



HAL
open science

Asymptotic ultimate regime of homogeneous Rayleigh–Bénard convection on logarithmic lattices

Amaury Barral, Berengere Dubrulle

► **To cite this version:**

Amaury Barral, Berengere Dubrulle. Asymptotic ultimate regime of homogeneous Rayleigh–Bénard convection on logarithmic lattices. *Journal of Fluid Mechanics*, 2023, 962, pp.A2. 10.1017/jfm.2023.204 . hal-04141097

HAL Id: hal-04141097

<https://hal.science/hal-04141097v1>

Submitted on 26 Jun 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

Banner appropriate to article type will appear here in typeset article

1 Asymptotic Ultimate Regime of Homogeneous 2 Rayleigh-Bénard Convection on Logarithmic 3 Lattices

4 Amaury Barral¹ and Berengere Dubrulle¹†

5 ¹SPEC/IRAMIS/DSM, CEA, CNRS, University Paris-Saclay, CEA Saclay, 91191 Gif-sur-Yvette, France

6 (Received xx; revised xx; accepted xx)

7 We investigate how the heat flux Nu scales with the imposed temperature gradient Ra in
8 Homogeneous Rayleigh-Bénard convection using 1, 2 and 3D simulations on logarithmic
9 lattices. Logarithmic lattices are a new spectral decimation framework which enables us to
10 span an unprecedented range of parameters (Ra , Re , Pr) and test existing theories using little
11 computational power. We first show that known diverging solutions can be suppressed with
12 a large-scale friction. In the turbulent regime, for $Pr \approx 1$, the heat flux becomes independent
13 of viscous processes (“asymptotic ultimate regime”, $Nu \sim Ra^{1/2}$ with no logarithmic
14 correction). We recover scalings coherent with the theory developed by Grossmann & Lohse,
15 for all situations where the large-scale frictions keep a constant magnitude with respect to
16 viscous and diffusive dissipation. We also identify another turbulent friction dominated
17 regime at $Pr \ll 1$, where deviations from GL prediction are observed. These two friction
18 dominated regimes may be relevant in some geophysical or astrophysical situations, where
19 large scale friction arises due to rotation, stratification or magnetic field.

20 1. Introduction

21 Convection is a dynamical process that governs heat transport and mixing in a variety of
22 systems ranging from planetary and astrophysical flows to industrial devices. In that respect,
23 a crucial question is how the heat flux in the system is connected with the temperature
24 gradient. Near equilibrium, where both quantities are small, Fourier laws apply, and the
25 heat flux is simply proportional to the temperature gradient. For larger values, the system
26 enters a non-linear then turbulent regime, where thermal energy is converted into mechanical
27 energy, and the relation becomes nonlinear. The deviations from linearity are quantified by
28 the relation between the Nusselt number, Nu , the ratio between the heat flux and its laminar
29 value, and the Rayleigh number Ra , the non-dimensional temperature gradient.

30 In fluid mechanics, the paradigmatic system describing convection is a fluid enclosed
31 in a volume, in which thermal energy is injected at the bottom via imposed heat flux or
32 temperature gradient. Its dynamics is described by the Rayleigh-Bénard (RB) equations.
33 Despite decades of theoretical, experimental and numerical developments, the scaling of the
34 heat transfer in RB remains a subject of discussion and active research. In bounded domains
35 at low Ra , a simple argument by Malkus & Chandrasekhar (1954) based on the criticality

† Email address for correspondence: berengere.dubrulle@cea.fr

Table 1: Scaling predictions for HRB observables in the turbulent regime with and without friction. The observables are given by Table (2). DNS stands for Direct Numerical Simulation using regular Fourier modes (Calzavarini *et al.* 2005) while LL refers to simulations using Fourier modes on a LogLattices (this paper). U_{ls}^2 and Θ_{ls}^2 are large scale kinetic and thermal energy. Exponents are computed by fitting over $Ra > 10^7$ (resp. $1 < Pr < 50$) for varying Ra (resp. Pr). Errors represent std of fit parameters.

GL Theory $f = 0$	DNS $f = 0$	LL $f = 1$
$Nu \sim \sqrt{Ra Pr}$	$Pr^{0.43} Ra^{0.50}$	$Pr^{0.51 \pm 0.01} Ra^{0.53 \pm 0.03}$
$Re \sim \sqrt{Ra/Pr}$	$Pr^{-0.55} Ra^{0.5}$	$Pr^{-0.54 \pm 0.01} Ra^{0.54 \pm 0.01}$
$\epsilon_\theta \sim c_1 \sqrt{Re/Ra} + c_2 Re \sqrt{Pr/Ra}$	$(Re Pr)^{-0.17}$	$Re^x Pr^{-x-0.5} / \sqrt{Ra}, 1 \lesssim x \lesssim 1.2$
$\epsilon_u \sim Re^3 (Pr/Ra)^{3/2}$	$Re^{2.77} (Pr/Ra)^{3/2}$	$Re^{2.88 \pm 0.03} (Pr^{0.95 \pm 0.01} / Ra)^{3/2}$

36 of the thermal boundary layer gives $Nu \sim Ra^{1/3}$, observed in many experiments (see Ahlers
37 *et al.* (2009) for review). As we increase $Ra \rightarrow \infty$, viscous processes (and their associated
38 boundary layers) are believed to become irrelevant, resulting in an “ultimate regime of
39 convection”, where $Nu \sim Ra^{1/2}$ (hereafter called “asymptotic ultimate regime”) (Spiegel
40 1963; Grossmann & Lohse 2000), with possible logarithmic corrections (Kraichnan 1962;
41 Grossmann & Lohse 2011) (hereafter called “ultimate regime”). Experimental or numerical
42 observations of the (asymptotic) ultimate regime prove to be very difficult, and no final
43 consensus has been reached so far about its existence in a pure RB setting (Chavanne *et al.*
44 1997; Urban *et al.* 2019; Doering & Constantin 1996; Zhu *et al.* 2018, 2019a; Roche 2020)
45 (see Ahlers *et al.* (2009) for a less recent but more synthetic review). “When the gravity
46 is artificially increased using centrifugal force, one can indeed observe hints of an ultimate
47 regime (Jiang *et al.* 2022). On the other hand, various modifications of the RB geometry
48 aiming at modifying the influence of the boundary layers result in experimental observation
49 of a regime where $Nu \sim Ra^{1/2}$: using highly elongated cells (Castaing *et al.* 2017; Pawar
50 & Arakeri 2016), using rough (Ciliberto & Laroche 1999; Rusaouën *et al.* 2018; Zhu *et al.*
51 2019b; Kawano *et al.* 2021) or porous“ (Zou & Yang 2021; Motoki *et al.* 2022) boundaries,
52 or radiatively heating the flow (Lepot *et al.* 2018; Bouillaut *et al.* 2019).

53 From a numerical point of view, a simple way to remove boundary layers is to consider
54 a triply periodic geometry, and heat the flow via an applied temperature gradient. This
55 setting was first explored by Borue & Orszag (1997); Lohse & Toschi (2003); Calzavarini
56 *et al.* (2005, 2007) and called Homogeneous Rayleigh-Bénard (HRB) convection. The
57 corresponding scalings and predictions are summarized in Table 1. Although the results
58 of those simulations are consistent with the predictions of Grossmann & Lohse (2000)
59 (hereafter called “GL theory”), they are undermined by several drawbacks: statistics polluted
60 by the growth of uncontrolled exponential instabilities (Calzavarini *et al.* 2006) of unclear
61 physical relevance, a small Ra and Pr range, sparse data points due to difficulties in running
62 numerically challenging simulations. Indeed, pushing the Rayleigh number to large values
63 increases the numerical burden beyond the capacity of present computers, as the number of
64 grid points needed to describe the flow usually scales like Re^3 with $Re \sim Ra^{1/2}$. In an attempt
65 to reduce the number of degrees of freedom, models based on sparse interacting Fourier
66 modes have been recently devised (Campolina & Mailybaev 2018, 2021). Those modes are
67 evenly spaced points in log space (thus thereafter called “log-lattice”) and are interacting
68 via nonlinear equations that are derived from the fluid equations by substituting for the

69 convolution product a new operator, that can be seen as a convolution on the log-lattice,
 70 and preserves all the main symmetries and conservation laws of the original equations. As
 71 such, log-lattices are likely to preserve properties of the original equations that are directly
 72 linked to these symmetries and conservation laws. This was indeed checked for the Burgers
 73 and Navier-Stokes equation in the Fourier space (energy spectrum, energy transfers), over
 74 an unprecedented wide range of scales (Campolina & Mailybaev 2021). Another interesting
 75 feature of log-lattices is that in 1D, they encompass classical shell models of turbulence for
 76 special values of the log-lattice spacing (Campolina & Mailybaev 2021), such as the Sabra
 77 shell model of turbulence (Gloaguen *et al.* 1985; Biferale 2003).

78 1D shell models of turbulence were used previously in the context of HRB (Ching &
 79 Ko 2008) in an effort to increase the Ra and Pr range of results. They successfully display
 80 the asymptotic ultimate regime of convection, at the price of tuning several parameters of
 81 the model to get rid of the uncontrolled exponential instabilities. This, combined with the
 82 1D nature of the model, renders the informative and conclusive nature of the observations
 83 questionable. The goal of the present letter is therefore to re-explore the HRB equation using
 84 the log-lattice framework, that allows both the exploration of a wide range of parameters
 85 on a large array of wavenumbers, and a flexibility of dimensionality from 1D to 3D, at low
 86 numerical cost, and without additional empirical parameters. Given that they preserve all
 87 main conservations laws and symmetry of the original HRB equation, many features of the
 88 original equation are still valid, like the exact conservation laws of Table 1. Whether the
 89 GL theory still applies, and what are the modifications of the asymptotic ultimate regime
 90 implied by the log-lattice geometry are interesting open questions that we investigate here. In
 91 that respect, the present paper offers an exploration of the analogy and differences between
 92 log-lattices and classical fluid dynamics in a more complex case (HRB) than previous
 93 examples (Campolina & Mailybaev 2018, 2021).

