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Abstract

We address the convergence of thresholding schemes for spectral optimisation problems and linear control problems.

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A Proof of Theorem II

As mentioned in the introduction, when the proofs are identical to those of Theorem I we omit them. We introduce the analog of Definition 7.

Definition 1 (Critical points)**.** *For any $V_0 \in (0;1)$, for any $f \in \mathcal{F}(V_0)$, we say f is a **critical point of** λ if:*

1. *There exists a unique $\mu_f = \mu_{u_f,V_0}$ such that*

$$\mathrm{Vol}(\{\eta_f > \mu_f\}) = V_0\mathrm{Vol}(\Omega) = \mathrm{Vol}(\{\eta_f \geqslant \mu_f\}),$$

2. *and $f = \mathbb{1}_{\{\eta_f > \mu_f\}}$.*

*A **critical set** is a subset E of Ω such that $\mathbb{1}_E \in \mathcal{F}(V_0)$ and such that $\mathbb{1}_E$ is a critical point of λ.*

Definition 2. *An admissible control $f^* \in \mathcal{F}(V_0)$ is called a stable local minimiser of λ in $\mathcal{F}(V_0)$ if there exist $C = C(f^*), \delta = \delta(f^*) > 0$ such that*

$$\forall f \in \mathcal{F}(V_0), \|f - f^*\|_{L^1(\Omega)} \leqslant \delta(f^*) \Rightarrow \lambda(f) \geqslant \lambda(f^*) + C(f^*)\|f - f^*\|^2_{L^1(\Omega)}.$$

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A.1 First consequences of a large volume constraint

We show regularity properties similar to those of Lemma 13. We let, for any $f \in \mathcal{F}(V_0)$, μ_{f,V_0} be the unique real number such that

$$\frac{\mathrm{Vol}(\{\eta_f > \mu_{f,V_0}\})}{\mathrm{Vol}(\Omega)} \leqslant V_0 \leqslant \frac{\mathrm{Vol}(\{\eta_f \geqslant \mu_{f,V_0}\})}{\mathrm{Vol}(\Omega)}.$$

We also define

$$\omega_{f,V_0} := \{\eta_f > \mu_{f,V_0}\}.$$

Lemma 3. *There exists $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that, for any $V_0 \in (1-\varepsilon_0; 1)$, for any $f \in \mathcal{F}(V_0)$, we have on the one hand*

$$\mathrm{Vol}(\{\eta_f = \mu_{f,V_0}\}) = 0$$

and, on the other hand, the following regularity properties:

1. *ω_{f,V_0} has a $\mathscr{C}^{1,\alpha}$ boundary (for any $\alpha \in (0;1)$),*
2. $$\min_{\partial\omega_{f,V_0}} \left|\frac{\partial \eta_f}{\partial \nu}\right| \geqslant \delta_0,$$
3. *$\partial\omega_{f,V_0}$ is locally a graph over $\partial\Omega$.*

Finally, for any $\alpha \in (0;1)$, $\partial\omega_{f,V_0}$ converges to $\partial\Omega$ in $\mathscr{C}^{1,\alpha}$ as $V_0 \to 1$, uniformly in $f \in \mathcal{F}(V_0)$.

Lemma 3. The proof is identical, except that the analog of (2.4) deals with the first Dirichlet eigenfunction ψ_1 of Ω. Let us explain in more details: let $(\lambda^D(\Omega), \psi_1)$ be the first Dirichlet eigenpair of the domain Ω:

$$\begin{cases} -\Delta\psi_1 = \lambda^D(\Omega)\psi_1 & \text{in } \Omega\,, \\ \psi_1 \in W_0^{1,2}(\Omega)\,, & \\ \psi_1 > 0 & \text{in } \Omega\,, \\ \int_\Omega \psi_1^2 = 1. & \end{cases} \tag{A.1}$$

From the variational formulation (1.3) of $\lambda(f)$ and the fact that $f \in \mathcal{F}(V_0)$ we have the bound

$$\lambda^D(\Omega) - 1 \leqslant \lambda(f) \leqslant \lambda^D(\Omega)$$

whence, from a standard bootstrapping argument there holds, for any $p \in [1; +\infty)$,

$$\sup_{V_0 \in (0;1)} \sup_{f \in \mathcal{F}(V_0)} \|\eta_f\|_{W^{2,p}(\Omega)} < \infty. \tag{A.2}$$

It is expected that as $V_0 \to 1$ we should have $\lambda(f)+1 \approx \lambda^D$. Let us prove this. From the variational formulation (1.3) we have

$$-\int_\Omega \eta_f^2\,(1-f) \leqslant \lambda(f) - (\lambda^D - 1) \leqslant \int_\Omega \psi_1^2 (1-f).$$

From (A.2) we deduce that

$$\lim_{V_0 \to 1} \sup_{f \in \mathcal{F}(V_0)} |\lambda(f) - (\lambda^D(\Omega) - 1)| = 0.$$

Let us now consider a sequence $\{V_k\}_{k\in\mathbb{N}}$ converging to 1 and, for any $k\in\mathbb{N}$, a function $f_k\in\mathcal{F}(V_k)$. Passing to the limit in the weak formulation of the equation (1.4) we deduce that any closure point η_∞ of $\{\eta_{f_k}\}_{k\in\mathbb{N}}$ solves the equation

$$\begin{cases} -\Delta\eta_\infty = \lambda^D(\Omega)\eta_\infty & \text{in } \Omega\,, \\ \eta_\infty \in W_0^{1,2}(\Omega)\,, & \\ \eta_\infty > 0 & \text{in } \Omega\,, \\ \int_\Omega \eta_\infty^2 = 1. & \end{cases} \tag{A.3}$$

From the simplicity of the first eigenvalue $\lambda^D(\Omega)$ it follows that $\eta_\infty=\psi_1$. From this fact it is then easy to show, as in Lemma 13, that for any $\alpha\in(0;1)$,

$$\lim_{V_0\to 1}\sup_{f\in\mathcal{V}_0}\|\psi_1-\eta_f\|_{\mathscr{C}^{1,\alpha}(\Omega)}=0. \tag{A.4}$$

We now use the maximum principle of Hopf: there exists a constant $\delta_\Omega>0$ such that

$$\inf_{\partial\Omega}|\nabla\psi_1|\geqslant\delta_\Omega>0$$

and we conclude the proof of Lemma 3 by the same arguments.

□

In particular we also obtain the existence of a constant M such that

$$\forall V_0\in(1-\varepsilon_0;1)\,,\forall f\in\mathcal{F}(V_0)\,,\mathrm{Lip}(\partial\omega_{f,V_0})+\mathrm{Per}(\omega_{f,V_0})\leqslant M. \tag{A.5}$$

We also have in a similar manner the following topological information

Lemma 4. *There exists $\varepsilon_0'>0$ such that, for any $V_0\in(1-\varepsilon_0';1)$, for any $f\in\mathcal{F}(V_0)$ the boundary $\partial\omega_{f,V_0}$ is connected.*

We may, analogously to Lemma 14, choose $\varepsilon_0=\varepsilon_0'$ where ε_0 is given in Lemma 13.

