



HAL
open science

The second neighborhood conjecture for oriented graphs missing a c_4 , c_4,s_3 , chair and chair-free graph

Darine Al-Mniny, Salman Ghazal

► **To cite this version:**

Darine Al-Mniny, Salman Ghazal. The second neighborhood conjecture for oriented graphs missing a c_4 , c_4,s_3 , chair and chair-free graph. The Australasian Journal of Combinatorics, 2021, 81, pp.58-88. 10.48550/arXiv.2010.10790 . hal-04136272

HAL Id: hal-04136272

<https://hal.science/hal-04136272>

Submitted on 15 Jul 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The Second Neighborhood Conjecture for Oriented Graphs Missing $\{C_4, \overline{C_4}, S_3, \text{chair}$ and co-chair $\}$ -Free Graph

Darine AL MNINY^{1, 2}, Salman GHAZAL^{3, 4}

Abstract

Seymour's Second Neighborhood Conjecture (SNC) asserts that every oriented graph has a vertex whose first out-neighborhood is at most as large as its second out-neighborhood. In this paper, we prove that if G is a graph containing no induced C_4 , $\overline{C_4}$, S_3 , chair and $\overline{\text{chair}}$, then every oriented graph missing G satisfies this conjecture. As a consequence, we deduce that the conjecture holds for every oriented graph missing a threshold graph, a generalized comb or a star.

1 Introduction

Throughout this paper, all graphs are considered to be simple, that is, there are no loops and no multiple edges. Given a graph G , the vertex set and edge set of G are denoted by $V(G)$ and $E(G)$ respectively. Given an edge xy of G , the vertices x and y are called the endpoints of xy and they are said to be adjacent. Two edges of G are said to be adjacent if they have a common endpoint. The neighborhood of a vertex v in G , denoted by $N_G(v)$, is the set of all vertices adjacent to v . The degree $d_G(v)$ of v in G is defined

¹KALMA Laboratory, Department of Mathematics, Faculty of Sciences I, Lebanese University, Beirut - Lebanon. (darinealmniny@gmail.com)

²Camille Jordan Institute, Claude Bernard University - Lyon 1, France. (mniny@math.univ-lyon1.fr)

³Department of Mathematics, Faculty of Sciences I, Lebanese University, Beirut - Lebanon. (salman.ghazal@ul.edu.lb)

⁴Department of Mathematics and Physics, School of Arts and Sciences, Beirut International University, Beirut - Lebanon. (salman.ghazal@liu.edu.lb)

to be $d_G(v) := |N_G(v)|$. Note that we may omit the subscript if the graph is clear from the context. Given two sets of vertices U and W of G , we denote by $E[U, W]$ the set of all edges in G that joins a vertex in U to a vertex in W . For $A \subseteq V(G)$, $G[A]$ denotes the subgraph of G induced by A . If $G[A]$ is an empty graph, then A is called a stable set, that is, there is no edge that joins any two distinct vertices of A . However, if $G[A]$ is a complete graph, then A is called a clique set, that is, any two distinct vertices of A are adjacent. The complement graph \overline{G} of G is defined as follows: $V(\overline{G}) = V(G)$ and $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$. A graph H is called forbidden subgraph of G if H is not (isomorphic to) an induced subgraph of G . In this case, we say that G is H -free graph.

A digraph is an orientation of a graph so that it contains neither loops nor parallel arcs. However, an oriented graph is a digraph without digons (directed cycles of length 2). Given a digraph D , the vertex set and arc set of D are denoted by $V(D)$ and $E(D)$ respectively. For $(x, y) \in E(D)$ with $x, y \in V(D)$, we say that y is an out-neighbor of x , x is an in-neighbor of y and x and y are adjacent. The (first) out-neighborhood (resp. in-neighborhood) $N_D^+(v)$ (resp. $N_D^-(v)$) of a vertex v in D is the set of all out-neighbors (resp. in-neighbors) of v . Moreover, the second out-neighborhood $N_D^{++}(v)$ of v in D is the set of vertices that are at distance 2 from v , that is, $N_D^{++}(v) := \{x \in V(D) - N_D^+(v); \exists y \in N_D^+(v) \mid (y, x) \in E(D)\}$. The out-degree, the in-degree and the second out-degree of v in D are defined as follows: $d_D^+(v) := |N_D^+(v)|$, $d_D^-(v) := |N_D^-(v)|$ and $d_D^{++}(v) := |N_D^{++}(v)|$, respectively. Note that we omit the subscript if the digraph is clear from the context. For short, we write $x \rightarrow y$ if the arc $(x, y) \in E(D)$. Also, we write $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_n$, if $x_i \rightarrow x_{i+1}$ for every $1 \leq i \leq n - 1$.

Let D be an oriented graph and let $v \in V(D)$, we say that v has the second neighborhood property SNP if $d^+(v) \leq d^{++}(v)$. In 1990, P. Seymour [1] conjectured the following:

Conjecture 1. *Every oriented graph has a vertex satisfying the SNP.*

The above conjecture is called "The Second Neighborhood Conjecture", and it is abbreviated by "SNC". The SNC on tournaments is called Dean's conjecture, where tournaments are orientations of complete graphs. In 1996, Fisher [2] proved Dean's Conjecture. In 2000, a shorter proof of Dean's conjecture was given by Havet and Thomassé [8] using a tool called the median order. In 2007, Fidler and Yuster [3] proved the SNC for tournaments missing a matching, using local median orders and dependency digraphs. In 2012, Ghazal [6] proved the weighted version of SNC for tournaments missing a

generalized star. Then in 2013 Ghazal [4] proved the SNC for tournaments missing a comb, cycle of length 4 or 5. In 2015, Ghazal [5] refined the result of [3] and he showed in particular that every tournament missing a matching has a certain "feed vertex" satisfying the SNP.

In this paper, we prove the SNC for any oriented graph missing a graph G , where G contains no C_4 , $\overline{C_4}$, S_3 , chair and $\overline{\text{chair}}$ as induced subgraphs. This generalizes the results of [6] and [4].

2 Definitions and Preliminaries

A chair is a graph G whose vertex set is $V(G) = \{x, y, z, t, v\}$ and whose edge set is $E(G) = \{xy, yz, zt, zv\}$. The co-chair or $\overline{\text{chair}}$ is defined to be the complement of a chair. We denote by $C_n = v_1v_2\dots v_nv_1$ the cycle on n vertices, by $P_n = v_1v_2\dots v_n$ the path on n vertices and by S_3 the graph on 6 vertices indicated in Figure 1.

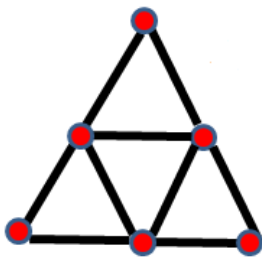


Figure 1: S_3

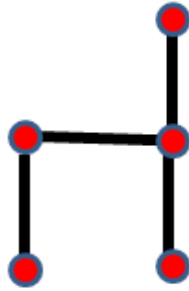


Figure 2: Chair

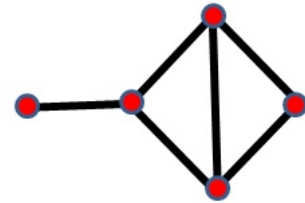


Figure 3: Co-chair

A graph G is called a split graph if its vertex set is the disjoint union of a stable set S and a clique set K . In this case, we write G is an $\{S, K\}$ -split graph. For an $\{S, K\}$ -split graph G , if $sx \in E(G) \forall s \in S$ and $\forall x \in K$, then G is called a complete split graph. Otherwise if $E[S, K]$ forms a perfect matching of G , then G is called a perfect split graph.

In [9] and [10], the notion of a threshold graph is introduced as follows:

Definition 1. A graph G is called a threshold graph if:

- 1) $V(G) = \bigcup_{i=1}^{n+1} (X_i \cup A_{i-1})$, where the A_i 's and the X_i 's are pairwise disjoint sets.

- 2) $K := \bigcup_{i=1}^{n+1} X_i$ is a clique and the X_i 's are nonempty, except possibly X_{n+1} .
- 3) $S := \bigcup_{i=0}^n A_i$ is a stable set and the A_i 's are nonempty, except possibly A_0 .
- 4) $\forall 1 \leq j \leq i \leq n$, $G[A_i \cup X_j]$ is a complete split graph.
- 5) The only edges of G are the edges of the subgraphs mentioned above.

In this case, G is called an $\{S, K\}$ -threshold graph.

On the structure of threshold graphs, Hammer and Chvátal noticed the following:

Theorem 2.1. (Hammer and Chvátal [9], [10]) G is a threshold graph if and only if C_4 , \overline{C}_4 and P_4 are forbidden subgraphs of G .

As a generalization of threshold graphs, Ghazal introduced the notion of generalized combs and he studied their structures and properties as follows:

Definition 2. (Ghazal [7]) A graph G is called a generalized comb if:

- 1) $V(G)$ is disjoint union of sets $A_0, \dots, A_n, M_1, \dots, M_l, X_1, \dots, X_{n+1}, Y_2, \dots, Y_{l+2}$ with $Y_1 = X_1$. These sets are called the sets of the generalized comb G .
- 2) $S := A \cup M$ is a stable set, where $M = \bigcup_{i=1}^l M_i$ and $A = \bigcup_{i=0}^n A_i$.
- 3) $K := X \cup Y$ is a clique, where $X = \bigcup_{i=1}^{n+1} X_i$ and $Y = \bigcup_{i=1}^{l+2} Y_i$.
- 4) $\forall 1 \leq j \leq i \leq n$, $G[A_i \cup X_j]$ is a complete split graph.
- 5) $G[A \cup Y]$ is a complete split graph.
- 6) $\forall 1 \leq i \leq l$, $G[Y_i \cup M_i]$ is a perfect split graph or $M_i = \phi$.
- 7) $\forall 1 \leq i < j \leq l+1$, $G[Y_j \cup M_i]$ is a complete split graph.
- 8) $X_{n+1}, Y_{l+2}, Y_{l+1}$ and A_0 are the only possibly empty sets among the X_i 's, Y_i 's and A_i 's.

9) *The only edges of G are the edges of the subgraphs mentioned above.*

In this case, we say that G is an $\{S, K\}$ -generalized comb.

Theorem 2.2. *(Ghazal [7]) G is a generalized comb if and only if C_4 , \overline{C}_4 , C_5 , S_3 , chair and co-chair are forbidden subgraphs of G .*

Corollary 2.1. *(Ghazal [7]) Every threshold graph is a generalized comb.*

Proposition 2.1. *Let G be a generalized comb defined as in Definition 2. Then $G' = G - \bigcup_{1 \leq i \leq l} E[Y_i, M_i]$ is a threshold graph.*

Proof. It is clear that G' contains no induced C_4 , \overline{C}_4 or P_4 . Consequently, Theorem 2.1 implies that G' is a threshold graph. \square

On the structure of graphs containing no C_4 , \overline{C}_4 , S_3 , chair and co-chair as induced subgraphs, Ghazal remarked the following:

Theorem 2.3. *(Ghazal [7]) C_4 , \overline{C}_4 , S_3 , chair and co-chair are forbidden subgraphs of a graph G if and only if $V(G)$ is disjoint union of three sets S , K and C such that:*

- 1) $G[S \cup K]$ is an $\{S, K\}$ -generalized comb.
- 2) $G[C]$ is empty or isomorphic to the cycle C_5 .
- 3) Every vertex in C is adjacent to every vertex in K but to no vertex in S .

From now on, if $G = C_5$, we set $G = xyzuvx$. If G is an $\{S, K\}$ -generalized comb, we follow the same notations as in Definition 2. Moreover, if G is a $\{C_4, \overline{C}_4, S_3, \text{chair and co-chair}\}$ -free graph, we use the notations in Theorem 2.3. Note that if G is defined as in Theorem 2.3 and $G[C]$ is empty, then G is a generalized comb.