94 2. Numerical simulations

95

2.1. Generalities

96 The dynamics of a homogeneous fluid, with coefficient of thermal dilation α , viscosity ν
 97 and diffusivity κ , subject to a temperature gradient ΔT over a length H and vertical gravity g
 98 is given by the HRB set of equations (Lohse & Toschi 2003; Calzavarini *et al.* 2005, 2006,
 100 2007),

$$\begin{aligned}
 \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho_0} \nabla p &= \nu \nabla^2 \mathbf{u} + \alpha g \theta \mathbf{z}, \\
 \partial_t \theta + \mathbf{u} \cdot \nabla \theta &= \kappa \nabla^2 \theta + u_z \frac{\Delta T}{H}, \\
 \nabla \cdot \mathbf{u} &= 0,
 \end{aligned}
 \tag{2.1}$$

102 where \mathbf{u} is the velocity, θ the temperature fluctuation, ρ_0 is the (constant) reference density
 103 and p is the pressure. Here, the mean temperature gradient ΔT acts as a forcing term. This
 104 gradient is non-dimensionalized into the Rayleigh number $\text{Ra} = \alpha g H^3 \Delta T / (\nu \kappa)$. The Prandtl
 105 number $\text{Pr} = \nu / \kappa$ is the ratio of the fluid viscosity to its thermal diffusivity. The mean total
 106 heat flux is the z direction is $J = \langle u_z \theta \rangle - \kappa \Delta T$ which is adimensionalized into $\text{Nu} = JH / \kappa \Delta T$.

107 Taking global space and time average of the equation (2.1), one can derive (Lohse &
 108 Toschi 2003; Calzavarini *et al.* 2005) two exact relation for the volume averaged kinetic and

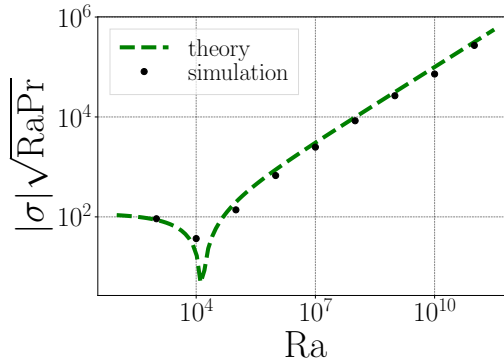


Figure 1: Absolute value of the rate of growth of instability $\sigma = d \log X / d \log t$ where $X = \langle u\theta \rangle$ without large-scale friction ($f = 0$), vs. Rayleigh number. The green dashed line is the theoretical growth rate for $k = k_c = 2\pi\sqrt{3}$, corresponding to Eq. (2.4). The interval $k < k_c$ corresponds to negative values of σ .

109 thermal dissipation, which respectively scale as

$$110 \quad \nu \langle (\partial_i u_j)^2 \rangle_V = \nu^3 H^{-4} \text{Nu Ra Pr}^{-2}, \quad (2.2)$$

$$111 \quad \kappa \langle (\partial_i \theta)^2 \rangle_V = \kappa H^{-2} (\Delta T)^2 \text{Nu}. \quad (2.3)$$

113 Additionally, to get rid of the pressure term, we take the rotational of the above equation
114 ($\boldsymbol{\omega} = \text{rot} \mathbf{u} = i\mathbf{k} \times \mathbf{u}$).

115 2.2. Quantities of interest

116 2.3. Adaptation on log-lattices: HRB with friction

117 2.3.1. Exponential instabilities in HRB

118 As first shown by Calzavarini *et al.* (2006), HRB equations are prone to exponential
119 instabilities, due to the conservation of the total energy. In the absence of large-scale friction,
120 we also observe those instabilities in our log-lattice simulations (Figure 2a). As shown
121 in Figure (1), the growth rate of the instability in the log-lattice simulations matches the
122 theoretical growth rate given by Calzavarini *et al.* (2006); Schmidt *et al.* (2012):

$$123 \quad \sigma \sqrt{\text{Ra Pr}} = \frac{1}{2} \left[\sqrt{((\text{Pr}+1)k^2)^2 + 4 \text{Pr}(\text{Ra} - k^4)} - (\text{Pr}+1)k^2 \right] \sim \sqrt{\text{Ra}}, \quad (2.4)$$

124 for $\theta, u \sim e^{\sigma t + i\mathbf{k} \cdot \vec{x}}$. This expression yields unstable solutions for $\text{Ra} > \text{Ra}_c = k_{\min}^4$ where
125 k_{\min} is the modulus of the smallest mode on the grid, which is $2\pi\sqrt{3}$ in our case.

126 However, the non-linear behavior of the instability in the log-lattice case is quite different
127 from the one reported by Calzavarini: instabilities tend to extend significantly further and
128 for longer times. Our interpretation is that in our log-lattice model, the modes are not
129 coupled enough to develop the nonlinear saturation. The instabilities widely interfere with
130 the statistical stability of observables and need to be removed for a meaningful analysis.
131 Physically, these exponential ramps originate for a lack of energy sink to absorb the constant
132 energy injection in the bulk by the (fixed) temperature gradient. Previous works on 1D
133 simulations (Ching & Ko 2008) have shown that without a large-scale sink to counteract
134 this source, energy diverges at large scales and scaling laws become incorrect. Therefore,
135 to get rid of the exponential instabilities, we include a large-scale friction f on both u and
136 θ . By doing so, the instability saturates, and we achieve a statistically stationary state for

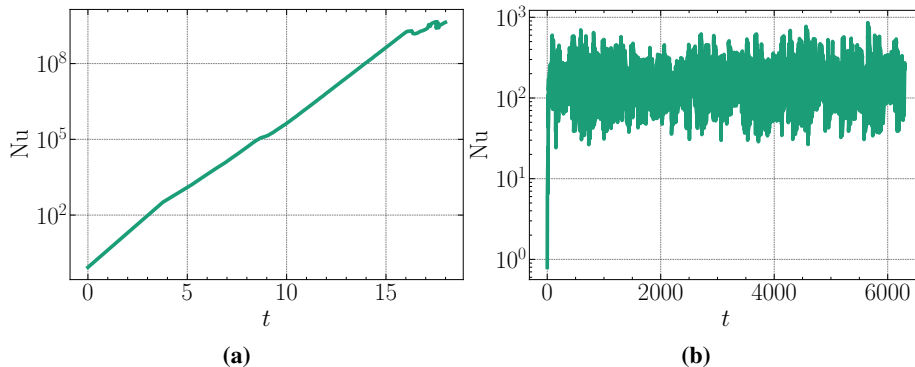


Figure 2: Influence of the large-scale friction on the time behavior of the Nusselt number Nu in 3D HRB. **(2a)** Without friction: we observe the growth of an exponential instability. **(2b)** With friction: the instability saturates and the dynamics become statistically convergent. Parameters: $Ra = 10^6$, $Pr = 1$, $N = 13$.

Table 2: Physical quantities expressed as a function of the non-dimensional variables of Eq. (2.5). $\langle \cdot \rangle$ denotes the temporal and spatial average.

$$Nu = \frac{JH}{\kappa\Delta T} - 1 \rightarrow \sqrt{Ra Pr} \cdot \langle u_z \theta \rangle - 1$$

$$Re = \frac{\sqrt{\langle U_i U_i \rangle} H}{\nu} \rightarrow \sqrt{\frac{Ra}{Pr}} \cdot \sqrt{\langle u_i u_i \rangle}$$

$$\epsilon_\theta = \kappa \langle (\partial_t \Theta)^2 \rangle \rightarrow \frac{\langle (\partial_i \theta)^2 \rangle}{\sqrt{Ra Pr}}$$

$$\epsilon_u = \nu \langle (\partial_i U_j)^2 \rangle \rightarrow \sqrt{\frac{Pr}{Ra}} \cdot \langle (\partial_i u_j)^2 \rangle$$

137 the heat transfer, as displayed in figure 2. Note however that the fluctuations of Nu around
 138 the stationary value are very broad, and extend over one or two orders of magnitudes. The
 139 same phenomenon was observed in the DNS of HRB (Calzavarini *et al.* 2005, 2006) and
 140 mentioned to be a source of difficulty to achieve reliable results (Borue & Orszag 1997). For
 141 this reason, very long simulations are necessary to get steady averages (Pumir & Shraiman
 142 1995; Calzavarini *et al.* 2006). In DNS, this cannot be achieved without cutting down the
 143 resolution, which may impact the reliability of dissipation estimates (Yeung *et al.* 2018). In
 144 the log-lattice framework, we do not have this problem, and we performed high resolution
 145 very long time averages on the log of Nu , and represent all quantities in log-log variables.
 146

147 2.3.2. Equations

148 To investigate the ultimate regime, it is natural to adimensionalize the equation in terms of
 149 “inertial quantities”, i.e. using the vertical width H as a unit of length, the free fall velocity
 150 $U_{ff} = \alpha g \Delta T H$ as a unit of velocity, and ΔT as a unit of temperature. Table 2 indicates the form
 151 taken by observables after rescaling as indicated. The equations including the temperature
 152 gradient and the friction can then be written in terms of velocity as (with the Einstein