A.2 Applications of the quantitative bathtub principle

We apply the quantitative bathtub principle ([39, Proposition 26]) to the thresholding algorithm (3) and, as in Lemma 16 we obtain the following result:

Lemma 5. *Let ε_0,δ_0 be in the conditions of Lemma 3. For any $V_0\in(1-\varepsilon_0;1)$, for any $f_0\in\mathcal{F}(V_0)$, the sequence $\{f_k\}_{k\in\mathbb{N}}$ is uniquely defined (in the sense that for any $k\in\mathbb{N}$ there holds $\mathrm{Vol}\left(\{\eta_{f_k}=\mu_{f_k,V_0}\}\right)=0$) and there holds*

$$\sum_{k=0}^{\infty}\|f_{k+1}-f_k\|_{L^1(\Omega)}^2<\infty. \tag{A.6}$$

The following result is adapted from Lemma 17.

Lemma 6. *Let ε_0 be in the conditions of Lemma 13. For any $V_0\in(1-\varepsilon_0;1)$, for any initialisation $f_0\in\mathcal{F}(V_0)$, any $L^\infty-*$ closure point f_∞ of the sequence $\{f_k\}_{k\in\mathbb{N}}$ generated by the thresholding algorithm is a bang-bang function:*

$$\exists E_\infty\subset\Omega\,,f_\infty=\mathbb{1}_{E_\infty}. \tag{A.7}$$

Furthermore $E_\infty=\{\eta_{f_\infty}>\mu_{u_{f_\infty},V_0}\}$, which is uniquely defined. In other words, E_∞ is a critical set in the sense of Definition 1.

Using this result we get the following dichotomy for the sequence $\{f_k\}_{k\in\mathbb{N}}$ (see Lemma 18 for a proof).

Lemma 7. *Let ε_0 be chosen as in Lemma 13. For any $V_0 \in (1-\varepsilon_0; 1)$, for any initialisation $f_0 \in \mathcal{F}(V_0)$, let $\{f_k\}_{k\in\mathbb{N}}$ be the sequence generated by the thresholding algorithm. Then:*

1. *Either $\{f_k\}_{k\in\mathbb{N}}$ has a unique closure point f_∞, in which case $f_k \underset{k\to\infty}{\to} f_\infty$ strongly in $L^1(\Omega)$,*
2. *Or $\{f_k\}_{k\in\mathbb{N}}$ has an infinite number of closure points.*

The goal of the next sections In the following section we study the shape hessians at critical shape. This is where the main differences with the proof of Theorem I occur.

A.3 Qualitative study of critical points

In this section we analyse the critical sets (in the sense of Definition 1). For the second order shape derivatives of the Lagrangians to be meaningful we need the critical sets to have an (at least) $\mathscr{C}^3$ boundary. Here we are exactly in the setting for which [11, Lemma 3] was introduced and, similar to Lemma 19, we have the following regularity result:

Lemma 8. *Let $\varepsilon_0 > 0$ be as in Lemma 3. Any critical set E (in the sense of Definition 7) has a compact analytic boundary and, furthermore, is uniformly bounded in the $\mathscr{C}^2$ topology.*

Shape derivative formalism We first identify the functional λ with a shape functional Λ, by defining

$$\Lambda : \Omega \supset E \mapsto \lambda(\mathbb{1}_E).$$

For any compact $\mathscr{C}^3$ subset E of Ω and for any compactly supported vector field $\Phi \in W^{2,\infty}(\Omega; \mathbb{R}^d)$, we can define, for any $t \in (-1; 1)$ small enough, the function

$$e_{E,\Phi}(t) := \Lambda\Big((\mathrm{Id} + t\Phi)E\Big).$$

It was proved in [39] (using the aforementioned implicit function method of [41]) that Λ is twice shape differentiable, and we define:

$$\Lambda'(E)[\Phi] = e'_{E,\Phi}(0)\,, \text{resp. } \Lambda''(E)[\Phi,\Phi] = e''_{E,\Phi}(0).$$

First order shape derivative and definition of the Lagrange multiplier As for the Dirichlet energy, we work with a Lagrangian and we shall use the expressions for the derivative of the volume functional given in (2.24).

We introduce the associated notion of shape criticality:

Definition 9 (Critical shape)**.** *A $\mathscr{C}^3$ shape E is* ***a critical shape*** *for Λ if, for any compactly supported $\Phi \in W^{2,\infty}(\Omega)$,*

$$\int_{\partial E} \langle \Phi, \nu_E \rangle = 0 \Rightarrow \Lambda'(E)[\Phi] = 0.$$

We now compute the Lagrange multiplier at a given critical shape. If a shape E is critical, then the condition given in Definition 22 rewrites as: there exists a constant μ_E such that, for any compactly supported vector field $\Phi \in W^{2,\infty}(\Omega; \mathbb{R}^d)$,

$$\left(\Lambda + \mu_E \mathrm{Vol}\right)'(E)[\Phi] = 0.$$

To compute μ_E we rely on the following lemma:

Lemma 10. *For any $\mathscr{C}^3$ shape, for any compactly supported vector field $\Phi \in W^{2,\infty}(\Omega;\mathbb{R}^d)$,*

$$\Lambda'(E)[\Phi] = -\int_{\partial E} \eta_{\mathbb{1}_E}^2 \langle \Phi, \nu_E \rangle.$$

Furthermore, the shape derivative of the map $E \mapsto \eta_{\mathbb{1}_E}$ at E in the direction Φ is the unique solution of

$$\begin{cases} -\Delta \eta'_\Phi - \lambda(\mathbb{1}_E)\eta'_\Phi - \mathbb{1}_E \eta'_\Phi = \Lambda'(E)[\Phi]\eta_{\mathbb{1}_E} & \text{in } \Omega\,, \\ [\![\partial_\nu \eta'_\Phi]\!] = -\eta_{\mathbb{1}_E}\langle \Phi, \nu_E\rangle & \text{on } \partial E, \\ \int_\Omega \eta_E \eta'_\Phi = 0 \end{cases} \tag{A.8}$$

where $[\![\cdot]\!]$ denotes the jump of a function across a hypersurface.

This lemma is proved in [35, Appendix (B.2)]. It allows to prove, excatly in the same way as Lemma 24, the following statement:

Lemma 11. *A shape $E \subset \Omega$ is a critical set in the sense of Definition 1 if, and only if it is a critical shape in the sense of Definition 9.*

To alleviate notations we introduce a notation for the shape Lagrangian:

Definition 12 (Shape Lagrangian)**.** *Let E be a critical shape for $\mathscr{F}$. The **associated shape Lagrangian** is*

$$\Theta_E := \Lambda + \mu_E \mathrm{Vol}$$

where $E = \{\eta_{\mathbb{1}_E} > \mu_E\}$.

With the above ingredients at hand, we now move to second-order shape derivatives.

