 and 2). Then, we show the existence of smooth best responses. To this end, we consider the consistency of the system of equations which determines such best responses. Hence, if the Jacobian matrix associated with the equations which defines implicitly the best responses is of full rank, then the strategy of any follower is a well defined continuously differentiable function of the sole strategy profile of leaders (Lemma 1). We also show the followers' reactions are bounded (Proposition 3). Then, by considering the subgame between leaders, and their reduced form payoffs, we study the optimal decision mappings of leaders (Proposition 4). In the second step, we prove the existence of a SNE of the perturbed market game. To this end, we show the optimal behavior of traders are mutually consistent in each perturbed subgame as well as in the entire perturbed game (Lemma 2). In the third step, we show that there exist some uniform bounds on the market price in a SNE of the perturbed game (Lemma 3). In the fourth step, we prove the SNE of the perturbed game is non-autarkic (Lemma 4). Finally, in the fifth step, we show that the SNE with trade is an equilibrium point of the market game (Lemma 5). The continuous differentiability of the utility functions and the behavior of the indifference curves near the boundary of the consumption sets play a critical role. Thus, some examples illustrate the role played by the assumptions. This inquiry reveals that the assumptions made on the utility functions provide a set of sufficient conditions for the existence of a SNE with trade.

The remaining part of the paper is organized as follows. In section 2, we describe the model. Section 3 is devoted to the existence of a SNE with trade. Section 4 provides some examples to discuss our assumptions and to buttress the working of our approach. In section 5 we conclude. Section 6 contains an Appendix which proves some intermediate results.

## 2. THE MODEL

The model is described in four steps. First, we give the basic framework and fix some notations. Second, we state some assumptions. Third, we describe the market game associated with the exchange economy. Fourth, we define the SNE.

### 2.1. Framework and notations

Consider an exchange economy,  $\mathcal{E}$ , with two divisible homogeneous commodities labeled  $X$  and  $Y$ . Let  $p_X$  and  $p_Y$  be their unit prices. Traders are of two types, namely 1 and 2, so the set of traders is partitioned into two subsets  $T_1$  and  $T_2$ , with  $T_1 \cap T_2 = \{\emptyset\}$ . We assume  $2 \leq |T_1| < \infty$  and  $2 \leq |T_2| < \infty$ , where  $|T|$  denotes the cardinality of the set  $T$ . Traders who belong to  $T_1$  (resp.  $T_2$ ) are indexed by  $i$  (resp. by  $j$ ). We assume there are  $M_1$  leaders and  $N_1 - M_1$  followers of type 1, with  $T_1 = \{1, \dots, M_1, M_1 + 1, \dots, N_1\}$ . Likely, we have  $T_2 = \{1, \dots, M_2, M_2 + 1, \dots, N_2\}$ . Then,  $M_1 \geq 1$  and  $N_1 - M_1 \geq 1$ . Likewise,  $M_2 \geq 1$  and  $N_2 - M_2 \geq 1$ .

In what follows, we adopt the following notational conventions. Vectors are in bold and capital letters denote either sets or summations. Let  $\mathbf{z} \in \mathbb{R}_+^n$ . Then,  $\mathbf{z} \geq \mathbf{0}$  means  $z_i \geq 0$ ,  $i = 1, \dots, n$ ;  $\mathbf{z} > \mathbf{0}$  means there is some  $i$  such that  $z_i > 0$ , with  $\mathbf{z} \neq \mathbf{0}$ , and  $\mathbf{z} \gg \mathbf{0}$  means  $z_i > 0$  for all  $i$ ,  $i = 1, \dots, n$ . The transpose of  $\mathbf{z}$  is denoted by  $\mathbf{z}'$ . Let  $\mathbf{z}_i \geq \mathbf{0}$  be an action. An action profile is given by  $\mathbf{z} = (z_1, \dots, z_i, \dots, z_n)$ , with  $\mathbf{z} \geq \mathbf{0}$ . In addition, let  $\mathbf{z}_{-i} = (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)$  be the action profile of all traders but trader  $i$ . We sometimes consider  $\mathbf{Z} \equiv \sum_{i=1}^n z_i$ , with  $\mathbf{Z}_{-i} \equiv \sum_{-i \neq i} z_{-i}$

$\mathbf{Z} - z_i$ . Let  $A$  be a finite set, with  $A = \{1, \dots, m, \dots, n\}$ . The restriction of  $A$  to a subset of  $m$  elements is denoted by  $A^m$ , with  $A^m = A \setminus \{m+1, \dots, n\}$ . The Cartesian product of sets  $A_i$  is denoted by  $\prod_{i \in \mathcal{I}} A_i$ , where  $\mathcal{I} = \{1, \dots, n\}$  is the index set.

Moreover,  $\prod_i A_i$  is the Cartesian product of all sets but  $i$ . Let  $f$  be a function defined by  $f : A \subseteq \mathbb{R}^n \rightarrow B \subseteq \mathbb{R}$ , with  $\mathbf{z} \mapsto f(\mathbf{z})$ . The Cartesian product of a set of functions  $f^j(\cdot)$ ,  $j = 1, \dots, m$ , is denoted by  $\times_j f^j(\cdot)$ . The partial derivative of  $f$  with respect to  $z_i$  at  $z = \bar{z}$  is  $\frac{\partial f}{\partial z_i}(\bar{z})$ ,  $i = 1, \dots, n$ . Likewise, when  $n = 1$ , the derivative of  $f$  with respect to  $z$  at  $z = \bar{z}$  is  $\frac{df}{dz}(\bar{z})$ . The second-order partial derivative of  $f$  with respect to  $z_i$  at  $z = \bar{z}$  is denoted by  $\frac{\partial^2 f}{(\partial z_i)^2}(\bar{z})$  (by  $\frac{d^2 f}{(dz)^2}(\bar{z})$  when  $n = 1$ ). The notation  $f \in \mathcal{C}^2$  means  $f$  is twice-continuously differentiable. A  $m$  dimensional vector function  $\mathbf{F}$  is defined by  $\mathbf{F} : A \subseteq \mathbb{R}^n \rightarrow B \subseteq \mathbb{R}^m$ , with  $\mathbf{F}(\mathbf{z}) = (f_1(\mathbf{z}), \dots, f_2(\mathbf{z}), \dots, f_m(\mathbf{z}))$ . The notation  $\mathbf{z}(\mathbf{e})$ , where  $\mathbf{e} \in \mathbb{R}^k$ , means that each  $z_i$  is a function of  $\mathbf{e}$ ,  $i = 1, \dots, n$ . When  $m = 1$ , the gradient vector of  $f$  is denoted by  $\nabla f = \left( \frac{\partial f(\mathbf{z})}{\partial z_1}, \dots, \frac{\partial f(\mathbf{z})}{\partial z_i}, \dots, \frac{\partial f(\mathbf{z})}{\partial z_n} \right)$ . The Jacobian matrix of  $\mathbf{F}(\mathbf{z})$  with respect to  $\mathbf{z}$  at  $\bar{\mathbf{z}}$  is denoted by  $\mathcal{J}_{\mathbf{F}}(\bar{\mathbf{z}})$ , with  $\mathcal{J}_{\mathbf{F}}(\bar{\mathbf{z}}) = \left[ \frac{\partial(f_1, \dots, f_j, \dots, f_m)}{\partial(z_1, \dots, z_i, \dots, z_n)} \right]$ . With a slight abuse of notation, let  $|\mathcal{J}_{\mathbf{F}}(\bar{\mathbf{z}})|$  be the determinant of  $\mathcal{J}_{\mathbf{F}}$  at  $\bar{\mathbf{z}}$ . The Hessian matrix of  $\mathbf{F}(\mathbf{z})$  with respect to  $\mathbf{z}$  at  $\bar{\mathbf{z}}$  is denoted by  $\mathcal{H}_{\mathbf{F}}(\bar{\mathbf{z}})$ , with  $\mathcal{H}_{\mathbf{F}}(\bar{\mathbf{z}}) = \left[ \frac{\partial^2 f}{\partial z_i \partial z_j}(\bar{z}) \right]$ ,  $i, j = 1, \dots, n$ . If  $\mathcal{A}_{(m,n)}$  is a matrix of dimension  $(m, n)$ , then its transpose, which we denote by  $\mathcal{A}'_{(m,n)}$ , is a matrix of dimension  $(n, m)$ . Finally, when the distinction matters, if we partition  $\mathbf{z}$  in such a way  $\mathbf{z} = (\mathbf{x}, \mathbf{y})$ , then  $\mathcal{J}_{\mathbf{F}_{\mathbf{x}}}(\bar{\mathbf{z}})$  is the Jacobian matrix of  $\mathbf{F}(\mathbf{z})$  at  $\bar{\mathbf{z}}$  when the differentiation is partial and made with respect to  $\mathbf{x}$  only. Its determinant is  $|\mathcal{J}_{\mathbf{F}_{\mathbf{x}}}(\bar{\mathbf{z}})|$ .

## 2.2. Assumptions on endowments and preferences

We now provide two kinds of assumptions regarding the fundamentals (endowments and preferences) for  $\mathcal{E}$ . First, there are fixed initial endowments which satisfy the following assumption.

**Assumption 1.**  $\mathbf{w}_i = (\alpha_i, 0)$ , with  $\alpha_i > 0$ , for each  $i \in T_1$ , and  $\mathbf{w}_j = (0, \beta_j)$ , with  $\beta_j > 0$ , for each  $j \in T_2$ .

Assumption 1 is standard in the bilateral oligopoly model with a finite number of traders (Gabszewicz and Michel, 1997, Dickson and Hartley, 2008). Indeed, as emphasized by Cordella and Gabszewicz (1998), it does not require the initial endowments to be strictly in the interior of the commodity space (Amir et al., 1990), or the traders sell their entire endowments (Shubik, 1973, Shapley, 1976).

Second, the preferences of each trader  $k$  are described by an utility function  $u_k : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ ,  $\mathbf{z}_k \mapsto u_k(\mathbf{z}_k)$ , with  $\mathbf{z}_k = (x_k, y_k)$ , and where  $x_k$  and  $y_k$  are the amounts of goods  $X$  and  $Y$  consumed by trader  $k$ ,  $k = i, j$ . We make the following set of assumptions, which we designate as Assumption 2.

**Assumption 2.** For all  $\mathbf{z}_k \in \mathbb{R}_+^2$ , the utility function  $u_k$  satisfies:

2a.  $\forall k, u_k \in \mathcal{C}^2(\mathbb{R}_{++}^2)$ ;

2b.  $\forall k, \nabla u_k(\mathbf{z}_k) \gg \mathbf{0}$ , where  $\nabla u_k(\mathbf{z}_k) = \left( \frac{\partial u_k(\mathbf{z}_k)}{\partial x_k}, \frac{\partial u_k(\mathbf{z}_k)}{\partial y_k} \right)$ ;

2c.  $\forall k, |\mathcal{A}(2,2)| = \left| \begin{bmatrix} 0 & \frac{\partial u_k}{\partial x_k} \\ \frac{\partial u_k}{\partial y_k} & \frac{\partial^2 u_k}{(\partial x_k)^2} \end{bmatrix} \right| < 0$ ,  $|\mathcal{A}(3,3)| = \left| \begin{bmatrix} 0 & \nabla u_k \\ (\nabla u_k)' & \mathcal{H}_{u_k} \end{bmatrix} \right| > 0$ .

2d. there are at least one leader and one follower of each type for whom  $\nabla u_k(\mathbf{z}_k)$  satisfies  $\lim_{\mathbf{z}_k \rightarrow 0} \nabla u_k(\mathbf{z}_k) = (\infty, \infty)$ .

Hypothesis (2a) says the utility functions are twice-continuously differentiable in the interior of the commodity space. And it includes the case of infinite partial derivatives along the boundary of the consumption set (see Kreps, 2012, p. 58). (2b) and (2c) say the utility functions are strictly monotonic and strictly quasi-concave. From (2d), the indifference curves of (at least) two traders of each type do not intersect the quantity axis. These assumptions are discussed in Section 4.

## 2.3. The market game

We introduce now the noncooperative game, namely  $\Gamma$ , associated with  $\mathcal{E}$ . Let  $\mathcal{S}_i$  and  $\mathcal{S}_j$  be the strategy sets of traders  $i$  and  $j$  respectively, with:

$$\mathcal{S}_i = \{(q_i, b_i) \in \mathbb{R}_+^2 : q_i \leq \alpha_i \text{ and } b_i = 0\}, i \in T_1 \quad (1)$$

$$\mathcal{S}_j = \{(q_j, b_j) \in \mathbb{R}_+^2 : q_j = 0 \text{ and } b_j \leq \beta_j\}, j \in T_2. \quad (2)$$

The quantity  $q_i$  in (1) denotes the pure strategy of trader  $i \in T_1$ . The strategy  $q_i$  represents the amount of commodity  $X$  trader  $i \in T_1$  sells in exchange for

commodity  $Y$ . Similarly,  $b_j$  is the pure strategy of trader  $j \in T_2$ . A strategy profile is represented by the vector  $(\mathbf{q}; \mathbf{b}) = (q_1, q_2, \dots, q_{N_1}; b_1, b_2, \dots, b_{N_2})$ , with  $(\mathbf{q}; \mathbf{b}) \in \prod_{i \in T_1} \mathcal{S}_i \times \prod_{j \in T_2} \mathcal{S}_j$ . Let  $\mathbf{q}_{-i}$  denote the strategy profile of all traders of type 1 but  $i$ .

The same holds for  $\mathbf{b}_{-j}$ . In addition, let  $\mathbf{q}^L$  and  $\mathbf{q}^F$  be respectively the strategy profiles of type 1 leaders and followers. The same holds for  $\mathbf{b}$ , with  $\mathbf{b} = (\mathbf{b}^L, \mathbf{b}^F)$ .

Given a price vector  $\mathbf{p} = (p_X, p_Y)$  and a feasible strategy profile  $(\mathbf{q}; \mathbf{b})$ , the market clearing price  $\frac{p_X}{p_Y}(\mathbf{q}; \mathbf{b})$  is determined according to the following mechanism:

$$\frac{p_X}{p_Y}(\mathbf{q}; \mathbf{b}) = \begin{cases} \frac{\mathbf{B}}{\mathbf{Q}}, & \text{if } \mathbf{B} > 0 \text{ and } \mathbf{Q} > 0 \\ 0, & \text{otherwise,} \end{cases} \quad (3)$$

where  $\mathbf{Q} \equiv \sum_{i \in T_1} q_i$  is equal to  $\frac{p_Y}{p_X} \mathbf{B}$ , with  $\mathbf{B} \equiv \sum_{j \in T_2} b_j$ .

The final allocations assign the following bundles for each type of traders:

$$\forall i \in T_1, \mathbf{z}_i = \begin{cases} \left( \alpha_i - q_i, \frac{\mathbf{B}}{q_j + \mathbf{Q}_{-j}} q_i \right), & \text{if } \frac{p_X}{p_Y} > 0 \\ (\alpha_i, 0), & \text{if } \frac{p_X}{p_Y} = 0, \end{cases} \quad (4)$$

$$\forall j \in T_2, \mathbf{z}_j = \begin{cases} \left( \frac{\mathbf{Q}}{b_j + \mathbf{B}_{-j}} b_j, \beta_j - b_j \right), & \text{if } \frac{p_Y}{p_X} > 0 \\ (0, \beta_j), & \text{if } \frac{p_Y}{p_X} = 0. \end{cases} \quad (5)$$

The corresponding utility levels may be written as payoffs:

$$\pi_i(q_i, \mathbf{q}_{-i}; \mathbf{b}) = u_i \left( \alpha_i - q_i, \frac{\mathbf{B}}{q_j + \mathbf{Q}_{-j}} q_i \right), \quad i \in T_1 \quad (6)$$

$$\pi_j(\mathbf{q}; b_j, \mathbf{b}_{-j}) = u_j \left( \frac{\mathbf{Q}}{b_j + \mathbf{B}_{-j}} b_j, \beta_j - b_j \right), \quad j \in T_2. \quad (7)$$

The finite game  $\mathbf{\Gamma} := \{T, (\mathcal{S}_i, \pi_i), (\mathcal{S}_j, \pi_j)\}_{i \in T_1}^{j \in T_2}$  represents a two-stage game where the players are the traders, the strategies are the supplies, and the payoffs are the utility levels they reach in the market outcome. This game displays two stages of decisions and no discounting. We also assume the timing of positions is given. Then, all leaders (followers) play only at stage 1 (2). In addition, traders meet once and cannot make binding agreements. By precluding binding agreements, we consider each trader acts independently and without communication with any of the others. Thus,  $\mathbf{\Gamma}$  is a two-stage hierarchical game with observable delays, which embodies two simultaneous move subgames: one between the leaders, namely  $\mathbf{\Gamma}_L$ , and one between the followers, namely  $\mathbf{\Gamma}_F$ . Indeed, the  $(M_1 + M_2)$  leaders play a two-stage game with the  $(N_1 - M_1) + (N_2 - M_2)$  followers, but any leader (follower) plays with each other leader (follower) a simultaneous move game. Finally, information is assumed to be complete and perfect. Information is perfect because any leader perfectly knows the behavior of all followers, and, each follower's information set is a single decision node.<sup>2</sup> In each decision node, any follower makes an optimal choice, so sequential rationality prevails. As sequential rationality is common knowledge, the game is solved by backward induction.

<sup>2</sup>It is worth noticing that the entire game includes two partial games with strategic uncertainty:  $\mathbf{\Gamma}_F$  is a simultaneous move subgame, and  $\mathbf{\Gamma}_L$  is a simultaneous move game. But the entire (sub)game  $\mathbf{\Gamma}$  is a sequential game with perfect information: every information set is a singleton, and every node initiates a subgame. Followers perfectly know the optimal strategies of leaders, and no trader makes a choice in two subgames.

## 2.4. SNE: definition

To define a SNE, we complete the description of the market game. To this end, we define some concepts which are related to the optimal behavior of traders in each subgame.

Consider the subgame  $\Gamma_F$ . For each  $(\mathbf{q}^L; \mathbf{b}^L)$ , with  $(\mathbf{q}^L; \mathbf{b}^L) \in \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{j=1}^{M_2} \mathcal{S}_j$ , the optimal decision mappings (ODM) of followers are defined as follows.

DEFINITION 1. (ODM). Let  $\phi_i : \prod_{-i \in T_1} \mathcal{S}_{-i} \times \prod_{j \in T_2} \mathcal{S}_j \rightarrow \mathcal{S}_i$ , with  $\phi_i(\mathbf{q}_{-i}; \mathbf{b}) = \{q_i \in \mathcal{S}_i : q_i \in \arg \max \pi_i(q_i, \mathbf{q}_{-i}; \mathbf{b})\}$ , be follower  $i$ 's optimal decision mapping,  $i = M_1 + 1, \dots, N_1$ . Similarly, let  $\psi_j : \prod_{i \in T_1} \mathcal{S}_i \times \prod_{-j \in T_2} \mathcal{S}_{-j} \rightarrow \mathcal{S}_j$ , with  $\psi_j(\mathbf{q}; \mathbf{b}_{-j}) = \{b_j \in \mathcal{S}_j : b_j \in \arg \max \pi_j(\mathbf{q}; b_j, \mathbf{b}_{-j})\}$ ,  $j = M_2 + 1, \dots, N_2$ .

Let us notice this market game displays a rich set of strategic interactions. Therefore, in contrast with the duopoly game the behavior of traders is more difficult to handle. The ODM's differ from best responses, whilst in the usual duopoly game, the optimal decision of the follower always coincides with his best response (Julien, 2017). But with several followers the best responses might be not well-defined (see Example 3 in the Appendix). In Section 3 we provide a criterion to show the existence of smooth best responses, which are defined as follows.

DEFINITION 2. (BR). Let  $\sigma_i : \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{j=1}^{M_2} \mathcal{S}_j \rightarrow \mathcal{S}_i$ , with  $q_i = \sigma_i(\mathbf{q}^L; \mathbf{b}^L)$ , be the best response of follower  $i$ ,  $i = M_1 + 1, \dots, N_1$ . Likewise, for all  $j = M_2 + 1, \dots, N_2$ , let  $\varphi_j : \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{j=1}^{M_2} \mathcal{S}_j \rightarrow \mathcal{S}_j$ , with  $b_j = \varphi_j(\mathbf{q}^L; \mathbf{b}^L)$ .

Consider now the subgame  $\Gamma_L$ . Define the function  $\sigma : \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{j=1}^{M_2} \mathcal{S}_j \rightarrow \prod_{i=M_1+1}^{N_1} \mathcal{S}_i$ , with  $\mathbf{q}^F = \sigma(\mathbf{q}^L; \mathbf{b}^L)$ , and the function  $\varphi : \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{j=1}^{M_2} \mathcal{S}_j \rightarrow \prod_{j=M_2+1}^{N_2} \mathcal{S}_j$ , with  $\mathbf{b}^F = \varphi(\mathbf{q}^L; \mathbf{b}^L)$ . Then, from (3), we deduce  $\frac{p_X}{p_Y}(\mathbf{q}^L, \sigma(\mathbf{q}^L; \mathbf{b}^L); \mathbf{b}^L, \varphi(\mathbf{q}^L; \mathbf{b}^L))$ , so we can define, for each leader, the optimal decision mapping as the solution to the maximization of her reduced form payoff.

DEFINITION 3. Let  $\phi_i : \prod_{-i \neq i} \mathcal{S}_{-i} \times \prod_{j=1}^{M_2} \mathcal{S}_j \rightarrow \mathcal{S}_i$ , with  $\phi_i(\mathbf{q}_{-i}^L; \mathbf{b}^L) = \{q_i \in \mathcal{S}_i : q_i \in \arg \max \pi_i(q_i, \mathbf{q}_{-i}^L, \sigma(q_i, \mathbf{q}_{-i}^L; \mathbf{b}^L); \mathbf{b}^L, \varphi(q_i, \mathbf{q}_{-i}^L; \mathbf{b}^L))\}$ , be leader  $i$ 's optimal decision mapping,  $i = 1, \dots, M_1$ . Likewise, let  $\psi_j : \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{-j \neq j} \mathcal{S}_{-j} \rightarrow \mathcal{S}_j$ , with  $\psi_j(\mathbf{q}^L; \mathbf{b}_{-j}^L) = \{b_j \in \mathcal{S}_j : b_j \in \arg \max \pi_j(\mathbf{q}^L, \sigma(\mathbf{q}^L; b_j, \mathbf{b}_{-j}^L); b_j, \mathbf{b}_{-j}^L, \varphi(\mathbf{q}^L; b_j, \mathbf{b}_{-j}^L))\}$ ,  $j = 1, \dots, M_2$ .

