



**HAL**  
open science

# Homotopy Transfer Theorem and minimal models for pre-Calabi-Yau algebras

Marion Boucrot

► **To cite this version:**

Marion Boucrot. Homotopy Transfer Theorem and minimal models for pre-Calabi-Yau algebras. 2023.  
hal-04135543v1

**HAL Id: hal-04135543**

**<https://hal.science/hal-04135543v1>**

Preprint submitted on 21 Jun 2023 (v1), last revised 5 Jul 2023 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Homotopy Transfer Theorem and minimal models for pre-Calabi-Yau algebras

Marion Boucrot

---

## Abstract

In this article we prove that given quasi-isomorphic dg vector spaces and a pre-Calabi-Yau structure on one of them there exists a pre-Calabi-Yau structure on the other as well as a pre-Calabi-Yau morphism between the two. This extends the Homotopy Transfer Theorem proved by T. Kadeishvili and a similar result of D. Petersen. Moreover, we show that a pre-Calabi-Yau morphism whose first component is an isomorphism of dg vector spaces is a pre-Calabi-Yau isomorphism, *i.e.* the inverse of the isomorphism of dg vector spaces extends to an inverse of the pre-Calabi-Yau morphism. We incidentally show that any pre-Calabi-Yau algebra has a minimal model. Finally, we prove that any quasi-isomorphism of pre-Calabi-Yau algebras admits a quasi-inverse.

**Mathematics subject classification 2020:** 16E45, 18G70, 14A22

**Keywords:**  $A_\infty$ -algebras, pre-Calabi-Yau algebras, homotopy transfer, minimal models

---

## 1 Introduction

Pre-Calabi-Yau algebras were introduced by M. Kontsevich and Y. Vlassopoulos in the last decade. In the finite dimensional case, these structures have also appeared in [9] under the name of  $V_\infty$ -algebras and P. Seidel called them  $A_\infty$ -algebras with boundary in [6]. They also appear under the name of weak Calabi-Yau structures in [3]. These references show that pre-Calabi-Yau structures play an important role in homological algebra, symplectic geometry, string topology, noncommutative geometry and even in Topological Quantum Field Theory.

The notion of pre-Calabi-Yau algebras is strongly related to the notion  $A_\infty$ -algebras. Actually, for  $d \in \mathbb{Z}$ , a  $d$ -pre-Calabi-Yau structure on a finite dimensional graded vector space  $A$  is equivalent to a cyclic  $A_\infty$ -structure on  $A \oplus A^*[d-1]$  that restricts to  $A$ , where  $A^*$  denotes the graded dual of  $A$ .

In 1980, T. Kadeishvili proved in [2] the following theorem:

**Theorem 1.1.** (*Homotopy Transfer Theorem*) *For any dg algebra  $C$  such that  $H_i(C)$  is free for  $i \geq 0$ , there exists an  $A_\infty$ -structure on  $H(C)$  with vanishing differential and an  $A_\infty$ -morphism  $f : H(C) \rightarrow C$  such that  $f^1$  is a quasi-isomorphism.*

In 2020, D. Petersen published in [5] a proof of the following theorem, which is some kind of Homotopy Transfer Theorem:

**Theorem 1.2.** *Let  $R$  be a ring,  $(V, d_V)$  and  $(W, d_W)$  dg  $R$ -modules and  $f : V \rightarrow W$  a chain map. Let  $\nu$  be a square-zero coderivation of  $C(W)$  of degree  $-1$  whose arity 1 term is  $d_W$ . Assume that  $f$  induces a quasi-isomorphism  $\text{Hom}_R(C(V), V) \rightarrow \text{Hom}_R(C(W), W)$ . Then, there exists noncanonically a square-zero coderivation  $\mu$  of  $C(V)$  of degree  $-1$  whose arity 1 term is  $d_V$  and a morphism  $F : C(V) \rightarrow C(W)$  of coalgebras whose arity 1 term is  $f$  and which is a chain map with respect to  $\mu$  and  $\nu$ .*

Using the bar construction relating  $A_\infty$ -algebras and coalgebras, the latter gives rise to the following theorem:

**Theorem 1.3.** *Consider dg vector spaces  $(V, d_V)$  and  $(W, d_W)$  and  $f : (V, d_V) \rightarrow (W, d_W)$  a quasi-isomorphism. Let  $m_W$  be an  $A_\infty$ -structure on  $W$  with  $m_W^1 = d_W[1]$ . Then, there exists an  $A_\infty$ -structure  $m_V$  on  $V$  such that  $m_V^1 = d_V[1]$  and an  $A_\infty$ -morphism  $F : (V, m_V) \rightarrow (W, d_W)$  with  $F^1 = f[1]$ .*

We also have the statement in the other direction, namely, if  $V$  is endowed with an  $A_\infty$ -structure it is possible to construct such a structure on  $W$ .

We prove in this paper similar results for pre-Calabi-Yau algebras. Let us briefly present the contents of this paper. Section 2 is devoted to present the conventions we follow and notations we will use throughout this paper. In section 3 we recall the notions of  $A_\infty$ -algebras and  $A_\infty$ -morphisms. We also recall well-known results including the ones appearing in [5], giving proofs entirely in terms of the structure of the  $A_\infty$ -algebra, instead of the associated bar constructions, as it is the case in the mentioned article. This is done in order to allow the comparison with the case of pre-Calabi-Yau algebras since the latter do not have any bar construction. We do not claim any originality in this regards. Section 4 is the core of the article. After recalling the definitions of  $d$ -pre-Calabi-Yau algebras given in [4], we prove that a  $d$ -pre-Calabi-Yau morphism whose first component is an isomorphism is an isomorphism in the category of  $d$ -pre-Calabi-Yau algebras (see Lemma 4.9). Then, we prove theorems 4.13 and 4.14 by generalizing the proofs of the case of  $A_\infty$ -algebras. Those theorems state that given dg vector spaces and a quasi-isomorphism between them, if one carries a  $d$ -pre-Calabi-Yau structure then we can construct such a structure on the other, as well as a  $d$ -pre-Calabi-Yau morphism between the two. We deduce that there exists minimal models for  $d$ -pre-Calabi-Yau algebras. Finally, using Lemma 4.9 and theorems 4.13 and 4.14, we show that a quasi-isomorphism of  $d$ -pre-Calabi-Yau algebras has a quasi-inverse (see Theorem 4.18).

**Acknowledgements.** This work is part of a PhD thesis supervised by Estanislao Herscovich and Hossein Abbaspour. The author thank them for the useful advices and comments about this paper. This work is supported by the French National Research Agency in the framework of the "France 2030" program (ANR-15-IDEX-0002) and by the LabEx PERSYVAL-Lab (ANR-11-LABX-0025-01).

## 2 Notations and conventions

In what follows  $\mathbb{k}$  will be a field of characteristic 0 and to simplify we will denote  $\otimes$  for  $\otimes_{\mathbb{k}}$ . We will denote by  $\mathbb{N} = \{0, 1, 2, \dots\}$  the set of natural numbers and define  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ . For  $i, j \in \mathbb{N}$ , we define the interval of integers  $\llbracket i, j \rrbracket = \{n \in \mathbb{N} \mid i \leq n \leq j\}$ . We also define  $\mathcal{T} = \bigsqcup_{k>0} \mathcal{T}_k$  where  $\mathcal{T}_k = \mathbb{N}^k$  for  $k > 1$  and  $\mathcal{T}_1 = \mathbb{N}^*$  and given  $\bar{n} = (n_1, \dots, n_k) \in \mathcal{T}_k$ , we denote  $N(\bar{n}) = n_1 + \dots + n_k$ .

Given a (cohomologically) graded vector space  $V = \bigoplus_{i \in \mathbb{Z}} V^i$ , we define the graded vector space  $V[n]$  given by  $V[n]^i = V^{n+i}$  for  $i \in \mathbb{Z}$  and consider, for  $n \in \mathbb{Z}$ , the map  $s_{V,n} : V \rightarrow V[n]$  whose

underlying set theoretic map is the identity. We will denote  $s_{V,n}$  simply by  $s_n$  when there is no possible confusion, and  $s_1$  just by  $s$ .

We now recall the Koszul sign rules, that are the ones we use to determine the signs appearing in this paper. If  $V, W$  are graded vector spaces, we have a map  $\tau_{V,W} : V \otimes W \rightarrow W \otimes V$  defined as

$$\tau_{V,W}(v \otimes w) = (-1)^{|w||v|} w \otimes v$$

where  $v \in V$  is a homogeneous element of degree  $|v|$  and  $w \in W$  is a homogeneous element of degree  $|w|$ . More generally, given graded vector spaces  $V_1, \dots, V_n$  and  $\sigma \in \mathcal{S}_n$ , we have a map

$$\tau_{V_1, \dots, V_n}^\sigma : V_1 \otimes \dots \otimes V_n \rightarrow V_{\sigma^{-1}(1)} \otimes \dots \otimes V_{\sigma^{-1}(n)}$$

sending an element  $v_1 \otimes \dots \otimes v_n$  to  $(-1)^\epsilon (v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(n)})$  with

$$\epsilon = \sum_{\substack{i>j \\ \sigma^{-1}(i) < \sigma^{-1}(j)}} |v_{\sigma^{-1}(i)}| |v_{\sigma^{-1}(j)}|$$

where  $v_i \in V_i$  is a homogeneous element of degree  $|v_i|$  for  $i \in \llbracket 1, n \rrbracket$ .

All the products in this paper will be products in the category of graded vector spaces. Given graded vector spaces  $(V_i)_{i \in I}$ , we thus have  $\prod_{i \in I} V_i = \bigoplus_{n \in \mathbb{Z}} \prod_{i \in I} V_i^n$  where the second product is the usual product of vector spaces.

Given two graded vector spaces  $V, W$  we denote by  $\text{Hom}_{\mathbb{k}}(V, W)$  the vector space of  $\mathbb{k}$ -linear maps  $f : V \rightarrow W$  and by  $\text{hom}_{\mathbb{k}}^d(V, W)$  the vector space of homogeneous  $\mathbb{k}$ -linear maps  $f : V \rightarrow W$  of degree  $d$ , *i.e.*  $f(v) \in W^{n+d}$  for all  $v \in V^n$ . We assemble them in the graded vector space  $\mathcal{H}om_{\mathbb{k}}(V, W) = \bigoplus_{d \in \mathbb{Z}} \text{hom}_{\mathbb{k}}^d(V, W) \subseteq \text{Hom}_{\mathbb{k}}(V, W)$ . We define the **graded dual** of a graded vector space  $V = \bigoplus_{n \in \mathbb{Z}} V^n$  as  $V^* = \mathcal{H}om_{\mathbb{k}}(V, \mathbb{k})$ . Moreover, given graded vector spaces  $V, V', W, W'$  and homogeneous elements  $f \in \mathcal{H}om_{\mathbb{k}}(V, V')$  and  $g \in \mathcal{H}om_{\mathbb{k}}(W, W')$ , we have that

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w)$$

for homogeneous elements  $v \in V$  and  $w \in W$ . Throughout this paper, when we consider an element  $v$  of degree  $|v|$  in a graded vector space  $V$ , we mean a homogeneous element  $v$  of  $V$ . Also, we will denote by  $\text{id}$  the identity map of every space of morphisms, without specifying it. Recall that given a dg vector space  $(A, d_A)$  its **cohomology** is the graded vector space

$$H(A) = \bigoplus_{n \in \mathbb{Z}} H^n(A) = \bigoplus_{n \in \mathbb{Z}} \ker(d_A^n) / \text{im}(d_A^{n-1})$$

In this paper, we will often make the use of the diagrammatic calculus introduced in [4]. We refer the reader to [1] for a detailed account of the involved definitions and terminology, which we shall follow in this article. Using the convention of that reference, we remark that, for this article, the type of a diagram will be just a tuple of integers instead of a tuple of objects, since we work with pre-Calabi-Yau algebras instead of pre-Calabi-Yau categories. Moreover, given a diagram, we will often omit the corresponding bold arrow, *i.e.* the bold arrow can be any of the outgoing arrows of the diagram.

### 3 Transferring $A_\infty$ -structures

This section is devoted to recall the notions of  $A_\infty$ -algebras and  $A_\infty$ -morphisms. For convenience of the reader, we also present the results appearing in [5], and we give proofs entirely in terms of the structure of the  $A_\infty$ -algebra, instead of the associated bar constructions, as it is the case in [5]. We do not claim any originality in this regards. This is merely done in order to allow the comparison with the case of pre-Calabi-Yau algebras appearing in the next section, since the latter do not have any bar construction.