3 Characterization Using Dependency Digraphs

Let D be an oriented graph. For two vertices x and y of D , we say that xy is a missing edge of D if $(x, y) \notin E(D)$ and $(y, x) \notin E(D)$. A vertex v of D is called a whole vertex if it is not incident to any missing edge, i.e., $N^+(v) \cup N^-(v) = V(D) - \{v\}$. Otherwise, we say that v is a non-whole vertex. The missing graph G of D is defined to be the graph formed by the missing edges of D , formally, G is the graph whose edge set is the set of all the missing edges of D and whose vertex set is the set of the non-whole vertices.

In this case, we say that D is missing G . Given two missing edges x_1y_1 and x_2y_2 , we say that x_1y_1 loses to x_2y_2 if: $x_1 \rightarrow x_2$ and $y_2 \notin N^+(x_1) \cup N^{++}(x_1)$, $y_1 \rightarrow y_2$ and $x_2 \notin N^+(y_1) \cup N^{++}(y_1)$.

The dependency digraph Δ_D (or simply Δ) of D is defined to be the digraph whose vertex set consists of all the missing edges of D , and whose arc set contains the arc (ab, cd) if and only if the missing edge ab loses to the missing edge cd . Note that Δ may contain digons. These digraphs were used in [3, 4] to prove SNC for some oriented graphs.

Definition 3. (Ghazal [6]) *A missing edge ab is called good if one of the following holds:*

- (i) $(\forall v \in V \setminus \{a, b\})[(v \rightarrow a) \Rightarrow (b \in N^+(v) \cup N^{++}(v))];$
- (ii) $(\forall v \in V \setminus \{a, b\})[(v \rightarrow b) \Rightarrow (a \in N^+(v) \cup N^{++}(v))].$

If ab satisfies (i) we say that (a, b) is a convenient orientation of ab . Else, we say that (b, a) is a convenient orientation of ab .

Lemma 3.1. (Ghazal [4]) *Let D be an oriented graph and let Δ denote its dependency digraph. A missing edge ab is good if and only if its in-degree in Δ is zero.*

In [6], threshold graphs are characterized using dependency digraphs as follows:

Theorem 3.1. (Ghazal [6]) *Let G be a graph. The following statements are equivalent:*

- i) G is a threshold graph;
- ii) Every missing edge of every oriented graph missing G is good;
- iii) The dependency digraph of every oriented graph missing G is empty.

Problem 1. *Let $\vec{\mathcal{P}}$ be the family of all digraphs consisting of vertex disjoint directed paths and let $\mathcal{F}(\vec{\mathcal{P}}) = \{G \text{ is a graph}; \forall D \text{ missing } G, \Delta_D \in \vec{\mathcal{P}}\}$. Characterize $\mathcal{F}(\vec{\mathcal{P}})$.*

Proposition 3.1. *$G \in \mathcal{F}(\vec{\mathcal{P}})$ if and only if $G' \in \mathcal{F}(\vec{\mathcal{P}})$, for every G' induced subgraph of G .*

Proof. Sufficient Condition. Simply take $G' = G$.

Necessary Condition. Assume first that $G' = G - v$ for some $v \in V(G)$. Let D' be any oriented graph missing G' . Let α and β be 2 distinct extra vertices neither in D' nor in G . Define D as follows: The missing graph of D is G , $V(D) = V(D') \cup \{v, \alpha, \beta\}$, that is, $D - \{v, \alpha, \beta\} = D'$. The arcs (α, v) , (v, β) and (α, β) are in D . For every $x \in V(D')$, if $xv \notin E(G)$, then $(x, v) \in E(D')$. Finally, for every $x \in V(D')$, the arcs (x, α) and (β, x) are in D . Then the addition of v, α and β to D' in this way neither affects the losing relations between the missing edges of D' nor creates new ones. Hence, Δ_D is equal to $\Delta_{D'}$ plus isolated vertices (these isolated vertices are the edges of G incident to v). Since D is missing G , then $\Delta_D \in \vec{\mathcal{P}}$. Whence, $\Delta_{D'} \in \vec{\mathcal{P}}$. Thus $G' \in \mathcal{F}(\vec{\mathcal{P}})$. Now, the proof follows by induction on the number of vertices removed from G to obtain the induced subgraph. \square

It is easy to observe the following:

Proposition 3.2. \overline{C}_4 , chair and co-chair are not in $\mathcal{F}(\vec{\mathcal{P}})$.

Proof. Let D be the oriented graph with vertex set $V(D) = \{a, b, c, d\}$ and arc set $E(D) = \{(a, c), (b, d), (d, a), (c, b)\}$. Then D is missing \overline{C}_4 , ab loses to cd and cd loses to ba . Thus $\Delta_D \notin \vec{\mathcal{P}}$.

Let D' be the oriented graph with vertex set $V(D') = \{a, b, c, d, x\}$ and arc set $E(D') = \{(a, d), (b, c), (c, a), (b, x), (x, a), (x, c)\}$. Then D' is missing a chair, ab loses to both dc and dx . Thus $\Delta_{D'} \notin \vec{\mathcal{P}}$.

Let D'' be the oriented graph with vertex set $V(D'') = \{a, b, c, d, x\}$ and arc set $E(D'') = \{(a, c), (b, d), (d, a), (a, x)\}$. Then D'' is missing a co-chair, ab loses to both dc and dx . Thus $\Delta_{D''} \notin \vec{\mathcal{P}}$. \square

Proposition 3.3. (Ghazal [4]) $C_5 \in \mathcal{F}(\vec{\mathcal{P}})$.

Proof. Let D be an oriented graph missing C_5 and let Δ denote its dependency digraph. Then we may check by cases that one of the following occurs up to isomorphism:

- (i) Δ has no arcs.
- (ii) Δ has exactly one arc, say $uv \rightarrow xy$.
- (iii) Δ has exactly two arcs, say $uv \rightarrow xy$ and $xv \rightarrow yz$.

(iv) Δ has exactly two arcs, say $uv \rightarrow xy \rightarrow zu$.

(v) Δ has exactly three arcs, say $uv \rightarrow xy \rightarrow zu \rightarrow vx$.

(vi) Δ has exactly three arcs, say $uv \rightarrow xy \rightarrow zu$ and $xv \rightarrow zy$.

In all the cases above, Δ consists of vertex disjoint directed paths of lengths at most 3, that is, $\Delta \in \vec{\mathcal{P}}$. Thus the result holds. \square

Theorem 3.2. *Let G be a graph having no induced C_4 , C_5 , nor S_3 . Then $G \in \mathcal{F}(\vec{\mathcal{P}})$ if and only if G is a generalized comb.*

Proof. Necessary Condition. Since $G \in \mathcal{F}(\vec{\mathcal{P}})$, then Proposition 3.1 together with Proposition 3.2 imply that $\overline{C_4}$, chair and co-chair are not induced subgraphs of G . But C_4 , C_5 , S_3 are not also induced subgraphs of G , thus due to Theorem 2.2 G is a generalized comb.

Sufficient Condition. Let D be an oriented graph missing a generalized comb G and let Δ denote its dependency digraph. Using the definition of G , each possible losing relation can occur between two edges in $E[Y_t, M_t]$, for some t . For $i = 1, 2, 3$, suppose that $a_i x_i \in E[Y_t, M_t]$ with $a_i \in M_t$ and $x_i \in Y_t$. Assume that $a_1 x_1$ loses to the two others. Then we have $a_1 \rightarrow x_3$, $x_1 \rightarrow a_2$, $a_2 \notin N^+(a_1) \cup N^{++}(a_1)$ and $x_3 \notin N^+(x_1) \cup N^{++}(x_1)$. By definition of G , the only edge of $G[Y_t \cup M_t]$ incident to a_2 is $a_2 x_2$. Thus $a_2 x_3$ is not a missing edge and so either $a_2 \rightarrow x_3$ or $x_3 \rightarrow a_2$. Whence, either $x_3 \in N^+(x_1) \cup N^{++}(x_1)$ or $a_2 \in N^+(a_1) \cup N^{++}(a_1)$, a contradiction. Thus the maximum out-degree in Δ is 1. Similarly, we can prove that the maximum in-degree in Δ is 1. This implies that Δ is composed of directed cycles and paths only.

Assume now that Δ contains a directed cycle $a_1 b_1 \rightarrow \dots \rightarrow a_n b_n \rightarrow a_1 b_1$, with $a_i \in M_t$ and $b_i \in Y_t$, for some t . Then by the losing relations we must have $a_{i+1} \rightarrow a_i \forall i < n$ and $a_1 \rightarrow a_n$ in D . We will show now by induction on i that $\forall 1 \leq i < n$, $a_i \rightarrow a_n$. It is true for $i = 1$. Assume it is true for $i - 1$. Then $a_{i-1} \rightarrow a_n$. Since $a_{i-1} b_{i-1}$ loses to $a_i b_i$, then $a_i \notin N^{++}(a_{i-1})$. But $a_i a_n$ is not a missing edge of D , then we must have $a_i \rightarrow a_n$, since otherwise $a_{i-1} \rightarrow a_n \rightarrow a_i$ in D , a contradiction. This proves that $\forall 1 \leq i < n$, $a_i \rightarrow a_n$. In particular, $a_{n-1} \rightarrow a_n$, a contradiction. Thus Δ has no directed cycles. This shows that $G \in \mathcal{F}(\vec{\mathcal{P}})$. \square

As a consequence, we deduce the following on the characterization of the graphs of our interest:

Corollary 3.1. *Given a graph G , the following statements are equivalent:*

- i) C_4 and S_3 are forbidden subgraphs of G and $G \in \mathcal{F}(\vec{\mathcal{P}})$;
- ii) C_4 , $\overline{C_4}$, S_3 , chair and co-chair are forbidden subgraphs of a graph G ;
- iii) $V(G)$ is disjoint union of three sets S , K and C such that:
 - 1) $G[S \cup K]$ is an $\{S, K\}$ -generalized comb;
 - 2) $G[C]$ is empty or isomorphic to the cycle C_5 ;
 - 3) every vertex in C is adjacent to every vertex in K but to no vertex in S .

Proof. Due to Theorem 2.3, ii) and iii) are equivalent. However, ii) follows from i) due to Proposition 3.1 and Proposition 3.2. Now assume that iii) holds and let D be an oriented graph missing G . Observe that every edge in $E(G) - E(C)$ is incident to a vertex in K . This implies that there is no losing relation between an edge in $E(C)$ and an edge in $E(G) - E(C)$, since otherwise there is an edge ab with $a \in C$, $b \in K$ and $ab \notin E(G)$. This contradicts the fact that every vertex in C is adjacent in G to every vertex in K . In the same way, we can prove that there is no losing relations between an edge in $E(G[S \cup K])$ and an edge in $E[K, C]$, or between two edges in $E[K, C]$. Thus the only possible losing relations hold either between two edges in $G[S \cup K]$ or between two edges in $G[C]$. However, $G[S \cup K]$ is a generalized comb, then by Theorem 3.2 it is in $\mathcal{F}(\vec{\mathcal{P}})$. Moreover, $G[C]$ is empty or isomorphic to the cycle C_5 , whence by Proposition 3.3 it is in $\mathcal{F}(\vec{\mathcal{P}})$. Therefore, $G \in \mathcal{F}(\vec{\mathcal{P}})$ and so i) holds. \square

We will use the following lemma frequently in the proof of our main theorem:

Lemma 3.2. *Suppose that rs loses to ab and $f \rightarrow a$ in an oriented graph D , namely with $s \rightarrow b$. If fs is not a missing edge of D , then $f \rightarrow s \rightarrow b$ in D and thus $b \in N^+(f) \cup N^{++}(f)$.*

Proof. Since fs is not a missing edge, then either $f \rightarrow s$ or $s \rightarrow f$ in D . If $s \rightarrow f$ in D , then $s \rightarrow f \rightarrow a$ in D and thus $a \in N^{++}(s)$, which contradicts the fact that rs loses to ab . Thus $f \rightarrow s$. Whence, the result follows. \square

4 Main Theorem

Let $L = v_1v_2\dots v_n$ be an ordering of the vertices of a digraph D . An arc $(v_i, v_j) \in E(D)$ is called forward with respect to L if $i < j$. Otherwise, it is called backward with respect to L . L is called a local median order of D if it maximizes the set of forward arcs of D w.r.t. L , that is, the set $\{(v_i, v_j) \in E(D); i < j\}$. In this case, L satisfies the feedback property: For all $1 \leq i \leq j \leq n$:

$$d_{]i,j]}^+(v_i) \geq d_{]i,j]}^-(v_i);$$

and

$$d_{[i,j[}^-(v_j) \geq d_{[i,j[}^+(v_j),$$

where $]i, j] := D[\{v_{i+1}, v_{i+2}, \dots, v_j\}]$ and $[i, j[:= D[\{v_i, v_{i+1}, \dots, v_{j-1}\}]$.

