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Performance Paradox of Dynamic Matching Models under Greedy Policies

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Abstract

We consider the stochastic matching model on a non-bipartite matching graph and analyze the impact of adding an edge to the expected number of items in the system. One may see adding an edge as increasing the flexibility of the system, for example asking a family registering for social housing to list less requirements in order to be compatible with more housing units. Therefore it may be natural to think that adding edges to the matching graph will lead to a decrease of the expected number of items in the system and the waiting time to be matched. In our previous work, and we proved this is not always true for the First Come First Matched discipline and provided sufficient conditions for the existence of the performance paradox: despite a new edge in the matching graph, the expected total number of items can increase. These sufficient conditions are related to the heavy-traffic assumptions in queueing systems. The intuition behind is that the performance paradox occurs when the added edge in the compatibility graph disrupts the draining of a bottleneck. In this paper, we generalize this performance paradox result to a family of so called greedy matching policies and explore the type of matching graphs where such paradoxes occur. Intuitively, a greedy matching policy never leaves compatible items unmatched, so the state space of the system consists of finite words of item classes that belong to an independent set of the matching graph. Some examples of greedy

matching policies are First Come First Match, Match the Longest, Match the Shortest, Random selection, Priority. We prove several results about the existence of performance paradoxes for greedy disciplines for some family of graphs. More precisely, we prove several results about the lifting of the paradox from one graph to another one. For a certain family of graphs we prove that there exists a paradox for the whole family of greedy policies. Most of these results are based on Strong Aggregation of Markov chains and graph theoretical properties.

Keywords: Matching models, Performance Paradox, Strong Aggregation, Separable solutions

1 Introduction

Braess showed that adding resources to a transportation networks can hurt the performance of the system [2]. Many authors have been interested in investigating the existence of such a paradox in several contexts related to queueing networks (see for instance [3–7]). In this work, we analyze whether such a phenomenon is achieved in dynamic matching models where the compatibility graph is non-bipartite.

In our previous work [1], we analyze the performance of the stochastic First Come First Match, (FCFM in the following) matching model with a general compatibility graph when the flexibility increases, i.e when an edge is added to the compatibility graph. We show that increasing the flexibility can increase the overall performance of the system, which is reminiscent of the Braess paradox. In this paper, we generalize the results of our previous work by investigating the existence of the performance paradox for greedy matching policies (which include the FCFM policy). We show that for a certain family of graphs, any greedy policy exhibits the performance paradox. This can be seen as a Braess paradox for stochastic matching: for a given family of graphs, any greedy matching policy leads to overall performance decrease when we add an additional resource (i.e. an edge in our case).

The technical part of the paper is as follows. In section 2, we describe the model and we summarize the results obtained in [1]. We also study a simple example of a matching graph with 4 nodes such that the paradox exists for the FCFM (the quasi-complete graph and the complete graph). Due to this simple graph structures and some lumpability arguments, we were able to numerically compute the expectation of the total number of items in the system and find the intervals of arrival probabilities which lead to a performance paradox.

The existence of such a performance paradox leads to many questions we now try to address:

- First, is it only due to the FCFM discipline? Are other disciplines like Match the Longest or Random also prone to this phenomenon?

- Is the paradox related to the matching graph? The result we proved in [1] are based on the Independent Set which is the main bottleneck of the system and a neighbor node of this set. This construction does not imply at first glance some constraints on the topology. But we are not able to analyze an arbitrary matching graph to find the expectation of total number of items. Indeed, even if we know that some models have a product form solution, it is rather difficult to find the normalization constant which remains necessary to compute the expected total number of items. Thus one may ask if some constraints on the size of the compatibility graph are necessary.
- Does the paradox only appear when we have only one bottleneck in the system (i.e. this is the technical condition of our main theorem in [1])? Is this condition only technical or is it mandatory?

To answer these questions, we develop some theoretical techniques to transfer the existence of a paradox from one discipline to any greedy discipline for the graph studied in Section 2. Then we provide other results to lift the paradox from one graph to a more complex graph which is built from the former one. We also give some examples of paradoxes.

To be more precise, in section 3, we give an abstract definition of many disciplines and we define greedy disciplines. Most of the disciplines considered in the literature are shown to be greedy with the notable exception of the threshold disciplines studied in [8] and [9]. Then we prove that all greedy disciplines exhibit performance paradox for quasi-complete matching graphs. The results are proved using strong aggregation of Markov chains. Intuitively we prove that all the greedy disciplines are equivalent for quasi-complete matching graph when we are interested in the total number of items. In Section 4, we introduce two operations on matching graphs, the JOIN (noted \bowtie) and the UNION (noted \cup). We prove that if the matching graph G is build by the JOIN of subgraphs $G1$ and $G2$, then one can find the steady-state distribution from two sub-models based on $G1$ and $G2$. Section 4 is also devoted to two designs of matching graphs: $G1 \bowtie IS_n$ (IS being an independent set with size n) and $G1 \bowtie (G2 \cup G3)$. For both constructions we prove a transfer result: if matching graph $G1$ associated with a greedy discipline satisfies the conditions of Theorem 2 that imply a performance paradox, then the matching graph $G1 \bowtie IS_n$ associated with the same discipline also satisfies the conditions of Theorem 2 and thus also exhibits a performance paradox.

1.1 Related Work

The model we consider in this work has been introduced in [10], where it is called general stochastic matching model to empathize that the matching graph is non-bipartite, and further studied in [11]. The works [10] and [11] present interesting properties of this model such as that the policies Match the Longest and FCFM disciplines have a maximal stability region. Recently, the author in [12] shows that this matching model with the FCFM discipline is related to the order-independent loss queueing networks. An extension of

FCFM stochastic matching to multigraphs with self-loops has been studied in [13].

A related model to ours is the bipartite dynamic matching model. In this model, the graph that determines compatibilities between items is bipartite and, thus, the nodes can be separated in two disjoint sets: server nodes and customer nodes. To the best of our knowledge, [14] was the first analyzing the fully dynamic setting of matching model, i.e., all the items arrive to the system according to a random process. He analyzed the problem of how to assign public houses to tenants. In that work, it is considered that an available public house is assigned to the longest waiting family among those that expressed their interest for that house. The First Come First Served infinite bipartite matching model, proposed by Caldentey et al. [15], considers an infinite sequence of server nodes, independent and identically distributed according to a probability distribution on the server classes, and an independent infinite sequence of job items, independent and identically distributed according to a probability distribution on the job classes. In [16], the authors proved that an alternative Markov description from [15] has a product form stationary distribution. The authors in [17] study the stability of the system for different matching policies. Optimal matching policies of bipartite matching models have been studied in an asymptotic regime in [8] and for the N-shaped model in [9]. The author in [18] studies a bipartite matching model, in which jobs and servers are matched according to the First Come First Served policy, but jobs are immediately lost if they do not find an available server upon arrival.

The impact of flexibility in redundancy networks has been analyzed in [19]. Their main result shows that providing more redundancy to a class leads to a performance improvement of this case but it might not be beneficial for the other classes. The authors in [16] show that, under the heavy traffic assumption on the arrivals, the First Come First Served infinite bipartite matching model has the same stationary distribution as the First Come First Served-Assign the Longest Idle Server queueing model. Furthermore, the authors in [20] show that the First Come First Served-Assign the Longest Idle Server model has the same stationary distribution as the redundancy service model. Therefore, from the work of [19], one might conclude that the performance paradox existence analysis in dynamic matching models has been already carried out. However, we would like to remark that these works assume a bipartite matching graph and strongly depend on the product form result for First Come First Served discipline so they cannot be generalized to any greedy discipline.

2 Matching Model and Performance Paradox

We consider a queueing system with n classes of items. Items of different classes arrive to the system according to independent Poisson processes, with rates $\lambda_i > 0$, $i = 1, \dots, n$. The compatibilities between item classes are described by a connected non-bipartite compatibility graph $G = (V, \mathcal{E})$, where $V = \{1, \dots, n\}$ is the set of item classes and \mathcal{E} is the set of allowed matching pairs:

items of classes i and j are compatible if and only if $(i, j) \in \mathcal{E}$. If an incoming item is compatible with one item present in the system, both items can be matched, in which case they disappear. Otherwise, the incoming item is placed at the end of the queue of unmatched items.

For a class $i \in V$, we denote by $\Gamma(i) = \{j \in V : (i, j) \in \mathcal{E}\}$ the set of item classes that are compatible with class i , i.e. the set of all the neighbors of the node i in the compatibility graph G . For any subset of item classes $V_1 \subseteq V$, let $\Gamma(V_1) = \bigcup_{i \in V_1} \Gamma(i)$, and $|\lambda_{V_1}| = \sum_{i \in V_1} \lambda_i$.

To more easily describe the models based on composition of matching graphs (see Section 4), we present the models in continuous time. Indeed it is easier to use race condition on continuous-time models to represent the interaction of items (see for instance [21] for an example on the compositions of stochastic automata). Therefore we describe the models in continuous time, we uniformize them and we analyze the discrete-time models using strong aggregation of DTMC and the existence of paradox is proved for discrete time models.

2.1 Markov Representation

Let \mathcal{V}^* be the set of finite words over the alphabet V and we denote the empty word by \emptyset . Let

$$\mathbb{W} = \{w = w_1 \cdots w_q \in \mathcal{V}^* : \forall (i, j) \in [1, q]^2, i \neq j, (w_i, w_j) \notin \mathcal{E}\}$$

be the subset of words without a compatible pair of letters, i.e. the set of ordered independent sets of \mathcal{G} . For any $w \in \mathbb{W}$ and any $x \in V$, let $|w|_x$ be the number of occurrences of letter x in word w . Let $|w|$ be the *length* of w (i.e., the number of letters of the word w).

A state of the system right after the new arrival (if any) has been matched or placed in the queue of unmatched items can be described by a word $w \in \mathbb{W}$. Each letter $w_i \in V$ represents the class of an unmatched item waiting in the system and the order of the letters represents their order of arrival.

Let us present some notation that we will use throughout the paper. Let $M(G, \lambda, D)$ be the continuous-time Markov chain associated with matching graph G , matching discipline D and arrival rates of letters λ . Let $\mathbb{E}[M(G, \lambda, D)]$ be the expectation of the total number of letters for this Markov chain. K_n will be a complete graphs with n nodes, IS_n will be an independent set with n nodes (i.e. n nodes without edges). $K_n - (a, b)$ will denote the complete graph with n nodes without edge (a, b) : it is called a quasi-complete graph in this paper.

Proposition 1 *In the models under consideration, the continuous-time Markov chain that represents the number of unmatched items is uniformizable. Indeed $\sum_i \lambda_i < \infty$.*

After applying standard uniformization technique, with a uniformization constant $\Lambda > \sum_{i=1}^n \lambda_i$, we obtain a matching model in discrete time. In each time slot $t \in N$, one item arrives to the system with probability $1 - \alpha_0$, and there are no arrivals with probability

$$\alpha_0 = 1 - \frac{1}{\Lambda} \sum_{i=1}^n \lambda_i > 0.$$

Thus each item belongs to a class within the set of item classes sampled from a conditional probability distribution over V given the event that there is an arrival: $\alpha = (\alpha_1, \dots, \alpha_n)$, with $\alpha_i = \frac{\lambda_i}{\sum_{i=1}^n \lambda_i}$, $\forall 1 \leq i \leq n$. It follows that $\alpha_i > 0$, $\forall i$.

