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Nonparametric density estimation for the small jumps of Lévy processes

Céline Duval ^{*}, Taher Jalal [†] and Ester Mariucci [‡]

Abstract

We consider the problem of estimating the density of the process associated with the small jumps of a pure jump Lévy process, possibly of infinite variation, from discrete observations of one trajectory. The interest of such a question relies on the observation that even when the Lévy measure is known, the density of the increments of the small jumps of the process cannot be computed. We discuss results both from low and high frequency observations. In a low frequency setting, assuming the Lévy density associated the jumps larger than 1 in absolute value is known, a spectral estimator relying on the deconvolution structure of the problem achieves minimax parametric rates of convergence with respect to the integrated L_2 loss. In a high frequency setting it is possible to remove the assumption that the Lévy measure of the large jumps is known. In that case the rate of convergence depends on the sampling scheme and on the behaviour of the Lévy measure in a neighborhood of zero. An adaptive penalized procedure is also proposed to select the cutoff parameter. The procedure is illustrated numerically for α -stable Lévy processes.

Keywords. Deconvolution; Density estimation; Lévy processes; Small jumps; Infinitely divisible distributions.

1 Introduction

1.1 Motivations

Jump processes have been extensively studied and widely used in the mathematical modeling of phenomena that may exhibit abrupt changes, and therefore are matters of interest in various fields such as mathematical finance, seismology, climatology, neuroscience, and so on. Among the most mathematically tractable examples of jump processes are Lévy processes (see e.g. [1, 4, 6, 11, 26] for reviews and other applications). These are stochastic processes that have a rather rigid structure (their increments are stationary and independent) but have often been used as proxies to establish results for jump processes with a more flexible structure, e.g., Itô's semi martingales.

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From a probabilistic point of view, the dynamics of the trajectories of a Lévy process X is well understood. The law of X is uniquely determined by the so-called Lévy triplet that contains a drift term, a diffusion coefficient and a Lévy measure (see e.g. [3, 30]). For any pure jump Lévy process X , the distribution of its increments is the convolution between a martingale X^S describing its small jumps and a compound Poisson process X^B gathering the large jumps (large than 1) of the process. For most Lévy processes whose Lévy measure is infinite, a closed form expression for the law of its increments is not known and the core of the problem lies in computing the distribution of the small jumps which is never known in closed form. We stress that even in very well known situations, for instance when X is an α -stable Lévy process, there are already a lot of results for controlling the law of X but nothing can be said for X^S which is not an α -stable Lévy process.

This renders these processes difficult to handle and simulate (see [18, 22]), the small jumps are therefore sometimes replaced with Gaussian distributions (see [13]), which can be theoretically justified in some cases (see [10]). However this approximation is valid only in specific regimes (see the lower bound result in [10]) and untrue in general.

The behaviour of the Lévy measure in a neighborhood of the origin is a key element in understanding the jump activity. It is essentially related to the law of the so-called small jump process of X^S . So far there are no results in the literature that estimate the density of X^S from discrete observations of X contrary to X^B which has been extensively studied (see e.g. [2, 19, 20]). Using the convolution structure of the Lévy process and that X^B is a compound Poisson process with intensity and jump law depending on the Lévy measure of X in an explicit way we can derive an estimation procedure for X^S . We can rely on the vast literature for deconvolution results to conduct our study (see e.g. [7–9, 12, 17, 21, 27, 31] for the study of the quadratic risk, other results for the multivariate anisotropic densities [16, 25, 29] also exist).

In this paper we consider pure jumps Lévy processes X with a Lévy measure absolutely continuous with respect to the Lebesgue measure and we focus on the estimation of the density of X^S at time Δ , denoted by g_Δ , from n equidistant observations of X with a sampling rate Δ . We propose several estimators based on a spectral approach to take into account both low frequency observations ($\Delta > 0$ fixed) and high frequency ($\Delta \rightarrow 0$). It appears that in the low frequency regime, if the Lévy measure of X^B is known, it appears that the inverse deconvolution problem is well posed and as the density g_Δ very regular (see [28]) our estimator attains parametric rates of convergence that are optimal. In the high frequency setting, without any knowledge on the distribution of X^B another estimator can be proposed and its rate of convergence depend on the behaviour of the Lévy density in a neighborhood of the origin. Adaptation is achieved adapting a penalized procedure presented in [14] Theorem 4.1. Finally, a short numerical study is conducted on α -stable Lévy processes.

1.2 Setting and notations

Consider a pure jump Lévy process X characterized by its Lévy triplet $(b_\nu, 0, \nu)$ where ν is a Borel measure on \mathbb{R} such that

$$\nu(\{0\}) = 0; \quad \nu(\mathbb{R}) = +\infty \quad \text{and} \quad \int_{\mathbb{R}} (y^2 \wedge 1) \nu(dy) < \infty$$

and

$$b_\nu := \begin{cases} \int_{|x| \leq 1} x \nu(dx) & \text{if } \int_{|x| \leq 1} |x| \nu(dx) < \infty, \\ 0 & \text{if } \int_{|x| \leq 1} |x| \nu(dx) = \infty. \end{cases} \quad (1)$$

Tanks to the Lévy-Itô decomposition (see [3]), X can be written as

$$X_t = tb_\nu + X_t^S + X_t^B,$$

where X^S is a centered martingale accounting for the jumps of X of size smaller than 1 and is independent of X^B which is a compound Poisson process independent of X^S with intensity $\lambda = \nu(\mathbb{R} \setminus [-1, 1])$ and jump density $f = p \mathbf{1}_{[-1, 1]^c} / \lambda$ where $p(x) = \frac{\nu(dx)}{dx}$.

In the following we write $X_t^B = \sum_{i=1}^{N_t} Y_i$ where N is a Poisson process of intensity λ independent of the sequence of i.i.d. (independent and identically distributed) random variables Y_i with common density f . We will denote by f_Δ^B the density of X_Δ^B given by

$$f_\Delta^B(x) = \sum_{k=0}^{\infty} \exp(-\lambda\Delta) \frac{(\lambda\Delta)^k}{k!} f^{*k}(x), \quad (2)$$

where f^{*k} is the k -th convolution of the density f and $f^{*0} = \delta_0$ is the Dirac measure at point 0.