In this case, the last vertex v_n is called a feed vertex.

We will need the following proposition:

Proposition 4.1. *Suppose that $L = v_1v_2\dots v_n$ is a local median order of a digraph D and $e = (v_j, v_i) \in E(D)$ with $i < j$. Then L is a local median order of the digraph D' obtained from D by reversing the orientation of e .*

We will use the following theorem:

Theorem 4.1. *(Havet et al. [8]) Every feed vertex of a tournament has the SNP.*

Now we are ready to prove our main theorem:

Theorem 4.2. *Let D be an oriented graph missing a $\{C_4, \overline{C_4}, S_3, \text{chair and co-chair}\}$ -free graph G . Then D satisfies the SNC.*

Proof. Let Δ denote the dependency digraph of D , and let $\Delta[E(C)]$ denote the subdigraph of Δ induced by the set of vertices that correspond to the edges of C . Corollary 3.1 follows that Δ consists of disjoint directed paths only and each of its arcs occurs only between two edges in the same set $E[Y_j, M_j]$ for some j or between two edges of C . Let $P = m_0y_0 \rightarrow \dots \rightarrow m_iy_i \rightarrow \dots \rightarrow m_ky_k$ be a maximal directed path in Δ , with $m_i \in M_j$ and $y_i \in Y_j$. Due to the maximality of P and due to Lemma 3.1, m_0y_0 is a good missing edge and so it has a convenient orientation. If (m_0, y_0) is a convenient orientation, we add the arcs (m_{2i}, y_{2i}) and the arcs (y_{2i+1}, m_{2i+1}) to D . Else, we add the arcs (y_{2i}, m_{2i}) and the arcs (m_{2i+1}, y_{2i+1}) . We do this for every

maximal directed path in Δ whose vertices are edges in $E[Y_j, M_j]$. We have many cases.

Case *i*: $G[C]$ is empty or $\Delta[E(C)]$ is empty.

In this case, the obtained oriented graph D' is missing $G' = G - \cup E[Y_j, M_j]$ which is a threshold graph by Proposition 2.1. We assign to every missing edge of D' (which is good by Theorem 3.1) a convenient orientation and we add it to D' . The obtained oriented graph T is a tournament. Let L be a local median order of T and let f denote its feed vertex. Then by Theorem 4.1 f has the SNP in T . Reorient all the missing edges incident to f towards f except those whose out-degree in Δ is not zero. The same order L is a local median order of the obtained tournament T' , f is also feed vertex of L and thus f has the SNP in T' . We will prove that f has the SNP in D also. For this aim, we consider many cases.

Case 1: f is a whole vertex. Clearly, f gains no new out-neighbor. We will prove that f gains no new second out-neighbor. Assume that $f \rightarrow a \rightarrow b \rightarrow f$ in T' . Then $f \rightarrow a$ and $b \rightarrow f$ in D . If $a \rightarrow b$ in $E(D)$ or (a, b) is a convenient orientation w.r.t. D , then $b \in N_D^{++}(f)$. If $a \rightarrow b$ in $E(D') - E(D)$ and (a, b) is not a convenient orientation w.r.t. D , then there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. But $f \rightarrow a$ in D and fs is a non-missing edge of D , then by Lemma 3.2 $b \in N_D^{++}(f)$. If $a \rightarrow b$ in $E(T') - E(D')$, then (a, b) is a convenient orientation w.r.t. D' . Hence $b \in N_{D'}^{++}(f)$ and so there is a' such that $f \rightarrow a' \rightarrow b$ in D' . Since $f \rightarrow a'$ in D' and f is whole vertex, then $f \rightarrow a'$ in D . But this is already treated above, thus $b \in N_D^{++}(f)$.

Case 2: $\exists 1 \leq t \leq l$ such that $f \in M_t$. There is a maximal directed path $P = m_0y_0 \rightarrow \dots \rightarrow m_iy_i \rightarrow \dots \rightarrow m_ky_k$ in Δ such that $f = m_i$.

Case 2.1: Assume $(y_i, m_i) \in E(D')$. Clearly, f gains no new first out-neighbor. We claim that f gains no new second out-neighbor. Assume $m_i \rightarrow a \rightarrow b \rightarrow m_i$ in T' . Then $(m_i, a) \in E(D)$ and $(a, b) \in E(T)$.

Subcase a: If $(a, b) \in E(D)$, then clearly $b \in N_D^{++}(f)$.

Subcase b: If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation w.r.t. D and hence $b \in N_D^{++}(f)$ or there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. There is j such that $rs, ab \in E[Y_j, M_j]$. Assume $m_i = r$, then $y_i = s$, $a = y_{i+1}$ and $b = m_{i+1}$. Since $(y_i, m_i) \in E(D')$, then $(m_{i+1}, y_{i+1}) \in E(D')$, that is, $(b, a) \in E(D')$, a contradiction. So $m_i \neq r$.

Assume now that $s = m_i$. Then $a = m_{i+1}$. However $(m_{i+1}, m_i) \in E(D)$, then $(a, m_i) \in E(D)$, a contradiction. So $s \neq m_i$. Now we prove that $m_i s$ is not a missing edge of D . If $b \in Y_j$, then $s \in M_j$ and thus $m_i s$ is not a missing edge. Else $b \in M_j$, whence $a \in Y_j$ and $s \in Y_j$. Since $m_i a$ is not a missing edge, then by definition of G , $f s = m_i s$ is also not missing edge. But $f \rightarrow a$ in $E(D)$ and $a \notin N_D^{++}(s)$, then due to Lemma 3.2 we get $b \in N_D^{++}(f)$.

Subcase c: If $(a, b) \in E(T) - E(D')$. Then (a, b) is a convenient orientation w.r.t D' . But $f \rightarrow a$ in $E(D)$ and so in $E(D')$, then $b \in N_{D'}^{++}(f)$. But this is already treated above in Subcase *i.2.1.a* and Subcase *i.2.1.b*.

Case 2.2: Assume $(m_i, y_i) \in E(D')$. Here there are two cases to be consider.

Case 2.2.1: Assume $i = k$, that is, $f = m_k$. Clearly, f gains no new out-neighbor. We will prove that f gains no new second out-neighbor. Suppose $f \rightarrow a \rightarrow b \rightarrow f$ in T' . Then $(f, a) \in E(D)$ and $(a, b) \in E(T)$.

Subcase a: If $(a, b) \in E(D)$, then clearly $b \in N_D^{++}(f)$.

Subcase b: If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation w.r.t. D and hence $b \in N_D^{++}(f)$ or there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. There is j such that $rs, uv \in E[Y_j, M_j]$. Since $f = m_k$, we have $r \neq m_k$ and $s \neq m_k$. If $b \in Y_j$, then $s \in M_j$. Then $m_k s$ is not a missing edge. Else $b \in M_j$. Whence, $a \in Y_j$ and $s \in Y_j$. Since $m_k a$ is not a missing edge, then by definition of G , $f s = m_k s$ is also not missing edge. But $f \rightarrow a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Subcase c: If $(a, b) \in E(T) - E(D')$, then (a, b) is a convenient orientation w.r.t. D' and so $b \in N_{D'}^{++}(f)$. Then there is a vertex a' such that $m_k \rightarrow a' \rightarrow b$ in D' . Since $(m_k, a) \in E(D)$, then $a \neq y_k$ and $\forall j > t, a \notin Y_j$. Assume $a' = y_k$. Then $(y_k, b) \in E(D')$ and $b \neq m_k$. Thus $(y_k, b) \in E(D)$. This means that $b \notin A \cup X \cup Y \cup C$. Then either b is a whole vertex or $b \in M$. If b is whole, then ab is not a missing edge, a contradiction. So $b \in M$. Whence, $\exists \alpha$ such that $b \in M_\alpha$. If $\alpha < t$, then by definition of G , $y_k b \in E(G)$, that is, $y_k b$ is a missing edge, a contradiction. Thus $\alpha \geq t$. Since $b \in M_\alpha$ with $\alpha \geq t$ and ab is a missing edge of D' , then by definition of G , $a \in Y_j$ for some $j > \alpha$. Thus $a \in Y_j$ for some $j > t$, a contradiction. So $a' \neq y_k$. Then $(m_k, a') \in E(D)$. But this is treated in Subcase *i.2.2.1.a* and Subcase *i.2.2.1.b*.

Case 2.2.2: Assume $i < k$. Then f gains only y_i as an out-neighbor. We will prove that f gains only m_{i+1} as a second out-neighbor.

Subcase a: Suppose that $m_i \rightarrow y_i \rightarrow b$ in T' such that $b \neq m_{i+1}$. Then $(y_i, b) \notin E(D') - E(D)$ and $(y_i, b) \in E(T)$.

Subcase a.1: If $(y_i, b) \in E(D)$. Since $y_i \in Y_t$, then $y_{i+1} \in Y_t$. Since $y_i b$ is not a missing edge, then by definition of G , $y_{i+1} b$ is not a missing edge. Since $y_i \rightarrow b$ in $E(D)$ and $y_{i+1} \notin N_D^{++}(y_i)$, then we must have $y_{i+1} \rightarrow b$. Then $m_i \rightarrow y_{i+1} \rightarrow b$ in D .

Subcase a.2: If $(y_i, b) \in E(T) - E(D')$. Then (y_i, b) is a convenient orientation w.r.t. D' . Then $b \in N_{D'}^{++}(m_i)$, that is, there is vertex a such that $m_i \rightarrow a \rightarrow b$ in D' .

Subcase a.2.1: If $a = y_i$, then by definition of G , $(y_i, b) \in E(D)$, a contradiction to the fact that $y_i b$ is a missing edge of D .

Subcase a.2.2: If $a \neq y_i$, then $(m_i, a) \in E(D)$. If $(a, b) \in E(D)$, then clearly $b \in N_D^{++}(f)$. Else $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation w.r.t. D and hence $b \in N_D^{++}(f)$ or there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. Thus $\exists j$ such that $rs, ab \in E[Y_j, M_j]$. Assume $r = m_i$. Then $y_i = s$, $a = y_{i+1}$ and $b = m_{i+1}$, a contradiction to the fact that $b \neq m_{i+1}$. So $r \neq m_i$. Assume $s = m_i$. Then $a = m_{i+1}$. However $(m_{i+1}, m_i) \in E(D)$, then $(a, m_i) \in E(D)$, a contradiction. So $s \neq m_i$. Now we will prove that $m_i s$ is not a missing edge. If $b \in Y_j$, then $s \in M_j$ and thus $m_i s$ is not a missing edge. Else $b \in M_j$. Whence, $a \in Y_j$ and $s \in Y_j$. Since $m_i a$ is not a missing edge, then by definition of G , $f s = m_i s$ is also not missing edge. But $f \rightarrow a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Subcase b: Suppose $m_i \rightarrow a \rightarrow b$ in T' with $a \neq y_i$ and $b \neq m_{i+1}$. Then $(m_i, a) \in E(D)$ and $(a, b) \in E(T)$.

Subcase b.1: If $(a, b) \in E(D')$. This is the same as Subcase *i.2.2.2.a.2.2*.