$\mathbb{E}[M(G, \alpha, D)]$ will denote the expectation of the total number of letters for this discrete-time Markov chain with arrival probabilities α .

A subset of nodes $\mathcal{I} \subseteq V$ is called an *independent set* if there is no edge between any two items in \mathcal{I} , i.e. for any $i, j \in \mathcal{I}$, $(i, j) \notin \mathcal{E}$. Let \mathbb{I} be the set of independent sets of G . We assume that

$$|\alpha_{\mathcal{I}}| < |\alpha_{\Gamma(\mathcal{I})}|, \quad \forall \mathcal{I} \in \mathbb{I}. \quad (1)$$

According to [10, Proposition 2], the above condition is a necessary condition for any matching policy to be stable. Thus if the compatibility graph is bipartite, the system is not stable, see [10]). Throughout the paper, we assume that the final compatibility graph is not bipartite, even if we consider bipartite graph as subgraphs.

For any $\mathcal{I} \in \mathbb{I}$, denote by $\Delta_{\mathcal{I}} = |\alpha_{\Gamma(\mathcal{I})}| - |\alpha_{\mathcal{I}}|$. In the following $\Delta_{\mathcal{I}}$ is called the stability gap of independent set \mathcal{I} .

2.2 Performance Paradox for First Come First Match Discipline

We say that there exists a performance paradox for matching graph G and discipline D if there exists an edge (i^*, j^*) such that

$$\mathbb{E}[M(G, \alpha, D)] > \mathbb{E}[M(G - (i^*, j^*), \alpha, D)] \quad (2)$$

that is, if the mean number of items can be increased by adding an edge to the compatibility graph.

The number of classes determines the number of nodes of the matching graph. Hence, the matching graphs we consider in this article are formed by, at least, four nodes. Clearly, all the connected matching graphs with less than four nodes are complete or bipartite (see Fig. 1)

Let us now show the link with our previous results [1]. We first present some notation to understand the main result of this section.

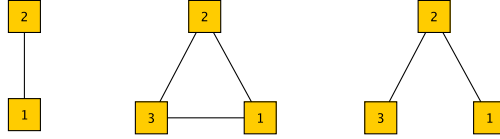


Fig. 1 All the connected compatibility graphs with less than four nodes are complete or bipartite.

Definition 1 (Bottleneck) A set $\mathcal{I} \in \operatorname{argmin}_{\mathcal{I} \in \mathbb{I}} \Delta_{\mathcal{I}}$ will be called a *bottleneck set* in the sense that it has the smallest maximal draining speed.

Let

$$\bar{\delta} = \min_{\mathcal{I} \in \mathbb{I}} \Delta_{\mathcal{I}} = \min_{\mathcal{I} \in \mathbb{I}} (|\alpha_{\Gamma(\mathcal{I})}| - |\alpha_{\mathcal{I}}|). \quad (3)$$

We have $\bar{\delta} > 0$ as we assume that α satisfies (1). We select a bottleneck set (or saturated independent set) $\hat{\mathcal{I}} \in \operatorname{argmin}_{\mathcal{I} \in \mathbb{I}} \Delta_{\mathcal{I}}$ with the highest cardinality, i.e.

$$|\hat{\mathcal{I}}| = \max_{\mathcal{I} \in \mathbb{I} \text{ s.t. } \Delta_{\mathcal{I}} = \bar{\delta}} |\mathcal{I}|. \quad (4)$$

We are interested in how the performance of the system evolves by adding an edge when $\Delta_{\hat{\mathcal{I}}}$ tends towards 0. First, we define a parameterized family of item class distributions:

$$\alpha_i^{\delta} = \begin{cases} \alpha_i + \frac{\bar{\delta}}{2} \frac{\alpha_i}{|\alpha_{\hat{\mathcal{I}}}|} - \frac{\delta}{2} \frac{\alpha_i}{|\alpha_{\hat{\mathcal{I}}}|} & \text{if } i \in \hat{\mathcal{I}} \\ \alpha_i - \frac{\bar{\delta}}{2} \frac{\alpha_i}{|\alpha_{\Gamma(\hat{\mathcal{I}})}|} + \frac{\delta}{2} \frac{\alpha_i}{|\alpha_{\Gamma(\hat{\mathcal{I}})}|} & \text{if } i \in \Gamma(\hat{\mathcal{I}}) \\ \alpha_i & \text{otherwise} \end{cases}$$

for all $0 < \delta \leq \bar{\delta}$, where $\bar{\delta}$ is defined in (3) and $\hat{\mathcal{I}}$ in (4). It is clear that $\alpha^{\bar{\delta}} = \alpha$. By definition of α^{δ} ,

$$|\alpha_{\Gamma(\hat{\mathcal{I}})}^{\delta}| - |\alpha_{\hat{\mathcal{I}}}^{\delta}| = |\alpha_{\Gamma(\hat{\mathcal{I}})}| - \frac{\bar{\delta}}{2} + \frac{\delta}{2} - |\alpha_{\hat{\mathcal{I}}}| - \frac{\bar{\delta}}{2} + \frac{\delta}{2} = \bar{\delta} - \bar{\delta} + \delta = \delta,$$

which tends to 0 when δ tends to 0.

We now consider the expectations of the total number of items for both models as a function of δ and analyze their difference when δ tends to 0.

Definition 2 (Saturated Independent Set) An independent set \mathcal{I} is called *saturated* if $\Delta_{\mathcal{I}}^{\delta} = |\alpha_{\Gamma(\mathcal{I})}^{\delta}| - |\alpha_{\mathcal{I}}^{\delta}|$ tends to 0 when δ tends to 0.

In our previous work [1], for technical reasons, we assume there is only one saturated independent set $\hat{\mathcal{I}}$. We now focus on the existence of a performance paradox. In Section 2.4. we show that, under FCFM, a performance paradox

exists when the saturated independent set is a neighbor of the nodes of the added edge.

Theorem 1 [Adapted from Theorem 2 of [1]] *If $\hat{\mathcal{I}}$ is uniquely defined, and if $\hat{\mathcal{I}}$ has both i^* and j^* as neighbors, then there exists a performance paradox for δ sufficiently small.*

We provide the full proof in the Appendix ?? as [1] was a short paper and did not contain the complete proof.

2.3 An example of Performance Paradox

In the following, we provide an example in which we can establish the exact value of δ for the performance paradox to occur. The example is based on a matching model formed by a quasi-complete graph with four nodes (denoted as $K_4 - (3, 4)$) (see Fig. 2). We also consider the graph where we add edge $(3, 4)$ (i.e. complete graph K_4). Therefore all the nodes different from $n - 1$ and n are independent sets that are connected to these nodes.

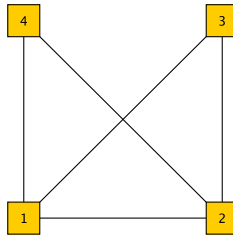


Fig. 2 Quasi-complete compatibility graph with 4 nodes: $K_4 - (3, 4)$.

We consider the following conditional probability distribution of arrivals: $\alpha_1 = 0.45$, $\alpha_2 = 0.11$ and $\alpha_3 = \alpha_4 = 0.22$. For this instance, the saturated independent set is $\hat{\mathcal{I}} = \{1\}$ with $\bar{\delta} = |\alpha_{\{2,3,4\}}| - |\alpha_1| = 0.1$. We define a new collection of conditional probability distributions α^δ for all $0 < \delta \leq 0.1$, i.e. $\alpha_1^\delta = 0.5 - \frac{\delta}{2}$, $\alpha_2^\delta = 0.1 + \frac{\delta}{10}$ and $\alpha_3^\delta = \alpha_4^\delta = 0.2 + \frac{\delta}{5}$. Since the saturated independent set has the nodes that connect the missing edge as neighbors (i.e., the node 1 is a neighbor of node 3 and node 4), we know from Theorem 1 that there exists a performance paradox for δ sufficiently small.

We study the exact value of the *saturation threshold*, i.e. the maximum δ value for the existence of the paradox under the FCFM policy. According to Lemma 3 and Lemma 6 (in the next section as they are proved for any greedy policy) we can conclude that, for FCFM policy, for the matching model under consideration here, the performance paradox exists if and only if $\delta < 0.0818369$. All the details of the computations are presented in Appendix A.

2.4 Multiple Saturated Independent Sets

The definition of the parametrized α^δ implies that the unique saturated independent set is $\hat{\mathcal{I}}$. We now discuss the situation where there are multiple saturated independent sets. This means that multiple independent sets have their stability gap which tends to 0 as δ tends to 0. In that case, the proof of Theorem 1 in [1] fails because we cannot prove the existence of a strictly positive (or strictly negative) constant at the numerator when we rewrite the difference in expectation as a rational fraction of δ . However, in this section, we present an example with two saturated independent sets where the performance paradox exists.

Again we consider compatibility graph K_4 and $K_4 - (3, 4)$. Let us define the conditional probability distribution of the arrivals as $\alpha_1 = \alpha_2 = 0.15$, $\alpha_3 = 0.4$ and $\alpha_4 = 0.3$. We choose the saturated independent sets to be $\{3\}$ and $\{1, 2\}$. We define a new parametrized family of conditional probability distributions α^δ , for all $0 < \delta \leq 0.1$, as $\alpha_1^\delta = \alpha_2^\delta = 0.25 - \delta$, $\alpha_3^\delta = 0.5 - \delta$ and $\alpha_4^\delta = 0. + 3\delta$. We prove that there exists a performance paradox and compute exactly for which value of δ it appears. The proof is available in Appendix B.

Proposition 2 *For the dynamic matching model under consideration, there exists a performance paradox if and only if $\delta \in (0, 0.0563)$.*

We conclude from the previous proof that the difference $\mathbb{E}[M(K_4, \alpha^\delta, FCFM)] - \mathbb{E}[M(K_4 - (3, 4), \alpha^\delta, FCFM)]$ is unbounded as δ tends to 0. In the following result, we quantify its growth rate. The proof is available in Appendix C.

Proposition 3 *When δ tends to 0,*

$$\delta(\mathbb{E}[M(K_4, \alpha^\delta, FCFM)] - \mathbb{E}[M(K_4 - (3, 4), \alpha^\delta, FCFM)]) \rightarrow \frac{1}{24}.$$

Remark 1 This example shows that the existence of the paradox is not related to the uniqueness of a saturated independent set. This was a technical condition used to prove Theorem 1 in [1].

3 Greedy Matching Disciplines

In the following, m will be a word (or a sequence of letters), x will be a letter, $\Gamma(x)$ is the neighbor of x in the matching graph, $\Gamma(m) = \cup_{x \in m} \Gamma(x)$. S is the set of states and V the set of letters.

Definition 3 (Matching) Let m be a state of the system and x a letter which arrives in the system. We say that x matches with m if $x \in \Gamma(m)$.

Remember that a subsequence of a word (or a sequence of letters) m is derived from m by deleting some or no letters without changing the order. Subsequence can contain consecutive letters which were not consecutive in the original word. Thus subsequences are not necessarily substrings.

Definition 4 (Effect of a matching) Let m be a state of the system and x a letter which arrives in the system, if there is no matching, letter x is appended at the end of word m . Otherwise, there is a matching and the matching discipline will explain the possible transitions and their respective probability.