Consider the i.i.d. observations $\mathbb{X} = (X_{i\Delta} - X_{(i-1)\Delta})_{i=1}^n$ with $X_0 = 0$. Our aim is to estimate the density g_Δ of $Z_\Delta := X_\Delta^S + \Delta b_\nu$ from \mathbb{X} both under the assumption $\Delta > 0$ fixed and $\Delta \rightarrow 0$, and compute the L^2 integrated risk. For that we need to assume that X is a Lévy process with with a Lévy density $p = \frac{d\nu}{dx}$ satisfying

$$p(x) \geq \frac{M}{|x|^{1+\alpha}}, \quad \forall |x| \leq 1 \quad (\mathcal{A}_{M,\alpha})$$

for some $0 < \alpha \leq 2$.

The estimation strategy that we analyse is based on a spectral approach, and we use the following notations. Given a random variable Z , $\phi_Z(u) = \mathbb{E}[e^{iuZ}]$ denotes the characteristic function of Z . For $g \in L^1(\mathbb{R})$, $\mathcal{F}g(u) = \int e^{iux} g(x) dx$ is understood to be the Fourier transform. Moreover, we denote by $\|\cdot\|$ the L^2 -norm of functions, $\|g\|^2 := \int |g(x)|^2 dx$. Given some function $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we denote by g_m the uniquely defined function with Fourier transform $\mathcal{F}g_m = (\mathcal{F}g) \mathbf{1}_{[-m, m]}$.

2 Main results

2.1 Estimation in the low frequency regime

Let $\Delta > 0$ and suppose that ν is known on $\mathbb{R} \setminus [-1, 1]$ such that in the decomposition: $X_t = b_\nu t + X_t^S + X_t^B = Z_t + X_t^B$ the density of X_t^B is entirely known. Thanks to the convolution structure of the law of X_Δ , it holds $\phi_{X_\Delta} = \phi_{Z_\Delta} \phi_{X_\Delta^B}$. In particular, for a fixed $\Delta > 0$, $\phi_{X_\Delta^B}$ is known and never vanishes as

$$|\phi_{X_\Delta^B}(u)| = |\exp(-\lambda\Delta(\phi_{Y_1}(u) - 1))| \geq e^{-2\lambda\Delta} > 0, \quad (3)$$

where $Y_1 \sim f$. Hence, the quantity

$$\phi_{Z_\Delta}(u) = \frac{\phi_{X_\Delta}(u)}{\phi_{X_\Delta^B}(u)}$$

is well defined for all $u \in \mathbb{R}$ and can be estimated by

$$\widehat{\phi}_{Z_\Delta}(u) = \frac{1}{n} \sum_{j=1}^n \frac{e^{iu(X_{j\Delta} - X_{(j-1)\Delta})}}{\phi_{X_\Delta^B}(u)}. \quad (4)$$

From (4) we derive an estimator of g_Δ , using a spectral cut-off as the latter quantity may not be in $L^1(\mathbb{R})$:

$$\widehat{g}_{\Delta,m}(x) = \frac{1}{2\pi} \int_{-m}^m \widehat{\phi}_{Z_\Delta}(u) e^{-iux} du. \quad (5)$$

The following result gives an upper bound for the integrated L^2 -risk of $\widehat{g}_{\Delta,m}$.

Theorem 1. *Let X be a Lévy process whose Lévy measure ν satisfies $(\mathcal{A}_{M,\alpha})$, for some $M > 0$ and $\alpha \in]0, 2]$. Let $\Delta > 0$ and g_Δ be the density of $\Delta b_\nu + X_\Delta^S$ and $\widehat{g}_{\Delta,m}$ the estimator defined in (5). Then, for all $m \geq 1$ it holds that*

$$\mathbb{E}[\|\widehat{g}_{\Delta,m} - g_\Delta\|^2] \leq \|g_{\Delta,m} - g_\Delta\|^2 + \frac{e^{4\lambda\Delta}}{\pi} \frac{m}{n},$$

and $\|g_{\Delta,m} - g_\Delta\|^2 \leq K\Delta^{-1/\alpha} e^{-c\Delta m^\alpha}$ for constants $c > 0$ and $K > 0$ (that may depend on α).

Proof. To control the integrated L^2 -risk we write the decomposition

$$\begin{aligned} \mathbb{E}[\|\widehat{g}_{\Delta,m} - g_\Delta\|^2] &= \|g_{\Delta,m} - g_\Delta\|^2 + \mathbb{E}[\|\widehat{g}_{\Delta,m} - g_{\Delta,m}\|^2] \\ &= \|g_{\Delta,m} - g_\Delta\|^2 + \frac{1}{2\pi} \int_{-m}^m \mathbb{E}[|\widehat{\phi}_{Z_\Delta}(u) - \phi_{Z_\Delta}(u)|^2] du. \end{aligned}$$

The first term is the standard bias term for which we can write using Plancherel equality, (12), the fact that $m \geq 1$ and $(\mathcal{A}_{M,\alpha})$, that for some constant $c > 0$:

$$\|g_{\Delta,m} - g_\Delta\|^2 = \frac{1}{2\pi} \int_{[-m,m]^c} |\phi_{Z_\Delta}(u)|^2 du \leq \frac{1}{\pi} \int_m^\infty e^{-2c\Delta u^\alpha} du \leq \tilde{c} e^{-c\Delta m^\alpha}, \quad (6)$$

where $\tilde{c} = \frac{1}{\pi} \int_0^\infty e^{-c\Delta u^\alpha} du = \frac{\Delta^{-\frac{1}{\alpha}}}{\pi} \int_0^\infty e^{-cv} dv < \infty$ as $\alpha > 0$.

