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Controllability with one scalar control of a system of interaction between the Navier-Stokes system and a damped beam equation

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Abstract

We consider the controllability of a fluid-structure interaction system, where the fluid is modeled by the Navier-Stokes system and where the structure is a damped beam located on a part of its boundary. The motion of the fluid is bi-dimensional whereas the deformation of the structure is one-dimensional and we use periodic boundary conditions in the horizontal direction. Our result is the local null-controllability of this free-boundary system by using only one scalar control acting on an arbitrary small part of the fluid domain. This improves a previous result obtained by the authors where three scalar controls were needed to achieve the local null-controllability. In order to show the result, we prove the final-state observability of a linear Stokes-beam interaction system in a cylindrical domain. This is done by using a Fourier decomposition, proving Carleman inequalities for the corresponding system for the low-frequencies solutions and in the case where the observation domain is an horizontal strip. Then we conclude this observability result by using a Lebeau-Robbiano strategy for the heat equation and a uniform exponential decay for the high-frequencies solutions. Then, the result on the nonlinear system can be obtained by a change of variables and a fixed-point argument.

Keywords: Null controllability, Navier-Stokes systems, Carleman estimates, fluid-structure interaction systems
2010 Mathematics Subject Classification. 76D05, 35Q30, 74F10, 76D55, 76D27, 93B05, 93B07, 93C10

1 Introduction

In this article, we study the controllability property of a fluid-structure interaction system, in the case where the fluid is incompressible and viscous and in the case where the structure is a deformable beam corresponding to a part of the boundary of the fluid domain. The fluid motion is bidimensional and the elastic displacement of the structure is one-dimensional. To be more precise, let us define the configuration of reference:

$$\mathcal{I} := \mathbb{R}/(2\pi\mathbb{Z}), \quad \Omega := \mathcal{I} \times (0, 1), \quad \Gamma_0 = \mathcal{I} \times \{0\}, \quad \Gamma_1 = \mathcal{I} \times \{1\}.$$

With this notation, Ω corresponds to the fluid domain if there is no displacement of the structure; Γ_1 represents the beam domain that is transformed into

$$\Gamma_{1+\zeta} := \{(x_1, 1 + \zeta(x_1)) ; x_1 \in \mathcal{I}\} \tag{1.1}$$

for an elastic deformation $\zeta : \mathcal{I} \rightarrow (-1, \infty)$. The fluid domain becomes (see, Figure 1).

$$\Omega_\zeta := \{(x_1, x_2) \in \mathcal{I} \times \mathbb{R} ; x_2 \in (0, 1 + \zeta(x_1))\}. \tag{1.2}$$

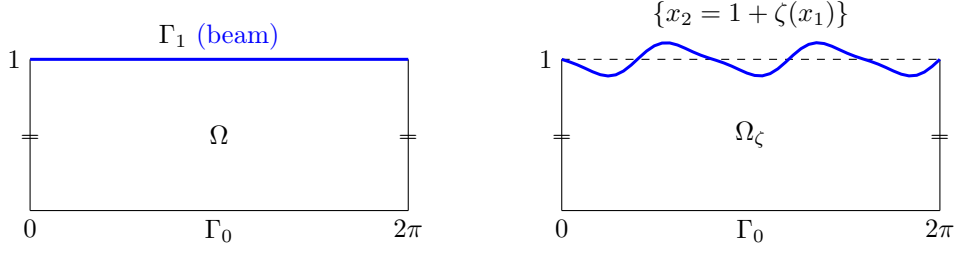


Figure 1: Our geometry

Note that we work with the torus \mathcal{I} in order to have periodic boundary conditions in the e_1 direction, if (e_1, e_2) denotes the canonical basis of \mathbb{R}^2 .

Let us consider an arbitrary small nonempty open subset ω of Ω . Our aim is to show the local null-controllability of the following fluid-structure interaction system

$$\left\{ \begin{array}{l} \partial_t w + (w \cdot \nabla)w - \operatorname{div} \mathbb{T}(w, \pi) = 1_\omega f e_1 \quad t > 0, x \in \Omega_{\zeta(t)}, \\ \operatorname{div} w = 0 \quad t > 0, x \in \Omega_{\zeta(t)}, \\ w(t, x_1, 1 + \zeta(t, x_1)) = (\partial_t \zeta)(t, x_1) e_2, \quad t > 0, x_1 \in \mathcal{I}, \\ w = 0 \quad t > 0, x \in \Gamma_0, \\ \partial_{tt} \zeta + \alpha_1 \partial_{x_1}^4 \zeta - \alpha_2 \partial_{x_1}^2 \zeta - \alpha_3 \partial_t \partial_{x_1}^2 \zeta = -\tilde{\mathbb{H}}_\zeta(w, \pi) \quad t > 0, x_1 \in \mathcal{I}, \\ w(0, \cdot) = w^0 \quad \text{in } \Omega_{\zeta_1^0}, \quad \zeta(0, \cdot) = \zeta_1^0, \quad \partial_t \zeta(0, \cdot) = \zeta_2^0 \quad \text{in } \mathcal{I}, \end{array} \right. \quad (1.3)$$

where

$$\alpha_1 > 0, \quad \alpha_2 \geq 0, \quad \alpha_3 > 0.$$

In the above system, the Cauchy stress tensor and the symmetric gradient are defined as follows:

$$\mathbb{T}(w, \pi) = 2\mu \mathbb{D}(w) - \pi I_2, \quad \mathbb{D}(w) = \frac{1}{2} (\nabla w + (\nabla w)^*), \quad (1.4)$$

with a viscosity $\mu > 0$, whereas the force of the fluid applied on the beam is given by

$$\tilde{\mathbb{H}}_\zeta(w, \pi)(t, x_1) = \left[(1 + |\partial_{x_1} \zeta|^2)^{1/2} [\mathbb{T}(w, \pi) \nu] (t, x_1, 1 + \zeta(t, x_1)) \cdot e_2 \right]. \quad (1.5)$$

In the above formula, ν is the unit exterior normal to $\Omega_{\zeta(t)}$. The first four equations of (1.3) are the Navier-Stokes system satisfied by the velocity w and the pressure π of the fluid. We have assumed the standard no-slip boundary conditions at the boundaries of the fluid domain. We also assume that the beam satisfies the damped beam corresponding to the fifth equation of (1.3). Finally the last line of (1.3) gives the initial conditions for this coupled nonlinear system. We want to control (1.3) by using the distributed control f localized in ω and that acts only in the e_1 direction.

In order to show the local null controllability of (1.3), a standard strategy consists in applying a change of variables to write the fluid system into the cylindrical domain $(0, T) \times \Omega$, and then in linearizing the transformed system. Then one can show that the corresponding linear system

$$\left\{ \begin{array}{l} \partial_t w - \mu \Delta w + \nabla \pi = 1_\omega f e_1 \quad \text{in } (0, T) \times \Omega, \\ \operatorname{div} w = 0 \quad \text{in } (0, T) \times \Omega, \\ w = 0 \quad \text{on } (0, T) \times \Gamma_0, \\ w = (\partial_t \zeta) e_2 \quad \text{on } (0, T) \times \Gamma_1, \\ \partial_{tt} \zeta + \alpha_1 \partial_{x_1}^4 \zeta - \alpha_2 \partial_{x_1}^2 \zeta - \alpha_3 \partial_t \partial_{x_1}^2 \zeta = -\mathbb{T}(w, \pi) \nu \cdot e_2 \quad \text{in } (0, T) \times \mathcal{I}, \\ w(0, \cdot) = w^0 \quad \text{in } \Omega, \quad \zeta(0, \cdot) = \zeta_1^0, \quad \partial_t \zeta(0, \cdot) = \zeta_2^0 \quad \text{in } \mathcal{I}, \end{array} \right. \quad (1.6)$$

is null-controllable. Using a general result in [41], one can deduce the null-controllability of above system with the source terms. Then, we can estimate the coefficients coming from the change of variables and use a fixed-point argument to conclude. We are thus reduced to show the null-controllability of (1.6) or equivalently (see, for instance, [58, Theorem 11.2.1, p.357]) to show the final-state observability of the following system (see, [12, Section 2.1] for more details)

$$\left\{ \begin{array}{l} \partial_t u - \Delta u + \nabla p = 0 \quad \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0 \quad \text{in } (0, T) \times \Omega, \\ u = 0 \quad \text{on } (0, T) \times \Gamma_0, \\ u = \partial_t \eta e_2 \quad \text{on } (0, T) \times \Gamma_1, \\ \partial_t^2 \eta + \partial_{x_1}^4 \eta - \partial_{x_1}^2 \partial_t \eta = p|_{\Gamma_1} \quad \text{in } (0, T) \times \mathcal{I}, \\ u(0, \cdot) = u^0 \quad \text{in } \Omega, \quad \eta(0, \cdot) = \eta_1^0, \quad \partial_t \eta(0, \cdot) = \eta_2^0 \quad \text{in } \mathcal{I}. \end{array} \right. \quad (1.7)$$

In the above system, we have assumed that

$$\alpha_1 = 1, \quad \alpha_2 = 0, \quad \alpha_3 = 1, \quad \mu = 1$$

since these constants will not play any role in our study. On the right-hand side of the beam equation, we should have

$$-\mathbb{T}(u, p)\nu \cdot e_2 = -\partial_{x_2} u_2 + p$$

instead of p , but for regular solutions, on Γ_1 , $-\partial_{x_2} u_2 = \partial_{x_1} u_1 = 0$. Moreover, for regular solutions, we have also the following computations:

$$0 = \int_{\Omega} \operatorname{div} u \, dx = \frac{d}{dt} \int_0^{2\pi} \eta \, dx_1$$

so that

$$\int_0^{2\pi} \eta(t, x_1) \, dx_1 = \int_0^{2\pi} \eta_1^0(x_1) \, dx_1 \quad (t \geq 0).$$

To simplify, we assume that $\eta_1^0, \eta_2^0 \in L_0^2(\mathcal{I})$ where

$$L_0^2(\mathcal{I}) := \left\{ f \in L^2(\mathcal{I}) ; \int_0^{2\pi} f(x_1) \, dx_1 = 0 \right\}$$

so that for any $t \geq 0$, $\eta(t, \cdot), \partial_t \eta(t, \cdot) \in L_0^2(\mathcal{I})$. Using these properties, and integrating the beam equation in (1.7) in \mathcal{I} yield the following condition on the pressure:

$$\int_0^{2\pi} p(t, x_1, 1) \, dx_1 = 0 \quad (t \in (0, T)). \quad (1.8)$$

Let us consider the following operators associated with the beam equation:

$$\mathcal{D}(A_1) := H^4(\mathcal{I}) \cap L_0^2(\mathcal{I}), \quad A_1 \eta := \partial_{x_1}^4 \eta, \quad (1.9)$$

$$\mathcal{D}(A_2) := H^2(\mathcal{I}) \cap L_0^2(\mathcal{I}), \quad A_2 \eta := -\partial_{x_1}^2 \eta. \quad (1.10)$$

We also define the Hilbert space of states for our system:

$$\mathcal{H} := \left\{ (u, \eta_1, \eta_2) \in L^2(\Omega) \times \mathcal{D}(A_1^{1/2}) \times L_0^2(\mathcal{I}) ; u_2 = \eta_2 \text{ on } \Gamma_1, \quad u_2 = 0 \text{ on } \Gamma_0, \quad \operatorname{div} u = 0 \text{ in } \Omega \right\}, \quad (1.11)$$

endowed with the canonical scalar product of $L^2(\Omega) \times \mathcal{D}(A_1^{1/2}) \times L^2(\mathcal{I})$. As we recall it in the next section, for any $[u^0, \eta_1^0, \eta_2^0] \in \mathcal{H}$, (1.7) admits a unique (weak) solution $[u, \eta, \partial_t \eta] \in C^0([0, T]; \mathcal{H})$.