Let us now consider the consistency of optimal behavior. The equilibrium of the two-stage game  $\Gamma$  is a pure strategy SPNE, while the equilibria in both subgames  $\Gamma_L$  and  $\Gamma_F$  are Nash equilibria. But such a SPNE is a Nash equilibrium (NE thereafter) of each subgame of  $\Gamma$  (Selten, 1975).<sup>3</sup>

<sup>3</sup>It requires the strategies of the leaders and the followers to constitute a NE of any subgame. In



Define the function  $\Lambda_L : \prod_{i=1}^{M_1} S_i \times \prod_{j=1}^{M_2} S_j \rightarrow \prod_{i=1}^{M_1} S_i \times \prod_{j=1}^{M_2} S_j$ , with  $\Lambda_L(\mathbf{q}^L; \mathbf{b}^L) = \times_{i=1}^{M_1} \phi_i \times_{j=1}^{M_2} \psi_j$ . A pure strategy NE of the subgame  $\Gamma_L$  is a fixed point  $(\tilde{\mathbf{q}}^L; \tilde{\mathbf{b}}^L)$  of  $\Lambda_L(\mathbf{q}^L; \mathbf{b}^L)$  such that no leader has an interest to deviate unilaterally from her decision. In addition, consider  $\Gamma_F$ . Define  $\Lambda_F : \prod_{i=1}^{N_1} S_i \times \prod_{j=1}^{N_2} S_j \rightarrow \prod_{i=M_1+1}^{N_1} S_i \times \prod_{j=M_2+1}^{N_2} S_j$ , with  $\Lambda_F(\tilde{\mathbf{q}}^L, \mathbf{q}^F; \tilde{\mathbf{b}}^L, \mathbf{b}^F) = \times_{i=M_1+1}^{N_1} \phi_i \times_{j=M_2+1}^{N_2} \psi_j$ . A pure strategy NE of the subgame  $\Gamma_F$  is a fixed point  $(\tilde{\mathbf{q}}^F; \tilde{\mathbf{b}}^F)$  of  $\Lambda_F$  such that no follower has an interest to deviate unilaterally from his decision.

Finally, consider the NE of the entire game  $\Gamma$ . A pure strategy SPNE of the entire game  $\Gamma$  is a fixed point  $(\tilde{\mathbf{q}}^L, \tilde{\mathbf{q}}^F; \tilde{\mathbf{b}}^L, \tilde{\mathbf{b}}^F)$ , with  $(\tilde{\mathbf{q}}^F; \tilde{\mathbf{b}}^F) = (\sigma(\tilde{\mathbf{q}}^L; \tilde{\mathbf{b}}^L); \varphi(\tilde{\mathbf{q}}^L; \tilde{\mathbf{b}}^L))$ . Therefore, at a SNE each leader (each follower) behaves optimally given her (his) conjecture, and the choice s/he makes is consistent with this conjecture. The market price (3) is given by  $\frac{p_X}{p_Y}(\tilde{\mathbf{q}}; \tilde{\mathbf{b}})$ . Therefore, the allocations corresponding to (4)-(5) are given by  $\tilde{\mathbf{z}}_i = \mathbf{z}_i(\tilde{q}_i, \frac{p_X}{p_Y}(\tilde{\mathbf{q}}^L; \tilde{\mathbf{b}}^L))$ , for  $i \in T_1$ , and  $\tilde{\mathbf{z}}_j = \mathbf{z}_j(\tilde{b}_j, \frac{p_X}{p_Y}(\tilde{\mathbf{q}}; \tilde{\mathbf{b}}))$ , for  $j \in T_2$ . Finally, the payoffs (6)-(7) are given by  $\pi_i(\tilde{q}_i, \tilde{\mathbf{q}}_{-i}; \tilde{\mathbf{b}}) = u_i(\mathbf{z}_i(\tilde{q}_i, \frac{p_X}{p_Y}(\tilde{\mathbf{q}}; \tilde{\mathbf{b}})))$  for  $i \in T_1$ , and by  $\pi_j(\tilde{\mathbf{q}}; \tilde{b}_j, \tilde{\mathbf{b}}_{-j}) = u_j(\mathbf{z}_j(\tilde{b}_j, \frac{p_X}{p_Y}(\tilde{\mathbf{q}}; \tilde{\mathbf{b}})))$  for  $j \in T_2$ . Therefore, a SNE is a noncooperative oligopoly equilibrium of  $\Gamma$  such that, on the one hand, the markets clear, and on the other hand, in each step of the game, no trader wants to deviate from her choice.

We are now able to define formally a SNE for the market game  $\Gamma$ .

DEFINITION 4. (SNE). A  $N_1 + N_2$ -tuple  $(\tilde{\mathbf{q}}; \tilde{\mathbf{b}})$ , consisting of a strategy profile  $(\tilde{\mathbf{q}}^L, \tilde{\mathbf{q}}^F; \tilde{\mathbf{b}}^L, \tilde{\mathbf{b}}^F) = (\tilde{q}_1, \dots, \tilde{q}_{M_1}, \tilde{q}_{M_1+1}, \dots, \tilde{q}_{N_1}; \tilde{b}_1, \dots, \tilde{b}_{M_2}, \tilde{b}_{M_2+1}, \dots, \tilde{b}_{N_2})$ , with  $\tilde{q}_i = \sigma_i(\tilde{\mathbf{q}}^L; \tilde{\mathbf{b}}^L)$ ,  $i = M_1 + 1, \dots, N_1$ , and  $\tilde{b}_j = \varphi_j(\tilde{\mathbf{q}}^L; \tilde{\mathbf{b}}^L)$ ,  $j = M_2 + 1, \dots, N_2$ , constitutes a Stackelberg-Nash equilibrium of  $\Gamma$  if:

- a.  $\forall i \in \{M_1 + 1, \dots, N_1\}$ ,  $\pi_i(\tilde{q}_i, \tilde{\mathbf{q}}_{-i}; \tilde{\mathbf{b}}) \geq \pi_i(q_i, \tilde{\mathbf{q}}_{-i}; \tilde{\mathbf{b}})$ , for all  $q_i \in \mathcal{S}_i$ ;
- b.  $\forall j \in \{M_2 + 1, \dots, N_2\}$ ,  $\pi_j(\tilde{\mathbf{q}}; \tilde{b}_j, \tilde{\mathbf{b}}_{-j}) \geq \pi_j(\tilde{\mathbf{q}}; b_j, \tilde{\mathbf{b}}_{-j})$ , for all  $b_j \in \mathcal{S}_j$ ;
- c.  $\forall i \in \{1, \dots, M_1\}$ ,  $\pi_i(\tilde{q}_i, \tilde{\mathbf{q}}_{-i}^L, \sigma(\tilde{q}_i, \tilde{\mathbf{q}}_{-i}^L; \tilde{\mathbf{b}}^L); \tilde{\mathbf{b}}^L, \varphi(\tilde{q}_i, \tilde{\mathbf{q}}_{-i}^L; \tilde{\mathbf{b}}^L)) \geq$

$$u_i(q_i, \tilde{\mathbf{q}}_{-i}^L, \sigma(q_i, \tilde{\mathbf{q}}_{-i}^L; \tilde{\mathbf{b}}^L); \tilde{\mathbf{b}}^L, \varphi(q_i, \tilde{\mathbf{q}}_{-i}^L; \tilde{\mathbf{b}}^L)), \text{ for all } \sigma(\mathbf{q}^L; \tilde{\mathbf{b}}^L) \in \prod_{i=M_1+1}^{N_1} \mathcal{S}_i \text{ and}$$

$$\text{all } \varphi(\mathbf{q}^L; \tilde{\mathbf{b}}^L) \in \prod_{j=M_2+1}^{N_2} \mathcal{S}_j;$$

- d.  $\forall j \in \{1, \dots, M_2\}$ ,  $u_j(\tilde{\mathbf{q}}^L, \sigma(\tilde{\mathbf{q}}^L; \tilde{b}_j, \tilde{\mathbf{b}}_{-j}^L); \tilde{b}_j, \tilde{\mathbf{b}}_{-j}^L, \varphi(\tilde{\mathbf{q}}^L; \tilde{b}_j, \tilde{\mathbf{b}}_{-j}^L)) \geq$

$$u_j(\tilde{\mathbf{q}}^L, \sigma(\tilde{\mathbf{q}}^L; b_j, \tilde{\mathbf{b}}_{-j}^L); b_j, \tilde{\mathbf{b}}_{-j}^L, \varphi(\tilde{\mathbf{q}}^L; b_j, \tilde{\mathbf{b}}_{-j}^L)), \text{ for all } \sigma(\tilde{\mathbf{q}}^L; \mathbf{b}^L) \in \prod_{i=M_1+1}^{N_1} \mathcal{S}_i$$

$$\text{and all } \varphi(\tilde{\mathbf{q}}^L; \mathbf{b}^L) \in \prod_{i=M_1+1}^{N_1} \mathcal{S}_j, \text{ for all } b_j \in \mathcal{S}_j.$$

addition, it is a SPNE without empty threats: it rules out incredible threats by the followers. The reason is the strategy of any follower is optimal for any supply set by the leaders. The followers can set their own supplies according to any possible function of the quantities set by the leaders, with the belief that the leaders will not counter-react. Similarly, the leaders expect the follower to conform to the decisions given by their best responses.

### 3. EXISTENCE OF A SNE WITH TRADE

Let us now consider the existence of a SNE with trade. It is well known that the autarkic equilibrium is always a NE in simultaneous move strategic market games (see, in particular, Cordella and Gabszewicz, 1998, Giraud, 2003, Busetto and Codognato, 2006). The next example, which is borrowed from Cordella and Gabszewicz (1998), illustrates this feature in the sequential game.

**EXAMPLE 1.** (Autarkic SNE). Let  $|T_1| = |T_2| = 2$ . Assumption 1 is  $\alpha_i = 1$ , for each  $i \in T_1$ , and  $\beta_j = 1$ , for each  $j \in T_2$ . Assumption 2 is  $u_i(x_i, y_i) = \gamma x_i + y_i$ ,  $i \in T_1$ , and  $u_i(x_i, y_i) = x_i + \gamma y_i$ ,  $j \in T_2$ , with  $\gamma \in (0, 1)$ , so (2d) does not hold. The unique competitive equilibrium is given by  $p^* = 1$  and  $z_i^* = (0, 1)$ ,  $i \in T_1$ , and  $z_j^* = (1, 0)$ ,  $j \in T_2$ . In addition, the Cournot-Nash equilibrium strategies are given by  $(\hat{q}_1, \hat{q}_2; \hat{b}_1, \hat{b}_2) = (0, 0; 0, 0)$ . Consider now the SNE. The followers' ODM are  $\phi_2(q_1, q_2; b_1) = -b_1 + \sqrt{\frac{1}{\gamma} b_1 (q_1 + q_2)}$  and  $\psi_2(q_1; b_1, b_2) = -q_1 + \sqrt{\frac{1}{\gamma} (b_1 + b_2) q_1}$ . The best responses are given by  $\sigma(q_1; b_1) = \frac{(1-2\gamma)b_1 + \sqrt{(1-4\gamma)(b_1)^2 + 4\gamma b_1 q_1}}{2\gamma}$  and  $\varphi(q_1; b_1) = \frac{(1-2\gamma)q_1 + \sqrt{(1-4\gamma)(q_1)^2 + 4\gamma b_1 q_1}}{2\gamma}$ . Then, the leaders' SNE strategies are  $\tilde{q}_1 = 0$  and  $\tilde{b}_1 = 0$ . Accordingly, the strategies of followers are  $\sigma(0; 0) = 0$  and  $\varphi(0; 0) = 0$ . Then, the only SNE is the trivial equilibrium  $(\tilde{q}_1, \tilde{q}_2; \tilde{b}_1, \tilde{b}_2) = (0, 0; 0, 0)$ .

Therefore, if one leader is making a positive supply, the other leader by deviating and making a positive supply can generate a subgame where at least one side of the market is making a null supply. No leader/follower finds it profitable to participate in exchange because no other leader/follower does. For any trader, the strategic advantage from trading, whichever is the stage of the game, is offset by the strategic advantage of reducing her supply to manipulate the market price. Nevertheless, we are able to state the following theorem.

**THEOREM 1.** (*Existence of a SNE with trade*). *Consider the market game  $\mathbf{\Gamma}$ , and let Assumptions 1 and 2 be satisfied. Then, there exists a Stackelberg-Nash equilibrium with trade.*

*Proof.* The approach of the proof is as follows. We consider a slight perturbation of the market game  $\mathbf{\Gamma}$  which is used by Dubey and Shubik (1978) for their existence proof. Therefore, consider a perturbed game  $\mathbf{\Gamma}^\epsilon$  in which some outside agency puts a fixed quantity  $\epsilon > 0$  of the two commodities on each side of the market. Given  $\epsilon > 0$ , the price (3) of  $\mathbf{\Gamma}^\epsilon$  is now given by:

$$\left( \frac{p_X}{p_Y} \right)^\epsilon = \frac{\mathbf{B} + \epsilon}{\mathbf{Q} + \epsilon}. \quad (8)$$

The allocations and payoffs are  $\tilde{\mathbf{z}}_{i,\epsilon} = \mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, (\frac{p_X}{p_Y})^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon))$  and  $\pi_i^\epsilon(\tilde{q}_{i,\epsilon}, \tilde{\mathbf{q}}_{-i,\epsilon}; \tilde{\mathbf{b}}_\epsilon)$  for  $i \in T_1$ , and  $\tilde{\mathbf{z}}_{j,\epsilon} = \mathbf{z}_{j,\epsilon}(\tilde{b}_{j,\epsilon}, (\frac{p_X}{p_Y})^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon))$  and  $\pi_j^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{b}_{j,\epsilon}, \tilde{\mathbf{b}}_{-j,\epsilon})$  for  $j \in T_2$ . The proof obtains by following five main steps. First, we study the optimal behavior of traders in each perturbed subgames. Second, we show the optimal behavior are mutually consistent (existence of a  $\epsilon$ -SNE). Third, we show the market price is bounded in a  $\epsilon$ -SNE. Fourth, we show the existence of an  $\epsilon$ -SNE with trade. Finally, we consider the sequence of  $\epsilon$ -SNE's with trade, and show the SNE is an equilibrium point of  $\mathbf{\Gamma}$ , a Nash equilibrium which is robust to slight perturbation of the market game. ■

Let us first define formally the concept of  $\epsilon$ -SNE.

DEFINITION 5. ( $\epsilon$ -SNE). For all  $\epsilon > 0$ , a  $N_1 + N_2$ -tuple  $(\tilde{\mathbf{q}}_\epsilon^L, \tilde{\mathbf{q}}_\epsilon^F; \tilde{\mathbf{b}}_\epsilon^L, \tilde{\mathbf{b}}_\epsilon^F)$  of feasible strategies  $(\tilde{q}_{1,\epsilon}, \dots, \tilde{q}_{M_1+1,\epsilon}, \tilde{q}_{M_1+1,\epsilon}, \dots, \tilde{q}_{N_1,\epsilon}; \tilde{b}_{1,\epsilon}, \dots, \tilde{b}_{M_2,\epsilon}, \tilde{b}_{M_2+1,\epsilon}, \dots, \tilde{b}_{N_2,\epsilon})$ , with  $\tilde{q}_{i,\epsilon} = \sigma_i^\epsilon(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L)$ ,  $i = M_1 + 1, \dots, N_1$ , and  $\tilde{b}_{j,\epsilon} = \varphi_j^\epsilon(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L)$ ,  $j = M_2 + 1, \dots, N_2$ , constitutes a Stackelberg-Nash equilibrium of  $\Gamma^\epsilon$  if like in Definition 4, conditions a., b., c. and d. hold, but where  $\pi_i$  is replaced by  $\pi_i^\epsilon$  for  $i \in \{1, \dots, N_1\}$ , and  $\pi_j$  is replaced by  $\pi_j^\epsilon$  for  $j \in \{1, \dots, N_2\}$  respectively.

To show the existence of an  $\epsilon$ -SNE (with trade) we need some intermediate results. First, we consider the behavior of traders in the perturbed game  $\Gamma^\epsilon$ .

Consider the perturbed subgame  $\Gamma_F^\epsilon$ . For any given strategy profile of the leaders, the program of follower  $i$  consists in maximizing his payoff  $\pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}; \mathbf{b}_\epsilon; \epsilon)$  given by (6). The next proposition echoes Definition 1.

PROPOSITION 1. *Let the utility functions  $u_k$  satisfy Assumption 2. Then, for all  $\epsilon > 0$ , the mappings  $\phi_i^\epsilon : \prod_{-i \in T_1} \mathcal{S}_{-i} \times \prod_{j \in T_2} \mathcal{S}_j \times \mathbb{R}_{++} \rightarrow \mathcal{S}_i$ , with  $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}; \mathbf{b}_\epsilon; \epsilon)$ ,  $i = M_1 + 1, \dots, N_1$ , and  $\psi_j^\epsilon : \prod_{i \in T_1} \mathcal{S}_{-i} \times \prod_{-j \in T_2} \mathcal{S}_j \times \mathbb{R}_{++} \rightarrow \mathcal{S}_j$ , with  $\psi_j^\epsilon(\mathbf{q}_\epsilon; \mathbf{b}_{-j,\epsilon}; \epsilon)$ ,  $j = M_2 + 1, \dots, N_2$  are well defined, point-valued (functions) and continuously differentiable.*

*Proof.* See Appendix A. ■

The next proposition provides monotonicity properties about followers' ODM.

PROPOSITION 2. *Let  $\phi^\epsilon = (\phi_{M_1+1}^\epsilon, \dots, \phi_{N_1}^\epsilon)$  and  $\psi^\epsilon = (\psi_{M_2+1}^\epsilon, \dots, \psi_{N_2}^\epsilon)$  be respectively  $(N_1 - M_1)$  and  $(N_2 - M_2)$  dimensional vector functions. Consider the Jacobian matrices  $\mathcal{J}_{\phi_{\mathbf{q}_\epsilon^F}^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) = \left[ \frac{\partial \phi^\epsilon(\cdot)}{\partial \mathbf{q}_\epsilon^F} \right]$  and  $\mathcal{J}_{\psi_{\mathbf{b}_\epsilon^F}^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) = \left[ \frac{\partial \psi^\epsilon(\cdot)}{\partial \mathbf{b}_\epsilon^F} \right]$ . Then,  $-\mathbf{I} \ll \mathcal{J}_{\phi_{\mathbf{q}_\epsilon^F}^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \leq \mathbf{I}$ , where  $\mathbf{I}$  is the  $(N_1 - M_1, N_1 - M_1)$  unit matrix, and  $-\mathbf{I} \ll \mathcal{J}_{\psi_{\mathbf{b}_\epsilon^F}^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \leq \mathbf{I}$ , where  $\mathbf{I}$  is the  $(N_2 - M_2, N_2 - M_2)$  unit matrix. In addition,  $\mathcal{J}_{\phi_{\mathbf{b}_\epsilon^F}^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) = \left[ \frac{\partial \phi^\epsilon(\cdot)}{\partial \mathbf{b}_\epsilon^F} \right] \in (-\mathbf{I}, \mathbf{I})$ , and  $\mathcal{J}_{\psi_{\mathbf{q}_\epsilon^F}^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) = \left[ \frac{\partial \psi^\epsilon(\cdot)}{\partial \mathbf{q}_\epsilon^F} \right] \in (-\mathbf{I}, \mathbf{I})$ , where the  $\mathbf{I}$ s are  $(N_1 - M_1, N_2 - M_2)$  and  $(N_2 - M_2, N_1 - M_1)$  unit matrices.*

*Proof.* See Appendix B. ■

Followers interact in a simultaneous move game. Thus, the followers optimal decisions must be consistent to solve the game. By "consistent" we mean that the best responses (see Definition 2) can be deduced from the collection of optimal decision mappings. It is worth noticing that the best responses might not exist (see Example 3). To introduce our criterion define the function  $\Phi_i^\epsilon : \prod_{i \in T_1} \mathcal{S}_i \times \prod_{j \in T_2} \mathcal{S}_j \times \mathbb{R}_{++} \rightarrow \mathcal{S}_i$ , with  $\Phi_i^\epsilon(\mathbf{q}_\epsilon; \mathbf{b}_\epsilon) := q_{i,\epsilon} - \phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_{-i,\epsilon}^F; \epsilon)$ ,  $i = M_1 + 1, \dots, N_1$ , and the function  $\Psi_j^\epsilon : \prod_{i \in T_1} \mathcal{S}_i \times \prod_{j \in T_2} \mathcal{S}_j \times \mathbb{R}_{++} \rightarrow \mathcal{S}_j$ , with  $\Psi_j^\epsilon(\mathbf{b}_\epsilon; \mathbf{q}_\epsilon) := b_{j,\epsilon} - \psi_j^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^L, \mathbf{b}_{-j,\epsilon}^F; \epsilon)$ ,  $j = M_2 + 1, \dots, N_2$ ,  $\epsilon > 0$ . This set of functions will be useful to build the system of equations that will implicitly define the best responses.

Let  $\Upsilon^\epsilon : \prod_{i \in T_1} \mathcal{S}_i \times \prod_{j \in T_2} \mathcal{S}_j \times \mathbb{R}_{++} \rightarrow \prod_{i=M_1+1}^{N_1} \mathcal{S}_i \times \prod_{j=M_2+1}^{N_2} \mathcal{S}_j$  the  $(N_1 - M_1) + (N_2 - M_2)$ -dimensional vector function defined by  $\Upsilon^\epsilon = (\Phi_{M_1+1}^\epsilon, \dots, \Phi_{N_1}^\epsilon; \Psi_{M_2+1}^\epsilon, \dots, \Psi_{N_2}^\epsilon)$ . Consider the  $(N_1 - M_1) + (N_2 - M_2)$ -dimensional vector equation  $\Upsilon^\epsilon(\mathbf{q}_\epsilon; \mathbf{b}_\epsilon; \epsilon) = \mathbf{0}$ . These  $(N_1 - M_1) + (N_2 - M_2)$  equations taken together consist in a system of

$(N_1 - M_1) + (N_2 - M_2)$  simultaneous equations with  $(N_1 - M_1) + (N_2 - M_2)$  unknowns  $(\mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^F)$  and  $M_1 + M_2$  parameters  $(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L)$ . This system defines implicitly (at least locally) best responses. Let  $(\bar{\mathbf{q}}_\epsilon^L, \bar{\mathbf{q}}_\epsilon^F; \bar{\mathbf{b}}_\epsilon^L, \bar{\mathbf{b}}_\epsilon^F)$  be an interior point of  $\prod_{i \in T_1} \mathcal{S}_i \times \prod_{j \in T_2} \mathcal{S}_j$ , so the identity  $\mathbf{Y}^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L); \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L)) \equiv \mathbf{0}$  holds in an open neighborhood of  $(\bar{\mathbf{q}}_\epsilon^L, \bar{\mathbf{q}}_\epsilon^F; \bar{\mathbf{b}}_\epsilon^L, \bar{\mathbf{b}}_\epsilon^F)$ . Implicit partial differentiation with respect to  $(\mathbf{b}_\epsilon^L; \mathbf{q}_\epsilon^L)$  of this identity leads to:

$$\mathcal{J}_{\mathbf{Y}^\epsilon_{(\mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^F)}}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \cdot \mathcal{A}^\epsilon = -\mathcal{B}^\epsilon, \text{ for each } \epsilon > 0, \quad (9)$$

where  $\mathcal{J}_{\mathbf{Y}^\epsilon_{(\mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^F)}}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon)$  is the  $((N_1 - M_1) + (N_2 - M_2); (N_1 - M_1) + (N_2 - M_2))$  matrix formed by all partial derivatives of  $\mathbf{Y}^\epsilon$  with respect to  $(\mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^F)$  at  $(\bar{\mathbf{q}}_\epsilon^L, \bar{\mathbf{q}}_\epsilon^F; \bar{\mathbf{b}}_\epsilon^L, \bar{\mathbf{b}}_\epsilon^F)$ , and  $\mathcal{A}^\epsilon$  and  $\mathcal{B}^\epsilon$  are matrices of dimension  $((N_1 - M_1) + (N_2 - M_2), M_1 + M_2)$  (see Appendix C). The next lemma says that the solution to (9), if it exists, determines the best responses.

