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Active Design of Diffuse Acoustic Fields in Enclosures

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1 This paper presents a numerical framework for designing diffuse fields in rooms of
2 any shape and size, driven at arbitrary frequencies. That is, we aim at overcoming
3 the Schroeder frequency limit for generating diffuse fields in an enclosed space. We
4 formulate the problem as a Tikhonov regularized inverse problem and propose a low-
5 rank approximation of the spatial correlation that results in significant computational
6 gains. Our approximation is applicable to arbitrary sets of target points and allows
7 us to produce an optimal design at a computational cost that grows only linearly
8 with the (potentially large) number of target points. We demonstrate the feasibility
9 of our approach through numerical examples where we approximate diffuse fields at
10 frequencies well below the Schroeder limit.

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11 I. INTRODUCTION

12 Many aerospace structures, satellites and internal components experience acoustic loads
13 that are diffuse in nature. That is, these loads are composed of a large number of waves
14 having random amplitude, propagation direction and phase. Ground-based qualification
15 testing of these structures are typically performed in acoustic reverberation chambers within
16 which a diffuse field arises naturally. The minimum frequency beyond which an acoustic
17 field in an enclosure is (naturally) diffuse, termed the Schroeder frequency^{1,2}, depends on
18 the chamber volume V and absorption as per the relation

$$f_s = \sqrt{\frac{c^3 T_{60}}{4 \ln(10)V}} \quad (1)$$

19 Here, c is the phase speed and T_{60} the reverberation time, the time required after source
20 termination for the energy in the room to attenuate 60 dB. This expression provides an
21 estimate of the frequency at which sufficient modal overlap first occurs. See Kuttruff³ for a
22 detailed derivation of (1).

23 The dimensions of a chamber determine the lowest frequency at which a diffuse field
24 would naturally occur, as f_s^2 is inversely proportional to the enclosure volume. Although
25 there are means to improve or fully develop a diffuse field within a given chamber such as
26 splayed walls, rotating panels, and moving vanes⁴, there are currently no known means to
27 induce a diffuse field at frequencies below f_s .

28 Traditionally, large air horns were used to generate the necessary high amplitude acoustic
29 levels within reverberation chambers. Recently, horns have been augmented or replaced
30 with concert-grade loudspeakers, allowing improved closed-loop control and the potential

31 to achieve a wider range of test spectra^{5,6}. The addition of loudspeaker sources provides
32 an opportunity for optimizing the acoustic field within a chamber, as will be shown in this
33 paper. Synthesizing diffuse fields below the Schroeder frequency can lead to significant
34 financial and time savings. For instance, physical dimensions of new chambers could be
35 relaxed and shipping of test articles across facilities could be eliminated.

36 The design of diffuse acoustic fields has been explored, to some extent, in the open
37 literature. Specific examples include the work of Bravo and Maury⁷ who designed various
38 types of acoustic random fields, including diffuse fields, using direct acoustic field testing.
39 To that end, they solved a quadratic optimization problem in which they found correlated
40 sources that minimized the misfit between the predicted field and the spatial correlation of
41 the target random field near the surface of a test body. In recent work, Alvarez-Blanco et
42 al.⁸ presented an approach to design controls for an array of loud speakers to create diffuse
43 fields in direct acoustic tests. For this, they also used a pseudo-inverse strategy to obtain
44 solutions of a quadratic optimization problem that directly provided the signal inputs. For
45 other relevant, recent work in this area see⁹⁻¹¹.

46 As in the aforementioned work, we are interested in designing diffuse fields. However, in
47 contrast to the existing literature, we investigate the synthesis of these fields in reverberant
48 rooms at arbitrary frequencies (even below the Schroeder limit). Hence, our main contribu-
49 tion is the development of a numerical approach for the design of acoustic diffuse fields in
50 enclosed rooms. To that end, we put forward a PDE-constrained optimization formulation
51 for the control problem. Although existing work has addressed the Multiple Input Multiple
52 Output (MIMO) control problem^{7,8}, we provide further insight into Tikhonov regularization

53 for these problems. Moreover, one of the numerical challenges in designing or producing
54 samples of a diffuse field in simulations is the need to capture its spatial correlation. The
55 latter may require fine discretizations (e.g. using finite elements) that lead to large dense
56 matrices.

57 The computational challenge of factorizing the spatial correlation (in the context of diffuse
58 fields) was recently addressed in¹², who proposed to generate diffuse fields given by expan-
59 sions on the eigenfunctions of the pressure correlation operator (i.e. using the Karhunen-
60 Loève decomposition of the latter). To avoid the computational bottleneck of factorizing a
61 dense correlation, they employed a low-order Taylor expansion in Cartesian components of
62 the relative position vector, approximating the continuous correlation function while ensur-
63 ing the vector does not exceed the operating wavelength significantly. This approximation
64 reduced the correlation to a product of univariate kernels, whose eigenfunctions are recog-
65 nized as spheroidal wave functions

66 In contrast, our approach exploits a more-accurate low-rank approximation of the cor-
67 relation function, applicable to arbitrary sets of target points. This allows us to solve the
68 least-squares problem producing an optimal design at a computational cost that grows only
69 linearly with the (potentially large) number of target points (instead of quadratically if us-
70 ing directly the dense correlation matrix). In addition, this proposed treatment allows us to
71 formulate simple and efficient regularized versions of the optimization problem, which cater
72 to it being possibly ill-conditioned and remain computationally economical.

73 The rest of the paper is organized as follows. We provide a summary of the plane-wave
74 model for diffuse fields and the ensuing statistics. We then formulate the forward problem

75 for modeling a random acoustic field in an enclosed room. Next, we provide the optimization
76 formulation for the control problem and the description of our approach for the low-rank
77 approximation of the spatial correlation. We also offer two Tikhonov regularization strategies
78 for the control problem. Next, we demonstrate the feasibility of our proposed strategy using
79 numerical examples in an enclosed room driven at frequencies below the Schroeder limit.
80 Finally, we provide some conclusions and future directions.

81 II. BACKGROUND

82 Here we summarize existing results on the theoretical modeling of diffuse fields. We will
83 adopt the plane wave model in which a random pressure field is conceived as the interaction
84 of an infinite number of plane waves with randomized direction and phase¹³. It is well
85 known that this model leads to a Gaussian spatio-temporal pressure field, which is fully
86 characterized by its mean and correlation function. In this work, for the sake of simplicity
87 and without loss of generality, we will concentrate on pure-tone fields.