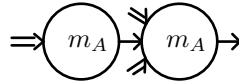
**Definition 3.1.** Given graded vector spaces  $A$  and  $B$ , we define the graded vector space  $C(A, B)$  as  $C(A, B) = \prod_{n \in \mathbb{N}^*} C^n(A, B)$  where  $C^n(A, B) = \mathcal{H}om_{\mathbb{k}}(A[1]^{\otimes n}, B[1])$ . We will denote  $C(A, A)$  simply by  $C(A)$ .

**Remark 3.2.** If  $(A, d_A)$  and  $(B, d_B)$  are dg vector spaces,  $C(A, B)$  becomes a dg vector space whose differential is given by  $d_{C(A, B)}F = d_{B[1]} \circ F - (-1)^{|F|} F \circ d_{A[1]^{\otimes n}}$  for a homogeneous element  $F \in C^n(A, B)$ .

**Definition 3.3.** An  $A_\infty$ -**structure** on a graded vector space  $A$  is an element  $m_A \in C(A)$  of degree 1 satisfying the Stasheff identities introduced by J. Stasheff in [7] and given by

$$\sum_{\substack{r+s+t=n \\ s>0}} m_A^{r+1+t} \circ (\text{id}^{\otimes r} \otimes m_A^s \otimes \text{id}^{\otimes t}) = 0 \quad (SI^n)$$

for each  $n \in \mathbb{N}^*$ . This is tantamount to  $[m_A, m_A]_G = 0$  where  $[-, -]_G$  denotes the Gerstenhaber bracket. In terms of diagrams, this reads  $\sum \mathcal{E}(\mathcal{D}) = 0$  where the sum is over all the filled diagrams  $\mathcal{D}$  of type  $n \in \mathbb{N}^*$  and of the form



A graded vector space endowed with an  $A_\infty$ -structure is called an  $A_\infty$ -**algebra**.

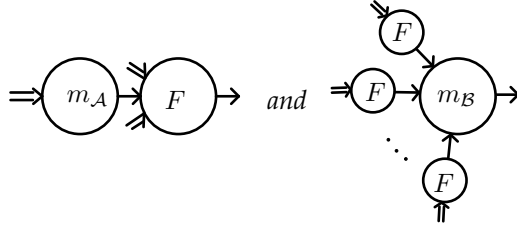
**Remark 3.4.** Note that the identity  $(SI^1)$  is simply  $m_A^1 \circ m_A^1 = 0$ , meaning that an  $A_\infty$ -algebra  $(A, m_A)$  gives rise to a dg vector space  $(A, m_A^1[-1])$ , whose cohomology will be denoted by  $H(A)$ .

The following definition is due to M. Sugawara (see [8]).

**Definition 3.5.** Let  $(A, m_A)$ ,  $(B, m_B)$  be two  $A_\infty$ -algebras. An  $A_\infty$ -**morphism**  $F$  from  $A$  to  $B$  is an element  $F \in C(A, B)$  of degree 0 satisfying the identities

$$\sum_{\substack{r+s+t=n \\ s>0}} F^{r+1+t} \circ (\text{id}^{\otimes r} \otimes m_A^s \otimes \text{id}^{\otimes t}) = \sum_{r_1+\dots+r_i=n} m_B^i \circ (F^{r_1} \otimes \dots \otimes F^{r_i}) \quad (MI^n)$$

for each  $n \in \mathbb{N}^*$ . This tantamount to  $F \circ_G m_A - m_B \circ_M F = 0$  where  $\circ_G$  denotes the Gerstenhaber product and  $\circ_M$  denotes the composition appearing on the right hand side of  $(MI^n)$ . In terms of diagrams, this reads  $\sum \mathcal{E}(\mathcal{D}) = \sum \mathcal{E}(\mathcal{D}')$  where the sums are over all the filled diagrams  $\mathcal{D}$  and  $\mathcal{D}'$  of type  $n \in \mathbb{N}^*$  and of the form



respectively.

**Definition 3.6.** Given  $A_\infty$ -algebras  $(A, m_A)$ ,  $(B, m_B)$ ,  $(C, m_C)$  and  $A_\infty$ -morphisms  $F : A \rightarrow B$  and  $G : B \rightarrow C$ , the **composition of F and G** is the element  $(G \circ F) \in C(A, C)$  defined by

$$(G \circ F)^n = \sum_{r_1 + \dots + r_i = n} G^i \circ (F^{r_1} \otimes \dots \otimes F^{r_i})$$

for every  $n \in \mathbb{N}^*$ .

**Lemma 3.7.** The data of  $A_\infty$ -algebras together with  $A_\infty$ -morphisms and their composition give rise to a category denoted  $A_\infty$ . Moreover, given an  $A_\infty$ -algebra  $(A, m_A)$ , the  **$A_\infty$ -identity morphism on A** is the one whose  $n$ -th component  $(\text{id}_A)^n : A[1]^{\otimes n} \rightarrow A[1]$  is  $\text{id}_{A[1]}$  if  $n = 1$  and 0 if  $n > 1$ . It will be denoted  $\text{id}_A$  or simply by  $\text{id}$  when there is no possible confusion.

**Definition 3.8.** Given  $A_\infty$ -algebras  $(A, m_A)$  and  $(B, m_B)$ , an  $A_\infty$ -morphism  $F : A \rightarrow B$  is a **quasi-isomorphism** if  $F^1 : (A[1], m_A^1) \rightarrow (B[1], m_B^1)$  is a quasi-isomorphism of dg vector spaces. Moreover,  $F$  is an **isomorphism** if there exists an  $A_\infty$ -morphism  $G : B \rightarrow A$  such that  $G \circ F = \text{id}_A$  and  $F \circ G = \text{id}_B$ .

**Lemma 3.9.** Let  $(A, m_A)$ ,  $(B, m_B)$  be  $A_\infty$ -algebras and  $F : A \rightarrow B$  be an  $A_\infty$ -morphism. Suppose that  $F^1 : (A[1], m_A^1) \rightarrow (B[1], m_B^1)$  is an isomorphism of dg vector spaces. Then,  $F$  is an isomorphism of  $A_\infty$ -algebras.

*Proof.* Since  $F^1 : (A[1], m_A^1) \rightarrow (B[1], m_B^1)$  is an isomorphism of dg vector spaces, it has an inverse  $G^1 : (B[1], m_B^1) \rightarrow (A[1], m_A^1)$  which is a chain map. To simplify, we will denote  $F^1 = f$  and  $G^1 = g$ . Consider  $n > 1$  and suppose that we have constructed maps  $G^i : B[1]^{\otimes i} \rightarrow A[1]$  satisfying that  $(G \circ F)^i = 0$  for  $i < n$ . We have that

$$(G \circ F)^n = G^n \circ f^{\otimes n} + \sum_{\substack{r_1 + \dots + r_i = n \\ i < n}} G^i \circ (F^{r_1} \otimes \dots \otimes F^{r_i})$$

We thus define

$$G^n = - \sum_{\substack{r_1 + \dots + r_i = n \\ i < n}} G^i \circ (F^{r_1} \otimes \dots \otimes F^{r_i}) \circ g^{\otimes n}$$

Therefore, we have constructed a collection  $G$  of maps  $G^i : B[1]^{\otimes i} \rightarrow A[1]$  for each  $i \in \mathbb{N}^*$  such that  $G \circ F = \text{id}_A$ . Note that we did not use the fact that  $F$  is an  $A_\infty$ -morphism. Thus, there exists a collection  $H$  of maps  $H^i : A[1]^{\otimes i} \rightarrow B[1]$  such that  $H \circ G = \text{id}_B$ . Using the associativity of the composition, we get that  $H = F$  and thus  $G \circ F = \text{id}_A$  and  $F \circ G = \text{id}_B$ .

It remains to show that  $G$  is an  $A_\infty$ -morphism. It is clear that  $g$  satisfies the identity  $(MI^1)$ . Consider  $n > 1$  and suppose that  $G$  satisfies the identities  $(MI^i)$  for all  $i < n$ . Since  $F$  is an  $A_\infty$ -morphism, we have that  $F \circ_G m_A = m_B \circ_M F$  and then  $(F \circ_G m_A) \circ_M G = (m_B \circ_M F) \circ_M G$ . Therefore, we

have that  $G \circ_G ((F \circ_G m_A) \circ_M G) = G \circ_G ((m_B \circ_M F) \circ_M G)$ . It is clear that the right member of this equality is  $G \circ_G m_B$  since  $F \circ G = \text{id}$ . We now show that  $G \circ_G ((F \circ_G m_A) \circ_M G) = m_A \circ_M G$ . We have that

$$\begin{aligned}
& G \circ_G ((F \circ_G m_A) \circ_M G)^n \\
&= \sum_{\substack{r+s+t=n \\ s>0}} \sum_{q_1+\dots+q_p=s} \sum_{\substack{u+v+w=p \\ v>0}} G^{r+1+t} \circ (\text{id}^{\otimes r} \otimes (F^{u+1+w} \circ (\text{id}^{\otimes u} \otimes m_A^v \otimes \text{id}^{\otimes w})) \circ (G^{q_1} \otimes \dots \otimes G^{q_p})) \otimes \text{id}^{\otimes t}) \\
&= \sum_{\substack{r+s+t=n \\ s>0}} \sum_{q_1+\dots+q_p=s} \sum_{\substack{u+v+w=p \\ v>0}} G^{r+1+t} \circ (\text{id}^{\otimes r} \otimes F^{u+1+w} \otimes \text{id}^{\otimes t}) \\
&\quad \circ (\text{id}^{\otimes r} \otimes G^{q_1} \otimes \dots \otimes G^{q_u} \otimes (m_A \circ_M G)^{q_{u+1}+\dots+q_{u+v}} \circ G^{q_{u+v+1}} \otimes \dots \otimes G^{q_p} \otimes \text{id}^{\otimes t}) \\
&= (m_A \circ_M G)^n + \sum_{\substack{r+s+t=n \\ s>0}} \sum_{\substack{q_1+\dots+q_p=s \\ q_{u+1}+\dots+q_{u+v}<n}} \sum_{\substack{u+v+w=p \\ v>0}} G^{r+1+t} \circ (\text{id}^{\otimes r} \otimes F^{u+1+w} \otimes \text{id}^{\otimes t}) \\
&\quad \circ (\text{id}^{\otimes r} \otimes G^{q_1} \otimes \dots \otimes G^{q_u} \otimes (G \circ_G m_B)^{q_{u+1}+\dots+q_{u+v}} \circ G^{q_{u+v+1}} \otimes \dots \otimes G^{q_p} \otimes \text{id}^{\otimes t}) \\
&= (m_A \circ_M G)^n + \sum_{\substack{r+s+t=n \\ s>0}} \sum_{\substack{q_1+\dots+q_p=s \\ q_{u+1}+\dots+q_{u+v}<n}} \sum_{\substack{u+v+w=p \\ =x+y+z}} \sum_{q_{u+1}+\dots+q_{u+v}} G^{r+1+t} \circ (\text{id}^{\otimes r} \otimes F^{u+1+w} \otimes \text{id}^{\otimes t}) \\
&\quad \circ (\text{id}^{\otimes r} \otimes G^{q_1} \otimes \dots \otimes G^{q_u} \otimes G^{x+1+z} \circ (\text{id}^{\otimes x} \otimes m_B^y \otimes \text{id}^{\otimes z}) \circ G^{q_{u+v+1}} \otimes \dots \otimes G^{q_p}) \otimes \text{id}^{\otimes t}
\end{aligned}$$

for every  $n \in \mathbb{N}^*$ . Finally, we have that

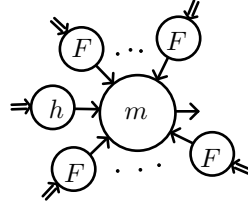
$$\begin{aligned}
& \sum_{\substack{r+s+t=n \\ s>0}} \sum_{\substack{q_1+\dots+q_p=s \\ q_{u+1}+\dots+q_{u+v}<n}} \sum_{\substack{u+v+w=p \\ v>0}} \sum_{\substack{q_{u+1}+\dots+q_{u+v} \\ =x+y+z}} G^{r+1+t} \circ (\text{id}^{\otimes r} \otimes F^{u+1+w} \otimes \text{id}^{\otimes t}) \\
& \quad \circ (\text{id}^{\otimes r} \otimes G^{q_1} \otimes \dots \otimes G^{q_u} \otimes G^{x+1+z} \circ (\text{id}^{\otimes x} \otimes m_B^y \otimes \text{id}^{\otimes z}) \circ G^{q_{u+v+1}} \otimes \dots \otimes G^{q_p}) \otimes \text{id}^{\otimes t} = 0
\end{aligned}$$

since  $F \circ G = \text{id}$ . This finishes the proof.  $\square$

**Definition 3.10.** Let  $A$  and  $B$  be two graded vector spaces and consider  $F \in C(A, B)$ . Given elements  $m \in C(B)$  and  $h \in C(A, B)$  we define the **composition of  $m$  and  $h$  with respect to  $F$**  as the element  $m \circ_F h \in C(A, B)$  given by

$$(m \circ_F h)^n = \sum_{u_1+\dots+u_i=n} \sum_{j=1}^i m^i \circ (F^{u_1} \otimes \dots \otimes F^{u_{j-1}} \otimes h^{u_j} \otimes F^{u_{j+1}} \otimes \dots \otimes F^{u_i})$$

for every  $n \in \mathbb{N}^*$ . In terms of diagrams,  $(m \circ_F h)^n = \sum \mathcal{E}(\mathcal{D})$  where the sum is over all the filled diagrams  $\mathcal{D}$  of type  $n$  and of the form