In particular we have that

$$\tilde{c} \leq \frac{\Delta^{-1/\alpha}}{\pi} \int_0^\infty \frac{e^{-cv}}{\alpha v^{1-\frac{1}{\alpha}}} dv =: \Delta^{-1/\alpha} K.$$

Also observe that $g_\Delta \in L^2(\mathbb{R})$ as $g_\Delta \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ by means of Lemma 2 in the Appendix. For the variance term, using that

$$\begin{aligned} \mathbb{E}[|\widehat{\phi}_{Z_\Delta}(u) - \phi_{Z_\Delta}(u)|^2] &= \frac{1}{|\phi_{X_\Delta^B}(u)|^2} \mathbb{E}[|\widehat{\phi}_{X_\Delta}(u) - \phi_{X_\Delta}(u)|^2] \\ &= \frac{1}{|\phi_{X_\Delta^B}(u)|^2} \mathbb{V} \left(\frac{1}{n} \sum_{j=1}^n e^{iu(X_{j\Delta} - X_{(j-1)\Delta})} \right) = \frac{1 - |\phi_{X_\Delta}(u)|^2}{|\phi_{X_\Delta^B}(u)|^2} \frac{1}{n}, \end{aligned}$$

we easily get

$$\mathbb{E}[\|\widehat{g}_{\Delta,m} - g_{\Delta,m}\|^2] \leq \frac{e^{4\lambda\Delta}}{\pi} \frac{m}{n}. \quad (7)$$

Gathering (6) and (7) we derive:

$$\mathbb{E}[\|\widehat{g}_{\Delta,m} - g_{\Delta}\|^2] \leq \frac{e^{4\lambda\Delta}}{\pi} \frac{m}{n} + K\Delta^{-1/\alpha} e^{-c\Delta m^\alpha},$$

which concludes the proof. \square

Remark 1. Making a bias variance compromise to select m we get to solve $me^{c\Delta m^\alpha} = n$, for that we use the properties of the Lambert W function and solve

$$\frac{m^*}{n} = (c\Delta)^{-1/\alpha} e^{-c\Delta(m^*)^\alpha} \iff m^* = \left(\frac{W(\alpha n^\alpha)}{\alpha c\Delta} \right)^{\frac{1}{\alpha}}.$$

Using asymptotic expansion of the Lambert W function, we derive that the optimal cutoff should be selected as follows $m^* \asymp \left(\frac{\log(\alpha n^\alpha)}{\alpha c\Delta} \right)^{\frac{1}{\alpha}} \asymp \left(\frac{\log(n)}{\Delta} \right)^{\frac{1}{\alpha}}$. In that case, the rate implied by Theorem 1 is

$$\mathbb{E}[\|\widehat{g}_{\Delta,m} - g_{\Delta}\|^2] \lesssim \frac{(\log n)^{\frac{1}{\alpha}}}{n\Delta^{\frac{1}{\alpha}}},$$

an almost (up to a log-loss) parametric rate (recall that $\Delta > 0$ is fixed), which is consistent with the fact that: *i)* we are in a well posed deconvolution problem (see (3)), *ii)* under the assumptions of Lemma 2, the Lévy density is C^∞ , morally g_{Δ} has a regularity ∞ .

The problem of finding a data driven way to select m is studied in Section 2.3. The optimal cutoff m^* depends on the unknown quantity α appearing in Assumption $(\mathcal{A}_{M,\alpha})$. Interestingly the adaptation problem of selecting m consists in estimating a possible α for condition $(\mathcal{A}_{M,\alpha})$. This is simpler than estimating the true Blumenthal-Gettoor index of X , all we need is a minorant of it.

2.2 Estimation in a high frequency regime

In this Section we consider the case where $\Delta \rightarrow 0$. Then, it remains possible to estimate the density of Z_{Δ} using the estimator $g_{\Delta,m}$ as defined in (5) and with similar arguments as those illustrated in the previous paragraph, one can show that it is consistent as soon as $n\Delta^{\frac{1}{\alpha}} \rightarrow \infty$ and that its L^2 rate of convergence is $n^{-1}\Delta^{-\frac{1}{\alpha}}$.

However, in the high frequency setting, it is possible to omit the assumption that $\phi_{X_{\Delta}^B}$ is known since in this asymptotic $\phi_{X_{\Delta}^B}$ is close to 1. We therefore propose to consider a second estimator of g_{Δ} , defined as follows

$$\widetilde{g}_{\Delta,m}(x) = \frac{1}{2\pi} \int_{-m}^m \widehat{\phi}_{X_{\Delta}}(u) e^{-iux} du, \quad (8)$$

whose L_2 risk is controlled in the following result. Note that if Δ is fixed (5) is an estimator of the density of X_{Δ} (see Section 4 of [23]).

Theorem 2. Let X be a Lévy process whose Lévy measure ν satisfies $(\mathcal{A}_{M,\alpha})$, for some $M > 0$ and $\alpha \in]0, 2]$. Let $\Delta \in (0, 1)$ be such that $\lambda\Delta \leq 1$, where $\lambda = \nu(\mathbb{R} \setminus [-1, 1])$. Then, there exist $c > 0$ and $K > 0$ (that may depend on α) such that for all $m \geq 1$ it holds:

$$\mathbb{E}[\|\tilde{g}_{\Delta,m} - g_{\Delta}\|^2] \leq K \left(\|g_{\Delta,m} - g_{\Delta}\|^2 + \frac{m}{n} + \Delta^{2-\frac{1}{\alpha}} \right),$$

and $\|g_{\Delta,m} - g_{\Delta}\|^2 \leq K\Gamma\left(\frac{1}{\alpha}, 2c\Delta m^{\alpha}\right) \Delta^{-\frac{1}{\alpha}}$ for constants $c > 0$ and $K > 0$ (that may depend on α) and where $\Gamma(s, x)$ denotes the incomplete Gamma function $\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt$.

Proof. To control the integrated L^2 -risk we write the decomposition ($f_{\Delta,m}$ denotes the projection on S_m of the density f_{Δ} of X_{Δ})

$$\begin{aligned} \mathbb{E}[\|\tilde{g}_{\Delta,m} - g_{\Delta}\|^2] &= \|f_{\Delta,m} - g_{\Delta}\|^2 + \mathbb{E}[\|\hat{g}_{\Delta,m} - f_{\Delta,m}\|^2] \\ &= \|f_{\Delta,m} - g_{\Delta}\|^2 + \frac{1}{2\pi} \int_{-m}^m \mathbb{E}[|\hat{\phi}_{X_{\Delta}}(u) - \phi_{X_{\Delta}}(u)|^2] du. \end{aligned}$$

The second variance term is easily bounded by $\frac{m}{\pi n}$. The first term is a bias term for which we can write

$$\|f_{\Delta,m} - g_{\Delta}\|^2 \leq 2\|f_{\Delta,m} - g_{\Delta,m}\|^2 + 2\|g_{\Delta,m} - g_{\Delta}\|^2.$$