88 A. Plane Wave Model

89 Following Jacobsen¹⁴, we model a pure-tone diffuse field as a random pressure field ex-
90 pressed as

$$P(\mathbf{x}, \omega; \theta) = \lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \sum_{n=1}^N A_n e^{-i(\kappa \mathbf{D}_n \cdot \mathbf{x} + \Phi_n)} \quad (2)$$

91 where $\kappa = \omega/c$ is the wave number, c is the speed of sound, \mathbf{D}_n are independent random
92 vectors (uniformly distributed over the unit sphere) describing the direction of a plane wave,

93 A_n are independent random variables, and Φ_n are random phases. The vector θ represents
 94 the collection of all the random variables in the model. Notice that, as a consequence of
 95 the Central Limit Theorem, the random field $P(\mathbf{x}, \omega; \theta)$ is Gaussian¹⁴. Also, without loss of
 96 generality, we use in the sequel a constant value $A_n := p_0$.

97 B. First and Second Order Statistics of a Diffuse Field

98 Here we summarize well-known results on the statistics of the stochastic field shown in
 99 (2). It is straightforward to show that

$$\mathbb{E}[P(\mathbf{x}, \omega; \theta)] = 0 \quad \forall \mathbf{x}, \omega$$

100 where $\mathbb{E}[\cdot]$ denotes expectation.

Let $r_{qs} = \|\mathbf{x}_q - \mathbf{x}_s\|$ be the distance between any two points \mathbf{x}_q and \mathbf{x}_s . Then, the correlation can be shown to be given by¹⁵

$$\begin{aligned} G(r_{qs}, \omega) &= \mathbb{E}[P(\mathbf{x}_q, \omega; \theta)\overline{P(\mathbf{x}_s, \omega; \theta)}] \\ &= \frac{p_0^2}{2} \frac{\sin(\kappa(\omega)r_{qs})}{\kappa(\omega)r_{qs}} \end{aligned} \quad (3)$$

101 where the overbar denotes complex conjugation.

102 As previously noted, the random field $P(\mathbf{x}, \omega; \theta)$ is Gaussian. Hence, we can completely
 103 characterize the diffuse field through its mean and correlation. Moreover, the field is weakly
 104 isotropic (i.e. the mean is constant and the correlation depends only on the distance between
 105 two points)¹⁶. A more extensive discussion on diffuse fields can be found in^{14,15,17,18}.

106 III. DESIGN OF DIFFUSE FIELDS

107 As stated before, we seek to develop a numerical framework for designing approximately
 108 diffuse fields for rooms of any shape and size at arbitrary frequencies. To this end, we
 109 first introduce the acoustic equations and numerical approximations, followed by the design
 110 problem, and a formal regularized treatment of the ensuing ill-posed design problem.

111 As we will see later in this section, one of the key challenges in the design problem is the
 112 computational expense that arises from the discretization of the target correlation function.
 113 One of our contributions in this work is an efficient low-rank representation of the target
 114 correlation that renders the design problem tractable.

115 A. Forward Problem

We model a room as a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary (walls) Γ . The part of
 the walls with impedance conditions is denoted as Γ_r , while that occupied by the speakers
 is denoted as $\Gamma_N = \cup_j \Gamma_{N_j}, j = 1 \dots d$, where d is the number of speakers in the room and
 $\Gamma = \Gamma_r \cup \Gamma_N$. The pressure field in the room satisfies¹⁹

$$\begin{aligned} \nabla^2 p + \kappa^2 p &= 0 \quad \text{in } \Omega \\ \nabla p \cdot \mathbf{n} + iZ^{-1}\omega\rho_0 p &= 0 \quad \text{on } \Gamma_r \\ \nabla p \cdot \mathbf{n}_j - \rho_0 s_j &= 0 \quad \text{on } \Gamma_{N_j} \quad \text{for } j = 1, \dots, d \end{aligned} \tag{4}$$

116 where ω is the angular frequency, ρ_0 is the fluid density, κ is the wavenumber, s_j is the
 117 normal acoustic acceleration over Γ_j , Z is the specific acoustic impedance, and \mathbf{n}_j is a unit
 118 vector normal to Γ_j and directed out of the room. Using the Finite Element Method²⁰ to

119 obtain a discrete representation of (4), we arrive at

$$R\mathbf{p} = F\mathbf{s}$$

120 where $R := K - \kappa^2 M + i\omega C$, K is the acoustic stiffness, M is the mathematical mass matrix,
 121 C is the damping matrix, $F \in \mathbb{R}^{N \times d}$ is a matrix that acts on the acoustic accelerations vector
 122 $\mathbf{s} \in \mathbb{R}^d$.

Let $\mathcal{T} := \{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ be a finite set of m target locations where we want the field to
 be diffuse. Now, let $B \in \mathbb{R}^{m \times d}$ be the Boolean matrix such that $\mathbf{p}_{\mathcal{T}} = B\mathbf{p} \in \mathbb{C}^m$ collects the
 nodal pressures at the target locations. We then have

$$\begin{aligned} \mathbf{p}_{\mathcal{T}} &= B\mathbf{p} \\ &= BR^{-1}F\mathbf{s} \\ &= T\mathbf{s} \end{aligned}$$

123 where we have introduced the transfer matrix $T := BR^{-1}F \in \mathbb{C}^{m \times d}$. Also, we point out that
 124 if $Z \neq 0$, R is invertible for any frequency²¹.

For random sources, the relationship between the cross-correlation of the target pressures
 denoted as G , and the cross-correlation of the sources, denoted as S , at a given frequency
 is given as

$$\begin{aligned} G &= \mathbb{E}[\mathbf{p}_{\mathcal{T}}\mathbf{p}_{\mathcal{T}}^h] \\ &= T \mathbb{E}[\mathbf{s}\mathbf{s}^h] T^h \\ &= TST^h \end{aligned} \tag{5}$$

125 where the superscript h denotes complex conjugation and transposition. So, given S , we
 126 can compute G at given locations. Next, we develop a design methodology for diffuse fields
 127 based on the forward model presented in this section.

128 B. Inverse or Design Problem

129 Our design problem can be described as: given a cross-correlation at a set \mathcal{T} of target
 130 nodes, determine the cross-correlation of the input sources. To this end, let $\mathbf{x}_i, \mathbf{x}_j \in \mathcal{T}$ be
 131 two target locations. Then, from (3), the components of the target cross-correlation matrix
 132 at a given frequency, \hat{G}_{ij} , are given as

$$\hat{G}_{ij} = \frac{p_o^2 \sin(\kappa r_{ij})}{2 \kappa r_{ij}} \quad (6)$$

133 We first introduce an un-regularized inverse problem for the sake of simplicity. This
 134 formulation leads to a decomposition of the inverse problem that strongly reduces compu-
 135 tational cost, as will be shown. After this, a regularized version of the problem follows
 136 naturally. Define an objective function as

$$J(S) := \frac{1}{2} \|G(S) - \hat{G}\|_F^2 \quad (7)$$

137 where $G(S)$ solves (5) and $\|G\|_F$ is the Frobenius matrix norm, which is associated with an
 138 inner product: $\|G\|_F^2 = (G, G)_F =: \text{tr}(G^h G)$. Then, the optimal cross-correlation, S_o , can
 139 be obtained as

$$S_o = \arg \min_{S \in \mathbb{C}^{d \times d}} J(S). \quad (8)$$

140 While the search space for the above minimization should *a priori* be restricted to positive
 141 definite and Hermitian matrices S , it turns out that the minimum-norm solution found by
 142 the unconstrained minimization (8) automatically satisfies those requirements, see Remark 3.