**Lemma 3.11.** Let  $A$  and  $B$  be two graded vector spaces. Consider  $m_A \in C(A)$  and  $m_B \in C(B)$  of degree 1 and  $F \in C(A, B)$  of degree 0. Then, the following identities hold.

$$m_A \circlearrowleft (m_A \circlearrowleft m_A) = (m_A \circlearrowleft m_A) \circlearrowleft m_A \quad (3.1)$$

$$(m_B \circlearrowleft_M F) \circlearrowleft_G m_A = m_B \circlearrowleft_F (F \circlearrowleft_G m_A) \quad (3.2)$$

*Proof.* We first show (3.1). For every  $n \in \mathbb{N}^*$ , we have that

$$\begin{aligned} ((m_A \circlearrowleft m_A) \circlearrowleft m_A)^n &= \sum_{\substack{r+s+t=n \\ s>0}} (m_A \circlearrowleft m_A)^{r+1+t} \circ (\text{id}^{\otimes r} \otimes m_A^s \otimes \text{id}^{\otimes t}) \\ &= \sum_{\substack{r+s+t=n \\ s>0}} \sum_{\substack{u+v+w=r+1+t \\ v>0}} m_A^{u+1+w} \circ (\text{id}^{\otimes u} \otimes m_A^v \otimes \text{id}^{\otimes w}) \circ (\text{id}^{\otimes r} \otimes m_A^s \otimes \text{id}^{\otimes t}) \\ &= \sum_{\substack{r+s+t=n \\ s>0}} \sum_{\substack{u+v+w=r+1+t \\ v>0 \\ r \geq u+v}} m_A^{u+1+w} \circ (\text{id}^{\otimes u} \otimes m_A^v \otimes \text{id}^{\otimes(r-u-v)} \otimes m_A^s \otimes \text{id}^{\otimes t}) \\ &\quad - \sum_{\substack{r+s+t=n \\ s>0}} \sum_{\substack{u+v+w=r+1+t \\ v>0 \\ r < u}} m_A^{u+1+w} \circ (\text{id}^{\otimes r} \otimes m_A^s \otimes \text{id}^{\otimes(u-r-1)} \otimes m_A^v \otimes \text{id}^{\otimes w}) \\ &\quad + \sum_{\substack{r+s+t=n \\ s>0}} \sum_{\substack{u+v+w=r+1+t \\ v>0 \\ r \geq u \\ r < u+v}} m_A^{u+1+w} \circ (\text{id}^{\otimes u} \otimes m_A^v \circ (\text{id}^{\otimes(r-u)} \otimes m_A^s \otimes \text{id}^{\otimes(t-w)}) \otimes \text{id}^{\otimes w}) \\ &= \sum_{\substack{r+s+t=n \\ s>0}} \sum_{\substack{u+v+w=r+1+t \\ v>0 \\ r \geq u \\ r < u+v}} m_A^{u+1+w} \circ (\text{id}^{\otimes u} \otimes m_A^v \circ (\text{id}^{\otimes(r-u)} \otimes m_A^s \otimes \text{id}^{\otimes(t-w)}) \otimes \text{id}^{\otimes w}) \end{aligned}$$

On the other hand, we have that

$$\begin{aligned} (m_A \circlearrowleft (m_A \circlearrowleft m_A))^n &= \sum_{\substack{r+s+t=n \\ s>0}} m_A^{r+1+t} \circ (\text{id}^{\otimes r} \otimes (m_A \circlearrowleft m_A)^s \otimes \text{id}^{\otimes t}) \\ &= \sum_{\substack{r+s+t=n \\ s>0}} \sum_{\substack{u+v+w=s \\ v>0}} m_A^{r+1+t} \circ (\text{id}^{\otimes r} \otimes (m_A^{u+1+w} \circ (\text{id}^{\otimes u} \otimes m_A^v \otimes \text{id}^{\otimes w}))) \otimes \text{id}^{\otimes t}) \\ &= \sum_{\substack{r+s+t=n \\ s>0}} \sum_{\substack{u+v+w=s \\ v>0}} m_A^{r+1+t} \circ (\text{id}^{\otimes r} \otimes m_A^{u+1+w} \otimes \text{id}^{\otimes t}) \circ (\text{id}^{\otimes r+u} \otimes m_A^v \otimes \text{id}^{\otimes(w+t)}) \end{aligned}$$

Now, we show (3.2). For every  $n \in \mathbb{N}^*$ , we have that

$$((m_B \circlearrowleft_M F) \circlearrowleft_G m_A)^n = \sum_{\substack{r+s+t=n \\ s>0}} (m_B \circlearrowleft_M F)^{r+1+t} \circ (\text{id}^{\otimes r} \otimes m_A^s \otimes \text{id}^{\otimes t})$$



$$\begin{aligned}
&= \sum_{\substack{r+s+t=n \\ s>0}} \sum_{r+1+t=u_1+\dots+u_i} m_B^i \circ (F^{u_1} \otimes \dots \otimes F^{u_i}) \circ (\text{id}^{\otimes r} \otimes m_A^s \otimes \text{id}^{\otimes t}) \\
&= \sum_{\substack{r+s+t=n \\ s>0}} \sum_{\substack{r=u_1+\dots+u_{j-1}+u_j^1 \\ t=u_j^2+\dots+u_i}} m_B^i \circ (F^{u_1} \otimes \dots \otimes F^{u_j} (\text{id}^{\otimes u_j^1} \otimes m_A^s \otimes \text{id}^{\otimes u_j^2}) \otimes \dots \otimes F^{u_i})
\end{aligned}$$

On the other hand, we have that

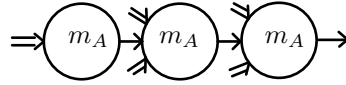
$$\begin{aligned}
(m_B \circ_M (F \circ_G m_A))^n &= \sum_{u_1+\dots+u_i=n} \sum_{j=1}^i m_B^i \circ (F^{u_1} \otimes \dots \otimes (F \circ_G m_A)^{u_j} \otimes \dots \otimes F^{u_i}) \\
&= \sum_{u_1+\dots+u_i=n} \sum_{j=1}^i \sum_{\substack{r+s+t=u_j \\ s \neq 0}} m_B^i \circ (F^{u_1} \otimes \dots \otimes F^{r+1+t} \circ (\text{id}^{\otimes r} \otimes m_A^s \otimes \text{id}^{\otimes t}) \otimes \dots \otimes F^{u_i})
\end{aligned}$$

This finishes the proof.  $\square$

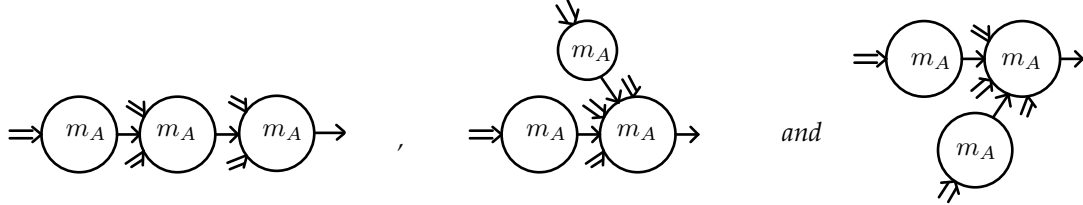
**Remark 3.12.** It is also possible to prove Lemma 3.11 using diagrammatic calculus. For the convenience of the reader, we present the method here since we will prove the results in the case of pre-Calabi-Yau algebras in a similar manner. For every  $n \in \mathbb{N}^*$ , we have that

$$(m_A \circ_G (m_A \circ_G m_A))^n = \sum \mathcal{E}(\mathcal{D})$$

where the sum is over all the filled diagrams  $\mathcal{D}$  of type  $n$  and of the form

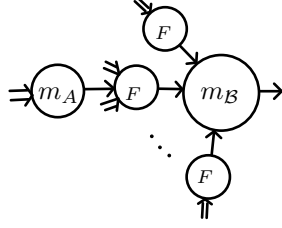


On the other hand,  $((m_A \circ_G m_A) \circ_G m_A)^n = \sum \mathcal{E}(\mathcal{D}_1) + \sum \mathcal{E}(\mathcal{D}_2) - \sum \mathcal{E}(\mathcal{D}_3)$  where the sum is over all the filled diagrams  $\mathcal{D}_1$ ,  $\mathcal{D}_2$  and  $\mathcal{D}_3$  of type  $n$  and of the form



respectively. The minus sign comes from the fact that the order of the first outgoing arrows of the two sources filled with  $m_A$  is reversed and  $|m_A| = 1$ . Thus, the identity (3.1) is satisfied since  $\sum \mathcal{E}(\mathcal{D}_2) = \sum \mathcal{E}(\mathcal{D}_3)$  and  $\sum \mathcal{E}(\mathcal{D}) = \sum \mathcal{E}(\mathcal{D}_1)$ .

We now show (3.2). For  $n \in \mathbb{N}^*$  we have that  $(m_B \circ_F (F \circ_G m_A))^n = \sum \mathcal{E}(\mathcal{D})$  where the sum is over all the filled diagrams  $\mathcal{D}$  of type  $n$  and of the form



It is clear that  $((m_B \circ_M F) \circ_G m_A)^n = \sum \mathcal{E}(\mathcal{D})$ .

**Corollary 3.13.** *Let  $A$  and  $B$  be two graded vector spaces. Consider  $m_A \in C(A)$  and  $m_B \in C(B)$  of degree 1 and  $F \in C(A, B)$  of degree 0. Suppose that  $m_B$  satisfies the Stasheff identities  $(SI^n)$  for every  $n \in \mathbb{N}^*$ . Then, the following holds.*

$$F \circ_G (m_A \circ_G m_A) = (F \circ_G m_A - m_B \circ_M F) \circ_G m_A + m_B \circ_F (F \circ_G m_A - m_B \circ_M F) \quad (3.3)$$

*Proof.* Rewrite the equation (3.3) as

$$F \circ_G (m_A \circ_G m_A) = (F \circ_G m_A) \circ_G m_A - (m_B \circ_M F) \circ_G m_A + m_B \circ_F (F \circ_G m_A) - m_B \circ_F (m_B \circ_M F)$$

By Lemma 3.11, we already have that  $(m_B \circ_M F) \circ_G m_A = m_B \circ_F (F \circ_G m_A)$ . Moreover, one can prove in the same way as we did for (3.1) that  $F \circ_G (m_A \circ_G m_A) = (F \circ_G m_A) \circ_G m_A$ . Finally, since  $m_B$  satisfies the Stasheff identities,  $m_B \circ_F (m_B \circ_M F) = (m_B \circ_G m_B) \circ_F F = 0$ .  $\square$

We now recall the following theorem appearing in [5] in the case of  $A_\infty$ -algebras and give a proof which do not use the bar construction.

**Theorem 3.14.** *Let  $(A, d_A), (B, d_B)$  be dg vector spaces and  $f : A \rightarrow B$  be a quasi-isomorphism of dg vector spaces. Let  $m_B \in C(B)$  be an element of degree 1 with  $m_B^1 = d_B[1]$  that turns  $B$  into an  $A_\infty$ -algebra. Then, there exists an  $A_\infty$ -structure  $m_A$  on  $A$  with  $m_A^1 = d_A[1]$  as well as an  $A_\infty$ -morphism  $F \in C(A, B)$  such that  $F^1 = f[1]$ .*

*Proof.* We will construct the  $A_\infty$ -structure and the  $A_\infty$ -morphism inductively. We already have that  $m_A^1 = d_A[1]$  and  $F^1 = f[1]$  satisfy the identities  $(SI^1)$  and  $(MI^1)$ . Consider  $n > 1$  and suppose that we have constructed  $m_A \in C(A)$  of degree 1 and  $F \in C(A, B)$  of degree 0 such that the identities  $(SI^i)$  and  $(MI^i)$  are satisfied for every  $i < n$ .