An upper bound for $\|g_{\Delta,m} - g_{\Delta}\|^2$ has already been provided in (6). Under the assumption $\lambda\Delta \leq 1$, by means of Plancherel equality and (3), it holds:

$$\begin{aligned} \|f_{\Delta,m} - g_{\Delta,m}\|^2 &= \frac{1}{2\pi} \int_{[-m,m]} |\phi_{X_{\Delta}}(u)|^2 \left| 1 - \frac{1}{\phi_{X_{\Delta}^B}(u)} \right|^2 du \\ &\leq \frac{e^4}{2\pi} \int_{[-m,m]} |\phi_{X_{\Delta}}(u)|^2 |\phi_{X_{\Delta}^B}(u) - 1|^2 du, \end{aligned}$$

where for some positive constant C

$$\phi_{X_{\Delta}^B}(u) - 1 = \exp(-\lambda\Delta(\phi_{Y_1}(u) - 1)) - 1 = \sum_{k \geq 1} \frac{(\lambda\Delta)^k}{k!} (1 - \phi_{Y_1}(u))^k \leq 2\Delta + C\Delta^2.$$

It follows that

$$\|f_{\Delta,m} - g_{\Delta,m}\|^2 \leq \frac{e^4(2\Delta + C\Delta^2)^2}{2\pi} \int_{\mathbb{R}} |\phi_{X_{\Delta}}(u)|^2 du$$

Regarding the term $\|\phi_{X_{\Delta}}\|_2^2$, using (12) in Lemma 2 we obtain

$$\begin{aligned} \|\phi_{X_{\Delta}}\|_2^2 &\leq \int_{|u| \geq 1} e^{-2c\Delta|u|^{\alpha}} du + \int_{|u| \leq 1} |\phi_{X_{\Delta}}(u)|^2 du \leq 2 \int_1^{\infty} e^{-2c\Delta u^{\alpha}} du + 2 \\ &= \frac{2\Delta^{-1/\alpha}}{\alpha} \int_{\Delta}^{\infty} e^{-2cv} v^{\frac{1}{\alpha}-1} dv + 2 \leq K\Delta^{-1/\alpha}, \end{aligned}$$

for some positive constant K , which may depend on α . Gathering both terms we derive that

$$\|f_{\Delta,m} - g_{\Delta,m}\|^2 \leq K\Delta^{2-\frac{1}{\alpha}}. \quad (9)$$

Gathering all terms we derive

$$\begin{aligned} \mathbb{E}[\|\tilde{g}_{\Delta,m} - g_{\Delta}\|^2] &= \|f_{\Delta,m} - g_{\Delta}\|^2 + \mathbb{E}[\|\hat{g}_{\Delta,m} - f_{\Delta,m}\|^2] \\ &\leq C \left(\|g_{\Delta,m} - g_{\Delta}\|^2 + \frac{m}{n} + \Delta^{2-\frac{1}{\alpha}} \right), \end{aligned}$$

for some positive constant C . Moreover, one can write using (6)

$$\|g_{\Delta,m} - g_{\Delta}\|^2 \leq \frac{1}{\pi} \int_m^{\infty} e^{-2c\Delta u^\alpha} du = \frac{1}{\pi(2c\alpha\Delta)^{\frac{1}{\alpha}}} \Gamma\left(\frac{1}{\alpha}, 2c\Delta m^\alpha\right)$$

which completes the proof. \square

As discussed in Remark 1, in order to establish the rate of convergence of $\tilde{g}_{\Delta,m}$ in L^2 norm one needs to choose a threshold m which realises a bias-variance tradeoff. Under the asymptotic $m\Delta^\alpha \rightarrow \infty$, it holds that $\Gamma(\frac{1}{\alpha}, 2c\Delta m^\alpha)\Delta^{-\frac{1}{\alpha}} \sim \Delta^{-1}m^{1-\alpha}e^{-2c\Delta m^\alpha}$. To find a trade-off between bias and variance we look for m solution of $\Delta^{-1}m^{1-\alpha}e^{-2c\Delta m^\alpha} = m/n$ which leads to $m_* = \frac{(\log(2cn))^{1/\alpha}}{(2c)^{1/\alpha}n\Delta^{1/\alpha}}$, using the properties of the function Lambert W . This choice of m_* allows to derive a rate of convergence of order of

$$\max \left\{ \frac{1}{n\Delta^{\frac{1}{\alpha}}}, \Delta^{2-\frac{1}{\alpha}} \right\}$$

which is of order of $(n\Delta^{1/\alpha})^{-1}$ if $\alpha > 1/2$. Furthermore notice that, for $\alpha \leq 1/2$, the consistency of $\tilde{g}_{\Delta,m}$ is not ensured. Finally, we observe that it is always possible to estimate g_{Δ} with a rate of order in L^2 norm for any $\alpha \in (0, 2)$ by means of the estimator $\hat{g}_{\Delta,m}$ defined in (5). However, such an estimator requires the knowledge the law of X_{Δ}^B , whereas this assumption is not needed to define $\tilde{g}_{\Delta,m}$.

2.3 Adaptation procedure

We propose an adaptive procedure to select m for the estimator $\hat{g}_{\Delta,m}$ defined in (5) and that enables to attain the bound of Theorem 1. This procedure is a penalization procedure inspired by the one proposed in [14]. Note that it can be straightforwardly adapted to select m for the estimator $\tilde{g}_{\Delta,m}$ defined in (8).

Consider the space $S_m = \{t \in L^2(\mathbb{R}), \text{supp}(\mathcal{F}(t)) \subset [-m, m]\}$. This space is generated by an orthonormal basis defined by

$$\psi_{m,j}(x) = \sqrt{\pi m} \psi(mx - j), \quad j \in \mathbb{Z} \quad \psi(x) = \frac{\sin(x)}{\pi x}. \quad (10)$$

Indeed $\mathcal{F}\psi_{m,j}(u) = \sqrt{\pi} \frac{e^{iuj/m}}{\sqrt{m}} \mathbf{1}_{[-m,m]}(u)$ and it holds using Plancherel

$$\langle \psi_{m,j}, \psi_{m,k} \rangle = \frac{1}{2\pi} \langle \mathcal{F}\psi_{m,j}, \mathcal{F}\psi_{m,k} \rangle = \frac{1}{2m} \int_{-m}^m e^{\frac{iu}{m}(j-k)} du = \delta_{jk}.$$