Definition 5 (Matching discipline) Let m be a state of the system and x a letter which arrives in the system and matches with the word m . A matching discipline D is a function from $S \times V$ which returns a non empty subset of states $N_D(m, x)$ and a discrete distribution of probability $\psi_{N_D(m, x)}$ on this subset such that:

1. All the words in $N_D(m, x)$ are subsequences of m
2. The distribution gives the probability to make a transition to any state in $N_D(m, x)$: $\alpha_x \psi_{N_D(m, x)}(p)$ is the probability to jump from state m to any state p in $N_D(m, x)$, due to the arrival of an item of type x . Of course, we have:

$$\sum_{p \in N_D(m, x)} \psi_{N_D(m, x)}(p) = 1$$

Of course if the set $N_D(m, x)$ is a singleton, the distribution of probability is a Dirac.

Remark 2 This definition allows to model disciplines where $N_D(m, x) = \{m\}$ (i.e. the matching has no effect on word m even when x matched a letter in m , for instance for a threshold based discipline where the deletion only occurs for long words). It is also possible to represent disciplines where the effect of a matching is to remove all the letters in m which match with letter x . It is why we now introduce the notion of greedy matching discipline.

Definition 6 (Greedy matching discipline) A matching discipline GD is greedy if for m being a state of the system and x a letter which arrives in the system such that $x \in \Gamma(m)$, for all state p in $N_{GD}(m, x)$, we have $|p| = |m| - 1$,

Example 1 Clearly, FCFM is a greedy discipline such that the set $N_D(m, x)$ is a singleton for all word m and matching letter x . Match the Longest with random drawing of ties is a greedy discipline but the set $N_D(m, x)$ may contain several words.

Definition 7 (RANDOM discipline) Assuming that a word and a letter match, we delete one matching letter in the word. And this letter is chosen with an uniform distribution among the matching letters.

Example 2 Consider state $m = (a, b, c, a, b, a)$. Suppose that a letter t arrives and that t matches with both a and b but it does not match with c .

$$N_D(m, t) = \{(b, c, a, b, a), (a, c, a, b, a), (a, b, c, b, a), (a, b, c, a, a), (a, b, c, a, b)\}$$

Each of these words has a probability of $1/5$ to be selected.

Proposition 4 *RANDOM is a greedy discipline.*

Proposition 5 *If the matching graph is such that the degree of every node is positive, the arrival rates of all items are positive and the discipline is greedy, then the Markov chain contains a state with 0 items which is reachable from any state.*

Proof Indeed all items can be deleted due to an arrival of an item of the desirable classes. \square

Now we establish some relations between First Come First Match (FCFM) discipline and any greedy discipline (GD).

3.1 Complete Graphs and Greedy Disciplines

The performance paradox analysis consists of comparing the mean number of items in a quasi-complete graph and a complete graph. In the following result, we provide an explicit expression of the mean number of items of a complete graph with an arbitrary greedy matching policy.

Lemma 1 *Let GD be a greedy matching discipline, the states of the Markov chain associated with K_n are words which are associated with the independent sets of the matching graph, i.e. the following words: $\emptyset, 1^*, 2^*, \dots, n^*$.*

Proof by construction. First, states with several types of item are not reachable as items which match, instantaneously disappear even if we do not know which ones are deleted as we do not know exactly the discipline. Assuming now that the initial state is the empty state. The first arrival (say a type x item) provokes a transition to word (x) . If the next arrival is again a type x item (with rate λ_x), the next state is (x, x) , otherwise the arriving letter (say y) matches with the x as the matching graph is a complete graph. For any greedy matching discipline, this matching provokes the destruction of both y and x and a transition to the empty word. Such an event occurs with rate λ_y . Thus, by induction on the transitions, we only obtain words with only one type of item or the empty word. As the valid states only contain one type of items, the discipline does not matter as soon as it is greedy and exactly one letter is deleted in the word. Furthermore, for this matching graph, the transition rates of the chain do not depend on the matching discipline. Note this is not true for sparser matching graph. The chains may be described as a collection of n birth and death processes connected through state \emptyset . \square

Corollary 1 *The two Markov chains associated with matching graph K_n and disciplines GD and FCFM are equal: $M(K_n, \vec{\lambda}, GD) = M(K_n, \vec{\lambda}, FCFM)$.*

Proof : Indeed, both chains have the same state space and the same transitions rates. Remember that FCFM is a greedy discipline. \square

Lemma 2 *We have: $\mathbb{E}[M(K_n, \vec{\lambda}, GD)] = \mathbb{E}[M(K_n, \vec{\lambda}, FCFM)]$.*

Proof : This is a corollary of the previous results. \square

Lemma 3 *Assume $\alpha_i < 0.5$ for all $i = 1, \dots, n$. For any greedy matching policy, the mean number of items in a complete graph with n nodes is*

$$\left(1 + \sum_{i=1}^n \frac{\alpha_i}{1 - 2\alpha_i}\right)^{-1} \left(\sum_{i=1}^n \frac{\alpha_i(1 - \alpha_i)}{(1 - 2\alpha_i)^2}\right). \quad (5)$$

Proof We note that all the words (different from the empty word) have a single letter. In fact, when the state is k times the letter x , with probability α_x the number of letters increases by one and with probability $1 - \alpha_x$ decreases by one. This means that the Markov chain consists of n birth and death processes which are connected through the empty word. As a result, since $\alpha_i < 0.5$ for all $i = 1, \dots, n$, we have that the steady-state distribution exists and it is unique. Let $\pi(x, k)$ be the steady-state probability of a word where the letter x appears k times. It is easy to see that $\pi(x, k) = \pi_\emptyset \left(\frac{\alpha_x}{1 - \alpha_x}\right)^k$, where π_\emptyset is the normalization constant. It is also straightforward that $\pi_\emptyset = \left(1 + \sum_{x=1}^n \frac{\alpha_x(1 - \alpha_x)}{(1 - 2\alpha_x)^2}\right)^{-1}$. Finally, the mean number of items is given by

$$\begin{aligned} \sum_{x=1}^n \sum_{k=1}^{\infty} k \pi(x, k) &= \left(1 + \sum_{x=1}^n \frac{\alpha_x(1 - \alpha_x)}{(1 - 2\alpha_x)^2}\right)^{-1} \sum_{x=1}^n \sum_{k=1}^{\infty} k \left(\frac{\alpha_x}{1 - \alpha_x}\right)^k \\ &= \left(1 + \sum_{x=1}^n \frac{\alpha_x(1 - \alpha_x)}{(1 - 2\alpha_x)^2}\right)^{-1} \sum_{x=1}^n \frac{\frac{\alpha_x}{1 - \alpha_x}}{\left(1 - \frac{\alpha_x}{1 - \alpha_x}\right)^2} \\ &= \left(1 + \sum_{x=1}^n \frac{\alpha_x(1 - \alpha_x)}{(1 - 2\alpha_x)^2}\right)^{-1} \sum_{x=1}^n \frac{\alpha_x(1 - \alpha_x)}{(1 - 2\alpha_x)^2}. \end{aligned}$$

\square

The above result provides an explicit expression for the expectation of the total number of items if the matching is a complete graph. In the next section we give a similar result when the matching graph is a quasi-complete graph.

3.2 Quasi-Complete Graphs and Greedy Disciplines

Let us now study the expectation of the number of items for matching graph K_n with an edge deleted: (noted $K_n - (n - 1, n)$). We first show how to derive two matching graphs which have the same expected size of words in steady-state. This construction is based on the exact aggregation of Markov chains and the strong lumpability property (see [22] for the initial definition of aggregation for finite state space chains, and [23] for a recent description of this subject for denumerable ones). The following presentation follows [23].

Let W be a Markov chain on set of states \mathbb{W} . Let (B_1, \dots, B_k) be a partition of \mathbb{W} . We define a new process Y as follows:

$$Y_n = m \leftrightarrow W_n \in B_m.$$

The question is to find conditions such that Y is also a Markov chain. Under these conditions, Y will be denoted as an exact aggregation of W for partition (B_1, \dots, B_k) . The strong lumpability condition (defined in the following) implies such a result (see [23] for the proof).

Definition 8 (Strong Lumpability) W is strongly lumpable for partition (B_1, \dots, B_k) of its state space if for all subset index i and j and for all state m_1 and m_2 in B_i , we have

$$Pr(W_{n+1} \in B_j | W_n = m_1) = Pr(W_{n+1} \in B_j | W_n = m_2).$$

B_i will be denoted as macro state i in the following.

We first consider an arbitrary matching graph and an arbitrary node x . Let us denote by \mathcal{G}_x this graph. Let W_x denote the Markov chain associated with matching \mathcal{G}_x .

Definition 9 (Decomposition matching and Aggregated matching) We define a new matching by a decomposition of x into two nodes y and z . The decomposition is defined by:

- $\Gamma(y) = \Gamma(z) = \Gamma(x)$
- $\alpha_y > 0$
- $\alpha_z > 0$
- $\alpha_y + \alpha_z = \alpha_x$.

Let G_{yz} be the matching graph where x is decomposed into y and z . G_{yz} will be denoted as the decomposition matching while G_x is called the aggregated matching. W_{yz} is the Markov chain associated with G_{yz} .

For instance we consider an aggregated matching in the right part Fig. 3. Node x is decomposed in nodes n and $n - 1$ to build the decomposition matching graph (left part of the same figure).

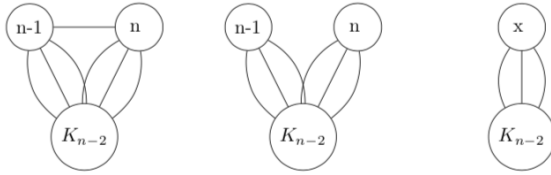


Fig. 3 Matching graphs: K_n (left), $K_n - (n - 1, n)$ (middle), K_{n-1} (right). In all three graphs, there is an edge between single nodes ($n - 1$ and n for the first two graphs, and x for the third graph) and any node in K_{n-2} . This is depicted as a triple edge.

Proposition 6 *If the aggregated matching is associated with a stable Markov chain, the same property holds for the decomposition matching.*

Note that by construction $y \notin \Gamma(z)$ and $z \notin \Gamma(y)$.

Lemma 4 *We consider the continuous time associated with matching graph $K_n - (n - 1, n)$, arrival rates $\bar{\lambda}$ and a greedy discipline GD : $M(K_n - (n - 1, n), \bar{\lambda}, GD)$. The states of the chain are the words $\emptyset, 1^*, 2^*, 3^*, \dots, (n - 2)^*$ and words written with letters $n - 1$ and n . The chain is lumpable according to the following partition.*

- states $\emptyset, 1^*, 2^*, 3^*$ remain in singletons.
- all the states m , written with letters n and $n - 1$ and such that $|m| = l > 0$, are gathered into a macro state called C_l .

Let Y be the lumped Markov chain.

Proof We consider the uniformized version of the chain with rate Λ as mentioned formerly to get $M(K_n - (n - 1, n), \bar{\alpha}, GD)$. Let P_{GD} the transition probability matrix of this DTMC. First, we will prove that this chain is lumpable for the partition. Let S be its state space. We have to prove the following equality for the macro states which are not singleton. Let i and j be macro-state indices

$$\sum_{k \in C_i} P_{GD}[k, m] = \sum_{k \in C_i} P_{GD}[k, n] \quad \forall m, n \in C_j.$$

If such a condition holds, we note $P_{GD}(C_i, C_j) = \sum_{k \in C_i} P_{GD}[k, m]$ where m is any state in C_j .