LEMMA 1. *If  $\left| \mathcal{J}_{\mathbf{Y}^\epsilon_{(\mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^F)}}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \right| \neq 0$ , then, for all  $\epsilon > 0$ , there exist well defined functions  $\sigma_i^\epsilon : \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{j=1}^{M_2} \mathcal{S}_j \times \mathbb{R}_{++} \rightarrow \mathcal{S}_i$ , with  $b_{i,\epsilon} = \sigma_i^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)$ ,  $i = M_1 + 1, \dots, N_1$ , and  $\varphi_j^\epsilon : \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{j=1}^{M_2} \mathcal{S}_j \times \mathbb{R}_{++} \rightarrow \mathcal{S}_j$ , with  $q_{j,\epsilon} = \varphi_j^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)$ ,  $j = M_2 + 1, \dots, N_2$ . Moreover,  $\sigma_i^\epsilon \subset \mathcal{C}^1$ ,  $i = M_1 + 1, \dots, N_1$ , and  $\varphi_j^\epsilon \subset \mathcal{C}^1$ ,  $j = M_2 + 1, \dots, N_2$ .*

*Proof.* See Appendix C. ■

Lemma 1 gives a sufficient condition for the existence of continuous differentiable best responses: the Jacobian of  $\mathbf{Y}^\epsilon(\cdot)$  is a linear map which is invertible. It is worth noticing that, as the perturbed best responses are differentiable, then, for all  $\epsilon > 0$ , the vector function  $(\sigma^\epsilon(\cdot; \epsilon); \varphi^\epsilon(\cdot; \epsilon))$  is differentiable. The next proposition gives some bounds on the optimal responses of followers.

PROPOSITION 3. *Let  $\sigma^\epsilon = (\sigma_{M_1+1}^\epsilon, \dots, \sigma_{N_1}^\epsilon)$  and  $\varphi^\epsilon = (\varphi_{M_2+1}^\epsilon, \dots, \varphi_{N_2}^\epsilon)$  be  $(N_1 - M_1)$  and  $(N_2 - M_2)$  dimensional vector functions. Consider  $\mathcal{J}_{\sigma_{\mathbf{q}_\epsilon^L}^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) = \left[ \frac{\partial \sigma^\epsilon(\cdot)}{\partial \mathbf{q}_\epsilon^L} \right]$  and  $\mathcal{J}_{\varphi_{\mathbf{q}_\epsilon^L}^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) = \left[ \frac{\partial \varphi^\epsilon(\cdot)}{\partial \mathbf{q}_\epsilon^L} \right]$ . Then,  $\mathcal{J}_{\sigma_{\mathbf{q}_\epsilon^L}^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \in [-\mathbf{I}_3, \mathbf{I}_3)$  and  $\mathcal{J}_{\varphi_{\mathbf{q}_\epsilon^L}^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \geq \mathbf{0}$ , where  $\mathbf{I}_3$  is the  $(N_1 - M_1, M_1)$  unit matrix and  $\mathbf{0}$  is the  $(N_2 - M_2, M_1)$  zero matrix. In addition,  $\mathcal{J}_{\varphi_{\mathbf{b}_\epsilon^L}^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \in [-\mathbf{I}_4, \mathbf{I}_4)$  and  $\mathcal{J}_{\sigma_{\mathbf{b}_\epsilon^L}^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \geq \mathbf{0}$ , where  $\mathbf{I}_4$  and  $\mathbf{0}$  are of dimension  $(N_2 - M_2, M_2)$  and  $(N_1 - M_1, M_2)$  respectively.*

*Proof.* See Appendix D. ■

Consider now the subgame  $\mathbf{\Gamma}_L^\epsilon$ . Each leader knows how the market price is affected by the followers' reactions. Indeed, let  $\sigma^\epsilon : \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{j=1}^{M_2} \mathcal{S}_j \times \mathbb{R}_{++} \rightarrow \prod_{i=M_1+1}^{N_1} \mathcal{S}_i$ , with  $\mathbf{q}_\epsilon^F = \sigma^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)$ , and let  $\varphi^\epsilon : \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{j=1}^{M_2} \mathcal{S}_j \times \mathbb{R}_{++} \rightarrow \prod_{j=M_2+1}^{N_2} \mathcal{S}_j$ , with  $\mathbf{b}_\epsilon^F = \varphi^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)$ . In particular,  $\sigma^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon) \subset \mathcal{C}^1$  and  $\varphi^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon) \subset \mathcal{C}^1$ . Therefore,  $\frac{\partial X}{\partial Y}((\mathbf{q}_\epsilon^L, \sigma^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon); \mathbf{b}_\epsilon^L, \varphi^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon))) \subset \mathcal{C}^1$ . Then, leader  $i$  maximizes her reduced payoff  $\pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L, \sigma^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon); \mathbf{b}_\epsilon^L, \varphi^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon))$ . The next proposition states the existence of ODM for leaders (see Definition 3).

PROPOSITION 4. *Let the utility functions  $u_k$  satisfy Assumption 2. Then, for all  $\epsilon > 0$ , the mappings  $\phi_i^\epsilon : \prod_{-i \neq i} \mathcal{S}_{-i} \times \prod_{j=1}^{M_2} \mathcal{S}_j \times \mathbb{R}_{++} \rightarrow \mathcal{S}_i$ , with  $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon)$ ,  $i = 1, \dots, M_1$ , and  $\psi_j^\epsilon : \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{-j=1} \mathcal{S}_{-j} \times \mathbb{R}_{++} \rightarrow \mathcal{S}_j$ , with  $\psi_j^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_{-j,\epsilon}^L; \epsilon)$ ,  $j = 1, \dots, M_2$ , are well defined, point-valued (functions) and continuously differentiable.*

*Proof.* See Appendix E. ■

We are now able to state the existence of an  $\epsilon$ -SNE.

LEMMA 2. (*Existence of  $\epsilon$ -SNE*). *Consider  $\Gamma^\epsilon$ , and let Assumptions 1 and 2 be satisfied. Then, for all  $\epsilon > 0$ , there exists an  $\epsilon$ -Stackelberg-Nash equilibrium of  $\Gamma^\epsilon$ .*

*Proof.* We have to show that the optimal strategic behavior are mutually consistent, i.e., there is a pure strategy SPNE for the entire perturbed game  $\Gamma^\epsilon$ , which constitutes a NE of each perturbed subgame  $\Gamma_L^\epsilon$  and  $\Gamma_F^\epsilon$ . We first show that  $\Gamma_L^\epsilon$  has a NE. To this end, define the function  $\Lambda_L^\epsilon : \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{j=1}^{M_2} \mathcal{S}_j \times \mathbb{R}_{++} \rightarrow \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{j=1}^{M_2} \mathcal{S}_j$ , with  $\Lambda_L^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon) = \times_{i=1}^{M_1} \phi_i^\epsilon \times \times_{j=1}^{M_2} \psi_j^\epsilon$ , where  $\phi_i^\epsilon$ ,  $i = 1, \dots, M_1$ , and  $\psi_j^\epsilon$ ,  $j = 1, \dots, M_2$ , are well-defined functions from Proposition 3 (see Appendix E). The function  $\Lambda_L^\epsilon$  is a continuous function (as the product of continuous functions  $\phi_i^\epsilon$ ,  $i = 1, \dots, M_1$ , and  $\psi_j^\epsilon$ ,  $j = 1, \dots, M_2$ , from Proposition 4) over a compact and convex subset of Euclidean space (as the product of compact and convex sets  $\mathcal{S}_i$ ,  $i = 1, \dots, M_1$ , and  $\mathcal{S}_j$ ,  $j = 1, \dots, M_2$ ). Then, by the Brouwer Fixed Point Theorem, the function  $\Lambda_L^\epsilon$  admits a fixed point, namely  $(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L)$ , which is a NE of  $\Gamma_L^\epsilon$ . Now, we show  $\Gamma_F^\epsilon$  has a NE. Define  $\Lambda_F^\epsilon : \prod_{i=1}^{N_1} \mathcal{S}_i \times \prod_{j=1}^{N_2} \mathcal{S}_j \times \mathbb{R}_{++} \rightarrow \prod_{i=M_1+1}^{N_1} \mathcal{S}_i \times \prod_{j=M_2+1}^{N_2} \mathcal{S}_j$ , with  $\Lambda_F^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F) = \times_{i=M_1+1}^{N_1} \phi_i^\epsilon \times \times_{j=M_2+1}^{N_2} \psi_j^\epsilon$ , where  $\phi_i^\epsilon$ ,  $i = M_1+1, \dots, N_1$ , and  $\psi_j^\epsilon$ ,  $j = M_2+1, \dots, N_2$ , are known to exist from Proposition 1. Fix  $(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L)$ . The function  $\Lambda_F^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F)$  is continuous on  $\prod_i \mathcal{S}_i \times \prod_j \mathcal{S}_j$ , a compact and convex set of Euclidean space. Then, it has a fixed point, namely  $(\tilde{\mathbf{q}}_\epsilon^F; \tilde{\mathbf{b}}_\epsilon^F)$ , which is a NE of  $\Gamma_F^\epsilon$ . Finally, from Lemma 1, for all  $\epsilon > 0$ , we can define  $(\mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^F) = (\sigma^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon); \varphi^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon))$ . If  $(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L)$  is a fixed point, then, by using Lemma 1, and by continuity of  $\sigma^\epsilon(\cdot)$  and  $\varphi^\epsilon(\cdot)$ , we deduce  $(\tilde{\mathbf{q}}_\epsilon^F; \tilde{\mathbf{b}}_\epsilon^F) = (\sigma^\epsilon(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L; \epsilon); \varphi^\epsilon(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L; \epsilon))$  is a fixed point of  $\Gamma_F^\epsilon$ , for all  $\epsilon > 0$ . Then,  $(\tilde{\mathbf{q}}_\epsilon^L, \tilde{\mathbf{q}}_\epsilon^F; \tilde{\mathbf{b}}_\epsilon^L, \tilde{\mathbf{b}}_\epsilon^F)$  is a fixed point of  $\Gamma^\epsilon$ . ■

The next lemma concerns the existence of bounds on market price in an  $\epsilon$ -SNE.

LEMMA 3. *Assume there are at least one leader and one follower of each type. Then, in an  $\epsilon$ -SNE, there exist uniform bounds  $\xi_1 > 0$  and  $\xi_2 > 0$  such that:*

$$\forall \epsilon > 0, \xi_1 < \left( \frac{p_X}{p_Y} \right)^\epsilon < \xi_2, \text{ with } \left( \frac{p_X}{p_Y} \right)^\epsilon = \frac{\mathbf{B} + \epsilon}{\mathbf{Q} + \epsilon}. \quad (10)$$

*Proof.* See Appendix F. ■

The following lemma is related to the existence of an  $\epsilon$ -SNE with trade.

LEMMA 4. (*Existence of  $\epsilon$ -SNE with trade*). *Consider  $\Gamma^\epsilon$ , and let Assumptions 1 and 2 be satisfied. Then, for all  $\epsilon > 0$ , there exists an  $\epsilon$ -SNE with trade of  $\Gamma^\epsilon$ .*

*Proof.* We have to show that there are non trivial equilibrium strategies in each stage, i.e., there exist lower and upper uniform bounds on equilibrium bids such that there are at least one leader and one follower of the first type (resp. second type) for whom  $0 < \tilde{q}_{i,\epsilon} < \alpha_i$  (resp.  $0 < \tilde{b}_{j,\epsilon} < \beta_j$ ).

*Follower  $i$ .* Consider the payoff given by (6). Let  $\sigma_i^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon) \in \mathcal{S}_i$ . We have to show that there are  $q_i, \bar{q}_i \in \mathcal{S}_i$  such that  $0 < q_i \leq \sigma_i^\epsilon(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L; \epsilon) \leq \bar{q}_i < \alpha_i$ , for at least one  $i$ ,  $i = M_1 + 1, \dots, N_1$ . Fix the strategies of all other traders in equilibrium. Follower  $i$ 's marginal payoff may be written (see (A2) in Appendix A):

$$\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}} = -\frac{\partial u_i}{\partial x_i} + \left(\frac{p_X}{p_Y}\right)^\epsilon \frac{\mathbf{Q}_{-i,\epsilon} + \epsilon}{q_{i,\epsilon} + \mathbf{Q}_{-i,\epsilon} + \epsilon} \frac{\partial u_i}{\partial y_i}, \text{ for all } \epsilon > 0. \quad (11)$$

From Proposition 1, there exists  $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon) \geq 0$ ,  $i = M_1 + 1, \dots, N_1$ . In addition, from Lemma 1, there exists  $q_{i,\epsilon} = \sigma_i^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon) \geq 0$ ,  $i = M_1 + 1, \dots, N_1$ . Then, in equilibrium we have  $\tilde{q}_{i,\epsilon} = \sigma_i^\epsilon(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L; \epsilon) \geq 0$ ,  $i = M_1 + 1, \dots, N_1$ . Let  $MRS_{X/Y}^i = \frac{\partial u_i / \partial x_i}{\partial u_i / \partial y_i}$ , so (11) may be written:

$$\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}} = \frac{\partial u_i}{\partial y_i} \left[ \left(\frac{p_X}{p_Y}\right)^\epsilon - MRS_{X/Y}^i \right], \text{ for all } \epsilon > 0. \quad (12)$$

Consider the case  $\sigma_i^\epsilon(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L; \epsilon) \geq b_i > 0$ . As  $(\frac{p_X}{p_Y})^\epsilon > \xi_1$  and  $\frac{\mathbf{Q}_{-i,\epsilon} + \epsilon}{q_{i,\epsilon} + \mathbf{Q}_{-i,\epsilon} + \epsilon} \leq 1$ , then (12) may be written:

$$\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}} > \frac{\partial u_i}{\partial y_i} (\xi_1 - MRS_{X/Y}^i), \text{ for all } \epsilon > 0. \quad (13)$$

From (2a)-(2c), we deduce  $\frac{\partial MRS_{X/Y}^i}{\partial q_{i,\epsilon}} > 0$ . Assume  $\sigma_i^\epsilon(\cdot; \epsilon) = 0$ . Then, from (2d),  $\lim_{q_{i,\epsilon} \rightarrow 0} MRS_{X/Y}^i = 0$ , so (12) becomes  $\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}} > \frac{\partial u_i}{\partial y_i} \xi_1$ . But, from (2d), we have  $\lim_{b_{i,\epsilon} \rightarrow 0} \frac{\partial u_i}{\partial y_i} = \lim_{y_i \rightarrow 0} \frac{\partial u_i}{\partial y_i} = \infty$ , so we deduce  $\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}} > \infty$ . A contradiction. Therefore, there must be  $q_i > 0$ , with  $q_i = \sigma_i^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)$  and  $q_i \in \mathcal{S}_i$ , such that  $(\frac{\partial MRS_{X/Y}^i}{\partial q_{i,\epsilon}})|_{q_{i,\epsilon}=q_i} = \xi_1$ . As  $(\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}})|_{q_{i,\epsilon}=q_i} > 0$ , then for all  $\sigma_i^\epsilon(\cdot; \epsilon) \in \mathcal{S}_i$ , we have  $\sigma_i^\epsilon(\cdot; \epsilon) \geq q_i > 0$ . Then,  $\sigma_i^\epsilon(\cdot; \epsilon) > 0$ , so  $\lambda_{i,\epsilon} = 0$  in (A2), for at least one  $i \in \{1, \dots, M_1\}$ . Likewise,  $0 < b_j \leq \varphi_j^\epsilon(\cdot; \epsilon)$ ,  $j = M_2 + 1, \dots, N_2$ .

Consider now the case  $q_{i,\epsilon} \leq \bar{q}_i < \alpha_i$ . As  $(\frac{p_X}{p_Y})^\epsilon < \xi_2$  and  $\frac{\mathbf{Q}_{-i,\epsilon} + \epsilon}{q_{i,\epsilon} + \mathbf{Q}_{-i,\epsilon} + \epsilon} \leq 1$ , then:

$$\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}} < \frac{\partial u_i}{\partial y_i} (\xi_2 - MRS_{X/Y}^i), \text{ for all } \epsilon > 0. \quad (14)$$

From (2a)-(2c),  $\frac{\partial MRS_{X/Y}^i}{\partial q_{i,\epsilon}} > 0$ . In addition, from (2d),  $\lim_{q_{i,\epsilon} \rightarrow 0} MRS_{X/Y}^i = 0$  and  $\lim_{q_{i,\epsilon} \rightarrow \alpha_i} MRS_{X/Y}^i = \infty$ . Then, there is  $\bar{q}_i < \alpha_i$ , with  $\bar{q}_i = \bar{\sigma}_i^\epsilon(\cdot; \epsilon)$  and  $\bar{\sigma}_i^\epsilon(\cdot; \epsilon) \in \mathcal{S}_i$ , such that  $(\frac{\partial MRS_{X/Y}^i}{\partial q_{i,\epsilon}})|_{q_{i,\epsilon}=\bar{q}_i} = \xi_2$ . Then, from (14),  $(\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}})|_{q_{i,\epsilon}=\bar{q}_i} < 0$ , where  $\pi_i^\epsilon$  is strictly concave in  $q_{i,\epsilon}$  on  $[0, \alpha_i]$ . Then, for all  $\tilde{q}_{i,\epsilon} \in \mathcal{S}_i$ , we get  $\tilde{q}_{i,\epsilon} \leq \bar{q}_i$ , so  $\mu_{i,\epsilon} = 0$  in (A2). But, then,  $\varphi_i^\epsilon(\cdot; \epsilon) \leq \bar{q}_i < \alpha_i$  for at least one follower  $i$ .

*Leader  $i$ .* Fix the strategies of all other leaders in equilibrium. Leader  $i$ 's marginal payoff may be written:

$$\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}} = \frac{\partial u_i}{\partial y_i} \left[ \left(\frac{p_X}{p_Y}\right)^\epsilon \chi - MRS_{X/Y}^i \right], \text{ for all } \epsilon > 0. \quad (15)$$

where  $\chi \equiv 1 - (1 + \nu_\epsilon) \frac{\tilde{q}_{i,\epsilon}}{\tilde{q}_{i,\epsilon} + \tilde{\mathbf{Q}}_{-i,\epsilon} + \epsilon} + \eta_\epsilon \frac{\tilde{q}_{i,\epsilon}}{\tilde{\mathbf{B}}_{\epsilon} + \epsilon}$ , with  $\chi \in [0, 1]$ , is the inverse of the markup (see (E2) in Appendix E).

Consider the case  $\tilde{q}_{i,\epsilon} \geq \underline{q}_i > 0$ . As  $(\frac{\partial X}{\partial Y})^\epsilon > \xi_1$  and  $\chi \leq 1$ , then (15) is  $\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}} > \frac{\partial u_i}{\partial y_i} (\xi_1 - MRS_{X/Y}^i)$ , for all  $\epsilon > 0$ . From (2a)-(2c),  $\frac{\partial MRS_{X/Y}^i}{\partial q_{i,\epsilon}} > 0$ . Assume  $\tilde{q}_{i,\epsilon} = 0$ . Then,  $\chi = 1$ , and, from (2d),  $\lim_{q_{i,\epsilon} \rightarrow 0} MRS_{X/Y}^i = 0$ , so (15) is  $\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}} > \frac{\partial u_i}{\partial y_i} \xi_1$ . But, from (2d),  $\lim_{q_{i,\epsilon} \rightarrow 0} \frac{\partial u_i}{\partial y_i} = \lim_{y_i \rightarrow 0} \frac{\partial u_i}{\partial y_i} = \infty$ , so  $\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}} > \infty$ . A contradiction. Therefore, there is  $\underline{q}_i > 0$ , with  $\underline{q}_i \in \mathcal{S}_i$ , such that  $(\frac{\partial MRS_{X/Y}^i}{\partial q_{i,\epsilon}})_{|q_{i,\epsilon} = \underline{q}_i} = \xi_1$ . As  $(\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}})_{|q_{i,\epsilon} = \underline{q}_i} > 0$ , then for all  $\tilde{q}_{i,\epsilon} \in \mathcal{S}_i$ ,  $\tilde{q}_{i,\epsilon} \geq \underline{q}_i > 0$ . Then,  $\tilde{q}_{i,\epsilon} > 0$ , so  $\lambda_{i,\epsilon} = 0$  in (E2), for at least one  $i \in \{1, \dots, M_1\}$ .

The proof of  $0 < \underline{b}_j \leq b_j \leq \tilde{b}_{j,\epsilon} < \beta_j$  for at least one  $j \in \{1, \dots, N_2\}$ , follows the same steps as the one provided for type 1 traders. ■

Finally, we show the SNE is an equilibrium point (EP), which we now define.