Our main result is the final-state observability for the system (1.7):

Theorem 1.1. *Assume $T > 0$ and ω is a nonempty open set such that $\omega \Subset \Omega$. There exists $C > 0$ such that for any $[u^0, \eta_1^0, \eta_2^0] \in \mathcal{H}$, the solution of (1.7) satisfies*

$$\| [u(T, \cdot), \eta(T, \cdot), \partial_t \eta(T, \cdot)] \|_{\mathcal{H}}^2 \leq C e^{C/T} \iint_{(0,T) \times \omega} |u_1|^2 dx dt. \quad (1.12)$$

Remark 1.2. *The above theorem is one of the first results concerning the observability of a fluid-structure interaction system with a deformable structure. It implies (see Theorem 1.4 below) the local null-controllability of the system (1.6) with a control acting only on the fluid. We also manage to control only on one component of the fluid. It is worth noting that we obtain an optimal cost of the form $e^{C/T}$, whereas in the literature, even without any structure, the cost for the controllability of fluid systems with controls with vanishing components is of the form e^{C/T^m} with $m > 1$ (see the references in Remark 1.3 below). Our method is quite general and could be applied to other systems: for instance one can replace the damped beam equation in (1.6) by a wave equation without damping. In fact in that case, the underlying semigroup is also analytic (see [4]). We could also extend our result in the three-dimensional case with a control acting on two components of the fluid velocity (see [18] for the well-posedness of a similar system).*

In a previous article [12] by the authors, we have obtained the final state observability

$$\| [u(T, \cdot), \eta(T, \cdot), \partial_t \eta(T, \cdot)] \|_{\mathcal{H}}^2 \leq C e^{C/T^2} \left(\iint_{(0,T) \times \omega} |u|^2 dx dt + \iint_{(0,T) \times \mathcal{J}} |\partial_t \eta|^2 dx_1 dt \right), \quad (1.13)$$

where $\mathcal{J} \Subset \mathcal{I}$ is a nonempty open set. The result obtained here improve this previous result by using only one scalar observation instead of three. By duality, we only need one scalar control in (1.6) instead of three scalar controls in [12]. We have also improved the estimate of the cost of the control.

Remark 1.3. *There are several other results in the literature concerning the controllability of N -dimensional Stokes or Navier-Stokes systems with $N - 1$ scalar controls, see for instance [16], [14], [17], etc. Here due to the geometry of Ω , the above observability inequality (1.12) does not hold if we replace u_1 by u_2 . This fact can be seen in the Fourier decomposition described below.*

We can obtain from Theorem 1.1 a local null-controllability of (1.3). To state this result, we first recall the notion of strong solutions for (1.3). Since $\omega \Subset \Omega$, there exists $\varepsilon > 0$ small enough such for any $\zeta \in H^3(\mathcal{I})$ with

$$\| \zeta \|_{H^3(\mathcal{I})} \leq \varepsilon, \quad (1.14)$$

then $\omega \Subset \Omega_\zeta$ and there exists a diffeomorphism $X_\zeta : \Omega \rightarrow \Omega_\zeta$ such that $X_\zeta(\omega) = \omega$. Now, if

$$\zeta \in L^2(0, T; H^4(\mathcal{I})) \cap C^0([0, T]; H^3(\mathcal{I})) \cap H^1(0, T; H^2(\mathcal{I})) \cap C^1([0, T]; H^1(\mathcal{I})) \cap H^2(0, T; L^2(\mathcal{I})) \quad (1.15)$$

is small enough (in the above space), $\zeta(t, \cdot)$ satisfies (1.14) for all $t \in [0, T]$ and we can define the following spaces as follows

$$\begin{aligned} w &\in L^2(0, T; H^2(\Omega_\zeta)) & \text{if } w \circ X_\zeta &\in L^2(0, T; H^2(\Omega)) \\ w &\in H^1(0, T; L^2(\Omega_\zeta)) & \text{if } w \circ X_\zeta &\in H^1(0, T; L^2(\Omega)) \\ \pi &\in L^2(0, T; H^1(\Omega_\zeta)) & \text{if } \pi \circ X_\zeta &\in L^2(0, T; H^1(\Omega)). \end{aligned}$$

A triplet (w, π, ζ) is a strong solution of (1.3) if ζ satisfies (1.15), if

$$w \in L^2(0, T; H^2(\Omega_\zeta)) \cap H^1(0, T; L^2(\Omega_\zeta)), \quad \pi \in L^2(0, T; H^1(\Omega_\zeta))$$

and if all the equations in (1.3) are satisfied a.e. in time and space. The existence and uniqueness of strong solutions have been obtained in [40] and [27] (see also [53]) for small data, that is if

$$\| \zeta_1^0 \|_{H^3(\mathcal{I})} + \| \zeta_2^0 \|_{H^1(\mathcal{I})} + \| w^0 \|_{H^1(\Omega_{\zeta_1^0})} + \| f \|_{L^2(0, T; L^2(\omega))} \quad (1.16)$$

is small enough. We are now in a position to state the following local null-controllability result for (1.3):

Theorem 1.4. *Assume $T > 0$ and ω is a nonempty open set such that $\omega \Subset \Omega$. There exists $R_0 > 0$ such that for any $(\zeta_1^0, \zeta_2^0, w^0) \in \mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4}) \times H^1(\Omega_{\zeta_1^0})$ with the compatibility conditions*

$$\operatorname{div} w^0 = 0 \text{ in } \Omega_{\zeta_1^0}, \quad w^0 = 0 \text{ on } \Gamma_0, \quad w^0(x_1, 1 + \zeta_1^0(x_1)) = \zeta_2^0(x_1)e_2 \quad (x_1 \in \mathcal{I}), \quad (1.17)$$

and with the smallness assumption

$$\|\zeta_1^0\|_{H^3(\mathcal{I})} + \|\zeta_2^0\|_{H^1(\mathcal{I})} + \|w^0\|_{H^1(\Omega_{\zeta_1^0})} \leq R_0, \quad (1.18)$$

there exists a control $f \in L^2(0, T; L^2(\omega))$ such that the solution of (1.3) satisfies

$$\zeta(T, \cdot) = \partial_t \zeta(T, \cdot) = 0 \quad \text{in } \mathcal{I}, \quad w(T, \cdot) = 0 \quad \text{in } \Omega_{\zeta(T, \cdot)}.$$

The proof of Theorem 1.4 is quite standard from Theorem 1.1 and is done in [12], we thus skip its proof here and only show Theorem 1.1.

The system (1.3) (without controls) has been introduced in [52] to model the blood flow in vessels and has been studied by many authors. We can quote the following articles that are only a part of the literature on the subject: [15] (existence of weak solutions), [6], [40], [27] and [44] (existence of strong solutions), [53] (stabilization of strong solutions), [1] (stabilization of weak solutions around a stationary state). In some works, the authors remove the damping on the structure (that is $\alpha_3 = 0$): [26], [47], [59] (weak solutions), [28], [2], [3], [4] (strong solutions). There are also many studies on more complex models: [39, 38] (linear elastic Koiter shell), [48] (dynamic pressure boundary conditions), [49, 50] (3D cylindrical domain with nonlinear elastic cylindrical Koiter shell), [56] and [57] (nonlinear elastic and thermoelastic plate equations), [42], [43] (compressible fluids), etc. Concerning the controllability of fluid-structure interaction systems, note that several works have been done in the case where the structure is a rigid body in [9], [10], [19], [29], [54]. The only result we know about the controllability of the system (1.6) is [12] where we obtain a result with controls on the fluid and on the structure. Here we improve this result as explained in Remark 1.2. We can also mention [46] devoted to an observability inequality for the adjoint of a linearized simplified compressible fluid-structure model similar to our system. Another work in the same topic is [13], where we replace the damped beam equation by a heat equation and we obtain the null-controllability of the corresponding system. As in our previous works, [13] and [12], our strategy consists in considering the Stokes system as heat equations where the pressure is a source term, (in the same spirit as [23] or as [20]). The pressure is estimated by using that it satisfies a Laplace equation, but the difficulty for these estimates is coming from the fact that for this Laplace equation, we don't have any boundary condition. The idea is that, for low frequencies, it is possible to obtain an estimate without boundary conditions. Then, one has to handle the high frequencies of the solution. It was done by using the microlocal analysis in [12], in the same spirit as in many other works on Carleman estimates near boundaries and interfaces (see, for instance, [8, 7, 11, 30, 33, 34, 35] and the recent books [31, 32] for elliptic counterparts).

In this article, we follow a different approach corresponding to the method of [36] for the heat equation. Such a strategy has been used successfully in a similar problem with a free boundary: the Stefan problem with surface tension, in the same geometry as here, see [25]. In particular, we use the same idea that consists to obtain first the observability for an horizontal strip of Ω instead of ω . Then using the Lebeau-Robbiano method and the Lebeau-Robbiano result for the heat equation on the torus \mathcal{I} , we can conclude to the observability in ω . Note that our method of proof, by separating the high frequencies and the low frequencies with Fourier series is similar to the method in [5] (see in particular Theorem 2.1 in this reference).

The outline of the article is as follows: in the next section, we present the functional framework and in particular the decomposition in high and low frequency of the solutions of (1.7) by using the Fourier decomposition in the x_1 variable. We also show a uniform exponential decay of the high-frequency part of the solution. Section 3 is devoted to the Carleman estimates of the low frequency part of the solution. We use the low frequency hypothesis to handle the fact that the pressure satisfies a Laplace equation with no boundary conditions. In this section, we have an observation in an horizontal strip of Ω and with several observations that can be expressed in terms of the derivatives of u_1 . Then in Section 4, we show the main result, that is, Theorem 1.1, by removing the additional observations and by reducing the region of control. Then we combine the estimates for low frequencies and high frequencies with a Lebeau-Robbiano strategy to deduce the result.

