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Pricing and Hedging Guaranteed Equity Securities

David Lee

ABSTRACT

Equity-linked securities with a guaranteed return gain popularity in financial market. The contract depends on the performance of a basket of equity components averaged over a certain period, and also guarantees the investor a minimum return. This paper presents a new method for valuing the guaranteed equity-linked securities. We compute the security's price, corresponding hedge ratios, and risk sensitivities. The model appears to be accurate over a wide range of underlying security parameters, based on numerical studies.

Key Words: Guaranteed equity security, hedge ratio, risk sensitivity, asset pricing, derivative valuation, risk management.

JEL Classification: E44, G21, G12, G24, G32, G33, G18, G28

1 Introduction

Equity-linked products with a guaranteed return become very popular in markets either as investment instruments or life insurance policies. All of these products have a common feature: the contract pays off a pre-determined amount plus a payment linked to an underlying equity, or basket or index. If the underlying performs well over the lifetime of the contract, a high return is paid back. This guarantee feature allows investors to benefit from the upside potential of equity growth without full exposure to the downside risk.

The market for these products has evolved significantly from traditional single stock equity derivatives to basket-based products. Traditional forms of equity derivatives were generally used to raise equity capital for companies, while these structured products are increasingly being used in the process of debt fund raising through new issue arbitrage techniques. Most forms of equity linked structured products have their performance linked to an underlying benchmark such as an equity index, real estate index, commodities, interest rates etc.

There is a rich literature on the subject of equity-linked securities. Kotadia (2021) provides a brief background on the pricing of equity-linked structured products and discusses issues around valuation of these products.

Henderson & Pearson (2011) argue that the issuers exploit the behavioral bias in investors mind to sell structured products and analyze US based structured products for returns compared to risk free deposits. Roger (2008) analyzes capital protected notes for loss averse investors and suggested that the reason for price premium is to account for service provided by the issuers.

Hens (2012) analyzes the reason for retail investors buying structured products and concluded the probability mis-estimation and behavioral biases play dominant role in investors mind while buying such products. Deng, Husson & McCann (2015) published a detailed valuation guide for various types of structured products and derived various payoff patterns for structured products.

There are risks embedded in structured notes. Johnson and Stulz (1987) consider the default risk for valuing options and conclude that the effect of credit risk can be significant. The value of a vulnerable European option can fall with the time to maturity, the interest rate, and the variance of the underlying asset.

Hull and White (1995) assume that other than the option, the option writer could have some equal ranking claims on their assets upon default. The default can happen at any time before the option expiration date. Default is set to occur once the assets of the option writer fall below that fixed level. The results of their numerical examples show that the impact of default risk on the price of an American option is less than that on the price of a European option.

Klein (1996) explained that there exists an endogenous variable represented by the proportion of nominal claims paid out in default. Default is not restricted to occur only at maturity, and the recovery rate setting will be effective if the option writer's asset value hasn't rebounded above the default boundary at maturity date of the European option.

Klein and Inglis (2001) incorporate a default boundary which depends on the potential liability of the written option and the option writer's liabilities. Instead of deriving an analytic solution, a three-dimensional binomial tree approach is employed.

Klein and Yang (2010) develop the pricing procedure for vulnerable American options using a numerical method. They combine a default boundary at the time of maturity and a default barrier before maturity which is variable and linked to the payoff on the option.

This article presents a new method for valuing equity-linked securities with a guaranteed return. There is an embedded Asian style option on the basket of indices within the security. In this paper we focus on valuing the embedded option and its sensitivity to different market conditions.

Using the model, we can price a type of segregated fund whose value at maturity is guaranteed to be greater than the starting invested principal. The fund holder incurs, in addition to a management fee, a protection fee towards the fund's guaranteed minimum value at maturity. We also show the numerical results of the valuation of the guaranteed funds.

The rest of this paper is organized as follows: The form of the guaranteed equity security is described in Section 2. Section 3 presents the valuation and risk models; Section 4 discusses the numerical results. Section 5 elaborates the valuation of a segregated fund. The conclusions are provided in Section 6.

2 Product

We consider a security whose payoff depends on the return from a finite number, M , of equity-linked indices. Let I_t^j , for $j = 1, \dots, M$, denote the price of the j^{th} index at time equal to t , and let ω_j denote a fixed, positive weight corresponding to this index. Next let T denote the security's maturity. Furthermore let $\{t_1, \dots, t_N\}$, where $N > 0$ and $0 < t_1 < \dots < t_N \leq T$, be a finite set of observation times.