**DEFINITION 6.** (EP). A Stackelberg-Nash equilibrium  $(\tilde{\mathbf{q}}; \tilde{\mathbf{b}})$  is an *equilibrium point* of  $\mathbf{\Gamma}$  if there exist sequences  $\{\epsilon_n\}_{n=1}^\infty$  and  $\{(\tilde{\mathbf{q}}_{\epsilon_n}; \tilde{\mathbf{b}}_{\epsilon_n})\}_{n=1}^\infty$  such that:

- i.  $\epsilon_n > 0$  and  $\lim_{n \rightarrow \infty} \{\epsilon_n\} = 0$ ;
- ii.  $(\tilde{\mathbf{q}}_{\epsilon_n}; \tilde{\mathbf{b}}_{\epsilon_n})$  is a Nash equilibrium of  $\mathbf{\Gamma}^{\epsilon_n}$ ;
- iii.  $\lim_{n \rightarrow \infty} \{(\tilde{\mathbf{q}}_{\epsilon_n}; \tilde{\mathbf{b}}_{\epsilon_n})\} = (\tilde{\mathbf{q}}; \tilde{\mathbf{b}})$ .

**LEMMA 5.** (SNE is an EP). Consider the market game  $\mathbf{\Gamma}$ , and let Assumptions 1 and 2 be satisfied. Then, the SNE with trade is an equilibrium point of  $\mathbf{\Gamma}$ .

*Proof.* Consider a sequence  $\{\epsilon_n\}$  such that  $\lim_{n \rightarrow \infty} \{\epsilon_n\} = 0$ . Pick a sequence  $\{(\tilde{q}_{i,\epsilon_n}; \tilde{b}_{j,\epsilon_n})\}$ ,  $i \in T_1$ ,  $j \in T_2$ ,  $n = 1, 2, \dots$ . Consider the subgame  $\mathbf{\Gamma}_L^{\epsilon_n}$ . From Lemma 4, we know that, for at least one leader of each type, we have  $\underline{q}_i \leq \tilde{q}_{i,\epsilon_n} \leq \bar{q}_i$ ,  $i \in \{1, \dots, M_1\}$ , and  $\underline{b}_j \leq \tilde{b}_{j,\epsilon_n} \leq \bar{b}_j$ ,  $j \in \{1, \dots, M_2\}$ , for  $n = 1, 2, \dots$ . Thus, the sequence  $\{(\tilde{q}_{i,\epsilon_n}; \tilde{b}_{j,\epsilon_n})\}$  is defined over a compact set. Then, from the Bolzano-Weierstrass Theorem (see Corollary 4.7, p. 25 in Aliprantis et al. (1998)), there exists a subsequence  $\{(\tilde{q}_{i,\epsilon_{k_n}}; \tilde{b}_{j,\epsilon_{k_n}})\}$  which converges to a limit point  $(\tilde{q}_i; \tilde{b}_j)$ , where  $\underline{q}_i \leq \tilde{q}_i \leq \bar{q}_i$ ,  $i \in \{1, \dots, M_1\}$ , and  $\underline{b}_j \leq \tilde{b}_j \leq \bar{b}_j$ ,  $j \in \{1, \dots, M_2\}$ , from Lemma 4. As the payoff functions of the leaders are strictly concave (see Appendix D), they are continuous, so  $(\tilde{q}_i; \tilde{b}_j)$  is an EP of  $\mathbf{\Gamma}_L$ . Consider now the subgame  $\mathbf{\Gamma}_F^{\epsilon_n}$ . From Lemma 1, there exist  $q_{i,\epsilon} = \sigma_i^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)$ , for all  $i = M_1 + 1, \dots, N_1$ , and  $b_{j,\epsilon} = \varphi_j^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)$ , for all  $j = M_2 + 1, \dots, N_2$ . Consider the sequence of best responses  $\{\sigma_i^{\epsilon_n}(\mathbf{q}_{\epsilon_n}^L; \mathbf{b}_{\epsilon_n}^L; \epsilon_n); \varphi_j^{\epsilon_n}(\mathbf{q}_{\epsilon_n}^L; \mathbf{b}_{\epsilon_n}^L; \epsilon_n)\}$ ,  $n = 1, 2, \dots$ , which are defined over compact sets. Let  $(\tilde{\mathbf{q}}_{\epsilon_{k_n}}^L; \tilde{\mathbf{b}}_{\epsilon_{k_n}}^L; \epsilon_{k_n})$  be a NE with trade of the subgame  $\mathbf{\Gamma}_L^{\epsilon_{k_n}}$ . Then, there is a subsequence  $\{\sigma_i^{\epsilon_{k_n}}(\tilde{\mathbf{q}}_{\epsilon_{k_n}}^L; \tilde{\mathbf{b}}_{\epsilon_{k_n}}^L; \epsilon_{k_n}); \varphi_j^{\epsilon_{k_n}}(\tilde{\mathbf{q}}_{\epsilon_{k_n}}^L; \tilde{\mathbf{b}}_{\epsilon_{k_n}}^L; \epsilon_{k_n})\}$  such that  $\lim_{n \rightarrow \infty} \{\sigma_i^{\epsilon_{k_n}}(\tilde{\mathbf{q}}_{\epsilon_{k_n}}^L; \tilde{\mathbf{b}}_{\epsilon_{k_n}}^L; \epsilon_{k_n}); \varphi_j^{\epsilon_{k_n}}(\tilde{\mathbf{q}}_{\epsilon_{k_n}}^L; \tilde{\mathbf{b}}_{\epsilon_{k_n}}^L; \epsilon_{k_n})\} = \{\sigma_i(\tilde{\mathbf{q}}^L; \tilde{\mathbf{b}}^L); \varphi_j(\tilde{\mathbf{q}}^L; \tilde{\mathbf{b}}^L)\}$  as  $\lim_{n \rightarrow \infty} \{\sigma_i^{\epsilon_{k_n}}(\cdot; \epsilon_{k_n}); \varphi_j^{\epsilon_{k_n}}(\cdot; \epsilon_{k_n})\} = \{\sigma_i(\cdot); \varphi_j(\cdot)\}$ . But  $\{\sigma_i(\tilde{\mathbf{q}}^L; \tilde{\mathbf{b}}^L); \varphi_j(\tilde{\mathbf{q}}^L; \tilde{\mathbf{b}}^L)\} = (\tilde{q}_i; \tilde{b}_j)$ . In addition, from Lemma 4, we have  $\underline{q}_i \leq \tilde{q}_i \leq \bar{q}_i$ ,  $i \in \{M_1 + 1, \dots, N_1\}$ , and  $\underline{b}_j \leq \tilde{b}_j \leq \bar{b}_j$ ,  $j \in \{M_2 + 1, \dots, N_2\}$ . By continuity of the payoff functions of the followers (see Appendix A), we deduce  $(\tilde{q}_i; \tilde{b}_j)$  is an EP of  $\mathbf{\Gamma}_F$ . As  $(\tilde{\mathbf{q}}^F; \tilde{\mathbf{b}}^F) = (\sigma(\tilde{\mathbf{q}}^L; \tilde{\mathbf{b}}^L); \varphi(\tilde{\mathbf{q}}^L; \tilde{\mathbf{b}}^L))$ , then  $(\tilde{\mathbf{q}}^L, \tilde{\mathbf{q}}^F; \tilde{\mathbf{b}}^L, \tilde{\mathbf{b}}^F)$  is an interior pure strategy SPNE of  $\mathbf{\Gamma}$ . Then, the SNE with trade is an EP of  $\mathbf{\Gamma}$ , which means there exists a strategy profile  $(\tilde{\mathbf{q}}; \tilde{\mathbf{b}})$ , which is a non autarkic SNE of  $\mathbf{\Gamma}$ . ■

#### 4. DISCUSSION: SOME EXAMPLES

We provide some examples to buttress the working of our approach as well as to discuss the assumptions made on the utility functions and the role they play in the proof of the existence Theorem. We discuss three properties: the differentiability, the strict quasi-concavity and the behavior of the indifference curves along the boundary of the consumption sets. Example 1 computes a SNE when Assumption 2 is satisfied. Example 2 illustrates that (2c) is not necessary. Example 3 illustrates existence failure. Example 4 shows a SNE may exist even if (2a), (2c) and (2d) do not hold for some traders. In each case, we also compute the Cournot-Nash equilibrium (CNE) supplies and the competitive equilibrium (CE) supplies. In all examples Assumption 1 is  $\alpha_i = 1$ , for each  $i \in T_1$ , and  $\beta_j = 1$ , for each  $j \in T_2$ .

##### 4.1. A SNE under Assumption 2

Let  $|T_1| = |T_2| = 4$ , with two leaders and two followers of each type. Assumption 2 is given by:

$$u_k(x_k, y_k) = x_k \cdot y_k, \quad k = i, j, \quad i, j = 1, 2. \quad (16)$$

The CE supplies are given by  $(q_1^*, q_2^*, q_3^*, q_4^*) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  and  $(b_1^*, b_2^*, b_3^*, b_4^*) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ . In addition, the CNE supplies are given by  $(\hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{q}_4) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$  and  $(\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ .

Let us now compute the SNE. The ODM (see Definition 1) are given by:

$$\phi_3(q_1, q_2, q_4) = -(q_1 + q_2 + q_4) + \sqrt{(q_1 + q_2 + q_4)^2 + (q_1 + q_2 + q_4)} \quad (17)$$

$$\phi_4(q_1, q_2, q_3) = -(q_1 + q_2 + q_3) + \sqrt{(q_1 + q_2 + q_3)^2 + (q_1 + q_2 + q_3)} \quad (18)$$

$$\psi_3(b_1, b_2, b_4) = -(b_1 + b_2 + b_4) + \sqrt{(b_1 + b_2 + b_4)^2 + (b_1 + b_2 + b_4)} \quad (19)$$

$$\psi_4(b_1, b_2, b_3) = -(b_1 + b_2 + b_3) + \sqrt{(b_1 + b_2 + b_3)^2 + (b_1 + b_2 + b_3)}. \quad (20)$$

Let  $\Phi_i(q_1, q_2, q_3, q_4) := q_i - \phi_i(q_1, q_2, \cdot)$ ,  $i = 3, 4$ , and  $\Psi_j(b_1, b_2, b_3, b_4) := b_j - \psi_j(b_1, b_2, \cdot)$ ,  $j = 3, 4$ . The Jacobian corresponding to (9) is given by:

$$\mathcal{J}_{\mathbf{r}_{(q^F, b^F)}} = \begin{bmatrix} 1 & g & 0 & 0 \\ h & 1 & 0 & 0 \\ 0 & 0 & 1 & g' \\ 0 & 0 & h' & 1 \end{bmatrix}, \quad (21)$$

where  $g \equiv 1 - \frac{q_1 + q_3 + \frac{1}{2}}{\sqrt{(q_1 + q_3)^2 + q_1 + q_3}}$ ,  $h \equiv 1 - \frac{q_1 + q_2 + \frac{1}{2}}{\sqrt{(q_1 + q_2)^2 + q_1 + q_2}}$ ,  $g' \equiv 1 - \frac{b_1 + b_3 + \frac{1}{2}}{\sqrt{(b_1 + b_3)^2 + b_1 + b_3}}$  and  $h' \equiv 1 - \frac{b_1 + b_2 + \frac{1}{2}}{\sqrt{(b_1 + b_2)^2 + b_1 + b_2}}$ . We get  $|\mathcal{J}_{\mathbf{r}_{(q^F, b^F)}}| = (1 - gh)(1 - g'h') \neq 0$  as  $gh \neq 1$  and  $g'h' \neq 1$ . Then, Lemma 1 holds, and the best responses are given by:



$$\sigma_i(q_1, q_2) = \frac{1}{6} - \frac{1}{3}(q_1 + q_2) + \sqrt{\frac{1}{3}(q_1 + q_2)^2 + 2(q_1 + q_2) + \frac{1}{4}}, \quad i = 3, 4 \quad (22)$$

$$\varphi_j(b_1, b_2) = \frac{1}{6} - \frac{1}{3}(b_1 + b_2) + \frac{1}{3}\sqrt{(b_1 + b_2)^2 + 2(b_1 + b_2) + \frac{1}{4}}, \quad j = 3, 4. \quad (23)$$

In the second step, any leader maximizes her reduced payoff:

$$\max \frac{(1 - q_i) \cdot q_i}{\frac{1}{3} + \frac{1}{3}(q_1 + q_2) + \frac{2}{3}\sqrt{(q_1 + q_2)^2 + 2(q_1 + q_2) + \frac{1}{4}}}, \quad i = 1, 2 \quad (24)$$

$$\max \frac{b_j \cdot (1 - b_j)}{\frac{1}{3} + \frac{1}{3}(b_1 + b_2) + \frac{2}{3}\sqrt{(b_1 + b_2)^2 + 2(b_1 + b_2) + \frac{1}{4}}}, \quad j = 1, 2. \quad (25)$$

The first and second-order conditions yield the unique solution  $\tilde{q}_i = 0.421907$ ,  $i = 1, 2$ , and  $\tilde{b}_j = 0.421907$ ,  $j = 1, 2$ . From (22)-(23), we deduce  $(\tilde{q}_3, \tilde{q}_4) = (\tilde{b}_3, \tilde{b}_4) = (0.427986, 0.427986)$ . The SNE supplies are given by the strategy profiles:

$$(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4) = (0.421907, 0.421907, 0.427986, 0.427986) \quad (26)$$

$$(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4) = (0.421907, 0.421907, 0.427986, 0.427986). \quad (27)$$

## 4.2. The boundary conditions

We assume  $|T_1| = |T_2| = 2$ . The utility functions of traders are given by:

$$u_i(x_i, y_i) = \gamma_i x_i + y_i, \quad \gamma_i \in (0, 1) \quad i = 1, 2 \quad (28)$$

$$u_j(x_j, y_j) = x_j \cdot y_j, \quad j = 1, 2. \quad (29)$$

The CE supplies are given by  $(q_1^*, q_2^*) = (\frac{1}{2\gamma}, \frac{1}{2\gamma})$  and  $(b_1^*, b_2^*) = (\frac{1}{2}, \frac{1}{2})$ . In addition, if  $\gamma_1 \neq \gamma_2$ , the CNE supplies are  $(\hat{q}_1, \hat{q}_2) = (\frac{2}{3} \frac{\gamma_2}{(\gamma_1 + \gamma_2)^2}, \frac{2}{3} \frac{\gamma_1}{(\gamma_1 + \gamma_2)^2})$  and  $(\hat{b}_1, \hat{b}_2) = (\frac{1}{3}, \frac{1}{3})$ , while if  $\gamma_1 = \gamma_2$ , then  $(\hat{q}_1, \hat{q}_2) = (\frac{1}{6\gamma}, \frac{1}{6\gamma})$  and  $(\hat{b}_1, \hat{b}_2) = (\frac{1}{3}, \frac{1}{3})$ .

Let us now compute the SNE. The ODM are given by:

$$\phi_2(q_1; b_1, b_2) = -q_1 + \sqrt{\frac{b_1 + b_2}{\gamma_2} q_1} \quad (30)$$

$$\psi_2(b_1) = -b_1 + \sqrt{(b_1)^2 + b_1}. \quad (31)$$

Let  $\Phi_2(q_1, q_2; b_1, b_2) := q_2 + q_1 - \sqrt{\frac{b_1 + b_2}{\gamma_2} q_1}$  and  $\Psi_2(q_1) := b_2 + b_1 - \sqrt{(b_1)^2 + b_1}$ . The Jacobian is given by:

$$\mathcal{J}_{\mathbf{r}_{(q^F; b^F)}} = \begin{bmatrix} 1 & -\frac{1}{2}\sqrt{\frac{q^1}{\gamma(b_1 + b_2)}} \\ 0 & 1 \end{bmatrix}. \quad (32)$$

We have  $|\mathcal{J}_{\mathbf{r}_{(q^F; b^F)}}| = 1$ . Then, the best responses are given by:

$$\sigma(q_1; b_1) = -q_1 + \sqrt{\frac{1}{\gamma_2} \sqrt{(b_1)^2 + b_1} q_1} \quad (33)$$

$$\varphi(b_1) = -b_1 + \sqrt{(b_1)^2 + b_1}. \quad (34)$$

Then, some computations yield the SNE supplies:

$$(\tilde{q}_1, \tilde{q}_2) = \left( \frac{\sqrt{2\sqrt{97} + 62}}{48} \frac{\gamma_2}{(\gamma_1)^2}, \frac{\sqrt{2\sqrt{97} + 62}}{24} \frac{1}{\gamma_1} \left( 1 - \frac{1}{2} \frac{\gamma_2}{\gamma_1} \right) \right) \quad (35)$$

$$(\tilde{b}_1, \tilde{b}_2) = \left( \frac{\sqrt{97} - 5}{12}, \frac{5 - \sqrt{97} + \sqrt{2\sqrt{97} + 62}}{12} \right). \quad (36)$$

It is worth noticing that there is a SNE with trade even if some traders have linear preferences. But there is at least one trader (here the leader of type 2 and the follower of type 2) who has never zero demand for her "own" commodity: the indifference curves of the traders who initially own commodity  $Y$  do not intersect the axis. In addition, it can be checked that if the leaders had linear utility functions, while followers had Cobb-Douglas utility functions, then there would be a SNE with trade. But if traders of the same type had preferences represented by the same linear utility function, then the SNE would be autarkic (see Cordella and Gabszewicz, 1998).

### 4.3. No SNE

Let  $|T_1| = |T_2| = 2$ . The utility functions are given by:

$$u_i(x_i, y_i) = \min \left( x_i, \sqrt{(y_i)^2 + 1} \right), \quad i = 1, 2 \quad (37)$$

$$u_j(x_j, y_j) = \min(\sqrt{(x_j)^2 + 1}, y_j), \quad j = 1, 2. \quad (38)$$

The CE supplies are given by  $(q_1^*, q_2^*) = (0, 0)$  and  $(b_1^*, b_2^*) = (0, 0)$  (so autarky is Pareto optimal). In addition, the CNE supplies are given by  $(\hat{q}_1, \hat{q}_2) = (0, 0)$  and  $(\hat{b}_1, \hat{b}_2) = (0, 0)$ .

Let us now compute the SNE. The ODM are given by:

$$\phi_2(q_1; b_1, b_2) = -q_1 + (b_1 + b_2) \quad (39)$$

$$\psi_2(q_1, q_2; b_1) = -b_1 + (q_1 + q_2). \quad (40)$$

Let  $\Phi_2(q_1, q_2; b_1, b_2) := q_2 + q_1 - (b_1 + b_2)$  and  $\Psi_2(q_1, q_2; b_1, b_2) := b_2 + b_1 - (q_1 + q_2)$ . The Jacobian corresponding to (9), namely  $\mathcal{J}_{\mathbf{r}_{(q^F; b^F)}} = \left[ \frac{\partial(\Phi_2; \Psi_2)}{\partial(q^F; b^F)} \right]$ , is given by:

$$\mathcal{J}_{\mathbf{r}_{(q^F; b^F)}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \quad (41)$$

As  $\left| \mathcal{J}_{\mathbf{r}_{(q^F; b^F)}} \right| = 0$  there are no best responses. Whilst optimal decision mappings exist, best responses are not defined. Therefore, the hierarchical game cannot be 'solved', i.e., there is no SNE. The reason why there are no best responses stems from the fact that the utility functions are not (continuously) differentiable. While

the optimal decision mappings vary continuously with all the strategies, the system (39)-(40) has no solution, and then no linear approximation of this system makes possible the determination of best response mappings. Nevertheless, as the next example illustrates, the (continuous) differentiability of *all* the utility functions is not necessary.

#### 4.4. SNE without A2

We assume  $|T_1| = |T_2| = 2$ . The utility functions are given by:

$$u_k(x_k, y_k) = \min \{x_k, y_k\}, \quad k = i, j, \quad i, j = 1 \quad (42)$$

$$u_k(x_k, y_k) = x_k + y_k, \quad k = i, j, \quad i, j = 2. \quad (43)$$

The CE supplies are given by  $(q_1^*, q_2^*) = (\frac{1}{2}, \frac{1}{2})$  and  $(b_1^*, b_2^*) = (\frac{1}{2}, \frac{1}{2})$ . In addition, the CNE supplies are given by  $(\hat{q}_1, \hat{q}_2) = (0, 0)$  and  $(\hat{b}_1, \hat{b}_2) = (0, 0)$ .

Let us now compute the SNE. The ODM are given by:

$$\phi_2(q_1; b_1, b_2) = -q_1 + \sqrt{(b_1 + b_2)q_1} \quad (44)$$

$$\psi_2(q_1, q_2; b_1) = -b_1 + \sqrt{b_1(q_1 + q_2)}. \quad (45)$$

Let  $\Phi_2(q_1, q_2; b_1, b_2) := q_2 + q_1 - \sqrt{(b_1 + b_2)q_1}$  and  $\Psi_2(q_1, q_2; b_1, b_2) := b_2 + b_1 - \sqrt{b_1(q_1 + q_2)}$ . The Jacobian is given by:

$$\mathcal{J}_{\mathbf{r}_{(q^F, b^F)}} = \begin{bmatrix} 1 & -\frac{1}{2}\sqrt{\frac{q_1}{b_1+b_2}} \\ -\frac{1}{2}\sqrt{\frac{b_1}{q_1+q_2}} & 1 \end{bmatrix}. \quad (46)$$

We get  $|\mathcal{J}_{\mathbf{r}_{(q^F, b^F)}}| = 1 - \frac{1}{4}\sqrt{\frac{b_1q_1}{(b_1+b_2)(q_1+q_2)}} \neq 0$ . The best responses are:

$$\sigma(q_1; b_1) = \frac{\sqrt{(4b_1 - 3q_1)q_1} - q_1}{2} \quad (47)$$

$$\varphi(q_1; b_1) = \frac{\sqrt{(4q_1 - 3b_1)b_1} - b_1}{2}. \quad (48)$$

Then, some computations lead to the unique SNE strategy profile:

$$(\tilde{q}_1, \tilde{q}_2) = \left(\frac{1}{2}, 0\right) \quad (49)$$

$$(\tilde{b}_1, \tilde{b}_2) = \left(\frac{1}{2}, 0\right). \quad (50)$$

Therefore, a SNE with trade may exist even if Assumptions (2a), (2c) and (2d) are not satisfied for all traders, so Assumption 2 must not necessary hold for all traders. In addition, it is worth noticing that the symmetric CNE is autarkic, whilst the SNE is non-autarkic. This example illustrates a feature which is specific to a two-stage game setting: it allows trade in the subgame between leaders whilst there is no trade in the subgame between followers, and thereby in the entire game between leaders and followers. Such a situation could be called a "partial trade equilibrium" or a "partial autarkic equilibrium".

## 5. CONCLUSION

This paper constitutes an attempt to extend the simultaneous move bilateral market game with two commodities and corner endowments. To this end, we consider a two-stage market game of complete and perfect information with a finite number of traders. As it provides a richer set of strategic interactions, the existence of a noncooperative equilibrium is more difficult to handle with. One salient feature of the model stems from the fact that the existence of a Nash equilibrium for the entire game also depends on whether optimal decision mappings of followers are consistent. Lemma 1 provides a criterion to show the existence of best responses, and thereby to show the existence of a SNE with trade.