Subcase b.2: If $(a, b) \in E(T) - E(D')$, then (a, b) is a convenient orientation w.r.t. D' and thus $b \in N_{D'}^{++}(f)$. This means that there is a vertex a' such that $m_i \rightarrow a' \rightarrow b$ in D' . If $a' = y_i$, then $(y_i, b) \in E(D)$. This case is already treated in Subcase *i.2.2.2.a.1*. Else, we proceed in the same method

as in Subcase *i.2.2.2.a.2.2*.

Case 3: $\exists 1 \leq t \leq l$ such that $f \in Y_t$ and $M_t \neq \emptyset$. There is a maximal directed path $P = m_0y_0 \rightarrow \dots \rightarrow m_iy_i \rightarrow \dots \rightarrow m_ky_k$ in Δ such that $f = y_i$.

Case 3.1: Assume $(m_i, y_i) \in E(D')$. Clearly, f gains no new first out-neighbor. We prove that f gains no new second out-neighbor. Assume $y_i \rightarrow a \rightarrow b \rightarrow y_i$ in T' . Then $(y_i, a) \in E(D)$ and $(a, b) \in E(T)$.

Subcase a: If $(a, b) \in E(D)$, then clearly $b \in N_D^{++}(f)$.

Subcase b: If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation w.r.t. D and hence $b \in N_D^{++}(f)$ or there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. Thus $\exists j$ such that $rs, ab \in E[Y_j, M_j]$. Assume $r = y_i$. Then $s = m_i$, $a = m_{i+1}$ and $b = y_{i+1}$. Since $(m_i, y_i) \in E(D')$, then $(y_{i+1}, m_{i+1}) \in E(D')$, that is, $(b, a) \in E(D')$, a contradiction. Then $r \neq y_i$. Assume $s = y_i$. Then $a = y_{i+1}$. Hence $y_i a = y_i y_{i+1}$ is a missing edge, contradiction. So $s \neq y_i$. Now we prove that $y_i s$ is not a missing edge. If $a \in Y_j$, then $y_i a$ is a missing edge, a contradiction. So $a \in M_j$ and hence $s \in M_j$. Since $y_i a$ is not a missing edge, then by definition of G , $y_i s$ is also not a missing edge. Since $y_i \rightarrow a$ in D and $a \notin N^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Subcase c: If $(a, b) \in E(T) - E(D')$, then (a, b) is a convenient orientation w.r.t. D' . Whence, $b \in N_{D'}^{++}(f)$. Then there is a vertex a' such that $y_i \rightarrow a' \rightarrow b$ in D' . Thus $(y_i, a') \in D$. This case is already treated in Subcase *i.3.1.a* and Subcase *i.3.1.b*.

Case 3.2: Assume $(y_i, m_i) \in E(D')$.

Case 3.2.1: Assume $i = k$, that is, $f = y_k$. Clearly, f gains no new out-neighbor. We prove that f gains no new second out-neighbor. Suppose that $f \rightarrow a \rightarrow b \rightarrow f$ in T' . Then $(f, a) \in E(D)$ and $(a, b) \in E(T)$.

Subcase a: If $(a, b) \in E(D)$, then clearly $b \in N_D^{++}(f)$.

Subcase b: If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation w.r.t. D and hence $b \in N_D^{++}(f)$ or there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. Then $\exists j$ such that $rs, ab \in E[Y_j, M_j]$. Since $f = y_k$, then $r \neq y_k$ and $s \neq y_k$. Since $(y_k, a) \in E(D)$, then $a \notin Y$. Whence, $a \in M$ and $s \in M$. Since $y_k a$ is not a missing edge, then by definition of G , $y_k s$ is

also not a missing edge. Since $f \rightarrow a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Subcase c: If $(a, b) \in E(T) - E(D')$, then it is a convenient orientation w.r.t. D' . But $f \rightarrow a$ in D and thus in D' , then $b \in N_{D'}^{++}(f)$. So there is a vertex a' such that $f \rightarrow a' \rightarrow b$ in D' .

Subcase c.1: Suppose $a' = m_k$. Since $y_k a$ is not a missing edge of D , then a is a whole vertex of D or $a \in M - \{m_k\}$. Since $(a, b) \in E(T) - E(D)$, then a is not whole. Thus $\exists j$ such that $a \in M_j - \{m_k\}$. The definition of G together with the facts that $f = y_k \in Y_t$, $a \in M_j - \{m_k\}$ and $y_k a$ is not missing edge imply that $j \geq t$. Since $(a, b) \notin E(D')$ and $a \in M_j$, then $\exists \alpha > j$ such that $b \in Y_\alpha$. Thus $ba' \in E[Y_\alpha, M_t]$ with $\alpha > t$. So, by using the definition of G , $a'b$ is a missing edge of D and D' , a contradiction since $(a', b) \in E(D')$.

Subcase c.2: Suppose $a' \neq m_k$. Whence $(y_k, a') \in E(D)$. But $(a', b) \in E(D')$, then this is already discussed in Subcase *i.3.2.1.a* and Subcase *i.3.2.1.b*.

Case 3.2.2: Assume $i < k$. Clearly, $f = y_i$ gains only m_i as an out-neighbor. We prove that f gains only y_{i+1} as a second out-neighbor.

Subcase a: Suppose that $f \rightarrow m_i \rightarrow b \rightarrow f$ in T' with $b \neq y_{i+1}$. Then $(y_i, m_i) \in E(D')$ and $(m_i, b) \in E(T)$.

Subcase a.1: If $(m_i, b) \in E(D)$. Since $b \neq y_{i+1}$ and $m_i b$ is not a missing edge of D , then by definition of G , also $m_{i+1} b$ is not a missing edge. Since $m_i \rightarrow b$ in D and $m_{i+1} \notin N^{++}(m_i)$, then we must have $m_{i+1} \rightarrow b$ in D . Thus $y_i \rightarrow m_{i+1} \rightarrow b$ in D .

Subcase a.2: If $(m_i, b) \in E(D') - E(D)$. Then $m_i b = m_i y_i$ and hence $b = y_i$, a contradiction. So this case does not hold.

Subcase a.3: If $(m_i, b) \in E(T) - E(D')$. Then (m_i, b) is a convenient orientation w.r.t. to D' . Since $y_i \rightarrow m_i$ in D' , then $b \in N_{D'}^{++}(y_i)$. Then there is a vertex a' such that $y_i \rightarrow a' \rightarrow b$ in D' . If $a' = m_i$, then $(m_i, b) \in E(D')$, a contradiction. Thus $a' \neq m_i$ and so $(y_i, a') \in E(D)$ and $(a', b) \in E(D')$.

Subcase a.3.1: If $(a', b) \in E(D)$, then $y_i \rightarrow a' \rightarrow b$ in D .

Subcase a.3.2: If $(a', b) \in E(D') - E(D)$, then either (a', b) is a convenient

orientation w.r.t. D and hence $b \in N_D^{++}(f)$ or there is $rs \rightarrow a'b$ in Δ , namely $s \rightarrow b$ and $a' \notin N_D^{++}(s)$. If $rs = m_i y_i$, then $a'b = m_{i+1} y_{i+1}$. But $b \neq y_{i+1}$, then $b = m_{i+1}$. Since $(y_i, m_i) \in E(D')$, then $(m_{i+1}, y_{i+1}) \in E(D')$, that is, $(b, a') \in E(D')$, a contradiction. Now we claim that $y_i s$ is not a missing edge of D . Since $y_i a'$ is not a missing edge, then $a' \notin Y$. Whence, $a' \in M$ and $s \in M$. Therefore, using the definition of G and the fact that $y_i a'$ is not a missing edge, we reach our claim. Since $y_i \rightarrow a'$ in D and $a' \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Subcase b: Assume $y_i \rightarrow a \rightarrow b \rightarrow y_i$ in T' with $a \neq m_i$ and $b \neq y_{i+1}$. Then $(y_i, a) \in E(D)$ and $(a, b) \in E(T)$. If $(a, b) \in E(D')$, then this is already treated in Subcases *a.3.1* and *a.3.2*. Else if $(a, b) \in E(T) - E(D')$, then it is a convenient orientation w.r.t. D' and hence $b \in N_{D'}^{++}(y_i)$. Then there is a vertex a' such that $y_i \rightarrow a' \rightarrow b$ in D' . If $a' = m_i$, this is already treated in Subcase *i.3.2.2.a.1* and Subcase *i.3.2.2.a.2*. Else if $a' \neq m_i$, this is already treated in Subcase *i.3.2.2.a.3.1* and Subcase *i.3.2.2.a.3.2*.

Case 4: $\exists 1 \leq t \leq l+1$ such that $f \in Y_t$ such that $M_t = \phi$. Clearly, f gains no new out-neighbor. We prove it gains no new second out-neighbor. Suppose $f \rightarrow a \rightarrow b \rightarrow f$ in T' . Then $(f, a) \in E(D)$ and $(a, b) \in E(T)$. We consider the following cases.

Subcase a: If $(a, b) \in E(D)$, then clearly $b \in N_D^{++}(f)$.

Subcase b: If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation w.r.t. D and hence $b \in N_D^{++}(f)$ or there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. Then $\exists j$ such that $rs, ab \in E[Y_j, M_j]$. Since fa is not a missing edge of D , then $a \notin Y$. Hence $a \in M_j$ and $s \in M_j$. Since $a, s \in M_j$ and fa is not a missing edge, then also fs is not a missing edge. Since $f \rightarrow a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Subcase c: If $(a, b) \in E(T) - E(D')$, then (a, b) is a convenient orientation w.r.t. D' . But $f \rightarrow a$ in D and D' , then $b \in N_{D'}^{++}(f)$. So there is a vertex a' such that $f \rightarrow a' \rightarrow b$ in D' . Then $(f, a') \in E(D)$. But this is already treated in Subcase *i.4.a* and Subcase *i.4.b*.

Case 5: $f \in Y_{l+2}$. Exactly same as Case *i.4*, with only one difference in Subcase *b*. The difference is that fs is not a missing edge in Subcase *i.5.b*, because $E[Y_{l+2}, M_j] = \phi$ by definition of G , while in Subcase *i.4.b* we had to prove it.

Case 6: $f \in V(G) - (Y \cup M)$. Clearly, f gains no new out-neighbor. We will prove that it gains no new second out-neighbor. Suppose $f \rightarrow a \rightarrow b \rightarrow f$ in T' . Then $(f, a) \in E(D)$ and $(a, b) \in E(T)$. We consider the following sub-cases.

Subcase a: If $(a, b) \in E(D)$, then clearly $b \in N_D^{++}(f)$.

Subcase b: If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation w.r.t. D and hence $b \in N_D^{++}(f)$ or there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. Thus $\exists j$ such that $rs, ab \in E[Y_j, M_j]$. If $a \in Y_j$, then fa is a missing edge of D , a contradiction. So $a \in M_j$ and hence $s \in M_j$. Then fs is not a missing edge. Since $f \rightarrow a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Subcase c: If $(a, b) \in E(T) - E(D')$. Then (a, b) is a convenient orientation w.r.t. D' . But $f \rightarrow a$ in D and so D' , then $b \in N_{D'}^{++}(f)$. Then there is a vertex a' such that $f \rightarrow a' \rightarrow b$ in D' . Then $(f, a') \in E(D)$ and $(a', b) \in E(D')$. But this is already discussed in Subcase *i.6.a* and Subcase *i.6.b*.

Therefore, f has the SNP in D in all cases. This completes the proof of Case *i*.

Case ii: $\Delta[E(C)]$ contains exactly one arc, say $uv \rightarrow xy$.

Assume without loss of generality that (u, v) is a convenient orientation of the good missing edge uv . We add to D the arcs (u, v) and (x, y) , we assign to the good missing edges xv, yz and zu a convenient orientation and then we add them to D . The obtained oriented graph D' is missing $G' = G - (\cup E[Y_j, M_j] \cup E(C))$ which is a threshold graph. We assign to the missing edges of D' convenient orientations and we add them to D' to get a tournament T . Let L be a local median order of T and let f denote its feed vertex. Reorient all the missing edges incident to f towards f , except those whose out-degree in Δ is not zero. The same order L is a local median order of the obtained tournament T' , f is also feed vertex of L and thus f has the SNP in T' . We will prove that f has the SNP in D also. For this purpose, we consider the following cases.