Finally let P denote an amount guaranteed to the security holder at maturity. The payoff at maturity depends on the weighted sum, over each index, of the relative change in the arithmetic average of the index's price, with respect to the set observation points above, from the index's initial level. Formally the payoff at maturity is given by

$$\max \left(P + \sum_{i=1}^M \omega_i \frac{\frac{1}{N} \sum_{j=1}^N I_{t_j}^i - I_0^i}{I_0^i}, P \right) = P + \max \left(\sum_{i=1}^M \omega_i \frac{\frac{1}{N} \sum_{j=1}^N I_{t_j}^i - I_0^i}{I_0^i}, 0 \right). \quad (2.1)$$

Next let

$$Z_t = \sum_{j=1}^M \alpha_j I_t^j$$

denote the price at time t of a basket of the equity-linked indices above; here $\alpha_j = \frac{\omega_j}{I_0^j}$ is the ratio of the j^{th} index's weight over the index's initial level. Then the payoff (2.1) is equivalent to

$$P + \max\left(\frac{1}{N} \sum_{i=1}^N Z_{t_i} - \sum_{i=1}^M \omega_i, 0\right),$$

which is the sum of the payoff from an Asian style option sampled at the discrete points above plus the guaranteed component.

3 Model Description

In this section we present our model for pricing an Asian style option with payoff at maturity, T , of the form

$$\max\left(\frac{1}{N} \sum_{i=1}^N Z_{t_i} - \sum_{i=1}^M \omega_i, 0\right). \quad (3.1)$$

Here we assume that each index follows geometric Brownian motion with drift under its respective risk-neutral probability measure. Each index is then expressed under the domestic risk-neutral probability measure by a corresponding change of measure. Observe that, under these assumptions, the random variable

$$Y = \frac{1}{N} \sum_{i=1}^N Z_{t_i} \quad (3.2)$$

is not log-normally distributed. This, then, makes it mathematically difficult to value the payoff (3.1) using analytical techniques.

The standard Levy approach towards valuing the payoff (3.1) is to approximate Y in (3.2) by a log-normally distributed random variable. Here the defining parameters for the log-normal random variable are uniquely determined by matching its first two moments with those of Y .

The option value is then given from an analytical formula by taking the expected value of the payoff (3.1), but where the underlying security value, Y , is replaced by that of the log-normally distributed random variable. As we will see in Section 4, however, this method may produce largely inaccurate pricing and hedging results depending on option's tenor and volatility parameters.

Our valuation approach aims to match more moments, and can be viewed as an extension of Levy's. Specifically, we approximate Y in (3.2) by a shifted log-normal random variable, of the form

$$a + e^{b+c\varepsilon}, \quad (3.3)$$

where a and b are constants, c is a positive constant, and ε is a standard, normally distributed random variable. Here a , b and c are uniquely determined, with analytical form, by matching the first three moments of Y with those of (3.3). An analytical, approximate option pricing formula is then derived by taking the expected value of the payoff (3.1), but where the underlying security's value is replaced by that of the shifted log-normal random variable.

Assume that, under the domestic risk neutral probability measure, the process $\{I_t^j | t \geq 0\}$, for $j = 1, \dots, M$, satisfies a stochastic differential equation of the form

$$dI_t^j = I_t^j (\mu_j dt + \sigma_j dW_t^j)$$

where μ_j is a constant drift parameter, σ_j is a constant volatility parameter, and $\{W_t^j | t \geq 0\}$ is a standard Brownian motion. Suppose also that the Brownian motions $\{W_t^j | t \geq 0\}$ and $\{W_t^k | t \geq 0\}$, for $j, k \in \{1, \dots, M\}$, have a constant instantaneous correlation coefficient, ρ_{jk} . The first moment of Y then equals

$$E(Y) = \frac{1}{N} \sum_{k=1}^N \sum_{i=1}^M \alpha_i I_0^i e^{\mu_i t_k}.$$

Furthermore, the second moment of Y is given by

$$E(Y^2) = \frac{1}{N^2} \sum_{k, l \in \{1, \dots, N\}} E(Z_{t_k} Z_{t_l}),$$

where

$$E(Z_{t_k} Z_{t_l}) = \sum_{m, n \in \{1, \dots, M\}} \alpha_m \alpha_n I_0^m I_0^n e^{\mu_m t_k + \mu_n t_l + \sigma_m \sigma_n \rho_{mn} \min(t_k, t_l)}.$$