Notation. In the whole paper, we use C as a generic positive constant that does not depend on the other terms of the inequality. The value of the constant C may change from one appearance to another. We also use the notation $X \lesssim Y$ if there exists a constant $C > 0$ such that we have the inequality $X \leq CY$. The notation $X \lesssim_k Y$ stands for $X \leq CY$, where C is a positive constant depending on k .

2 Functional framework and Fourier decomposition

2.1 Functional framework

In the introduction, we have already introduced the operators A_1 and A_2 in (1.9) and (1.10) and the Hilbert space \mathcal{H} in (1.11). We define the orthogonal projection on the space \mathcal{H} :

$$\mathcal{P} : L^2(\Omega) \times \mathcal{D}(A_1^{1/2}) \times L_0^2(\mathcal{I}) \rightarrow \mathcal{H}.$$

We recall (see, for instance, [1, Proposition 3.1]) that the orthogonal of \mathcal{H} in $L^2(\Omega) \times \mathcal{D}(A_1^{1/2}) \times L_0^2(\mathcal{I})$ is given by

$$\mathcal{H}^\perp = \left\{ (\nabla p, 0, -p|_{\Gamma_1}) ; p \in H^1(\Omega), \int_0^{2\pi} p(x_1, 1) dx_1 = 0 \right\}. \quad (2.1)$$

Then we define the space

$$\mathcal{V} := \left\{ (u, \eta_1, \eta_2) \in H^1(\Omega) \times \mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4}) ; u = \eta_2 e_2 \text{ on } \Gamma_1, u = 0 \text{ on } \Gamma_0, \operatorname{div} u = 0 \text{ in } \Omega \right\},$$

and the unbounded operator \mathcal{A} associated with (1.7):

$$\mathcal{D}(\mathcal{A}) := \mathcal{V} \cap \left[H^2(\Omega) \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2}) \right], \quad \mathcal{A} \begin{bmatrix} u \\ \eta_1 \\ \eta_2 \end{bmatrix} := \mathcal{P} \begin{bmatrix} \Delta u \\ \eta_2 \\ -A_1 \eta_1 - A_2 \eta_2 \end{bmatrix}. \quad (2.2)$$

More precisely, we can write (1.7) in the form

$$\frac{d}{dt} \begin{bmatrix} u \\ \eta \\ \partial_t \eta \end{bmatrix} = \mathcal{A} \begin{bmatrix} u \\ \eta \\ \partial_t \eta \end{bmatrix} \quad (t \geq 0), \quad \begin{bmatrix} u \\ \eta \\ \partial_t \eta \end{bmatrix} (0) = \begin{bmatrix} u^0 \\ \eta_1^0 \\ \eta_2^0 \end{bmatrix}. \quad (2.3)$$

It is shown (see, for instance, [1, Proposition 3.4 and Proposition 3.11]) that \mathcal{A} is the infinitesimal generator of an analytic semigroup of contractions on \mathcal{H} and this shows the existence and uniqueness of a weak solution $[u, \eta, \partial_t \eta] \in C^0([0, T]; \mathcal{H})$ of (1.7) for any $[u^0, \eta_1^0, \eta_2^0] \in \mathcal{H}$. Moreover, it is shown in [1] the existence of $\delta \in \left(0, \frac{\pi}{2}\right)$ such that

$$\rho(\mathcal{A}) \subset \Sigma := \left\{ \lambda \in \mathbb{C}^* ; |\arg \lambda| < \frac{\pi}{2} + \delta \right\} \cup \{0\} \quad (2.4)$$

and

$$\left\| (\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \lesssim \frac{1}{|\lambda|} \quad (\lambda \in \Sigma_\delta \setminus \{0\}). \quad (2.5)$$

We need this property for the study of the exponential decay of the high frequencies of the solutions of (1.7).

Finally, we have the following result:

Proposition 2.1. *The operator \mathcal{A} defined above satisfies for any $k \geq 1$,*

$$\mathcal{D}(\mathcal{A}^k) \subset H^{2k}(\Omega) \times H^{2(k+1)}(\mathcal{I}) \times H^{2k}(\mathcal{I})$$

and

$$\|u\|_{H^{2k}(\Omega)} + \|\eta_1\|_{H^{2(k+1)}(\mathcal{I})} + \|\eta_2\|_{H^{2k}(\mathcal{I})} \lesssim_k \|(u, \eta_1, \eta_2)\|_{\mathcal{D}(\mathcal{A}^k)} \quad ((u, \eta_1, \eta_2) \in \mathcal{D}(\mathcal{A}^k)).$$

Proof. The proof can be done by induction, we have already the case $k = 1$. If $(u, \eta_1, \eta_2) \in \mathcal{D}(\mathcal{A}^{k+1})$, then

$$(f, g_1, g_2) := -\mathcal{A}(u, \eta_1, \eta_2) \in \mathcal{D}(\mathcal{A}^k)$$

In particular, from (2.1) and (2.2),

$$\left\{ \begin{array}{ll} -\Delta u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u|_{\Gamma_0} = 0 & \text{on } \Gamma_0 \\ u|_{\Gamma_1} = \eta_2 e_2 & \text{on } \Gamma_1 \\ -\eta_2 = g_1 & \text{in } \mathcal{I} \\ A_2 \eta_2 + A_1 \eta_1 = p + g_2 & \text{in } \mathcal{I}. \end{array} \right.$$

We thus deduce that $\eta_2 \in \mathcal{D}(A_1^{(k+1)/2}) \subset H^{2(k+1)}(\mathcal{I})$, and applying the elliptic regularity of the Stokes system (see, [55, Proposition 2.2, p. 33]), we deduce $u \in H^{2k+2}(\Omega)$, $p \in H^{2k+1}(\Omega)$. From the last equation of the above system, we finally obtain $A_1 \eta_1 \in \mathcal{D}(A_1^{k/2})$ and thus $\eta_1 \in \mathcal{D}(A_1^{(k+2)/2})$. \square

To complete this functional framework, we introduce the space $\mathcal{U} := L^2(\omega)$ and the observation operator $\mathcal{C} \in \mathcal{L}(\mathcal{H}, \mathcal{U})$ defined by

$$\mathcal{C} \begin{bmatrix} u \\ \eta_1 \\ \eta_2 \end{bmatrix} := u|_{\omega}.$$

Then our main result, that is Theorem 1.1 writes

$$\left\| e^{TA} \begin{bmatrix} u^0 \\ \eta_1^0 \\ \eta_2^0 \end{bmatrix} \right\|_{\mathcal{H}} \leq C e^{C/T} \int_0^T \left\| \mathcal{C} e^{tA} \begin{bmatrix} u^0 \\ \eta_1^0 \\ \eta_2^0 \end{bmatrix} \right\|_{\mathcal{U}} dt \quad \left(\begin{bmatrix} u^0 \\ \eta_1^0 \\ \eta_2^0 \end{bmatrix} \in \mathcal{H} \right). \quad (2.6)$$

As it is standard, we are going to prove the above observability inequality for $(u^0, \eta_1^0, \eta_2^0) \in \bigcap_{k \geq 1} \mathcal{D}(\mathcal{A}^k)$ that is dense in \mathcal{H} (see, for instance, [58, Proposition 2.3.6, p.30]) and we obtain the result by using the continuity of the operators involved in the inequality. With this choice, we deduce from Proposition 2.1 that the corresponding solution of (1.7) (or equivalently (2.3)) satisfies for any $k, \ell \in \mathbb{N}$,

$$u \in C^\ell([0, \infty); H^{2k}(\Omega)), \quad \eta \in C^\ell([0, \infty); H^{2k}(\mathcal{I})).$$

2.2 Fourier series

In order to show Theorem 1.1, we decompose the solution of (1.7) using the Fourier series: we set for all $n \in \mathbb{Z}$,

$$E_n(x_1) := \frac{e^{inx_1}}{\sqrt{2\pi}}, \quad (2.7)$$

and

$$u^{(n)}(t, x_2) = (u(t, \cdot, x_2), E_n)_{L^2(\mathcal{I})}, \quad p^{(n)}(t, x_2) = (p(t, \cdot, x_2), E_n)_{L^2(\mathcal{I})}, \quad \eta^{(n)}(t) = (\eta(t, \cdot), E_n)_{L^2(\mathcal{I})},$$

where $(\cdot, \cdot)_{L^2(\mathcal{I})}$ is the standard hermitian product. We can check that $(\eta^{(n)}, u^{(n)}, p^{(n)})$ satisfies, for $n \in \mathbb{Z}$, the following system

$$\left\{ \begin{array}{ll} \partial_t u^{(n)} + n^2 u^{(n)} - \partial_{x_2}^2 u^{(n)} + \begin{bmatrix} in p^{(n)} \\ \partial_{x_2} p^{(n)} \end{bmatrix} = 0 & \text{in } (0, T) \times (0, 1), \\ in u_1^{(n)} + \partial_{x_2} u_2^{(n)} = 0 & \text{in } (0, T) \times (0, 1), \\ u^{(n)}(t, 0) = 0 & t \in (0, T), \\ u^{(n)}(t, 1) = \partial_t \eta^{(n)}(t) e_2 & t \in (0, T), \\ \partial_t^2 \eta^{(n)} + n^4 \eta^{(n)} + n^2 \partial_t \eta^{(n)} = p^{(n)}(\cdot, 1) & \text{in } (0, T). \end{array} \right. \quad (2.8)$$

In the case $n = 0$, using (1.11) and the above system yields

$$\eta^{(0)} \equiv 0, \quad u_2^{(0)} \equiv 0, \quad p^{(0)} \equiv 0, \quad (2.9)$$

whereas $u_1^{(0)}$ satisfies the heat equation:

$$\begin{cases} \partial_t u_1^{(0)} - \partial_{x_2}^2 u_1^{(0)} = 0 & \text{in } (0, T) \times (0, 1), \\ u_1^{(0)} = 0 & \text{in } (0, T) \times \{0, 1\}. \end{cases} \quad (2.10)$$

Therefore, we can write

$$u(t, x_1, x_2) = \sum_{n \in \mathbb{Z}} u^{(n)}(t, x_2) E_n(x_1), \quad p(t, x_1, x_2) = \sum_{n \in \mathbb{Z}^*} p^{(n)}(t, x_2) E_n(x_1), \quad (2.11)$$

$$\eta(t, x_1) = \sum_{n \in \mathbb{Z}^*} \eta^{(n)}(t) E_n(x_1). \quad (2.12)$$

Remark 2.2. *This decomposition and in particular (2.10) justify that one can not expect to obtain an observability inequality such as (1.12) with u_2 instead of u_1 (see Remark 1.3).*