143 C. Optimality Condition and minimum-norm least-squares solution

144 To simplify our derivations, define a linear operator $\mathcal{A} : \mathbb{C}^{d \times d} \rightarrow \mathbb{C}^{m \times m}$ as

$$\mathcal{A}S := TST^h \tag{9}$$

145 Substituting this expression into the objective (7), we get

$$J(S) := \frac{1}{2} \|\mathcal{A}S - \hat{G}\|_F^2$$

The first-order optimality condition for Problem (8) is that the gradient of the objective be zero at the minimizer. Using the inner product associated with the Frobenius norm, the directional derivative of the objective at $S \in \mathbb{C}^{d \times d}$ in an arbitrary direction $H \in \mathbb{C}^{d \times d}$ is obtained as

$$\begin{aligned} \langle J'(S), H \rangle &= \left. \frac{d}{d\epsilon} J(S + \epsilon H) \right|_{\epsilon=0} \\ &= \operatorname{Re}(\mathcal{A}S - \hat{G}, \mathcal{A}H)_F \\ &= \operatorname{Re}(\mathcal{A}^* \mathcal{A}S - \mathcal{A}^* \hat{G}, H)_F \end{aligned}$$

146 where \mathcal{A}^* denotes the adjoint of \mathcal{A} , defined by $(W, \mathcal{A}V)_F = (\mathcal{A}^*W, V)_F$ for any $W \in \mathbb{C}^{m \times m}$
 147 and $V \in \mathbb{C}^{d \times d}$. Then, from the above expression, the first-order optimality condition
 148 $J'(S_o) = 0$ verified by S_o is obtained as

$$\mathcal{A}^* \mathcal{A}S_o - \mathcal{A}^* \hat{G} = 0, \tag{10}$$

149 which is in essence the normal equation for the least squares problem (8). Then, using (9)
 150 in (10) and simplifying, we get

$$T^h T S_o T^h T = T^h \hat{G} T$$

151 If $T^h T$ is not invertible (i.e. if T does not have full column rank d), the above equation fails
 152 to provide a solution $S_o \in \mathbb{C}^{d \times d}$ that is invertible (let alone positive definite). The transfer
 153 matrix T is therefore assumed henceforth to have rank d .

154 To solve equations (10), and also to later address regularized versions of Problem (8), it
 155 is convenient and computationally reasonable (under the present operating conditions) to
 156 introduce and use the reduced singular value decomposition (SVD) of T :

$$T = X \Sigma Y^h, \tag{11}$$

157 where $\Sigma \in \mathbb{R}^{d \times d}$ is a diagonal matrix holding the d nonzero singular values $\sigma_1 \geq \sigma_2 \geq$
 158 $\dots \sigma_d > 0$ of T while $X \in \mathbb{C}^{m \times d}$ and $Y \in \mathbb{C}^{d \times d}$ hold the d left and right associated singular
 159 vectors (arranged columnwise), respectively. In particular, the matrices X and Y have the
 160 orthonormality properties $X^h X = I_d$ and $Y^h Y = Y Y^h = I_d$, with I_d the $d \times d$ identity
 161 matrix. On introducing the SVD (11) in (10) and using the latter properties of X and Y ,
 162 the minimum-norm least-squares solution S_o is found as

$$S_o = Y \Sigma^{-1} (X^h \hat{G} X) \Sigma^{-1} Y^h. \tag{12}$$

163 and is clearly Hermitian and positive definite. Moreover, for any $\mathbf{s} \in \mathbb{C}^d$, we have $\mathbf{s}^h S_o \mathbf{s} =$
 164 $(X \Sigma^{-1} Y^h \mathbf{s})^h \hat{G} (X \Sigma^{-1} Y^h \mathbf{s}) > 0$ by virtue of the positive definiteness of \hat{G} , showing that S_o
 165 is in fact positive definite.

166 The expression (12) of S_o entails the evaluation of $X^h \hat{G} X$, whose $O(dm^2 + d^2m/2)$ cost
 167 constitutes a potential computational bottleneck as \hat{G} is a dense $m \times m$ matrix that may
 168 be large in realistic problems (e.g. $m = O(10^4)$ to $O(10^6)$). To reduce the computational
 169 complexity in m of the design solution method, we now introduce a low-rank approximation
 170 of \hat{G} .

171 D. Low-rank approximation of \hat{G}

172 The target correlation \hat{G} being symmetric and positive definite, we have $\hat{G} = \sum_{k=1}^m \psi_k \psi_k^h \lambda_k$,
 173 where ψ_k, λ_k are the eigenpairs of \hat{G} numbered so that $\lambda_1 \geq \lambda_2 \dots \geq \lambda_m > 0$. By the Eckart-
 174 Young theorem, this expansion, truncated to its first P terms, yields the best rank- P
 175 approximation of \hat{G} , denoted as \hat{G}_P , in the sense of the Frobenius norm. Specifically, the
 176 relative truncation error is given as

$$\mathcal{E}^2(P) := \frac{\|\hat{G} - \hat{G}_P\|_F^2}{\|\hat{G}\|_F^2} = \frac{\sum_{k=P+1}^m \lambda_k^2}{\sum_{j=1}^m \lambda_j^2} \quad (13)$$

177 The rate of decay of the eigenvalues of a correlation matrix depends on the correlation
 178 length. The latter is usually high, which allows a truncation order $Q \ll m$. However, setting
 179 up this approximation still entails obtaining a large enough number of eigenpairs of the
 180 $m \times m$ matrix \hat{G} to achieve and verify a sufficiently low truncation error, and this remains
 181 often impractical.