Also, the third moment of Y equals

$$E(Y^3) = \frac{1}{N^3} \sum_{l, m, n \in \{1, \dots, N\}} E(Z_{t_l} Z_{t_m} Z_{t_n}),$$

where

$$E(Z_{t_1} Z_{t_m} Z_{t_n}) = \sum_{i,j,k \in \{1, \dots, M\}} \alpha_i \alpha_j \alpha_k I_0^i I_0^j I_0^k e^{\mu_i t_1 + \mu_j t_m + \mu_k t_n + \sigma_i \sigma_j \rho_{ij} \min(t_1, t_m) + \sigma_i \sigma_k \rho_{ik} \min(t_1, t_n) + \sigma_j \sigma_k \rho_{jk} \min(t_m, t_n)}.$$

By matching the first three moments of Y with those of the shifted log-normal random variable (3.3), we obtain the system of nonlinear equations

$$E(Y) = a + e^{b + \frac{c^2}{2}}, \quad (3.4a)$$

$$E(Y^2) = a^2 + 2ae^{b + \frac{c^2}{2}} + e^{2b + 2c^2}, \quad (3.4b)$$

$$E(Y^3) = a^3 + 3a^2 e^{b + \frac{c^2}{2}} + 3ae^{2b + 2c^2} + e^{3b + \frac{9}{2}c^2}, \quad (3.4c)$$

for the unknowns a , b and c . It can be shown that, under certain conditions, the nonlinear system of equations above has a closed-form, unique, real solution.

Let r denote the risk-free interest rate (see <https://finpricing.com/lib/IrCurveIntroduction.html>) for a term equal to the option maturity, T . The Asian style option with payoff (3.1) then has value

$$\Omega = e^{-rT} E \left(\max \left(\frac{1}{N} \sum_{i=1}^N Z_{t_i} - \sum_{i=1}^M \omega_i, 0 \right) \right), \quad (3.7a)$$

which we approximate by

$$\tilde{\Omega} = e^{-rT} E \left(\max \left(a + e^{b+c\varepsilon} - \sum_{i=1}^M \omega_i, 0 \right) \right). \quad (3.7b)$$

Let X denote $\sum_{i=1}^M \omega_i$. If $X > a$, then (3.7b) equals

$$\begin{aligned}
& e^{-rT} \int_{\frac{\log(X-a)-b}{c}}^{+\infty} (a + e^{b+cy} - X) f(y) dy \\
& = e^{-rT} \left((a - X) n\left(\frac{b - \log(X - a)}{c}\right) + e^{\frac{b+c^2}{2}} n\left(c + \frac{b - \log(X - a)}{c}\right) \right),
\end{aligned}$$

where $f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$ is the probability density function for a standard, normally distributed random variable and $n(z) = \int_{-\infty}^z f(y) dy$ is the corresponding cumulative distribution function. If $X \leq a$, then (3.7b) equals

$$e^{-rT} \left(a - X + e^{\frac{b+c^2}{2}} \right).$$

We may be interested in the option delta, $\frac{\partial \Omega}{\partial I_0^j}$, and the option Vega, $\frac{\partial \Omega}{\partial \sigma_j}$, for $j = 1, \dots, M$. These hedge ratios are respectively approximated by $\frac{\partial \tilde{\Omega}}{\partial I_0^j}$ and $\frac{\partial \tilde{\Omega}}{\partial \sigma_j}$, for $j = 1, \dots, M$, and are obtained from direct differentiation of $\tilde{\Omega}$ using the chain rule.

Risk sensitivities or risk Greeks are the measures of the value of a financial product reaction to changes in underlying factors. Greeks are vital for risk management. They can help financial market participants isolating risk, hedging risk and explaining profit & loss.

The delta, $\frac{\partial f}{\partial S_0} X_0$ can be computed using the $O(\varepsilon)$ approximation,

$$\frac{f(\bar{F}_q(S_0 + \varepsilon S_0)) - f(\bar{F}_q(S_0))}{\varepsilon S_0} X_0,$$

where

- X_0 is the spot value of the domestic to foreign exchange rate for delivery at t_{s_1} ,
- and
- $\varepsilon=10^{-4}$ is a small relative shift.

The gamma $\frac{\partial^2 f}{\partial S_0^2} X_0$ can be calculated using the $O(\varepsilon)$ finite difference approximation,

$$\frac{f(\bar{F}_q(S_0 + 2\varepsilon S_0)) - 2f(\bar{F}_q(S_0 + \varepsilon S_0)) + f(\bar{F}_q(S_0))}{(\varepsilon S_0)^2} X_0,$$

where $\varepsilon = 10^{-4}$. Onscreen representations of gamma multiply the Murex/MX approximation by S_0 .