In order to study (1.7), we are going to decompose the solution into low and high frequencies: for $N \in \mathbb{N}^*$, we set

$$u^-(t, x_1, x_2) = \sum_{|n| \leq N} u^{(n)}(t, x_2) E_n(x_1), \quad p^-(t, x_1, x_2) = \sum_{0 < |n| \leq N} p^{(n)}(t, x_2) E_n(x_1), \quad (2.13)$$

$$\eta^-(t, x_1) = \sum_{0 < |n| \leq N} \eta^{(n)}(t) E_n(x_1), \quad (2.14)$$

$$u^+(t, x_1, x_2) = \sum_{|n| > N} u^{(n)}(t, x_2) E_n(x_1), \quad p^+(t, x_1, x_2) = \sum_{|n| > N} p^{(n)}(t, x_2) E_n(x_1), \quad (2.15)$$

$$\eta^+(t, x_1) = \sum_{|n| > N} \eta^{(n)}(t) E_n(x_1). \quad (2.16)$$

We will show the observability for the low frequency part, and obtain an exponential decay for the high-frequency part. Let us set for all $n \in \mathbb{Z}$,

$$\Pi_n f := (f, E_n)_{L^2(\mathcal{I})} E_n. \quad (2.17)$$

In particular, Π_n is an orthogonal projection in $L^2(\mathcal{I})$ and in $L^2(\Omega)$. We keep the notation Π_n to denote the following orthogonal projection

$$\Pi_n : L^2(\Omega) \times \mathcal{D}(A_1^{1/2}) \times L_0^2(\mathcal{I}) \rightarrow L^2(\Omega) \times \mathcal{D}(A_1^{1/2}) \times L_0^2(\mathcal{I}), \quad \Pi_n(f, g, h) := (\Pi_n f, \Pi_n g, \Pi_n h).$$

We have the preliminary result

Lemma 2.3. *For all $n \in \mathbb{Z}$,*

$$\Pi_n(\mathcal{H}) \subset \mathcal{H}, \quad \Pi_n(\mathcal{H}^\perp) \subset \mathcal{H}^\perp.$$

In particular, $\Pi_n \mathcal{P} = \mathcal{P} \Pi_n$.

Proof. Assume $(u, \eta_1, \eta_2) \in \mathcal{D}(\mathcal{A})$ (see (2.2)). Then one can check that $(\Pi_n u, \Pi_n \eta_1, \Pi_n \eta_2)$ satisfies all the conditions in (1.11). Then, since Π_n is a bounded operator in $L^2(\Omega) \times H^2(\mathcal{I}) \times L^2(\mathcal{I})$ and since \mathcal{H} is a closed subspace of $L^2(\Omega) \times H^2(\mathcal{I}) \times L^2(\mathcal{I})$, we deduce that $\Pi_n(\mathcal{H}) \subset \mathcal{H}$. The remaining properties can be obtained from this one from standard algebra results. \square

Let us set $\mathcal{H}_n := \Pi_n \mathcal{H}$. We deduce from the above lemma the following result:

Lemma 2.4. *Assume $n \in \mathbb{Z}$. Then, we have*

$$\begin{aligned} \mathcal{A}(\mathcal{D}(\mathcal{A}) \cap \mathcal{H}_n) &\subset \mathcal{H}_n, \\ U \in \mathcal{D}(\mathcal{A}) &\implies \Pi_n U \in \mathcal{D}(\mathcal{A}) \text{ and } \mathcal{A}\Pi_n U = \Pi_n \mathcal{A}U, \\ (\lambda I - \mathcal{A})^{-1} \Pi_n &= \Pi_n (\lambda I - \mathcal{A})^{-1} \quad (\lambda \in \rho(\mathcal{A})), \quad e^{t\mathcal{A}} \Pi_n = \Pi_n e^{t\mathcal{A}} \quad (t \geq 0). \end{aligned}$$

For $N \in \mathbb{N}^*$, we define

$$\Pi^{[N]} := \sum_{|n| \leq N} \Pi_n \tag{2.18}$$

and we deduce from the above lemma that for $t \geq 0$ and $\lambda \in \rho(\mathcal{A})$

$$(\lambda I - \mathcal{A})^{-1} \Pi^{[N]} = \Pi^{[N]} (\lambda I - \mathcal{A})^{-1}, \quad e^{t\mathcal{A}} \Pi^{[N]} = \Pi^{[N]} e^{t\mathcal{A}}. \tag{2.19}$$

2.3 Exponential decay of the high-frequency solutions

The aim of this section is to show the following results

Theorem 2.5. *There exists a constant $C > 0$ such that for any $N \in \mathbb{N}^*$, $t \geq 0$ and $U \in \mathcal{H}$,*

$$\left\| e^{t\mathcal{A}} (\mathbf{I} - \Pi^{[N]}) U \right\|_{\mathcal{H}} \leq \frac{1}{C} e^{-C(N+1)^2 t} \left\| (\mathbf{I} - \Pi^{[N]}) U \right\|_{\mathcal{H}}. \tag{2.20}$$

The theorem will be a direct consequence of Proposition 2.7 below and of (2.19). First, we use Lemma 2.3 to define

$$\mathcal{D}(\mathcal{A}_n) := \mathcal{D}(\mathcal{A}) \cap \mathcal{H}_n, \quad \mathcal{A}_n : \mathcal{D}(\mathcal{A}_n) \rightarrow \mathcal{H}_n, \quad U \mapsto \mathcal{A}U.$$

Moreover, $\rho(\mathcal{A}) \subset \rho(\mathcal{A}_n)$ and

$$(\lambda I - \mathcal{A})^{-1} \Pi_n = (\lambda I - \mathcal{A}_n)^{-1} \Pi_n \quad (\lambda \in \rho(\mathcal{A})). \tag{2.21}$$

Let us show the following result on \mathcal{A}_n :

Proposition 2.6. *There exists $\varepsilon > 0$ such that for all $n \in \mathbb{Z}^*$, if $|\lambda| \leq \varepsilon n^2$ then*

$$\lambda \in \rho(\mathcal{A}_n) \quad \text{and} \quad \left\| (\lambda I - \mathcal{A}_n)^{-1} \right\|_{\mathcal{L}(\mathcal{H}_n)} \lesssim \frac{1}{n^2}. \tag{2.22}$$

Proof. From [1, Proposition 3.5], for any $(u, \eta_1, \eta_2) \in \mathcal{D}(\mathcal{A})$, we have that

$$\left\| \begin{bmatrix} u \\ \eta_1 \\ \eta_2 \end{bmatrix} \right\|_{H^2(\Omega) \times \mathcal{D}(\mathcal{A}_1) \times \mathcal{D}(\mathcal{A}_1^{1/2})} \lesssim \left\| \mathcal{A} \begin{bmatrix} u \\ \eta_1 \\ \eta_2 \end{bmatrix} \right\|_{\mathcal{H}}.$$

In particular, if $(u^{(n)}, \eta_1^{(n)}, \eta_2^{(n)}) \in \mathcal{D}(\mathcal{A}_n)$ then

$$n^2 \left\| \begin{bmatrix} u^{(n)} \\ \eta_1^{(n)} \\ \eta_2^{(n)} \end{bmatrix} \right\|_{\mathcal{H}} \lesssim \left\| \mathcal{A}_n \begin{bmatrix} u^{(n)} \\ \eta_1^{(n)} \\ \eta_2^{(n)} \end{bmatrix} \right\|_{\mathcal{H}}.$$

This shows that $\left\| \mathcal{A}_n^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \lesssim 1/n^2$. Moreover, by writing $\lambda I - \mathcal{A}_n = \mathcal{A}_n (\lambda \mathcal{A}_n^{-1} - I)$ we deduce that if $\left\| \lambda \mathcal{A}_n^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \leq 1/2$, then $\lambda \in \rho(\mathcal{A}_n)$ and (2.22) holds. This concludes the proposition. \square

We deduce from the above result that

Proposition 2.7. *There exists $C > 0$ such that for any $n \in \mathbb{Z}^*$, $t \geq 0$ and $U \in \mathcal{H}$,*

$$\|e^{t\mathcal{A}}\Pi_n U\|_{\mathcal{H}} \leq \frac{1}{C} e^{-Cn^2 t} \|\Pi_n U\|_{\mathcal{H}}. \quad (2.23)$$

Proof. We recall that the operator \mathcal{A} satisfies (2.4) and (2.5). Applying [51, Theorem 7.7, p.30], we can write for any $t > 0$

$$e^{t\mathcal{A}}\Pi_n = \frac{1}{2\pi i} \left(\int_{\gamma_1} e^{\lambda t} (\lambda I - \mathcal{A})^{-1} \Pi_n d\lambda + \int_{\gamma_0} e^{\lambda t} (\lambda I - \mathcal{A})^{-1} \Pi_n d\lambda + \int_{\gamma_{-1}} e^{\lambda t} (\lambda I - \mathcal{A})^{-1} \Pi_n d\lambda \right) \quad (2.24)$$

where

$$\begin{aligned} \gamma_1 &:= \left\{ r e^{i(\pi/2+\delta/2)} ; r \in [\varepsilon n^2, \infty) \right\}, & \gamma_{-1} &:= \left\{ r e^{-i(\pi/2+\delta/2)} ; r \in [\varepsilon n^2, \infty) \right\}, \\ \gamma_0 &:= \left\{ \varepsilon n^2 e^{i\theta} ; \theta \in \left[-\frac{\pi+\delta}{2}, \frac{\pi+\delta}{2} \right] \right\}, \end{aligned}$$

with $\varepsilon > 0$ given by Proposition 2.6. First, we use (2.5) to obtain that

$$\left\| \int_{\gamma_1} e^{\lambda t} (\lambda I - \mathcal{A})^{-1} \Pi_n d\lambda \right\|_{\mathcal{L}(\mathcal{H})} \lesssim \int_{\varepsilon n^2}^{\infty} e^{-r \sin \frac{\delta}{2} t} \frac{dr}{r} \lesssim \frac{1}{\varepsilon n^2 \sin \frac{\delta}{2} t} e^{-\varepsilon n^2 \sin \frac{\delta}{2} t}.$$

In particular, for $t \geq \frac{1}{n^2}$,

$$\left\| \int_{\gamma_1} e^{\lambda t} (\lambda I - \mathcal{A})^{-1} \Pi_n d\lambda \right\|_{\mathcal{L}(\mathcal{H})} \lesssim \frac{1}{\varepsilon \sin \frac{\delta}{2}} e^{-\varepsilon n^2 \sin \frac{\delta}{2} t}. \quad (2.25)$$

We have a similar estimate for γ_{-1} .