First computations and elements for shape hessians In the subsequent paragraphs we use indifferently the expressions "shape hessians" and "second order shape derivatives". We begin with an expression of the shape hessian at a critical shape E in the direction Φ.

Lemma 13. *Let E be a critical shape in the sense of Definition 9 and let $\Phi \in W^{2,\infty}(\Omega;\mathbb{R}^d)$ be a compactly supported vector field. The second-order shape derivative of the shape Lagrangian Θ_E (defined in Definition 12) is given by the expression*

$$\begin{aligned} \frac{1}{2}\Theta''_E(E)[\Phi,\Phi] &= -\int_{\partial E} \eta_E \eta'_\Phi \langle \Phi, \nu_E\rangle - \int_{\partial E} \eta_E \frac{\partial \eta_E}{\partial \nu}\langle \Phi, \nu_E\rangle^2 \\ &= \mu_E \left(-\int_{\partial E} \eta'_\Phi \langle \Phi, \nu_E\rangle - \int_{\partial E} \frac{\partial \eta_E}{\partial \nu}\langle \Phi, \nu_E\rangle^2 \right) \end{aligned} \tag{A.9}$$

where η'_Φ solves (A.8) and where we used the fact that $\eta_E \equiv \mu_E$ on ∂E.

We want to prove that critical shapes are stable in the following sense:

Definition 14 (L^2-stability)**.** *A critical shape E is said to be L^2-**stable** if there exists a constant $c_E > 0$ such that for any compactly supported vector field $\Phi \in W^{2,\infty}(\Omega;\mathbb{R}^d)$,*

$$\Theta''_E(E)[\Phi,\Phi] \geqslant c_E \|\langle \Phi, \nu\rangle\|^2_{L^2(\partial E)}.$$

Our main result is the following proposition:

Proposition 15. *There exists $\varepsilon_1 > 0\,, \varepsilon_1 \leqslant \varepsilon_0\,, \underline{c} > 0$ such that, for any $V_0 \in (1-\varepsilon_1; 1)$, for any critical set E (in the sense of Definition 9), for any compactly supported vector field $\Phi \in W^{2,\infty}(\Omega; \mathbb{R}^d)$,*

$$\Theta''_E(E)[\Phi, \Phi] \geqslant \underline{c}\mu_E \|\langle \Phi, \nu\rangle\|^2_{L^2(\partial E)}.$$

Before we prove this proposition we need to introduce the diagonalisation basis. We consider a fixed critical shape $E \subset \Omega$ and work with $V_0 \in (1-\varepsilon_0; 1)$, where ε_0 is given by Lemma 3. Heuristically, given that η'_Φ satisfies (A.8) and that $\Lambda'(E)[\Phi] = 0$ as we are working at a critical shape, it is natural to solve the following eigenvalue problem: find $(\lambda, \phi) \in \mathbb{R} \times W^{1,2}(\Omega)$ such that

$$\begin{cases} -\Delta\phi - \Lambda(E)\phi - \mathbb{1}_E\phi = 0 & \text{in } E \cup (E)^c \\ [\![\partial_\nu \phi]\!]_E = -\lambda\rho\phi & \text{on } \partial E\,, \\ \int_{\partial E} \phi^2 > 0\,, & \\ \phi \in W^{1,2}_0(\Omega)\,, & \\ \int_\Omega \phi\eta_E = 0. & \end{cases} \tag{A.10}$$

where $\rho \in L^\infty(\partial E; [\delta; 1/\delta])$ $(\delta > 0)$ is a weight to be determined. Let us simply consider the eigenvalue problem (A.10) with a weight $\rho \in L^\infty(\partial E; [\delta, 1/\delta])$ for some small $\delta > 0$. Note that as we work with a critical shape E,

$$\eta_E \equiv \mu_E \text{ on } \partial E.$$

We consider the weighted space

$$L^2_\rho(\partial E) := \left\{ f \in L^2(\partial E), \int_{\partial E} \rho f^2 < \infty \right\} = L^2(\partial E).$$

It is endowed with the scalar product

$$\langle \cdot, \cdot \rangle_\rho : (f, g) \mapsto \int_{\partial E} \rho f g.$$

As we need to take care of the orthogonality condition $\int_\Omega \eta_E \eta'_\Phi = 0$ we introduce the space

$$X := \left\{ u \in W^{1,2}_0(\Omega) : \int_\Omega \eta_E u = 0 \right\} = \langle \eta_E \rangle^\perp. \tag{A.11}$$

To use the spectral theorem we need to ensure that the solution mapping (analogous to T_ρ) we consider is well-defined. This relies on the simplicity of the first eigenvalue $\Lambda(E) = \lambda(\mathbb{1}_E)$.

Lemma 16. *Let E be a critical shape in the sense of Definition 9. For any $f \in L^2(\partial E)$, for any $\rho \in L^\infty(\partial E; [\delta; 1/\delta])$ $(\delta > 0)$ there exists a unique solution $v_{f,\rho}$ to the equation*

$$\begin{cases} -\Delta v_{f,\rho} - \Lambda(E) v_{f,\rho} - \mathbb{1}_E v_{f,\rho} = 0 & \text{in } \Omega\,, \\ [\![\partial_\nu v_{f,\rho}]\!] = -\rho f & \text{on } \partial E\,, \\ v_{f,\rho} \in X \text{ defined in (A.11): } v_{f,\rho} \in W^{1,2}_0(\Omega) \text{ and } \int_\Omega v_{f,\rho}\eta_E = 0. & \end{cases} \tag{A.12}$$

Lemma 16. The first eigenvalue $\lambda(\mathbb{1}_E) = \Lambda(E)$ is simple. Consequently, there exists a parameter $\underline{m} > 0$ such that, X being the space introduced in (A.11),

$$\forall u \in X\,, \int_\Omega |\nabla u|^2 - \int_\Omega \mathbb{1}_E u^2 - \Lambda(E) \int_\Omega u^2 \geqslant \underline{m} \int_\Omega u^2. \tag{A.13}$$

As a consequence, for any $f \in L^2(\partial E)$, the energy functional

$$W_{\rho,f} : X \ni u \mapsto \frac{1}{2}\left(\int_\Omega |\nabla u|^2 - \int_\Omega \mathbb{1}_E u^2 - \Lambda(E)\int_\Omega u^2\right) - \int_{\partial E} \rho f u$$

is coercive. Consequently, $W_{\rho,f}$ admits a minimiser, which is a solution of (A.12). Furthermore, once again by simplicity of $\Lambda(E)$, there is at most one solution of (A.12). The proof is concluded. □

Define the operator $T_\rho : L^2_\rho(\partial E) \to L^2(\partial E)$ as follows: for any $f \in L^2_\rho(\partial E)$ let $v_{f,\rho} \in W^{1,2}(\Omega)$ be the unique solution of (A.12). Let $\mathrm{Tr}_E : W^{1,2}(\Omega) \to L^2(\partial E)$ be the trace operator. E is analytic (Lemma 8) so that this operator is well-defined. Introduce

$$T_\rho(f) := \mathrm{Tr}_E(v_{f,\rho}). \tag{A.14}$$

While the symmetry and compactness of the operator T_ρ are immediate, the positivity uses once more the simplicity of $\Lambda(E)$. Indeed, for any $f \in L^2_\rho(\partial E)$ we have

$$\begin{aligned}\int_{\partial E} \rho f T_\rho(f) &= \int_{\partial E} \rho f v_{f,\rho} \\ &= \int_\Omega |\nabla v_{f,\rho}|^2 - \int_\Omega \mathbb{1}_E v_{f,\rho}^2 - \Lambda(E)\int_\Omega v_{f,\rho}^2 \\ &\geqslant \underline{m}\int_\Omega v_{f,\rho}^2 \geqslant 0 \text{ from (A.13)},\end{aligned}$$

and the last inequality is strict unless $f \equiv 0$.