154 convention on summed repeated indices):

$$\begin{aligned}
 \partial_t u_i &= \mathbb{P} \left[-u_j \partial_j u_i + \theta \delta_{i=z} + \sqrt{\frac{\text{Pr}}{\text{Ra}}} \nabla^2 u_i - f u_i \delta_{k \approx k_{\min}} \right]_i, \\
 \partial_t \theta &= -u_i \partial_i \theta + u_z + \frac{\nabla^2 \theta}{\sqrt{\text{Ra Pr}}} - f \theta \delta_{k \approx k_{\min}},
 \end{aligned} \tag{2.5}$$

156 where the Dirac $\delta_{k \approx k_{\min}}$ filters out the small scales, and the projector, given in the Fourier
 157 space by $\mathbb{P}(\mathbf{A}) = \mathbf{A} - \frac{k_i}{k^2} k_j A_j$, accounts for the pressure term under the divergence-free
 158 condition. We also looked at those equations expressed in terms of the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$:

$$\begin{aligned}
 \partial_t \omega_i &= -\omega_j \partial_j u_i - u_j \partial_j \omega_i + \theta [\nabla \times \mathbf{z}]_i + \sqrt{\frac{\text{Pr}}{\text{Ra}}} \nabla^2 \omega_i - f \omega_i \delta_{k \approx k_{\min}}, \\
 \partial_t \theta &= -u_i \partial_i \theta + u_z + \frac{\nabla^2 \theta}{\sqrt{\text{Ra Pr}}} - f \theta \delta_{k \approx k_{\min}},
 \end{aligned} \tag{2.6}$$

161 Adding a large-scale friction to damp the inverse cascade is a classical trick-it is e.g.
 162 routinely used numerical simulations of 2D turbulence to avoid Bose condensation at $k = 0$
 163 and enable stationarity (Sukoriansky *et al.* 1999). The present case is 3D, but we interpret
 164 the formation of exponential ramps as a signature of back-scattering of energy, a feature that
 165 was already mentioned previously in shell models of Rayleigh-Bénard convection (Ching
 166 & Ko 2008). The addition of the friction is therefore a convenient way to damp the large-
 167 scale modes that are generated by the large-scale instability. Such friction is also added in
 168 many models of climate, as a subgrid model to account for the friction at the boundary
 169 layer that cannot be resolved in the stratified case. The hand waving argument is that, within
 170 boundary layers, a shear profile develops, with extraction of energy at the boundaries, which
 171 is proportional to the square of the shear. Assuming the shear to be constant in the boundary
 172 layer, we can then estimate it by the difference between the velocity at the top of the layer,
 173 minus the velocity at the boundary which is zero. In total, the energy pumped by friction is
 174 proportional to the square of the velocity, which is exactly the law we have implemented.
 175 Such friction is termed Rayleigh friction in the climate community (Stevens *et al.* 2002) and
 176 can actually be seen as a way to take into account the boundary conditions that we have
 177 removed in the HRB setting.

178 2.3.3. *Conservations laws for HRB with and without friction*

179 In the absence of friction, the conservation laws for HRB are given by Eqs. (2.2) and by (2.3).
 180 The presence of the friction just adds a supplementary term proportional to f in each equation.
 181 The result can be made non-dimensional using U_{ff} , H and ΔT as units of velocity, length and
 182 temperature, resulting in :

$$f \langle u^2 \delta_{k \approx k_{\min}} \rangle + \epsilon_u = \frac{\text{Nu} + 1}{\sqrt{\text{Ra Pr}}}, \tag{2.7}$$

$$f \langle \theta^2 \delta_{k \approx k_{\min}} \rangle + \epsilon_\theta = \frac{\text{Nu} + 1}{\sqrt{\text{Ra Pr}}}, \tag{2.8}$$

186 From now on, we define $U_{\text{ls}}^2 = \langle u^2 \delta_{k \approx k_{\min}} \rangle$ and $\Theta_{\text{ls}}^2 = \langle \theta^2 \delta_{k \approx k_{\min}} \rangle$.

2.4. Log-lattices

187

188 Log-lattice models fit into the more general framework of REduced Wavenumber set
 189 Approximation (REWA) (Grossmann *et al.* 1994) or fractal decimated models (Frisch *et al.*
 190 2012; Lanotte *et al.* 2015). The spirit of these methods is to use a reduced subset of modes
 191 obeying a well-defined hierarchy, so as to stick closer to the observed organized nature of
 192 turbulence. In the original REWA models (Grossmann *et al.* 1994), non-linear-interactions
 193 are projectively decreased either in a random manner or such that they are distributed over
 194 a fractal set (Frisch *et al.* 2012; Lanotte *et al.* 2015). In log-lattice models, the modes
 195 reduction is achieved by keeping modes following a geometric progression, thereby allowing
 196 to reach very small scales with a very small number of modes. The construction is detailed
 197 in Campolina & Mailybaev (2021), where it is shown that fluid equations on log-lattices
 198 respect all symmetries of the Euler equations, and retain classical and basic properties of the
 199 Navier-Stokes equation, such as constancy of energy flux in the inertial range.

200 There are several key differences compared to shell models (Brandenburg 1992; Ching
 201 & Ko 2008) or the original REWA model. Like in a shell model, simulations are carried
 202 out in Fourier space on an logarithmically-decimated grid. Unlike shell models, log-lattices
 203 are truly multidimensional, and unlike the original REWA model, the decimation does not
 204 have a fixed number of points per shell: $\mathbf{k}(n_1, \dots, n_d) = \sum_i \lambda^{n_i} \mathbf{e}_i$, $n_i \in \mathbb{Z}$ with d the spatial
 205 dimension and $\mathbf{e}_i = \mathbf{x}, \mathbf{y}, \mathbf{z}, \dots$. Log-lattices are endowed with a scalar product:

$$206 \quad (f, g) = \Re \left(\sum_{\mathbf{k}} f(\mathbf{k}) \overline{g(\mathbf{k})} \right), \quad (2.9)$$

207 and a convolution operator:

$$208 \quad (f * g)(\mathbf{k}) = \sum_{\substack{\mathbf{p}, \mathbf{q} \\ \mathbf{p} + \mathbf{q} = \mathbf{k}}} f(\mathbf{p}) g(\mathbf{q}), \quad (2.10)$$

209 that naturally extend the corresponding operators on regular Fourier grids. This ensures
 210 that the log-lattice operators respect the symmetries of the Navier-Stokes equation, which
 211 ensures the conservation of energy, helicity (3D) or enstrophy (2D), etc. provided that they
 212 are conserved in the original equation. The constrain on the interacting triads on log-lattices
 213 $\exists p, q \in \lambda^{\mathbb{Z}} : p + q \in \lambda^{\mathbb{Z}}$ restricts the acceptable values of λ to three main families: $\lambda = 2$,
 214 the plastic number $\lambda = \rho \approx 1.324$, and $\lambda^b - \lambda^a = 1$, $(a, b) \in \mathbb{N}^2$, whose biggest solution is
 215 the golden number $\lambda = \phi \approx 1.618$. From a numerical point of view, $\lambda = 2$ is the “fastest”
 216 option, as it has both a maximal span for a given number of points, and the least interactions
 217 per point. However, as outlined in the next part, we believe that $\lambda = 2$ should be avoided for
 218 incompressible simulations. We thereafter perform all our simulations with $\lambda = \phi$, which is
 219 the second biggest value of λ , and has the second least number of interactions per grid point.

220

2.5. Numerical details

2.5.1. Configuration

222 The minimum wave vector of the grid is set to $k_{\min} = 2\pi$ to match a simulation on a box
 223 of size $\tilde{L} = 1$. The grid size N is then set so as to reach the dissipative scale both for
 224 velocity and temperature. We alternate between several initial condition (IC) choices for
 225 our simulations: large-scale initialization, Kolmogorov spectrum, flat-spectrum. All those
 226 choices are modulated by a weak multiplicative complex noise. We find no significant
 227 influence of those initial conditions on the scaling laws. As Ra or Pr increase, the simulations
 228 become slower and slower. This sets the upper bound on the range of parameters we can

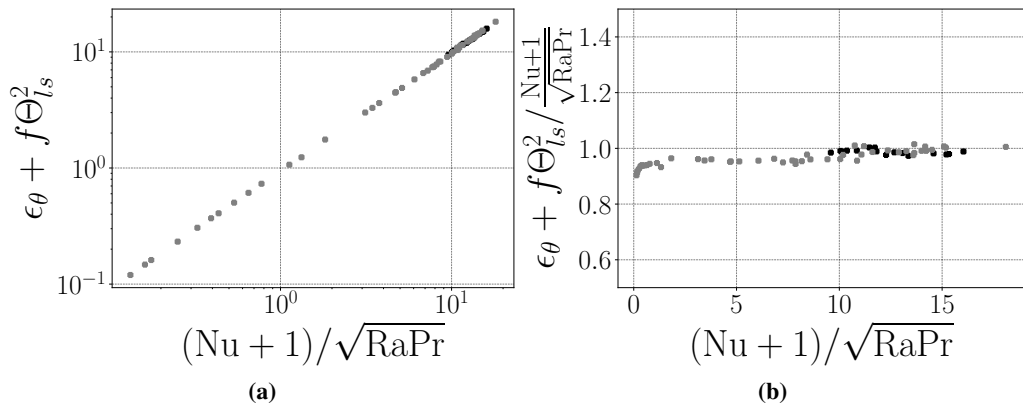


Figure 3: Exact conservation laws for ϵ_θ in 3D results. Black points correspond to varying Ra, gray points correspond to varying Pr. (3a) $\epsilon_\theta + f\Theta_{ls}^2$ vs $(Nu+1)/\sqrt{RaPr}$.