Since  $(m_A \circ_G m_A)^i$  vanishes for  $i < n$  by assumption, the  $n$ -th term of the identity (3.1) is simply

$$d_{A[1]} \circ (m_A \circ_G m_A)^n = (m_A \circ_G m_A)^n \circ d_{A[1]^{\otimes n}}$$

which gives  $d_{C(A)}(m_A \circ_G m_A)^n = 0$  i.e.  $(m_A \circ_G m_A)^n$  is a cycle.

Moreover, since  $(F \circ_G m_A - m_B \circ_M F)^i$  vanishes for  $i < n$  the  $n$ -th term of the identity (3.3) is

$$F^1 \circ (m_A \circ_G m_A)^n = (F \circ_G m_A - m_B \circ_M F)^n \circ_M d_{A[1]^{\otimes n}} + d_{B[1]} \circ (F \circ_G m_A - m_B \circ_M F)^n$$

so  $F^1 \circ (m_A \circ_G m_A)^n$  is a boundary. Since  $(m_A \circ_G m_A)^n$  is a cycle and  $F^1$  is a quasi-isomorphism,

$(m_A \circ_G m_A)^n$  is itself a boundary, meaning that there exists  $e \in C^n(A)$  such that  $(m_A \circ_G m_A)^n = \partial e$  where  $\partial$  denotes the differential of  $C(A, B)$ . We thus define the element  $\mu_A \in C(A)$  by  $\mu_A^i = m_A^i$  for  $i < n$  and  $\mu_A^n = m_A^n - e$ . It satisfies that  $(\mu_A \circ_G \mu_A)^i = 0$  for  $i \leq n$ .

Moreover, by (3.3) we have that  $(F \circ_G m_A - m_B \circ_M F)^n$  is a cycle so it can be written as

$$(F \circ_G m_A - m_B \circ_M F)^n = f \circ e' + \partial e''$$

where  $e'$  is a cycle of  $C^n(A)$  and  $e'' \in C^n(A, B)$ .

We set  $\nu_A^i = \mu_A^i$ ,  $G^i = F^i$  for  $i < n$  and  $\nu_A^n = \mu_A^n - e'$ ,  $G^n = F^n - e''$ . Therefore, since we have only modified  $\mu_A$  by adding a cycle, we have  $(\nu_A \circ_G \nu_A)^i = 0$  for  $i \leq n$ . Moreover, we have that

$$\begin{aligned} (G \circ_G \nu_A - m_B \circ_M G)^n &= \sum_{\substack{n=r+s+t \\ 1 < s < n}} F^{r+1+t} \circ (\text{id}^{\otimes r} \otimes m_A^s \otimes \text{id}^{\otimes t}) + F^1 \circ \nu_A^n + \sum_{n=r+1+t} G^n \circ (\text{id}^{\otimes r} \otimes m_A^1 \otimes \text{id}^{\otimes t}) \\ &+ \sum_{\substack{n=u_1+\dots+u_i \\ i > 1}} m_B^i \circ (F^{u_1} \otimes \dots \otimes F^{u_i}) + m_B^1 \circ G^n \\ &= \sum_{\substack{n=r+s+t \\ 1 < s < n}} F^{r+1+t} \circ (\text{id}^{\otimes r} \otimes m_A^s \otimes \text{id}^{\otimes t}) + F^1 \circ (\mu_A^n - e') + (F^n - e'') \circ d_{A[1]^{\otimes n}} \\ &+ \sum_{\substack{n=u_1+\dots+u_i \\ i > 1}} m_B^i \circ (F^{u_1} \otimes \dots \otimes F^{u_i}) + d_{B[1]} \circ (F^n - e'') \\ &= \sum_{\substack{n=r+s+t \\ s \neq 0}} F^{r+1+t} \circ (\text{id}^{\otimes r} \otimes \mu_A^s \otimes \text{id}^{\otimes t}) - F^1 \circ e' - e'' \circ d_{A[1]^{\otimes n}} \\ &+ \sum_{\substack{n=u_1+\dots+u_i \\ i > 1}} m_B^i \circ (F^{u_1} \otimes \dots \otimes F^{u_i}) - d_{B[1]} \circ e'' \\ &= (F \circ_G m_A - m_B \circ_M F)^n - F^1 \circ e' - \partial e'' = 0 \end{aligned}$$

so  $G$  satisfies the morphism identities  $(MI^i)$  for every  $i \leq n$ . This finishes the proof.  $\square$

As proved in [5] we also have the statement in the other direction, namely, we have the following theorem:

**Theorem 3.15.** *Let  $(A, d_A)$ ,  $(B, d_B)$  be dg vector spaces and  $f : A \rightarrow B$  be a quasi-isomorphism of dg vector spaces. Let  $m_A \in C(A)$  be an element of degree 1 with  $m_A^1 = d_A[1]$  that turns  $A$  into an  $A_\infty$ -algebra. Then, there exists an  $A_\infty$ -structure  $m_B \in C(B)$  on  $B$  with  $m_B^1 = d_B[1]$  as well as an  $A_\infty$ -morphism  $F \in C(A, B)$  such that  $F^1 = f[1]$ .*

*Proof.* As in the proof of Theorem 3.14, we will construct the  $A_\infty$ -structure and the  $A_\infty$ -morphism inductively. We already have that  $m_B^1 = d_B[1]$  and  $F^1 = f[1]$  satisfy the identities  $(SI^1)$  and  $(MI^1)$ . Suppose that we have constructed  $m_B$  of degree 1 and  $F \in C(A, B)$  of degree 0 such that the identities  $(SI^i)$  and  $(MI^i)$  are satisfied for every  $i < n$ .

The  $n$ -th term of the identity (3.1) applied with  $m_B$  instead of  $m_A$  is just

$$d_{B[1]} \circ (m_B \circ_G m_B)^n = (m_B \circ_G m_B)^n \circ d_{B[1]^{\otimes n}}$$

since  $(m_B \circ_G m_B)^i$  vanishes for  $i < n$  which gives that  $(m_B \circ_G m_B)^n$  is a cycle in  $C(B)$ . Moreover, the  $n$ -th term of the identity

$$(m_B \circ_G m_B) \circ_M F = (m_B \circ_M F - F \circ_G m_A) \circ_G m_A + m_B \circ_F (m_B \circ_M F - F \circ_G m_A)$$

gives that  $(m_B \circ_G m_B)^n \circ_M F^1$  is a boundary. As before, since  $(m_B \circ_G m_B)^n$  is a cycle and  $F^1$  is a quasi-isomorphism,  $(m_B \circ_G m_B)^n$  is itself a boundary, meaning that there exists  $e \in C^n(B)$  such that  $(m_B \circ_G m_B)^n = \partial e$  where  $\partial$  denotes the differential of  $C(A, B)$ . We thus define  $\mu_B^i = m_B^i$  for  $i < n$  and  $\mu_B^n = m_B^n - e$  and get  $(\mu_B \circ_G \mu_B)^i = 0$  for  $i \leq n$ . Again,  $(\mu_B \circ_M F - F \circ_G m_A)^n$  is a cycle and we can write

$$(\mu_B \circ_M F - F \circ_G m_A)^n = e' \circ (F^1)^{\otimes n} + \partial e''$$

with  $e'$  a cycle of  $C^n(B)$  and  $e'' \in C^n(A, B)$ . Define  $\nu_B^i = \mu_B^i$ ,  $G^i = F^i$  for  $i < n$  and  $\nu_B^n = \mu_B^n - e'$ ,  $G^n = F^n - e''$ . Therefore, since we have only modified  $\mu$  by adding a cycle, we have  $(\nu_A \circ_G \nu_A)^i = 0$  for  $i \leq n$ . By the same argument as for Theorem 3.14, we have that  $(G \circ_G m_A - \nu_B \circ_M G)^n = 0$  so  $G$  satisfies the morphism identities  $(MI^i)$  for every  $i \leq n$ . This finishes the proof.  $\square$

We now recall the definition of a minimal model of an  $A_\infty$ -algebra.

**Definition 3.16.** An  $A_\infty$ -algebra  $(B, m_B)$  is *minimal* if  $m_B^1 = 0$ . Given an  $A_\infty$ -algebra  $(A, m_A)$ , a *minimal model of  $A$*  is a minimal  $A_\infty$ -algebra  $(B, m_B)$  together with a quasi-isomorphism  $P : A \rightarrow B$  of  $A_\infty$ -algebras.

We remark the well-known fact that minimal models of  $A_\infty$ -algebras always exist. Indeed, given an  $A_\infty$ -algebra  $(A, m_A)$ , let  $p : A \rightarrow H(A)$  be any quasi-isomorphism of dg vector spaces. By Theorem 3.15, there exists an  $A_\infty$ -structure on  $H(A)$  and a morphism of  $A_\infty$ -algebras  $P : A \rightarrow H(A)$  such that  $P^1 = p[1]$ . Moreover, we have the following result.

**Lemma 3.17.** Let  $(A, m_A)$  be an  $A_\infty$ -algebra. Given any two quasi-isomorphisms of dg vector spaces  $p : A \rightarrow H(A)$  and  $i : H(A) \rightarrow A$ , the  $A_\infty$ -structures  $m_i$  and  $m_p$  on  $H(A)$  given in theorems 3.14 and 3.15 are isomorphic. In particular, given quasi-isomorphisms  $p, p' : A \rightarrow H(A)$  (resp.  $i, i' : H(A) \rightarrow A$ ) the  $A_\infty$ -structures  $m_p$  and  $m_{p'}$  (resp.  $m_i, m_{i'}$ ) on  $H(A)$  given in Theorem 3.15 (resp. in Theorem 3.14) are isomorphic.

*Proof.* Consider quasi-isomorphisms of dg vector spaces  $p : A \rightarrow H(A)$  and  $i : H(A) \rightarrow A$ . By Theorem 3.15, there is an  $A_\infty$ -structure  $m_p$  on  $H(A)$  and an  $A_\infty$ -morphism  $P : A \rightarrow H(A)$  such that  $P^1 = p[1]$ . Moreover, by Theorem 3.14, one can construct an  $A_\infty$ -structure  $m_i$  on  $H(A)$  as well as an  $A_\infty$ -morphism  $I : H(A) \rightarrow A$  such that  $I^1 = i[1]$ . Those two constructions give isomorphic structures. Indeed,  $P \circ I : (H(A), m_i) \rightarrow (H(A), m_p)$  is an  $A_\infty$ -morphism such that  $(P \circ I)^1$  is a quasi-isomorphism as the composition of quasi-isomorphism. In fact,  $p \circ i$  is an isomorphism since  $H(A)$  is minimal so  $P \circ I$  is an isomorphism of  $A_\infty$ -algebras by Lemma 3.9.  $\square$

**Definition 3.18.** Let  $(A, m_A)$  and  $(B, m_B)$  be  $A_\infty$ -algebras. We consider quasi-isomorphisms of complexes  $i_A : H(A) \rightarrow A$  and  $p_B : B \rightarrow H(B)$ . Then, by Theorem 3.14  $H(A)$  is endowed with an  $A_\infty$ -structure  $m_{i_A}$  and  $i_A$  extends to an  $A_\infty$ -morphism  $I_A : H(A) \rightarrow A$ . Moreover, by Theorem 3.15  $H(B)$  is endowed with an  $A_\infty$ -structure  $m_{p_B}$  and  $p_B$  extends to an  $A_\infty$ -morphism  $P_B : B \rightarrow H(B)$ . Given an  $A_\infty$ -morphism  $F : A \rightarrow B$ , we define an  $A_\infty$ -morphism  $H(F) : H(A) \rightarrow H(B)$  as  $H(F) = P_B \circ F \circ I_A$ .

Using the previous results, we deduce the following well-known theorem.