The main conclusions of the paper may be stated as follows. First, the failure of existence of a SNE stems from the fact that the system of equations which defines implicitly the best responses has not fixed point. Under Assumptions 1 and 2 such a system of equations is always consistent. Second, Assumptions 2 constitutes a set of sufficient conditions to guarantee the existence of a SNE with trade.

Further theoretical issues could be explored. First, the existence of a SNE should be extended to the case of best response correspondences. Second, the endogeneization of the order of moves should be undertaken. Third, further generalizations could consider a game with more than two stages, and/or an exchange economy with a number of commodities larger than two.

## 6. APPENDIX

Within this Appendix, we prove some intermediate results needed to prove the Theorem. Appendices A to E deal with the optimal behavior of traders. Appendix A (resp. D) is devoted to the characterization of the optimal decision mappings of followers (resp. leaders). Appendix B concerns the monotonicity properties of such mappings. Appendix C shows the existence of best responses. Appendix E show the reactions of followers are bounded. Appendix F shows the price is bounded in an  $\epsilon$ -SNE. To save notations, let  $p^\epsilon \equiv \left(\frac{p_X}{p_Y}\right)^\epsilon$ .

### 6.1. Appendix A: Proof of Proposition 1

Consider a follower of type 1 (the same holds for a follower of type 2). First, we show the mapping  $\phi_{i,\epsilon}(\mathbf{q}_{-i,\epsilon}; \mathbf{b}_\epsilon; \epsilon)$  is *well defined*. The program of follower  $i$  consists in maximizing  $\pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}; \mathbf{b}_\epsilon; \epsilon)$ , a continuous function, with respect to  $q_{i,\epsilon}$  subject to  $q_{i,\epsilon} \in [0, \alpha_i]$ , a nonempty and compact convex set. Then, from the Weierstrass Theorem, the set  $\arg \max\{\pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}; \mathbf{b}_\epsilon; \epsilon) : q_{i,\epsilon} \in \mathcal{S}_i\}$  is nonempty, so there exists  $\phi_{i,\epsilon} : \prod_{-i \in T_1} \mathcal{S}_{-i} \times \prod_{j \in T_2} \mathcal{S}_j \rightarrow \mathcal{S}_i$ , with  $q_{i,\epsilon} = \phi_{i,\epsilon}(\mathbf{q}_{-i,\epsilon}; \mathbf{b}_\epsilon; \epsilon)$ ,  $i = M_1 + 1, \dots, N_1$ ,  $\epsilon > 0$ . To characterize his optimal behavior, for all  $\epsilon > 0$ , let  $\mathcal{L}_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}; \mathbf{b}_\epsilon; \lambda_{i,\epsilon}, \mu_{i,\epsilon}; \epsilon) := \pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}; \mathbf{b}_\epsilon; \epsilon) + \lambda_{i,\epsilon} q_{i,\epsilon} + \mu_{i,\epsilon} (\alpha_i - q_{i,\epsilon})$  be the Lagrangian, where  $\lambda_{i,\epsilon} \geq 0$  and  $\mu_{i,\epsilon} \geq 0$  are the Kuhn-Tucker multipliers. Then, for all  $\epsilon > 0$ , and given  $(\mathbf{q}_{-i,\epsilon}; \mathbf{b}_\epsilon) \in \prod_{-i \in T_1} \mathcal{S}_{-i} \times \prod_{j \in T_2} \mathcal{S}_j$ , follower  $i$ 's optimal decision, i.e.,  $\phi_{i,\epsilon}(\mathbf{q}_{-i,\epsilon}^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon)$ , is the solution to:

$$\max \mathcal{L}_i^\epsilon(\cdot; \epsilon) = u_i \left( \alpha_i - q_{i,\epsilon}, \frac{\mathbf{B}_\epsilon + \epsilon}{q_{i,\epsilon} + \mathbf{Q}_{-i,\epsilon} + \epsilon} b_{i,\epsilon} \right) + \lambda_{i,\epsilon} q_{i,\epsilon} + \mu_{i,\epsilon} (\alpha_i - q_{i,\epsilon}). \quad (\text{A1})$$

For all  $\epsilon > 0$ , the Kuhn-Tucker conditions may be written:

$$\begin{aligned} \frac{\partial \mathcal{L}_i^\epsilon}{\partial q_i} &= -\frac{\partial u_i}{\partial x_i} + p^\epsilon \frac{\mathbf{Q}_{-i,\epsilon} + \epsilon}{q_{i,\epsilon} + \mathbf{Q}_{-i,\epsilon} + \epsilon} \frac{\partial u_i}{\partial y_i} + \lambda_{i,\epsilon} - \mu_{i,\epsilon} = 0 \\ \lambda_{i,\epsilon} &\geq 0, q_{i,\epsilon} \geq 0, \text{ with } \lambda_{i,\epsilon} q_{i,\epsilon} = 0 \\ \mu_{i,\epsilon} &\geq 0, (\alpha_i - q_{i,\epsilon}) \geq 0, \text{ with } \mu_{i,\epsilon}(\alpha_i - q_{i,\epsilon}) = 0, i = M_1 + 1, \dots, N_1. \end{aligned} \quad (\text{A2})$$

We have either  $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon) = 0$  or  $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon) > 0$ . Therefore, if  $q_{i,\epsilon} > 0$ , then  $\lambda_{i,\epsilon} = 0$ , where  $b_{i,\epsilon}$  is the solution to  $-\frac{\partial u_i}{\partial x_i} + p^\epsilon \frac{\mathbf{Q}_{-i,\epsilon} + \epsilon}{q_{i,\epsilon} + \mathbf{Q}_{-i,\epsilon} + \epsilon} \frac{\partial u_i}{\partial y_i} = \mu_{i,\epsilon}$ , which yields  $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^F, \mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon; \epsilon) > 0$ . In addition, if  $\mu_{i,\epsilon} > 0$ , then  $q_{i,\epsilon} = \phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon) = \alpha_i$ , while if  $\mu_{i,\epsilon} = 0$ , then  $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon) \in (0, \alpha_i)$ . Now, if  $\lambda_{i,\epsilon} > 0$ , then  $q_{i,\epsilon} = 0$ , which means that  $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon) = 0$  and  $\mu_{i,\epsilon} = 0$  since  $q_{i,\epsilon} < \alpha_i$ . Therefore, either we have  $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon) > 0$  when  $q_{i,\epsilon} \in (0, \alpha_i]$  or  $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon) = 0$ . Then,  $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon) \geq 0$ . In addition, from (2b), we have  $\nabla u_i(x_i, y_i) \gg \mathbf{0}$ , then we deduce  $\nabla \pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}; \mathbf{b}_\epsilon; \epsilon) \neq \mathbf{0}$ , when  $q_{i,\epsilon} \in [0, \alpha_i]$ . But then, the set  $q_{i,\epsilon} \in \{\arg \max \pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}; \mathbf{b}_\epsilon; \epsilon) : q_{i,\epsilon} \in [0, \alpha_i]\}$  is nonempty, so there exists  $\phi_i^\epsilon : \prod_{-i \in T_1} \mathcal{S}_{-i} \times \prod_{j \in T_2} \mathcal{S}_j \rightarrow \mathcal{S}_i$  such that  $q_{i,\epsilon} = \phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon)$ ,  $i = M_1 + 1, \dots, N_1$ .

Second, we show that  $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon)$  is *point-valued*. Consider an interior solution to (A2), i.e.,  $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon) \in (0, \alpha_i)$ , in which case we get  $\lambda_{i,\epsilon} = \mu_{i,\epsilon} = 0$ . Differentiating  $\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}} = -\frac{\partial u_i}{\partial x_i} + p^\epsilon \frac{\mathbf{Q}_{-i,\epsilon} + \epsilon}{q_{i,\epsilon} + \mathbf{Q}_{-i,\epsilon} + \epsilon} \frac{\partial u_i}{\partial y_i}$  with respect to  $q_{i,\epsilon}$  leads to  $\frac{\partial^2 \pi_i^\epsilon}{(\partial q_{i,\epsilon})^2} = \frac{\partial^2 u_i}{(\partial x_i)^2} - 2p^\epsilon \frac{\mathbf{Q}_{-i,\epsilon} + \epsilon}{\mathbf{Q}_\epsilon + \epsilon} \frac{\partial^2 u_i}{\partial x_i \partial y_i} + \left[ p^\epsilon \frac{\mathbf{Q}_{-i,\epsilon} + \epsilon}{\mathbf{Q}_\epsilon + \epsilon} \right]^2 \frac{\partial^2 u_i}{(\partial y_i)^2} - 2p^\epsilon \frac{\mathbf{Q}_{-i,\epsilon} + \epsilon}{(\mathbf{Q}_\epsilon + \epsilon)^2} \frac{\partial u_i}{\partial y_i}$ . As  $p^\epsilon \frac{\mathbf{Q}_{-i,\epsilon} + \epsilon}{\mathbf{Q}_\epsilon + \epsilon} = \frac{\partial u_i / \partial x_i}{\partial u_i / \partial y_i}$ , the first three terms on the right hand side of this equation are equal to the negative of the determinant of the bordered Hessian matrix of  $u_i$ , which is positive from (2c), and as the last term is negative, then  $\frac{\partial^2 \pi_i^\epsilon}{(\partial q_{i,\epsilon})^2} < 0$ .

Third, we show  $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon)$  is *continuously differentiable*. From Berge Maximum Theorem (1959),  $q_{i,\epsilon} = \phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon)$ ,  $i = M_1 + 1, \dots, N_1$ , is  $\mathcal{C}^1$ .

## 6.2. Appendix B: Proof of Proposition 2

First, consider the case:  $-\mathbf{I} \ll \mathcal{J}_{\phi_{\mathbf{q}_\epsilon^F}}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \leq \mathbf{I}$ , where  $\mathbf{I}$  is the  $(N_1 - M_1, N_1 - M_1)$  unit matrix.

The matrix  $\mathcal{J}_{\phi_{\mathbf{q}_\epsilon^F}}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon)$  has unit terms on the main diagonal. Next, consider the partial effects of a change in the strategy of any other follower (of any type), i.e.,  $q_{-i,\epsilon}$ ,  $-i \neq i$ ,  $-i = M_1 + 1, \dots, N_1$ , and  $b_{j,\epsilon}$ ,  $j = M_2 + 1, \dots, N_2$ . To this end, let  $\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}}(\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon), \phi_{-i}^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon); \mathbf{b}_\epsilon; \epsilon) \equiv 0$ , where, for each  $i = M_1 + 1, \dots, N_1$ , and  $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon)$  is the solution to (A2). Implicit partial differentiation of the identity  $\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}}(\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon), \phi_{-i}^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{i,\epsilon}^F; \mathbf{b}_\epsilon; \epsilon); \mathbf{b}_\epsilon; \epsilon) \equiv 0$

with respect to  $q_{-i,\epsilon}$ , with  $-i \neq i$ , leads to  $\frac{\partial \phi_i^\epsilon(\cdot)}{\partial q_{-i,\epsilon}} = -\frac{\frac{\partial^2 \pi_i^\epsilon}{\partial q_{i,\epsilon} \partial q_{-i,\epsilon}}}{\frac{\partial^2 \pi_i^\epsilon}{(\partial q_{i,\epsilon})^2}}$ , so we deduce:

$$\frac{\partial \phi_i^\epsilon(\cdot)}{\partial q_{-i,\epsilon}} = \frac{p^\epsilon \left( \frac{q_{i,\epsilon}}{\mathbf{Q}_\epsilon + \epsilon} \frac{\partial^2 u_i}{\partial x_i \partial y_i} + \frac{q_{i,\epsilon} - (\mathbf{Q}_{-i,\epsilon} + \epsilon)}{(\mathbf{Q}_\epsilon + \epsilon)^2} \frac{\partial u_i}{\partial y_i} - p^\epsilon \frac{(\mathbf{Q}_{-i,\epsilon} + \epsilon) q_{i,\epsilon}}{(\mathbf{Q}_\epsilon + \epsilon)^2} \frac{\partial^2 u_i}{(\partial y_i)^2} \right)}{\frac{\partial^2 u_i}{(\partial x_i)^2} - p^\epsilon \left( 2 \frac{\mathbf{Q}_{-i,\epsilon} + \epsilon}{\mathbf{Q}_\epsilon + \epsilon} \frac{\partial^2 u_i}{\partial x_i \partial y_i} - p^\epsilon \left( \frac{\mathbf{Q}_{-i,\epsilon} + \epsilon}{\mathbf{Q}_\epsilon + \epsilon} \right)^2 \frac{\partial^2 u_i}{(\partial y_i)^2} + 2 \frac{\mathbf{Q}_{-i,\epsilon} + \epsilon}{(\mathbf{Q}_\epsilon + \epsilon)^2} \frac{\partial u_i}{\partial y_i} \right)}. \quad (\text{B1})$$

As  $\left| \frac{(\mathbf{Q}_{-i,\epsilon+\epsilon} q_{i,\epsilon})}{(\mathbf{Q}_{\epsilon+\epsilon})^2} \right| < \left| \left( \frac{\mathbf{Q}_{-i,\epsilon+\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}} \right)^2 \right|$ ,  $2 \frac{\mathbf{Q}_{-i,\epsilon+\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}} > \frac{q_{i,\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}}$ , and  $\frac{q_{i,\epsilon} - (\mathbf{Q}_{-i,\epsilon+\epsilon})}{(\mathbf{Q}_{\epsilon+\epsilon})^2} < 2 \frac{\mathbf{Q}_{-i,\epsilon+\epsilon}}{(\mathbf{Q}_{\epsilon+\epsilon})^2}$ , then we deduce  $\left| \frac{\partial \phi_i^\epsilon(\cdot)}{\partial q_{-i,\epsilon}} \right| < 1$ .

Second, consider the case:  $-\mathbf{I} \ll \mathcal{J}_{\phi_{\mathbf{b}_\epsilon^F}}^\epsilon(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \ll \mathbf{I}$ . Implicit partial differentiation with respect to  $b_{j,\epsilon}$ ,  $j = M_2 + 1, \dots, N_2$ , leads to:

$$\frac{\partial \phi_i^\epsilon(\cdot)}{\partial b_{j,\epsilon}} = \frac{\frac{q_{i,\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}} \frac{\partial^2 u_i}{(\partial x_i)^2} - \frac{\mathbf{Q}_{-i,\epsilon+\epsilon}}{(\mathbf{Q}_{\epsilon+\epsilon})^2} \frac{\partial u_i}{\partial y_i} - p^\epsilon \frac{(\mathbf{Q}_{-i,\epsilon+\epsilon}) q_{i,\epsilon}}{(\mathbf{Q}_{\epsilon+\epsilon})^2} \frac{\partial^2 u_i}{(\partial y_i)^2}}{\frac{\partial^2 u_i}{(\partial x_i)^2} - p^\epsilon \left( 2 \frac{\mathbf{Q}_{-i,\epsilon+\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}} \frac{\partial^2 u_i}{\partial x_i \partial y_i} - p^\epsilon \left( \frac{\mathbf{Q}_{-i,\epsilon+\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}} \right)^2 \frac{\partial^2 u_i}{(\partial y_i)^2} + 2 \frac{\mathbf{Q}_{-i,\epsilon+\epsilon}}{(\mathbf{Q}_{\epsilon+\epsilon})^2} \frac{\partial u_i}{\partial y_i} \right)}. \quad (\text{B2})$$

Then, a similar reasoning leads to the conclusion  $\left| \frac{\partial \phi_i^\epsilon(\cdot)}{\partial b_{j,\epsilon}} \right| < 1$ , for all  $i \in \{M_1 + 1, \dots, N_1\}$ , and all  $j \in \{M_2 + 1, \dots, N_2\}$ .

The cases  $-\mathbf{I} \ll \mathcal{J}_{\psi_{\mathbf{b}_\epsilon^F}}^\epsilon(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \leq \mathbf{I}$  and  $-\mathbf{I} \ll \mathcal{J}_{\psi_{\mathbf{q}_\epsilon^F}}^\epsilon(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \ll \mathbf{I}$  may be handled in the same way.

### 6.3. Appendix C: Proof of Lemma 1

Consider the set of optimal decision mappings specified in Definition 1. We consider a set of functions which will be useful to build the system of equations that will implicitly define the best responses for the perturbed game. Define the function  $\Phi_i^\epsilon: \prod_{i \in T_1} \mathcal{S}_i \times \prod_{j \in T_2} \mathcal{S}_j \times \mathbb{R}_{++} \rightarrow \mathcal{S}_i$ , with  $\Phi_i^\epsilon(\mathbf{q}_\epsilon; \mathbf{b}_\epsilon; \epsilon) := b_{i,\epsilon} - \phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon)$ ,  $i = M_1 + 1, \dots, N_1$ , and the function  $\Psi_j^\epsilon: \prod_{i \in T_1} \mathcal{S}_i \times \prod_{j \in T_2} \mathcal{S}_j \times \mathbb{R}_{++} \rightarrow \mathcal{S}_j$ , with  $\Psi_j^\epsilon(\mathbf{q}_\epsilon; \mathbf{b}_\epsilon; \epsilon) := q_{j,\epsilon} - \psi_j^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^L, \mathbf{b}_{-j,\epsilon}^F; \epsilon)$ ,  $j = M_2 + 1, \dots, N_2$ . For all  $\epsilon > 0$ , consider the following system of equations of the perturbed game:

$$\begin{aligned} \Phi_i^\epsilon(\mathbf{q}_\epsilon; \mathbf{b}_\epsilon; \epsilon) &= 0, \quad i = M_1 + 1, \dots, N_1, \\ \Psi_j^\epsilon(\mathbf{q}_\epsilon; \mathbf{b}_\epsilon; \epsilon) &= 0, \quad j = M_2 + 1, \dots, N_2. \end{aligned} \quad (\text{C1})$$

For all  $\epsilon > 0$ , let the  $(N_1 - M_1) + (N_2 - M_2)$ -dimensional vector function  $\Upsilon^\epsilon$  be defined as  $\Upsilon^\epsilon: \prod_{i \in T_1} \mathcal{S}_i \times \prod_{j \in T_2} \mathcal{S}_j \times \mathbb{R}_{++} \rightarrow \prod_{i=M_1+1}^{N_1} \mathcal{S}_i \times \prod_{j=M_2+1}^{N_2} \mathcal{S}_j$ , with  $\Upsilon^\epsilon = \Upsilon^\epsilon(\mathbf{q}_\epsilon; \mathbf{b}_\epsilon; \epsilon) = (\Phi_{M_1+1}^\epsilon(\cdot; \epsilon), \dots, \Phi_{N_1}^\epsilon(\cdot; \epsilon); \Psi_{M_2+1}^\epsilon(\cdot; \epsilon), \dots, \Psi_{N_2}^\epsilon(\cdot; \epsilon))$ . Thus, (C1) may be written as a  $(N_1 - M_1) + (N_2 - M_2)$ -dimensional vector equation  $\Upsilon^\epsilon(\mathbf{q}_\epsilon; \mathbf{b}_\epsilon; \epsilon) = \mathbf{0}$ . Since we focus on inner solutions, consider the restriction of  $\prod_{i=1}^{N_1} \mathcal{S}_i \times \prod_{j=1}^{N_2} \mathcal{S}_j \times \mathbb{R}_{++}$  to the open set  $\prod_{i=1}^{N_1} \bar{\mathcal{S}}_i \times \prod_{j=1}^{N_2} \bar{\mathcal{S}}_j \times \mathbb{R}_{++}$ , with  $\bar{\mathcal{S}}_i \subset \mathcal{S}_i$ ,  $i = 1, \dots, N_1$ , and  $\bar{\mathcal{S}}_j \subset \mathcal{S}_j$ ,  $j = 1, \dots, N_2$ . The vector function  $\Upsilon^\epsilon(\mathbf{q}_\epsilon; \mathbf{b}_\epsilon)$  is  $C^1$  on the open set  $\prod_{i=1}^{N_1} \bar{\mathcal{S}}_i \times \prod_{j=1}^{N_2} \bar{\mathcal{S}}_j \times \mathbb{R}_{++}$  as each  $\Phi_i^\epsilon$  and each  $\Psi_j^\epsilon$  are  $C^1$  functions of  $(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L)$  on the open set  $\prod_{i=1}^{N_1} \bar{\mathcal{S}}_i \times \prod_{j=1}^{N_2} \bar{\mathcal{S}}_j \times \mathbb{R}_{++}$ . Let  $(\bar{\mathbf{q}}_\epsilon^L, \bar{\mathbf{q}}_\epsilon^F; \bar{\mathbf{b}}_\epsilon^L, \bar{\mathbf{b}}_\epsilon^F)$  be an interior point of  $\prod_{i \in T_1} \bar{\mathcal{S}}_i \times \prod_{j \in T_2} \bar{\mathcal{S}}_j$ , where  $(\bar{\mathbf{q}}_\epsilon^L; \bar{\mathbf{b}}_\epsilon^L)$  corresponds to a parameter configuration. Therefore, the following identity, which defines implicitly (at least locally) best responses, holds in an open neighborhood of  $(\bar{\mathbf{q}}_\epsilon^L, \bar{\mathbf{q}}_\epsilon^F; \bar{\mathbf{b}}_\epsilon^L, \bar{\mathbf{b}}_\epsilon^F)$ :

$$\Upsilon^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F(\mathbf{b}_\epsilon^L; \mathbf{q}_\epsilon^L; \epsilon); \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(\mathbf{b}_\epsilon^L; \mathbf{q}_\epsilon^L; \epsilon)) \equiv \mathbf{0}. \quad (\text{C2})$$