(3b) Compensated plot $(\epsilon_\theta + f\Theta_{ls}^2) / \left(\frac{Nu+1}{\sqrt{RaPr}} \right)$ vs $(Nu+1)/\sqrt{RaPr}$.

229 integrate while retaining statistically relevant observables in a reasonable simulation time
 230 (one CPU days at most). In 3D, this yields $Ra_{\max} \approx 10^{10}$ for $Pr = 1$ and $Pr_{\max} \approx 5 \cdot 10^4$
 231 for $Ra = 10^8$. The lower bound is set by the value of the Nusselt number, which must obey
 232 $Nu \gg 1$, the value $Nu \approx 1$ corresponding to the laminar regime with trivial scaling laws.
 233 Finally, integrating equations on log-lattices yields interesting and new numerical challenges.
 234 We built our own ODE integrator to solve them, as detailed in Supplementary Materials.
 235 Once we have ran a simulation for a long enough time, we compute $Nu, \epsilon_\theta, \epsilon_u$ by taking
 236 long time and space averages (with $\langle ab \rangle = \frac{1}{T} \int_t dt(a, b)$) according to table 2. The accuracy
 237 of our results is controlled by checking that we recover the exact laws of HRB convection
 238 Eqs (2.8) and (2.7). This is shown in Fig. 3 and 4, for all 3D data sets used in the present
 239 paper (see Table 3). Furthermore, the ratio between the friction term and the dissipation is
 240 shown in Fig. 5.

241 2.5.2. Simulation sets

242 The results we obtained come from seven types of simulation that are described in the Table 3.
 243 For comparison, we also included in some graphs the results by (Calzavarini *et al.* 2005),
 244 obtained using DNS of the same equations, but at $f = 0$.

245 Historically, we performed first vorticity simulations, then velocity simulations, improving
 246 the integrator scheme in between to be able to better handle various numerical challenges
 247 raised by simulating wavenumbers as high as $k \sim 10^5$ in 3D. For transparency reasons,
 248 we decided to include all datasets we had at our disposal, but we believe that the velocity
 249 simulations are the more faithful ones, in the sense that they deal better with the small
 250 scales at large Rayleigh or Reynolds number. This sensitivity to small scale modeling (and
 251 resolution) is also a well-known feature of direct numerical simulations, especially when it
 252 comes to statistics of gradients or energy dissipation (Yeung *et al.* 2018).

253 We have verified that the size of the grid for 3D simulations ($N = 13$) does not affect
 254 the mean value of the observables Nu, Re, \dots , which is already converged for grids of size
 255 $N \geq 6$. However the tail of the pdfs does depend on N . Another 3D simulation set at $N = 20$
 256 (not shown here, both vs Ra and Pr) displays the same scaling laws as the $N = 13$ case,
 257 confirming this analysis.

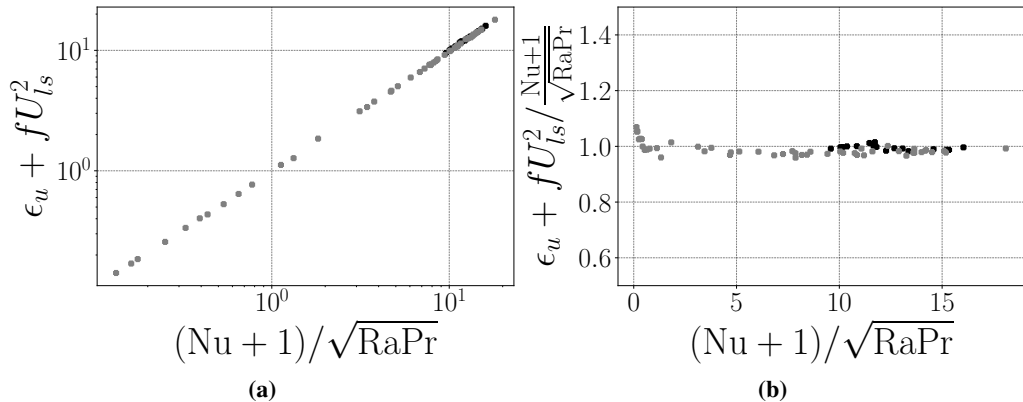


Figure 4: Exact conservation laws for ϵ_u in 3D results. Black points correspond to varying Ra, gray points correspond to varying Pr. (4a) $\epsilon_u + fU_{ls}^2$ vs $(Nu+1)/\sqrt{RaPr}$. (4b) Compensated plot $(\epsilon_u + fU_{ls}^2)/((Nu+1)/\sqrt{RaPr})$ vs $(Nu+1)/\sqrt{RaPr}$.

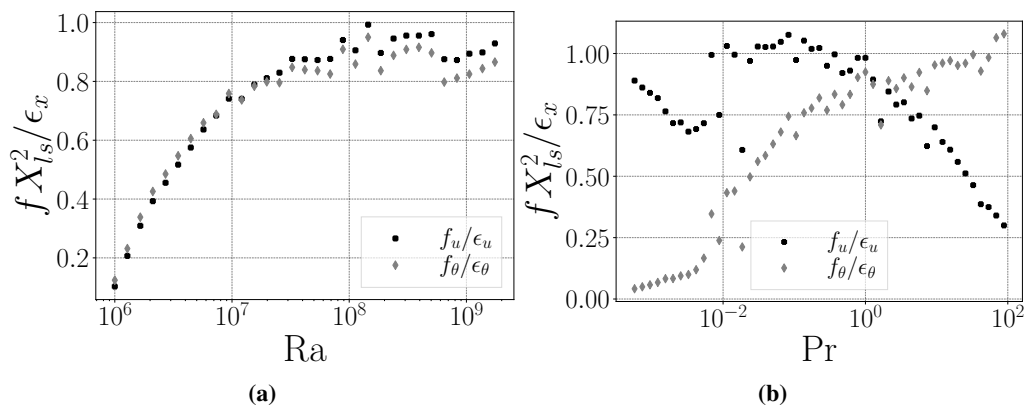


Figure 5: Ratio between friction $f_x = fX_{ls}^2$ and dissipation ϵ_x for $x = u, \theta$ (5a) versus Ra at Pr = 1 and (5b) versus Pr at Ra = 10^8 .

258 2.5.3. A case against $\lambda = 2$

259 This section explains why the log-lattice parameter $\lambda = 2$ is ill-suited to simulating
 260 divergence-free equations. It is not specific to HRB simulations, however we believe this
 261 issue has not been reported in a publication before.

262 $\lambda = 2$ is the biggest grid parameter that can be accommodated on a log-lattice. For a fixed
 263 grid size N in dimension D , it is therefore very tempting to use $\lambda = 2$, since among all the λ s
 264 it spans the greatest range of wavenumbers (the convolution's complexity rises as $O(N^D)$).
 265 However, $\lambda = 2$ misrepresents the convection term $u_j \partial_j u_i$.

266 The heart of the problem is easily understood through a simple 2D example. Consider the
 267 convection term $u_x \partial_x \omega + u_y \partial_y \omega$ of a divergence-free flow, with a large-scale initialization
 268 $u(k > k_0) = \omega(k > k_0) = 0$ for some k_0 . From a physical point of view, we expect convection
 269 to populate the $k \geq k_0$ region as time advances. However, with $\lambda = 2$, this does not happen,
 270 as is demonstrated below.

Table 3: Parameters of the data sets used in the present paper. D is the dimension. The "velocity" datasets are obtained by integration of Eq. (2.5), while the "vorticity" datasets are obtained by integration of Eq. (2.6). *DNS* refers to direct simulations of (Calzavarini *et al.* 2005), using a classical spectral Fourier code (on a regular grid). The ++ label refers to an integration using an improved integrator, using a reshuffling of variable matrices that allows faster simulations. The Ra and Pr column provides the Rayleigh and Prandtl number range of the simulations. f is the large scale friction, $N = 1 + \log k_{\max}/\log(\phi)$, where k_{\max} is the maximal wavenumber of the simulation and ϕ , the golden mean, is a measure of the spatial resolution. For log-lattices, it corresponds to the number of modes in each direction. N_{av} is the length of the simulation, divided by the large eddy turnover time. It provides the number of decorrelated frames that can be used to estimate statistical averages. The tolerance refers to the absolute and relative tolerances that are fixed equal in all the simulations.

Name	D	Type	Ra	Pr	f	N	N_{av}	Tolerance	Symbol
(I)	1D	Velocity	$[10^5, 10^{50}]$	1	1	120	-	10^{-3}	●
(II)	2D	Vorticity	$[10^5, 10^{50}]$	1	1	20	-	10^{-3}	◆
(III)	3D	Velocity	$[1, 10^{10}]$	1	1	13	> 480	10^{-6}	●
(IV)	3D	Vorticity	$[1, 10^{10}]$	1	1	13	> 480	10^{-6}	■
(V)	3D	Velocity	10^8	$[5 \cdot 10^{-4}, 10^2]$	1	13	> 50	10^{-6}	●
(VI)	3D	Vorticity	10^8	$[5 \cdot 10^{-4}, 10^2]$	1	13	> 50	10^{-6}	■
(VII)	3D	++Velocity	$\{10^9, 10^{10}, 10^{11}\}$	$[5 \cdot 10^{-4}, 10^2]$	1	13	> 80	10^{-6}	-
(VIII)	3D	Velocity	$[10^6, 10^{10}]$	1	1	13	> 50	10^{-6}	●
Calzavarini	3D	DNS	$[10^5, 10^8]$	$[10^{-1}, 10]$	0	-	> 64	-	◆

271 In a divergence-free flow, $u_x * \partial_x \omega = -i \left(\frac{\omega k_y}{k^2} * k_x \omega \right)$, $u_y * \partial_y \omega = i \left(\frac{\omega k_x}{k^2} * k_y \omega \right)$ where
 272 * denotes a convolution. In a $\lambda = 2$ log-lattice, convolutions are defined as[†] $f * g(\lambda^n, \lambda^m) =$
 273 $f(\lambda^{n-1}, \lambda^{m-1}) \cdot g(\lambda^{n-1}, \lambda^{m-1})$. Due to the initial conditions, this yields:

$$274 \quad (u_x * \partial_x \omega + u_y * \partial_y \omega) (k \approx k_0) = 0$$

275 There is no forward convection at all, therefore there can be no forward cascade in such
 276 case.

277 This does not happen for other values of λ , for which the convolution is evaluated at
 278 asymmetric positions. We therefore advise against using $\lambda = 2$ in divergence-free fluids, and
 279 suggest to rather use $\lambda = \phi$ (the second-biggest grid parameter).