182 We therefore propose an alternative strategy for deriving low-rank approximations of \hat{G} .
 183 It is based on observing that the generic entry \hat{G}_{ij} of \hat{G} , see (6), is in fact equivalently given

$$\hat{G}_{ij} = \frac{p_o^2}{2} j_0(\kappa |\mathbf{r}_{ij}|)$$

185 where j_0 is the spherical Bessel function of first kind and order zero and $\mathbf{r}_{ij} := \mathbf{x}_i - \mathbf{x}_j$ is
 186 the position vector joining two generic target points. Now, for any $\mathbf{z} \in \mathbb{R}^3$, the function j_0
 187 admits the integral representation

$$j_0(|\mathbf{z}|) = \frac{1}{4\pi} \int_{\hat{S}} e^{i\mathbf{z} \cdot \hat{\boldsymbol{\theta}}} dS(\hat{\boldsymbol{\theta}}), \quad (14)$$

188 where \hat{S} is the unit sphere (spanned by unit vectors $\hat{\boldsymbol{\theta}}$). For instance, expressing the above
 189 integral using spherical angular coordinates reduces it to the one-dimensional integral repre-
 190 sentation formula (10.54.1) given in²². Let the above integral be approximated by a Q -point
 191 quadrature rule with nodes $\hat{\boldsymbol{\theta}}_q \in \hat{S}$ and positive weights w_q ($1 \leq q \leq Q$), yielding

$$j_0(|\mathbf{z}|) = \left(\frac{1}{4\pi} \sum_{q=1}^Q w_q e^{i\mathbf{z} \cdot \hat{\boldsymbol{\theta}}_q} \right) + \varepsilon_Q,$$

192 ε_Q being the quadrature error. Setting $\mathbf{z} = \kappa(\mathbf{x}_i - \mathbf{x}_j)$, we thus approximate \hat{G}_{ij} as

$$\hat{G}_{ij} \approx \frac{p_o^2}{8\pi} \sum_{q=1}^Q w_q e^{i\mathbf{x}_i \cdot \hat{\boldsymbol{\theta}}_q} e^{-i\mathbf{x}_j \cdot \hat{\boldsymbol{\theta}}_q},$$

193 a result which in turn yields, upon application to all pairs of target points, the following
 194 (approximate) decomposition of the target correlation matrix \hat{G} :

$$\hat{G} \approx \Phi \Phi^h = \sum_{q=1}^Q \boldsymbol{\phi}_q \boldsymbol{\phi}_q^h, \quad \Phi = [\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_Q] \in \mathbb{C}^{m \times Q}, \quad \Phi_{jq} = (\boldsymbol{\phi}_q)_j = \sqrt{w_q} e^{i\mathbf{x}_j \cdot \hat{\boldsymbol{\theta}}_q}. \quad (15)$$

195 The size Q of the quadrature rule ensuring a desired (small enough) quadrature error depends
 196 on the oscillatory character of the integral (14), and hence on the magnitude of the argument
 197 $|\mathbf{z}|$ of j_0 there. In this study, the latter is bounded from above by the largest spatial

198 separation between target points and the operating frequency. Consequently, Q does not
 199 depend on the number m of target points once their maximum spatial separation is fixed.
 200 In the forthcoming examples, $Q = O(10^2)$ whereas $m = O(10^4)$, so that (15) accomplishes a
 201 low-rank approximation of \hat{G} , whose computation is moreover economical as the m -vectors
 202 ϕ_q are given explicitly.

203 The low-rank approximation (15) greatly reduces the computational load in evaluating
 204 S_o . Indeed, using (15) in (12) gives

$$S_o = (Y\Sigma^{-1}Z)(Y\Sigma^{-1}Z)^h, \quad Z := X^h\Phi \in \mathbb{C}^{d \times Q},$$

205 whose evaluation needs only $O(m)$ computational work and memory, down from $O(m^2)$ if
 206 using the full matrix \hat{G} .

207 **Remark 1** *Unlike in the expansion $\hat{G} = \sum_{q=1}^m \psi_q \psi_q^T \lambda_q$ in terms of eigenpairs, the vectors*
 208 *ϕ_q in (15) are not orthogonal.*

209 E. Regularized least-squares solution

210 We now address the case where $T^h T$ may be ill-conditioned (i.e. have a large condition
 211 number), with T still assumed to have full column rank. Let

$$\mathbf{s}_q = Y\Sigma^{-1}X^h\phi_q \tag{16}$$

212 Then, the minimum-norm solution to (8) is given by

$$S_o = \sum_{q=1}^Q \mathbf{s}_q \mathbf{s}_q^h \tag{17}$$

213 We notice that \mathbf{s}_q given by (16) is (by our assumption on T) the unique solution of the
 214 least-squares problem

$$\mathbf{s}_q = \arg \min_{\mathbf{u} \in \mathbb{C}^d} \frac{1}{2} \|T\mathbf{u} - \phi_q\|_2^2 \quad (18)$$

215 where $\|\cdot\|_2$ is the Euclidean norm. Therefore, to obtain S_o given by (17), we have to solve
 216 at most Q problems of the type (18). To cater for $T^h T$ being possibly ill-conditioned, we
 217 add a regularization term in (18) to get the problem

$$\mathbf{s}_q(\alpha) = \arg \min_{\mathbf{u} \in \mathbb{C}^d} \frac{1}{2} \left(\|T\mathbf{u} - \phi_q\|_2^2 + \alpha \|\mathbf{u}\|_2^2 \right), \quad (19)$$

218 (where $\alpha > 0$ is a regularization parameter), whose unique minimizer is given in closed form
 219 as

$$\mathbf{s}_q(\alpha) = Y(\Sigma + \alpha I)^{-1} X^h \phi_q. \quad (20)$$

220 The resulting input correlation matrix is then given by

$$S_{o1}(\alpha) := \sum_{q=1}^Q \mathbf{s}_q(\alpha) \mathbf{s}_q^h(\alpha) = (Y(\Sigma + \alpha I)^{-1} Z) (Y(\Sigma + \alpha I)^{-1} Z)^h. \quad (21)$$

We can show that $S_{o1}(\alpha)$ converges to the minimum-norm solution S_{o1} of our original problem (8) as $\alpha \rightarrow 0$. Indeed, from (16) and (20), we have

$$\begin{aligned} \mathbf{s}_q(\alpha) - \mathbf{s}_q &= Y[(\Sigma + \alpha I)^{-1} - \Sigma^{-1}] X^h \phi_q \\ &= Y(\Sigma + \alpha I)^{-1} [I - (\Sigma + \alpha I)\Sigma^{-1}] X^h \phi_q \\ &= -\alpha Y(\Sigma + \alpha I)^{-1} \Sigma^{-1} X^h \phi_q \end{aligned}$$

221 Therefore, $\|\mathbf{s}_q(\alpha) - \mathbf{s}_q\| \rightarrow 0$ and $\|S_{o1}(\alpha) - S_o\|_F \rightarrow 0$ as $\alpha \rightarrow 0$ (since $S_{o1}(\alpha)$ and S_o are
 222 both given by finite sums of tensor products)