Therefore, we have the following decomposition of

$$\widehat{g}_{\Delta,m} = \sum_{j \in \mathbb{Z}} \widehat{a}_{m,j} \psi_{m,j}, \quad \widehat{a}_{m,j} = \langle \widehat{g}_{\Delta,m}, \psi_{m,j} \rangle = \frac{\sqrt{\pi}}{2\sqrt{m}} \int_{-m}^m \widehat{\phi}_{Z_{\Delta}}(u) e^{-\frac{iuj}{m}} du.$$

Using either Plancherel or this series representation, we get

$$\|\widehat{g}_{\Delta,m}\|^2 = \frac{1}{2\pi} \int_{-m}^m |\widehat{\phi}_{Z_{\Delta}}(u)|^2 du = \sum_{j \in \mathbb{Z}} |\widehat{a}_{m,j}|^2.$$

The adaptive procedure is build using penalization techniques, define the contrast for $t \in S_m$,

$$\gamma_n(t) = \|t\|^2 - 2\langle \widehat{g}_{\Delta,m}, t \rangle = \|t\|^2 - \frac{1}{\pi} \int \widehat{\phi}_{Z_{\Delta}}(u) \mathcal{F}t(-u) du$$

for which we easily check that $\widehat{g}_{\Delta,m} = \arg \min_{t \in S_m} \gamma_n(t)$ and $\gamma_n(\widehat{g}_{\Delta,m}) = -\|\widehat{g}_{\Delta,m}\|^2$. Considering a collection $(S_m, m = 1, \dots, n)$ we select adaptively m satisfying

$$\widehat{m} = \arg \min_{m \in \{1, \dots, n\}} (\gamma_n(\widehat{g}_{\Delta,m}) + \text{pen}(m)), \quad \text{with } \text{pen}(m) = \kappa e^{4\lambda\Delta} \frac{m}{n}. \quad (11)$$

Theorem 3. *The adaptive estimator $\widehat{g}_{\Delta, \widehat{m}}$ defined in (5) with \widehat{m} defined in (11) for $\kappa > 32/(3\pi)$ satisfies for a positive constant C*

$$\mathbb{E}[\|\widehat{g}_{\Delta, \widehat{m}} - g_{\Delta}\|^2] \leq 3 \inf_{m \in \{1, \dots, n\}} (\mathbb{E}[\|\widehat{g}_{\Delta,m} - g_{\Delta}\|^2] + \text{pen}(m)) + \frac{C}{n}.$$

Proof of Theorem 3. By definition we get for $g_{\Delta,m}$ the orthogonal projection of g_{Δ} on S_m that

$$\gamma_n(\widehat{g}_{\Delta, \widehat{m}}) + \text{pen}(\widehat{m}) \leq \gamma_n(\widehat{g}_{\Delta,m}) + \text{pen}(m).$$

Using that

$$\begin{aligned} \gamma_n(t) - \gamma_n(s) &= \|t - g_{\Delta}\|^2 - \|s - g_{\Delta}\|^2 - 2\langle g_{\Delta}, t - s \rangle - \frac{1}{\pi} \langle \widehat{\phi}_{Z_{\Delta}}, \mathcal{F}(t - s) \rangle \\ &= \|t - g_{\Delta}\|^2 - \|s - g_{\Delta}\|^2 - 2\nu_n(t - s), \end{aligned}$$

where using Plancherel

$$\nu_n(t) = \frac{1}{2\pi} \langle \widehat{\phi}_{Z_{\Delta}} - \phi_{Z_{\Delta}}, \mathcal{F}(t) \rangle = \frac{1}{2\pi n} \sum_{j=1}^n \int \left(\frac{e^{iu(X_{j\Delta} - X_{(j-1)\Delta})}}{\phi_{X_{\Delta}^B}(u)} - \frac{\mathbb{E}[e^{iuX_{\Delta}}]}{\phi_{X_{\Delta}^B}(u)} \right) \mathcal{F}t(-u) du.$$

Combining these results implies that

$$\begin{aligned}
\|\widehat{g}_{\Delta, \widehat{m}} - g_{\Delta}\|^2 &\leq \|\widehat{g}_{\Delta, m} - g_{\Delta}\|^2 + 2\nu_n(\widehat{g}_{\Delta, \widehat{m}} - \widehat{g}_{\Delta, m}) + \text{pen}(m) - \text{pen}(\widehat{m}) \\
&= \|\widehat{g}_{\Delta, m} - g_{\Delta}\|^2 + 2\|\widehat{g}_{\Delta, \widehat{m}} - \widehat{g}_{\Delta, m}\| \nu_n \left(\frac{\widehat{g}_{\Delta, \widehat{m}} - \widehat{g}_{\Delta, m}}{\|\widehat{g}_{\Delta, \widehat{m}} - \widehat{g}_{\Delta, m}\|} \right) + \text{pen}(m) - \text{pen}(\widehat{m}) \\
&\leq \|\widehat{g}_{\Delta, m} - g_{\Delta}\|^2 + 2\|\widehat{g}_{\Delta, \widehat{m}} - \widehat{g}_{\Delta, m}\| \sup_{t \in S_m + S_{\widehat{m}}, \|t\|=1} \nu_n(t) + \text{pen}(m) - \text{pen}(\widehat{m}) \\
&\leq \|\widehat{g}_{\Delta, m} - g_{\Delta}\|^2 + \frac{1}{4}\|\widehat{g}_{\Delta, \widehat{m}} - \widehat{g}_{\Delta, m}\|^2 + 4 \sup_{t \in S_m + S_{\widehat{m}}, \|t\|=1} \nu_n(t)^2 + \text{pen}(m) - \text{pen}(\widehat{m}) \\
&\leq \frac{3}{2}\|\widehat{g}_{\Delta, m} - g_{\Delta}\|^2 + \frac{1}{2}\|\widehat{g}_{\Delta, \widehat{m}} - g_{\Delta}\|^2 + 4 \left(\sup_{t \in S_m + S_{\widehat{m}}, \|t\|=1} \nu_n(t)^2 - p(m, \widehat{m}) \right) \\
&\quad + 4p(m, \widehat{m}) + \text{pen}(m) - \text{pen}(\widehat{m}),
\end{aligned}$$

where $p(m, m') = \frac{4}{\pi} e^{4\lambda\Delta} (m \vee m')/n$ is fixed by applying the Talagrand inequality to ν_n (see the following Lemma 1). Note that $S_m + S_{m'} = S_{m \vee m'}$.