From (2.21), we have

$$\int_{\gamma_0} e^{\lambda t} (\lambda I - \mathcal{A})^{-1} \Pi_n d\lambda = \left(\int_{\gamma_0} e^{\lambda t} (\lambda I - \mathcal{A}_n)^{-1} d\lambda \right) \Pi_n \quad (2.26)$$

and from Proposition 2.6, we can write

$$\int_{\gamma_0} e^{\lambda t} (\lambda I - \mathcal{A}_n)^{-1} d\lambda = \int_{\tilde{\gamma}_0} e^{\lambda t} (\lambda I - \mathcal{A}_n)^{-1} d\lambda \quad (2.27)$$

where

$$\tilde{\gamma}_0 := \left\{ -\varepsilon n^2 \sin \frac{\delta}{2} + ir ; r \in \left[-\varepsilon n^2 \cos \frac{\delta}{2}, \varepsilon n^2 \cos \frac{\delta}{2} \right] \right\}.$$

Combining this with (2.22) and (2.26), we deduce that

$$\left\| \int_{\gamma_0} e^{\lambda t} (\lambda I - \mathcal{A})^{-1} \Pi_n d\lambda \right\|_{\mathcal{L}(\mathcal{H})} \lesssim 2\varepsilon \cos \frac{\delta}{2} e^{-\varepsilon n^2 \sin \frac{\delta}{2} t}. \quad (2.28)$$

Gathering (2.24) and (2.25) with the above estimate, we deduce the existence of a constant $C = C(\varepsilon, \delta) > 0$ such that for all $t \geq \frac{1}{n^2}$,

$$\|e^{t\mathcal{A}}\Pi_n\|_{\mathcal{L}(\mathcal{H})} \leq \frac{1}{C} e^{-Cn^2 t}.$$

On the other hand, using standard properties on strongly continuous semigroups, there exists a constant $C > 0$ such that for any $t \in [0, 1]$,

$$\|e^{t\mathcal{A}}\Pi_n\|_{\mathcal{L}(\mathcal{H})} \leq \|e^{t\mathcal{A}}\|_{\mathcal{L}(\mathcal{H})} \leq C.$$

Combining the last two estimates, we deduce (2.23). \square

3 Carleman estimates with an horizontal strip observation

We are going to show Carleman estimates for the low frequency part of the solutions of (1.7). More precisely, for $N \in \mathbb{N}^*$, we consider here (η^-, u^-, p^-) given by (2.13), (2.14). It is the solution of (1.7) associated with the initial condition $\Pi^{[N]}(u^0, \eta_1^0, \eta_2^0)$ where $\Pi^{[N]}$ is defined by (2.18). Using Lemma 2.4, we can assume that the above initial condition is in $\bigcap_{k \geq 1} \mathcal{D}(\mathcal{A}^k)$ and thus, with Proposition 2.1, that (η^-, u^-, p^-) is a smooth solution of (1.7). For sake of clarity, we drop here the exponent “-” and assume that (η, u, p) is a smooth solution of (1.7) with initial data of the form

$$u^0(x_1, x_2) = \sum_{|n| \leq N} u^{0,(n)}(x_2) E_n(x_1), \quad \eta_1^0(x_1) = \sum_{0 < |n| \leq N} \eta_1^{0,(n)} E_n(x_1), \quad (3.1)$$

$$\eta_2^0(x_1) = \sum_{0 < |n| \leq N} \eta_2^{0,(n)} E_n(x_1). \quad (3.2)$$

In this section, we consider that the region of observation is an horizontal strip: since ω is nonempty open subset of Ω and we can assume the existence of $a_1, a_2 \in \mathcal{I}$ and $b_1, b_2 \in (0, 1)$ such that $a_1 < a_2, b_1 < b_2$ and

$$(a_1, a_2) \times (b_1, b_2) \subset \omega \quad (3.3)$$

We also consider $\tilde{b}_1, \tilde{b}_2 \in (b_1, b_2), \tilde{b}_1 < \tilde{b}_2$ and we set

$$\mathbb{I} := \mathcal{I} \times (b_1, b_2), \quad \tilde{\mathbb{I}} := \mathcal{I} \times (\tilde{b}_1, \tilde{b}_2). \quad (3.4)$$

Let us introduce the corresponding Carleman weights. We consider

$$\begin{aligned} \psi \in C^\infty([0, 1], [0, 1]), \quad \psi > 0 \text{ in } (0, 1), \quad \psi'(x_2) = 0 \implies x_2 \in (\tilde{b}_1, \tilde{b}_2), \\ \psi(x_2) = x_2 \quad \left(x_2 \in \left(0, \frac{\tilde{b}_1}{2} \right) \right), \quad \psi(x_2) = 1 - x_2 \quad \left(x_2 \in \left(\frac{1 + \tilde{b}_2}{2}, 1 \right) \right). \end{aligned}$$

Then we set for $\lambda \geq 1$

$$\varphi(t, x_2) := \frac{e^{\lambda(2+\psi(x_2))} - e^{4\lambda}}{t(T-t)}, \quad \xi(t, x_2) := \frac{e^{\lambda(2+\psi(x_2))}}{t(T-t)}. \quad (3.5)$$

We set

$$\varphi_0(t) := \frac{e^{2\lambda} - e^{4\lambda}}{t(T-t)}, \quad \varphi_1(t) := \frac{e^{3\lambda} - e^{4\lambda}}{t(T-t)}, \quad \xi_0(t) := \frac{e^{2\lambda}}{t(T-t)}, \quad \xi_1(t) := \frac{e^{3\lambda}}{t(T-t)} \quad (3.6)$$

so that from (3.5)

$$\varphi_0(t) \leq \varphi(t, x_2) \leq \varphi_1(t) \quad \text{and} \quad \xi_0(t) \leq \xi(t, x_2) \leq \xi_1(t) \quad (t \in (0, T), x_2 \in (0, 1)). \quad (3.7)$$

Assume $s_0 > 0$. Then, for any $s \geq s_0 N T^2$, we have

$$s \lambda \xi_0(t) \geq 4 s_0 N. \quad (3.8)$$

Notation. In all that follows, we use s_0, λ_0 as generic positive constants that may change from one appearance to another, but always in an increasing way.

If (η, u, p) is a strong solution of (1.7), then u_1 and p satisfy the following equations

$$\begin{cases} \partial_t u_1 - \Delta u_1 = -\partial_{x_1} p & \text{in } (0, T) \times \Omega, \\ u_1 = 0 & \text{on } (0, T) \times \partial\Omega, \end{cases} \quad \begin{cases} \partial_t^2 u_1 - \Delta \partial_t u_1 = -\partial_{x_1} \partial_t p & \text{in } (0, T) \times \Omega, \\ \partial_t u_1 = 0 & \text{on } (0, T) \times \partial\Omega, \end{cases} \quad (3.9)$$

$$\Delta p = 0 \quad \text{in } (0, T) \times \Omega, \quad \Delta \partial_t p = 0 \quad \text{in } (0, T) \times \Omega. \quad (3.10)$$

Let us introduce

$$\begin{aligned} I_1(s, \lambda, u, \eta) := & \iint_{(0, T) \times \Omega} e^{2s\varphi} \left((s\lambda\xi)^8 |u_1|^2 + (s\lambda\xi)^6 |\nabla u_1|^2 + (s\lambda\xi)^4 \left[|\nabla^2 u_1|^2 + |\partial_t u_1|^2 \right] \right) dt dx \\ & + \iint_{(0, T) \times \Omega} e^{2s\varphi} \left((s\lambda\xi)^2 |\nabla \partial_t u_1|^2 + |\nabla^2 \partial_t u_1|^2 + |\partial_t^2 u_1|^2 \right) dt dx \\ & + \iint_{(0, T) \times \Omega} e^{2s\varphi_0} \left((s\lambda\xi_0)^6 \left(|u_2|^2 + |\partial_{x_2} u_2|^2 \right) + (s\lambda\xi_0)^2 \left(|\partial_t u_2|^2 + |\partial_{x_2} \partial_t u_2|^2 \right) \right) dt dx \\ & + \iint_{(0, T) \times \mathcal{I}} e^{2s\varphi_0} \left((s\lambda\xi_0)^6 |\partial_t \eta|^2 + (s\lambda\xi_0)^2 |\partial_t^2 \eta|^2 \right) dt dx_1. \end{aligned} \quad (3.11)$$

We also recall that $\tilde{\Gamma}$ is introduced in (3.4). We first estimate the velocities of the fluid and on the structure.

Lemma 3.1. *There exist $\lambda_0, s_0 > 0$ such that for any $T > 0$, $\lambda \geq \lambda_0$, $s \geq s_0(T + T^2)$ and for any strong solution (η, u, p) of (1.7), we have*

$$\begin{aligned} I_1(s, \lambda, u, \eta) \lesssim & \iint_{(0, T) \times \tilde{\Gamma}} (s\lambda\xi)^8 e^{2s\varphi} |u_1|^2 dt dx + \iint_{(0, T) \times \tilde{\Gamma}} (s\lambda\xi)^2 e^{2s\varphi} |\partial_t u_1|^2 dt dx \\ & + \iint_{(0, T) \times \Omega} e^{2s\varphi} \left(s^5 \lambda^4 \xi^5 |\partial_{x_1} p|^2 + s\xi |\partial_{x_1} \partial_t p|^2 \right) dt dx. \end{aligned} \quad (3.12)$$

Proof. We can apply a standard parabolic Carleman estimates on (3.9) (see, for instance, [24] or [22]): there exist $\lambda_0, s_0 > 0$ such that for any $T > 0$, $\lambda \geq \lambda_0$, $s \geq s_0(T + T^2)$

$$\begin{aligned} & \iint_{(0, T) \times \Omega} e^{2s\varphi} \left((s\lambda\xi)^8 |u_1|^2 + (s\lambda\xi)^6 |\nabla u_1|^2 + (s\lambda\xi)^4 \left[|\nabla^2 u_1|^2 + |\partial_t u_1|^2 \right] \right) dt dx \\ & \lesssim \iint_{(0, T) \times \Omega} e^{2s\varphi} s^5 \lambda^4 \xi^5 |\partial_{x_1} p|^2 dt dx + \iint_{(0, T) \times \tilde{\Gamma}} e^{2s\varphi} (s\xi\lambda)^8 |u_1|^2 dt dx. \end{aligned} \quad (3.13)$$

Then, using that $\partial_{x_2} u_2 = -\partial_{x_1} u_1$, (3.7) and the Poincaré inequality, we deduce that

$$\begin{aligned} & \iint_{(0, T) \times \Omega} e^{2s\varphi_0} (s\lambda\xi_0)^6 \left(|u_2|^2 + |\partial_{x_2} u_2|^2 \right) dt dx \\ & \lesssim \iint_{(0, T) \times \Omega} e^{2s\varphi} s^5 \lambda^4 \xi^5 |\partial_{x_1} p|^2 dt dx + \iint_{(0, T) \times \tilde{\Gamma}} e^{2s\varphi} (s\xi\lambda)^8 |u_1|^2 dt dx. \end{aligned} \quad (3.14)$$

From the trace theorem and the boundary conditions of (1.7), we also deduce

$$\iint_{(0, T) \times \Gamma_1} e^{2s\varphi_0} (s\lambda\xi_0)^6 |\partial_t \eta|^2 dt dx_1 \lesssim \iint_{(0, T) \times \Omega} e^{2s\varphi} s^5 \lambda^4 \xi^5 |\partial_{x_1} p|^2 dt dx + \iint_{(0, T) \times \tilde{\Gamma}} e^{2s\varphi} (s\xi\lambda)^8 |u_1|^2 dt dx.$$