223 *Second regularization method*

224 Alternatively, we can consider the regularized version

$$S_{o2}(\alpha) := \arg \min_{S \in \mathbb{C}^{d \times d}} J_\alpha(S), \quad J_\alpha(S) := \frac{1}{2} \|\mathcal{A}S - \hat{G}\|_F^2 + \frac{\alpha}{2} \|S\|_F^2 \quad (22)$$

225 of the original least-squares problem (8), whose stationarity condition is

$$T^h T S_{o2}(\alpha) T^h T + \alpha S_{o2}(\alpha) = T \hat{G} T^h.$$

226 Invoking again the reduced SVD (11) of T , the above equation becomes

$$\Sigma^2 H(\alpha) \Sigma^2 + \alpha H(\alpha) = \Sigma \hat{H} \Sigma, \quad \text{with } H := Y^h S_{o2}(\alpha) Y. \quad (23)$$

227 Since Σ is diagonal, the above equation decouples into componentwise scalar equations

228 whereby

$$H_{ij}(\alpha) = \frac{\sigma_i \sigma_j}{\sigma_i^2 \sigma_j^2 + \alpha} (Z Z^h)_{ij} \quad 1 \leq i, j \leq d, \quad (24)$$

229 and $S_{o2}(\alpha) = Y H(\alpha) Y^h$ is readily found once $H(\alpha)$ is evaluated using the above formula.

230 Moreover, it is easy to verify that (24) with $\alpha = 0$ yields $S_o = Y H(0) Y^h$ through (12), and

231 that we have

$$H_{ij}(\alpha) - H_{ij}(0) = -\frac{\alpha}{\sigma_i \sigma_j (\sigma_i^2 \sigma_j^2 + \alpha)} (Z Z^h)_{ij} \quad 1 \leq i, j \leq d.$$

232 Consequently, this second regularization approach also verifies $\|S_{o2}(\alpha) - S_o\| \rightarrow 0$ as $\alpha \rightarrow 0$.

233 **Remark 2** *The first regularization yields $H_{ij}(\alpha) = (Z Z^h)_{ij} / (\sigma_i + \alpha)(\sigma_j + \alpha)$ instead of (24),*

234 *with H_{ij} as in (23). This shows that the two regularization approaches are not identical,*

235 *although the next remarks 3, 4 show that they are similar in several ways.*

236 **Remark 3** *The minimum-norm solution S_o to problem (8), as well as its regularized ap-*
237 *proximations $S_{o1}(\alpha)$ and $S_{o2}(\alpha)$, are Hermitian and positive definite (and hence acceptable*
238 *as correlation matrices), without those restrictions needing to be explicitly enforced (e.g.*
239 *through constraints). In fact, any matrix $S \in \mathbb{C}^{d \times d}$ can be additively decomposed into its*
240 *Hermitian and skew-Hermitian parts: $S = S_1 + S_2$ with $S_1^h = S_1$ and $S_2^h = -S_2$, and we have*
241 *$\|S\|_F^2 = \|S_1\|_F^2 + \|S_2\|_F^2$. Moreover, it is easy to verify that $G_1 := \mathcal{A}(S_1)$ and $G_2 := \mathcal{A}(S_2)$*
242 *are respectively Hermitian and skew-Hermitian. For the objective functional $J(S)$, this gives*
243 *(since \hat{G} is Hermitian)*

$$2J(S) = \|G_1 - \hat{G}\|_F^2 + \|G_2\|_F^2,$$

244 *so that optimality implies $G_2 = 0$, hence $S_2 = 0$ since by assumption T has full (column)*
245 *rank. A similar line of reasoning applies to the regularized versions of problem (8).*

246 **Remark 4** *From a computational complexity standpoint, both regularizations require $O(md^2) +$*
247 *$O(Qd^2) + O(d^3)$ complex arithmetic operations, with $m \gg Q \geq d$ in the present context. The*
248 *leading $C \times md^2$ amount of arithmetic operations (where C is a method-dependent constant)*
249 *arises from the decomposition of the transfer matrix T . Both regularization methods may (as*
250 *explained) use the reduced SVD of T , in which case $C = 6^{23}$ (Sec. 8.6). Alternatively, the*
251 *first regularization may as easily be carried out using a thin QR factorization of $[T; \sqrt{\alpha}I]$,*
252 *resulting in $C = 2^{23}$ (Sec. 5.2).*

253 IV. NUMERICAL RESULTS

254 A. Problem Description

255 In this section, we demonstrate how we can construct approximate diffuse fields in en-
256 closed rooms for arbitrary frequencies (e.g. below the Schroeder frequency). That is, by
257 driving the room with signals drawn from a multivariate Gaussian vector with zero mean
258 and an optimal speaker correlation, we can obtain a random field in the target region whose
259 spatial cross-correlation is close to (3) in the sense described by (8).

260 We consider a room with dimensions $5.75 \times 4.25 \times 3 \text{ m}^3$. The impedance for the wall,
261 roof, and floor is taken constant for all simulations and set to $Z = 4.25 \times 10^5 \text{ kg}/(\text{m}^2 \text{ s})$,
262 while the speed of sound and mass density are $c = 340 \text{ m/s}$ and $\rho_0 = 1 \text{ kg}/\text{m}^3$. The side
263 walls of the room contain uniformly spaced speakers each with an area of $0.25 \times 0.25 \text{ m}^2$. We
264 consider three different speaker configurations in this study: 1) 9 speakers/wall ($d = 36$), 2)
265 16 speakers/wall ($d = 64$), and 3) 25 speakers/wall ($d = 100$). For each case, we compute
266 optimal speaker correlations for three frequencies: 150, 250, and 300 Hz. A representative
267 geometry for the room with 36 speakers is shown in Figure 1. The target region, also shown,
268 has dimensions $2 \times 2 \times 2 \text{ m}^3$ and is located in the center of the room.

269 For each of the studied cases, we first compute a low-rank approximation of the target
270 correlation \hat{G} as per (15), using $Q = 200$. This quadrature rule results in a relative ap-
271 proximation error of less than 0.1% on \hat{G} for all cases studied herein. We then solve the
272 least-squares problems (19) for $1 \leq q \leq Q$. Finally, the optimal speaker cross-correlation S_o
273 is obtained as per (21). The Tikhonov regularization parameter α is determined using an

274 L-curve approach^{24,25}. To this end, we use the objective and regularization term in (22) for
 275 either of the regularization strategies described in Section III E. We point out that the com-
 276 putational cost of solving all the aforementioned optimization problems is negligible when
 277 compared to the computational cost of building the transfer matrix T .

278 We use an in-house Finite Element code developed using the FEniCS library²⁶ in conjunc-
 279 tion with the parallel direct solver MUMPS for all the calculations presented herein. The
 280 models are meshed with four-node tetrahedral elements. All the results shown are generated
 281 with a mesh containing approximately 130,000 nodes and 750,000 elements, which is fine
 282 enough to achieve a low discretization error in all calculations.