**Theorem 3.19.** *Let  $(A, m_A), (B, m_B)$  be two  $A_\infty$ -algebras and  $F : (A, m_A) \rightarrow (B, m_B)$  be an  $A_\infty$ -morphism. Consider quasi-isomorphisms  $i_A : H(A) \rightarrow A$  and  $p_B : B \rightarrow H(B)$  and the associated  $A_\infty$ -morphism  $H(F) : (H(A), m_{i_A}) \rightarrow (H(B), m_{p_B})$  given in Definition 3.18. Then, if  $F$  is a quasi-isomorphism, there exists a quasi-isomorphism  $G : (B, m_B) \rightarrow (A, m_A)$  of  $A_\infty$ -algebras and quasi-isomorphisms  $i_B : H(B) \rightarrow B, p_A : A \rightarrow H(A)$  such that  $p_A \circ i_A = \text{id}_{H(A)}, p_B \circ i_B = \text{id}_{H(B)}$  and whose associated  $H(G)$  in the sense of Definition 3.18 is the inverse of  $H(F)$ .*

*Proof.* Since  $F$  is a quasi-isomorphism, it induces by Lemma 3.9 an isomorphism of  $A_\infty$ -algebras  $H(F) : (H(A), m_{i_A}) \rightarrow (H(B), m_{p_B})$  which has an inverse  $K : (H(B), m_{p_B}) \rightarrow (H(A), m_{i_A})$ . Consider  $i_B : H(B) \rightarrow B, p_A : A \rightarrow H(A)$  such that  $p_A \circ i_A = \text{id}_{H(A)}$  and  $p_B \circ i_B = \text{id}_{H(B)}$ . By Theorem 3.14 and Theorem 3.15, those four maps extend to  $A_\infty$ -morphisms  $I_A : H(A) \rightarrow A, P_A : A \rightarrow H(A), I_B : H(B) \rightarrow B$  and  $P_B : B \rightarrow H(B)$  such that  $(P_A \circ I_A)^1 = \text{id}_{H(A)[1]}$  and  $(P_B \circ I_B)^1 = \text{id}_{H(B)[1]}$ . Moreover, up to an automorphism of the  $A_\infty$ -algebra  $H(A)$  (resp.  $H(B)$ ), it is possible to assume that  $P_A \circ I_A = \text{id}$  (resp.  $P_B \circ I_B = \text{id}$ ). Then  $G = I_A \circ K \circ P_B$  gives a quasi-inverse for  $F$ . Indeed,  $H(G) = P_A \circ G \circ I_B = P_A \circ I_A \circ K \circ P_B \circ I_B = K$  is an inverse of  $H(F)$  by definition.  $\square$

## 4 Transferring pre-Calabi-Yau structures

### 4.1 Pre-Calabi-Yau algebras

In this subsection, we recall the notions of pre-Calabi-Yau structures and morphisms introduced in [4] and show that a  $d$ -pre-Calabi-Yau morphism whose first component is an isomorphism is an isomorphism in the category of  $d$ -pre-Calabi-Yau algebras.

**Definition 4.1.** *Given graded vector spaces  $A$  and  $B$  and  $d \in \mathbb{Z}$ , we denote by  $\text{Multi}_d(A, B)$  the graded vector space*

$$\text{Multi}_d(A, B) = \prod_{k \in \mathbb{N}^*} \prod_{(n_1, \dots, n_k) \in \mathcal{T}_k} \text{Multi}_d^{n_1, \dots, n_k}(A, B)$$

where  $\text{Multi}_d^{n_1, \dots, n_k}(A, B) = \mathcal{H}om_{\mathbb{k}}(A[1]^{\otimes n_1} \otimes \dots \otimes A[1]^{\otimes n_k}, B[-d]^{\otimes k})$ . To simplify, we will denote  $\text{Multi}_d(A, A)$  simply by  $\text{Multi}_d(A)$ .

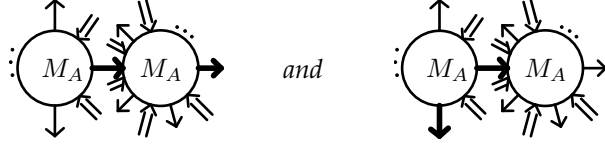
The action of  $\sigma = (\sigma_n)_{n \in \mathbb{N}^*} \in \prod_{n \in \mathbb{N}^*} C_n$  on  $F = (F^{n_1, \dots, n_k})_{(n_1, \dots, n_k) \in \mathcal{T}_k} \in \text{Multi}_d(A, B)$  is the element  $\sigma \cdot F \in \text{Multi}_d(A, B)$  given by

$$(\sigma \cdot F)^{n_1, \dots, n_k} = \tau_{B[-d], \dots, B[-d]}^{\sigma^{-1}} \circ F^{\sigma(n_1, \dots, n_k)} \circ \tau_{A[1]^{\otimes n_1}, \dots, A[1]^{\otimes n_k}}^{\sigma}$$

for  $(n_1, \dots, n_k) \in \mathcal{T}_k$ . We denote  $\text{Multi}_d(A, B)^{C_\bullet}$  the space of elements of  $\text{Multi}_d(A, B)$  that are invariant under the action of  $\prod_{n \in \mathbb{N}^*} C_n$ .

**Remark 4.2.** *If  $(A, d_A)$  and  $(B, d_B)$  are dg vector spaces,  $\text{Multi}_d(A, B)$  becomes a dg vector space, with differential given by  $d_{\text{Multi}_d(A, B)} F = d_{B[-d]^{\otimes k}} \circ F - (-1)^{|F|} F \circ d_{A[1]^{\otimes (n_1 + \dots + n_k)}}$  for  $F \in \text{Multi}_d^{n_1, \dots, n_k}(A, B)$ .*

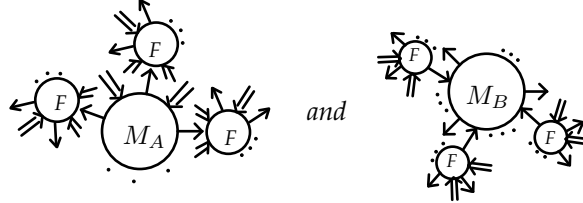
**Definition 4.3.** *A  $d$ -pre-Calabi-Yau structure on a graded vector space  $A$  is an element of degree 1  $s_{d+1} M_A \in \text{Multi}_d(A)^{C_\bullet}[d+1]$  satisfying the identities  $(SI^{n_1, \dots, n_k})$  given by  $\sum \mathcal{E}(\mathcal{D}) + \sum \mathcal{E}(\mathcal{D}') = 0$  where the sums are over all the filled diagrams  $\mathcal{D}$  and  $\mathcal{D}'$  of type  $(n_1, \dots, n_k)$  and of the form*



for every  $(n_1, \dots, n_k) \in \mathcal{T}_k$ . This is tantamount to  $[s_{d+1}M_A, s_{d+1}M_A]_{\text{nec}} = 0$  where  $[-, -]_{\text{nec}}$  denotes the necklace bracket given as the graded commutator of the necklace product  $\circ_{\text{nec}}$  introduced in [4]. A graded vector space endowed with a  $d$ -pre-Calabi-Yau structure is called a  **$d$ -pre-Calabi-Yau algebra**.

We now recall the definition of pre-Calabi-Yau morphisms given in [4].

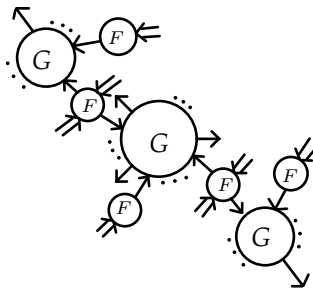
**Definition 4.4.** Let  $(A, s_{d+1}M_A), (B, s_{d+1}M_B)$  be two  $d$ -pre-Calabi-Yau algebras. A  **$d$ -pre-Calabi-Yau morphism**  $F : (A, s_{d+1}M_A) \rightarrow (B, s_{d+1}M_B)$  is an element  $s_{d+1}F \in \text{Multi}_d(A, B)^{C \bullet}[d+1]$  of degree 0 satisfying the identities  $(MI^{n_1, \dots, n_k})$  given by  $\sum \mathcal{E}(\mathcal{D}) = \sum \mathcal{E}(\mathcal{D}')$  where the sums are over all the filled diagrams  $\mathcal{D}$  and  $\mathcal{D}'$  of type  $(n_1, \dots, n_k)$  and of the form



for every  $(n_1, \dots, n_k) \in \mathcal{T}_k$ . This is tantamount to  $s_{d+1}F \circ_{\text{multinec}} s_{d+1}M_A - s_{d+1}M_B \circ_{\text{pre}} s_{d+1}F = 0$  where the compositions  $\circ_{\text{multinec}}$  and  $\circ_{\text{pre}}$  denote the right hand side and left hand side of the equation respectively.

**Remark 4.5.** The identities  $(SI^n)$  for  $n \in \mathbb{N}^*$  are just the Stasheff identities given in Definition 3.3. Then, if  $(A, s_{d+1}M_A)$  is a  $d$ -pre-Calabi-Yau algebra the collection of maps  $(M_A^n)_{n \in \mathbb{N}^*}$  turns  $A$  into an  $A_\infty$ -algebra and as before,  $(A[1], M_A^1)$  is a dg vector space. Moreover, a  $d$ -pre-Calabi-Yau morphism  $F : (A, s_{d+1}M_A) \rightarrow (B, s_{d+1}M_B)$  induces an  $A_\infty$ -morphism between the underlying  $A_\infty$ -algebras.

**Definition 4.6.** Let  $(A, s_{d+1}M_A), (B, s_{d+1}M_B)$  and  $(C, s_{d+1}M_C)$  be  $d$ -pre-Calabi-Yau algebras and consider  $s_{d+1}F : (A, s_{d+1}M_A) \rightarrow (B, s_{d+1}M_B)$  and  $s_{d+1}G : (B, s_{d+1}M_B) \rightarrow (C, s_{d+1}M_C)$  two  $d$ -pre-Calabi-Yau morphisms. The **composition of  $s_{d+1}F$  and  $s_{d+1}G$**  is by definition the element  $s_{d+1}G \circ s_{d+1}F$  of  $\text{Multi}_d(A, B)^{C \bullet}[d+1]$  given by  $(s_{d+1}G \circ s_{d+1}F)^{n_1, \dots, n_k} = \sum \mathcal{E}(\mathcal{D})$  where the sum is over all the filled diagrams  $\mathcal{D}$  of type  $(n_1, \dots, n_k)$  and of the form



**Lemma 4.7.** *The data of  $d$ -pre-Calabi-Yau algebras together with  $d$ -pre-Calabi-Yau morphisms and their composition give rise to a category denoted  $\text{pCY}_d$ . Given a  $d$ -pre-Calabi-Yau algebra  $(A, s_{d+1}M_A)$ , the  **$d$ -pre-Calabi-Yau identity morphism on  $A$**  is the element of  $\text{Multi}_d(A)[d+1]$  given for  $(n_1, \dots, n_k) \in \mathcal{T}_k$  by  $\text{id}_{A[1]}$  if  $k = 1, n_1 = 1$  and by 0 otherwise. It will be denoted  $\text{id}_A$  or simply by  $\text{id}$  when there is no possible confusion.*

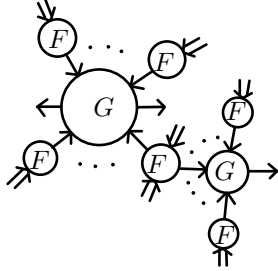
**Definition 4.8.** *Given  $d$ -pre-Calabi-Yau algebras  $(A, s_{d+1}M_A)$  and  $(B, s_{d+1}M_B)$ , we say that a  $d$ -pre-Calabi-Yau morphism  $s_{d+1}F : (A, s_{d+1}M_A) \rightarrow (B, s_{d+1}M_B)$  is a **quasi-isomorphism** if the map  $s_{d+1}F^1 : (A[1], M_A^1) \rightarrow (B[1], M_B^1)$  is a quasi-isomorphism of dg vector spaces. Moreover,  $s_{d+1}F$  is an **isomorphism** if there exists a  $d$ -pre-Calabi-Yau morphism  $s_{d+1}G : (B, s_{d+1}M_B) \rightarrow (A, s_{d+1}M_A)$  such that  $s_{d+1}G \circ s_{d+1}F = \text{id}_A$ , and  $s_{d+1}F \circ s_{d+1}G = \text{id}_B$ .*

**Lemma 4.9.** *Let  $(A, s_{d+1}M_A), (B, s_{d+1}M_B)$  be  $d$ -pre-Calabi-Yau algebras and consider a  $d$ -pre-Calabi-Yau morphism  $s_{d+1}F : (A, s_{d+1}M_A) \rightarrow (B, s_{d+1}M_B)$ . If  $s_{d+1}F^1 : (A[1], s_{d+1}M_A^1) \rightarrow (B[1], s_{d+1}M_B^1)$  is an isomorphism of dg vector spaces, then  $s_{d+1}F$  is an isomorphism of  $d$ -pre-Calabi-Yau algebras.*