280 2.5.4. Zero-divergence problem in 1D

281 In the 1D case, we cannot impose the zero-divergence condition, so that quantities like $u_x \partial_x \theta$
 282 and $\partial_x(u_x \theta)$ are not equivalent. Here, we have followed the same choice than Ching & Ko
 284 (2008), and wrote the equation as:

$$285 \quad \begin{aligned} \partial_t u &= -u \partial_x u + \theta + \sqrt{\frac{\text{Pr}}{\text{Ra}}} \nabla^2 u - f u \delta_{k \approx k_{\min}}, \\ \partial_t \theta &= -u \partial_x \theta + u + \frac{\nabla^2 \theta}{\sqrt{\text{Ra Pr}}} - f \theta \delta_{k \approx k_{\min}}. \end{aligned} \quad (2.11)$$

[†] excluding the $k = 0$ mode, which is not used in this paper

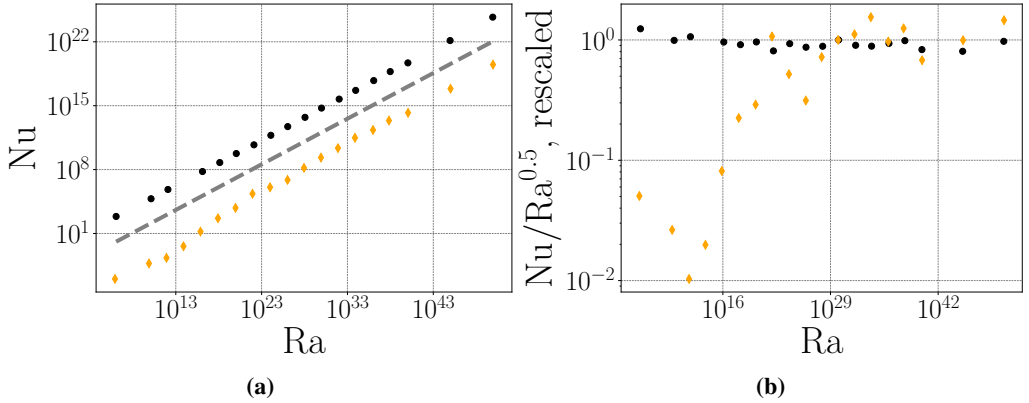


Figure 6: Non-dimensional heat transfer Nu vs Rayleigh number Ra in 1D and 2D. Correspondence between symbols and datasets are given in Table 3. (6a) Nu vs Ra . The gray dashed line corresponds to $Nu \sim \sqrt{Ra}$, corresponding to ultimate regime scaling. (6b) Compensated plot $A Nu/\sqrt{Ra}$ vs Ra , where A is adjusted to collapse the 1D and 2D data in the ultimate regime.

286 3. Results and Discussion

287

3.1. One and two-dimensional cases

288 Figure 6 presents the Nu vs Ra scaling in 1D and 2D. The 1D Nu scaling law extends over
 289 50 orders of magnitude in Ra (Fig. 6a), and follows closely the law $Nu \sim Ra^{1/2}$, as can
 290 be checked by the compensated plot in Fig. (6b), in agreement with Ching & Ko (2008).
 291 In 2D, the scaling also extends approximately over 30 orders of magnitudes for $Ra > 10^{23}$.
 292 Moreover, the compensated plot highlights small fluctuations around this law, see Fig. (6b),
 293 due to statistical noise.

294

3.2. In 3D

295 In 3D, the simulations get significantly more turbulent and results are subject to more
 296 statistical fluctuations. Another source of fluctuations comes from a physical phenomenon,
 297 associated with the existence of friction. To showcase this effect, we plot in Fig. 5a and Fig. 5b
 298 the ratio between the energy dissipated by friction and the energy dissipated by viscosity or
 299 diffusivity for both the kinetic energy and the thermal energy.

300 Fixing $Pr = 1$ and varying Ra between 10^3 and 10^8 , we observe in Fig. 5a that both
 301 $f_u = fU_{ls}^2/\epsilon_u$ or $f_\theta = f\Theta_{ls}^2/\epsilon_\theta$ behave in the same way as a function of Ra at low Ra , the
 302 dissipation due to friction is small, and gradually increases towards reaching a plateau around
 303 $Ra \sim 10^7$, where energy dissipated by frictions reach about 90% of the energy dissipated by
 304 viscosity or diffusivity. We can thus define a “non-universal” regime where f/ϵ depends on
 305 Ra , Pr and a “universal” regime where f/ϵ does not depend on Ra , Pr .

306 The critical Rayleigh number where the plateau occurs is likely to depend on the Prandtl
 307 number. To check this; we now fix $Ra = 10^8$ and vary Pr from several order of magnitude.
 308 In Fig. 5b, we then observe an interesting symmetrical behaviour, with respect to $Pr = 1$:
 309 decreasing Pr , we observe that the energy dissipated by the velocity friction remains of the
 310 same order of magnitude than the dissipation by viscosity, while the energy dissipated by
 311 thermal friction strongly decays and become negligible. As Pr shifts away from 1, we observe
 312 the symmetrical behavior, with velocity friction becoming negligible, while thermal friction
 313 remains of the same order of magnitude than the thermal energy dissipation. As we will see,

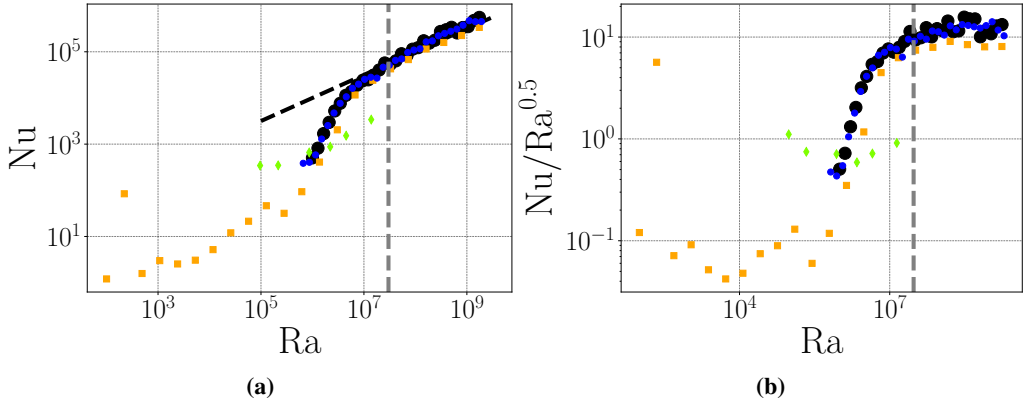


Figure 7: Non-dimensional heat transfer Nu vs Rayleigh number Ra in 3D for $Pr = 1$. Correspondence between symbols and datasets are given in Table 3. The gray dashed line separates the non-universal (left) and the universal (right) friction dominated regimes for data corresponding to Fig.5. (7a) Nu vs Ra . The black dashed line corresponds to $Nu \sim \sqrt{Ra}$, corresponding to asymptotic ultimate regime scaling. (7b) Compensated plot Nu/\sqrt{Ra} vs Ra .

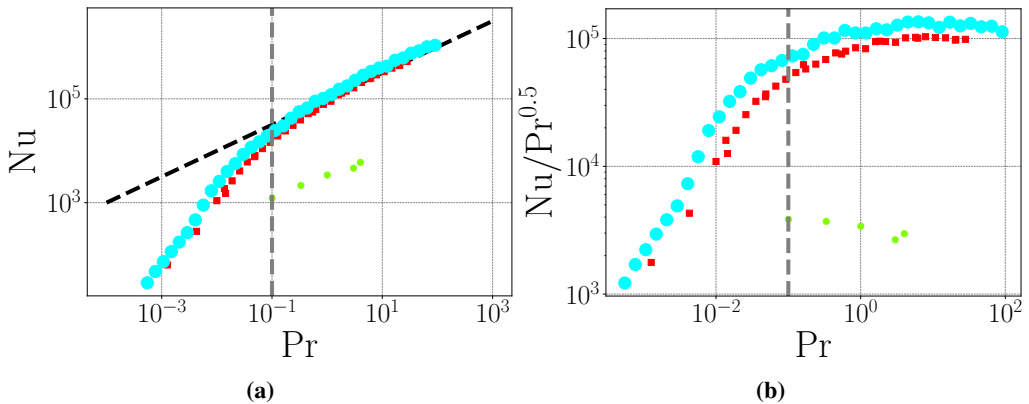


Figure 8: Scaling of non-dimensional heat transfer Nu as a function of Prandtl number Pr in 3D for $Ra = 10^8$. Correspondence between symbols and datasets are given in Table 3. The gray dashed line separates the non-universal (left) and the universal (right) friction dominated regimes for data corresponding to Fig.5. (8a) Nu vs Pr . The black dashed line corresponds to $Nu \sim \sqrt{Pr}$, corresponding to asymptotic ultimate regime scaling. (8b) Compensated plot Nu/\sqrt{Pr} vs Pr .