Gathering (3.13), (3.14), the above relation and similar relations for the time derivatives of u and $\partial_t \eta$, we deduce (3.12). \square

On the other hand, we have the following Carleman estimate for the Laplace operator

Lemma 3.2. *Assume $r \in \mathbb{R}$. There exist $\lambda_0, s_0 > 0$ such that for any $T > 0$, $\lambda \geq \lambda_0$, $s \geq s_0(T + T^2)$ and for any $p \in L^2(0, T; H^2(\Omega))$,*

$$\begin{aligned} & \lambda \iint_{(0, T) \times \Omega} e^{2s\varphi} \left((s\lambda\xi)^{2r+3} |p|^2 + (s\lambda\xi)^{2r+1} |\nabla p|^2 \right) dt dx \\ & \quad + \iint_{(0, T) \times \partial\Omega} e^{2s\varphi_0} (s\lambda\xi_0)^{2r+1} \left((s\lambda\xi_0)^2 |p|_{\partial\Omega}|^2 - 2 |\partial_{x_1} p|_{\partial\Omega}|^2 \right) dt dx_1 \\ & \lesssim \iint_{(0, T) \times \Omega} |-\Delta p|^2 (s\lambda\xi)^{2r} e^{2s\varphi} dt dx + \iint_{(0, T) \times \bar{\Omega}} e^{2s\varphi} (s\lambda\xi)^{2r+3} |p|^2 dx. \end{aligned} \quad (3.15)$$

Proof. We set

$$q = (s\lambda\xi)^r e^{s\varphi} p \quad (3.16)$$

and we perform standard computations (see, for instance, [21], [36], [31, pp.106–117]), to obtain the existence of positive constants C, s_0 such that for $s \geq s_0(T + T^2)$,

$$\begin{aligned} & \frac{1}{C} \iint_{(0, T) \times \Omega} \left(s^3 \lambda^4 \xi^3 |q|^2 + s \lambda^2 \xi |\nabla q|^2 + \frac{1}{s\xi} |\Delta q|^2 \right) dt dx \\ & \quad + \operatorname{Re} \iint_{(0, T) \times \partial\Omega} \left(s^3 \lambda^3 \xi_0^3 |q|^2 - 2s \lambda^2 \xi_0 \frac{\partial q}{\partial n} \bar{q} + 2s \lambda \xi_0 \left| \frac{\partial q}{\partial n} \right|^2 - s \lambda \xi_0 |\nabla q|^2 \right) dt dx_1 \\ & \leq C \left(\iint_{(0, T) \times \Omega} |-\Delta p|^2 (s\lambda\xi)^{2r} e^{2s\varphi} dt dx + \iint_{(0, T) \times \bar{\Omega}} s^3 \lambda^4 \xi^3 |q|^2 dx \right). \end{aligned} \quad (3.17)$$

In particular, there exists $s_0 > 0$ such that for $s \geq s_0(T + T^2)$,

$$\left| 2s \lambda^2 \xi_0 \frac{\partial q}{\partial n} \bar{q} \right| \leq \frac{1}{2} s^3 \lambda^3 \xi_0^3 |q|^2 + \frac{1}{2} s \lambda \xi_0 |\partial_{x_2} q|^2$$

and combining this with (3.17) yields

$$\begin{aligned} & \frac{1}{C} \iint_{(0, T) \times \Omega} \left(s^3 \lambda^4 \xi^3 |q|^2 + s \lambda^2 \xi |\nabla q|^2 + \frac{1}{s\xi} |\Delta q|^2 \right) dt dx \\ & \quad + \iint_{(0, T) \times \partial\Omega} \left(\frac{1}{2} s^3 \lambda^3 \xi_0^3 |q|^2 + \frac{1}{2} s \lambda \xi_0 |\partial_{x_2} q|^2 - s \lambda \xi_0 |\partial_{x_1} q|^2 \right) dt dx_1 \\ & \leq C \left(\iint_{(0, T) \times \Omega} |-\Delta p|^2 (s\lambda\xi)^{2r} e^{2s\varphi} dt dx + \iint_{(0, T) \times \bar{\Omega}} s^3 \lambda^4 \xi^3 |q|^2 dx \right). \end{aligned} \quad (3.18)$$

Using (3.16) in the above estimate, we obtain (3.15). \square

The main result of this section is the following Carleman estimate for low-frequencies solutions with respect to the horizontal variable. In this frequency regime, we can handle the boundary terms of the pressure in Lemma 3.2.

Theorem 3.3. *There exist $\lambda_0, s_0 > 0$ such that for any*

$$T > 0, \quad N \in \mathbb{N}^*, \quad \lambda \geq \lambda_0, \quad s \geq s_0(T + NT^2)$$

and for any strong solution (η, u, p) of (1.7) with (3.1)–(3.2), we have

$$\begin{aligned} I_1(s, \lambda, u, \eta) + \iint_{(0,T) \times \mathcal{I}} e^{2s\varphi} (s\lambda\xi_0)^2 |\partial_{x_1}^4 \eta|^2 dt dx_1 &\lesssim \iint_{(0,T) \times \tilde{\mathcal{I}}} (s\lambda\xi)^8 e^{2s\varphi} |u_1|^2 dt dx \\ &+ \iint_{(0,T) \times \tilde{\mathcal{I}}} e^{2s\varphi} s^7 \lambda^6 \xi^7 \left(|\partial_t u_1|^2 + |\partial_{x_1}^2 u_1|^2 + |\partial_{x_2}^2 u_1|^2 \right) dt dx \\ &+ \iint_{(0,T) \times \tilde{\mathcal{I}}} e^{2s\varphi} s^3 \lambda^2 \xi^3 \left(|\partial_t^2 u_1|^2 + |\partial_t \partial_{x_1}^2 u_1|^2 + |\partial_t \partial_{x_2}^2 u_1|^2 \right) dt dx. \end{aligned} \quad (3.19)$$

Proof. From Lemma 2.4, if

$$(u^0, \eta_1^0, \eta_2^0) \in \mathcal{H}^{[N]},$$

then the solution (η, u, p) of (1.7) satisfies

$$(u(t, \cdot), \eta(t, \cdot), \partial_t \eta(t, \cdot)) \in \mathcal{H}^{[N]} \quad t \geq 0.$$

We thus deduce that (η, u, p) satisfies (2.13) and (2.14). Then, using (3.8), we obtain the existence of $s_0 > 0$ such that if $s \geq s_0 NT^2$, then a.e. in $(0, T)$,

$$\int_{\partial\Omega} |\partial_{x_1} p|_{\partial\Omega}(t, x_1)|^2 dx_1 \leq N^2 \int_{\partial\Omega} |p|_{\partial\Omega}(t, x_1)|^2 dx_1 \leq \frac{1}{4} \int_{\partial\Omega} (s\lambda\xi_0)^2 |p|_{\partial\Omega}(t, x_1)|^2 dx_1. \quad (3.20)$$

Applying Lemma 3.2 to p with $r = 2$ and using (3.10), we deduce

$$\begin{aligned} &\iint_{(0,T) \times \Omega} e^{2s\varphi} \left(s^7 \lambda^6 \xi^7 |p|^2 + s^5 \lambda^4 \xi^5 |\nabla p|^2 \right) dt dx \\ &+ \iint_{(0,T) \times \partial\Omega} e^{2s\varphi_0} s^5 \lambda^3 \xi_0^5 \left((s\lambda\xi_0)^2 |p|_{\partial\Omega}|^2 - 2 |\partial_{x_1} p|_{\partial\Omega}|^2 \right) dt dx_1 \lesssim \iint_{(0,T) \times \tilde{\mathcal{I}}} e^{2s\varphi} s^7 \lambda^6 \xi^7 |p|^2 dt dx. \end{aligned} \quad (3.21)$$

Combining the above estimate with (3.20), we obtain

$$\begin{aligned} &\iint_{(0,T) \times \Omega} e^{2s\varphi} \left(s^7 \lambda^6 \xi^7 |p|^2 + s^5 \lambda^4 \xi^5 |\nabla p|^2 \right) dt dx + \iint_{(0,T) \times \partial\Omega} e^{2s\varphi_0} s^7 \lambda^5 \xi_0^7 |p|_{\partial\Omega}|^2 dt dx_1 \\ &\lesssim \iint_{(0,T) \times \tilde{\mathcal{I}}} e^{2s\varphi} s^7 \lambda^6 \xi^7 |p|^2 dt dx. \end{aligned} \quad (3.22)$$

Similarly, applying Lemma 3.2 to $\partial_t p$ with $r = 0$ and using (3.10), we deduce

$$\begin{aligned} &\iint_{(0,T) \times \Omega} e^{2s\varphi} \left(s^3 \lambda^2 \xi^3 |\partial_t p|^2 + s \xi |\nabla \partial_t p|^2 \right) dt dx + \iint_{(0,T) \times \partial\Omega} e^{2s\varphi_0} s^3 \lambda \xi_0^3 |\partial_t p|_{\partial\Omega}|^2 dt dx_1 \\ &\lesssim \iint_{(0,T) \times \tilde{\mathcal{I}}} e^{2s\varphi} s^3 \lambda^2 \xi^3 |\partial_t p|^2 dt dx. \end{aligned} \quad (3.23)$$

Combining (3.12), (3.22) and (3.23) gives

$$\begin{aligned} I_1(s, \lambda, u, \eta) &\lesssim \iint_{(0,T) \times \tilde{\mathcal{I}}} (s\lambda\xi)^8 e^{2s\varphi} |u_1|^2 dt dx + \iint_{(0,T) \times \tilde{\mathcal{I}}} (s\lambda\xi)^2 e^{2s\varphi} |\partial_t u_1|^2 dt dx \\ &+ \iint_{(0,T) \times \tilde{\mathcal{I}}} e^{2s\varphi} s^7 \lambda^6 \xi^7 |p|^2 dt dx + \iint_{(0,T) \times \tilde{\mathcal{I}}} e^{2s\varphi} s^3 \lambda^2 \xi^3 |\partial_t p|^2 dt dx. \end{aligned} \quad (3.24)$$

It remains to estimate η , without time derivatives as in $I_1(s, \lambda, u, \eta)$ (see (3.11)). For this, we use the beam equation in (1.7)

$$\partial_{x_1}^4 \eta(t, x_1) = p(t, x_1, 1) - \partial_t^2 \eta(t, x_1) + \partial_{x_1}^2 \partial_t \eta(t, x_1), \quad t \in (0, T), \quad x_1 \in \mathcal{I}.$$