We may thus apply the spectral theorem: there exists a sequence of eigenpairs $\{(\sigma_{k,\rho}\,, \psi_{k,\rho})\}_{k\in\mathbb{N}} \in \left(\mathbb{R}_+^* \times L^2_\rho(\partial E)\right)^{\mathbb{N}}$ such that

$$\sigma_{k,\rho} \text{ is non-increasing in } k \text{ and } \sigma_{k,\rho} \underset{k\to\infty}{\longrightarrow} 0,$$

$$\{\psi_{k,\rho}\}_{k\in\mathbb{N}} \text{ is an orthonormal basis of } L^2_\rho(\partial E)$$

and

$$\forall k \in \mathbb{N}\,, \quad T_\rho(\psi_{k,\rho}) = \sigma_{k,\rho}\psi_{k,\rho}.$$

We set $\phi_{k,\rho} := v_{\psi_{k,\rho}}$ to extend it to a function on Ω. Hence,

$$\begin{cases} -\Delta\phi_{k,\rho} - \Lambda(E)\phi_{k,\rho} - \mathbb{1}_E\phi_{k,\rho} = 0 & \text{in } \Omega\,, \\ [\![\partial_\nu \phi_{k,\rho}]\!] = -\rho\psi_{k,\rho} = -\frac{1}{\sigma_{k,\rho}}\rho\phi_{k,\rho} & \text{on } \partial E\,, \\ \phi_{k,\rho} \in X : \ \phi_{k,\rho} \in W_0^{1,2}(\Omega)\,, \int_\Omega \phi_{k,\rho}\eta_E = 0. \end{cases} \tag{A.15}$$

We thus obtain, defining

$$\lambda_{k,\rho} := \frac{1}{\sigma_{k,\rho}}$$

the system

$$\forall k \in \mathbb{N}^*\,, \begin{cases} -\Delta\phi_{k,\rho} - \Lambda(E)\phi_{k,\rho} - \mathbb{1}_E\phi_{k,\rho} = 0 & \text{in } \Omega\,, \\ [\![\partial_\nu \phi_{k,\rho}]\!] = -\lambda_{k,\rho}\rho\phi_{k,\rho} & \text{on } \partial E\,, \\ \phi_{k,\rho} \in X : \ \phi_{k,\rho} \in W_0^{1,2}(\Omega)\,, \int_\Omega \phi_{k,\rho}\eta_E = 0 \end{cases}$$

$$\text{and } \forall k\,,k' \in \mathbb{N}^*\,, \int_{\partial E} \rho\phi_{k,\rho}\phi_{k',\rho} = \delta_{k,k'}. \quad \text{(A.16)}$$

Thus the family $\{\phi_{0,\rho}\} \cup \{\phi_{k,\rho}\}_{k\in\mathbb{N}^*} = \{\phi_{k,\rho}\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2_\rho(\partial E)$ for the scalar product $\langle\cdot,\cdot\rangle_\rho$.

From the Courant-Fisher principle we have, furthermore, the following characterisation of the first eigenvalue:

$$\lambda_{0,\rho} = \min_{u\in X\,,\int_{\partial E} u^2>0} \frac{\int_\Omega |\nabla u|^2 - \int_\Omega \mathbb{1}_E u^2 - \Lambda(E)\int_\Omega u^2}{\int_{\partial E}\rho u^2}. \quad \text{(A.17)}$$

Diagonalisation of $\Theta''_E(E)$ We start from (B.7) and we seek to determine the weight ρ. We use the following notational convention: for any $\Phi \in W^{2,\infty}(\Omega;\mathbb{R}^d)$,

$$\varphi := \langle \Phi\,,\nu\rangle\,, \tilde{\varphi} := \frac{1}{\rho}\varphi.$$

The first-order shape derivative η'_Φ satisfies

$$\begin{cases} -\Delta\eta'_\Phi - \Lambda(E)\eta'_\Phi - \mathbb{1}_E\eta'_\Phi = 0 & \text{in } \Omega\,, \\ \left[\!\left[\frac{\partial\eta'_\Phi}{\partial\nu}\right]\!\right] = -\rho\tilde{\varphi} & \text{on } \partial E\,, \\ \eta'_\Phi \in X:\ \eta'_\Phi \in W^{1,2}_0(\Omega) \text{ and } \int_\Omega \eta'_\Phi\eta_E = 0. \end{cases} \quad \text{(A.18)}$$

We decompose $\tilde{\varphi}$ in the basis $\{\phi_{k,\rho}\}_{k\in\mathbb{N}}$ as

$$\tilde{\varphi} = \sum_{k=0}^{\infty} \alpha_{k,\rho}\phi_{k,\rho}. \quad \text{(A.19)}$$

Thus,

$$\eta'_\Phi = \sum_{k=0}^{\infty} \frac{\alpha_{k,\rho}}{\lambda_{k,\rho}}\phi_{k,\rho}.$$

As a conclusion

$$\begin{aligned} \frac{1}{2\mu_E}\Theta''_E(E)[\Phi,\Phi] &= -\int_{\partial E}\eta'_\Phi\langle\Phi\,,\nu_E\rangle - \int_{\partial E}\frac{\partial\eta_E}{\partial\nu}\langle\Phi,\nu_E\rangle^2 \\ &= -\int_{\partial E}\rho\tilde{\varphi}\eta'_\Phi - \int_{\partial E}\rho^2\frac{\partial\eta_E}{\partial\nu}\left(\tilde{\varphi}\right)^2. \end{aligned}$$

Let us choose

$$\rho_E = -\frac{1}{\partial_\nu\eta_E}.$$

From Lemma 3, we have $\rho_E \in L^\infty([\delta;1/\delta])$ for δ small enough. Finally, this yields

$$\begin{aligned} \frac{1}{2\mu_E}\Theta''_E(E)[\Phi,\Phi] &= -\int_{\partial E}\eta'_\Phi\langle\Phi\,,\nu_E\rangle - \int_{\partial E}\frac{\partial\eta_E}{\partial\nu}\langle\Phi,\nu_E\rangle^2 \\ &= -\int_{\partial E}\rho_E\tilde{\varphi}\eta'_\Phi - \int_{\partial E}\rho_E^2\frac{\partial\eta_E}{\partial\nu}\left(\tilde{\varphi}\right)^2 \\ &= \sum_{k=0}^{\infty}\alpha^2_{k,\rho_E}\left(1 - \frac{1}{\lambda_{k,\rho_E}}\right). \end{aligned}$$