*Proof.* By Lemma 3.9, we have already constructed maps  $G^n : B[1]^{\otimes n} \rightarrow A[1]$  for every  $n \in \mathbb{N}^*$ . We now construct maps  $G^{n_1, \dots, n_k} : B[1]^{\otimes n_1} \otimes \dots \otimes B[1]^{\otimes n_k} \rightarrow A[-d]^{\otimes k}$  for every  $k > 1$  and  $(n_1, \dots, n_k) \in \mathcal{T}_k$ . Consider  $k > 1$  and suppose that we have defined the maps  $G^{i_1, \dots, i_j} : B[1]^{\otimes i_1} \otimes \dots \otimes B[1]^{\otimes i_j} \rightarrow A[-d]^{\otimes k}$  satisfying that  $(s_{d+1}G \circ s_{d+1}F)^{i_1, \dots, i_j} = 0 = (s_{d+1}F \circ s_{d+1}G)^{i_1, \dots, i_j}$  for every  $(i_1, \dots, i_j) \in \mathcal{T}_j, j < k$ . We have

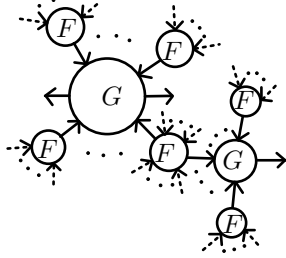
$$(s_{d+1}F \circ s_{d+1}G)^{0, \dots, 0} = (s_{d+1}F^1)^{\otimes k} \circ s_{d+1}G^{0, \dots, 0} + s_{d+1}\chi^{0, \dots, 0}$$

where  $\chi^{0, \dots, 0}$  is a sum of maps  $\mathbb{k} \rightarrow A[-d]^{\otimes k}$  such that the only components of  $s_{d+1}G$  that appear are  $s_{d+1}G^{u_1, \dots, u_r}$  where  $r < k$ . We thus define  $s_{d+1}G^{0, \dots, 0} = -(s_{d+1}F^1)^{\otimes k} \circ s_{d+1}\chi^{0, \dots, 0}$ . Consider  $\bar{n} \in \mathcal{T}_k$  and suppose that we have defined the maps  $G^{i_1, \dots, i_k} : B[1]^{\otimes i_1} \otimes \dots \otimes B[1]^{\otimes i_k} \rightarrow A[-d]^{\otimes k}$  satisfying that  $(s_{d+1}G \circ s_{d+1}F)^{i_1, \dots, i_k} = 0 = (s_{d+1}F \circ s_{d+1}G)^{i_1, \dots, i_k}$  for every  $\bar{i} = (i_1, \dots, i_k) \in \mathcal{T}_k$  with  $N(\bar{i}) < N(\bar{n})$ . We have that  $(s_{d+1}G \circ s_{d+1}F)^{\bar{n}} = \mathcal{E}(\mathcal{D}_1) + \sum \mathcal{E}(\mathcal{D}_2)$  where  $\mathcal{D}_1$  is the unique diagram of type  $\bar{n}$  containing a disc of type  $\bar{n}$  filled with  $G$  whose incoming arrows are shared with discs of type 1 filled with  $F$  and where the sum is over all the filled diagrams  $\mathcal{D}_2$  of type  $\bar{n}$  and of the form



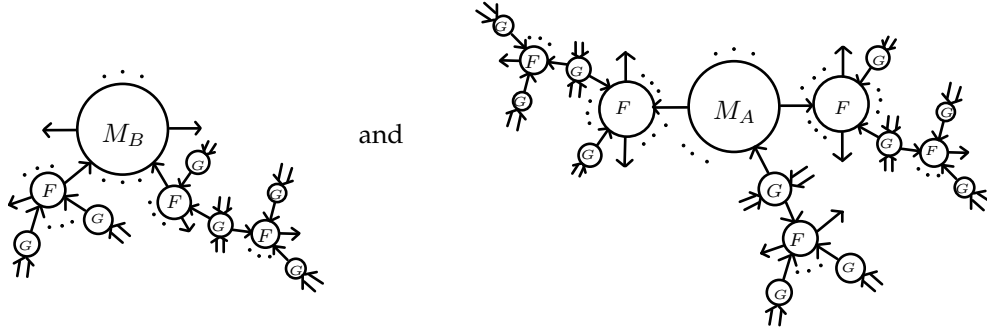
respectively. Note that  $\mathcal{D}_1$  is the only diagram containing a disc of type  $\bar{n}$  filled with  $G$ . Moreover, diagrams involved in the previous sum  $\sum \mathcal{E}(\mathcal{D}_2)$  contain either only discs filled with  $F$  having one outgoing arrow or at least one disc filled with  $F$  with several outgoing arrows. In the first case, the terms only contain discs filled with  $G$  of type  $\bar{i} \in \mathcal{T}_k$  with  $N(\bar{i}) < N(\bar{n})$ . Indeed, the discs filled with  $F$  only have one outgoing arrow but there is at least one of them having more than one incoming arrow. Then, there is at least one  $j \in \llbracket 1, k \rrbracket$  such that  $i_j < n_j$ . In the second case, they

only contain discs filled with  $G$  of type  $(i_1, \dots, i_j) \in \mathcal{T}_j$  and  $j < k$  since the total number of outputs is  $k$ . This allows us to define  $s_{d+1}G^{\bar{n}} = -\sum \mathcal{E}(\mathcal{D})$  where the sum is over all the filled diagrams  $\mathcal{D}$  of type  $\bar{n}$  and of the form



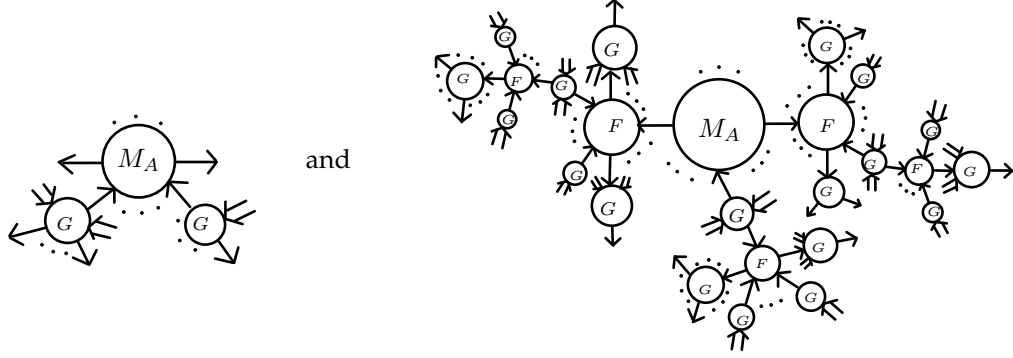
where the dotted arrows represent the map  $g$ . Note that we did not use the fact that  $F$  is a  $d$ -pre-Calabi-Yau morphism. Thus, there exists an element  $s_{d+1}H \in \text{Multi}_d(A, B)^{C \cdot [d+1]}$  such that  $s_{d+1}H \circ s_{d+1}G = \text{id}_B$ . Using the associativity of the composition, we get that  $H = F$  and thus  $s_{d+1}G \circ s_{d+1}F = \text{id}_A$  and  $s_{d+1}F \circ s_{d+1}G = \text{id}_B$ .

It remains to check that  $s_{d+1}G$  is a  $d$ -pre-Calabi-Yau morphism. By Lemma 3.9 the identities  $(MI^n)$  are satisfied for  $n \in \mathbb{N}^*$ . We now consider  $k > 1$  and suppose that the identities  $(MI^{\bar{i}})$  are satisfied for every  $\bar{i} \in \mathcal{T}_j$  for  $j < k$  and consider  $\bar{n} \in \mathcal{T}_k$ . Since  $s_{d+1}F$  is a  $d$ -pre-Calabi-Yau morphism, we have that  $s_{d+1}F \circ_{\text{multinec}} s_{d+1}M_A = s_{d+1}M_B \circ_{\text{pre}} s_{d+1}F$  which gives  $\sum \mathcal{E}(\mathcal{D}) = \sum \mathcal{E}(\mathcal{D}')$  where the sums are over all the diagrams  $\mathcal{D}$  and  $\mathcal{D}'$  of type  $\bar{n}$  and of the form

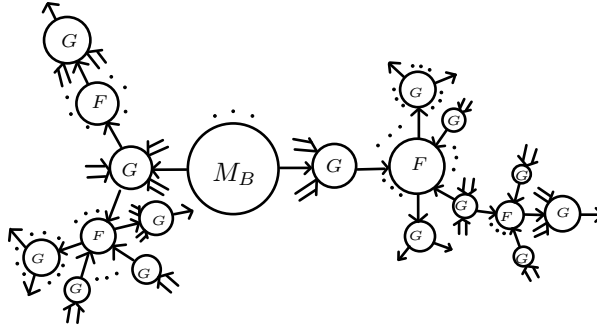


respectively. Since  $s_{d+1}F \circ s_{d+1}G = \text{id}$ , we have that  $\sum \mathcal{E}(\mathcal{D}) = M_B^{\bar{n}}$ . Therefore, we have that  $(s_{d+1}G \circ_{\text{multinec}} s_{d+1}M_B)^{\bar{n}} = \sum \mathcal{E}(\mathcal{D}'_1) + \sum \mathcal{E}(\mathcal{D}'_2)$  where the sums are over all the diagrams of type  $\bar{n}$  and of the form





respectively. We now show that  $\sum \mathcal{E}(\mathcal{D}'_2) = 0$ . Suppose that the identity  $(MI^{\bar{i}})$  is also satisfied for every  $\bar{i} \in \mathcal{T}_j$  with  $j = k$  and  $N(\bar{i}) < N(\bar{n})$ . We then have that  $\sum \mathcal{E}(\mathcal{D}'_2) = \sum \mathcal{E}(\mathcal{D}'_3)$  where the second sum is over all the filled diagrams  $\mathcal{D}'_3$  of type  $\bar{n}$  and of the form



Note that for the diagrams of type  $(0, \dots, 0) \in \mathcal{T}_k$  we only used the identities  $(MI^{\bar{i}})$  for  $\bar{i} \in \mathcal{T}_j$  with  $j < k$ . Then, by induction and since  $(s_{d+1}F \circ s_{d+1}G)^{\bar{i}} = 0$  for  $\bar{i} \neq 1$ , we have that  $\sum \mathcal{E}(\mathcal{D}'_3) = 0$  so  $s_{d+1}G$  is a  $d$ -pre-Calabi-Yau morphism.  $\square$

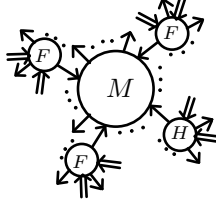
## 4.2 Transferring pre-Calabi-Yau structures

In this subsection, we generalize the results of [5] to the case of  $d$ -pre-Calabi-Yau algebras. We first define two different ways of composing maps and then give useful identities for the proof of Theorem 4.13 and Theorem 4.14.

**Definition 4.10.** Let  $A$  and  $B$  be two graded vector spaces and consider  $s_{d+1}F \in \text{Multi}_d(A, B)[d+1]$ . Given  $s_{d+1}M \in \text{Multi}_d(B)[d+1]$  and  $s_{d+1}H \in \text{Multi}_d(A, B)[d+1]$  we define the **lower composition of  $s_{d+1}M$  and  $s_{d+1}H$  with respect to  $s_{d+1}F$**  as the element  $(s_{d+1}M \underset{s_{d+1}F}{\circ} s_{d+1}H) \in \text{Multi}_d(A, B)[d+1]$  such that

$$(s_{d+1}M \underset{s_{d+1}F}{\circ} s_{d+1}H)^{n_1, \dots, n_k} = \sum \mathcal{E}(\mathcal{D})$$

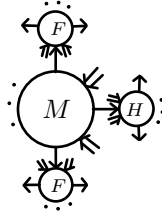
where the sum is over all the filled diagrams  $\mathcal{D}$  of type  $(n_1, \dots, n_k)$  and of the form



Moreover, given  $s_{d+1}M' \in \text{Multi}_d(A)[d+1]$  we define the **upper composition of  $s_{d+1}M'$  and  $s_{d+1}H$  with respect to  $s_{d+1}F$**  as the element  $s_{d+1}H \overset{s_{d+1}F}{\circ} s_{d+1}M' \in \text{Multi}_d(A, B)[d+1]$  such that

$$(s_{d+1}H \overset{s_{d+1}F}{\circ} s_{d+1}M')^{n_1, \dots, n_k} = \sum \mathcal{E}(\mathcal{D})$$

where the sum is over all the filled diagrams  $\mathcal{D}$  of type  $(n_1, \dots, n_k)$  and of the form