Implicit partial differentiation with respect to each component of  $(\bar{\mathbf{q}}_\epsilon^L; \bar{\mathbf{b}}_\epsilon^L)$  leads to the equation:

$$\mathcal{J}_{\Upsilon^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \cdot \mathcal{A}^\epsilon + \mathcal{B}^\epsilon = \mathbf{0}, \text{ for each } \epsilon > 0, \quad (\text{C3})$$

where:

$$\mathcal{J}_{\Upsilon^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) = \begin{bmatrix} 1 & \cdots & \frac{\partial \Phi_{M_1+1}^\epsilon}{\partial q_{N_1, \epsilon}} & \frac{\partial \Phi_{M_1+1}^\epsilon}{\partial b_{M_2+1, \epsilon}} & \cdots & \frac{\partial \Phi_{M_1+1}^\epsilon}{\partial b_{N_2, \epsilon}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \Phi_{N_1}^\epsilon}{\partial q_{M_1+1, \epsilon}} & \cdots & 1 & \frac{\partial \Phi_{N_1}^\epsilon}{\partial b_{M_2+1, \epsilon}} & \cdots & \frac{\partial \Phi_{N_1}^\epsilon}{\partial b_{N_2, \epsilon}} \\ \frac{\partial \Psi_{M_2+1}^\epsilon}{\partial q_{M_1+1, \epsilon}} & \cdots & \frac{\partial \Psi_{M_2+1}^\epsilon}{\partial q_{N_1, \epsilon}} & 1 & \cdots & \frac{\partial \Psi_{M_2+1}^\epsilon}{\partial b_{N_2, \epsilon}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \Psi_{N_2}^\epsilon}{\partial q_{M_1+1, \epsilon}} & \cdots & \frac{\partial \Psi_{N_2}^\epsilon}{\partial q_{N_1, \epsilon}} & \frac{\partial \Psi_{N_2}^\epsilon}{\partial b_{M_2+1, \epsilon}} & \cdots & 1 \end{bmatrix}$$

is a  $((N_1 - M_1) + (N_2 - M_2), (N_1 - M_1) + (N_2 - M_2))$  matrix, while the two matrices:

$$\mathcal{A}^\epsilon = \begin{bmatrix} \frac{\partial q_{M_1+1, \epsilon}}{\partial q_{1, \epsilon}} & \cdots & \frac{\partial q_{M_1+1, \epsilon}}{\partial q_{M_1, \epsilon}} & \frac{\partial q_{M_1+1, \epsilon}}{\partial b_{1, \epsilon}} & \cdots & \frac{\partial q_{M_1+1, \epsilon}}{\partial b_{M_2, \epsilon}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial q_{N_1, \epsilon}}{\partial q_{1, \epsilon}} & \cdots & \frac{\partial q_{N_1, \epsilon}}{\partial q_{M_1, \epsilon}} & \frac{\partial q_{N_1, \epsilon}}{\partial b_{1, \epsilon}} & \cdots & \frac{\partial q_{N_1, \epsilon}}{\partial b_{M_2, \epsilon}} \\ \frac{\partial b_{M_2+1, \epsilon}}{\partial q_{1, \epsilon}} & \cdots & \frac{\partial b_{M_2+1, \epsilon}}{\partial q_{M_1, \epsilon}} & \frac{\partial b_{M_2+1, \epsilon}}{\partial b_{1, \epsilon}} & \cdots & \frac{\partial b_{M_2+1, \epsilon}}{\partial b_{M_2, \epsilon}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial b_{N_2, \epsilon}}{\partial q_{1, \epsilon}} & \cdots & \frac{\partial b_{N_2, \epsilon}}{\partial q_{M_1, \epsilon}} & \frac{\partial b_{N_2, \epsilon}}{\partial b_{1, \epsilon}} & \cdots & \frac{\partial b_{N_2, \epsilon}}{\partial b_{M_2, \epsilon}} \end{bmatrix}$$

and

$$\mathcal{B}^\epsilon = \begin{bmatrix} \frac{\partial \Phi_{M_1+1}^\epsilon}{\partial q_{1, \epsilon}} & \cdots & \frac{\partial \Phi_{M_1+1}^\epsilon}{\partial q_{M_1, \epsilon}} & \frac{\partial \Phi_{M_1+1}^\epsilon}{\partial b_{1, \epsilon}} & \cdots & \frac{\partial \Phi_{M_1+1}^\epsilon}{\partial b_{M_2, \epsilon}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \Phi_{N_1}^\epsilon}{\partial q_{1, \epsilon}} & \cdots & \frac{\partial \Phi_{N_1}^\epsilon}{\partial q_{M_1, \epsilon}} & \frac{\partial \Phi_{N_1}^\epsilon}{\partial b_{1, \epsilon}} & \cdots & \frac{\partial \Phi_{N_1}^\epsilon}{\partial b_{M_2, \epsilon}} \\ \frac{\partial \Psi_{M_2+1}^\epsilon}{\partial q_{1, \epsilon}} & \cdots & \frac{\partial \Psi_{M_2+1}^\epsilon}{\partial q_{M_1, \epsilon}} & \frac{\partial \Psi_{M_2+1}^\epsilon}{\partial b_{1, \epsilon}} & \cdots & \frac{\partial \Psi_{M_2+1}^\epsilon}{\partial b_{M_2, \epsilon}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \Psi_{N_2}^\epsilon}{\partial q_{1, \epsilon}} & \cdots & \frac{\partial \Psi_{N_2}^\epsilon}{\partial q_{M_1, \epsilon}} & \frac{\partial \Psi_{N_2}^\epsilon}{\partial b_{1, \epsilon}} & \cdots & \frac{\partial \Psi_{N_2}^\epsilon}{\partial b_{M_2, \epsilon}} \end{bmatrix}.$$

are of dimension  $((N_1 - M_1) + (N_2 - M_2), M_1 + M_2)$ .

The square matrix  $\mathcal{J}_{\Upsilon^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon)$  has unit terms on the main diagonal and off-diagonal terms bounded below by  $-1$  and above by  $1$  as from Proposition 2, we have that  $-\mathbf{I} \ll \mathcal{J}_{\Upsilon^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \ll \mathbf{I}$  and  $-\mathbf{I} \ll \mathcal{J}_{\Psi^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \ll \mathbf{I}$ . Then,  $\frac{\partial \Phi_i^\epsilon(\cdot)}{\partial q_{-i, \epsilon}} = -\frac{\partial \phi_i^\epsilon(\cdot)}{\partial q_{-i, \epsilon}} \in (-1, 1)$ , with  $-i \neq i$ , and  $\left| \frac{\partial \Phi_i^\epsilon(\cdot)}{\partial b_{j, \epsilon}} \right| = \left| -\frac{\partial \phi_i^\epsilon(\cdot)}{\partial b_{j, \epsilon}} \right| < 1$ ,  $i = M_1 + 1, \dots, N_1$ ; and  $\frac{\partial \Psi_j^\epsilon(\cdot)}{\partial b_{-j, \epsilon}} = -\frac{\partial \psi_j^\epsilon(\cdot)}{\partial b_{-j, \epsilon}} \in (-1, 1)$ , with  $-j \neq j$ , and  $\left| \frac{\partial \Psi_j^\epsilon(\cdot)}{\partial q_{i, \epsilon}} \right| = \left| -\frac{\partial \psi_j^\epsilon(\cdot)}{\partial q_{i, \epsilon}} \right| < 1$ ,  $j = M_2 + 1, \dots, N_2$ . The signs of the off diagonal terms depend on whether strategies of followers within each side and/or between both sides are complements or substitutes. But, in any case, for all  $\epsilon > 0$ , the rows of the matrix  $\mathcal{J}_{\Upsilon^\epsilon}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon)$

are linearly independent, so the matrix  $\mathcal{J}_{\mathbf{r}^\epsilon_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon)$  is of full rank, and then invertible. Then, for all  $\epsilon > 0$ ,  $\left| \mathcal{J}_{\mathbf{r}^\epsilon_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \right| \neq 0$ . Then, by the Implicit Function Theorem, there exist open sets  $\mathcal{U} \times \mathcal{V}$  in  $\prod_{i=1}^{N_1} \bar{\mathcal{S}}_i \times \prod_{j=1}^{N_2} \bar{\mathcal{S}}_j$  and  $(\mathcal{U} \times \mathcal{V})_{(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L)}$  in  $\prod_{i=1}^{M_1} \bar{\mathcal{S}}_i \times \prod_{j=1}^{M_2} \bar{\mathcal{S}}_j$ , with  $(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \subseteq \mathcal{U} \times \mathcal{V}$  and  $(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L) \subseteq (\mathcal{U} \times \mathcal{V})_{(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L)}$  such that for each  $(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L)$  in  $(\mathcal{U} \times \mathcal{V})_{(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L)}$ , there exists (at least locally) some unique  $((N_1 - M_1) + (N_2 - M_2))$  dimensional vector function  $(\mathbf{q}_\epsilon^F(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon); \mathbf{b}_\epsilon^F(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon))$  in some neighborhood of  $(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L)$  such that  $(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon); \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)) \in \mathcal{U} \times \mathcal{V}$  and  $\mathbf{r}^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon); \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)) \equiv \mathbf{0}$ . Indeed, the unique solution  $(\sigma^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon); \varphi^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon))$  to  $(\mathbf{q}_\epsilon^F(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon); \mathbf{b}_\epsilon^F(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)) = \mathbf{r}^{-1}(\mathbf{0})$  is defined by  $\sigma^\epsilon : \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{j=1}^{M_2} \mathcal{S}_j \times \mathbb{R}_{++} \supset (\mathcal{U} \times \mathcal{V})_{(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L)} \rightarrow \prod_{i=M_1+1}^{N_1} \mathcal{S}_i$ , with  $\mathbf{q}_\epsilon^F = \sigma^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)$ , and by  $\varphi^\epsilon : \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{j=1}^{M_2} \mathcal{S}_j \times \mathbb{R}_{++} \supset (\mathcal{U} \times \mathcal{V})_{(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L)} \rightarrow \prod_{j=M_2+1}^{N_2} \mathcal{S}_j$ , with  $\mathbf{b}_\epsilon^F = \varphi^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)$ . For all  $\epsilon > 0$ , each component function  $\sigma_i^\epsilon(\cdot)$  is defined as  $\sigma_i^\epsilon : \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{j=1}^{M_2} \mathcal{S}_j \times \mathbb{R}_{++} \supset (\mathcal{U} \times \mathcal{V})_{(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L)} \rightarrow \mathcal{S}_i$ , with  $q_{i,\epsilon} = \sigma_i^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)$ ,  $i = M_1 + 1, \dots, N_1$ , and each component function  $\varphi_j^\epsilon(\cdot)$  is defined as  $\varphi_j^\epsilon : \prod_{i=1}^{M_1} \mathcal{S}_i \times \prod_{j=1}^{M_2} \mathcal{S}_j \times \mathbb{R}_{++} \supset (\mathcal{U} \times \mathcal{V})_{(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L)}$ , with  $b_{j,\epsilon} = \varphi_j^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)$   $j = M_2 + 1, \dots, N_2$ . In addition, for all  $\epsilon > 0$ ,  $\sigma_i^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon) \in C^1$ , for each  $i \in \{M_1 + 1, \dots, N_1\}$ , and  $\varphi_j^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon) \in C^1$ , for each  $j \in \{M_2 + 1, \dots, N_2\}$ .

#### 6.4. Appendix D: Proof of Proposition 3

We must show that, for  $i \in \{M_1 + 1, \dots, N_1\}$ , we have  $-1 \leq \frac{\partial \sigma_i^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)}{\partial q_{-i,\epsilon}} < 1$ ,  $-i \neq i$ ,  $-i = 1, \dots, M_1$ , and, for  $j \in \{M_2 + 1, \dots, N_2\}$ , we have  $\frac{\partial \varphi_j^\epsilon(\cdot)}{\partial q_{i,\epsilon}} \geq 0$ ,  $i = 1, \dots, M_1$ . The same analysis will hold for  $j \in \{M_2 + 1, \dots, N_2\}$ , with  $0 \leq \frac{\partial \varphi_j^\epsilon(\cdot)}{\partial b_{-j,\epsilon}} < 1$ ,  $-j \neq j =$ ,  $-j = 1, \dots, M_2$ , and for  $i \in \{M_1 + 1, \dots, N_1\}$ , with  $\frac{\partial \sigma_i^\epsilon(\cdot)}{\partial b_{j,\epsilon}} \geq 0$ ,  $j = 1, \dots, M_2$ .

First, we show  $-1 \leq \frac{\partial \sigma_i^\epsilon(\cdot)}{\partial q_{-i,\epsilon}} < 1$ ,  $i = M_1 + 1, \dots, N_1$ ,  $-i \neq i$ ,  $-i = 1, \dots, M_1$ . Consider the system given by (9). Without loss of generality, we want to determine  $\frac{\partial q_{M_1+1,\epsilon}}{\partial q_{1,\epsilon}}$ . Then, by using Cramer's rule, we deduce

$$\frac{\partial q_{M_1+1,\epsilon}}{\partial q_{1,\epsilon}} = - \frac{\left| \mathcal{J}'_{\mathbf{r}^\epsilon_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \right|}{\left| \mathcal{J}_{\mathbf{r}^\epsilon_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \right|}, \quad (\text{D1})$$

where  $\mathcal{J}'_{\mathbf{r}^\epsilon_{(\mathbf{b}_\epsilon^F, \mathbf{q}_\epsilon^F)}}(\bar{\mathbf{b}}_\epsilon; \bar{\mathbf{q}}_\epsilon)$  is the  $((N_1 - M_1) + (N_2 - M_2), (N_1 - M_1) + (N_2 - M_2))$  square matrix obtained by replacing the first column in  $\mathcal{J}_{\mathbf{r}^\epsilon_{(\mathbf{b}_\epsilon^F, \mathbf{q}_\epsilon^F)}}(\bar{\mathbf{b}}_\epsilon; \bar{\mathbf{q}}_\epsilon)$  by the first column of  $\mathcal{B}^\epsilon$ , so that:



$$\mathcal{J}'_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) = \begin{bmatrix} \frac{\partial \Phi_{M_1+1}^\epsilon}{\partial q_{1,\epsilon}} & \cdots & \frac{\partial \Phi_{M_1+1}^\epsilon}{\partial q_{N_1,\epsilon}} & \frac{\partial \Phi_{M_1+1}^\epsilon}{\partial b_{M_2+1,\epsilon}} & \cdots & \frac{\partial \Phi_{M_1+1}^\epsilon}{\partial b_{N_2,\epsilon}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \Phi_{N_1}^\epsilon}{\partial q_{1,\epsilon}} & \cdots & 1 & \frac{\partial \Phi_{N_1}^\epsilon}{\partial b_{M_2+1,\epsilon}} & \cdots & \frac{\partial \Phi_{N_1}^\epsilon}{\partial b_{N_2,\epsilon}} \\ \frac{\partial \Psi_{M_2+1}^\epsilon}{\partial q_{1,\epsilon}} & \cdots & \frac{\partial \Psi_{M_2+1}^\epsilon}{\partial q_{N_1,\epsilon}} & 1 & \cdots & \frac{\partial \Psi_{M_2+1}^\epsilon}{\partial b_{N_2,\epsilon}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \Psi_{N_2}^\epsilon}{\partial q_{1,\epsilon}} & \cdots & \frac{\partial \Psi_{N_2}^\epsilon}{\partial q_{N_1,\epsilon}} & \frac{\partial \Psi_{N_2}^\epsilon}{\partial b_{M_2+1,\epsilon}} & \cdots & 1 \end{bmatrix} \quad (\text{D2})$$

Note that (D1) is well-defined as from Lemma 1, we have  $\left| \mathcal{J}_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \right| \neq 0$ . Let  $\frac{\partial \Phi_i^\epsilon}{\partial q_{1,\epsilon}} = 0$ ,  $i = M_1 + 1, \dots, N_1$ , and  $\frac{\partial \Psi_{M_2+1}^\epsilon}{\partial q_{1,\epsilon}} = 0$ ,  $j = M_2 + 1, \dots, N_2$ , in (D2). The matrices  $\mathcal{J}'_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon)$  and  $\mathcal{J}_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon)$  have common terms: the off-diagonal terms of the matrix  $\mathcal{B}^\epsilon$  coincide with the off-diagonal terms of the matrix  $\mathcal{J}_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon)$  as  $\mathbf{Q} \equiv \sum_{i \in T_1} q_i$  and  $\mathbf{B} \equiv \sum_{j \in T_2} b_j$ . Assume that  $\frac{\partial b_{M_1+1,\epsilon}}{\partial q_{1,\epsilon}} < -1$ .

Then, we deduce  $\left| \mathcal{J}'_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \right| > \left| \mathcal{J}_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \right|$ . Expansion by cofactors of the both sides of the inequality  $\left| \mathcal{J}'_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \right| > \left| \mathcal{J}_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \right|$ , and cancellation among common terms on both sides, lead to:

$$\frac{\partial \Phi_{M_1+1}^\epsilon}{\partial q_{1,\epsilon}} \left| \bar{\mathcal{J}}'_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \right| > \left| \bar{\mathcal{J}}_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \right|, \quad (\text{D3})$$

where  $\left| \bar{\mathcal{J}}'_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \right|$  (resp.  $\left| \bar{\mathcal{J}}_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \right|$ ) stands for the principal minor of order  $((N_1 - M_1) + (N_2 - M_2) - 1, (N_1 - M_1) + (N_2 - M_2) - 1)$  of  $\mathcal{J}'_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon)$

(resp.  $\mathcal{J}_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon)$ ). But  $\left| \bar{\mathcal{J}}'_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \right| > \left| \bar{\mathcal{J}}_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon) \right|$  by construction. Then, we deduce  $\frac{\partial \Phi_{M_1+1}^\epsilon}{\partial q_{1,\epsilon}} > 1$ , which is false as we assumed  $\frac{\partial \Phi_{M_1+1}^\epsilon}{\partial q_{1,\epsilon}} < -1$ . A

contradiction. Then, we have  $-\frac{\mathcal{J}'_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon)}{\mathcal{J}_{\mathbf{r}^\epsilon}_{(\mathbf{q}_\epsilon^F, \mathbf{b}_\epsilon^F)}(\bar{\mathbf{q}}_\epsilon; \bar{\mathbf{b}}_\epsilon)} \leq 1$ , so we deduce  $\frac{\partial \Phi_{M_1+1}^\epsilon}{\partial q_{1,\epsilon}} \geq -1$ .

Next, assume  $\frac{\partial b_{M_1+1,\epsilon}}{\partial q_{1,\epsilon}} > 1$ . A similar reasoning leads to  $\frac{\partial \Phi_{M_1+1}^\epsilon}{\partial q_{1,\epsilon}} < -1$ , a contradiction. Then we deduce  $\frac{\partial \Phi_{M_1+1}^\epsilon}{\partial q_{1,\epsilon}} < 1$ . The same argument holds for all the other best responses. The same reasoning holds for any  $i = 1, \dots, M_1$ , so we have  $-\mathbf{I} \leq \frac{\partial \boldsymbol{\sigma}^\epsilon(\mathbf{b}_\epsilon^L; \mathbf{q}_\epsilon^L; \epsilon)}{\partial \mathbf{q}_\epsilon^L} \ll \mathbf{I}$ , where  $\mathbf{I}$  is the  $(N_1 - M_1, M_1)$  unit matrix.

Second, we show  $\frac{\partial \varphi_j^\epsilon(\cdot)}{\partial q_{i,\epsilon}} \geq 0$ ,  $j = M_2 + 1, \dots, N_2$ ,  $i = 1, \dots, M_1$ . Assume  $\frac{\partial \varphi_j^\epsilon(\cdot)}{\partial q_{i,\epsilon}} < 0$ ,  $j = M_2 + 1, \dots, N_2$ ,  $i = 1, \dots, M_1$ : any type two follower decreases his supply when any leader increases her supply. This means that commodities are complements for these followers, and thereby, that their utility functions are not differentiable, which contradicts Assumption (2a). Then, we must have  $\frac{\partial \varphi_j^\epsilon(\cdot)}{\partial \mathbf{q}_\epsilon^L} \geq \mathbf{0}$ , where  $\mathbf{0}$  is the  $(N_2 - M_2, M_1)$  zero matrix. Likewise,  $\frac{\partial \boldsymbol{\sigma}^\epsilon(\cdot)}{\partial \mathbf{b}_\epsilon^L} \geq \mathbf{0}$ , where  $\mathbf{0}$  is the  $(N_1 - M_1, M_2)$  zero matrix.

## 6.5. Appendix E: Proof of Proposition 4

Consider a leader of type 1 (the same holds for a leader of type 2). First, we show  $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon)$  is *well defined*. For all  $\epsilon > 0$ , leader  $i$ 's problem is to maximize her reduced payoff  $\pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L, \boldsymbol{\sigma}^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon); \mathbf{b}_\epsilon^L, \boldsymbol{\varphi}^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon))$ , a continuous function (as  $\boldsymbol{\sigma}^\epsilon(\cdot; \epsilon)$  and  $\boldsymbol{\varphi}^\epsilon(\cdot; \epsilon)$  are continuous) with respect to  $q_{i,\epsilon}$  subject to  $q_{i,\epsilon} \in [0, \alpha_i]$ , a nonempty and compact convex set. Then, there exists  $\phi_i^\epsilon : \prod_{-i=1}^{M_1} \mathcal{S}_{-i} \times \prod_{j=1}^{M_2} \mathcal{S}_j \rightarrow \mathcal{S}_i$ , with  $q_{i,\epsilon} = \phi_i^\epsilon(\mathbf{q}_{-i}^L; \mathbf{b}_\epsilon^L; \epsilon)$ ,  $i = M_1 + 1, \dots, N_1$ . Let  $\mathcal{L}_i^\epsilon(q_{i,\epsilon}; \mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \lambda_{i,\epsilon}; \mu_{i,\epsilon}; \epsilon) := \pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L, \boldsymbol{\sigma}^\epsilon(q_{i,\epsilon}, \cdot; \epsilon); \mathbf{b}_\epsilon^L, \boldsymbol{\varphi}^\epsilon(q_{i,\epsilon}, \cdot; \epsilon)) + \lambda_{i,\epsilon} q_{i,\epsilon} + \mu_{i,\epsilon}(\alpha_i - q_{i,\epsilon})$ ,  $\epsilon > 0$ , with  $\lambda_{i,\epsilon}, \mu_{i,\epsilon} \geq 0$ . Then,  $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon)$  is the solution to:

$$\max \mathcal{L}_i^\epsilon(\cdot; \epsilon) = u_i \left( \alpha_i - q_{i,\epsilon}, \frac{\mathbf{B}_\epsilon^L + \sum_j \varphi_j^\epsilon(\cdot; \epsilon) + \epsilon}{q_{i,\epsilon} + \mathbf{Q}_{-i,\epsilon}^L + \sum_{k \neq i} \sigma_k^\epsilon(\cdot; \epsilon) + \epsilon} q_{i,\epsilon} \right) + \lambda_{i,\epsilon} q_{i,\epsilon} + \mu_{i,\epsilon}(\alpha_i - q_{i,\epsilon}). \quad (\text{E1})$$

For all  $\epsilon > 0$ , the Kuhn-Tucker conditions may be written:

$$\begin{aligned} \frac{\partial \mathcal{L}_i^\epsilon}{\partial q_{i,\epsilon}} &= -\frac{\partial u_i}{\partial x_i} + p^\epsilon \chi \frac{\partial u_i}{\partial y_i} + \lambda_{i,\epsilon} - \mu_{i,\epsilon} = 0 \\ \lambda_{i,\epsilon} &\geq 0, b_{i,\epsilon} \geq 0, \text{ with } \lambda_{i,\epsilon} b_{i,\epsilon} = 0 \\ \mu_{i,\epsilon} &\geq 0, (\alpha_i - b_{i,\epsilon}) \geq 0, \text{ with } \mu_{i,\epsilon}(\alpha_i - b_{i,\epsilon}) = 0, \end{aligned} \quad (\text{E2})$$

where  $\chi \equiv 1 - (1 + \nu_{i,\epsilon}^X) \frac{q_{i,\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}} + \eta_{i,\epsilon}^X \frac{q_{i,\epsilon}}{\mathbf{B}_{\epsilon+\epsilon}}$ ,  $\nu_{i,\epsilon}^X = \frac{\partial \sum_i \sigma_i^\epsilon(\cdot)}{\partial q_{i,\epsilon}}$ , and  $\eta_{i,\epsilon}^X = \frac{\partial \sum_j \varphi_j^\epsilon(\cdot)}{\partial q_{i,\epsilon}}$ . By construction  $\nu_{i,\epsilon}^X = \nu_{i,\epsilon}^X$ , and  $\eta_{i,\epsilon}^X = \eta_{i,\epsilon}^X$ , with  $\nu_{i,\epsilon}^X \in [-1, 1]$  and  $\eta_{i,\epsilon}^X \geq 0$ . Indeed, as  $\chi \in [0, 1]$ , then  $0 \leq -(1 + \nu_{i,\epsilon}^X) \frac{q_{i,\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}} + \eta_{i,\epsilon}^X \frac{q_{i,\epsilon}}{\mathbf{B}_{\epsilon+\epsilon}} \leq 1$ , which leads to  $\frac{\eta_{i,\epsilon}^X}{2 - \nu_{i,\epsilon}^X} \leq \frac{\mathbf{B}_{\epsilon+\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}} \leq \frac{\eta_{i,\epsilon}^X}{1 + \nu_{i,\epsilon}^X}$ . Then,  $\nu_{i,\epsilon}^X \leq \frac{1}{2}$ . In addition, from Proposition 3, we get  $\nu_{i,\epsilon}^X \geq -1$  as for  $i \in \{1, \dots, M_1\}$ ,  $\frac{\partial \varphi_j^\epsilon(\cdot)}{\partial q_{i,\epsilon}} \geq -1$ ,  $i = M_1 + 1, \dots, N_1$ . Next, from (2a), we have  $\eta_{i,\epsilon}^X \geq 0$ .