In particular, given that the sequence $\{\lambda_{k,\rho_E}\}_{k\in\mathbb{N}}$ is non-decreasing, we have the estimate from below:

$$\frac{1}{2\mu_E}\Theta''_E(E)[\Phi,\Phi] \geqslant \left(1-\frac{1}{\lambda_{0,\rho_E}}\right)\sum_{k=0}^{\infty}\alpha^2_{k,\rho_E} = \left(1-\frac{1}{\lambda_{0,\rho_E}}\right)\|\langle\Phi,\nu_E\rangle\|^2_{L^2(\partial E)}. \tag{A.20}$$

Hence we have the following sufficient condition for the stability of a critical shape:

Lemma 17. *If*

$$\lambda_{0,\rho_E} > 1$$

then E is stable in the sense of Definition 14.

To prove Proposition 15 we study the asymptotic behaviour of λ_{0,ρ_E} as $V_0 \to 1$.

Asymptotic behaviour of λ_{0,ρ_E} as $V_0 \to 1$ We prove here the following lemma:

Lemma 18. *There holds*

$$\min_{E \text{ critical shape}} \lambda_{0,E} \underset{V_0\to 1}{\to} \infty.$$

Proof of Lemma 18. From Lemma 3 and the Rayleigh quotient formulation (A.17) of λ_{0,ρ_E} we have, for $V_0 \in (1-\varepsilon_0;\varepsilon)$,

$$\lambda_{0,\rho_E} \geqslant \delta_0 \min_{u\in X\,,\int_{\partial E}u^2>0} \frac{\int_\Omega |\nabla u|^2 - \int_\Omega \mathbb{1}_E u^2 - \Lambda(E)\int_\Omega u^2}{\int_{\partial E} u^2}.$$

As in Lemma 31 we argue by contradiction: assume that there exists a sequence $\varepsilon \to 0$, a constant $M_0 > 0$, and, for any $\varepsilon > 0$, a critical set E_ε such that $\mathbb{1}_{E_\varepsilon} \in \mathcal{F}(V_0)$ with $V_0 \in (1-\varepsilon;1)$ that satisfies

$$\lambda_{0,\rho_{E_\varepsilon}} \leqslant M_0.$$

Define, for any $\varepsilon > 0$,

$$\lambda(\varepsilon) := \min_{u\in X\,,\int_{\partial E_\varepsilon}u^2\neq 0} \frac{\int_\Omega |\nabla u|^2 - \int_\Omega \mathbb{1}_E u^2 - \Lambda(E)\int_\Omega u^2}{\int_{\partial E_\varepsilon} u^2}.$$

By assumption the sequence $\{\lambda(\varepsilon)\}_{\varepsilon\to 0}$ is uniformly bounded. In other words we have

$$0 < \lambda(\varepsilon) \leqslant M_1 < \infty \tag{A.21}$$

for some suitable constant M_1. We define, for any $\varepsilon > 0$, φ_ε as the first eigenfunction associated with $\lambda(\varepsilon)$; in other words, φ_ε solves the equation

$$\begin{cases} -\Delta\varphi_\varepsilon - \Lambda(E)\varphi_\varepsilon - \mathbb{1}_E\varphi_\varepsilon = 0 & \text{in } \Omega\,, \\ \left[\!\left[\frac{\partial\varphi_\varepsilon}{\partial\nu}\right]\!\right] = -\lambda(\varepsilon)\varphi_\varepsilon & \text{on } \partial E_\varepsilon\,, \\ \varphi_\varepsilon \in X\,, \\ \int_{\partial E}\varphi_\varepsilon^2 = 1. \end{cases} \tag{A.22}$$

In the same way we proved that (A.21) implied (2.41), it is easy to show that if (A.21) holds, then:

$$\forall\beta\in(0;1)\,,\exists c_\beta\,,\sup_{\varepsilon>0}\|\varphi_\varepsilon\|_{\mathscr{C}^{0,\beta}} \leqslant c_\beta. \tag{A.23}$$

The only difficulty here is to initialise the bootstrap to obtain higher integrability of φ_ε but this relies once again on the spectral gap estimate (A.13), as we immediately obtain

$$\int_\Omega |\nabla \varphi_\varepsilon|^2 \leqslant M \left(\lambda(\varepsilon) + \int_\Omega \varphi_\varepsilon^2 \right) \leqslant M''$$

for some M'' under the assumption (A.21). The rest of the proof is identical. □

We then prove the coercivity result (Proposition 15) with the same proof as that of Proposition 29.

Local quantitative inequalities around critical shapes As in [39] we have the following results:

Lemma 19. *Let $\varepsilon_1 > 0$ be given by Proposition 15. For any $V_0 \in (1-\varepsilon_1; 1)$ the critical shapes are isolated in the sense that*

$$\inf_{E, E' \text{ critical}, \ E' \neq E} |E' \Delta E| = \delta(V_0) > 0.$$

Lemma 20. *Let ε_1 be given by Proposition 15. For any $V_0 \in (1-\varepsilon_1; 1)$, any critical shape E is a local L^1 minimiser: there exist $c_E > 0$ and $r_E > 0$ such that, for any $f \in \mathcal{F}(V_0)$ with $\|f - \mathbb{1}_E\|_{L^1(\Omega)} \leqslant r_E$, there holds*

$$\lambda(f) - \lambda(\mathbb{1}_E) \geqslant c_E \|f - \mathbb{1}_E\|^2_{L^1(\Omega)}.$$

A.4 Conclusion of the proof of Theorem II

Proof of Theorem II. Finally, the proof of Theorem II is identical to that of Theorem I. □

B Proof of Theorem III

Similar to the proof of Theorem II, when the proofs are identical to those of Theorem I we omit them. We introduce the analog of Definitions 7-8.