We only need to prove that this property holds for macro-states with more than one state: i.e. the macro states which contain the words m which contains letters $n - 1$ and n . Let C_l be such a macro-state. From the definition, all the words in C_l have length equal to l . We have to consider all the events:

- Arrival of a type n item. A type n item does not match any letter in word m (remember that m only contains letters n and $n - 1$). Thus the letter n is appended to word m and the state becomes $(m|n)$. This state is in macro-state C_{l+1} . And the probability of this transition is α_n . Thus from any state in C_l , we have a single transition with probability α_n to a state in C_{l+1} .

- Arrival of a type $n - 1$ item: same property, from any state in C_l , we have a transition with probability α_{n-1} to a state in C_{l+1} .
- Arrival of a type k item (with $k < n - 1$): such an arrival deletes a type n or type $n - 1$ item. In the chain we have a transition from state m to all the states p in set $N_{GD}(m, k)$ with probability $\alpha_k \psi_{N_D(m,x)}(p)$. As the discipline is greedy, set $N_{GD}(m, k)$ is not empty. And all the states in $N_{GD}(m, k)$ have a length equal to $l - 1$. Thus there are all included into macro-state C_{l-1} . As

$$\sum_{p \in N_D(m,x)} \psi_{N_D(m,x)}(p) = 1$$

we clearly have

$$P(C_l, C_{l-1}) = \sum_{p \in N_D(m,x)} \alpha_k \psi_{N_D(m,x)}(p) = \alpha_k$$

Thus, we have one transition with probability α_k from any state in C_l , to a state in C_{l-1} . □

From the above result, the following one follows directly.

Proposition 7 *Y can be identified with the Markov chain associated with matching graph K_{n-1} and the same discipline.*

We now present the following result that proves that the expected total number of items coincides for all greedy disciplines in a quasi-complete graph.

Lemma 5 *For matching graph $K_n - (n - 1, n)$, all greedy disciplines provide the same expectation for the total number of items.*

Proof : Let GD be an arbitrary greedy discipline and $\vec{\mu}$ defined by $\mu_i = \lambda_i$ for $i < n - 1$ and $\mu_{n-1} = \lambda_n + \lambda_{n-1}$. By Lemma 2:

$$\mathbb{E}[M(K_{n-1}, \vec{\mu}, GD)] = \mathbb{E}[M(K_{n-1}, \vec{\mu}, FCFM)]$$

and by Proposition 7:

$$\mathbb{E}[M(K_{n-1}, \vec{\mu}, FCFM)] = \mathbb{E}[M((K_n - (n - 1, n)), \vec{\mu}, FCFM)]$$

□

Thus, one can obtain from Lemma 3 the expected number of items with a simple modification of the arrival rates for all greedy disciplines.

Lemma 6 *Assume $\alpha_i < 0.5$ for all $i = 1, \dots, n-2$ and $\alpha_{n-1} + \alpha_n < 0.5$. For any greedy matching policy, the mean number of items in a quasi-complete graph with n nodes is*

$$\left(1 + \sum_{i=1}^{n-2} \frac{\alpha_i}{1-2\alpha_i} + \frac{\alpha_{n-1} + \alpha_n}{1-2(\alpha_{n-1} + \alpha_n)}\right)^{-1} \left(\sum_{i=1}^{n-2} \frac{\alpha_i(1-\alpha_i)}{(1-2\alpha_i)^2} + \frac{(\alpha_{n-1} + \alpha_n)(1-\alpha_{n-1}-\alpha_n)}{(1-\alpha_{n-1}-\alpha_n)^2}\right). \quad (6)$$

3.3 Existence of a performance paradox for any greedy discipline for $K_n - (n-1, n)$ matching graph

As a paradox occur for graphs K_n and $K_n - (n-1, n)$, and FCFM discipline it also occurs for any greedy discipline.

Theorem 2 *For any greedy discipline there is a paradox between K_n and $K_n - (n-1, n)$.*

Proof As we have a paradox for K_n and $K_n - (a, b)$ for FCFM, there exists a vector of arrivals $\vec{\lambda}$ such that:

$$\mathbb{E}[M(K_n, \vec{\alpha}, FCFM)] > \mathbb{E}[M(K_n - (n-1, n), \vec{\alpha}, FCFM)].$$

By Lemma 5, we have:

$$\mathbb{E}[M(K_n - (n-1, n), \vec{\alpha}, GD)] = \mathbb{E}[M(K_n - (n-1, n), \vec{\alpha}, FCFM)],$$

and due to Lemma 2 we also have

$$\mathbb{E}[M(K_n, \vec{\alpha}, GD)] = \mathbb{E}[M(K_n, \vec{\alpha}, FCFM)].$$

Therefore the paradox exists for all greedy discipline. □

Corollary 2 *If a discipline does not exhibit such a paradox for graph $K_n - (1, 2)$, this discipline is not greedy.*

4 The JOIN operation on matching graphs

First we have to restrict ourselves to a subset of matching disciplines.

Definition 10 A discipline is consistent if for all word m , if two letters x and y have the same neighborhood within m , they also have the same subset of states $N_D(m, x)$ and the same discrete distribution of probability $\psi_{N_D(m, x)}$ on this subset. More formally

If $m \cap \Gamma(x) = m \cap \Gamma(y)$, then $N_D(m, x) = N_D(m, y)$ and $\psi_{N_D(m, x)} = \psi_{N_D(m, y)}$.

We now present some matching disciplines that are consistent and one matching discipline which is not.

Remark 3 FCFM, Match the Longest and Random are consistent.

Remark 4 We remark that there exist greedy disciplines which are not consistent. For instance, consider the word $m = (a, b, c)$ and two items x and y such that

$$\Gamma(x) \cap m = \{a, c\} = \Gamma(y) \cap m.$$

One can design a discipline D such that $N_D(m, x) = a$ and $N_D(m, y) = c$. The greedy assumption is satisfied for this word and these items and the discipline is not consistent.

In the following, we assume that the matching disciplines are greedy and consistent.

We consider the following operations to build matching graphs: the JOIN operation and the UNION operation. These two operations allow to have a modular presentation of matching graphs and how to combine them. Again this is easier in continuous time thanks to the race condition. For a composition of two matching graphs G_1 and G_2 associated with transition rates vectors $\vec{\lambda}$ and $\vec{\mu}$ we denote as $(\vec{\lambda}, \vec{\mu})$ the rate vectors associated with the composition of G_1 and G_2 . Let us now begin with the definition of both operations.

Definition 11 (JOIN operation) We consider two arbitrary graphs $G_1 = (V_1, \mathcal{E}_1)$ and $G_2 = (V_2, \mathcal{E}_2)$. The JOIN of G_1 and G_2 is graph $G = (V, E)$ defined as follows:

- Nodes: $V = V_1 \cup V_2$,
- Edges: $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \{(x, y), x \in V_1, y \in V_2\}$.

Intuitively, we keep the nodes and edges of G_1 and G_2 and we add all the edges between V_1 and V_2 . The JOIN operation will be noted \bowtie .

We depict in the next figure the JOIN of an independent set (nodes 4 to 7) and a complete graph (nodes 1 to 3).

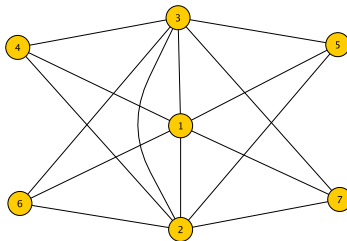


Fig. 4 Graph $IS_4 \bowtie K_3$.

Remark 5 If $\mathcal{E}_1 = \mathcal{E}_2 = \emptyset$, then G is a complete bipartite graph. Remember that we do not study model associated to bipartite matching graphs as their chains are not ergodic.

Definition 12 (\cup operation) We consider two arbitrary graphs $G_1 = (V_1, \mathcal{E}_1)$ and $G_2 = (V_2, \mathcal{E}_2)$. The UNION of G_1 and G_2 is graph $G = (V, E)$ defined as follows:

- Nodes: $V = V_1 \cup V_2$,
- Edges: $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$.

Assumption 1 (Discipline and Operations) We assume that the discipline is the same on both subgraphs when we build a JOIN or a UNION of matching graphs G_1 and G_2 .

4.1 Matching graphs $G_1 \bowtie G_2$

We consider any greedy and consistent discipline D . The results are not based on any particular discipline. When we give examples to illustrate the results, the matching discipline is mentioned.

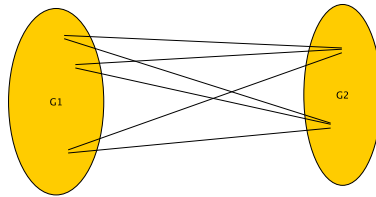


Fig. 5 Graph $G_1 \bowtie G_2$.

Lemma 7 Consider a matching graph G which is the JOIN of graph G_1 and G_2 . Let M_1 (resp. M_2) the continuous-time Markov chain associated with G_1 (resp. G_2) and vector of arrival rates $\vec{\lambda}$ (resp. $\text{vec}\mu$). Both chains have an empty state because the discipline is greedy. Then the continuous time Markov chain associated to G has the following states which are a couple of words:

- The states associated of M_1 (no items of G_2 are in the system) : the states will be denoted as (x, \emptyset) where x is a state of M_1 (*i.e.* a word). This set of states will be denoted as S_1 .
- The states associated with M_2 (no items of G_1 are in the system) : the states will be denoted as (\emptyset, y) where y is a node of M_2 . Similarly, we denote as S_2 this set of states.
- Both chains are connected through their empty states which are merged.

And the transitions are the same as in the Markov chains associated with G_1 or G_2 . More precisely : we have a transition from (x_1, \emptyset) to (x_2, \emptyset) if there is a transition

from x_1 to x_2 in M_1 . Similarly there is a transition from (\emptyset, y_1) to (\emptyset, y_2) if there is a transition from y_1 to y_2 in M_2 . Thus the set of states of the chain is $S_1 \cup S_2$. Let E be the empty state: $E = (\emptyset, \emptyset)$.

The proof is trivial as it is not possible to reach a state (x, y) with both $x \neq \emptyset$ and $y \neq \emptyset$ due to the edges between G_1 and G_2 after the JOIN. See Fig 6 for an example.

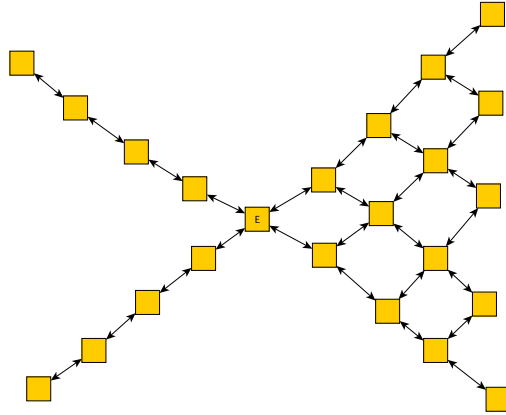


Fig. 6 Markov chain of the matching graph $IS_2 \bowtie K_2$ for RANDOM discipline. The chain is truncated to words smaller than 4 letters. The labels (except E) and the rates are omitted for the sake of readability.