**Lemma 4.11.** Let  $s_{d+1}M_A \in \text{Multi}_d(A)[d+1]$ ,  $s_{d+1}M_B \in \text{Multi}_d(B)[d+1]$  be two elements of degree 1 and consider  $s_{d+1}F \in \text{Multi}_d(A, B)[d+1]$ . Then,

$$s_{d+1}M_A \underset{\text{nec}}{\circ} (s_{d+1}M_A \underset{\text{nec}}{\circ} s_{d+1}M_A) = (s_{d+1}M_A \underset{\text{nec}}{\circ} s_{d+1}M_A) \underset{\text{nec}}{\circ} s_{d+1}M_A \quad (4.1)$$

Moreover, we have that

$$(s_{d+1}M_B \underset{\text{pre}}{\circ} s_{d+1}F) \overset{s_{d+1}F}{\circ} s_{d+1}M_A = s_{d+1}M_B \underset{s_{d+1}F}{\circ} (s_{d+1}F \underset{\text{multinec}}{\circ} s_{d+1}M_A) \quad (4.2)$$

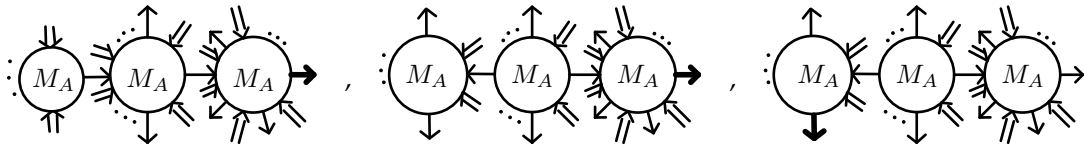
and

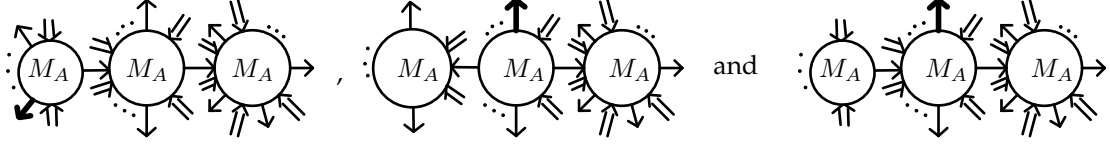
$$(s_{d+1}F \underset{\text{multinec}}{\circ} s_{d+1}M_A) \overset{s_{d+1}F}{\circ} s_{d+1}M_A = s_{d+1}F \underset{\text{multinec}}{\circ} (s_{d+1}M_A \underset{\text{nec}}{\circ} s_{d+1}M_A)$$

*Proof.* We first show (4.1). For every  $k \in \mathbb{N}^*$  and  $\bar{n} \in \mathcal{T}_k$  left hand term is given by

$$\begin{aligned} & (s_{d+1}M_A \underset{\text{nec}}{\circ} (s_{d+1}M_A \underset{\text{nec}}{\circ} s_{d+1}M_A))^{\bar{n}} \\ &= \sum \mathcal{E}(\mathcal{D}_1) + \sum \mathcal{E}(\mathcal{D}_2) - \sum \mathcal{E}(\mathcal{D}_3) + \sum \mathcal{E}(\mathcal{D}_4) + \sum \mathcal{E}(\mathcal{D}_5) + \sum \mathcal{E}(\mathcal{D}_6) \end{aligned}$$

where the sums are over all the filled diagrams  $\mathcal{D}_i$  of type  $\bar{n}$  and of the form





respectively. The minus sign before  $\sum \mathcal{E}(\mathcal{D}_3)$  comes from the fact that the order of the first outgoing arrows of the discs filled with  $M_A$  have changed their order and that  $s_{d+1}M_A$  of degree 1. We have that  $\sum \mathcal{E}(\mathcal{D}_3) = \sum \mathcal{E}(\mathcal{D}_2)$ . Moreover,  $\sum \mathcal{E}(\mathcal{D}_5) = 0$ . Indeed, the two discs sharing an arrow with the one with the bold arrow are filled with  $M_A$  and are ordered by the labeling of their outgoing arrows. There are two different cases: either the first disc is filled with the first  $M_A$  appearing in the identity (4.1) or with the second one. Since  $s_{d+1}M_A$  is of degree 1, the second case gives rise to a minus sign and the two cases then cancel each other.

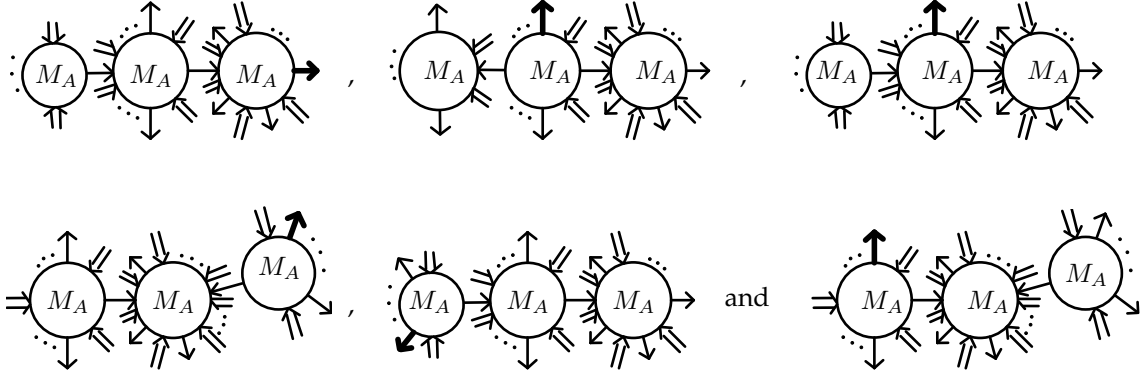
We thus get

$$(s_{d+1}M_A \circ_{\text{nec}} (s_{d+1}M_A \circ_{\text{nec}} s_{d+1}M_A))^{\bar{n}} = \sum \mathcal{E}(\mathcal{D}_1) + \sum \mathcal{E}(\mathcal{D}_4) + \sum \mathcal{E}(\mathcal{D}_6)$$

On the other hand, the right hand term is given by

$$\begin{aligned} & ((s_{d+1}M_A \circ_{\text{nec}} s_{d+1}M_A) \circ_{\text{nec}} s_{d+1}M_A)^{\bar{n}} \\ &= \sum \mathcal{E}(\mathcal{D}'_1) + \sum \mathcal{E}(\mathcal{D}'_2) + \sum \mathcal{E}(\mathcal{D}'_3) + \sum \mathcal{E}(\mathcal{D}'_4) + \sum \mathcal{E}(\mathcal{D}'_5) - \sum \mathcal{E}(\mathcal{D}'_6) \end{aligned}$$

where the sums are over all the filled diagrams  $\mathcal{D}'_i$  of type  $\bar{n}$  and of the form

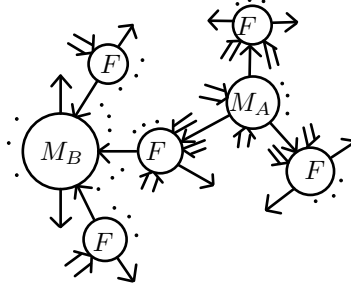


Again, the minus before  $\mathcal{E}(\mathcal{D}'_6)$  comes from the exchange of the order of two  $s_{d+1}M_A$ . Moreover,  $\mathcal{E}(\mathcal{D}'_6) = \mathcal{E}(\mathcal{D}'_4)$  and  $\mathcal{E}(\mathcal{D}'_2) = 0$  for the same reason as  $\mathcal{E}(\mathcal{D}_5) = 0$ . We thus get

$$((s_{d+1}M_A \circ_{\text{nec}} s_{d+1}M_A) \circ_{\text{nec}} s_{d+1}M_A)^{\bar{n}} = \sum \mathcal{E}(\mathcal{D}'_1) + \sum \mathcal{E}(\mathcal{D}'_3) + \sum \mathcal{E}(\mathcal{D}'_5)$$

Since  $\mathcal{E}(\mathcal{D}_1) = \mathcal{E}(\mathcal{D}'_1)$ ,  $\mathcal{E}(\mathcal{D}_4) = \mathcal{E}(\mathcal{D}'_5)$  and  $\mathcal{E}(\mathcal{D}_6) = \mathcal{E}(\mathcal{D}'_3)$ , the identity (4.1) is proved.

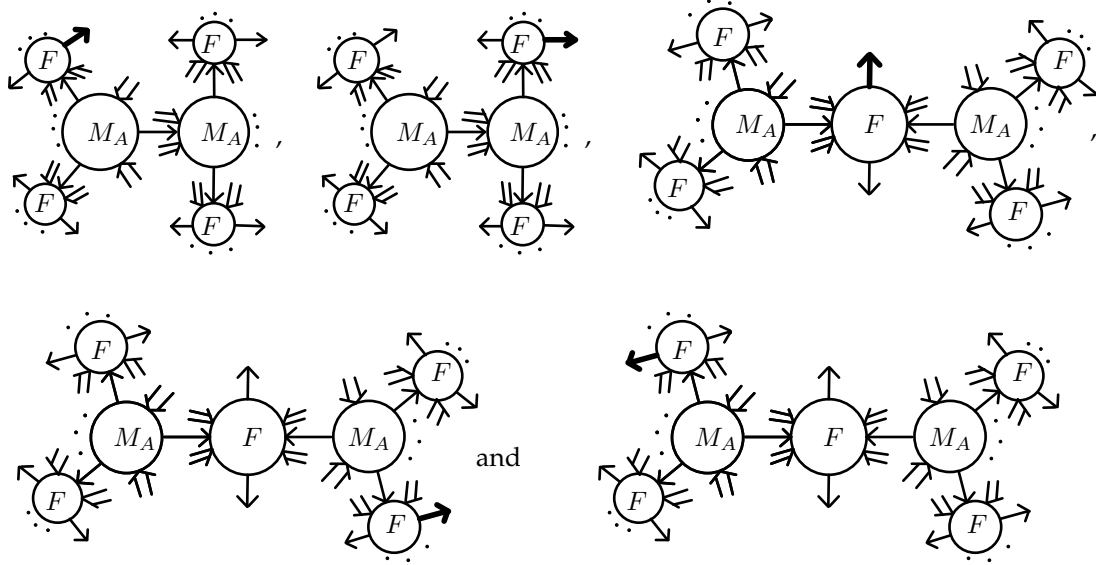
We now prove (4.2). First, note that  $((s_{d+1}M_B \circ_{\text{pre}} s_{d+1}F) \circ_{\text{pre}} s_{d+1}M_A)^{\bar{n}} = \sum \mathcal{E}(\mathcal{D})$  where the sum is over all the filled diagrams  $\mathcal{D}$  of type  $\bar{n}$  and of the form



It is easy to check that  $(s_{d+1}M_B \underset{s_{d+1}F}{\circ} (s_{d+1}F \underset{\text{multinec}}{\circ} s_{d+1}M_A))^{\bar{n}} = \sum \mathcal{E}(\mathcal{D})$  which proves (4.2). Finally, we have that

$$((s_{d+1}F \underset{\text{multinec}}{\circ} s_{d+1}M_A) \underset{s_{d+1}F}{\circ} s_{d+1}M_A)^{\bar{n}} = \sum \mathcal{E}(\mathcal{D}_1) + \sum \mathcal{E}(\mathcal{D}_2) + \sum \mathcal{E}(\mathcal{D}_3) + \sum \mathcal{E}(\mathcal{D}_4) - \sum \mathcal{E}(\mathcal{D}_5)$$

where the sums are over all the filled diagrams of type  $\bar{n}$  and of the form



respectively. Note that  $\sum \mathcal{E}(\mathcal{D}_4) = \sum \mathcal{E}(\mathcal{D}_5)$  and  $\sum \mathcal{E}(\mathcal{D}_3) = 0$ , so that

$$((s_{d+1}F \underset{\text{multinec}}{\circ} s_{d+1}M_A) \underset{s_{d+1}F}{\circ} s_{d+1}M_A)^{\bar{n}} = \sum \mathcal{E}(\mathcal{D}_1) + \sum \mathcal{E}(\mathcal{D}_2)$$

It is clear that

$$(s_{d+1}F \underset{\text{multinec}}{\circ} (s_{d+1}M_A \underset{\text{nec}}{\circ} s_{d+1}M_A))^{\bar{n}} = \sum \mathcal{E}(\mathcal{D}_1) + \sum \mathcal{E}(\mathcal{D}_2)$$

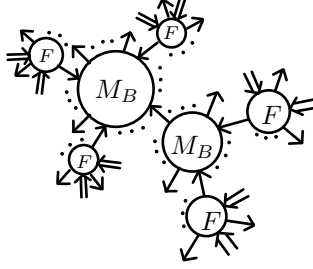
This finishes the proof.  $\square$

**Corollary 4.12.** *Let  $s_{d+1}M_A \in \text{Multi}_d(A)[d+1]$ ,  $s_{d+1}M_B \in \text{Multi}_d(B)[d+1]$  be two elements of degree 1 and consider  $s_{d+1}F \in \text{Multi}_d(A, B)[d+1]$  of degree 0. Suppose that  $s_{d+1}M_B$  satisfies the identities*