Combining this equation and (3.8), we obtain

$$\begin{aligned} \iint_{(0,T) \times \Gamma_1} e^{2s\varphi_0} (s\lambda\xi_0)^2 |\partial_{x_1}^4 \eta|^2 dt dx_1 &\lesssim \iint_{(0,T) \times \mathcal{I}} e^{2s\varphi_0} \left((s\lambda\xi_0)^6 |\partial_t \eta|^2 + (s\lambda\xi_0)^2 |\partial_t^2 \eta|^2 \right) dt dx_1 \\ &\quad + \iint_{(0,T) \times \Gamma_1} e^{2s\varphi_0} (s\lambda\xi_0)^2 |p|_{\Gamma_1}|^2 dt dx_1. \end{aligned}$$

Then, from (3.11), (3.12), (3.22) and (3.24), we find

$$\begin{aligned} I_1(s, \lambda, u, \eta) + \iint_{(0,T) \times \Gamma_1} e^{2s\varphi_0} (s\lambda\xi_0)^2 |\partial_{x_1}^4 \eta|^2 dt dx_1 \\ \lesssim \iint_{(0,T) \times \tilde{\mathbb{I}}} (s\lambda\xi)^8 e^{2s\varphi} |u_1|^2 dt dx + \iint_{(0,T) \times \tilde{\mathbb{I}}} (s\lambda\xi)^2 e^{2s\varphi} |\partial_t u_1|^2 dt dx \\ + \iint_{(0,T) \times \tilde{\mathbb{I}}} e^{2s\varphi} s^7 \lambda^6 \xi^7 |p|^2 dt dx + \iint_{(0,T) \times \tilde{\mathbb{I}}} e^{2s\varphi} s^3 \lambda^2 \xi^3 |\partial_t p|^2 dt dx. \end{aligned} \quad (3.25)$$

From the first equation of (2.8), we have for $0 < |n| \leq N$,

$$p^{(n)} = \frac{i}{n} \left(\partial_t u_1^{(n)} + n^2 u_1^{(n)} - \partial_{x_2}^2 u_1^{(n)} \right)$$

and thus

$$\int_{\mathcal{I}} |p|^2 dx_1 \leq \int_{\mathcal{I}} \left(|\partial_t u_1|^2 + |\partial_{x_1}^2 u_1|^2 + |\partial_{x_2}^2 u_1|^2 \right) dx_1.$$

We have a similar estimate for $\partial_t p$ and combining these relations with (3.25), we deduce (3.19). \square

4 Proof of the main result

To obtain Theorem 1.1, we need to remove in (3.19) the observations associated with derivatives of u_1 . This is done in the next section by using the parabolic regularity of the system. Then, we reduce the region of control to ω by using a spectral inequality for sums of eigenfunctions of the one-dimensional Laplace operator on the torus. Finally, we use a Lebeau-Robbiano strategy combined with Theorem 2.5 to prove the main result.

4.1 Parabolic regularity

Using (3.6), there exists $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$,

$$2\varphi_1 - \varphi_0 < 0. \quad (4.1)$$

We assume below that $\lambda \geq \lambda_0$. There exists $s_0 > 0$ such that for $s \geq s_0 T$,

$$|\partial_t \varphi_0| + |\partial_t \varphi_1| + |\partial_t \xi_0| \lesssim s \xi_0^2 \quad \text{in } (0, T). \quad (4.2)$$

In particular, there exists $s_0 > 0$ such that for $s \geq s_0(T + T^2)$ and for any $r \in \mathbb{R}$,

$$\left| \frac{d}{dt} [(s\xi_0)^r e^{s\varphi_0}] \right| \lesssim_r (s\xi_0)^{r+2} e^{s\varphi_0}. \quad (4.3)$$

Now, we set

$$\rho_0 := s\lambda\xi_0 e^{s\varphi_0} \quad (4.4)$$

and we show in this section the following result:

Proposition 4.1. *There exist $\lambda_0, s_0 > 0$ such that for any*

$$T > 0, \quad N \in \mathbb{N}^*, \quad \lambda \geq \lambda_0, \quad s \geq s_0(T + NT^2)$$

and for any strong solution (η, u, p) of (1.7) with (3.1)–(3.2), we have

$$\|\rho_0(u, \eta, \partial_t \eta)\|_{L^2(0, T; \mathcal{H})}^2 \lesssim \iint_{(0, T) \times \bar{\mathbb{I}}} \lambda^{12} (s\xi_1)^{20} e^{4s\varphi_1 - 2s\varphi_0} |u_1|^2 dt dx. \quad (4.5)$$

Proof. Combining (4.4), (3.7) and (3.11), and applying Theorem 3.3, we first obtain

$$\begin{aligned} \|\rho_0(u, \eta, \partial_t \eta)\|_{L^2(0, T; \mathcal{H})}^2 &\lesssim \iint_{(0, T) \times \bar{\mathbb{I}}} (s\lambda\xi_1)^8 e^{2s\varphi_1} |u_1|^2 dt dx \\ &+ \iint_{(0, T) \times \bar{\mathbb{I}}} e^{2s\varphi_1} s^7 \lambda^6 \xi_1^7 \left(|\partial_t u_1|^2 + |\partial_{x_1}^2 u_1|^2 + |\partial_{x_2}^2 u_1|^2 \right) dt dx \\ &+ \iint_{(0, T) \times \bar{\mathbb{I}}} e^{2s\varphi_1} s^3 \lambda^2 \xi_1^3 \left(|\partial_t^2 u_1|^2 + |\partial_t \partial_{x_1}^2 u_1|^2 + |\partial_t \partial_{x_2}^2 u_1|^2 \right) dt dx. \end{aligned} \quad (4.6)$$

Then we set

$$\begin{aligned} \rho_1 &:= \lambda (s\xi_0)^{-1} e^{s\varphi_0}, & \rho_2 &:= \lambda (s\xi_0)^{-3} e^{s\varphi_0}, \\ \rho_3 &:= \lambda (s\xi_0)^{-5} e^{s\varphi_0}, & \rho_4 &:= \lambda (s\xi_0)^{-7} e^{s\varphi_0}. \end{aligned}$$

We deduce from (4.3) that

$$|\rho'_{j+1}| \lesssim \rho_j \quad (j \in \{0, \dots, 3\})$$

and we can use the maximal regularity of (2.3) and (4.6) to deduce that

$$\begin{aligned} \sum_{j=0}^4 \sum_{i=0}^j \|\rho_j \partial_t^i(u, \eta, \partial_t \eta)\|_{L^2(0, T; \mathcal{D}(\mathcal{A}^{j-i}))} &\lesssim \iint_{(0, T) \times \bar{\mathbb{I}}} (s\lambda\xi_1)^8 e^{2s\varphi_1} |u_1|^2 dt dx \\ &+ \iint_{(0, T) \times \bar{\mathbb{I}}} e^{2s\varphi_1} s^7 \lambda^6 \xi_1^7 \left(|\partial_t u_1|^2 + |\partial_{x_1}^2 u_1|^2 + |\partial_{x_2}^2 u_1|^2 \right) dt dx \\ &+ \iint_{(0, T) \times \bar{\mathbb{I}}} e^{2s\varphi_1} s^3 \lambda^2 \xi_1^3 \left(|\partial_t^2 u_1|^2 + |\partial_t \partial_{x_1}^2 u_1|^2 + |\partial_t \partial_{x_2}^2 u_1|^2 \right) dt dx. \end{aligned} \quad (4.7)$$

Now we integrate by parts to estimate the right-hand side of the above relation:

$$\begin{aligned} &\iint_{(0, T) \times \bar{\mathbb{I}}} e^{2s\varphi_1} s^3 \lambda^2 \xi_1^3 |\partial_t^2 u_1|^2 dt dx \\ &= \iint_{(0, T) \times \bar{\mathbb{I}}} \frac{1}{2} \frac{d^2}{dt^2} \left(e^{2s\varphi_1} s^3 \lambda^2 \xi_1^3 \right) |\partial_t u_1|^2 dt dx - \operatorname{Re} \iint_{(0, T) \times \bar{\mathbb{I}}} \frac{d}{dt} \left(e^{2s\varphi_1} s^3 \lambda^2 \xi_1^3 \right) \partial_t^3 u_1 \bar{u}_1 dt dx \\ &\quad - \operatorname{Re} \iint_{(0, T) \times \bar{\mathbb{I}}} e^{2s\varphi_1} s^3 \lambda^2 \xi_1^3 \partial_t^4 u_1 \bar{u}_1 dt dx. \end{aligned} \quad (4.8)$$

In particular, for any $\varepsilon > 0$,

$$\begin{aligned} &\iint_{(0, T) \times \bar{\mathbb{I}}} e^{2s\varphi_1} s^3 \lambda^2 \xi_1^3 |\partial_t^2 u|^2 dt dx \\ &\lesssim \iint_{(0, T) \times \bar{\mathbb{I}}} e^{2s\varphi_1} s^7 \lambda^2 \xi_1^7 |\partial_t u_1|^2 dt dx + \frac{1}{\varepsilon} \iint_{(0, T) \times \bar{\mathbb{I}}} e^{2s(2\varphi_1 - \varphi_0)} \lambda^2 (s\xi_1)^{20} |u_1|^2 dt dx \\ &\quad + \varepsilon \left(\|\rho_3 \partial_t^3 u\|_{L^2(0, T; L^2(\Omega))}^2 + \|\rho_4 \partial_t^4 u\|_{L^2(0, T; L^2(\Omega))}^2 \right). \end{aligned} \quad (4.9)$$

Similarly,

$$\begin{aligned} \iint_{(0,T) \times \tilde{\mathbb{I}}} e^{2s\varphi_1} s^3 \lambda^2 \xi_1^3 |\partial_t \partial_{x_1}^2 u_1|^2 dt dx &= \operatorname{Re} \iint_{(0,T) \times \tilde{\mathbb{I}}} \frac{1}{2} \frac{d^2}{dt^2} (e^{2s\varphi_1} s^3 \lambda^2 \xi_1^3) \partial_{x_1}^4 u_1 \bar{u}_1 dt dx \\ &\quad - \operatorname{Re} \iint_{(0,T) \times \tilde{\mathbb{I}}} e^{2s\varphi_1} s^3 \lambda^2 \xi_1^3 (\partial_t^2 \partial_{x_1}^4 u_1) \bar{u}_1 dt dx. \end{aligned} \quad (4.10)$$

In particular, for any $\varepsilon > 0$,

$$\begin{aligned} \iint_{(0,T) \times \tilde{\mathbb{I}}} e^{2s\varphi_1} s^3 \lambda^2 \xi_1^3 |\partial_t \partial_{x_1}^2 u_1|^2 dt dx &\lesssim \frac{1}{\varepsilon} \iint_{(0,T) \times \tilde{\mathbb{I}}} e^{2s(2\varphi_1 - \varphi_0)} \lambda^2 (s\xi_1)^{20} |u_1|^2 dt dx \\ &\quad + \varepsilon \left(\|\rho_2 \partial_{x_1}^4 u\|_{L^2(0,T;L^2(\Omega))}^2 + \|\rho_4 \partial_t^2 \partial_{x_1}^4 u\|_{L^2(0,T;L^2(\Omega))}^2 \right). \end{aligned} \quad (4.11)$$