Case 1: f is a whole vertex. This is the same as Case *i.1*.

Case 2: $\exists 1 \leq t \leq l$ such that $f \in M_t$. Exactly same as Case *i.2*, with

only one difference in the subcases when $f \rightarrow a \rightarrow b$ with $(f, a) \in E(D)$ and $(a, b) \in E(D') - E(D)$ and it is not convenient w.r.t. D . Such subcase we call it unsteady. As usual, since (a, b) is not convenient w.r.t. D , then there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. The difference is that in the unsteady subcases of Case *ii.2* either $rs, ab \in E[Y_j, M_j]$ for some j or $rs = uv, ab = xy$. If $rs, ab \in E[Y_j, M_j]$ for some j , we proceed exactly in the same way as in the unsteady subcases of Case *i.2*. Else if $(r, s) = (u, v)$ and $(a, b) = (x, y)$, then fs is not a missing edge because $E[M, C] = \phi$. Since $f \rightarrow a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Case 3: $\exists 1 \leq t \leq l$ such that $f \in Y_t$ and $M_t \neq \phi$. Exactly same as Case *i.3*, with only one difference in the subcases when $f \rightarrow a \rightarrow b$ with $(f, a) \in E(D)$ and $(a, b) \in E(D') - E(D)$ and it is not convenient w.r.t. D . As usual, since (a, b) is not convenient w.r.t. D , then there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. The difference is that in the unsteady subcases of Case *ii.3* there are two cases to be consider: Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs = uv$ and $ab = xy$. If $rs, ab \in E[Y_j, M_j]$ for some j , we proceed exactly in the same way as in the unsteady subcases of Case *i.3*. Else if $(r, s) = (u, v)$ and $(a, b) = (x, y)$, then $fa = fx$ is a missing edge because $G[Y \cup C]$ is a complete split graph, a contradiction.

Case 4: $\exists 1 \leq t \leq l + 1$ such that $f \in Y_t$ such that $M_t = \phi$. Exactly same as Case *i.4*, with only one difference in Subcase *b*. The difference is that in Subcase *ii.4.b* there are two possibilities for the edges rs, ab : Either $rs, ab \in E[Y_j, M_j]$ for some j , or $(r, s) = (u, v)$ and $(a, b) = (x, y)$. The first case is treated in Subcase *i.4.b*. However, the second case does not exist since otherwise $fa = fx$ is a missing edge because $G[Y \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$.

Case 5: $f \in Y_{l+2}$. Exactly same as Case *i.4*, with two differences in Subcase *b*. The first difference is that in Subcase *ii.5.b* there are two possibilities for the edges rs, ab : Either $rs, ab \in E[Y_j, M_j]$ for some j , or $(r, s) = (u, v)$ and $(a, b) = (x, y)$. The first case is already treated in Subcase *i.4.b*. However, the second case does not exist since otherwise $fa = fx$ is a missing edge because $G[Y \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$. The second difference is that fs is not a missing edge in Subcase *ii.5.b*, because $E[Y_{l+2}, M_j] = \phi$ by definition of G , while in Subcase *i.4.b* we had to prove it.

Case 6: $f = u$. Clearly, u gains only v as a new first out-neighbor. We prove that it gains only y as a new second out-neighbor.

Case 6.1: Suppose that $u \rightarrow v \rightarrow b \rightarrow u$ in T' with $b \neq y$. Then $(v, b) \in E(T)$. Note that $b \neq x$ because $u \rightarrow x$ in D .

Subcase a: If $(v, b) \in E(D)$, then either $b = z$, $b \in S$ or b is a whole vertex. Then by the losing relation $uv \rightarrow xy$, we get $b \in N_D^{++}(u)$.

Subcase b: If $(v, b) \in E(D') - E(D)$, then $b = x$ or $b = u$, a contradiction. Thus this case does not exist.

Subcase c: If $(v, b) \in E(T) - E(D')$, then $b \in K$ and (v, b) is a convenient orientation w.r.t. D' . Then there exists v' such that $u \rightarrow v' \rightarrow b \rightarrow u$ in D' . Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(f, v') \in E(D')$, then $(f, v') \in E(D)$.

Subcase c.1: If $(v', b) \in E(D)$, then $b \in N_D^{++}(f)$.

Subcase c.2: If $(v', b) \in E(D') - E(D)$, then (v', b) is a convenient orientation w.r.t. D and hence $b \in N_D^{++}(f)$ or there is $rs \rightarrow v'b$ in Δ , namely $s \rightarrow b$ and $v' \notin N_D^{++}(s)$. Since $v' \notin C$, then $\exists j$ such that $rs, v'b \in E[Y_j, M_j]$. Since $b \in K$, then $b \in Y_j$. Whence, $v' \in M_j$ and $s \in M_j$. Thus fs is not a missing edge of D by definition of G . But $(f, v') \in E(D)$ and $v' \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Case 6.2: Suppose that $u \rightarrow a \rightarrow b \rightarrow u$ in T' with $a \neq v$ and $b \neq y$. Then $(u, a) \in E(D)$ and $(a, b) \in E(T)$. Note that $a \notin K \cup \{u, v, y, z\}$ and $b \notin \{u, v, x, y\}$.

Subcase a: If $(a, b) \in E(D)$, $b \in N_D^{++}(u)$.

Subcase b: If $(a, b) \in E(D') - E(D)$, then either $ab \in E(C)$ or $ab \in E[Y_j, M_j]$ for some j . If $ab \in E(C)$, then $(a, b) = (x, z)$ because $a \notin \{u, v, y, z\}$ and $b \notin \{u, v, y, x\}$. Thus xz is a missing edge of D , a contradiction. It follows that $ab \in E[Y_j, M_j]$ for some j . Then either (a, b) is a convenient orientation w.r.t. D and hence $b \in N_D^{++}(u)$ or there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. So by definition of G , us is not a missing edge of D . But $(u, a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(u)$.

Subcase c: If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation w.r.t. D' . Since $(u, a) \in E(D)$ and so in D' , there exists v' such

that $u \rightarrow v' \rightarrow b$ in D' . Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(u, v') \in E(D')$, then $(u, v') \in E(D)$. But this is already treated in Subcase *ii.6.1.c.1* and Subcase *ii.6.1.c.2*.

Case 7: $f \in C - \{u\}$. It is clear that f gains no new first out-neighbor. We will prove that it gains no new second out-neighbor. Suppose that $f \rightarrow a \rightarrow b \rightarrow f$ in T' . Then $(f, a) \in E(D)$, $(a, b) \in E(T)$ and $a \notin K$.

Subcase a: If $(a, b) \in E(D)$, $b \in N_D^{++}(u)$.

Subcase b: If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation w.r.t. D and hence $b \in N_D^{++}(f)$ or there is there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. If $(r, s) = (u, v)$ and $(a, b) = (x, y)$, then $a = x$ and $b = y$ and so $f \notin \{x, y\}$. Note that $f \neq v$, since otherwise $fa = vx$ is a missing edge of D , a contradiction. Since $f \in C - \{u, v, x, y\}$, then $f = z$ and hence $fs = zv$ is not a missing edge of D . Else if $rs, ab \in E[Y_j, M_j]$ for some j , then $r \neq f$ and $s \neq f$. Since $a \notin K$, then $a \notin Y_j$ and so $a \in M_j$. Whence, $s \in M_j$. Thus fs is not missing edge of D by definition of G . Therefore, by the losing relation $rs \rightarrow ab$ in Δ , we get $b \in N_D^{++}(f)$.

Subcase c: If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation w.r.t. D' . So there is v' such that $f \rightarrow v' \rightarrow b$ in D' . Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(f, v') \in E(D')$, then $(f, v') \in E(D)$. But this is already treated in Subcase *ii.6.1.c.1* and Subcase *ii.6.1.c.2*.

Case 8: $f \in V(G) - (Y \cup M \cup C) = A \cup (X - X_1) = A \cup (X - Y_1)$. Exactly same as Case *i.6*, with only one difference in Subcase *b*. The difference is that in Subcase *ii.8.b* there are two possibilities for the edges rs, ab : Either $rs, ab \in E[Y_j, M_j]$ for some j , or $(r, s) = (u, v)$ and $(a, b) = (x, y)$. The first case is treated in Subcase *i.6.b*. However, for the case $(r, s) = (u, v)$ and $(a, b) = (x, y)$, f must belong to A since otherwise $fa = fx$ is a missing edge because $G[X \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$. Thus $fs = fv$ is not missing edge of D because $E[A, C] = \phi$ by definition of G . Since $f \rightarrow a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Therefore, f has the SNP in D when Δ has exactly one arc between the edges of C .

Case iii: Suppose that $\Delta[E(C)]$ has exactly two arcs, say $uv \rightarrow xy$ and

$vx \rightarrow yz$.

Then $u \rightarrow x \rightarrow z \rightarrow v \rightarrow y$ in D and uv, vx, uz are good missing edges. Assume without loss of generality that (u, v) is a convenient orientation w.r.t. D . Add the arcs (u, v) and (x, y) to D . If (v, x) is a convenient orientation of vx , then add the arcs (v, x) and (y, z) . Otherwise, add the arcs (x, v) and (z, y) . Assign to uz a convenient orientation and add it to D . The obtained oriented graph D' is missing $G' = G - (\cup E[Y_j, M_j] \cup E(C))$ which is a threshold graph. So all the missing edges of D' are good. We assign to them a convenient orientation and we add them to get a tournament T . Let L be a local median order of T and let f denote its feed vertex. Reorient all the missing edges incident to f towards f except those whose out-degree in Δ is not zero. The same order L is again a local median order of the obtained tournament T' and f has the SNP in T' . We will prove that f has the SNP in D also. We have the following cases.

Case 1: f is a whole vertex. This is the same as Case *i.1*.

Case 2: $\exists 1 \leq t \leq l$ such that $f \in M_t$. Exactly same as Case *i.2*, with only one difference in the unsteady subcases, that is, in the subcases where $f \rightarrow a \rightarrow b$ with $(f, a) \in E(D)$ and $(a, b) \in E(D') - E(D)$ and it is not convenient w.r.t. D . As usual, since (a, b) is not convenient w.r.t. D , then there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. The difference is that in the unsteady subcases of Case *iii.2* either $rs, ab \in E[Y_j, M_j]$ for some j or $rs, ab \in E(C)$. If $rs, ab \in E[Y_j, M_j]$ for some j , we proceed exactly in the same way as in the unsteady subcases of Case *i.2*. Else if $rs, ab \in E(C)$ (the possible cases are: $(r, s) = (u, v)$ and $(a, b) = (x, y)$, $(r, s) = (v, x)$ and $(a, b) = (y, z)$ if (v, x) is a convenient orientation of vx or $(r, s) = (x, v)$ and $(a, b) = (z, y)$ if (x, v) is a convenient orientation of vx), then fs is not a missing edge because $E[M, C] = \phi$. Since $f \rightarrow a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Case 3: $\exists 1 \leq t \leq l$ such that $f \in Y_t$ and $M_t \neq \phi$. Exactly same as Case *i.3*, with only one difference in the subcases when $f \rightarrow a \rightarrow b$ with $(f, a) \in E(D)$ and $(a, b) \in E(D') - E(D)$ and it is not convenient w.r.t. D . As usual, since (a, b) is not convenient w.r.t. D , then there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. The difference is that in the unsteady subcases of Case *iii.3* there are two cases to be consider: Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs, ab \in E(C)$. If $rs, ab \in E[Y_j, M_j]$ for some j , we proceed exactly in the same way as in the unsteady subcases of Case *i.3*. Else if $rs, ab \in E(C)$, then fa is a missing edge because $G[Y \cup C]$ is a complete split

graph, a contradiction. Thus this case does not exist.

Case 4: $\exists 1 \leq t \leq l+1$ such that $f \in Y_t$ such that $M_t = \phi$. Exactly same as Case *i.4*, with only one difference in Subcase *b*. The difference is that in Subcase *iii.4.b* there are two possibilities for the edges rs, ab : Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs, ab \in E(C)$. The first case is treated in Subcase *i.4.b*. However, the second case does not exist since otherwise fa is a missing edge because $G[Y \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$. This means that this case does not exist.