Definition 21 (Critical points)**.** *For any $V_0 \in (0; 1)$, for any $f \in \mathcal{F}(V_0)$, we say f is a **critical point of** J if:*

1. *There exists a unique $\mu_f = \mu_{u_f, V_0}$ such that*

$$\mathrm{Vol}(\{p_f > \mu_f\}) = V_0 \mathrm{Vol}(\Omega) = \mathrm{Vol}(\{p_f \geqslant \mu_f\}),$$

2. *and $f = \mathbb{1}_{\{p_f > \mu_f\}}$.*

*A **critical set** is a subset E of Ω such that $\mathbb{1}_E \in \mathcal{F}(V_0)$ and such that $\mathbb{1}_E$ is a critical point of J.*

Definition 22. *An admissible control $f^* \in \mathcal{F}(V_0)$ is called a stable local minimiser of J in $\mathcal{F}(V_0)$ if there exist $C = C(f^*), \delta = \delta(f^*) > 0$ such that*

$$\forall f \in \mathcal{F}(V_0), \|f - f^*\|_{L^1(\Omega)} \leqslant \delta(f^*) \Rightarrow J(f) \geqslant J(f^*) + C(f^*) \|f - f^*\|^2_{L^1(\Omega)}.$$

B.1 First consequences of a large volume constraint

We show regularity properties similar to those of Lemma 13. We let, for any $f \in \mathcal{F}(V_0)$, μ_{f,V_0} be the unique real number such that

$$\frac{\mathrm{Vol}(\{p_f > \mu_{f,V_0}\})}{\mathrm{Vol}(\Omega)} \leqslant V_0 \leqslant \frac{\mathrm{Vol}(\{p_f \geqslant \mu_{f,V_0}\})}{\mathrm{Vol}(\Omega)}.$$

We also define

$$\omega_{f,V_0} := \{p_f > \mu_{f,V_0}\}.$$

Lemma 23. *There exists $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that, for any $V_0 \in (1-\varepsilon_0; 1)$, for any $f \in \mathcal{F}(V_0)$, we have on the one hand*

$$\mathrm{Vol}(\{p_f = \mu_{f,V_0}\}) = 0$$

and, on the other hand, the following regularity properties:

1. *ω_{f,V_0} has a $\mathscr{C}^{3,\alpha}$ boundary (for any $\alpha \in (0;1)$),*
2. $$\min_{\partial\omega_{f,V_0}} \left|\frac{\partial p_f}{\partial \nu}\right| \geqslant \delta_0,$$
3. *$\partial\omega_{f,V_0}$ is locally a graph over $\partial\Omega$.*

Finally, for any $\alpha \in (0;1)$, $\partial\omega_{f,V_0}$ converges to $\partial\Omega$ in $\mathscr{C}^{3,\alpha}$ as $V_0 \to 1$, uniformly in $f \in \mathcal{F}(V_0)$.

Lemma 23. Here the proof is slightly different; by using an adjoint state, we get more regularity on the free boundary of the set ω_{f,V_0}. We already observed in the proof of Lemma 13 that for any $\alpha \in (0;1)$ there held $\sup_{V_0 \in (0;1)\,, f \in \mathcal{F}(V_0)} \|u_f\|_{\mathscr{C}^{1,\alpha}(\overline{\Omega})} < \infty$. By standard Schauder estimates applied to equation (1.5) we get

$$\sup_{V_0 \in (0;1)\,, f \in \mathcal{F}(V_0)} \|p_f\|_{\mathscr{C}^{3,\alpha}(\overline{\Omega})} < \infty.$$

From (2.4) is is also clear that

$$\forall \alpha \in (0;1)\,, \lim_{V_0 \to 1} \sup_{f \in \mathcal{F}(V_0)} \|p_f - p_\Omega\|_{\mathscr{C}^{3,\alpha}(\Omega)} = 0 \tag{B.1}$$

where p_Ω is the unique solution to

$$\begin{cases} -\Delta p_\Omega = \frac{\partial j}{\partial u}(x, w_\Omega) & \text{in } \Omega\,, \\ p_\Omega = 0 & \text{on } \partial\Omega. \end{cases} \tag{B.2}$$

In this equation w_Ω is the torsion function (2.1). As $w_\Omega > 0$ and as $\partial_u j(x,\cdot) > 0$ the maximum principle of Hopf entails that $\inf_{\Omega(r_0)} |\nabla p_\Omega| > 0$ in the tubular neighbourhood $\Omega(r_0)$ defined in (2.8) for $r_0 > 0$ small enough. The rest of the proof is identical and the implicit function theorem implies that for V_0 close enough to 1 $\partial\omega_{f,V_0}$ is $\mathscr{C}^{3,\alpha}$.

□

Similarly we obtain the existence of a constant M such that

$$\forall V_0 \in (1-\varepsilon_0; 1)\,, \forall f \in \mathcal{F}(V_0)\,, \mathrm{Lip}(\partial\omega_{f,V_0}) + \mathrm{Per}(\omega_{f,V_0}) \leqslant M. \tag{B.3}$$

We also give the following information:

Lemma 24. *There exists $\varepsilon_0' > 0$ such that, for any $V_0 \in (1-\varepsilon_0'; 1)$, for any $f \in \mathcal{F}(V_0)$ the boundary $\partial\omega_{f,V_0}$ is connected.*

We may, analogously to Lemma 14, take $\varepsilon_0 = \varepsilon_0'$ where ε_0 is given in Lemma 23.

B.2 Applications of the quantitative bathtub principle

We apply the quantitative bathtub principle ([39, Proposition 26]) and obtain the following results for the thresholding algorithm (4).

Lemma 25. *Let ε_0, δ_0 be in the conditions of Lemma 23. For any $V_0 \in (1-\varepsilon_0; 1)$, for any $f_0 \in \mathcal{F}(V_0)$, the sequence $\{f_k\}_{k\in\mathbb{N}}$ is uniquely defined (in the sense that for any $k \in \mathbb{N}$ there holds $\mathrm{Vol}(\{p_{f_k} = \mu_{f_k,V_0}\}) = 0$) and there holds*

$$\sum_{k=0}^{\infty} \|f_{k+1} - f_k\|^2_{L^1(\Omega)} < \infty. \tag{B.4}$$

Lemma 26. *Let ε_0 be in the conditions of Lemma 13. For any $V_0 \in (1-\varepsilon_0; 1)$, for any initialisation $f_0 \in \mathcal{F}(V_0)$, any $L^\infty - *$ closure point f_∞ of the sequence $\{f_k\}_{k\in\mathbb{N}}$ generated by the thresholding algorithm is a bang-bang function:*

$$\exists E_\infty \subset \Omega\,, f_\infty = \mathbb{1}_{E_\infty}. \tag{B.5}$$

Furthermore $E_\infty = \{p_{f_\infty} > \mu_{u_{f_\infty},V_0}\}$, which is uniquely defined. In other words, E_∞ is a critical set in the sense of Definition 21.

We obtain the following dichotomy (see Lemma 18).

Lemma 27. *Let ε_0 be chosen as in Lemma 23. For any $V_0 \in (1-\varepsilon_0; 1)$, for any initialisation $f_0 \in \mathcal{F}(V_0)$, let $\{f_k\}_{k\in\mathbb{N}}$ be the sequence generated by the thresholding algorithm. Then:*

1. *Either $\{f_k\}_{k\in\mathbb{N}}$ has a unique closure point f_∞, in which case $f_k \underset{k\to\infty}{\to} f_\infty$ strongly in $L^1(\Omega)$,*
2. *Or $\{f_k\}_{k\in\mathbb{N}}$ has an infinite number of closure points.*

The goal of the next sections In the following section we study the shape hessians at critical shape. This is where the main differences with the proofs of Theorems I-II occur.

B.3 Qualitative study of critical points

In this section we analyse the critical sets (in the sense of Definition 21). For the second order shape derivatives of the Lagrangians to be meaningful we need the critical sets to have an (at least) $\mathscr{C}^3$ boundary, which is guaranteed by Lemma 23.