The Vega $\sum_{i=1}^n \frac{\partial f(\bar{v})}{\partial v_i}$ can be computed using the finite difference approximation,

$$\frac{f(\bar{v} + \bar{1}\varepsilon) - f(\bar{v})}{\varepsilon}, \quad (6)$$

where $\varepsilon=10^{-3}$ and $\bar{1} = [1, \dots, 1]^T \in R^n$. In the onscreen representation of Vega, the result (6) is shown divided by 100.

4 Numerical Results

It is interesting to compare the accuracy of our approximate option pricing formula, as well as that of a Levy based pricing formula, against a Monte Carlo benchmark. To this end we consider the following Levy based pricing approach. Let U be a log-normally distributed random variable, of the form

$$U = e^{a+b\varepsilon},$$

where a is a constant, b is a positive constant, and ε is a standard, normally distributed random variable. We choose a and b by matching the first two moments of the basket's price at maturity with those of the log-normal random variable U . We then approximate the option's price, (3.7a), by

$$e^{-rT} E \left(\max \left(U - \sum_{i=1}^M \omega_i, 0 \right) \right).$$

We have implemented both our pricing model, described in Section 3, and the Levy based approach above.

As an example, we consider the Asian style option arising from a security dependent on the return from a basket of five indices. Here the payoff, of the form (3.1), depends on the arithmetic average of the basket's price at twelve observation points.

These points are respectively set to the last business day in each of the eleven months that precede the month in which the security matures and the business day that immediately precedes the maturity date. Here the security was issued on July 23, 1998, and matures on July 23, 2003; the valuation date is on June 2, 1999. In Figure 4.1 we show the initial level and corresponding weight for each index, as well as the observation points.

The screenshot shows an Excel spreadsheet with the following data:

Basket Description						
Index #	1	2	3	4	5	
Initial Level	2421.04	391.64	1147.27	15944.36	2913.59	
Weight, %	25	30	10	2.5	2.5	

Observation Points			
Number	Term	Date	
1	3.25	08/31/2002	
2	3.33	09/30/2002	
3	3.41	10/31/2002	
4	3.50	11/30/2002	
5	3.58	12/31/2002	
6	3.67	01/31/2003	
7	3.74	02/28/2003	
8	3.83	03/31/2003	
9	3.91	04/30/2003	
10	3.99	05/31/2003	
11	4.08	06/30/2003	
12	4.14	07/22/2003	

Figure 4.1. Snapshot of basket description screen (here we display two significant digits).

In Figure 4.2 we show pricing results for various embedded options specified from the parameters in Figure 4.2. Here the Volatility Shift parameter indicates a respective relative shift to all original volatility parameter values. The corresponding benchmark option prices, shown in Figure 4.3, are numerical values for Formula (3.7a), which were computed using crude Monte Carlo simulation based on four million sample paths with .7% standard error. We note that, for the case of zero shift to the volatilities, the parameter values for the shifted log-normal random variable,

$$a + e^{b+c\varepsilon},$$

were computed as $a = .3259$, $b = -.6821$ and $c = 0.3332$.

In Figure 4.3 we display various hedge ratios, with respect to the first index, for the option specified from the original parameters shown in Figure 4.2. The hedge ratios based on Formula (3.7b) are from the direct, analytical differentiation of (3.7b). Benchmark hedge ratios are computed using a one-sided finite difference approximation applied to the true option pricing formula, (3.7a); here numerical values for (3.7a) were obtained using crude Monte Carlo simulation based on 4 million sample paths.

Levy based hedge ratios are based on a finite difference approximation. Observe that the relative error in the Levy based vega value from the Monte Carlo benchmark is approximately 37%, while the vega from Formula (3.7b) differs from the benchmark only by 5.8%.

	I	J	K	L	M
24					
25		Volatility	Model Price	Monte-Carlo	Levy
26		Shift, %	based on	Benchmark Price	Price
27			Formula (3.7b)		
28					
29		-50	12.59	12.61	12.61
30		0	13.48	13.50	13.74
31		50	15.10	15.13	15.80
32		100	16.99	17.06	18.37
33					
34					

Figure 4.2. Numerical option pricing results (expressed as a percentage of the notional amount).

	J	K	L	M
38				
39	Hedge Ratio	Model Value based on Formula (3.7b)	Monte-Carlo Benchmark Value	Levy Value
40				
41				
42				
43	Price Delta	0.0083	0.0083	0.0082
44	Vega	2.76	2.94	1.85
45				
46				

Figure 4.3. Hedge ratios with respect to the first index.