If  $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon) > 0$ , then  $\lambda_{i,\epsilon} = 0$ , where  $b_{i,\epsilon}$  is the solution to the equation  $-\frac{\partial u_i}{\partial x_i} + p^\epsilon \chi \frac{\partial u_i}{\partial y_i} = \mu_{i,\epsilon}$ . If  $\mu_{i,\epsilon} > 0$ , then  $q_{i,\epsilon} = \phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon) = \alpha_i$ , while if  $\mu_{i,\epsilon} = 0$ , then  $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon) \in (0, \alpha_i)$ . Now, if  $\lambda_{i,\epsilon} > 0$ , then  $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon) = 0$  and  $\mu_{i,\epsilon} = 0$  since  $q_{i,\epsilon} < \alpha_i$ . Then, either  $\phi_i^\epsilon(\mathbf{b}_{-i,\epsilon}^L; \mathbf{q}_\epsilon^L; \epsilon) > 0$  when  $b_{i,\epsilon} \in (0, \alpha_i]$  or  $\phi_i^\epsilon(\mathbf{b}_{-i,\epsilon}^L; \mathbf{q}_\epsilon^L; \epsilon) = 0$ . Then,  $\phi_i^\epsilon(\mathbf{b}_{-i,\epsilon}^L; \mathbf{q}_\epsilon^L; \epsilon) \geq 0$ . In addition, as under (2b), we have  $\nabla u_i(x_i, y_i) \gg \mathbf{0}$ , then  $\nabla \pi_i^\epsilon(b_{i,\epsilon}, \mathbf{b}_{-i,\epsilon}^L; \mathbf{q}_\epsilon^L; \epsilon) \neq \mathbf{0}$ , when  $b_{i,\epsilon} \in [0, \alpha_i]$ . But then, the set  $\{\arg \max \pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L, \boldsymbol{\sigma}^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon); \mathbf{b}_\epsilon^L, \boldsymbol{\varphi}^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon)) : q_{i,\epsilon} \in [0, \alpha_i]\}$  is nonempty, so there exists  $q_{i,\epsilon} = \phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon)$ ,  $i = 1, \dots, M_1$ .

Next,  $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon)$  is *point-valued*. The cases  $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon) = \{0, \alpha_i\}$  are trivial. If  $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon) > 0$  and differentiating (E2) with respect to  $q_{i,\epsilon}$  yields:

$$\frac{\partial^2 \pi_i^\epsilon}{(\partial q_{i,\epsilon})^2} = \frac{\partial^2 u_i}{(\partial x_i)^2} - 2p^\epsilon \chi \frac{\partial u_i}{\partial x_i \partial y_i} + (p^\epsilon \chi)^2 \frac{\partial^2 u_i}{(\partial y_i)^2} - \kappa \frac{\partial u_i}{\partial y_i}, \quad (\text{E3})$$

where  $\kappa \equiv \frac{(1 + \nu_{i,\epsilon}^X)(\mathbf{B}_{\epsilon+\epsilon})}{(\mathbf{Q}_{\epsilon+\epsilon})^2} (2 - \frac{(1 + \nu_{i,\epsilon}^X) q_{i,\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}}) - \frac{2\eta_{i,\epsilon}^X}{\mathbf{Q}_{\epsilon+\epsilon}} (1 - \frac{(1 + \nu_{i,\epsilon}^X) q_{i,\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}})$ . The first three terms on the right hand side of (E3) are equal to the negative of the determinant of the bordered Hessian matrix of  $u_i$ , which is positive from (2c). For (E3) to be strictly negative, it is sufficient that  $\kappa > 0$ , that is,  $\frac{\mathbf{B}_{\epsilon+\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}} > \frac{2\eta_{i,\epsilon}^X}{1 + \nu_{i,\epsilon}^X} \frac{1 - \frac{(1 + \nu_{i,\epsilon}^X) q_{i,\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}}}{2 - \frac{(1 + \nu_{i,\epsilon}^X) q_{i,\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}}}$ .

But, as  $\chi \leq 1$ , then  $\frac{\mathbf{B}_{\epsilon+\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}} \leq \frac{\eta_{\epsilon}^X}{1+\nu_{\epsilon}^X}$ . Assume that  $\frac{\eta_{\epsilon}^X}{1+\nu_{\epsilon}^X} < \frac{2\eta_{\epsilon}^X}{1+\nu_{\epsilon}^X} \frac{1-\frac{(1+\nu_{\epsilon}^X)q_{i,\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}}}{2-\frac{(1+\nu_{\epsilon}^X)q_{i,\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}}}$ . Then,  $\frac{1}{2} < \frac{1-\frac{(1+\nu_{\epsilon}^X)q_{i,\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}}}{2-\frac{(1+\nu_{\epsilon}^X)q_{i,\epsilon}}{\mathbf{Q}_{\epsilon+\epsilon}}}$ . A contradiction. Therefore,  $\kappa > 0$ . Then,  $\frac{\partial^2 \pi_{i,\epsilon}^{\epsilon}}{(\partial q_{i,\epsilon})^2} < 0$ .

Finally, from Berge Maximum Theorem,  $\phi_i^{\epsilon}(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_{\epsilon}^L; \epsilon) \subset \mathcal{C}^1$ .

## 6.6. Appendix F: Proof of Lemma 3

To show Lemma 3, we need one intermediate result based on the Uniform Monotonicity Lemma proved by Dubey and Shubik (1978) (see their Lemma C, p. 8), and which we adapt to our sequential framework.

LEMMA 6. (*Uniform monotonicity*). Let  $c \in \{X, Y\}$ , let  $u_k : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ ,  $\mathbf{z}_k \rightarrow u_k(\mathbf{z}_k)$ ,  $k = i, j$ ,  $i \in T_1$ ,  $j \in T_2$ , be a continuous and increasing function, and let  $H$  be a positive constant. Then, there exists a positive real number  $h(u_k(\cdot), c, H) \in (0, 1)$  such that, for all  $\mathbf{s}_k, \mathbf{z}_k \in \mathbb{R}_+^2$ , if  $\|\mathbf{z}_k\| \leq H$  and  $\|\mathbf{s}_k - \mathbf{z}_k\| \leq h(u_k(\cdot), c, H)$ , then  $u_k(\mathbf{s}_k + \mathbf{e}^c) > u_k(\mathbf{z}_k)$ , where  $\|\cdot\|$  denotes the Euclidean norm, and  $\mathbf{e}^c$  denotes the vector in  $\mathbb{R}_+^2$  whose  $c$ -th component is 1 and the other 0.

*Proof.* The Lemma is an immediate consequence of Lemma C in Dubey and Shubik (1978) (see their Appendix B, p. 19) as the utility functions satisfy notably Assumptions (2a) and (2b). ■

First, we show the existence of  $\xi_1 > 0$  such that  $p^{\epsilon} > \xi_1$ . Let  $(\tilde{\mathbf{b}}_{\epsilon}; \tilde{\mathbf{q}}_{\epsilon})$  be an  $\epsilon$ -SNE. Consider one leader  $j$  and one follower  $j'$ . Let:

$$\begin{aligned} H &= \max\{\bar{\alpha}, \bar{\beta}\}, \text{ with } \bar{\alpha} \equiv \sum_{i=1}^{N_1} \alpha_i \text{ and } \bar{\beta} \equiv \sum_{j=1}^{N_2} \beta_j; & (F1) \\ h &= \min\{h(u_j, Y, H), h(u_{j'}, Y, H)\}; \\ A &= \frac{1}{2} \min\{\beta_j, \beta_{j'}\}, j \neq j'. \end{aligned}$$

Assume, without loss of generality, that, for at least one leader  $j$  or one follower  $j'$  we have  $\tilde{b}_{j,\epsilon} \leq \frac{\tilde{\mathbf{B}}_{\epsilon}}{2}$  or  $\tilde{b}_{j',\epsilon} \leq \frac{\tilde{\mathbf{B}}_{\epsilon}}{2}$  (otherwise we would have  $\tilde{b}_{j,\epsilon} + \tilde{b}_{j',\epsilon} > \tilde{\mathbf{B}}_{\epsilon}$ ). Consider an increase of the strategy of one trader in each stage. First, assume that  $\beta_{j'} - \tilde{b}_{j',\epsilon} \geq A$ . Then, an increase  $\delta$  in follower  $j'$ 's strategy such that  $b_{j',\epsilon}(\delta) = \tilde{b}_{j',\epsilon} + \delta$  is feasible if it is sufficiently small, i.e., if  $\delta \in (0, \frac{1}{2} \min\{\epsilon, A\}]$ . Such an increase has the following incremental effect on his final holding:

$$\begin{aligned} x_{j',\epsilon}(\delta) - x_{j',\epsilon} &= \frac{\tilde{\mathbf{Q}}_{\epsilon} + \epsilon}{\tilde{\mathbf{B}}_{\epsilon} + \epsilon + \delta} (\tilde{b}_{j',\epsilon} + \delta) - \frac{\tilde{\mathbf{Q}}_{\epsilon} + \epsilon}{\tilde{\mathbf{B}}_{\epsilon} + \epsilon} \tilde{b}_{j',\epsilon} & (F2) \\ &= \delta \frac{\tilde{\mathbf{B}}_{\epsilon} + \epsilon - \tilde{b}_{j',\epsilon} \tilde{\mathbf{Q}}_{\epsilon} + \epsilon}{\tilde{\mathbf{B}}_{\epsilon} + \epsilon + \delta \tilde{\mathbf{B}}_{\epsilon} + \epsilon} \\ &> \delta \frac{\frac{\tilde{\mathbf{B}}_{\epsilon}}{2} + \frac{\epsilon}{2} + \frac{\delta}{2} \tilde{\mathbf{Q}}_{\epsilon} + \epsilon}{\tilde{\mathbf{B}}_{\epsilon} + \epsilon + \delta \tilde{\mathbf{B}}_{\epsilon} + \epsilon} = \frac{\delta}{2} \frac{1}{\tilde{p}^{\epsilon}}, \end{aligned}$$

and:

$$y_{j',\epsilon}(\delta) - y_{j',\epsilon} = (\beta_{j'} - \tilde{q}_{j',\epsilon} - \delta) - (\beta_{j'} - \tilde{q}_{j',\epsilon}) = -\delta, \quad (\text{F3})$$

where the strict inequality in (F2) results from  $\tilde{\mathbf{B}}_\epsilon + \epsilon - \tilde{b}_{j',\epsilon} \geq \frac{\tilde{\mathbf{B}}_\epsilon}{2} + \epsilon > \frac{\tilde{\mathbf{B}}_\epsilon}{2} + \frac{\epsilon}{2} + \frac{\delta}{2}$  (as  $\tilde{b}_{j',\epsilon} \leq \frac{\tilde{\mathbf{B}}_\epsilon}{2}$  and  $\delta \leq \frac{1}{2}\epsilon$ ). Let us define:

$$t = -2\tilde{p}^\epsilon \mathbf{e}^Y, \text{ where } \mathbf{e}^Y = (0, 1). \quad (\text{F4})$$

Then, the following vector inequality holds:

$$\mathbf{z}_{j',\epsilon}(b_{j',\epsilon}(\delta), p^\epsilon(\tilde{\mathbf{q}}_\epsilon; b_{j',\epsilon}(\delta), \tilde{\mathbf{b}}_{-j',\epsilon})) \geq \mathbf{z}_{j',\epsilon}(b_{j',\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon)) + \frac{\delta}{2} \frac{1}{\tilde{p}^\epsilon} (\mathbf{e}^X + t), \quad (\text{F5})$$

where  $\mathbf{e}^X = (1, 0)$ . We apply Lemma 6, with  $c = X$ ,  $\mathbf{z}_{j',\epsilon} = \mathbf{z}_{j',\epsilon}(b_{j',\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon))$  and  $\mathbf{s}_{j',\epsilon} = \mathbf{z}_{j',\epsilon}(b_{j',\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon)) + t$ . We know that  $\mathbf{z}_{j',\epsilon}(b_{j',\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon)) \in \mathbb{R}_+^2$  and  $\left\| \mathbf{z}_{j',\epsilon}(b_{j',\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon)) \right\| \leq H$ . Suppose that  $\mathbf{s}_{j',\epsilon} \in \mathbb{R}_+^2$  and  $\|t\| \leq h$ . Then, by Lemma 6, we deduce:

$$u_{j'}(\mathbf{z}_{j',\epsilon}(b_{j',\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon)) + \mathbf{e}^X + t) > u_{j'}(\mathbf{z}_{j',\epsilon}(b_{j',\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon))). \quad (\text{F6})$$

As from Assumptions (2b) and (2c)  $u_{j'}$  is strictly increasing and strictly quasi-concave, and as  $0 < \frac{\delta}{2} \frac{1}{\tilde{p}^\epsilon} < 1$ , then we deduce:

$$u_{j'}(\mathbf{z}_{j',\epsilon}(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) + \frac{\delta}{2} \frac{1}{\tilde{p}^\epsilon} (\mathbf{e}^X + t)) > u_{j'}(\mathbf{z}_{j',\epsilon}(b_{j',\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon))). \quad (\text{F7})$$

Since in  $(\tilde{\mathbf{q}}_\epsilon + \delta; \tilde{\mathbf{b}}_\epsilon)$  the strategy  $\tilde{q}_{j',\epsilon}$  increases, the amount of commodity 2 obtained by any trader  $i \in T_1$  increases. Then, from (2b) and (2c), we have that:

$$u_{j'}(\mathbf{z}_{j',\epsilon}(b_{j',\epsilon}(\delta), p^\epsilon(\tilde{\mathbf{q}}_\epsilon; b_{j',\epsilon}(\delta), \tilde{\mathbf{b}}_{-j',\epsilon}))) > u_{j'}(\mathbf{z}_{j',\epsilon}(b_{j',\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon))). \quad (\text{F8})$$

A contradiction. Hence, either  $\mathbf{z}_{j',\epsilon}(b_{j',\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon)) + t < \mathbf{0}$  or  $\|t\| > h$ . Therefore, if  $\mathbf{z}_{j',\epsilon}(b_{j',\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon)) + t < \mathbf{0}$ , then,  $\tilde{y}_{j',\epsilon} - 2\tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) < 0$ . As  $\tilde{y}_{j',\epsilon} = \beta_{j'} - \tilde{b}_{j',\epsilon} \geq A$ , we deduce:

$$\tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) > \frac{A}{2}. \quad (\text{F9})$$

Suppose now we have  $\|t\| > h$ . Then, we deduce:

$$\tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) > \frac{h}{2}. \quad (\text{F10})$$

Finally, assume that the inequality  $\beta_{j'} - \tilde{b}_{j',\epsilon} \geq A$  does not hold, which means that  $\beta_{j'} - \tilde{b}_{j',\epsilon} < A$ . Then, we have  $\tilde{b}_{j',\epsilon} > \beta_{j'} - A \geq A$ . Then  $\tilde{b}_{j',\epsilon} > A$ , so, we get:

$$\tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) > \frac{A}{\alpha}. \quad (\text{F11})$$

Therefore, it suffices to take for follower  $j'$ :

$$\xi_1^{j'} = \min \left\{ \frac{A}{2}, \frac{h}{2}, \frac{A}{\alpha} \right\}. \quad (\text{F12})$$

Consider now leader  $j$ , with  $\tilde{p}^\epsilon = \frac{\tilde{\mathbf{B}}_\epsilon^L + \sum_j \varphi_j^\epsilon(\mathbf{b}_\epsilon^L; \mathbf{q}_\epsilon^L) + \epsilon}{\tilde{\mathbf{Q}}_\epsilon^L + \sum_i \sigma_i^\epsilon(\mathbf{b}_\epsilon^L; \mathbf{q}_\epsilon^L) + \epsilon}$ . Assume  $\beta_j - \tilde{b}_{j,\epsilon} \geq A$ . We have to show that inequalities similar to (F9)-(F12) also hold for leader  $j$ . Consider an increase  $\delta$  in leader  $j$ 's strategy such that  $b_{j,\epsilon}(\delta) = \tilde{b}_{j,\epsilon} + \delta$ , with  $\delta \in (0, \frac{1}{2} \min\{\epsilon, A\}]$ . This increase has the following effect on her final holding:

$$\begin{aligned}
x_{j,\epsilon}(\delta) - x_{j,\epsilon} &= \frac{\tilde{\mathbf{Q}}_\epsilon^L + \sum_i \sigma_i^\epsilon(\mathbf{q}_\epsilon^L + \delta; \mathbf{b}_\epsilon^L) + \epsilon}{\tilde{\mathbf{B}}_\epsilon^L + \delta + \sum_j \varphi_j^\epsilon(\mathbf{q}_\epsilon^L + \delta; \mathbf{b}_\epsilon^L) + \epsilon} (\tilde{b}_{j,\epsilon} + \delta) - \frac{1}{\tilde{p}^\epsilon} \tilde{b}_{j,\epsilon} \quad (\text{F13}) \\
&= \delta \frac{\tilde{\mathbf{B}}_\epsilon + \epsilon - (1 + \nu_\epsilon^Y) \tilde{b}_{j,\epsilon}}{\tilde{\mathbf{B}}_\epsilon + \epsilon + (1 + \nu_\epsilon^Y) \delta} \frac{\tilde{\mathbf{Q}}_\epsilon + \epsilon}{\tilde{\mathbf{B}}_\epsilon + \epsilon} + \delta \eta_\epsilon^Y \frac{\tilde{b}_{j,\epsilon} + \delta}{\tilde{\mathbf{B}}_\epsilon + \epsilon + (1 + \nu_\epsilon^Y) \delta} \\
&> \delta (1 - \nu_\epsilon^Y) \frac{\frac{\tilde{\mathbf{B}}_\epsilon}{2} + \frac{\epsilon}{2} + (1 + \nu_\epsilon^Y) \frac{\delta}{2}}{\tilde{\mathbf{B}}_\epsilon + \epsilon + (1 + \nu_\epsilon^Y) \delta} \frac{\tilde{\mathbf{Q}}_\epsilon + \epsilon}{\tilde{\mathbf{B}}_\epsilon + \epsilon} + \delta a \eta_\epsilon^Y \\
&= \frac{\delta}{2} \left( \frac{1 - \nu_\epsilon^Y}{\tilde{p}^\epsilon} + 2a \eta_\epsilon^Y \right),
\end{aligned}$$

and:

$$y_{j,\epsilon}(\delta) - y_{j,\epsilon} = -\delta, \quad (\text{F14})$$

where  $a \equiv \frac{\tilde{b}_{j,\epsilon} + \delta}{\tilde{\mathbf{B}}_\epsilon + \epsilon + (1 + \nu_\epsilon^Y) \delta}$ , with  $0 < a \leq 1$ ,  $\nu_\epsilon^Y = \frac{\partial \sum_j \varphi_j^\epsilon(\cdot)}{\partial b_{j,\epsilon}}$  and  $\eta_\epsilon^Y = \frac{\partial \sum_i \sigma_i^\epsilon(\cdot)}{\partial b_{j,\epsilon}}$  for  $\delta$  sufficiently small, and where the strict inequality results from  $\tilde{\mathbf{B}}_\epsilon + \epsilon - (1 + \nu_\epsilon^Y) \tilde{b}_{j,\epsilon} \geq (1 - \nu_\epsilon^Y) \frac{\tilde{\mathbf{B}}_\epsilon}{2} + \epsilon > (1 - \nu_\epsilon^Y) (\frac{\tilde{\mathbf{B}}_\epsilon}{2} + \frac{\epsilon}{2} + (1 + \nu_\epsilon^Y) \frac{\delta}{2})$  as  $\tilde{b}_{j,\epsilon} \leq \frac{\tilde{\mathbf{B}}_\epsilon}{2}$ ,  $\delta \leq \frac{1}{2} \epsilon$  and  $\nu_\epsilon^Y \in [-1, \frac{1}{2}]$ . Let us define:

$$t = -2 \frac{\tilde{p}^\epsilon}{1 - \nu_\epsilon^Y + 2a \eta_\epsilon^Y \tilde{p}^\epsilon} \mathbf{e}^Y. \quad (\text{F15})$$

Then, the following vector inequality holds:

$$\mathbf{z}_{j,\epsilon}(q_{j,\epsilon}(\delta), p^\epsilon(q_{j,\epsilon}(\delta), \tilde{\mathbf{q}}_{-j,\epsilon}; \tilde{\mathbf{b}}_\epsilon)) \geq \mathbf{z}_{j,\epsilon}(\tilde{q}_{j,\epsilon}, \tilde{p}^\epsilon) + \frac{\delta}{2} \frac{1 - \nu_\epsilon^Y + 2a \eta_\epsilon^Y \tilde{p}^\epsilon}{\tilde{p}^\epsilon} (\mathbf{e}^X + t). \quad (\text{F16})$$