283 The Schroeder frequency for this room is calculated from (1). The reverberation time is
 284 estimated using the Norris-Eyring relation²⁷:

$$T_{60} = \frac{-24V \ln(10)}{A c \ln(1 - \alpha_{rand})},$$

285 where $A = 2(L_x L_y + L_x L_z + L_y L_z)$ is the room surface area and α_{rand} the random-incidence
 286 absorption coefficient¹⁹

$$\alpha_{rand} = 1 - \int_0^{\pi/2} \left| \frac{Z \cos(\theta) - \rho_0 c}{Z \cos(\theta) + \rho_0 c} \right|^2 \sin(2\theta) d\theta.$$

287 Using the above expressions in (1), we obtain $f_s = 1001$ Hz. Hence, we point out the
 288 frequencies used in the examples (150, 250, and 300 Hz) are well below the Schroeder limit
 289 for this room.

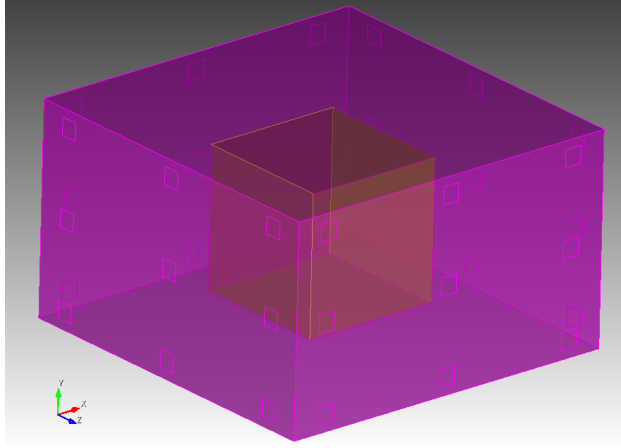


FIG. 1. Room Geometry

290 **B. Results**

291 Let's first define the relative error in correlation (or residual) as

$$\epsilon_G := \frac{\|\mathcal{A}S_o - \hat{G}\|_F}{\|\hat{G}\|_F}$$

292 Without loss of generality, we use $\frac{p_o^2}{2} = 1$ in all the examples. Hence, a useful metric to
 293 explore is how much the diagonal entries of $G^o = \mathcal{A}S_o$ depart from unity. To this end, we
 294 define the error

$$\epsilon_{ms} := \frac{\sqrt{\sum_i |G_{ii}^o - 1|^2}}{m}$$

295 An equivalent interpretation of this error is how much the mean square pressure (nor-
 296 malized to unity in our case) departs from being spatially constant. This is a widely used
 297 metric to judge the level of sound diffusion in laboratory experiments^{14,15,17} and proved to
 298 be very useful for assessing the quality of our numerical solutions.

299 Table I contains a summary of the results along with the Tikhonov parameter (α) used in
 300 each case. We notice that for all frequencies both errors, ϵ_G and ϵ_{ms} , decrease as the number

TABLE I. Results Summary

	150 Hz			250 Hz			300 Hz		
Speakers	ϵ_{ms}	ϵ_G	α	ϵ_{ms}	ϵ_G	α	ϵ_{ms}	ϵ_G	α
36	0.17	0.33	2×10^{-3}	0.35	0.51	1×10^{-2}	0.43	0.64	2×10^{-3}
64	0.09	0.17	1×10^{-3}	0.16	0.30	1×10^{-2}	0.18	0.37	2×10^{-3}
100	0.08	0.16	5×10^{-4}	0.08	0.23	1×10^{-3}	0.08	0.24	5×10^{-4}

301 of speakers increases, as expected. Another noticeable trend is that these errors increase as
 302 frequency increases.

303 The increasing trend in approximation errors can be explained by studying how the
 304 eigenvalues of the target correlation decay as functions of frequency. First, notice that
 305 the rank of the approximate correlation, $G_o = TS_oT^h$, is at most d (number of speakers).
 306 Hence, using the best approximation error in Eq. (13), we obtain a lower bound for the
 307 relative residual of our inverse problem as

$$\frac{\|G_o(d) - \hat{G}\|_F}{\|\hat{G}\|_F} \geq \mathcal{E}(d) \equiv \sqrt{\frac{\sum_{k=d+1}^m \lambda_k^2}{\sum_{j=1}^m \lambda_j^2}} \quad (25)$$

308 Hence, we see that the lower bound depends on the decay of the eigenvalue spectrum. To
 309 illustrate this behavior in our problem, define a participation factor as

$$P_f^2(d) \equiv \frac{\sum_{k=1}^d \lambda_k^2}{\sum_{j=1}^m \lambda_j^2}$$

310 and notice that $\mathcal{E}(d)^2 = 1 - P_f^2(d)$. Now, observe in Fig. 2 how $P_f(d)$ increases with
 311 decreasing frequency, in general. The latter trend indicates that the lower bound in Eq. (25)

312 increases with increasing frequency for a fixed truncation level, which is in agreement with
 313 the error trend reported in Table I.

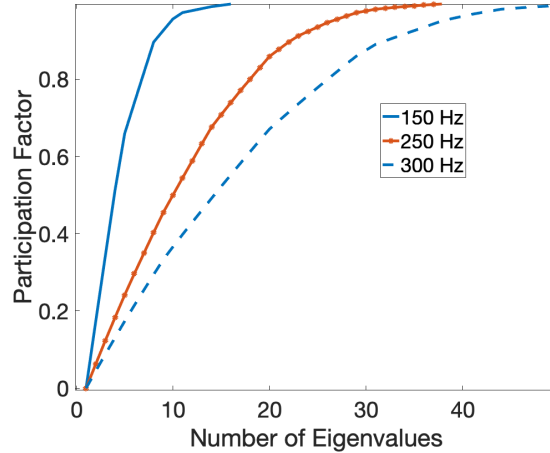


FIG. 2. Participation factor of eigenpairs of the correlation matrix at different frequencies

314

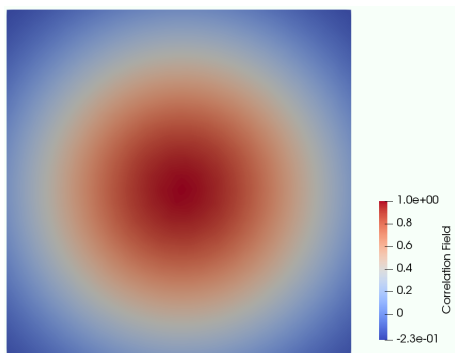
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316 To better illustrate the quality of our solutions, we plot the correlation field with respect
 317 to the center of the target region for different combinations of speakers and frequencies as
 318 shown in Figure 3. We can observe that the optimized correlation closely resembles that of
 319 a diffuse field. Furthermore, we compare the target and estimated fields along a diagonal
 320 across the target region in Figure 4. We can again observe a close match of the computed and
 321 target correlations for different frequency and speaker combinations. It is important to point
 322 out that we are showing only the target region at the center of the room in these images.
 323 As we will show later, the acoustic field departs from being purely diffuse at locations near
 324 the walls, as expected.