Shape derivative formalism We associate to the functional J a shape functional $\mathcal{J}$, by defining

$$\mathcal{J} : \Omega \supset E \mapsto J(\mathbb{1}_E).$$

For any compact $\mathscr{C}^3$ subset E of Ω and for any compactly supported vector field $\Phi \in W^{2,\infty}(\Omega; \mathbb{R}^d)$, we can define, for any $t \in (-1; 1)$ small enough, the function

$$e_{E,\Phi}(t) := \mathcal{J}\Big((\mathrm{Id} + t\Phi)E\Big).$$

From the same arguments that established the differentiability of $E \mapsto u_{\mathbb{1}_E}$ (Lemma 23) that $\mathcal{J}$ is twice shape differentiable, and we define:

$$\mathcal{J}'(E)[\Phi] = e'_{E,\Phi}(0)\,, \text{resp. } \mathcal{J}''(E)[\Phi,\Phi] = e''_{E,\Phi}(0).$$

First order shape derivative and definition of the Lagrange multiplier We use the formulas given in (2.24).

Definition 28 (Critical shape)**.** *A $\mathscr{C}^3$ shape E is* ***a critical shape*** *for $\mathcal{J}$: if, for any compactly supported $\Phi \in W^{2,\infty}(\Omega)$,*

$$\int_{\partial E} \langle \Phi, \nu_E \rangle = 0 \Rightarrow \mathcal{J}'(E)[\Phi] = 0.$$

We can now compute the Lagrange multiplier. Let E be a critical shape; then there exists μ_E such that, for any compactly supported vector field $\Phi \in W^{2,\infty}(\Omega; \mathbb{R}^d)$,

$$\left(\mathcal{J} + \mu_E \mathrm{Vol}\right)'(E)[\Phi] = 0.$$

We need, as in Lemma 23, the expression of the first-order shape derivative; we give it in the following statement:

Lemma 29. *For any $\mathscr{C}^3$ shape, for any compactly supported vector field $\Phi \in W^{2,\infty}(\Omega; \mathbb{R}^d)$,*

$$\mathcal{J}'(E)[\Phi] = \int_{\partial E} p_{\mathbb{1}_E} \langle \Phi, \nu_E \rangle.$$

Furthermore, the shape derivative of the map $E \mapsto u_{\mathbb{1}_E}$ at E in the direction Φ is the solution of (2.25).

Similar to Lemma 24 we have the following statement:

Lemma 30. *A shape $E \subset \Omega$ is a critical set in the sense of Definition 21 if, and only if it is a critical shape in the sense of Definition 28.*

We now define the shape Lagrangian

Definition 31 (Shape Lagrangian)**.** *Let E be a critical shape for $\mathscr{F}$. The* ***associated shape Lagrangian*** *is*

$$\mathcal{K}_E := \mathcal{J} + \mu_E \mathrm{Vol}$$

where $E = \{p_{\mathbb{1}_E} > \mu_E\}$.

With the above ingredients at hand, we now move to second-order shape derivatives.

First computations and elements for shape hessians We first express the shape hessian of the Lagrangian. To express it we need to compute the derivative of p_E; through the same argument as in [41] we obtain that the map $E \mapsto p_E$ is differentiable and that, for any compactly supported vector field $\Phi \in W^{2,\infty}(\Omega; \mathbb{R}^d)$, its shape derivative p'_Φ satisfies

$$\begin{cases} -\Delta p'_\Phi = u'_\Phi \partial^2_{uu} j(x, u_\Phi) & \text{in } \Omega\,, \\ p'_\Phi \in W^{1,2}_0(\Omega). \end{cases} \tag{B.6}$$

Lemma 32. *Let E be a critical shape in the sense of Definition 28 and let $\Phi \in W^{2,\infty}(\Omega; \mathbb{R}^d)$ be a compactly supported vector field. The second-order shape derivative of the shape Lagrangian $\mathcal{K}_E$ (defined in Definition 31) is given by the expression*

$$\mathcal{K}''_E(E)[\Phi, \Phi] = \int_{\partial E} p'_\Phi \langle \Phi, \nu_E \rangle + \int_{\partial E} \frac{\partial p_E}{\partial \nu} \langle \Phi, \nu_E \rangle^2 \tag{B.7}$$

where p'_Φ solves (B.6).

We want to prove that critical shapes are stable in the following sense:

Definition 33 (L^2-stability)**.** *A critical shape E is said to be L^2-**stable** if there exists a constant $c_E > 0$ such that for any compactly supported vector field $\Phi \in W^{2,\infty}(\Omega; \mathbb{R}^d)$,*

$$\mathcal{K}''_E(E)[\Phi, \Phi] \geqslant c_E \|\langle \Phi, \nu \rangle\|^2_{L^2(\partial E)}.$$

Our main result is the following proposition:

Proposition 34. *There exists $\varepsilon_1 > 0\,, \varepsilon_1 \leqslant \varepsilon_0\,, \underline{c} > 0$ such that, for any $V_0 \in (1 - \varepsilon_1; 1)$, for any critical set E (in the sense of Definition 28), for any compactly supported vector field $\Phi \in W^{2,\infty}(\Omega; \mathbb{R}^d)$,*

$$\mathcal{K}''_E(E)[\Phi, \Phi] \geqslant \underline{c} \|\langle \Phi, \nu \rangle\|^2_{L^2(\partial E)}.$$

The diagonalisation basis is defined in the same way it was introduced when studying ($\mathbf{P}_{\text{Dir}}$)-($\mathbf{P}_{\text{Eigen}}$). We consider a fixed critical shape $E \subset \Omega$ and work with $V_0 \in (1 - \varepsilon_0; 1)$, where ε_0 is given by Lemma 23. As p'_Φ satisfies (B.6) and as $\mathcal{K}'(E)[\Phi] = 0$, the operator the eigenvalues of which we should be considering is the following: consider a weight $\rho \in L^\infty(\partial E\,, [\delta; 1/\delta])\,, \delta > 0$ and for any function $f \in L^2(\partial E)$, let v_f and q_f be, respectively, the unique solutions of

$$\begin{cases} -\Delta v_f = 0 & \text{in } \Omega\,, \\ [\![\partial_\nu v_f]\!] = -\rho f & \text{on } \partial E\,, \\ v_f \in W^{1,2}_0(\Omega) \end{cases} \tag{B.8}$$

and

$$\begin{cases} -\Delta q_f = v_f \partial^2_{uu} j(x, u_E) & \text{in } \Omega \\ q_f \in W^{1,2}_0(\Omega). \end{cases} \tag{B.9}$$