Lemma 1. *There exists a positive constant C such that*

$$\mathbb{E} \left(\sup_{t \in S_{m \vee \widehat{m}}, \|t\|=1} |\nu_n(t)|^2 - 4e^{4\lambda\Delta} \frac{m \vee \widehat{m}}{\pi n} \right)_+ \leq \frac{C}{n}.$$

Plugging this result in above inequalities implies that

$$\frac{1}{2} \mathbb{E}[\|\widehat{g}_{\Delta, \widehat{m}} - g_{\Delta}\|^2] \leq \frac{3}{2} (\mathbb{E}[\|\widehat{g}_{\Delta, m} - g_{\Delta}\|^2] + \text{pen}(m)) + \frac{4C}{n} + \mathbb{E}[4p(m, \widehat{m}) - \text{pen}(\widehat{m})] - \frac{1}{2} \text{pen}(m),$$

using that for $\kappa > 32/(3\pi)$

$$4p(m, \widehat{m}) - \text{pen}(\widehat{m}) - \frac{1}{2} \text{pen}(m) = \frac{e^{4\lambda\Delta}}{n} \left(\frac{16}{\pi} (m \vee \widehat{m}) - \kappa \left(\frac{1}{2} m + \widehat{m} \right) \right) \leq 0.$$

Taking the infimum over m complete the proof. \square

Proof of Lemma 1. We apply the Talagrand inequality Lemma 3. Note that we can write

$$\nu_n(t) = \frac{1}{n} \sum_{j=1}^n (f_t(X_{j\Delta} - X_{(j-1)\Delta}) - \mathbb{E}[f_t(X_{\Delta})])$$

where for $t \in S_{m \vee m'}$,

$$f_t(x) = \frac{1}{2\pi} \int_{-m \vee m'}^{m \vee m'} \frac{e^{iux}}{\phi_{X_{\Delta}^B}(u)} \mathcal{F}t(-u) du.$$

For that we compute the three positive constants M , H and v introduced in Lemma 3. First note that as $\|t\| = 1$ we get using Cauchy-Schwarz and (3) that

$$\sup_{t \in S_{m \vee m'}, \|t\|=1} \|f_t\|_{\infty} \leq \left\| \frac{1}{\phi_{X_{\Delta}^B}} \right\|_{\infty} \frac{1}{2\pi} \sqrt{2(m \vee m')} \sqrt{\int_{-m \vee m'}^{m \vee m'} |\mathcal{F}t(u)|^2 du} \leq e^{2\lambda\Delta} \frac{\sqrt{m \vee m'}}{\sqrt{\pi}} =: M.$$

Using similar arguments we get

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in S_{m \vee m'}, \|t\|=1} |\nu_n(t)| \right)^2 &\leq \mathbb{E} \left[\sup_{t \in S_{m \vee m'}, \|t\|=1} \nu_n(t)^2 \right] \leq \frac{1}{2\pi} \int_{-m \vee m'}^{m \vee m'} \frac{\mathbb{E} [|\widehat{\phi}_{X_\Delta}(u) - \phi_{X_\Delta}(u)|^2]}{|\phi_{X_\Delta^B}(u)|^2} du \\ &\leq \frac{e^{4\lambda\Delta}}{\pi n} (m \vee m') =: H^2 \end{aligned}$$

Finally for the last term, following [15] we use the basis representation of the estimator to compute v^2 . Indeed, using the basis (10) it holds $t = \sum_{j \in \mathbb{Z}} b_j \psi_{j, m \vee m'}$ with $b_j = \langle t, \psi_{j, m \vee m'} \rangle$ such that $\sum_{j \in \mathbb{Z}} b_j^2 = 1$, and we can write

$$\begin{aligned} \mathbb{V}(f_t(X_\Delta)) &\leq \frac{1}{4\pi^2} \iint_{-m \vee m'}^{m \vee m'} \frac{\phi_{X_\Delta}(u-v)}{\phi_{X_\Delta^B}(u)\phi_{X_\Delta^B}(-v)} \mathcal{F}t(-u) \mathcal{F}t(v) dudv \\ &= \frac{1}{4\pi^2} \sum_{j,k \in \mathbb{Z}} b_j b_k \iint_{-m \vee m'}^{m \vee m'} \frac{\phi_{X_\Delta}(u-v)}{\phi_{X_\Delta^B}(u)\phi_{X_\Delta^B}(-v)} \mathcal{F}\psi_{j, m \vee m'}(-u) \mathcal{F}\psi_{k, m \vee m'}(v) dudv \\ &\leq \frac{1}{4\pi^2} \sqrt{\sum_{j,k \in \mathbb{Z}} b_j^2 b_k^2 \sum_{j,k \in \mathbb{Z}} \left| \iint_{-m \vee m'}^{m \vee m'} \frac{\phi_{X_\Delta}(u-v)}{\phi_{X_\Delta^B}(u)\phi_{X_\Delta^B}(-v)} \mathcal{F}\psi_{j, m \vee m'}(-u) \mathcal{F}\psi_{k, m \vee m'}(v) dudv \right|^2} \\ &= \frac{1}{4\pi^2} \sqrt{\iint_{-m \vee m'}^{m \vee m'} \left| \frac{\phi_{X_\Delta}(u-v)}{\phi_{X_\Delta^B}(u)\phi_{X_\Delta^B}(-v)} \right|^2 dudv} \\ &\leq \frac{e^{4\lambda\Delta}}{4\pi^2} \sqrt{\iint_{-m \vee m'}^{m \vee m'} |\phi_{X_\Delta}(u-v)|^2 dudv} \leq \frac{\sqrt{\pi} e^{4\lambda\Delta}}{2\pi^2} \sqrt{m \vee m'} \|f_{X_\Delta}\|, \end{aligned}$$

where we used at the third line the Cauchy-Schwarz inequality on the index $\lambda = (j, k)$ and at the penultimate equality that for a bi-variate function $\phi(u, v)$ its norm can be computed as $\|\phi\|^2 = \sum_{j,k \in \mathbb{Z}} \langle \phi, \psi_{j, m \vee m'} \otimes \psi_{k, m \vee m'} \rangle^2$. Therefore,

$$\sup_{t \in S_{m \vee m'}, \|t\|=1} \frac{1}{n} \sum_{j=1}^n \mathbb{V}(f_t(X_{j\Delta} - X_{(j-1)\Delta})) \leq \frac{\sqrt{\pi} e^{4\lambda\Delta}}{2\pi^2} \sqrt{m \vee m'} \|f_{X_\Delta}\| =: v^2.$$