314 this will have an impact on the thermal transport. Note that at small (resp. large) Pr , all the
 315 thermal (resp. velocity) modes become concentrated at large scale, where the friction occurs.
 316 Therefore, in the large Pr regime, the kinetic friction and viscous dissipation compete, while
 317 at small Pr the same remark holds for the thermal friction and diffusive dissipation. This may
 318 then explain the vanishing of the friction in those regime.

319 We now focus on the regimes where the ratio of friction to dissipation is approximately
 320 constant. These regimes are friction dominated, but, as we will see, are characterized by
 321 interesting universal scaling regimes.

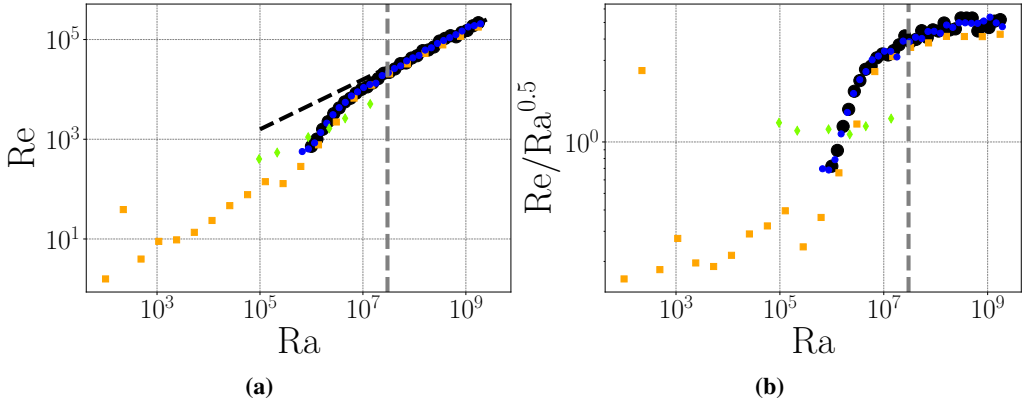


Figure 9: Scaling of Reynolds number Re as a function of Rayleigh number Ra in 3D for $Pr = 1$. Correspondence between symbols and datasets are given in Table 3. The gray dashed line separates the non-universal (left) and the universal (right) friction dominated regimes for data corresponding to Fig.5. (9a) Re vs Ra . The black dashed line corresponds to $Re \sim \sqrt{Ra}$, corresponding to asymptotic ultimate regime scaling. (9b) Compensated plot Re/\sqrt{Ra} vs Ra .

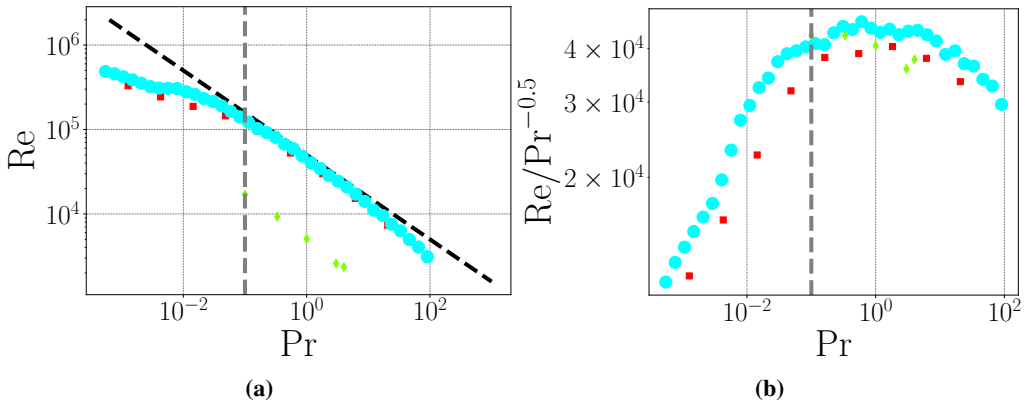


Figure 10: Scaling of Reynolds number Re as a function of Prandtl number Pr in 3D for $Ra = 10^8$. Correspondence between symbols and datasets are given in Table 3. The gray dashed line separates the non-universal (left) and the universal (right) friction dominated regimes for data corresponding to Fig.5. (10a) Re vs Pr . The black dashed line corresponds to $Re \sim 1/\sqrt{Pr}$, corresponding to ultimate regime scaling. (10b) Compensated plot $Re/(1/\sqrt{Pr})$ vs Pr .

322 Figures 7 and 8 presents the 3D Nu vs Ra , Pr scalings. Figures 9 and 10 presents the 3D
 323 Re vs Ra , Pr scalings. Scaling are always displayed both directly and in compensated form.

324 At low Ra , we first observe a transition from a laminar regime, where $Nu = 1$ up to a
 325 turbulent regime starting around $Ra \sim 10^7$ at $Pr = 1$. In this transition regime, the Nusselt
 326 number varies approximately like $Nu \sim Ra^{2/3}$, while the Reynolds number remains less than
 327 10^4 , but follows approximates laws $Re \sim Ra^{1/2}$. In this regime, the friction is negligible, as
 328 we saw, so that it corresponds to a laminar, frictionless regime.

329 After this laminar regime, we obtain a turbulent regime around $10^7 < Ra$ for $Pr = 1$ in

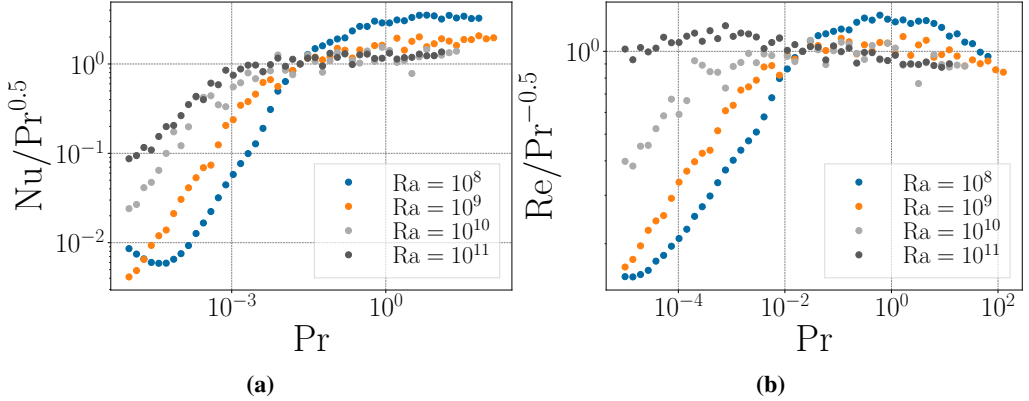


Figure 11: Scaling of heat transfer Nu as a function of Prandtl number Pr in 3D results at fixed Ra , dataset VII (Table 3) (11a) Nu/\sqrt{Pr} vs Pr for various Ra . (11b) $Re/(1/\sqrt{Pr})$ vs Pr for various Ra .

330 which $Nu \sim Ra^{1/2}$ and $Re \sim Ra^{1/2}$, like GL theory. The exact value of the exponent is
 331 provided in Table 1. In this regime, the friction are non-negligible, so that it is a “turbulent
 332 friction dominated regime” However, as both ratio $f_u = fU_{ls}^2/\epsilon_u$ or $f_\theta = f\Theta_{ls}^2/\epsilon_\theta$ remain
 333 independent of Ra , they do not change the scaling of the total kinetic and thermal energy
 334 dissipation. Therefore, the argument developed by GL theory should still apply in this
 335 situation, as is indeed observed, with minor corrections due to the small variations of the
 336 ratios.

337 In that respect, it is not surprising that the the extent of this regime varies with Pr , as is
 338 shown in Fig. 11 for various Ra . At $Ra = 10^8$, the “universal GL” regime stops for $Pr < \sim 10^{-1}$.
 339 In this range of parameters, Re is still large, so that the flow is turbulent. However, Nu drops
 340 quicker with decreasing Pr than in the universal GL regime, as can be seen from the filled data
 341 points in Fig. 8, in parallel with a similar drop for the thermal friction observed in Fig. 5b.
 342 This regime seems therefore dependent of the variation on the friction, and is non-universal.
 343 In this regime, the Reynolds number variation with Pr is milder than in the universal regime,
 344 as can be seen in Fig. 10.

345 As the Rayleigh number increases, we nevertheless observe in Fig. 11 that the extent of
 346 the universal turbulent regime extends towards smaller and smaller values of Pr , so that
 347 the universal scaling regime corresponds to an “asymptotic scaling regime” at low value of
 348 $Pr < 1$, valid in the limit of infinite Ra .

349 Figures 12 and 13 plot the kinetic and thermal dissipation rates $\epsilon_u, \epsilon_\theta$ against GL
 350 predictions. In agreement with what has been observed previously, we observe agreement
 351 with GL theory in the range of parameters where the friction ratios are approximately constant
 352 with the parameters, i.e. at large value of $Re Pr$. Overall, it is interesting to note that even
 353 when the friction is dominant, we can recover the ultimate regime scaling, as long as the
 354 velocity friction ratio remain relatively constant as a function of the parameters and neither
 355 there is not too big an asymmetry between the two frictions. In regimes where the asymmetry
 356 prevails, there are no clear scaling laws that emerge, meaning that the scaling are probably not
 357 universal in Ra and Pr only, and that friction depending corrections need to be implemented.