Let  $c = X$ ,  $\mathbf{z}_{j,\epsilon}(\tilde{q}_{j,\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon))$  and  $\mathbf{s}_{j,\epsilon} = \mathbf{z}_{j,\epsilon}(\tilde{q}_{j,\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon)) + t$ . We know that  $\mathbf{z}_{j,\epsilon}(\tilde{q}_{j,\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon)) \in \mathbb{R}_+^2$  and  $\|\mathbf{z}_{j,\epsilon}(\tilde{q}_{j,\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon))\| \leq H$ . Suppose that  $\mathbf{s}_{j,\epsilon} \in \mathbb{R}_+^2$  and  $\|t\| \leq h$ . Then, by Lemma 6, we deduce:

$$u_j(\mathbf{z}_{j,\epsilon}(\tilde{q}_{j,\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon)) + \mathbf{e}^X + t) > u_j(\mathbf{z}_{j,\epsilon}(\tilde{q}_{j,\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon))). \quad (\text{F17})$$

From Assumptions (2b) and (2c) and as  $0 < \delta (\frac{1 - \nu_\epsilon^Y}{2 \tilde{p}^\epsilon} + a \eta_\epsilon^Y) < 1$ , we deduce:

$$u_j(\mathbf{z}_{j,\epsilon}(\tilde{q}_{j,\epsilon}, \tilde{p}^\epsilon) + \frac{\delta}{2} \frac{1 - \nu_\epsilon^Y + 2a \eta_\epsilon^Y \tilde{p}^\epsilon}{\tilde{p}^\epsilon} (\mathbf{e}^X + t)) > u_j(\mathbf{z}_{j,\epsilon}(\tilde{q}_{j,\epsilon}, \tilde{p}^\epsilon)). \quad (\text{F18})$$

But then, by Assumptions (2b) and (2c), we have that:

$$u_j(\mathbf{z}_{j,\epsilon}(b_{j,\epsilon}(\delta), p^\epsilon(\tilde{\mathbf{q}}_\epsilon; b_{j,\epsilon}(\delta), \tilde{\mathbf{b}}_{-j,\epsilon}))) > u_j(\mathbf{z}_{j,\epsilon}(\tilde{q}_{j,\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon))). \quad (\text{F19})$$

A contradiction. Hence, either  $\mathbf{z}_{j,\epsilon}(\tilde{q}_{j,\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon)) + t < \mathbf{0}$  or  $\|t\| > h$ . Therefore, if  $\mathbf{z}_{j,\epsilon}(\tilde{q}_{j,\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon)) + t < \mathbf{0}$ , then,  $\tilde{y}_{j,\epsilon} - 2\frac{\tilde{p}^\epsilon}{1-\nu_\epsilon^Y + 2a\eta_\epsilon^Y \tilde{p}^\epsilon} < 0$ . As  $\tilde{y}_{j,\epsilon} = \beta_j - \tilde{b}_{j,\epsilon} \geq A$ , we deduce:

$$\tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) > \frac{A}{2} \left( \frac{1 - \nu_\epsilon^Y}{1 - a\eta_\epsilon^Y A} \right), \quad (\text{F20})$$

where  $\frac{A}{2} \frac{1-\nu_\epsilon^Y}{1-a\eta_\epsilon^Y A} > 0$ . Reason:  $\frac{A}{2} \frac{1-\nu_\epsilon^Y}{1-a\eta_\epsilon^Y A} \geq \frac{A}{2}(1-\nu_\epsilon^Y) > 0$ . The strict inequality holds as  $\frac{A}{2} > 0$  and  $\nu_\epsilon^Y < 1$ , while the weak inequality results from  $a\eta_\epsilon^Y A \geq 0$  since  $0 < a \leq 1$ ,  $A > 0$ , and  $\eta_\epsilon^Y \geq 0$  ( $u_j$  is differentiable so  $\eta_\epsilon^Y$  is never negative, and  $\chi \in [-1, 1]$  in (E2)). Next, if  $\|t\| > h$ , then:

$$\tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) > \frac{h}{2} \left( \frac{1 - \nu_\epsilon^Y}{1 - a\eta_\epsilon^Y h} \right), \quad (\text{F21})$$

where  $\frac{h}{2} \frac{1-\nu_\epsilon^Y}{1-a\eta_\epsilon^Y h} > 0$ . Reason:  $\frac{h}{2} \frac{1-\nu_\epsilon^Y}{1-a\eta_\epsilon^Y h} \geq \frac{h}{2}(1-\nu_\epsilon^Y) > 0$ . The strict inequality holds as  $\frac{h}{2} \in (0, \frac{1}{2})$  and  $\nu_\epsilon^Y < 1$ , while the weak inequality results from  $a\eta_\epsilon^Y h \geq 0$  since  $0 < a \leq 1$ ,  $h \in (0, 1)$ , and  $\eta_\epsilon^Y \geq 0$ . Finally, assume that the inequality  $\beta_j - \tilde{b}_{j,\epsilon} \geq A$  does not hold, i.e.,  $\beta_j - \tilde{b}_{j,\epsilon} < A$ . Then, we have  $\tilde{b}_{j,\epsilon} > \beta_j - A \geq A$ . Then,  $\tilde{b}_{j,\epsilon} > A$ , so we deduce:

$$\tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) > \frac{A}{\alpha}. \quad (\text{F22})$$

Therefore, it suffices to take for leader  $j$ :

$$\xi_1^j = \min \left\{ \frac{A}{2} \left( \frac{1 - \nu_\epsilon^Y}{1 - a\eta_\epsilon^Y A} \right), \frac{h}{2} \left( \frac{1 - \nu_\epsilon^Y}{1 - a\eta_\epsilon^Y h} \right), \frac{A}{\alpha} \right\}. \quad (\text{F23})$$

Then, by taking  $\xi_1 = \min(\xi_1^j, \xi_1^{j'})$ , where  $\xi_1 > 0$ , we conclude that:

$$\tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) > 0. \quad (\text{F24})$$

Second, we show the existence of  $\xi_2 > 0$  such that  $p^\epsilon < \xi_2$ . Consider one leader  $i$  and one follower  $i'$ . Let:

$$\begin{aligned} \hat{h} &= \min\{h(u_j, Y, H), h(u_{j'}, Y, H)\}; \\ \hat{A} &= \frac{1}{2} \min\{\alpha_i, \alpha_{i'}\}, i \neq i'. \end{aligned} \quad (\text{F25})$$

Assume that, for at least one leader  $i$  or one follower  $i'$ , we have  $\tilde{q}_{i,\epsilon} \leq \frac{\tilde{\mathbf{Q}}_\epsilon}{2}$  or  $\tilde{q}_{i',\epsilon} \leq \frac{\tilde{\mathbf{Q}}_\epsilon}{2}$ . Consider follower  $i'$ . Assume that  $\alpha_{i'} - \tilde{q}_{i',\epsilon} \geq \hat{A}$ . Then, it can be shown that an increase  $\delta$  in follower  $i'$ 's strategy such that  $q_{i',\epsilon}(\delta) = \tilde{q}_{i',\epsilon} + \delta$ , with  $\delta \in (0, \frac{1}{2} \min\{\epsilon, \hat{A}\}]$ , has the following incremental effect on his final holding:

$$x_{i',\epsilon}(\delta) - x_{i',\epsilon} = -\delta, \quad (\text{F26})$$

and:

$$y_{i',\epsilon}(\delta) - y_{i',\epsilon} > \frac{\delta}{2} \tilde{p}^\epsilon. \quad (\text{F27})$$

where the strict inequality in (F27) results from  $\tilde{\mathbf{Q}}_\epsilon + \epsilon - \tilde{q}_{i',\epsilon} \geq \frac{\tilde{\mathbf{Q}}_\epsilon}{2} + \epsilon \geq \frac{\tilde{\mathbf{Q}}_\epsilon + \epsilon - \tilde{q}_{i',\epsilon}}{2} + \frac{\epsilon}{2} + \frac{\delta}{2}$  (as  $\delta < \epsilon$ ). Let us define:

$$t = -\frac{2}{\tilde{p}^\epsilon} \mathbf{e}^X. \quad (\text{F28})$$

Then, we have the vector inequality:

$$\mathbf{z}_{i',\epsilon}(q_{i',\epsilon}(\delta), p^\epsilon(q_{i',\epsilon}(\delta), \tilde{\mathbf{q}}_{-i',\epsilon}; \tilde{\mathbf{b}}_\epsilon)) \geq \mathbf{z}_{i',\epsilon}(\tilde{q}_{i',\epsilon}, p^\epsilon(\tilde{\mathbf{q}}_{i,\epsilon}; \tilde{\mathbf{b}}_\epsilon)) + \frac{\delta}{2} \tilde{p}^\epsilon (t + \mathbf{e}^Y). \quad (\text{F29})$$

We apply once again Lemma 6, with  $c = Y$ . Suppose that  $\mathbf{r}_{i,\epsilon} \in \mathbb{R}_+^2$  and  $\|t\| \leq h$ . Then, by Lemma 6, we deduce:

$$u_{i'}(\mathbf{z}_{i',\epsilon}(\tilde{q}_{i',\epsilon}, p^\epsilon(\tilde{\mathbf{q}}_{i,\epsilon}; \tilde{\mathbf{b}}_\epsilon)) + t + \mathbf{e}^Y) > u_{i'}(\mathbf{z}_{i',\epsilon}(\tilde{q}_{i',\epsilon}, p^\epsilon(\tilde{\mathbf{q}}_{i,\epsilon}; \tilde{\mathbf{b}}_\epsilon))). \quad (\text{F30})$$

As  $u_i$  is strictly increasing and strictly quasi-concave, and as  $0 < \frac{\delta}{2} \tilde{p}^\epsilon < 1$ , then:

$$u_{i'}(\mathbf{z}_{i',\epsilon}(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) + \frac{\delta}{2} \tilde{p}^\epsilon (t + \mathbf{e}^Y)) > u_{i'}(\mathbf{z}_{i',\epsilon}(\tilde{q}_{i',\epsilon}, p^\epsilon(\tilde{\mathbf{q}}_{i,\epsilon}; \tilde{\mathbf{b}}_\epsilon))). \quad (\text{F31})$$

As the strategy  $b_{i',\epsilon}$  increases, the amount of commodity 2 obtained by any trader  $j \in T_2$  increases. But then, by Assumptions (2b) and (2c), we have that:

$$u_{i'}(\mathbf{z}_{i',\epsilon}(q_{i',\epsilon}(\delta), p^\epsilon(q_{i',\epsilon}(\delta), \tilde{\mathbf{q}}_{-i',\epsilon}; \tilde{\mathbf{b}}_\epsilon))) > u_{i'}(\mathbf{z}_{i',\epsilon}(\tilde{q}_{i',\epsilon}, p^\epsilon(\tilde{\mathbf{q}}_{i,\epsilon}; \tilde{\mathbf{b}}_\epsilon))), \quad (\text{F32})$$

a contradiction. Then, either  $\mathbf{z}_{i',\epsilon}(\tilde{q}_{i',\epsilon}, p^\epsilon) + t < \mathbf{0}$  or  $\|t\| > h$ . Thus, if  $\mathbf{z}_{i',\epsilon}(\tilde{q}_{i',\epsilon}, p^\epsilon) + t < \mathbf{0}$ , then,  $\tilde{x}_{i',\epsilon} - \frac{2}{\tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon)} < 0$ . As  $\tilde{x}_{i',\epsilon} = \alpha_{i'} - \tilde{q}_{i',\epsilon} \geq \hat{A}$ , we deduce:

$$\tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) < \frac{2}{\hat{A}}. \quad (\text{F33})$$

Suppose now we have  $\|t\| > h$ . Then, we deduce:

$$\tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) < \frac{2}{h}. \quad (\text{F34})$$

Finally, assume  $\alpha_i - \tilde{q}_{i',\epsilon} < \hat{A}$ . Then, we have  $\tilde{q}_{i',\epsilon} > \alpha_i - \hat{A} \geq \hat{A}$ , so  $\tilde{q}_{i',\epsilon} > \hat{A}$ . Then, we deduce:

$$\tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) < \frac{\tilde{\beta}}{\hat{A}}. \quad (\text{F35})$$

Therefore, it suffices to take:

$$\xi_2^{i'} = \max \left\{ \frac{2}{\hat{A}}, \frac{2}{h}, \frac{\tilde{\beta}}{\hat{A}} \right\}. \quad (\text{F36})$$

Consider now leader  $i$ . Assume that  $\alpha_i - \tilde{q}_{i,\epsilon} \geq \hat{A}$ . Let  $q_{i,\epsilon}(\delta) = \tilde{q}_{i,\epsilon} + \delta$ , with  $\delta \in (0, \frac{1}{2} \min\{\epsilon, \hat{A}\}]$ . Such an increase has the following effect on her final holding:

$$x_{i,\epsilon}(\delta) - x_{i,\epsilon} = -\delta, \quad (\text{F37})$$

and:

$$\begin{aligned}
y_{i,\epsilon}(\delta) - y_{i,\epsilon} &= \frac{\tilde{\mathbf{B}}_\epsilon^L + \sum_j \varphi_j^\epsilon(\mathbf{b}_\epsilon^L; \mathbf{q}_\epsilon^L + \delta) + \epsilon}{\tilde{\mathbf{Q}}_\epsilon^L + \delta + \sum_i \sigma_i^\epsilon(\mathbf{b}_\epsilon^L; \mathbf{q}_\epsilon^L + \delta) + \epsilon} (\tilde{q}_{i,\epsilon} + \delta) - \tilde{p}^\epsilon \tilde{q}_{i,\epsilon} \quad (\text{F38}) \\
&= \delta \frac{\tilde{\mathbf{Q}}_\epsilon + \epsilon - (1 + \nu_\epsilon^X) \tilde{q}_{i,\epsilon} \tilde{\mathbf{B}}_\epsilon + \epsilon}{\tilde{\mathbf{Q}}_\epsilon + (1 + \nu_\epsilon^X) \delta + \epsilon} + \delta \eta_\epsilon^X \frac{\tilde{q}_{i,\epsilon} + \delta}{\tilde{\mathbf{Q}}_\epsilon + (1 + \nu_\epsilon^X) \delta + \epsilon} \\
&> \delta (1 - \nu_\epsilon^X) \frac{\frac{\tilde{\mathbf{Q}}_\epsilon}{2} + (1 + \nu_\epsilon^X) \frac{\delta}{2} + \frac{\epsilon}{2} \tilde{\mathbf{B}}_\epsilon + \epsilon}{\tilde{\mathbf{Q}}_\epsilon + (1 + \nu_\epsilon^X) \delta + \epsilon} + \delta d \eta_\epsilon^X \\
&= \frac{\delta}{2} ((1 - \nu_\epsilon^X) \tilde{p}^\epsilon + 2d \eta_\epsilon^X),
\end{aligned}$$

where  $d \equiv \frac{\tilde{q}_{i,\epsilon} + \delta}{\tilde{\mathbf{Q}}_\epsilon + (1 + \nu_\epsilon^X) \delta + \epsilon}$ , with  $0 < d \leq 1$ ,  $\nu_\epsilon^X = \frac{\partial \sum_i \sigma_i^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)}{\partial q_{i,\epsilon}}$  and  $\eta_\epsilon^X = \frac{\partial \sum_j \varphi_j^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)}{\partial q_{i,\epsilon}}$  for  $\delta$  sufficiently small, and where the strict inequality results from  $\tilde{\mathbf{Q}}_\epsilon + \epsilon - (1 + \nu_\epsilon^X) \tilde{q}_{i,\epsilon} \geq (1 - \nu_\epsilon^X) (\frac{\tilde{\mathbf{Q}}_\epsilon}{2} + \frac{\epsilon}{2} + (1 + \nu_\epsilon^X) \frac{\delta}{2})$  as  $\tilde{q}_{i,\epsilon} \leq \frac{\tilde{\mathbf{Q}}_\epsilon}{2}$  always holds, and as  $\delta < \epsilon$ , with  $\nu_\epsilon^X \in [-1, 1)$ . Let us define:

$$t = -2 \frac{1}{(1 - \nu_\epsilon^X) \tilde{p}^\epsilon + 2d \eta_\epsilon^X} \mathbf{e}^X. \quad (\text{F39})$$

Then, the following vector inequality holds:

$$\mathbf{z}_{i,\epsilon}(q_{i,\epsilon}(\delta), p^\epsilon(q_{i,\epsilon}(\delta), \tilde{\mathbf{q}}_{-i,\epsilon}; \tilde{\mathbf{b}}_\epsilon)) \geq \mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon)) + \frac{\delta}{2} ((1 - \nu_\epsilon^X) \tilde{p}^\epsilon + 2d \eta_\epsilon^X) (t + \mathbf{e}^Y). \quad (\text{F40})$$

Suppose that  $\mathbf{s}_{i,\epsilon} \in \mathbb{R}_+^2$  and  $\|t\| \leq h$ . Then, by Lemma 6:

$$u_i(\mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon)) + t + \mathbf{e}^Y) > u_i(\mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon))). \quad (\text{F41})$$

From (2b) and (2c) and as  $0 < \delta (\frac{1 - \nu_\epsilon^X}{2} \tilde{p}^\epsilon + d \eta_\epsilon^X) < 1$ , we deduce:

$$u_i(\mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, \tilde{p}^\epsilon)) + \frac{\delta}{2} ((1 - \nu_\epsilon^X) \tilde{p}^\epsilon + 2d \eta_\epsilon^X) (t + \mathbf{e}^Y) > u_i(\mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, \tilde{p}^\epsilon)). \quad (\text{F42})$$

But then, by Assumptions (2b) and (2c), we have that:

$$u_i(\mathbf{z}_{i,\epsilon}(q_{i,\epsilon}(\delta), p^\epsilon(q_{i,\epsilon}(\delta), \tilde{\mathbf{q}}_{-i,\epsilon}; \tilde{\mathbf{b}}_\epsilon))) > u_i(\mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, \tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon))), \quad (\text{F43})$$

a contradiction. Then, either  $\mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, \tilde{p}^\epsilon) + t < \mathbf{0}$  or  $\|t\| > h$ . Therefore, if  $\mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, \tilde{p}^\epsilon) + t < \mathbf{0}$ , then,  $\tilde{x}_{i,\epsilon} - 2 \frac{1}{(1 - \nu_\epsilon^X) \tilde{p}^\epsilon + 2d \eta_\epsilon^X} < 0$ . As  $\tilde{x}_{i,\epsilon} = \alpha_i - \tilde{q}_{i,\epsilon} \geq \hat{A}$ , we deduce:

$$\tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) < \frac{2}{\hat{A}} \left( \frac{1 - d \eta_\epsilon^X \hat{A}}{1 - \nu_\epsilon^X} \right), \quad (\text{F44})$$

where  $\frac{2}{\hat{A}} \frac{1 - d \eta_\epsilon^X \hat{A}}{1 - \nu_\epsilon^X} > 0$ . Reason:  $\frac{2}{\hat{A}} \frac{1 - d \eta_\epsilon^X \hat{A}}{1 - \nu_\epsilon^X} \geq \frac{2}{\hat{A}} \frac{d^2}{1 - \nu_\epsilon^X} > 0$ . The strict inequality holds as  $d \in (0, 1]$  and  $\nu_\epsilon^X < 1$ . The weak one leads to  $d^2 + d \eta_\epsilon^X \hat{A} - 1 \leq 0$ , so  $d \leq -\frac{\eta_\epsilon^X \hat{A}}{2} + \frac{\sqrt{(\eta_\epsilon^X \hat{A})^2 + 4}}{2}$ , with  $d \leq 1$ . Then we must have  $-\frac{\eta_\epsilon^X \hat{A}}{2} + \frac{\sqrt{(\eta_\epsilon^X \hat{A})^2 + 4}}{2} \leq 1$ , which holds as  $\eta_\epsilon^X \hat{A} \geq 0$ .



Next, if  $\|t\| > h$ , then:

$$\tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) < \frac{2}{\hat{h}} \left( \frac{1 - d\eta_\epsilon^X \hat{h}}{1 - \nu_\epsilon^X} \right), \quad (\text{F45})$$

where  $\frac{2}{\hat{h}} \frac{1 - d\eta_\epsilon^X \hat{h}}{1 - \nu_\epsilon^X} > 0$ . Reason:  $\frac{2}{\hat{h}} \frac{1 - d\eta_\epsilon^X \hat{h}}{1 - \nu_\epsilon^X} \geq \frac{2}{\hat{h}} \frac{d^2}{1 - \nu_\epsilon^X} > 0$ . The weak inequality leads to  $d^2 + d\eta_\epsilon^X \hat{h} - 1 \leq 0$ , which yields  $d \leq -\frac{\eta_\epsilon^X \hat{h}}{2} + \frac{\sqrt{(\eta_\epsilon^X \hat{h})^2 + 4}}{2}$ , with  $d > 0$ . As  $d \leq 1$ , we must have  $-\frac{\eta_\epsilon^X \hat{h}}{2} + \frac{\sqrt{(\eta_\epsilon^X \hat{h})^2 + 4}}{2} \leq 1$ , which is satisfied as  $\eta_\epsilon^X \hat{h} \geq 0$ .

Finally, assume that the inequality  $\alpha_i - \tilde{q}_{i,\epsilon} \geq \hat{A}$  does not hold, i.e.,  $\alpha_i - \tilde{q}_{i,\epsilon} < \hat{A}$ . Then, we have  $\tilde{q}_{i,\epsilon} > \alpha_i - \hat{A} \geq \hat{A}$ . Then, we have  $\tilde{q}_{i,\epsilon} > \hat{A}$ , so we deduce:

$$\tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) < \frac{\bar{\beta}}{\hat{A}}. \quad (\text{F46})$$

Therefore, it suffices to take for leader  $i$ :

$$\xi_1^i = \max \left\{ \frac{2}{\hat{A}} \left( \frac{1 - d\eta_\epsilon^X \hat{A}}{1 - \nu_\epsilon^X} \right), \frac{2}{\hat{h}} \left( \frac{1 - d\eta_\epsilon^X \hat{h}}{1 - \nu_\epsilon^X} \right), \frac{\bar{\beta}}{\hat{A}} \right\}. \quad (\text{F47})$$

Then, by taking  $\xi_2 = \max(\xi_2^i, \xi_2^{i'})$ , we conclude that:

$$\tilde{p}^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon) < \xi_2. \quad (\text{F48})$$

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