It follows from the Talagrand inequality (see Lemma 3, $\delta = 1/2$) that there exist positive constants C_1 , C_2 and C_3 such that

$$\mathbb{E} \left(\sup_{t \in S_{m \vee m'}, \|t\|=1} |\nu_n(t)|^2 - 4e^{4\lambda\Delta} \frac{m \vee m'}{\pi n} \right)_+ \leq C_1 \left(\frac{\sqrt{m \vee m'}}{n} e^{-C_2 \sqrt{m \vee m'}} + \frac{m \vee m'}{n^2} e^{-C_3 \sqrt{n}} \right).$$

To complete the proof, we write

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in S_{m \vee \widehat{m}}, \|t\|=1} |\nu_n(t)|^2 - 4e^{4\lambda\Delta} \frac{m \vee \widehat{m}}{\pi n} \right)_+ &\leq \sum_{m'=1}^n \mathbb{E} \left(\sup_{t \in S_{m \vee m'}, \|t\|=1} |\nu_n(t)|^2 - 4e^{4\lambda\Delta} \frac{m \vee m'}{\pi n} \right)_+ \\ &\leq \frac{C_1}{n} \sum_{m'=1}^n \left(\sqrt{m \vee m'} e^{-C_2 \sqrt{m \vee m'}} + \frac{m \vee m'}{n} e^{-C_3 \sqrt{n}} \right) \leq \frac{C}{n}, \end{aligned}$$

which completes the proof. \square

3 Numerical example

We briefly illustrate our adaptive estimator $\hat{g}_{\Delta, \hat{m}}$ defined in (5) with \hat{m} defined in (11) on the symmetric α -stable Lévy process whose Lévy density is given by

$$\frac{\nu(dx)}{dx} = \frac{1}{|x|^{1+\alpha}} \mathbf{1}_{x \neq 0}.$$

Note that $(\mathcal{A}_{M, \alpha})$ is met for $(M, \alpha) = (1, \alpha)$. In that case realizations of X_Δ can be obtained using that $X_1 \sim S_\alpha \left((2\Gamma(1-\alpha) \cos(\frac{\pi\alpha}{2}))^{\frac{1}{\alpha}}, 0, 0 \right)$ and that $X_\Delta \stackrel{d}{=} \Delta^{1/\alpha} X_1$. The calibration of the constant κ in the penalty is done by preliminary simulation experiments. This constant is selected as $\kappa = 4 > 32/(3\pi)$.

In the sequel we consider cases where Δ is small and $\alpha \geq \frac{1}{2}$ as in that case (9) (making $m = \infty$ which is allowed) suggests that g_Δ should resemble the density of increments f_Δ . It is not possible to evaluate the associated L_2 loss easily. Indeed as has already been pointed out, even knowing the Lévy density, it is not possible to obtain an explicit formula for g_Δ . Sharper procedure to asses the performances of $\hat{g}_{\Delta, m}$ are under study.

Hereafter, we illustrate our procedure by plotting a cluster of 50 estimators in different cases, allowing us to appreciate its variability. To provide points of comparison, we also plot the histogram of the increments X_Δ , which in these regimes should be close to our curves, as well as the histogram of the increments $X_\Delta \{|X_\Delta| \leq 1\}$ which can be drawn by filtering the increments X_Δ that are such $|X_\Delta| \leq 1$. The different estimators and the histograms are computed on the same datasets.

Results and comments The results are displayed in Figures 1 and 2. All 50 density estimates are of similar shape which confirms the stability of the procedure. We observe that the estimates are closer to each other for $\alpha = 1.8$ than for $\alpha = 1.1$ and $\alpha = 0.7$ which can be explained as the approximation error, that is of order $(n\Delta^{1/\alpha})^{-1}$, decreases when α increases. For all the three values of α the histogram of X_Δ is close to the estimated densities far from 0, but is less precise in a neighborhood of 0. For $\alpha = 0.7, 1.1$ the histogram is less accurate since $\mathbb{P}(N_\Delta = 0) = e^{-\frac{2\Delta}{\alpha}}$ is smaller. The histogram of $X_\Delta \{|X_\Delta| \leq 1\}$ is close to the estimates for $\alpha = 0.7, 1.1$ and seems to be a better fit. But it is completely off for $\alpha = 1.8$ because in a high activity framework increments X_Δ are more likely to go above 1.

We also observe that the larger is α the larger is the support of g_Δ and the smaller is the estimated cutoff \hat{m} in average. When Δ gets close to 2 the estimated curve is visually similar to the density of a Gaussian random variable. Note also that when Δ gets smaller the estimated $\|\hat{g}_{\Delta, \hat{m}}\|_\infty$ increases. This is consistent with Lemma 2 which states that $\|g_\Delta\|_\infty \leq C\Delta^{-1/\alpha}$. Numerically, this upper bound in $O(\Delta^{-1/\alpha})$ seems to be of the correct order up to a constant.

4 Appendix

Smoothness of the Lévy density. The result below directly follows from Lemma 2.3 in Picard [28].