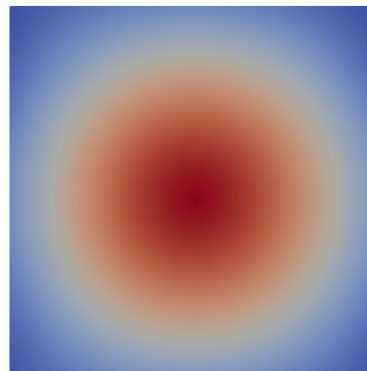
326

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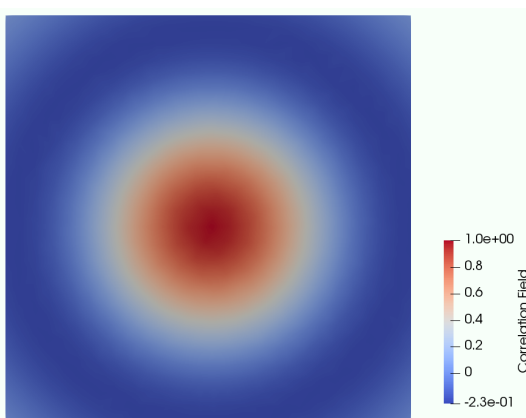
Recall that the mean square pressure (i.e. diagonal entries of the correlation) is spatially
 constant in a diffuse field (as captured by the metric ϵ_{ms} shown in Table I). We now provide



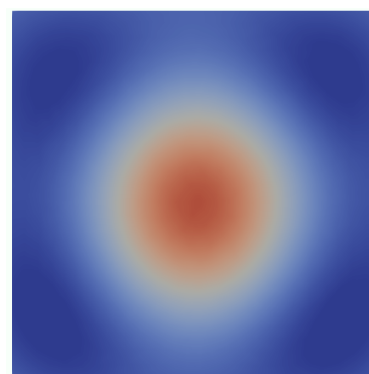
Sinc Function: 150 Hz



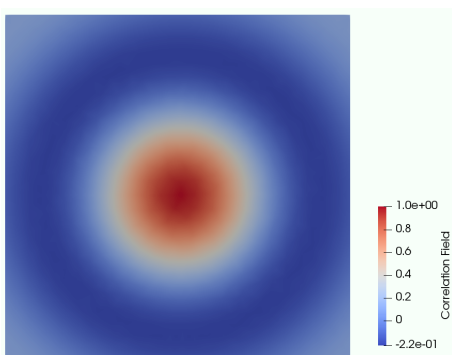
Optimal solution: 150 Hz, 36 speakers



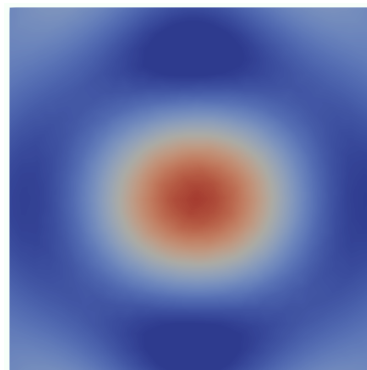
Sinc Function: 250 Hz



Optimal solution: 250 Hz, 64 speakers



Sinc Function: 300 Hz]



Optimal Solution: 300 Hz, 100 speakers

FIG. 3. Correlation field with respect to the center of the target region.