Finally, define the operator

$$T_\rho : f \mapsto \text{Tr}_{\partial E}(q_f).$$

It is easily checked that T_ρ is symmetric, compact and positive. We can thus use the spectral decomposition theorem and obtain the existence of an increasing, diverging sequence of positive eigenvalues $\{\lambda_{k,\rho}\}_{k \in \mathbb{N}}$ and a sequence of couples $\{\phi_{k,\rho}, \psi_{k,\rho}\}_{k \in \mathbb{N}}$ such that, for any $k \in \mathbb{N}$, there holds

$$\begin{cases} -\Delta \phi_{k,\rho} = 0 & \text{in } \Omega\,, \\ -\Delta \psi_{k,\rho} = \partial^2_{uu} j(x, u_{\mathbb{1}_E}) \phi_{k,\rho} & \text{in } \Omega\,, \\ [\![\partial_\nu \phi_{k,\rho}]\!] = -\lambda_{k,\rho} \rho \psi_{k,\rho} & \text{on } \partial E\,, \\ \psi_{k,\rho}\,, \phi_{k,\rho} \in W^{1,2}_0(\Omega)\,, \\ \int_{\partial E} \rho \psi^2_{k,\rho} = 1. \end{cases} \tag{B.10}$$

But now observe that the eigen-system (B.10) may be recase in a simpler form (that involves a fourth-order operator). Indeed, by noticing that the strict convexity of j in u allows us to write

$$\phi_{k,\rho} = -\frac{1}{j''(x, u_{\mathbb{1}_E})} \Delta \psi_{k,\rho}$$

(B.10) rewrites

$$\begin{cases} \Delta \left(\frac{1}{j''(x, \mathbb{1}_E)} \Delta \psi_{k,\rho} \right) = 0 & \text{in } \Omega\,, \\ \left[\!\!\left[\partial_\nu \frac{1}{\partial^2_{uu} j(x, u_{\mathbb{1}_E})} \Delta \psi_{k,\rho} \right]\!\!\right] = \lambda_{k,\rho} \rho \psi_{k,\rho} & \text{on } \partial E\,, \\ \psi_{k,\rho} \in W^{1,2}_0(\Omega)\,, \Delta \psi_{k,\rho} = 0 \text{ on } \partial\Omega. \end{cases} \tag{B.11}$$

The form (B.11) has one advantage, which is that it makes the associated Rayleigh quotient more easily read; it is indeed simply given by

$$R_\rho : W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \ni \psi \mapsto \frac{\int_\Omega \frac{1}{\partial^2_{uu} j(x, u_{\mathbb{1}_E})} (\Delta\psi)^2}{\int_{\partial E} \rho \psi^2}.$$

Now, as before, choosing

$$\rho_E := \frac{1}{\left| \frac{\partial p_E}{\partial \nu} \right|}$$

and decomposing $\tilde{\phi} := \frac{1}{\rho_E} \langle \Phi, \nu \rangle$ as

$$\tilde{\phi} = \sum_{k=0}^{\infty} a_k \psi_k$$

it appears that we have the diagonalised expression

$$\mathcal{K}''_E(E)[\Phi, \Phi] \leqslant \left(\frac{1}{\lambda_{0,\rho_E}} - 1 \right) \sum_{k=0}^{\infty} a_k^2$$

whence the question of the coercivity of $\mathcal{K}''_E$ boils down to studying the asymptotic behaviour of $\lambda_{0,\rho}$ as $V_0 \to 1$.

In other words, we have the following stability criterion:

Lemma 35. *If*

$$\lambda_{0,\rho_E} > 1$$

then E is stable in the sense of Definition 33.

Asymptotic behaviour of λ_{0,ρ_E} as $V_0 \to 1$ The following lemma is then almost the conclusion of this section:

Lemma 36. *There holds*

$$\min_{E \text{ critical shape}} \lambda_{0,\rho_E} \underset{V_0 \to 1}{\to} \infty.$$

Proof of Lemma 36. Notice that λ_{0,ρ_E} has the variational formulation

$$\lambda_{0,\rho_E} = \min_{\psi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \int_{\partial E} \psi^2 > 0} \frac{\int_\Omega \frac{1}{\partial^2_{uu} j(x, u_{\mathbb{1}_E})} (\Delta\psi)^2}{\int_{\partial E} \rho_E \psi^2}. \tag{B.12}$$

From Lemma 23, the strict convexity of j and the fact that $j''(x, \cdot)$ is bounded, and, finally, from the Rayleigh quotient formulation (B.12) of λ_{0,ρ_E} there exists a constant $c_0 > 0$ such that for $V_0 \in (1 - \varepsilon_0; \varepsilon)$,

$$\lambda_{0,\rho_E} \geqslant c_0 \min_{\psi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \int_{\partial E} \psi^2 > 0} \frac{\int_\Omega (\Delta\psi)^2}{\int_{\partial E} \psi^2}.$$

Following the strategy we introduced in Lemma 31 we argue by contradiction: assume that there exists a sequence $\varepsilon \to 0$, a constant $M_0 > 0$, and, for any $\varepsilon > 0$, a critical set E_ε such that $\mathbb{1}_{E_\varepsilon} \in \mathcal{F}(V_0)$ with $V_0 \in (1 - \varepsilon; 1)$ that satisfies

$$\lambda_{0,\rho_{E_\varepsilon}} \leqslant M_0.$$

Let for any $\varepsilon > 0$

$$\xi(\varepsilon) := \min_{\psi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)\,,\int_{\partial E_\varepsilon} \psi^2 > 0} \frac{\int_\Omega (\Delta\psi)^2}{\int_{\partial E_\varepsilon} \psi^2}.$$

However notice now that by the Poincaré inequality, for any $\psi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ there holds

$$\int_\Omega (\Delta\psi)^2 \geqslant c_1 \int_\Omega |\nabla\psi|^2$$

for some constant $c_1 > 0$, so that

$$\xi(\varepsilon) \geqslant c_1 \lambda(\varepsilon)$$

where $\lambda(\varepsilon)$ was defined in the proof of Theorem I. From Lemma 31 we deduce that $\xi(\varepsilon) \underset{\varepsilon \to 0}{\to} \infty$, the desired contradiction.

□

We then prove the coercivity result (Proposition 34) with the same proof as that of Proposition 29.

Local quantitative inequalities around critical shapes Similarly, we have:

Lemma 37. *Let $\varepsilon_1 > 0$ be given by Proposition 34. For any $V_0 \in (1 - \varepsilon_1; 1)$ the critical shapes are isolated in the sense that*

$$\inf_{E, E' \text{ critical, } E' \neq E} |E' \Delta E| = \delta(V_0) > 0.$$

Lemma 38. *Let ε_1 be given by Proposition 34. For any $V_0 \in (1 - \varepsilon_1; 1)$, any critical shape E is a local L^1 minimiser: there exist $c_E > 0$ and $r_E > 0$ such that, for any $f \in \mathcal{F}(V_0)$ with $\|f - \mathbb{1}_E\|_{L^1(\Omega)} \leqslant r_E$, there holds*

$$J(f) - J(\mathbb{1}_E) \geqslant c_E \|f - \mathbb{1}_E\|^2_{L^1(\Omega)}.$$

B.4 Conclusion of the proof of Theorem III

The conclusion of the proof is identical to the conclusion of the proof of Theorem I.

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