We also consider $\kappa \in C^\infty(\mathbb{R}, [0, 1])$, $\kappa \equiv 1$ in $(\tilde{b}_1, \tilde{b}_2)$ with compact support in (b_1, b_2) . Then

$$\iint_{(0,T) \times \tilde{\mathbb{I}}} e^{2s\varphi_1} s^3 \lambda^2 \xi_1^3 |\partial_t \partial_{x_2}^2 u_1|^2 dt dx = \iint_{(0,T) \times \mathbb{I}} \kappa e^{2s\varphi_1} s^3 \lambda^2 \xi_1^3 |\partial_t \partial_{x_2}^2 u_1|^2 dt dx$$

and integrating by parts as above, we find for any $\varepsilon > 0$,

$$\begin{aligned} \iint_{(0,T) \times \tilde{\mathbb{I}}} e^{2s\varphi_1} s^3 \lambda^2 \xi_1^3 |\partial_t \partial_{x_2}^2 u_1|^2 dt dx &\lesssim \frac{1}{\varepsilon} \iint_{(0,T) \times \mathbb{I}} e^{2s(2\varphi_1 - \varphi_0)} \lambda^2 (s\xi_1)^{20} |u_1|^2 dt dx \\ &\quad + \varepsilon \left(\|\rho_1 \partial_{x_2}^2 u\|_{L^2(0,T;L^2(\Omega))}^2 + \|\rho_2 \partial_{x_2}^4 u\|_{L^2(0,T;L^2(\Omega))}^2 \right. \\ &\quad \left. + \|\rho_3 \partial_t^2 \partial_{x_2}^2 u\|_{L^2(0,T;L^2(\Omega))}^2 + \|\rho_4 \partial_t^2 \partial_{x_1}^4 u\|_{L^2(0,T;L^2(\Omega))}^2 \right). \end{aligned} \quad (4.12)$$

Proceeding similarly for all the other terms, we deduce (4.5) after standard but technical computations (see for instance [23]). \square

4.2 Reducing the observation domain

Proposition 4.2. *There exists $C > 0$ such that for any $T > 0$, $N \in \mathbb{N}^*$ and for any strong solution (η, u, p) of (1.7) with (3.1)–(3.2), we have*

$$\|(u(T, \cdot), \eta(T, \cdot), \partial_t \eta(T, \cdot))\|_{\mathcal{H}}^2 \leq C e^{C(\frac{1}{T} + N)} \iint_{(0,T) \times \omega} |u|^2 dt dx. \quad (4.13)$$

Proof. We apply Proposition 4.1 and take $\lambda = \lambda_0$, $s = s_0(T + NT^2)$. Then, we have

$$\iint_{(0,T) \times \mathbb{I}} \lambda^{12} (s\xi_1)^{20} e^{4s\varphi_1 - 2s\varphi_0} |u|^2 dt dx \lesssim \iint_{(0,T) \times \mathbb{I}} |u|^2 dt dx$$

and there exists a constant C such that

$$e^{-C(\frac{1}{T} + N)} \|(u(T, \cdot), \eta(T, \cdot), \partial_t \eta(T, \cdot))\|_{\mathcal{H}}^2 \lesssim \|\rho_0(u, \eta, \partial_t \eta)\|_{L^2(T/4, 3T/4; \mathcal{H})}^2 \leq \|\rho_0(u, \eta, \partial_t \eta)\|_{L^2(0, T; \mathcal{H})}^2.$$

We deduce from the above estimate and (4.5) that

$$\|(u(T, \cdot), \eta(T, \cdot), \partial_t \eta(T, \cdot))\|_{\mathcal{H}}^2 \leq C e^{C(\frac{1}{T} + N)} \iint_{(0,T) \times \mathbb{I}} |u|^2 dt dx. \quad (4.14)$$

From (3.4) and (2.13),

$$\iint_{(0,T) \times \mathbb{I}} |u|^2 dt dx = \int_{(0,T) \times (b_1, b_2)} \sum_{|n| \leq N} \left| u^{(n)}(t, x_2) \right|^2 dt dx_2. \quad (4.15)$$

Using [37] and the fact that E_n are eigenvectors of the Laplace operator on \mathcal{I} with eigenvalues n^2 , there exists a constant $C > 0$ such that for a.e. $x_2 \in (0, 1)$ and a.e. $t \in (0, T)$,

$$\sum_{|n| \leq N} \left| u^{(n)}(t, x_2) \right|^2 \leq C e^{CN} \int_{(a_1, a_2)} \left| \sum_{|n| \leq N} u^{(n)}(t, x_2) E_n(x_1) \right|^2 dx_1.$$

Gathering (4.14), (4.15) and the above equation yields (4.13). \square

4.3 The Lebeau-Robbiano method

In order to prove Theorem 1.1, it remains to combine Proposition 4.2 and Theorem 2.5. This can be done in a general way by following the method developed in [36] for the controllability heat equation. More precisely, we consider the dual version of this method written in [45] where the author shows directly the final state observability and make explicit the dependency of the cost of control with respect to the time and we adapt it to our case. Let us point out that a similar abstract result is done in [5].

We assume that \mathcal{H} and \mathcal{U} are Hilbert spaces, that \mathcal{A} is the infinitesimal generator of a C^0 semigroup $(e^{t\mathcal{A}})_{t \geq 0}$, and that $\mathcal{C} \in \mathcal{L}(\mathcal{H}, \mathcal{U})$ an observation operator. We also suppose that for any $N \in \mathbb{N}^*$, there exists a continuous projection $\Pi^{[N]}$ of \mathcal{H} such that

$$\left\| (I - \Pi^{[N]}) \right\|_{\mathcal{L}(\mathcal{H})} \leq C_0$$

for some constant $C_0 > 0$ independent of N . Finally, we assume the existence of two positive constants C_1, C_2 such that for any $N \in \mathbb{N}^*$, $t \geq 0$, $T > 0$ and for any $U \in \mathcal{H}$,

$$\left\| \Pi^{[N]} e^{T\mathcal{A}} U \right\|_{\mathcal{H}}^2 \leq C_1 e^{C_1(\frac{1}{T} + N)} \int_0^T \left\| \mathcal{C} \Pi^{[N]} e^{t\mathcal{A}} U \right\|_{\mathcal{U}}^2 dt \quad (4.16)$$

and

$$\left\| (I - \Pi^{[N]}) e^{t\mathcal{A}} U \right\|_{\mathcal{H}}^2 \leq \frac{1}{C_2} e^{-C_2(N+1)^2 t} \left\| (I - \Pi^{[N]}) U \right\|_{\mathcal{H}}^2. \quad (4.17)$$

Then we have the following result:

Lemma 4.3. *Assume the above hypotheses and in particular (4.16), (4.17). Then we have the following final state observability: there exists $C > 0$ such that for any $T > 0$, and for any $U \in \mathcal{H}$,*

$$\left\| e^{T\mathcal{A}} U \right\|_{\mathcal{H}} \leq e^{C/T} \int_0^T \left\| \mathcal{C} e^{t\mathcal{A}} U \right\|_{\mathcal{U}}^2 dt. \quad (4.18)$$

Proof. First, we consider $\varepsilon > 0$ small enough such that for any $\tau \in (0, T)$,

$$C_0^2 \left(\frac{4C_1}{C_2} \left\| \mathcal{C} \right\|_{\mathcal{L}(\mathcal{H}, \mathcal{U})}^2 \varepsilon \tau e^{2C_1 \frac{1}{\varepsilon \tau} - C_2 \frac{1-\varepsilon}{\varepsilon^2 \tau}} + \frac{2}{C_2} e^{-C_2 \frac{1}{\varepsilon^2 \tau}} \right) \leq e^{-\frac{2C_1}{\varepsilon \tau}}. \quad (4.19)$$

Assume

$$T_1 > 0, \tau > 0, T_2 := T_1 + \tau$$

and let us take

$$N = \left\lfloor \frac{1}{\varepsilon \tau} \right\rfloor.$$

Then, using (4.16) and (4.17), we deduce

$$\begin{aligned} \|e^{T_2 \mathcal{A}U}\|_{\mathcal{H}}^2 &\leq 2 \left\| \Pi^{[N]} e^{T_2 \mathcal{A}U} \right\|_{\mathcal{H}}^2 + 2 \left\| (I - \Pi^{[N]}) e^{T_2 \mathcal{A}U} \right\|_{\mathcal{H}}^2 \leq 4C_1 e^{2C_1 \frac{1}{\varepsilon\tau}} \int_{T_2 - \varepsilon\tau}^{T_2} \|C e^{t\mathcal{A}U}\|_{\mathcal{U}}^2 dt \\ &\quad + \left(\frac{4C_1}{C_2} \|C\|_{\mathcal{L}(\mathcal{H}, \mathcal{U})}^2 \varepsilon\tau e^{2C_1 \frac{1}{\varepsilon\tau} - C_2 \frac{1-\varepsilon}{\varepsilon^2\tau}} + \frac{2}{C_2} e^{-C_2 \frac{1}{\varepsilon^2\tau}} \right) C_0^2 \|e^{T_1 \mathcal{A}U}\|_{\mathcal{H}}^2. \end{aligned} \quad (4.20)$$

We set

$$f_\varepsilon(\tau) := \frac{1}{4C_1} e^{-\frac{4C_1}{\varepsilon\tau}}$$

so that

$$\frac{f_\varepsilon(\tau)}{f_\varepsilon(2\tau)} = e^{-\frac{2C_1}{\varepsilon\tau}}$$

and we deduce from (4.19) and (4.20) that

$$f_\varepsilon(2\tau) \|e^{T_2 \mathcal{A}U}\|_{\mathcal{H}}^2 \leq \int_{T_1}^{T_2} \|C e^{t\mathcal{A}U}\|_{\mathcal{U}}^2 dt + f_\varepsilon(\tau) \|e^{T_1 \mathcal{A}U}\|_{\mathcal{H}}^2. \quad (4.21)$$

Now we take $k \in \mathbb{N}$ and we apply the above result to

$$T_1 := \frac{T}{2^{k+1}}, \quad T_2 := \frac{T}{2^k}, \quad \tau := \frac{T}{2^{k+1}}.$$

We obtain

$$f_\varepsilon\left(\frac{T}{2^k}\right) \left\| e^{\frac{T}{2^k} \mathcal{A}U} \right\|_{\mathcal{H}}^2 \leq \int_{\frac{T}{2^{k+1}}}^{\frac{T}{2^k}} \|C e^{t\mathcal{A}U}\|_{\mathcal{U}}^2 dt + f_\varepsilon\left(\frac{T}{2^{k+1}}\right) \left\| e^{\frac{T}{2^{k+1}} \mathcal{A}U} \right\|_{\mathcal{H}}^2. \quad (4.22)$$

Summing the above relation over $k \in \mathbb{N}$, we deduce the result. \square

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