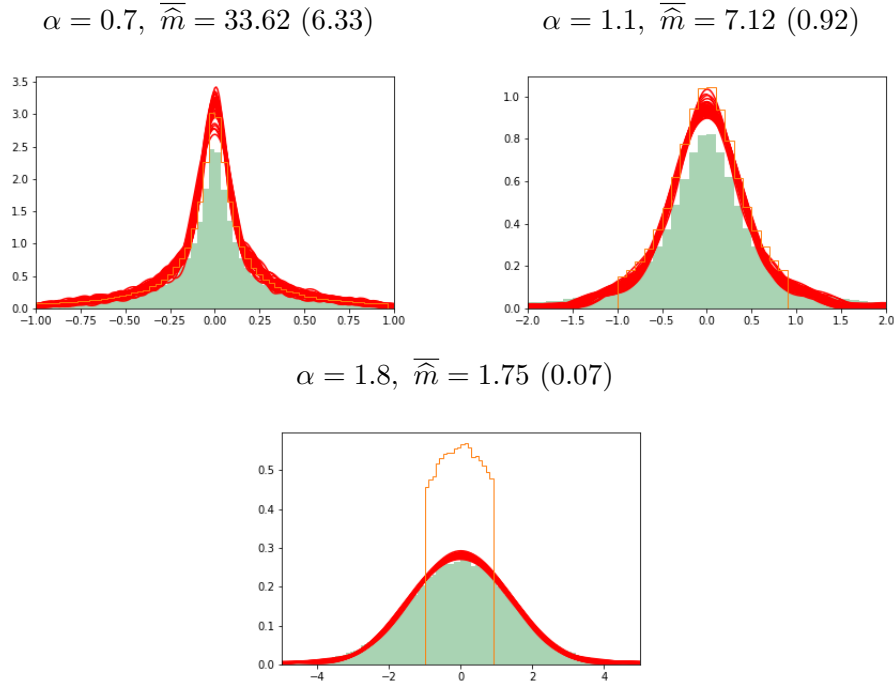


Figure 1: Estimation of g_Δ with $\Delta = 0.1$ from $n = 5000$ increments of X (red lines, average estimated \widehat{m} and associated standard deviation given in parenthesis) with the histogram of X_Δ (green) and the histogram of $X_\Delta/|X_\Delta| \leq 1$ (yellow).

Lemma 2. *Let X be a Lévy process satisfying $(\mathcal{A}_{M,\alpha})$. Then, denoting by $\phi_t(u) = \mathbb{E}[e^{iuX_t}]$, it holds*

$$|\phi_t(u)| \leq e^{-ct|u|^\alpha}, \quad \forall |u| \geq 1, \quad \forall t > 0, \quad (12)$$

for some $c > 0$. Furthermore, X_t has a smooth density g_t with all the derivatives uniformly bounded :

$$\sup_{x \in \mathbb{R}} |g_t^{(k)}(x)| \leq \frac{1}{2} \int_{\mathbb{R}} |\omega|^k |\phi_t(\omega)| d\omega \leq Ct^{-\frac{k+1}{\alpha}}, \quad \forall k \geq 0, \quad \forall t > 0, \quad (13)$$

where C is a positive constant only depending on k and c appearing in (12).

The Talagrand inequality. The result below follows from the Talagrand concentration inequality given in [24] and arguments in [5] (see the proof of their Corollary 2 page 354).

Lemma 3. *(Talagrand Inequality) Let Y_1, \dots, Y_n be independent random variables and let \mathcal{F} be a countable class of uniformly bounded measurable functions. Consider ν_n , the centered empirical process defined by*

$$\nu_n(f) = \frac{1}{n} \sum_{i=1}^n [f(Y_i) - \mathbb{E}(f(Y_i))]$$

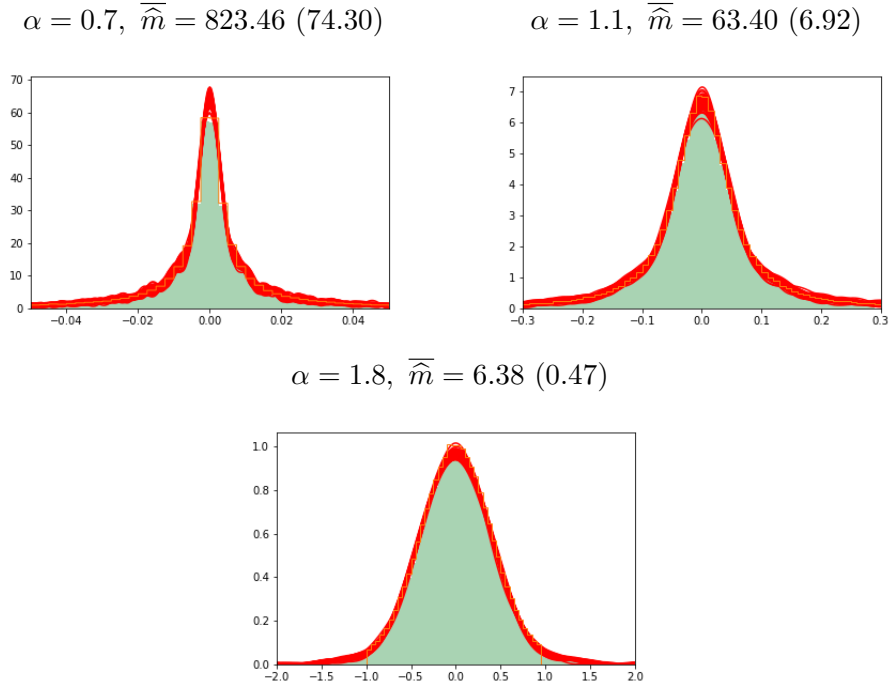


Figure 2: Estimation of g_Δ with $\Delta = 0.01$ from $n = 5000$ increments of X (red lines, average estimated \widehat{m} and associated standard deviation given in parenthesis) with the histogram of X_Δ (green) and the histogram of $X_\Delta/|X_\Delta| \leq 1$ (yellow).

for $f \in \mathcal{F}$. Assume there exists three positive constants M , H and v such that

$$\sup_{f \in \mathcal{F}} \|f\|_\infty \leq M, \quad \mathbb{E} \left[\sup_{f \in \mathcal{F}} |\nu_n(f)| \right] \leq H, \quad \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{k=1}^n \text{Var}(f(Y_k)) \leq v^2.$$

Then, for any $\delta > 0$ the following holds

$$\mathbb{E} \left[\sup_{f \in \mathcal{F}} |\nu_n(f)|^2 - 2(1 + 2\delta)H^2 \right]_+ \leq \frac{4}{K_1} \left(\frac{v^2}{n} \exp \left(-K_1 \delta \frac{nH^2}{v^2} \right) + \frac{49M^2}{K_1 n^2 C^2(\delta)} \exp \left(-\frac{K_1 C(\delta) \sqrt{2\delta} nH}{7M} \right) \right),$$

with $C(\delta) = \sqrt{1 + \delta} - 1$ and $K_1 = 1/6$.

By standard density arguments, this result can be extended to the case where \mathcal{F} is a unit ball of a linear normed space, after checking that $f \mapsto \nu_n(f)$ is continuous and \mathcal{F} contains a countable dense family.

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