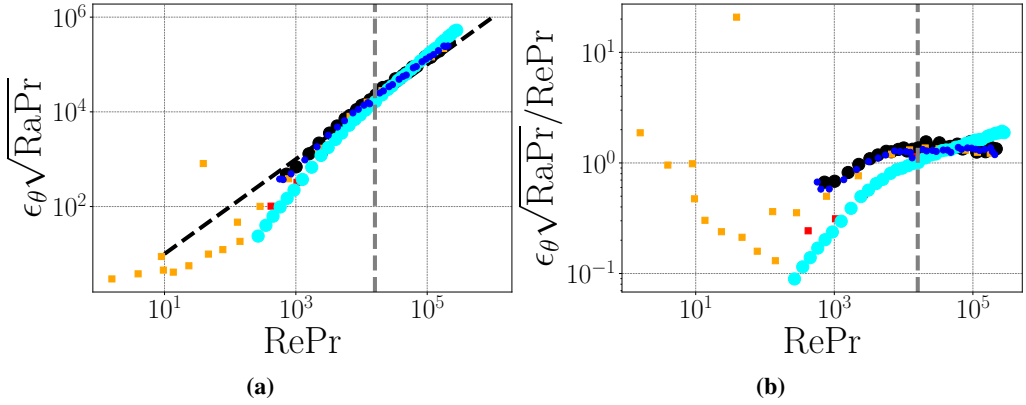


Figure 12: Scaling of thermal dissipation rate ϵ_θ compared to the GL prediction $\text{Re} \sqrt{\text{Pr}/\text{Ra}}$ in 3D results. Correspondence between symbols and datasets are given in Table 3). The gray dashed line separates the non-universal (left) and the universal (right) friction dominated regimes for data corresponding to Fig.5. (12a) $\epsilon_\theta \sqrt{\text{Ra Pr}}$ vs Re Pr . The black dashed line corresponds to the GL prediction $\epsilon_\theta \sim \text{Re}(\text{Pr}/\text{Ra})^{1/2}$. (12b) Compensated plot $\epsilon_\theta \sqrt{\text{Ra Pr}}/\sqrt{\text{Re Pr}}$ vs $\sqrt{\text{Re Pr}}$.

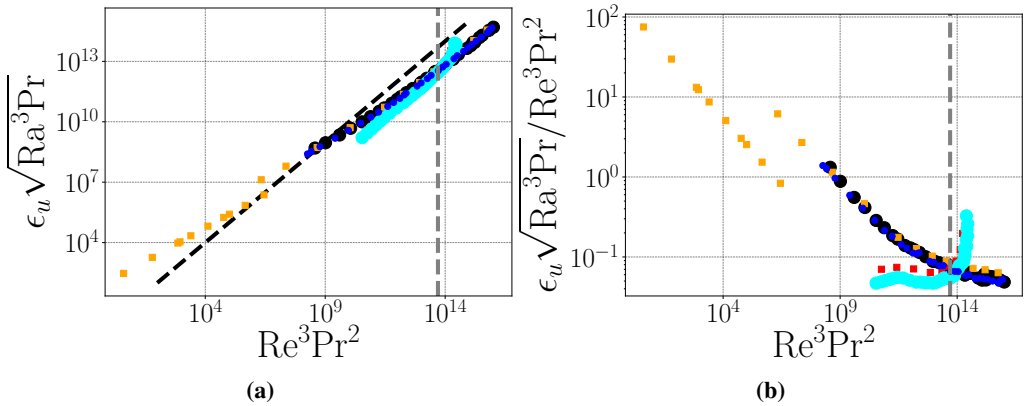


Figure 13: Scaling of kinetic dissipation rate ϵ_u compared to the GL prediction $\text{Re}^3 (\text{Pr}/\text{Ra})^{3/2}$ in 3D results. Correspondence between symbols and datasets are given in Table 3). The gray dashed line separates the non-universal (left) and the universal (right) friction dominated regimes for data corresponding to Fig.5. (13a) $\epsilon_u \sqrt{\text{Ra}^3 \text{Pr}}$ vs $\text{Re}^3 \text{Pr}^2$. The black dashed line corresponds to the GL prediction $\epsilon_u \sim \text{Re}^3 (\text{Pr}/\text{Ra})^{3/2}$. (13b) Compensated plot $\epsilon_u \sqrt{\text{Ra}^3 \text{Pr}}/\text{Re}^3 \text{Pr}^2$ vs $\text{Re}^3 \text{Pr}^2$.

358 4. Conclusion

359 In this letter, we investigated scaling laws in the Homogeneous Rayleigh Bénard (HRB)
 360 equations through a new mathematical framework (“log-lattice”). Using a modified DOPRI
 361 solver, we are able to explore a range of parameters and wave-numbers way beyond what is
 362 accessible in direct numerical simulations of the equations. By adding a large-scale friction
 363 to the HRB equations, we are able to solve the issue of exponentially diverging solutions.
 364 This large scale friction become non-negligible when the fluid become turbulent enough, so
 365 that total energy balance depart from the energy balance considered in GL theory, where no

friction is present. Despite this, we still observe scaling law for Nu and Pr that are very close to the universal turbulent predictions of Grossmann-Lohse (GL) theory: $Nu \sim Ra^{1/2} Pr^{1/2}$, $Re \sim Ra^{1/2} Pr^{-1/2}$, $\epsilon_\theta \sim Re(Pr/Ra)^{1/2}$, $\epsilon_u \sim Re^3(Pr/Ra)^{3/2}$ for an important range of parameters, corresponding to situations where the thermal friction is non-negligible and the kinetic friction does not vary significantly as a function of the parameters. This is obtained at large enough Ra and for Pr depending on the value of Ra.

In addition to this regime, we also observe another turbulent friction dominated regime at $Pr \ll 1$. This regime has no simple and universal dependence with the parameter, and depends on the variations of the kinetic friction with the parameters.

Our observation show that the inclusion of friction, which is necessary to obtain stationary regimes in the HRB framework, complexifies the phase space but nevertheless allows for the existence of a universal turbulent regime, where scaling laws are very close to the GL friction-less theoretical laws. In some geophysical or astrophysical situations, large scale friction arises due to rotation (Ekman friction), stratification (Rayleigh friction) or magnetic field (Hartman friction), and the two scaling regimes we find (one universal, and one non-universal) may be relevant and could be explored within the log-lattice framework.

More generally, we believe that log-lattices, with their unique performances in terms of numerical complexity, due to a spectrally sparse representation and strong mathematical qualities, have a great potential in numerical simulations of geophysical or astrophysical flows. However, as they are still in their infancy, many different paths would benefit from being explored to better understand their strengths and weaknesses. This in particular includes a better understanding of the influence of the numerical scheme which, as discussed in the supplementary materials, may misrepresent the viscosity at high wavenumbers. We believe that methods such as Whalen *et al.* (2015) could prove most useful in that regard. Other topics of interest include the behavior of observables when $\lambda \rightarrow 1$ and the addition of the $k = 0$ mode would prove very interesting to study. Likewise, in a similar spirit as was done for the REWA model in Grossmann *et al.* (1996), a detailed comparison of DNS and log-lattice results (which is far from trivial, as there is room for interpretation as to the mathematical meaning of the fields simulated on a log-lattice) would be highly useful.

Acknowledgements. We thank A. Mailybaev and C. Campolina for initiating us to the log-lattice computations, and for stimulating discussions.

Funding. This work received funding from the Ecole Polytechnique, from ANR EXPLOIT, grant agreement no. ANR-16-CE06-0006-01 and ANR TILT grant agreement no. ANR-20-CE30-0035.

Declaration of interests. The authors report no conflict of interest.

REFERENCES

- AHLERS, GUENTER, GROSSMANN, SIEGFRIED & LOHSE, DETLEF 2009 Heat transfer and large scale dynamics in turbulent Rayleigh-Bénard convection. *Rev. Mod. Phys.* **81**, 503–537.
- BIFERALE, LUCA 2003 Shell models of energy cascade in turbulence. *Annual Review of Fluid Mechanics* **35**, 441–468.
- BORUE, VADIM & ORSZAG, STEVEN A 1997 Turbulent convection driven by a constant temperature gradient. *Journal of Scientific computing* **12** (3), 305–351.
- BOUILLAUT, VINCENT, LEPOT, SIMON, AUMAÎTRE, SÉBASTIEN & GALLET, BASILE 2019 Transition to the ultimate regime in a radiatively driven convection experiment. *Journal of Fluid Mechanics* **861**, R5.
- BRANDENBURG, AXEL 1992 Energy spectra in a model for convective turbulence. *Phys. Rev. Lett.* **69**, 605–608.
- CALZAVARINI, E., DOERING, C. R., GIBBON, J. D., LOHSE, D., TANABE, A. & TOSCHI, F. 2006 Exponentially growing solutions in homogeneous Rayleigh-Bénard convection. *Phys. Rev. E* **73**, 035301.
- CALZAVARINI, E., LOHSE, D. & TOSCHI, F. 2007 Homogeneous Rayleigh-Bénard Convection. In *Progress in*