Case 5: $f \in Y_{l+2}$. Exactly same as Case *i.4*, with two differences in Subcase *b*. The first difference is that in Subcase *iii.5.b* there are two possibilities for the edges rs, ab : Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs, ab \in E(C)$. The first case is already treated in Subcase *i.4.b*. However, the second case does not exist since otherwise fa is a missing edge because $G[Y \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$. The second difference is that fs is not a missing edge in Subcase *iii.5.b*, because $E[Y_{l+2}, M_j] = \phi$ by definition of G , while in Subcase *i.4.b* we had to prove it.

Case 6: $f = u$. Clearly, f gains only v as a new first out-neighbor and y as a new second out-neighbor. We prove it gains only y as a second out-neighbor.

Case 6.1: Suppose that $u \rightarrow v \rightarrow b \rightarrow u$ in T' with $b \neq y$. Then $(v, b) \in E(T)$. Since x and z are first and second out-neighbors of u in D respectively, then we may assume that $b \notin C$ and hence $(v, b) \notin E(D') - E(D)$.

Subcase a: If $(v, b) \in E(D)$, then either $b \in S$ or b is a whole vertex. Then by the losing relation $uv \rightarrow xy$, we get $b \in N_D^{++}(u)$.

Subcase b: If $(v, b) \in E(T) - E(D')$, then $b \in K$ and (v, b) is a convenient orientation w.r.t. D' . But this is exactly the same as Subcase *ii.6.1.c*.

Case 6.2: Suppose that $u \rightarrow a \rightarrow b \rightarrow u$ in T' with $a \neq v$ and $b \neq y$. Thus $(a, b) \in E(T)$. Since $a \neq v$, then $(u, a) \in E(D)$ and hence $a \notin K$. Since x and z are first and second out-neighbors of u in D and $u \rightarrow v$ in T' , then we may assume $b \notin C$.

Subcase a: If $(a, b) \in E(D)$, $b \in N_D^{++}(u)$.

Subcase b: If $(a, b) \in E(D') - E(D)$, then either $ab \in E(C)$ or $ab \in$

$E[Y_j, M_j]$ for some j . If $ab \in E(C)$, then $b \in C$, a contradiction. Thus $ab \in E[Y_j, M_j]$ for some j . It follows that either (a, b) is a convenient orientation w.r.t. D and hence $b \in N_D^{++}(u)$ or there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. So by definition of G , us is not a missing edge of D . But $(u, a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(u)$.

Subcase c: If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation w.r.t. D' . Since $(u, a) \in E(D)$ and so in D' , there exists v' such that $u \rightarrow v' \rightarrow b$ in D' . Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(u, v') \in E(D')$, then $(u, v') \in E(D)$. But this is already treated in Subcase *iii.6.2.a* and Subcase *iii.6.2.b*.

Case 7: $f = v$. Here there are two cases to be consider.

Case 7.1: Assume $(v, x) \in E(D')$. Then v gains x as first out-neighbor and z as a second out-neighbor. This case is similar to Case *iii.6*.

Case 7.2: Assume $(x, v) \in E(D')$. Then $(z, y) \in E(D')$. Clearly, v gains no new first out-neighbor. We prove that it gains no new second out-neighbor. Suppose $f \rightarrow a \rightarrow b \rightarrow f$ in T' . Then $(f, a) \in E(D)$ and hence $a \notin K$.

Subcase a: If $(a, b) \in E(D)$, $b \in N_D^{++}(f)$.

Subcase b: If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation w.r.t. D and hence $b \in N_D^{++}(v)$ or there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. So either $rs, ab \in E(C)$ or $rs, ab \in E[Y_j, M_j]$ for some j . If $ab \in E(C)$, then $(a, b) = (x, y)$ or $(a, b) = (z, y)$ and hence $b = y$, which is impossible because $b \rightarrow f$ in T' while $f \rightarrow y$ in D . Thus $rs, ab \in E[Y_j, M_j]$ for some j . Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. So by definition of G , fs is not a missing edge of D . But $(f, a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Subcase c: If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation w.r.t. D' . Since $(f, a) \in E(D)$ and so in D' , there exists v' such that $f \rightarrow v' \rightarrow b$ in D' . Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(f, v') \in E(D')$, then $(f, v') \in E(D)$. But this is already treated in Subcase *iii.7.2.a* and Subcase *iii.7.2.b*.

Case 8: $f = x$. Here there are two cases to be consider.

Case 8.1: Assume $(v, x) \in E(D')$. Clearly, x gains no new first out-neighbor. We prove it gains no new second out-neighbor. Suppose $x \rightarrow a \rightarrow b \rightarrow x$ in T' . Then $(x, a) \in E(D)$, $(a, b) \in E(T)$ and $a \notin K$. Note that $a \notin \{x, y, u, v\}$ and we may assume $b \notin \{x, z, v\}$.

Subcase a: If $(a, b) \in E(D)$, $b \in N_D^{++}(x)$.

Subcase b: If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation w.r.t. D and hence $b \in N_D^{++}(x)$ or there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. So either $rs, ab \in E(C)$ or $rs, ab \in E[Y_j, M_j]$ for some j . If $ab \in E(C)$, then $(a, b) = (x, y)$ or $(a, b) = (y, z)$ and hence $a \in \{x, y\}$, a contradiction. Thus $rs, ab \in E[Y_j, M_j]$ for some j . Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. So by definition of G , fs is not a missing edge of D . But $(x, a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Subcase c: If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation w.r.t. D' . Since $(x, a) \in E(D)$ and so in D' , there exists v' such that $x \rightarrow v' \rightarrow b$ in D' . Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(x, v') \in E(D')$, then $(x, v') \in E(D)$. But this is already treated in Subcase *iii.8.1.a* and Subcase *iii.8.1.b*.

Case 8.2: Assume $(x, v) \in E(D')$. Clearly, x gains only v as a first out-neighbor and y as a second out-neighbor. We prove it gains only y as a second out-neighbor. Note that z and v are first and second out-neighbors of x in D , $(v, u) \notin E(T)$ and $(x, u) \notin E(T)$.

Subcase a: Suppose that $x \rightarrow v \rightarrow b \rightarrow x$ in T' with $b \neq y$. Then $(v, b) \in E(T)$. By the previous note, we may assume that $b \notin C$ and hence $(v, b) \notin E(D') - E(D)$.

Subcase a.1: If $(v, b) \in E(D)$, then either $b \in S$ or b is a whole vertex. Then by the losing relation $xv \rightarrow zy$, we get $b \in N_D^{++}(x)$.

Subcase a.2: If $(v, b) \in E(T) - E(D')$, then $b \in K$ and (v, b) is a convenient orientation w.r.t. D' . But this exactly the same as Subcase *ii.6.a.3*.

Subcase b: Suppose $x \rightarrow a \rightarrow b \rightarrow x$ in T' with $b \neq y$ and $a \neq v$. Then $(x, a) \in E(D)$ and thus $a \notin K \cup \{x, y, v, u\}$. Note that suppose $b \notin \{x, y, v, z\}$. We argue exactly as in Subcase *iii.7.2*.

Case 9: $f = y$. Clearly, f gains no new first out-neighbor. We prove that it gains no new second out-neighbor. Note that $(z, y) \in E(T')$ and u and x are first and second out-neighbors of y , respectively. Suppose that $f \rightarrow a \rightarrow b \rightarrow f$ in T' . Then $(f, a) \in E(D)$ and $a \notin K \cup \{x, y, v, z\}$.

Subcase a: If $(a, b) \in E(D)$, then $b \in N_D^{++}(f)$.

Subcase b: If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation w.r.t. D and hence $b \in N_D^{++}(f)$ or there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. So either $rs, ab \in E(C)$ or $rs, ab \in E[Y_j, M_j]$ for some j . If $ab \in E(C)$, then $(a, b) = (x, y)$ or $(a, b) = (y, z)$ or $(a, b) = (z, y)$ and hence $a \in \{x, y, z\}$, a contradiction. Thus $rs, ab \in E[Y_j, M_j]$ for some j . Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. So by definition of G , fs is not a missing edge of D . But $(f, a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Subcase c: If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation w.r.t. D' . Since $(f, a) \in E(D)$ and so in D' , there exists v' such that $f \rightarrow v' \rightarrow b$ in D' . Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(f, v') \in E(D')$, then $(f, v') \in E(D)$. But this is already treated in Subcase *iii.9.a* and Subcase *iii.9.b*.

Case 10: $f = z$. Exactly same as Case *iii.9* with difference that yz is reoriented so that $(y, z) \in E(T')$.

Case 11: $f \in V(G) - (Y \cup M \cup C) = A \cup (X - X_1) = A \cup (X - Y_1)$. Exactly same as Case *i.6*, with only one difference in Subcase *b*. The difference is that in Subcase *iii.11.b* there are two possibilities for the edges rs, ab : Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs, ab \in E(C)$. The first case is treated in Subcase *i.6.b*. However, for the second case, f must belong to A since otherwise fa is a missing edge because $G[X \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$. Thus fs is not missing edge of D because $E[A, C] = \phi$ by definition of G . Since $f \rightarrow a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Therefore, due to all above discussions, f has the SNP in D .

Case iv: Suppose that $\Delta[E(C)]$ has exactly two arcs, say $uv \rightarrow xy \rightarrow zu$.

Then $v \rightarrow y \rightarrow u \rightarrow x \rightarrow z$ in D and uv, yz, vx are good missing edges. Assume without loss of generality that (u, v) is a convenient orientation w.r.t.

D . Add the arcs (u, v) , (x, y) and (z, u) to D . Assign to the good missing edges vx and zy a convenient orientation and add them to D . The obtained oriented graph D' is missing $G' = G - (\cup E[Y_j, M_j] \cup E(C))$ which is a threshold graph. So all the missing edges of D' are good. We assign to them a convenient orientation and we add them to get a tournament T . Let L be a local median order of T and let f denote its feed vertex. Reorient all the missing edges incident to f towards f except those whose out-degree in Δ is not zero. The same order L is again a local median order of the obtained tournament T' and f has the SNP in T' . We will prove that f has the SNP in D . We have the following cases.

Case 1: f is a whole vertex. This is the same as Case $i.1$.

Case 2: $\exists 1 \leq t \leq l$ such that $f \in M_t$. Exactly same as Case $i.2$, with only one difference in the unsteady subcases, that is, in the subcases where $f \rightarrow a \rightarrow b$ with $(f, a) \in E(D)$ and $(a, b) \in E(D') - E(D)$ and it is not convenient w.r.t. D . As usual, since (a, b) is not convenient w.r.t. D , then there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. The difference is that in the unsteady subcases of Case $iv.2$ either $rs, ab \in E[Y_j, M_j]$ for some j or $rs, ab \in E(C)$. If $rs, ab \in E[Y_j, M_j]$ for some j , we proceed exactly in the same way as in the unsteady subcases of Case $i.2$. Else if $rs, ab \in E(C)$ (the possible cases are: $(r, s) = (u, v)$ and $(a, b) = (x, y)$ or $(r, s) = (x, y)$ and $(a, b) = (z, u)$), then fs is not a missing edge because $E[M, C] = \phi$. Since $f \rightarrow a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Case 3: $\exists 1 \leq t \leq l$ such that $f \in Y_t$ and $M_t \neq \phi$. Exactly same as Case $i.3$, with only one difference in the subcases when $f \rightarrow a \rightarrow b$ with $(f, a) \in E(D)$ and $(a, b) \in E(D') - E(D)$ and it is not convenient w.r.t. D . As usual, since (a, b) is not convenient w.r.t. D , then there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. The difference is that in the unsteady subcases of Case $iv.3$ there are two cases to be consider: Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs, ab \in E(C)$. If $rs, ab \in E[Y_j, M_j]$ for some j , we proceed exactly in the same way as in the unsteady subcases of Case $i.3$. Else if $rs, ab \in E(C)$, then fa is a missing edge because $G[Y \cup C]$ is a complete split graph, a contradiction. Thus this case does not exist.