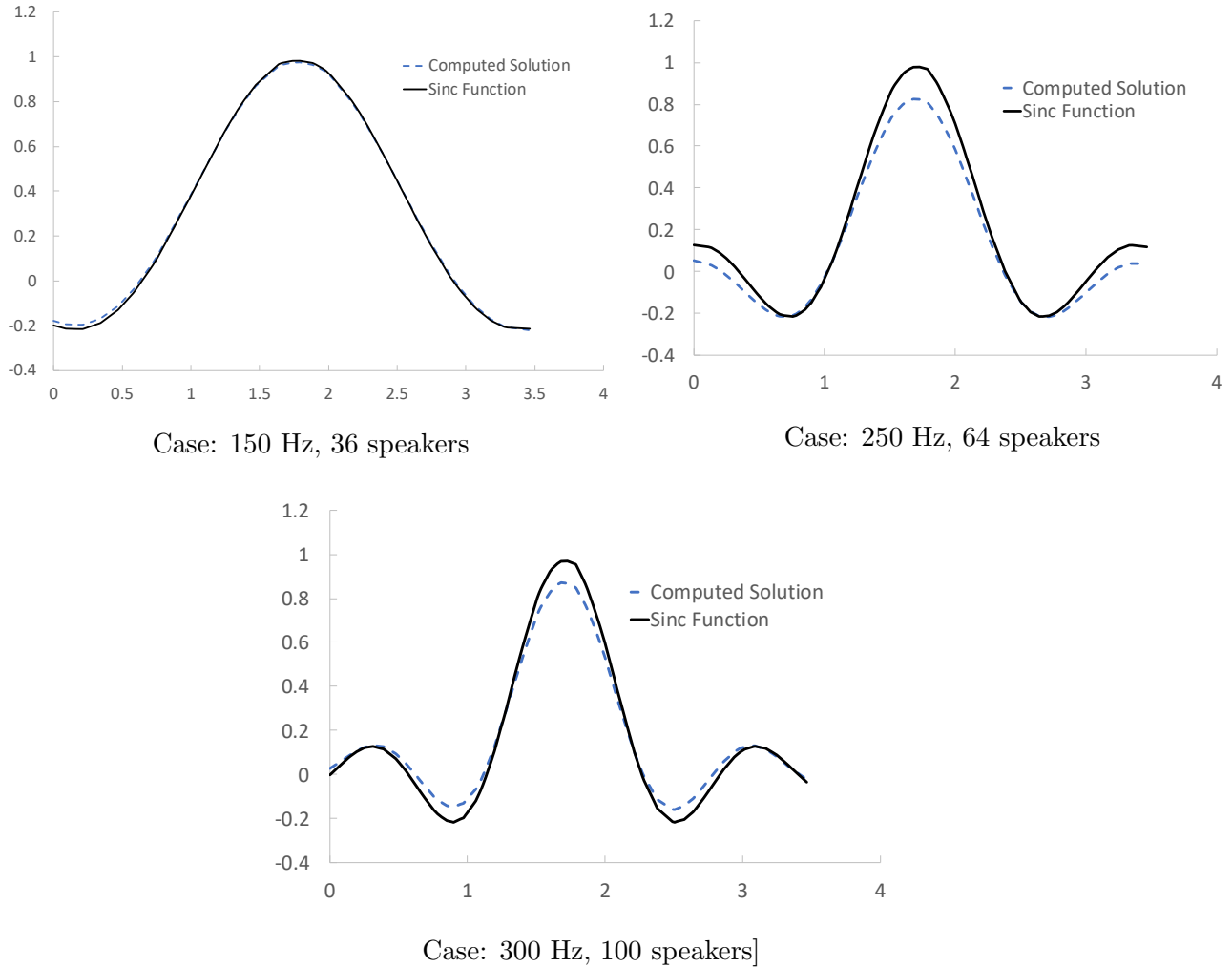


FIG. 4. Correlation field along a diagonal across the target region.

328 more global representations of this behavior. Figure 5(a) shows the mean square pressure
 329 along a line through the center of the room in the X direction for all cases. We can see that
 330 the field is close to unity and constant in a region slightly larger than the target domain
 331 and departs from pure diffusion close to the walls. Furthermore, for comparison purposes,
 332 we show in Figure 5(b) the mean square pressure obtained using random realizations from
 333 uncorrelated speakers assuming a standard normal Gaussian distribution. Notice that indeed

334 uncorrelated speakers cannot produce the desired constant field at this given frequency,
 335 reinforcing the success of the proposed optimization approach.

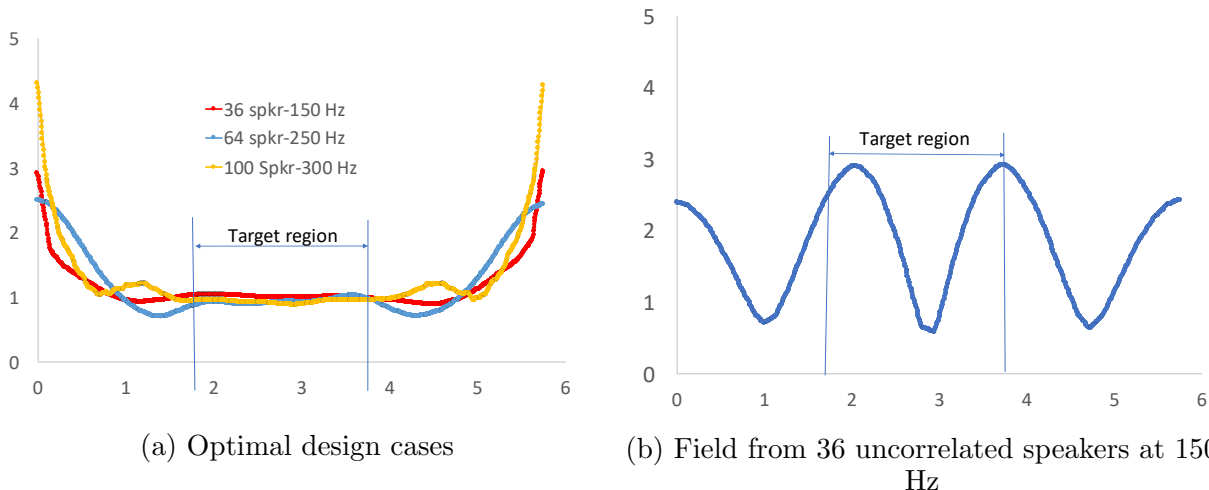


FIG. 5. Mean square pressure in the entire room along X-axis. Results are normalized to be on the same scale as the optimal case.

336 Lastly, Figure 6 shows the mean square pressure field for the three studied cases. Again,
 337 we notice that, in all cases, the field is constant over a region larger than the target one, but
 338 departs from diffuse behavior away from the target region, as expected.

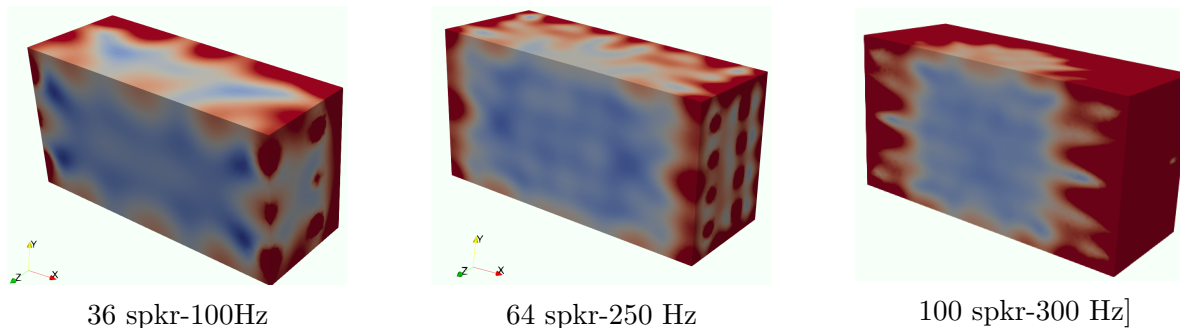


FIG. 6. Mean square pressure in the entire room.

339

340

341 V. SUMMARY AND CONCLUSIONS

342 We presented how an optimization approach can be used to produce diffuse fields in
343 enclosed rooms at arbitrary frequencies (even below the Schroeder limit). To this end, we
344 characterized a diffuse field solely by its mean and correlation as these are stochastic ho-
345 mogeneous and isotropic Gaussian processes. Then, we postulated an optimization problem
346 in which we sought the correlation structure of input speakers that minimized the distance
347 between the output correlation and that of a diffuse field over a target region. We addressed
348 the large computational cost that arises from the discretization of the target correlation us-
349 ing a low-rank expansion based on the integral representation of spherical Bessel functions.
350 Moreover, we formally showed how to regularize the ensuing ill-posed inverse problem. Our
351 results demonstrated that it is possible to obtain approximate diffuse fields in enclosed rooms
352 even at frequencies below the Schroeder limit by driving correlated speakers in an optimal
353 way. Also, we found that there is a limitation in the quality of the approximation that
354 strongly depends on the number of speakers and the frequencies of interest. As frequency
355 increases a larger number of speakers is needed to maintain a given level of error in the
356 diffuse field approximation. A direction for future work is to study the influence of speaker
357 location on the approximation error. Furthermore, it is possible to devise optimization al-
358 gorithms to find such locations. Also, incorporating uncertainty in boundary conditions,
359 material properties, etc. in the optimization formulation would be highly desirable.

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