- 413 *Turbulence II* (ed. Martin Oberlack, George Khujadze, Silke Günther, Tanja Weller, Michael Frewer,
414 Joachim Peinke & Stephan Barth), pp. 181–184. Berlin, Heidelberg: Springer Berlin Heidelberg.
- 415 CALZAVARINI, ENRICO, LOHSE, DETLEF, TOSCHI, FEDERICO & TRIPICCIONE, RAFFAELE 2005 Rayleigh and
416 Prandtl number scaling in the bulk of Rayleigh–Bénard turbulence. *Physics of Fluids* **17** (5), 055107,
417 arXiv: <https://doi.org/10.1063/1.1884165>.
- 418 CAMPOLINA, CIRO S. & MAILYBAEV, ALEXEI A. 2018 Chaotic Blowup in the 3D Incompressible Euler
419 Equations on a Logarithmic Lattice. *Phys. Rev. Lett.* **121**, 064501.
- 420 CAMPOLINA, CIRO S & MAILYBAEV, ALEXEI A 2021 Fluid dynamics on logarithmic lattices. *Nonlinearity*
421 **34** (7), 4684–4715.
- 422 CASTAING, B., RUSAOUEN, E., SALORT, J. & CHILLA, F. 2017 Turbulent heat transport regimes in a channel.
423 *Phys. Rev. Fluids* **2**, 062801.
- 424 CHAVANNE, X., CHILLÀ, F., CASTAING, B., HÉBRAL, B., CHABAUD, B. & CHAUSSY, J. 1997 Observation of
425 the Ultimate Regime in Rayleigh–Bénard Convection. *Phys. Rev. Lett.* **79**, 3648–3651.
- 426 CHING, EMILY S. C. & KO, T. C. 2008 Ultimate-state scaling in a shell model for homogeneous turbulent
427 convection. *Phys. Rev. E* **78**, 036309.
- 428 CILIBERTO, S. & LAROCHE, C. 1999 Random Roughness of Boundary Increases the Turbulent Convection
429 Scaling Exponent. *Phys. Rev. Lett.* **82**, 3998–4001.
- 430 DOERING, CHARLES R. & CONSTANTIN, PETER 1996 Variational bounds on energy dissipation in
431 incompressible flows. III. Convection. *Phys. Rev. E* **53**, 5957–5981.
- 432 FRISCH, URIEL, POMYALOV, ANNA, PROCACCIA, ITAMAR & RAY, SAMRIDDIH SANKAR 2012 Turbulence in
433 Noninteger Dimensions by Fractal Fourier Decimation. *Phys. Rev. Lett.* **108**, 074501.
- 434 GLOAGUEN, C., LÉORAT, J., POUQUET, A. & GRAPPIN, R. 1985 A scalar model for MHD turbulence. *Physica*
435 *D: Nonlinear Phenomena* **17** (2), 154–182.
- 436 GROSSMANN, SIEGFRIED & LOHSE, DETLEF 2000 Scaling in thermal convection: a unifying theory. *Journal*
437 *of Fluid Mechanics* **407**, 27–56.
- 438 GROSSMANN, SIEGFRIED & LOHSE, DETLEF 2011 Multiple scaling in the ultimate regime of thermal
439 convection. *Physics of Fluids* **23** (4), 045108, arXiv: <https://doi.org/10.1063/1.3582362>.
- 440 GROSSMANN, SIEGFRIED, LOHSE, DETLEF, L'VOV, VICTOR & PROCACCIA, ITAMAR 1994 Finite size corrections
441 to scaling in high Reynolds number turbulence. *Phys. Rev. Lett.* **73**, 432–435.
- 442 GROSSMANN, SIEGFRIED, LOHSE, DETLEF & REEH, ACHIM 1996 Developed Turbulence: From Full
443 Simulations to Full Mode Reductions. *Phys. Rev. Lett.* **77**, 5369–5372.
- 444 JIANG, HECHUAN, WANG, DONGPU, LIU, SHUANG & SUN, CHAO 2022 Experimental Evidence for the Existence
445 of the Ultimate Regime in Rapidly Rotating Turbulent Thermal Convection. *Phys. Rev. Lett.* **129**,
446 204502.
- 447 KAWANO, KOKI, MOTOKI, SHINGO, SHIMIZU, MASAKI & KAWAHARA, GENTA 2021 Ultimate heat transfer in
448 ‘wall-bounded’ convective turbulence. *Journal of Fluid Mechanics* **914**, A13.
- 449 KRAICHNAN, ROBERT H. 1962 Turbulent Thermal Convection at Arbitrary Prandtl Number. *The Physics of*
450 *Fluids* **5** (11), 1374–1389, arXiv: <https://aip.scitation.org/doi/pdf/10.1063/1.1706533>.
- 451 LANOTTE, ALESSANDRA S., BENZI, ROBERTO, MALAPAKA, SHIVA K., TOSCHI, FEDERICO & BIFERALE, LUCA
452 2015 Turbulence on a Fractal Fourier Set. *Phys. Rev. Lett.* **115**, 264502.
- 453 LEPOT, SIMON, AUMAÎTRE, SÉBASTIEN & GALLET, BASILE 2018 Radiative heating achieves the ultimate regime
454 of thermal convection. *Proceedings of the National Academy of Sciences* **115** (36), 8937–8941, arXiv:
455 <https://www.pnas.org/doi/pdf/10.1073/pnas.1806823115>.
- 456 LOHSE, DETLEF & TOSCHI, FEDERICO 2003 Ultimate State of Thermal Convection. *Phys. Rev. Lett.* **90**,
457 034502.
- 458 MALKUS, W. V. R. & CHANDRASEKHAR, SUBRAHMANYAN 1954 The heat transport and
459 spectrum of thermal turbulence. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* **225** (1161), 196–212, arXiv:
460 <https://royalsocietypublishing.org/doi/pdf/10.1098/rspa.1954.0197>.
- 461 MOTOKI, SHINGO, KAWAHARA, GENTA & SHIMIZU, MASAKI 2022 Steady thermal convection
462 representing the ultimate scaling. *Philosophical Transactions of the Royal Society*
463 *A: Mathematical, Physical and Engineering Sciences* **380** (2225), 20210037, arXiv:
464 <https://royalsocietypublishing.org/doi/pdf/10.1098/rsta.2021.0037>.
- 465 PAWAR, SHASHIKANT S. & ARAKERI, JAYWANT H. 2016 Two regimes of flux scaling in axially homogeneous
466 turbulent convection in vertical tube. *Phys. Rev. Fluids* **1**, 042401.
- 467 PUMIR, ALAIN & SHRAIMAN, BORIS I. 1995 Persistent Small Scale Anisotropy in Homogeneous Shear Flows.
468 *Phys. Rev. Lett.* **75**, 3114–3117.
- 469

- 470 ROCHE, PHILIPPE-E 2020 The ultimate state of convection: a unifying picture of very high Rayleigh numbers
471 experiments. *New Journal of Physics* **22** (7), 073056.
- 472 RUSAOUËN, E., LIOT, O., CASTAING, B., SALORT, J. & CHILLA, F. 2018 Thermal transfer in Rayleigh–Bénard
473 cell with smooth or rough boundaries. *Journal of Fluid Mechanics* **837**, 443–460.
- 474 SCHMIDT, LAURA E., CALZAVARINI, ENRICO, LOHSE, DETLEF, TOSCHI, FEDERICO & VERZICCO, ROBERTO 2012
475 Axially homogeneous Rayleigh–Bénard convection in a cylindrical cell. *Journal of Fluid Mechanics*
476 **691**, 52–68.
- 477 SPIEGEL, EDWARD A 1963 A Generalization of the Mixing-Length Theory of Turbulent Convection. *The
478 Astrophysical Journal* **138**, 216.
- 479 STEVENS, BJORN, DUAN, JIANJUN, MCWILLIAMS, JAMES C., MÜNNICH, MATTHIAS & NEELIN, J. DAVID 2002
480 Entrainment, Rayleigh Friction, and Boundary Layer Winds over the Tropical Pacific. *Journal of
481 Climate* **15** (1), 30–44.
- 482 SUKORIANSKY, SEMION, GALPERIN, BORIS & CHEKHLOV, ALEXEI 1999 Large scale drag representation
483 in simulations of two-dimensional turbulence. *Physics of Fluids* **11** (10), 3043–3053, arXiv:
484 <https://doi.org/10.1063/1.870163>.
- 485 URBAN, P., HANZELKA, P., KRÁLÍK, T., MACEK, M., MUSILOVÁ, V. & SKRBEK, L. 2019 Elusive transition to
486 the ultimate regime of turbulent Rayleigh–Bénard convection. *Phys. Rev. E* **99**, 011101.
- 487 WHALEN, PATRICK, BRIO, MOYSEY & MOLONEY, JEROME V 2015 Exponential time-differencing with
488 embedded Runge–Kutta adaptive step control. *Journal of Computational Physics* **280**, 579–601.
- 489 YEUNG, P. K., SREENIVASAN, K. R. & POPE, S. B. 2018 Effects of finite spatial and temporal resolution in
490 direct numerical simulations of incompressible isotropic turbulence. *Phys. Rev. Fluids* **3**, 064603.
- 491 ZHU, XIAOJUE, MATHAI, VARGHESE, STEVENS, RICHARD JAM, VERZICCO, ROBERTO & LOHSE, DETLEF 2019a
492 Reply to "Absence of Evidence for the Ultimate Regime in Two-Dimensional Rayleigh-Benard
493 Convection". *Physical Review Letters* **123** (25), 259402.
- 494 ZHU, XIAOJUE, MATHAI, VARGHESE, STEVENS, RICHARD J. A. M., VERZICCO, ROBERTO & LOHSE, DETLEF
495 2018 Transition to the Ultimate Regime in Two-Dimensional Rayleigh–Bénard Convection. *Phys.
496 Rev. Lett.* **120**, 144502.
- 497 ZHU, XIAOJUE, STEVENS, RICHARD J. A. M., SHISHKINA, OLGA, VERZICCO, ROBERTO & LOHSE, DETLEF
498 2019b $Nu \sim Ra^{1/2}$ scaling enabled by multiscale wall roughness in Rayleigh–Bénard turbulence.
499 *Journal of Fluid Mechanics* **869**, R4.
- 500 ZOU, SHUFAN & YANG, YANTAO 2021 Realizing the ultimate scaling in convection turbulence by spatially
501 decoupling the thermal and viscous boundary layers. *Journal of Fluid Mechanics* **919**, R3.