Lemma 8 Let $\vec{\lambda}$ be the vector of arrival rates for matching graph G_1 and let $\vec{\mu}$ be the vector of rates for items in G_2 . After uniformization with rate $\Lambda = 2(\|\vec{\lambda}\|_1 + \|\vec{\mu}\|_1)$, the transition probability matrix of discrete-time Markov chain associated with $G_1 \bowtie G_2$ and vector of arrivals $(\vec{\lambda}, \vec{\mu})$ has the following block decomposition associated to the partition of the states $(\{E\}, S_1 \setminus \{E\}, S_2 \setminus \{E\})$:

$$M = \begin{pmatrix} 1/2 & L1 & L2 \\ C1 & P1 & 0 \\ C2 & 0 & P2 \end{pmatrix},$$

where $L1$ and $L2$ are row vectors, $C1$ and $C2$ are column vectors and $P1$ and $P2$ are sub-stochastic matrices. Similarly, if the Markov chain is ergodic, we decompose the steady-state distribution π^M as $(\pi_0^M, \pi_{S_1}^M, \pi_{S_2}^M)$.

Proof : We have:

- due to the uniformization rate, $\|L1\|_1 = \frac{\|\vec{\lambda}\|_1}{\Lambda}$ and $\|L2\|_1 = \frac{\|\vec{\mu}\|_1}{\Lambda}$. Therefore the transition probability between E and E is $1/2$.
- There is no transition between a state (\emptyset, y) to (x, \emptyset) if y and x are not equal to \emptyset . Therefore these two blocks are null.

□

We now prove that the steady-state distribution of the chain associated with G have a closed form solution based on the steady-state solutions of chains associated with matching graphs we now describe. We call this solution a separable solution for the steady-state distribution. Let us begin by the description of the sub-models.

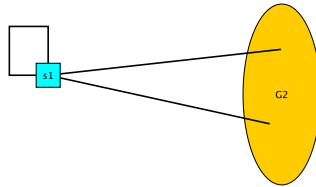


Fig. 7 $G'1$: New matching graph for the first sub-model.

Proposition 8 *We build a simpler matching graph as follows: we replace $G1$ by a single node $s1$, we add a loop on $s1$, we keep $G2$ unchanged and we add the edges between $s1$ and all the nodes in $G2$ (see Fig. 7). Matching graphs with self loops were recently introduced by Moyal et al. in [13] and independently in [24]. Let $G'2$ be this matching graph. Assume that the Markov chain associated $G'2$ and vector of arrival rates $(\sum_i \lambda_i, \vec{\mu})$ is ergodic. Let $M2$ be its transition probability matrix after uniformization with rate Λ and let π^{M2} be its steady state distribution. The state space of the Markov chain associated to $G'2$ is the following (see Fig. 8):*

- The state $(1, \emptyset)$ with one item of type $s1$ and no items of $G2$. Note that this is the only state with a positive number of items of type $s1$ as $G'2$ contains a loop on $s1$.
- The state $(0, \emptyset) = E$ which represents an empty system.
- The states associated with items of $G2$ (no $s1$ items are present in the system) : the states will be denoted as $(0, y)$ where y is a node of $G2$. Clearly, there is a one to one mapping between states $(0, y)$ in Markov chain $M2$ and states (\emptyset, y) in Markov chain P . Therefore we also denote as S_2 this set of states.

$M2$ has the following block decomposition associated to the partition of the states $(\{(1, \emptyset)\}, \{E\}, S_2 \setminus \{E\})$.

$$M2 = \left(\begin{array}{c|c|c} 1/2 & 1/2 & \vec{0} \\ \hline a & 1/2 & L2 \\ \hline 0 & C2 & P2 \end{array} \right),$$

where blocks $L2$, $C2$ and $P2$ have already been obtained in the decomposition of matrix M . We also consider a decomposition of the steady state distribution of $M2$: $(\pi_1^{M2}, \pi_0^{M2}, \pi_{S1}^{M2})$.

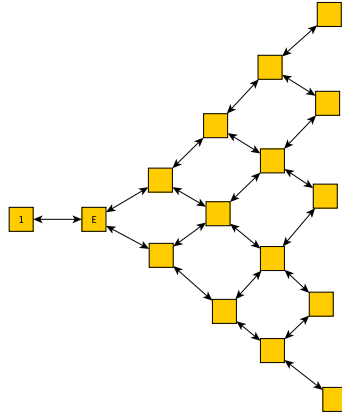


Fig. 8 Markov chain for $G'1$ for $G1 = K2$ and Random discipline.

Proof : We have to prove that the blocks $L2$, $C2$ and $P2$ are the same as in block decomposition of matrix M and give the value of a .

- From state $(1, \emptyset)$ all the arrivals match letter $s1$ and the transitions leads to $(0, \emptyset)$. Thus due to the uniformization rate, this transition has probability $1/2$ and there is a loop on state $(1, \emptyset)$ with probability $1/2$.
- The transition from $(0, \emptyset)$ leads to $(1, \emptyset)$ for an arrival of letter $s1$ (i.e. with a probability equal to $\frac{\|\vec{\lambda}\|_1}{\Lambda}$). Therefore $a = \frac{\|\vec{\mu}\|_1}{\Lambda}$ and there is a loop with probability $1/2$ on this state.
- The transition from $(0, \emptyset)$ to a state in $S2$ for an arrival of a letter of $G2$ (i.e. block $L2$). This is the same transition probability as in Matrix M as it is based on the same arrival rates, the same uniformization rate and the same discipline.
- The transition from a state in $S2$ to state $(0, \emptyset)$ for an arrival of a letter of $G1$ or $G2$ (i.e. block $C2$). All letters of $G1$ has the same effect as item $s1$ due to a consistent discipline and all letters of $G2$ have the same effect in M and in $M2$. The uniformization rate is the same in both models. Therefore both matrices have the same block $C2$ to model these transitions.

□

We do a similar construction for $G'1$ which is depicted in Fig. 9 by aggregation all the nodes of $G2$ into a single node $s2$ and adding a self loop on $s2$. Let $M1$ be the Markov chain associated with $G'1$ after uniformization with rate $\|\vec{\lambda}\|_1 + \|\vec{\mu}\|_1$ and (π^{M1}) its steady state distribution after assuming that $M1$ is ergodic. The state space is partitioned as follows: $(\{(1, \emptyset)\}, \{E\}, S1 \setminus \{E\})$. Similarly, we have a block decomposition for $M1$ based on this partition:

$$M1 = \left(\begin{array}{c|c|c} 1/2 & 1/2 & \vec{0} \\ \hline b & 1/2 & L1 \\ \hline 0 & C1 & P1 \end{array} \right),$$

with

$$b = \frac{\|\vec{\lambda}\|_1}{\|\vec{\lambda}\|_1 + \|\vec{\mu}\|_1}.$$

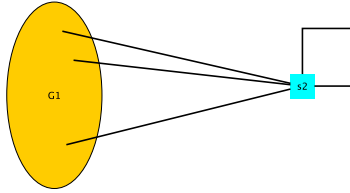


Fig. 9 $G'1$: New matching graph for the other sub-model.

Theorem 3 Let G is the JOIN of graph $G1$ and $G2$, and M the continuous-time Markov chain associated with G and arrival rates $(\vec{\alpha}, \vec{\beta})$. Assume that M , $M1$ and $M2$ are associated with ergodic discrete-time Markov chains. Let π_M be the steady state distribution of the chain associated with M . We decompose this distribution into three parts: the probability of the empty state, the probability of the states of S_1 and the probability of the states of S_2 .

$$\pi_M = (\pi_0^M, \pi_{S_1}^M, \pi_{S_2}^M),$$

where these elements are obtained through the steps detailed in the following. The steady-state distribution of $M1$ is:

$$\pi_{M1} = (\pi_1^{M1}, \pi_0^{M1}, \pi_{S_1}^{M1}),$$

with an abuse of notation here. Indeed $S1$ is the set of states of M (not of $M1$) but there is a one to one mapping between the set of non zero states of $M1$ and $S1$. Therefore we use the same index for the decomposition of the probability vector. Similarly, we note:

$$\pi_{M2} = (\pi_1^{M2}, \pi_0^{M2}, \pi_{S_2}^{M2}).$$

Then we have :

$$\pi_{S_1}^M = \pi_{S_1}^{M1} \frac{\pi_0^M}{\pi_0^{M1}},$$

and

$$\pi_{S_2}^M = \pi_{S_2}^{M2} \frac{\pi_0^M}{\pi_0^{M2}},$$

and finally

$$\pi_0^M = \left(\frac{1}{\pi_0^{M1}} + \frac{1}{\pi_0^{M2}} - 2 \right)^{-1}.$$

Proof It is based on some decomposition and matrix formulation for discrete time Markov chains and censored Markov chains (see [25] for censored Markov chains). First we write the global balance equation for M at the block level:

$$\pi_0^M = \pi_0^M / 2 + \pi_{S_1}^M C1 + \pi_{S_2}^M C2, \quad (7)$$

and

$$\pi_{S1}^M = \pi_0^M L1 + \pi_{S1}^M P1, \quad (8)$$

and finally

$$\pi_{S2}^M = \pi_0^M L2 + \pi_{S2}^M P2. \quad (9)$$

We do the same for $M1$:

$$\pi_1^{M1} = \pi_0^{M1} b + \pi_1^{M1} / 2, \quad (10)$$

$$\pi_0^{M1} = \pi_1^{M1} / 2 + \pi_0^{M1} / 2 + \pi_{S1}^{M1} C1, \quad (11)$$

and finally,

$$\pi_{S1}^{M1} = \pi_0^{M1} L1 + \pi_{S1}^{M1} P1 \quad (12)$$

One can consider the censored Markov chain extracted from $M1$ with censored set $\{s2\}$. According to Lemma 6.6 in [25], $\sum_{i=0}^{\infty} P1^i$ converges and we obtain:

$$\pi_{S1}^{M1} = \pi_0^{M1} L1 \sum_{i=0}^{\infty} P1^i, \quad (13)$$

and as π_0^{M1} is a scalar, we get:

$$\frac{\pi_{S1}^{M1}}{\pi_0^{M1}} = L1 \sum_{i=0}^{\infty} P1^i. \quad (14)$$

With the same argument, Eq. 8 gives:

$$\frac{\pi_{S1}^M}{\pi_0^M} = L1 \sum_{i=0}^{\infty} P1^i. \quad (15)$$

Taking into account Eq. 14 and Eq. 15 we obtain after substitution:

$$\pi_{S1}^M = \frac{\pi_0^M}{\pi_0^{M1}} \pi_{S1}^{M1}.$$

With a similar approach we have for matrix $P2$:

$$\frac{\pi_{S2}^{M2}}{\pi_0^{M2}} = L2(Id - P2)^{-1}, \quad (16)$$

and

$$\pi_{S2}^M = \frac{\pi_0^M}{\pi_0^{M2}} \pi_{S2}^{M2}.$$

The first two results of the theorem are now established. For the computation of π_0^M it is not possible to use Eq. 7 because it is not independent. Therefore we use normalization.

$$\pi_0^M + \|\pi_{S1}^M\|_1 + \|\pi_{S2}^M\|_1 = 1.$$

After substitution:

$$\pi_0^M \left(1 + \frac{1}{\pi_0^{M1}} \|\pi_{S1}^{M1}\|_1 + \frac{1}{\pi_0^{M2}} \|\pi_{S2}^{M2}\|_1 \right) = 1.$$