5 Segregated Fund Valuation

We consider a type of segregated fund that invests in various foreign and domestic equities and bonds. We assume that the fund provides a maturity guarantee, that is, the fund's price at maturity is assured to be greater than the original invested amount. We also assume that the fund has no dynamic lapse or reset features, and that the holder pays periodic management and protection fees.

We model the fund's value by the price of basket of representative equity and bond-linked indices; the maturity guarantee then measures the net shortfall from the basket's constituent indices.

Specifically suppose that the basket contains a fixed number, N , of indices. Furthermore let I_t^i , for $i = 1, \dots, N$, denote the price of the i^{th} index at time t . Next let P denote the principal amount originally invested in the fund. Assume also that at the fund's outset a percentage, v_i , of the principal is invested in the i^{th} ($i = 1, \dots, N$) index; the initial number of units, u_i , associated with the i^{th} index then equal

$$u_i = \frac{v_i P}{I_0^i}.$$

Let T denote the fund's maturity. Assume that protection and management fees are both collected at a set of times, $\{t_1, \dots, t_M\}$, where $0 < t_1 < \dots < t_M < T$. Suppose also that the protection and management fee are taken, at time t_i ($i=1, \dots, M$), as respective percentages, p_i and m_i , of the fund's price at t_i . The fund's price at maturity then equals

$$\sum_{i=1}^N \omega_i I_T^i \tag{5.1}$$

where $\omega_i = u_i \prod_{j=1}^M [1 - (m_j + p_j)]$.

Suppose that the invested principal, P , is 100% guaranteed at maturity. The payoff at maturity from this guarantee then equals

$$\max\left(P - \sum_{i=1}^N \omega_i I_T^i, 0\right), \tag{5.2}$$

which has the same form as that of a European style put option. Our approach towards valuing the payoff above is based on that presented in Section 3. Specifically, we assume that the i^{th} ($i=1, \dots, N$) index's price process, $\{I_t^i \mid t > 0\}$, follows geometric Brownian motion with drift under the domestic risk-neutral probability measure. We then approximate the basket's price at maturity, $\sum_{i=1}^N \omega_i I_T^i$, by a shifted log-normal random variable, of the form

$$a + e^{b+c\varepsilon}. \tag{5.3}$$

Here the parameters a , b and c are uniquely determined, as described in Section 3, by matching the first three moments of the shifted log-normal random variable with those of the basket's price at maturity. We next approximate the payoff (5.2) by replacing the basket's value at maturity with that of the shifted log-normal random variable, that is,

$$\max\left(P - \left(a + e^{b+c\varepsilon}\right), 0\right). \quad (5.4)$$

Let r denote the constant risk-free rate for a period equal to the fund's maturity, T . The payoff (5.4) then has value

$$e^{-rT} E\left(\max\left(P - \left(a + e^{b+c\varepsilon}\right), 0\right)\right) \quad (5.5)$$

where E denotes the domestic risk-neutral probability measure. If $a < P$, then (5.5) equals

$$e^{-rT} \left((P - a) n\left(\frac{\log(P - a) - b}{c}\right) - e^{\frac{b+c^2}{2}} n\left(\frac{\log(P - a) - b}{c} - c\right) \right)$$

where n is the cumulative distribution function for a standard, normally distributed random variable. If $a \geq P$, then (5.5) equals zero.

We note that our analytical method above for valuing the European style put option provides for a significant speed-up over alternative Monte Carlo or quasi-Monte Carlo pricing methods.

6 Conclusion

This paper presents a new model for valuing equity-linked derivatives. The approach towards pricing the embedded option was to approximate the option's underlying security value using a shifted log-normal random variable. The defining parameters for this random variable were given from the analytical, unique solution to a system of non-linear equations arising from a moment matching technique.

An analytical pricing formula was then derived by taking the expected value of the payoff, but where the underlying security's value was replaced by that of the shifted log-normal random variable.

The new analytic pricing model was numerically compared against both a Monte Carlo benchmark and a Levy based approximate option pricing formula. Our pricing model showed close agreement with the Monte Carlo benchmark and Levy based benchmark over a wide range of option parameter values.

We also derive analytical formulas for hedge ratios, from the direct differentiation of the approximate option pricing formula. These formulas were numerically compared against benchmark Monte Carlo based hedge ratios. We find that delta hedge ratios are in close agreement with the corresponding benchmark, but that the Vega hedge ratios differ by approximately 6% under average parameter values.

We also consider a type of segregated fund with a maturity guarantee. The value of the guaranteed return is modeled as the price of a certain European style put option on a basket of indices. The price can be computed using the method described in Section 3.

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