Case 4: $\exists 1 \leq t \leq l + 1$ such that $f \in Y_t$ such that $M_t = \phi$. Exactly same as Case $i.4$, with only one difference in Subcase b . The difference is that in Subcase $iv.4.b$ there are two possibilities for the edges rs, ab : Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs, ab \in E(C)$. The first case is treated in Subcase $i.4.b$. However, the second case does not exist since otherwise

fa is a missing edge because $G[Y \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$. This means that this case does not exist.

Case 5: $f \in Y_{l+2}$. Exactly same as Case *i.4*, with two differences in Subcase *b*. The first difference is that in Subcase *iv.5.b* there are two possibilities for the edges rs, ab : Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs, ab \in E(C)$. The first case is already treated in Subcase *i.4.b*. However, the second case does not exist since otherwise fa is a missing edge because $G[Y \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$. The second difference is that fs is not a missing edge in Subcase *iv.5.b*, because $E[Y_{l+2}, M_j] = \phi$ by definition of G , while in Subcase *i.4.b* we had to prove it.

Case 6: $f = u$. Clearly, u gains only v as a first out-neighbor and gains y as a second out-neighbor. We prove that u gains only y as a second out-neighbor. But this is exactly same as Case *iii.6*.

Case 7: $f = v$. It is clear that v gains no new first out-neighbor. We prove that it gains no new second out-neighbor. Suppose that $v \rightarrow a \rightarrow b \rightarrow v$ in T' . Then $(v, a) \in E(D)$ and $a \notin K \cup \{x, v, u\}$. Since $v \rightarrow y \rightarrow u$ in D , we may assume that $b \notin \{u, v, y\}$ and $a \neq y$.

Subcase a: If $(a, b) \in E(D)$ or is a convenient orientation w.r.t. D , then $b \in N_D^{++}(f)$.

Subcase b: If $(a, b) \in E(D) - E(D')$ and it is not convenient w.r.t. D , then there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. So either $rs, ab \in E(C)$ or $rs, ab \in E[Y_j, M_j]$ for some j . If $ab \in E(C)$, then $(a, b) = (x, y)$ or $(a, b) = (z, u)$ and hence $b \in \{u, y\}$, a contradiction. Thus $rs, ab \in E[Y_j, M_j]$ for some j . Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. So by definition of G , fs is not a missing edge of D . But $(f, a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Subcase c: If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation w.r.t. D' . Since $(f, a) \in E(D)$ and so in D' , there exists v' such that $f \rightarrow v' \rightarrow b$ in D' . Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C \cup k$. Since $v' \notin C$ and $(f, v') \in E(D')$, then $(f, v') \in E(D)$. But this is already treated in Subcase *iv.7.a* and Subcase *iv.7.b*.

Case 8: $f = x$. Then x gains only y as a first out-neighbor and gains u as a second out-neighbor. We prove that it gains only u as a new second out-neighbor.

Subcase a: Suppose that $x \rightarrow y \rightarrow b \rightarrow x$ in T' with $b \neq u$. Then $(y, b) \in E(T)$. Since $x \rightarrow z$ and $v \rightarrow y$ in D , then we may assume $b \notin C$ and hence $(y, b) \notin E(D') - E(D)$. If $(y, b) \in E(D)$, then b is a whole vertex or $b \in S$. Thus by the losing relation $xy \rightarrow zu$, we get $b \in N_D^{++}(x)$. Else if $(y, b) \in E(T) - E(D')$, then $b \in K$ and (y, b) is a convenient orientation w.r.t. D' . So there is v' such that $x \rightarrow v' \rightarrow b$ in D' . Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C \cup K$. Since $v' \notin C$ and $(f, v') \in E(D')$, then $(f, v') \in E(D)$. As usual we can prove that $b \in N_D^{++}(x)$ in D .

Subcase b: Suppose that $x \rightarrow a \rightarrow b \rightarrow x$ in T' with $a \neq y$ and $b \neq u$. Then $(x, a) \in D$ and thus $a \notin K$. Since $x \rightarrow z$ in D and $x \rightarrow y$ in T , then $b \notin \{x, y, u, z\}$. We proceed exactly as in Case *iv.7*.

Case 9: $f = y$. Clearly, y gains no new first out-neighbor. We prove that it gains no new second out-neighbor. Suppose that $y \rightarrow a \rightarrow b \rightarrow y$ in T' . Then $(y, a) \in E(D)$ and $a \notin K$. Since $y \rightarrow u \rightarrow x$ in D , then we may assume $b \notin \{x, y, u\}$. We continue exactly as in Case *iv.7*.

Case 10: $f = z$. It is clear that z gains no new vertex as a first out-neighbor. We prove that it gains no new vertex as a second out-neighbor. Suppose that $z \rightarrow a \rightarrow b \rightarrow z$ in T' . Then $(z, a) \in E(D)$ and $a \notin K$. Since $x \rightarrow z$ in D , we may assume that $a \notin K \cup \{x, z\}$.

Subcase a: If $(a, b) \in E(D)$ or it is a convenient orientation w.r.t. D , then $b \in N_D^{++}(z)$.

Subcase b: If $(a, b) \in E(D) - E(D')$ and it is not convenient w.r.t. D , then there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. So either $rs, ab \in E(C)$ or $rs, ab \in E[Y_j, M_j]$ for some j . If $ab \in E(C)$, then $(a, b) = (x, y)$ or $(a, b) = (z, u)$ and hence $a \in \{x, z\}$, a contradiction. Thus $rs, ab \in E[Y_j, M_j]$ for some j . Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. So by definition of G , fs is not a missing edge of D . But $(f, a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Subcase c: If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation w.r.t. D' . Since $(f, a) \in E(D)$ and so in D' , there exists v' such that $f \rightarrow v' \rightarrow b$ in D' . Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C \cup k$. Since $v' \notin C$ and $(f, v') \in E(D')$, then $(f, v') \in E(D)$. But this is already treated in Subcase *iv.10.a* and Subcase *iv.10.b*.

Case 11: $f \in V(G) - (Y \cup M \cup C) = A \cup (X - X_1) = A \cup (X - Y_1)$. Exactly same as Case *i.6*, with only one difference in Subcase *b*. The difference is that in Subcase *iv.11.b* there are two possibilities for the edges rs, ab : Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs, ab \in E(C)$. The first case is treated in Subcase *i.6.b*. However, for the second case, f must belong to A since otherwise fa is a missing edge because $G[X \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$. Thus fs is not missing edge of D because $E[A, C] = \phi$ by definition of G . Since $f \rightarrow a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Therefore, f satisfies the SNP in D .

Case *v*: Suppose that $\Delta[E(C)]$ has exactly three arcs, say $wv \rightarrow xy \rightarrow zu \rightarrow vx$.

Then $u \rightarrow x \rightarrow z \rightarrow v \rightarrow y$ in D and uv, yz are good missing edges. Assume without loss of generality that (u, v) is a convenient orientation w.r.t. D . Add the arcs (u, v) , (x, y) , (z, u) and (v, x) to D . Assign to yz a convenient orientation and add it to D . The obtained oriented graph D' is missing $G' = G - (\cup E[Y_j, M_j] \cup E(C))$ which is a threshold graph. So all the missing edges of D' are good. We assign to them a convenient orientation and we add them to get a tournament T . Let L be a local median order of T and let f denote its feed vertex. Reorient all the missing edges incident to f towards f except those whose out-degree in Δ is not zero. The same order L is again a local median order of the obtained tournament T' and f has the SNP in T' . We will prove that f has the SNP in D . We have the following cases.

Case 1: f is a whole vertex. This is the same as Case *i.1*.

Case 2: $\exists 1 \leq t \leq l$ such that $f \in M_t$. Exactly same as Case *i.2*, with only one difference in the unsteady subcases, that is, in the subcases where $f \rightarrow a \rightarrow b$ with $(f, a) \in E(D)$ and $(a, b) \in E(D') - E(D)$ and it is not convenient w.r.t. D . As usual, since (a, b) is not convenient w.r.t. D , then there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. The difference is that in the unsteady subcases of Case *v.2* either $rs, ab \in E[Y_j, M_j]$ for some j or $rs, ab \in E(C)$. If $rs, ab \in E[Y_j, M_j]$ for some j , we proceed exactly in the same way as in the unsteady subcases of Case *i.2*. Else if $rs, ab \in E(C)$ (the possible cases are: $(r, s) = (u, v)$ and $(a, b) = (x, y)$ or $(r, s) = (x, y)$ and $(a, b) = (z, u)$ or $(r, s) = (z, u)$ and $(a, b) = (v, x)$), then fs is not a missing edge because $E[M, C] = \phi$. Since $f \rightarrow a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Case 3: $\exists 1 \leq t \leq l$ such that $f \in Y_t$ and $M_t \neq \phi$. Exactly same as Case *i.3*, with only one difference in the subcases when $f \rightarrow a \rightarrow b$ with $(f, a) \in E(D)$ and $(a, b) \in E(D') - E(D)$ and it is not convenient w.r.t. D . As usual, since (a, b) is not convenient w.r.t. D , then there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. The difference is that in the unsteady subcases of Case *v.3* there are two cases to be consider: Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs, ab \in E(C)$. If $rs, ab \in E[Y_j, M_j]$ for some j , we proceed exactly in the same way as in the unsteady subcases of Case *i.3*. Else if $rs, ab \in E(C)$, then fa is a missing edge because $G[Y \cup C]$ is a complete split graph, a contradiction. Thus this case does not exist.

Case 4: $\exists 1 \leq t \leq l + 1$ such that $f \in Y_t$ such that $M_t = \phi$. Exactly same as Case *i.4*, with only one difference in Subcase *b*. The difference is that in Subcase *v.4.b* there are two possibilities for the edges rs, ab : Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs, ab \in E(C)$. The first case is treated in Subcase *i.4.b*. However, the second case does not exist since otherwise fa is a missing edge because $G[Y \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$. This means that this case does not exist.

Case 5: $f \in Y_{l+2}$. Exactly same as Case *i.4*, with two differences in Subcase *b*. The first difference is that in Subcase *v.5.b* there are two possibilities for the edges rs, ab : Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs, ab \in E(C)$. The first case is already treated in Subcase *i.4.b*. However, the second case does not exist since otherwise fa is a missing edge because $G[Y \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$. The second difference is that fs is not a missing edge in Subcase *v.5.b*, because $E[Y_{l+2}, M_j] = \phi$ by definition of G , while in Subcase *i.4.b* we had to prove it.

Case 6: $f = u$. Exactly same as Case *iv.6*.

Case 7: $f \in \{x, z\}$. Similar to the case $f = u$, that is to Case *v.6*.

Case 8: $f = y$. Exactly same as Case *iii.9* with only one difference in Subcase *b*. The difference is that in Subcase *v.8.b* when $ab \in E(C)$ then (a, b) can be either (x, y) or (z, u) or (v, x) and so $a \in \{x, z, v\}$, a contradiction.

Case 9: $f = v$. Similar to the case $f = y$, that is to Case *v.8*.

Case 10: $f \in V(G) - (Y \cup M \cup C) = A \cup (X - X_1) = A \cup (X - Y_1)$. Exactly same as Case *i.6*, with only one difference in Subcase *b*. The differ-

ence is that in Subcase *v.10.b* there are two possibilities for the edges rs, ab : Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs, ab \in E(C)$. The first case is treated in Subcase *i.6.b*. However, for the second case, f must belong to A since otherwise fa is a missing edge because $G[X \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$. Thus fs is not missing edge of D because $E[A, C] = \phi$ by definition of G . Since $f \rightarrow a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Therefore, in view of all above observations, f satisfies the SNP in D .

Case *vi*: Suppose that $\Delta[E(C)]$ has exactly three arcs, say $uv \rightarrow xy \rightarrow zu$ and $xv \rightarrow zy$.