Clearly, $\|\pi_{S1}^{M1}\|_1 = 1 - \pi_0^{M1} - \pi_1^{M1}$ and $\|\pi_{S2}^{M2}\|_1 = 1 - \pi_0^{M2} - \pi_1^{M2}$. Thus

$$\pi_0^M = \left(1 + \frac{1 - \pi_0^{M1}(1 + 2b)}{\pi_0^{M1}} + \frac{1 - \pi_0^{M2}(1 + 2a)}{\pi_0^{M2}} \right)^{-1}.$$

After simple algebraic manipulation, taking into account that $2(a + b) = 1$, we finally obtain the relation for π_0^M . And this concludes the proof. \square

Corollary 3 We can compute the expectation of the total number items for the JOIN matching graphs from these expectations for the sub-models:

$$\mathbb{E}[M((G1 \bowtie G2), (\vec{\lambda}, \vec{\mu}), D)] = \frac{\pi_0^M}{\pi_0^{M1}} \mathbb{E}[M(G'1, (\vec{\lambda}),)] + \frac{\pi_0^M}{\pi_0^{M2}} \mathbb{E}[M(G'2, \vec{\mu})] - 1.$$

Proof $\pi(m1, m2)$ will be the steady-state probability of state $(m1, m2)$. We decompose the summation on the state space:

$$\mathbb{E}[M(G1 \bowtie G2), (\vec{\lambda}, \vec{\mu}), D] = \sum_{x \in S1} |x| \pi^M(x, \emptyset) + \sum_{y \in S2} |y| \pi^M(\emptyset, y).$$

And

$$\mathbb{E}[M(G'1, \vec{\lambda}, D)] = \sum_{x \in S1} |x| \pi^{M1}(x, \emptyset) + \pi_1^{M1},$$

with a similar relation for $\mathbb{E}[M(G'2, \vec{\mu}, D)]$. After substitution and factorization, we get:

$$\mathbb{E}[M(G1 \bowtie G2), (\vec{\lambda}, \vec{\mu}), D] = \frac{\pi_0^M}{\pi_0^{M1}} \sum_{x \in S1} |x| \pi^{M1}(x, \emptyset) + \frac{\pi_0^M}{\pi_0^{M2}} \sum_{y \in S2} |y| \pi^{M2}(\emptyset, y).$$

As $\pi_1^{M1} = 2b\pi_0^{M1}$, $\pi_1^{M2} = 2a\pi_0^{M2}$ and $2(a + b) = 1$, we get the results after simple algebraic manipulations. \square

It is important to remark that the Markov chain associated with $G'1$ is not a lumped version of the DTMC associated with G . In general this Markov chain is not lumpable due to the possible matching between the items of $G2$. In the next subsection we study a simpler case where we have a strong aggregation of the DTMC and this leads to a lifting theorem for paradox.

4.2 Matching graphs $IS_{n1} \bowtie K_{n2}$

We consider a matching graph $G = IS_{n1} \bowtie K_{n2}$ with $n1 \geq 1$ and $n2 \geq 2$, the JOIN of an independent set with size $n1$ and a complete graph with size $n2$. We assume that the matching discipline is greedy. Let us first mention some graph theoretical results.

Proposition 9 If $n2 \geq 2$, then G is not bipartite. Indeed the graph contains at least one triangle.

Proposition 10 Clearly: $IS_1 \bowtie K_n = K_{n+1}$.

One can generalize this result to composition of quasi-complete graphs.

Lemma 9 Remember that $K_n - (a, b)$ is the complete graph with size n without the edge between node a and b . We have: $IS_1 \bowtie (K_n - (a, b)) = K_{n+1} - (a, b)$.

Proof both graphs have $n + 1$ vertices. $IS_1 \bowtie (K_n - (a, b))$ has n edges between the node in IS_1 and the nodes in $(K_n - (a, b))$ and $n(n - 1)/2 - 1$ vertices inside $(K_n - (a, b))$. $K_{n+1} - (a, b)$ contains $n(n + 1)/2 - 1$ edges and it is the only graph with $n + 1$ vertices and such a number of edges. As both graphs also have the same number of vertices, and as such a graph is unique, both graphs are equal. \square

We begin with a result even more general for the state space of $IS_{n_1} \bowtie G_{n_2}$ where G_{n_2} is an arbitrary graph with size n_2 .

Theorem 4 *Consider a matching graph $G = IS_{n_1} \bowtie G_{n_2}$, arrival rates of letters $(\vec{\lambda}, \vec{\mu})$ and an arbitrary consistent and greedy discipline D . Assume that $n_1 > 1$ and $K_2 \subset G_{n_2}$. We have uniformized the continuous time Markov chain with rate $\Lambda > \|\vec{\lambda}\|_1 + \|\vec{\mu}\|_1$ to obtain a discrete time model. This DTMC is lumpable for the following partition:*

1. State E remains unchanged
2. States (\emptyset, Y) remains unchanged.
3. States (X, \emptyset) are aggregated into macro-state $(|X|, \emptyset)$.

We aggregate all the nodes of the independent set into one macro state which is the length of the word. Let $\alpha_y = \lambda_y/\Lambda$ be the probability of an arrival of letters y for the letters in the independent set. z will denote the aggregated letters of the independent set. The arrival of letter z is $\alpha_z = \sum_{y=1}^{n_1} \alpha_y$. The lumped Markov chain is associated with the matching graph $IS_1 \bowtie G_{n_2}$.

Proof As states E and (\emptyset, Y) for all word Y remain unchanged, we just have to consider the macro states which contains the states (X, \emptyset) . Consider two arbitrary states $(X1, \emptyset)$ and $(X2, \emptyset)$ such that $|X1| = |X2| = k$. They are member of the same macro state (k, \emptyset) . We have two cases: arrival of a letter in $V(G1)$ or in $V(G2)$.

- Consider the arrival of an arbitrary letter in $V(G1)$. The letter does not match with $X1$ or $X2$ as this part of the matching graph is an independent set. Therefore it provokes a transition to a state which is aggregated into macro-state $(k + 1, \emptyset)$ and the matching discipline is irrelevant here as no matching occurs.
- Assume now an arrival of an arbitrary letter in $V(G2)$. A matching takes place and as the discipline is greedy and consistent, exactly one letter is deleted among the letters of $X1$ or $X2$. Therefore both states belong to macro state $(k - 1, \emptyset)$.

Thus the lumpability conditions hold. \square

As usual, both chains gives the same expectation.

Corollary 4 *Consider an arbitrary greedy and consistent discipline D . We have the same expectation for the total number of items for model $IS_1 \bowtie G_{n_2}$ and model*

$IS_{n1} \bowtie G_{n2}$ assuming that $\beta_z = \sum_y \alpha_y$ for the arrivals of letters z in IS_1 and $\beta_y = \alpha_y$ for y in G_{n2} .

$$\mathbb{E}[M((IS_{n1} \bowtie G_{n2}, \vec{\alpha}, D)] = \mathbb{E}[M((IS_1 \bowtie G_{n2}, \vec{\beta}, D)]$$

In general, we are not able to analyze chain $M((IS_1 \bowtie G_{n2}, \vec{\beta}, D)$. However in some simple cases, the results of the previous sections may help for the next two corollaries:

Corollary 5 *Consider an arbitrary greedy and consistent discipline D . We have the same expectation for the total number of items for model K_{n2+1} and model $IS_{n1} \bowtie K_{n2}$ assuming that $\alpha_x = \sum_y \alpha_y$ for the arrivals of letters initially in IS_{n1} .*

$$\mathbb{E}[M((IS_{n1} \bowtie K_{n2}, \vec{\alpha}, D)] = \mathbb{E}[M(K_{n2+1}, \vec{\beta}, D)].$$

Proof Taking $G2 = K2$ in Theorem 4 and Corollary 4 and using finally Lemma 11. \square

Corollary 6 *Consider an arbitrary greedy and consistent discipline D . We have the same expectation for the total number of items for model $K_{n2+1} - (a, b)$ and model $IS_{n1} \bowtie (K_{n2} - (a, b))$ assuming that $\alpha_z = \sum_y \alpha_y$ for the arrivals of letters initially in IS_{n1} .*

$$\mathbb{E}[M((IS_{n1} \bowtie K_{n2}, \vec{\alpha}, D)] = \mathbb{E}[M(K_{n2+1}, \vec{\beta}, D)].$$

Proof First we use Theorem 4 with $G2 = (K_{n2} - (a, b))$ to prove the lumpability and Corollary 4 to prove that the models associated with $IS_{n1} \bowtie (K_{n2} - (a, b))$ and $IS_1 \bowtie (K_{n2} - (a, b))$ have the same expectation of total number of items for any greedy discipline. Then we use Lemma 9 to complete the proof. \square

And the existence of a paradox follows:

Theorem 5 *If there exists a paradox between matching graph K_{n2+1} and $K_{n2+1} - (a, b)$ for a greedy and consistent discipline, the same (i.e. we keep the same arrival rates for letters in K_{n2}) paradox exists between $IS_{n1} \bowtie K_{n2}$ and $IS_{n1} \bowtie (K_{n2} - (a, b))$ for the same discipline.*

Proof based on the corollaries. Corollary 5 proves that K_{n2+1} and $IS_{n1} \bowtie K_{n2}$ have the same expectation for total number of items, while Corollary 6 establish the same relation for chains associated with matching graphs $K_{n2+1} - (a, b)$ and $IS_{n1} \bowtie (K_{n2} - (a, b))$. Thus if a paradox exists between K_{n2+1} and $K_{n2+1} - (a, b)$, it also exists between $K_{n2+1} - (a, b)$ and $IS_{n1} \bowtie (K_{n2} - (a, b))$. \square

Corollary 7 *For all $n \geq 4$, there exist some matching graph with size n which exhibits a paradox for any discipline D .*

Proof For $n = 4$, we give the example for FCFM discipline for matching graph $K_4 - (3, 4)$ in section 2.2 and we generalize to in Section 3 to any discipline D . For larger n , we consider matching graph $IS_{n-4} \bowtie (K_4 - (3, 4))$ and Theorem 5 proves the claim with the previous results on $K_4 - (3, 4)$. \square

Lemma 10 *If matching graph G with discipline FCFM satisfies the assumption of Theorem 1, then matching graph $G \bowtie IS_1$ with discipline FCFM also satisfies the same assumptions when the arrival probability of the letter in IS_1 is sufficiently small. Therefore matching graph $G \bowtie IS_1$ also exhibits a performance paradox.*

Proof Let x be the letter of IS_1 . And α^δ the arrival probabilities for graph G . We have to prove that the assumptions of Theorem 1 on the saturated independent sets are still valid for $G \bowtie \{x\}$. First, the independent sets of $G \bowtie IS_1$ are the independent sets of G and $\{x\}$. Let us consider the following arrival probabilities for graph $G \bowtie \{x\}$:

$$\beta_0^\delta = \alpha_0^\delta(1 - \delta), \quad \beta_x^\delta = \alpha_0^\delta\delta, \quad \beta_y^\delta = \alpha_y^\delta \quad \forall y \in V(G).$$

As α^δ is a probability distribution, β^δ is also a probability distribution when $\delta < 1$. Now we have to compute the stability gap for all independent sets of $G \bowtie \{x\}$. We slightly change the notation about the stability graph to add the graph as we build the graph. $\Delta_G(\mathcal{I})$ will denote the stability gap of independent set \mathcal{I} in graph G . Let us begin with $\{x\}$. All the nodes of G are neighbors of x . Thus:

$$\Delta_{\{x\}}^{G \bowtie \{x\}} = \sum_{y \in V(G)} \beta_y - \beta_x = \sum_{y \in V(G)} \alpha_y - \alpha_0\delta = 1 - \alpha_0 - \alpha_0\delta.$$

Therefore, $\{x\}$ is not a saturated independent set. Now consider any independent set \mathcal{I} of $G \bowtie \{x\}$ which is also an independent set of G . \mathcal{I} does not contain x , unlike $\Gamma(\mathcal{I})$. Therefore, due to the arrival probabilities we considered:

$$\Delta_{\mathcal{I}}^{G \bowtie \{x\}} = \Delta_{\mathcal{I}}^G + \alpha_0\delta.$$

Clearly, as the transformation of the gaps keep unchanged the saturated independent set and if we have only one saturated independent set in G , we have only one saturated independent set in $G \bowtie \{x\}$. Therefore the assumptions of Theorem 1 are still valid for the graph we have designed. \square

4.3 Matching graph $IS_{n1} \bowtie (G2 \cup G1)$

Assume now the following decomposition for the matching graph into three subgraphs:

$$G = IS_{n1} \bowtie (G1 \cup G2),$$

where $G1$ and $G2$ are non empty. We now establish that G may have some paradox if one of the sub-models exhibits performance paradox.

Proposition 11 *Consider an arbitrary greedy matching discipline D . Assume that the matching graph G is the UNION of $G1$ and $G2$. Let $\bar{\lambda}$ and $\bar{\mu}$ be respectively*

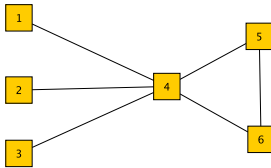


Fig. 10 Graph $G = IS_1 \bowtie (K_2 \cup IS_3)$, IS_1 contains node 4. Remark that G is connected while the subgraphs IS_1 and IS_3 are not.

the arrival rates of times associated with $G1$ and $G2$. Assume that Markov chains $M(G1, \vec{\lambda}, D)$ and $M(G2, \vec{\mu}, D)$ are ergodic. Then:

$$M(G1 \cup G2, (\vec{\lambda}, \vec{\mu}), D) = M(G1, \vec{\lambda}, D) \oplus M(G2, \vec{\mu}, D),$$

where \oplus is the tensor (or Kronecker sum).

Proof As the items in $G1$ and $G2$ does not interact, we have built two independent continuous-time Markov chains. And it is well known (see for instance the literature on Stochastic Automata Networks (see for example [21] and references therein) that the resulting Markov chains is the Kronecker sum of the components. \square

Corollary 8 *Because of independence of the Markov chains associated with $G1$ and $G2$, we clearly have:*

$$\mathbb{E}[(G1 \cup G2), (\vec{\lambda}, \vec{\mu}), D] = \mathbb{E}[M(G1, \vec{\lambda}, D)] \times \mathbb{E}[M(G2, \vec{\mu}, D)].$$

Therefore if a paradox exist for matching graph $G1$ and discipline D , it also exists for matching graph $G1 \cup G2$ and discipline D .

Lemma 11 *Assume that $G1$ with FCFM matching discipline satisfies the assumptions of Theorem 1 for vector of arrival rates $\vec{\mu}$. Assume also that $G2$ with vector of arrival rates $\vec{\nu}$ does not contain any saturated independent set for FCFM matching discipline. Then $G1 \cup G2$ with FCFM discipline and vector of arrivals $(\vec{\mu}, \vec{\nu})$ satisfies the assumptions of Theorem 1.*

Proof The independent sets of $G1 \cup G2$ are:

- the independent sets of $G1$,
- the independent sets of $G2$,
- all the sets which are the union of an independent set of $G1$ and an independent set of $G2$.

Therefore the saturated independent set of $G1$ is also a saturated independent set of $G1 \cup G2$. We now have to check that $G1 \cup G2$ contains only one saturated independent set. As $G2$ does not have a saturated independent set, there exists $\Delta > 0$ such that all the stability gap of independent sets of $G2$ are larger than Δ . Let $\mathcal{I}1$ be

an independent set of G_1 and \mathcal{I}_2 an independent set of G_2 . \mathcal{I}_2 is not saturated by assumption and clearly, due to the UNION operation:

$$\Delta_{\mathcal{I}_1 \cup \mathcal{I}_2}^{G_1 \bowtie G_2} = \Delta_{\mathcal{I}_1}^{G_1} + \Delta_{\mathcal{I}_2}^{G_2}.$$

Therefore as $\Delta_{\mathcal{I}_2}^{G_2} \geq \Delta$, $\mathcal{I}_1 \cup \mathcal{I}_2$ cannot be saturated. Thus the saturated independent set of G_1 is the saturated independent set of $G_1 \cup G_2$. \square

Remember that $G_1 \cup G_2$ is not connected. However we can build connected matching graphs with UNION and JOIN operations. The following result is a simple corollary of Lemma 7.

Lemma 12 (State Space) *Let $G_3 \bowtie (G_1 \cup G_2)$, then the states of the Markov chain associated with G are*

- State E ,
- states $(\vec{X}, \vec{0}, \vec{0})$ where \vec{X} is a state of the chain associated with G_2 ,
- states $(\vec{0}, \vec{Y}, \vec{Z})$ where \vec{Y} (resp. \vec{Z}) is a state of the chain associated with G_1 (resp. G_2),

Theorem 6 *We assume FCFM matching discipline and that matching graph G_1 satisfies the assumptions of Theorem 1. Let $\vec{\mu}$ the arrival rates associated with this paradox for G_1 . Then, we also have a performance paradox for matching graph $IS_{n_1} \bowtie (G_1 \cup G_2)$ associated with arrival rate vectors $(\vec{\lambda}, \vec{\mu}, \vec{\nu})$ and FCFM discipline.*

Proof We first aggregate all the nodes of IS_{n_1} into one single node because the associated chain is lumpable. According to Theorem 4, using discipline FCFM

$$\mathbb{E}[M((IS_{n_1} \bowtie (G_1 \cup G_2)), (\vec{\lambda}, \vec{\mu}, \vec{\nu}), FCFM)] = \mathbb{E}[M((IS_1 \bowtie (G_1 \cup G_2)), \vec{\beta}, FCFM)],$$

with $\vec{\beta} = (\sum \lambda, \vec{\mu}, \vec{\nu})$. Lemma 11 shows that as soon as the assumptions of Theorem 1 hold for G_1 , $G_1 \cup G_2$ also satisfies the same assumptions and the matching graph $IS_1 \bowtie (G_1 \cup G_2)$ also exhibits a paradox for FCFM discipline according to Lemma 10. \square

5 Conclusions

We consider the dynamic matching model with a non-bipartite matching graph and we analyze the existence of a performance degradation when the flexibility increases, i.e., when we add an edge to the matching graph. This analysis can be seen as an analogous to the Braess paradox in matching models. We first focus on the FCFM discipline and an arbitrary matching graph. For this case, we provide sufficient conditions on the existence of a performance paradox. Then, the performance paradox existence study is extended to greedy disciplines, which is a large family of matching disciplines that includes, not only FCFM, but also other popular disciplines such as Match the Longest and Random. For this instance, we provide conditions on the matching graph for the existence of a performance paradox.

As future work, we are interested in studying the existence of a performance paradox for a broader family of matching disciplines than the greedy disciplines. We are also interested in exploring the existence of a performance paradox in matching graphs with less number of edges. Finally, another interesting extension of this work consists of analyzing the existence of a performance paradox when the matching in bipartite in a general setting.

Appendix A Computations for the example

We first note that, using that $\alpha_1^\delta = 0.5 - \frac{\delta}{2}$, $\alpha_2^\delta = 0.1 + \frac{\delta}{10}$ and $\alpha_3^\delta = \alpha_4^\delta = 0.2 + \frac{\delta}{5}$, we obtain from (5) that the mean number of items of the complete graph under any greedy policy is

$$\mathbb{E}[M(K_4, \alpha^\delta, FCFM)] = \frac{2 \frac{(0.8 - \frac{\delta}{5})(0.2 + \frac{\delta}{5})}{(0.6 - \frac{2\delta}{5})^2} + \frac{(0.5 + \frac{\delta}{2})(0.5 - \frac{\delta}{2})}{\delta^2} + \frac{(0.9 - \frac{\delta}{10})(0.1 + \frac{\delta}{10})}{(0.8 - \frac{\delta}{5})^2}}{1 + 2 \frac{0.2 + \frac{\delta}{5}}{0.6 - \frac{2\delta}{5}} + \frac{0.5 - \frac{\delta}{2}}{\delta} + \frac{0.1 + \frac{\delta}{10}}{0.8 - \frac{\delta}{5}}}.$$

And using (6), we get that the mean number of items of the quasi complete graph under any greedy policy is

$$\mathbb{E}[M(K_4 - (3, 4), \alpha^\delta, FCFM)] = \frac{\frac{(0.6 - \frac{2\delta}{5})(0.4 + \frac{2\delta}{5})}{(0.2 - \frac{4\delta}{5})^2} + \frac{(0.5 + \frac{\delta}{2})(0.5 - \frac{\delta}{2})}{\delta^2} + \frac{(0.9 - \frac{\delta}{10})(0.1 + \frac{\delta}{10})}{(0.8 - \frac{\delta}{5})^2}}{1 + \frac{0.4 + \frac{2\delta}{5}}{0.2 - \frac{4\delta}{5}} + \frac{0.5 - \frac{\delta}{2}}{\delta} + \frac{0.1 + \frac{\delta}{10}}{0.8 - \frac{\delta}{5}}}.$$

We study the sign of $\mathbb{E}[M(K_4, \alpha^\delta, FCFM)] - \mathbb{E}[M(K_4 - (3, 4), \alpha^\delta, FCFM)]$, which is the same as the sign of

$$\begin{aligned} & \left(2 \frac{(0.8 - \frac{\delta}{5})(0.2 + \frac{\delta}{5})}{(0.6 - \frac{2\delta}{5})^2} + \frac{(0.5 + \frac{\delta}{2})(0.5 - \frac{\delta}{2})}{\delta^2} + \frac{(0.9 - \frac{\delta}{10})(0.1 + \frac{\delta}{10})}{(0.8 - \frac{\delta}{5})^2} \right) \\ & \left(1 + \frac{0.4 + \frac{2\delta}{5}}{0.2 - \frac{4\delta}{5}} + \frac{0.5 - \frac{\delta}{2}}{\delta} + \frac{0.1 + \frac{\delta}{10}}{0.8 - \frac{\delta}{5}} \right) \\ & - \left(\frac{(0.6 - \frac{2\delta}{5})(0.4 + \frac{2\delta}{5})}{(0.2 - \frac{4\delta}{5})^2} + \frac{(0.5 + \frac{\delta}{2})(0.5 - \frac{\delta}{2})}{\delta^2} + \frac{(0.9 - \frac{\delta}{10})(0.1 + \frac{\delta}{10})}{(0.8 - \frac{\delta}{5})^2} \right) \\ & \left(1 + 2 \frac{0.2 + \frac{\delta}{5}}{0.6 - \frac{2\delta}{5}} + \frac{0.5 - \frac{\delta}{2}}{\delta} + \frac{0.1 + \frac{\delta}{10}}{0.8 - \frac{\delta}{5}} \right). \end{aligned}$$

Simplifying this expression using Wolfram Mathematica, we obtain the following:

$$\frac{1}{\delta^3(-4 + \delta)^3(0.375 - 1.75\delta + \delta^2)^2}$$

$$\begin{aligned} &\times (-2.22045 \cdot 10^{-16} - 3\delta + 33.25\delta^2 + 45.25\delta^3 - 41.0937\delta^4 - 38.0625\delta^5 \\ &\quad + 27\delta^6 + 1.6875\delta^7 - 9.90625\delta^8 + 3.125\delta^9 - 0.25\delta^{10}). \end{aligned}$$

We note that $(0.375 - 1.75\delta + \delta^2)^2 > 0$ when $0 < \delta \leq 0.1$. Thus, since

$$\frac{1}{\delta^3(-4 + \delta)^3(0.375 - 1.75\delta + \delta^2)^2}$$

is negative when $0 < \delta \leq 0.1$, the sign of $\mathbb{E}[M(K_4, \alpha^\delta, FCFM)] - \mathbb{E}[M(K_4 - (3, 4), \alpha^\delta, FCFM)]$ is the opposite of the sign of the polynomial of degree 10:

$$\begin{aligned} &- 2.22045 \cdot 10^{-16} - 3\delta + 33.25\delta^2 + 45.25\delta^3 - 41.0937\delta^4 - 38.0625\delta^5 \\ &\quad + 27\delta^6 + 1.6875\delta^7 - 9.90625\delta^8 + 3.125\delta^9 - 0.25\delta^{10} \end{aligned}$$

Using Wolfram Mathematica, we know that this polynomial has one real root between 0 and 0.1, which is 0.0818369. Besides, if $0 < \delta \leq 0.0818369$, the polynomial is negative and positive otherwise. Therefore, the desired result follows.