Then $u \rightarrow x \rightarrow z \rightarrow v \rightarrow y \rightarrow u$ in D and uv and xv are good missing edges. Assume without loss of generality that (u, v) is a convenient orientation of uv . Add (u, v) , (x, y) and (z, u) to D . If (x, v) is a convenient orientation, then add (x, v) and (z, y) to D , otherwise add (v, x) and (y, z) to D . The obtained oriented graph D' is missing $G' = G - (\cup E[Y_j, M_j] \cup E(C))$ which is a threshold graph. So all the missing edges of D' are good. We assign to them a convenient orientation and we add them to get a tournament T . Let L be a local median order of T and let f denote its feed vertex. Reorient all the missing edges incident to f towards f except those whose out-degree in Δ is not zero. The same order L is again a local median order of the obtained tournament T' and f has the SNP in T' . We will prove that f has the SNP in D . We have the following cases.

Case 1: f is a whole vertex. This is the same as Case *i.1*.

Case 2: $\exists 1 \leq t \leq l$ such that $f \in M_t$. Exactly same as Case *i.2*, with only one difference in the unsteady subcases, that is, in the subcases where $f \rightarrow a \rightarrow b$ with $(f, a) \in E(D)$ and $(a, b) \in E(D') - E(D)$ and it is not convenient w.r.t. D . As usual, since (a, b) is not convenient w.r.t. D , then there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. The difference is that in the unsteady subcases of Case *vi.2* either $rs, ab \in E[Y_j, M_j]$ for some j or $rs, ab \in E(C)$. If $rs, ab \in E[Y_j, M_j]$ for some j , we proceed exactly in the same way as in the unsteady subcases of Case *i.2*. Else if $rs, ab \in E(C)$ (the possible cases are: $(r, s) = (u, v)$ and $(a, b) = (x, y)$, $(r, s) = (x, y)$ and $(a, b) = (z, u)$, $(r, s) = (v, x)$ and $(a, b) = (y, z)$ if (v, x) is a convenient orientation of vx or $(r, s) = (x, v)$ and $(a, b) = (z, y)$ if (x, v) is a convenient orientation of vx), then fs is not a missing edge because $E[M, C] = \phi$. Since $f \rightarrow a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Case 3: $\exists 1 \leq t \leq l$ such that $f \in Y_t$ and $M_t \neq \phi$. Exactly same as Case *i.3*, with only one difference in the subcases when $f \rightarrow a \rightarrow b$ with $(f, a) \in E(D)$ and $(a, b) \in E(D') - E(D)$ and it is not convenient w.r.t. D . As usual, since (a, b) is not convenient w.r.t. D , then there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. The difference is that in the unsteady subcases of Case *vi.3* there are two cases to be consider: Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs, ab \in E(C)$. If $rs, ab \in E[Y_j, M_j]$ for some j , we proceed exactly in the same way as in the unsteady subcases of Case *i.3*. Else if $rs, ab \in E(C)$, then fa is a missing edge because $G[Y \cup C]$ is a complete split graph, a contradiction. Thus this case does not exist.

Case 4: $\exists 1 \leq t \leq l + 1$ such that $f \in Y_t$ such that $M_t = \phi$. Exactly same as Case *i.4*, with only one difference in Subcase *b*. The difference is that in Subcase *vi.4.b* there are two possibilities for the edges rs, ab : Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs, ab \in E(C)$. The first case is treated in Subcase *i.4.b*. However, the second case does not exist since otherwise fa is a missing edge because $G[Y \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$. This means that this case does not exist.

Case 5: $f \in Y_{l+2}$. Exactly same as Case *i.4*, with two differences in Subcase *b*. The first difference is that in Subcase *vi.5.b* there are two possibilities for the edges rs, ab : Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs, ab \in E(C)$. The first case is already treated in Subcase *i.4.b*. However, the second case does not exist since otherwise fa is a missing edge because $G[Y \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$. The second difference is that fs is not a missing edge in Subcase *vi.5.b*, because $E[Y_{l+2}, M_j] = \phi$ by definition of G , while in Subcase *i.4.b* we had to prove it.

Case 6: $f = u$. Exactly same as Case *iv.6*.

Case 7: $f = y$. Exactly same as Case *iii.9* with only one difference in Subcase *b*. The difference is that in Subcase *vi.7.b* if $ab \in E(C)$ then (a, b) can be either $(x, y), (z, u), (y, z)$ or (z, y) and so $a \in \{x, y, z\}$, a contradiction.

Case 8: $f = z$. Exactly same as Case *vi.7* with difference that yz and uz are reoriented so that $(y, z) \in E(T')$ and $(u, z) \in E(T')$, respectively.

Case 9: $f = v$. Exactly same as Case *iii.7*, with only one difference: In Subcase *vi.9.2.b* if $ab \in E(C)$ then (a, b) can be either $(x, y), (z, u)$ or (z, y) and so $a \in \{x, z\}$, a contradiction because $z \rightarrow v$ in D and $x \rightarrow v$ in D' while

$v \rightarrow a$ in D .

Case 10: $f = x$. Here we consider two main cases:

Case 10.1: Assume $(x, v) \in E(D')$. Clearly, x gains only v and y as new first out-neighbors. However, x loses v as a second out-neighbor and gains u as a new second out-neighbor. We prove that x gains only u as a new second out-neighbor.

Subcase a: Suppose that $x \rightarrow a \rightarrow b \rightarrow x$ in T' , with $a \neq y$, $a \neq v$ and $b \neq u$. Then $(a, b) \in E(T)$, $(x, a) \in E(D)$ and thus $a \notin K$. Since $x \rightarrow z \rightarrow v$ in D and $x \rightarrow y$ in T , then we may assume that $b \notin C$.

Subcase a.1: If $(a, b) \in E(D)$, $b \in N_D^{++}(x)$.

Subcase a.2: If $(a, b) \in E(D') - E(D)$, then either $ab \in E(C)$ or $ab \in E[Y_j, M_j]$ for some j . If $ab \in E(C)$, then $b \in C$, a contradiction. Thus $ab \in E[Y_j, M_j]$ for some j . It follows that either (a, b) is a convenient orientation w.r.t. D and hence $b \in N_D^{++}(x)$ or there is $rs \rightarrow ab$ in Δ , namely $s \rightarrow b$ and $a \notin N_D^{++}(s)$. Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. So by definition of G , xs is not a missing edge of D . But $(x, a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(x)$.

Subcase a.3: If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation w.r.t. D' . Since $x \rightarrow a$ in D and so in D' , there exists v' such that $x \rightarrow v' \rightarrow b$ in D' . Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(x, v') \in E(D')$, then $(x, v') \in E(D)$. But this is already treated in Subcase *vi.10.1.a.1* and Subcase *vi.10.1.a.2*.

Subcase b: Suppose that $x \rightarrow y \rightarrow b \rightarrow x$ in T' , with $b \neq u$. Since $x \rightarrow z \rightarrow v$ in D , then we may assume that $b \notin C$ and hence $(y, b) \notin E(D') - E(D)$.

Subcase b.1: If $(y, b) \in E(D)$, then either $b \in S$ or b is a whole vertex. Then by the losing relation $xy \rightarrow zu$, we get $b \in N_D^{++}(x)$.

Subcase b.2: If $(y, b) \in E(T) - E(D')$, then $b \in K$ and (y, b) is a convenient orientation w.r.t. D' . But this is exactly the same as Subcase *vi.10.1.a.1* and Subcase *vi.10.1.a.2*.

Subcase c: Suppose that $x \rightarrow v \rightarrow b \rightarrow x$ in T' , with $b \neq u$. Since $x \rightarrow z \rightarrow v$ in D and $x \rightarrow y$ in T , then we may assume that $b \notin C$. We pro-

ceed similarly to Subcase *vi.10.1.b* by replacing the losing relation $xy \rightarrow zu$ in Subcase *vi.10.1.b* by the losing relation $xv \rightarrow zy$ in Subcase *vi.10.1.c.1*.

Case 10.2: Assume $(v, x) \in E(D')$. Clearly, x gains only y as a new first out-neighbor and u as a new second out-neighbor. We prove it gains only u as a new second out-neighbor.

Subcase a: Suppose that $x \rightarrow y \rightarrow b \rightarrow x$ in T' with $b \neq u$. Then $(y, b) \in E(T)$. Since $x \rightarrow z \rightarrow V$ in D , then we may assume $b \notin C$ and hence $(y, b) \notin E(D') - E(D)$. This is exactly as Subcase *vi.10.1.b*.

Subcase b: Suppose that $x \rightarrow a \rightarrow b \rightarrow x$ in T' with $a \neq y$ and $b \neq u$. Then $(a, b) \in E(T)$, $(x, a) \in E(D)$ and thus $a \notin K$. Since $x \rightarrow z \rightarrow v$ in D and $x \rightarrow y$ in T , then $b \notin C$. We proceed exactly as in Case *vi.10.1.a*.

Case 11: $f \in V(G) - (Y \cup M \cup C) = A \cup (X - X_1) = A \cup (X - Y_1)$. Exactly same as Case *i.6*, with only one difference in Subcase *b*. The difference is that in Subcase *vi.11.b* there are two possibilities for the edges rs, ab : Either $rs, ab \in E[Y_j, M_j]$ for some j , or $rs, ab \in E(C)$. The first case is treated in Subcase *i.6.b*. However, for the second case, f must belong to A since otherwise fa is a missing edge because $G[X \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$. Thus fs is not missing edge of D because $E[A, C] = \phi$ by definition of G . Since $f \rightarrow a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 3.2 we get $b \in N_D^{++}(f)$.

Therefore, all what precede prove that f has the SNP in D . This completes the proof. □

As immediate consequences of the previous theorem, we may conclude the following:

Corollary 4.1. *Every oriented graph missing a generalized comb satisfies the SNC.*

Corollary 4.2. *(Ghazal [4]) Every oriented graph missing a comb satisfies the SNC.*

Corollary 4.3. *(Ghazal [6]) Every oriented graph missing a threshold graph satisfies the SNC.*

Since threshold graphs, C_5 , generalized combs and $\{C_4, \overline{C_4}, S_3, \text{chair and co-chair}\}$ -free graphs are in $\mathcal{F}(\vec{\mathcal{P}})$ and any oriented graph missing one of the graphs mentioned before satisfies the SNC, we end this article by wondering the following:

Problem 2. *Does every oriented graph missing a graph in $\mathcal{F}(\vec{\mathcal{P}})$ satisfies SNC?*

References

- [1] N. Dean and B. J. Latka, *Squaring the tournament: an open problem*, Congress Numerantium 109 (1995), 73-80.
- [2] D. Fisher, *Squaring a tournament: a proof of Dean's conjecture*, J. Graph Theory 23 (1996), 43-48.
- [3] D. Fidler and R. Yuster, *Remarks on the second neighborhood problem*, J. Graph Theory 55 (2007), 208-220.
- [4] S. Ghazal, *A contribution to the second neighborhood problem*, Graphs and Combinatorics, **29** (2013), 1365–1375.
- [5] S. Ghazal, *A remark on the second neighborhood conjecture*. Electronic Journal of Graph Theory and Applications 3 (2) (2015), 182-190.
- [6] S. Ghazal, *Seymour's second neighborhood conjecture in tournaments missing a generalized star*, J. Graph Theory, **71** (2012), 89-94.
- [7] S. Ghazal, *The structure of graphs with forbidden induced C_4 , \overline{C}_4 , C_5 , S_3 , chair and co-chair* Electronic Journal of Graph Theory and Applications 3 (2) (2018), 182-190.
- [8] F. Havet and S. Thomassé, *Median Orders of Tournaments: A Tool for the Second Neighborhood Problem and Sumner's Conjecture*, J. Graph Theory 35 (2000), 244-256.
- [9] P.L. Hammer, V. Chvátal, *Aggregation of inequalities in integer programming*, Annals of Discrete Mathematics 1 (1977), 145-162.
- [10] *Set-packing and threshold graphs*, Set-packing and threshold graphs, Research Report, Comp. Sci. Dept. University of Waterloo, Canada CORR 73-21 (1973). available on arXiv <https://arxiv.org/abs/1607.00686>