Appendix B Proof of Proposition 2

Using (5), we obtain from the definition of α^δ that the mean number of items of the complete graph under any greedy policy is

$$\mathbb{E}[M(K_4 - (3, 4), \alpha^\delta, FCFM)] = \frac{\frac{(0.5-2\delta)(0.5+2\delta)}{(4\delta)^2} + \frac{(0.5-\delta)(0.5+\delta)}{(2\delta)^2} + \frac{3\delta(1-3\delta)}{(1-6\delta)^2}}{1 + \frac{0.5-2\delta}{4\delta} + \frac{0.5-\delta}{2\delta} + \frac{3\delta}{1-6\delta}}.$$

Likewise, using (6) and the definition α^δ , it results

$$\mathbb{E}[M(K_4, \alpha^\delta, FCFM)] = \frac{2\frac{(0.25-\delta)(0.75+\delta)}{(0.5+2\delta)^2} + \frac{(0.5-\delta)(0.5+\delta)}{(2\delta)^2} + \frac{3\delta(1-3\delta)}{(1-6\delta)^2}}{1 + 2\frac{0.25-\delta}{0.5+2\delta} + \frac{0.5-\delta}{2\delta} + \frac{3\delta}{1-6\delta}}$$

We now study the sign of $\mathbb{E}[M(K_4 - (3, 4), \alpha^\delta, FCFM)] - \mathbb{E}[M(K_4, \alpha^\delta, FCFM)]$, which is the same as the sign of

$$\begin{aligned} &\left(\frac{(0.5-2\delta)(0.5+2\delta)}{(4\delta)^2} + \frac{(0.5-\delta)(0.5+\delta)}{(2\delta)^2} + \frac{3\delta(1-3\delta)}{(1-6\delta)^2} \right) \\ &\quad \left(1 + 2\frac{0.25-\delta}{(0.5+2\delta)} + \frac{0.5-\delta}{2\delta} + \frac{3\delta}{1-6\delta} \right) - \\ &\quad \left(2\frac{(0.25-\delta)(0.75+\delta)}{(0.5+2\delta)^2} + \frac{(0.5-\delta)(0.5+\delta)}{(2\delta)^2} + \frac{3\delta(1-3\delta)}{(1-6\delta)^2} \right) \\ &\quad \left(1 + \frac{0.5-2\delta}{4\delta} + \frac{0.5-\delta}{2\delta} + \frac{3\delta}{1-6\delta} \right). \quad (\text{B1}) \end{aligned}$$

Simplifying this expression using Wolfram Mathematica, we obtain the following equivalent one:

$$\frac{54}{\delta^3(0.25 + \delta)^2(-1 + 6\delta)^3} \times \\ (4.52112 \cdot 10^{-6} - 0.0001808\delta + 0.0026222\delta^2 - 0.0180845\delta^3 \\ + 0.0640914\delta^4 - 0.133102\delta^5 + 0.30556\delta^6 - 0.8333\delta^7 + \delta^8.)$$

We note that $-1 + 6\delta < 0$ when $0 < \delta \leq 0.1$. Thus, since

$$\frac{54}{\delta^3(0.25 + \delta)^2(-1 + 6\delta)^3}$$

is negative when $0 < \delta \leq 0.1$, the sign of $\mathbb{E}[M(K_4 - (3, 4), \alpha^\delta, FCFM)] - \mathbb{E}[M(K_4, \alpha^\delta, FCFM)]$ is the opposite of the sign of the polynomial of degree 8:

$$4.52112 \cdot 10^{-6} - 0.0001808\delta + 0.0026222\delta^2 - 0.0180845\delta^3 + 0.0640914\delta^4 \\ - 0.133102\delta^5 + 0.30556\delta^6 - 0.8333\delta^7 + \delta^8.$$

Using Wolfram Mathematica, we know that this polynomial has one real root between 0 and 0.1, which is 0.0563. Besides, if $0 < \delta \leq 0.0563$, the polynomial is positive and negative otherwise. Therefore, the desired result follows.

Appendix C Proof of Proposition 3

$$\mathbb{E}[M(K_4 - (3, 4), \alpha^\delta, FCFM)] - \mathbb{E}[M(K_4, \alpha^\delta, FCFM)]$$

In Appendix B, we show that, for this matching model, we have that

$$\mathbb{E}[M(K_4 - (3, 4), \alpha^\delta, FCFM)] = \frac{\frac{(0.5-2\delta)(0.5+2\delta)}{(4\delta)^2} + \frac{(0.5-\delta)(0.5+\delta)}{(2\delta)^2} + \frac{3\delta(1-3\delta)}{(1-6\delta)^2}}{1 + \frac{0.5-2\delta}{4\delta} + \frac{0.5-\delta}{2\delta} + \frac{3\delta}{1-6\delta}}.$$

and

$$\mathbb{E}[M(K_4, \alpha^\delta, FCFM)] = \frac{2\frac{(0.25-\delta)(0.75+\delta)}{(0.5+2\delta)^2} + \frac{(0.5-\delta)(0.5+\delta)}{(2\delta)^2} + \frac{3\delta(1-3\delta)}{(1-6\delta)^2}}{1 + 2\frac{0.25-\delta}{0.5+2\delta} + \frac{0.5-\delta}{2\delta} + \frac{3\delta}{1-6\delta}}.$$

The desired result follows if we show that

- $\delta^2 \left(\frac{(0.5-2\delta)(0.5+2\delta)}{(4\delta)^2} + \frac{(0.5-\delta)(0.5+\delta)}{(2\delta)^2} + \frac{3\delta(1-3\delta)}{(1-6\delta)^2} \right)$ tends to $5 \cdot 0.5^6$ when $\delta \rightarrow 0$,
- $\delta^2 \left(2\frac{(0.25-\delta)(0.75+\delta)}{(0.5+2\delta)^2} + \frac{(0.5-\delta)(0.5+\delta)}{(2\delta)^2} + \frac{3\delta(1-3\delta)}{(1-6\delta)^2} \right)$ tends to 0.5^4 when $\delta \rightarrow 0$,
- $\frac{1}{\delta} \left(1 + \frac{0.5-2\delta}{4\delta} + \frac{0.5-\delta}{2\delta} + \frac{3\delta}{1-6\delta} \right)^{-1}$ tends to $\frac{8}{3}$ when $\delta \rightarrow 0$,

$$(d) \frac{1}{\delta} \left(1 + 2 \frac{0.25 - \delta}{0.5 + 2\delta} + \frac{0.5 - \delta}{2\delta} + \frac{3\delta}{1 - 6\delta} \right)^{-1} \text{ tends to } 4 \text{ when } \delta \rightarrow 0,$$

since when $\delta \rightarrow 0$,

$$\delta (\mathbb{E}[M(K_4 - (3, 4), \alpha^\delta, FCFM)] - \mathbb{E}[M(K_4, \alpha^\delta, FCFM)]) \rightarrow 0.5^4 \cdot 4 - 5 \cdot 0.5^6 \cdot \frac{8}{3} = \frac{1}{24}.$$

We first show (a).

$$\begin{aligned} \delta^2 \left(\frac{(0.5 - 2\delta)(0.5 + 2\delta)}{(4\delta)^2} + \frac{(0.5 - \delta)(0.5 + \delta)}{(2\delta)^2} + \frac{3\delta(1 - 3\delta)}{(1 - 6\delta)^2} \right) \\ = \frac{(0.5 - \delta)(0.5 + 2\delta)}{16} + \frac{(0.5 - \delta)(0.5 + \delta)}{4} + \frac{3\delta^3(1 - 3\delta)}{(1 - 6\delta)^2}, \end{aligned}$$

and the first and second terms tend, respectively, to 0.5^6 and $0.5^4 = 4 \cdot 0.5^6$ when $\delta \rightarrow 0$, whereas the third one tends to zero.

We now show (b).

$$\begin{aligned} \delta^2 \left(2 \frac{(0.25 - \delta)(0.75 + \delta)}{(0.5 + 2\delta)^2} + \frac{(0.5 - \delta)(0.5 + \delta)}{(2\delta)^2} + \frac{3\delta(1 - 3\delta)}{(1 - 6\delta)^2} \right) \\ = 2\delta^2 \frac{(0.25 - \delta)(0.75 + \delta)}{(0.5 + 2\delta)^2} + \frac{(0.5 - \delta)(0.5 + \delta)}{4} + \frac{3\delta^3(1 - 3\delta)}{(1 - 6\delta)^2}, \end{aligned}$$

and the first and third terms tend to zero when $\delta \rightarrow 0$, whereas the second one tends to 0.5^4 .

We also show (c).

$$\begin{aligned} \frac{1}{\delta} \left(1 + \frac{0.5 - 2\delta}{4\delta} + \frac{(0.5 - \delta)}{2\delta} + \frac{3\delta}{1 - 6\delta} \right)^{-1} = \\ \left(\delta + \frac{(0.5 - 2\delta)}{4} + \frac{(0.5 - \delta)}{2} + \frac{3\delta^2}{1 - 6\delta} \right)^{-1}, \end{aligned}$$

and when $\delta \rightarrow 0$, the last expression tends to $(\frac{1}{8} + \frac{1}{4})^{-1}$, which equals $\frac{8}{3}$. Finally, we show (d).

$$\begin{aligned} \frac{1}{\delta} \left(1 + 2 \frac{(0.25 - \delta)}{0.5 + 2\delta} + \frac{(0.5 - \delta)}{2\delta} + \frac{3\delta}{1 - 6\delta} \right)^{-1} = \\ \left(\delta + 2\delta \frac{(0.25 - \delta)}{(0.5 + 2\delta)} + \frac{(0.5 - \delta)}{2} + \frac{3\delta^2}{1 - 6\delta} \right)^{-1}, \end{aligned}$$

where all the terms tend to zero when $\delta \rightarrow 0$, except for $\frac{0.5 - \delta}{2}$, which tends to 0.25.

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