$(SI^{\bar{n}})$  for each  $\bar{n} \in \mathcal{T}$ . Then, the following holds.

$$\begin{aligned} s_{d+1}F \underset{\text{multinec}}{\circ} (s_{d+1}M_A \underset{\text{nec}}{\circ} s_{d+1}M_A) &= (s_{d+1}F \underset{\text{multinec}}{\circ} s_{d+1}M_A - s_{d+1}M_B \underset{\text{pre}}{\circ} s_{d+1}F) \overset{s_{d+1}F}{\circ} s_{d+1}M_A \\ &+ s_{d+1}M_B \underset{s_{d+1}F}{\circ} (s_{d+1}F \underset{\text{multinec}}{\circ} s_{d+1}M_A - s_{d+1}M_B \underset{\text{pre}}{\circ} s_{d+1}F) \end{aligned} \quad (4.3)$$

*Proof.* It suffices to show that  $s_{d+1}M_B \underset{s_{d+1}F}{\circ} (s_{d+1}M_B \underset{\text{pre}}{\circ} s_{d+1}F) = 0$ . By definition, for  $\bar{n} \in \mathcal{T}$  we have  $(s_{d+1}M_B \underset{s_{d+1}F}{\circ} (s_{d+1}M_B \underset{\text{pre}}{\circ} s_{d+1}F))^{\bar{n}} = \sum \mathcal{E}(\mathcal{D})$  where the sum is over all the filled diagrams  $\mathcal{D}$  of type  $\bar{n}$  and of the form



Then,  $s_{d+1}M_B \underset{s_{d+1}F}{\circ} (s_{d+1}M_B \underset{\text{pre}}{\circ} s_{d+1}F) = (s_{d+1}M_B \underset{\circ}{\circ} s_{d+1}M_B) \underset{\text{pre}}{\circ} s_{d+1}F$  which vanishes since  $s_{d+1}M_B$  satisfies the Stasheff identities by assumption.  $\square$

**Theorem 4.13.** Let  $(A, d_A), (B, d_B)$  be dg vector spaces and  $f : A \rightarrow B$  be a quasi-isomorphism of dg vector spaces. Let  $s_{d+1}M_B \in \text{Multi}_d(B)[d+1]$  be an element of degree 1 with  $s_{d+1}M_B^1 = d_B[1]$  that turns  $B$  into a  $d$ -pre-Calabi-Yau algebra. Then, there exists a  $d$ -pre-Calabi-Yau structure  $s_{d+1}M_A$  on  $A$  such that  $s_{d+1}M_A^1 = d_A[1]$  as well as a  $d$ -pre-Calabi-Yau morphism  $s_{d+1}F \in \text{Multi}_d(A, B)[d+1]$  such that  $s_{d+1}F^1 = f[1]$ .

*Proof.* We will construct the  $d$ -pre-Calabi-Yau structure and the  $d$ -pre-Calabi-Yau morphism inductively. We already have the maps for  $k = 1$  by Theorem 3.14. Consider  $k > 1$ ,  $\bar{n} \in \mathcal{T}_k$  and suppose that we have constructed  $s_{d+1}M_A \in \text{Multi}_d(A)[d+1]$  of degree 1 and  $s_{d+1}F \in \text{Multi}_d(A, B)[d+1]$  of degree 0 such that the identities  $(SI^{\bar{i}})$  and  $(MI^{\bar{i}})$  are satisfied for every  $\bar{i} \in \mathcal{T}_j$  for  $j < k$  and such that  $N(\bar{i}) < N(\bar{n})$  if  $j = k$ . Note that if  $\bar{n} = (0, \dots, 0)$  we only assume that the identities  $(SI^{\bar{i}})$  and  $(MI^{\bar{i}})$  are satisfied for every  $\bar{i} \in \mathcal{T}_j$  for  $j < k$ . Since  $(s_{d+1}M_A \underset{\text{nec}}{\circ} s_{d+1}M_A)^{\bar{i}}$  vanishes for  $\bar{i} \in \mathcal{T}_j$  if  $j < k$  or  $j = k$  and  $N(\bar{i}) < N(\bar{n})$ , the term indexed by  $\bar{n}$  in the identity (4.1) is simply

$$d_{A[-d] \otimes (k-1) \otimes A[1]} \circ (s_{d+1}M_A \underset{\text{nec}}{\circ} s_{d+1}M_A)^{\bar{n}} = (s_{d+1}M_A \underset{\text{nec}}{\circ} s_{d+1}M_A)^{\bar{n}} \circ d_{A[1] \otimes N(n)}$$

and we thus have that  $\partial(s_{d+1}M_A \underset{\text{nec}}{\circ} s_{d+1}M_A)^{\bar{n}} = 0$ , where we have denoted by  $\partial$  the differential of  $\text{Multi}_d(A, B)[d+1]$ , i.e.  $(s_{d+1}M_A \underset{\text{nec}}{\circ} s_{d+1}M_A)^{\bar{n}}$  is a cycle.

Moreover, the term indexed by  $\bar{n}$  of the identity (4.3) is

$$\begin{aligned} (f[1])^{\otimes k} \circ (s_{d+1}M_A \underset{\text{nec}}{\circ} s_{d+1}M_A)^{\bar{n}} &= (s_{d+1}F \underset{\text{multinec}}{\circ} s_{d+1}M_A - s_{d+1}M_B \underset{\text{pre}}{\circ} s_{d+1}F)^{\bar{n}} \circ d_{A[1] \otimes N(n)} \\ &+ d_{A[-d] \otimes (k-1) \otimes A[1]} \circ (s_{d+1}F \underset{\text{multinec}}{\circ} s_{d+1}M_A - s_{d+1}M_B \underset{\text{pre}}{\circ} s_{d+1}F)^{\bar{n}} \end{aligned}$$

since  $(s_{d+1}F \underset{\text{multinec}}{\circ} s_{d+1}M_A - s_{d+1}M_B \underset{\text{pre}}{\circ} s_{d+1}F)^{\bar{i}}$  vanishes for  $\bar{i} \in \mathcal{T}_k$  if  $N(\bar{i}) < N(\bar{n})$ .

Then,  $(f[1])^{\otimes k} \circ (s_{d+1}M_A \circ_{\text{nec}} s_{d+1}M_A)^{\bar{n}}$  is a boundary, and since  $(s_{d+1}M_A \circ_{\text{nec}} s_{d+1}M_A)^{\bar{n}}$  is a cycle and  $f$  is a quasi-isomorphism,  $(s_{d+1}M_A \circ_{\text{nec}} s_{d+1}M_A)^{\bar{n}}$  is itself a boundary, meaning that there exists  $E \in \text{Multi}_d^{\bar{n}}(A)$  such that  $(s_{d+1}M_A \circ_{\text{nec}} s_{d+1}M_A)^{\bar{n}} = \partial(s_{d+1}E)$ . We thus define  $\mu_A^{\bar{i}} = M_A^{\bar{i}}$  for  $\bar{i} \in \mathcal{T}_j$  with  $j < k$  or  $j = k$  and  $N(\bar{i}) < N(\bar{n})$  and  $\mu_A^{\bar{n}} = M_A^{\bar{n}} - E$ . Then,  $(s_{d+1}\mu_A \circ_{\text{nec}} s_{d+1}\mu_A)^{\bar{i}} = 0$  for  $\bar{i} \in \mathcal{T}_j$  with  $j < k$  or  $j = k$  and  $N(\bar{i}) \leq N(\bar{n})$ .

Moreover, by (3.3) we have that  $(s_{d+1}F \circ_{\text{multinec}} s_{d+1}M_A - s_{d+1}M_B \circ_{\text{pre}} s_{d+1}F)^{\bar{n}}$  is a cycle so it can be written as

$$(s_{d+1}F \circ_{\text{multinec}} s_{d+1}M_A - s_{d+1}M_B \circ_{\text{pre}} s_{d+1}F)^{\bar{n}} = (f[1])^{\otimes n} \circ s_{d+1}E' + \partial(s_{d+1}E'')$$

where  $E'$  is a cycle of  $\text{Multi}_d(A)$  and  $E'' \in \text{Multi}_d(A, B)$ .

We set  $\nu_A^{\bar{i}} = \mu_A^{\bar{i}}$ ,  $G^{\bar{i}} = F^{\bar{i}}$  for  $\bar{i} \in \mathcal{T}_j$  with  $j < k$  and for  $j = k$  and  $N(\bar{i}) < N(\bar{n})$  and set  $\nu_A^{\bar{n}} = \nu_A^{\bar{n}} - E'$ ,  $G^{\bar{n}} = F^{\bar{n}} - E''$ . Therefore, since we have only modified  $\mu_A$  by adding a cycle, we have  $(s_{d+1}\nu_A \circ_{\text{nec}} s_{d+1}\nu_A)^{\bar{i}} = 0$  for  $\bar{i} \in \mathcal{T}_j$  with  $j < k$  or  $j = k$  and  $N(\bar{i}) \leq N(\bar{n})$ . It is straightforward to check that  $(s_{d+1}G \circ_{\text{multinec}} s_{d+1}\nu_A - s_{d+1}M_B \circ_{\text{pre}} s_{d+1}G)^{\bar{n}} = 0$ .  $\square$

**Theorem 4.14.** *Let  $(A, d_A)$ ,  $(B, d_B)$  be dg vector spaces and  $f : A \rightarrow B$  be a quasi-isomorphism of dg vector spaces. Let  $s_{d+1}M_A \in \text{Multi}_d(A)[d+1]$  be an element of degree 1 with  $s_{d+1}M_A^1 = d_A[1]$  that turns  $A$  into a  $d$ -pre-Calabi-Yau algebra. Then, there exists a  $d$ -pre-Calabi-Yau structure  $s_{d+1}M_B$  on  $B$  such that  $s_{d+1}M_B^1 = d_B[1]$  as well as a  $d$ -pre-Calabi-Yau morphism  $s_{d+1}F \in \text{Multi}_d(A, B)[d+1]$  such that  $s_{d+1}F^1 = f[1]$ .*

*Proof.* We will construct the  $d$ -pre-Calabi-Yau structure and the  $d$ -pre-Calabi-Yau morphism inductively. We already have the maps for  $k = 1$  by Theorem 3.15. Consider  $k > 1$ .

Suppose that we have constructed an element  $s_{d+1}M_B \in \text{Multi}_d(B)[d+1]$  of degree 1 and  $s_{d+1}F \in \text{Multi}_d(A, B)[d+1]$  such that the identities  $(ST^{\bar{i}})$  and  $(MT^{\bar{i}})$  are satisfied for every  $\bar{i} \in \mathcal{T}_j$  such that  $j < k$  or  $j = k$  and  $N(\bar{i}) < N(\bar{n})$ .

The term indexed by  $\bar{n}$  of the identity (4.1) applied with  $s_{d+1}M_B$  shows as before that the element  $(s_{d+1}M_B \circ_{\text{nec}} s_{d+1}M_B)^{\bar{n}}$  is a cycle.

We now consider the following identity.

$$(s_{d+1}M_B \circ_{\text{nec}} s_{d+1}M_B) \circ_{\text{pre}} s_{d+1}F = (s_{d+1}M_B \circ_{\text{pre}} s_{d+1}F - s_{d+1}F \circ_{\text{multinec}} s_{d+1}M_A) \circ_{\text{pre}} s_{d+1}M_A + s_{d+1}M_B \circ_{s_{d+1}F} (s_{d+1}M_B \circ_{\text{pre}} s_{d+1}F - s_{d+1}F \circ_{\text{multinec}} s_{d+1}M_A) \quad (4.4)$$

Its term indexed by  $\bar{n}$  is

$$(s_{d+1}M_B \circ_{\text{nec}} s_{d+1}M_B)^{\bar{n}} \circ (f[1])^{\otimes(N(\bar{n}))} = (s_{d+1}M_B \circ_{\text{pre}} s_{d+1}F - s_{d+1}F \circ_{\text{multinec}} s_{d+1}M_A)^{\bar{n}} \circ d_{A[1] \otimes N(\bar{n})} + d_{A[-d] \otimes (k-1) \otimes A[1]} \circ (s_{d+1}M_B \circ_{\text{pre}} s_{d+1}F - s_{d+1}F \circ_{\text{multinec}} s_{d+1}M_A)^{\bar{n}}$$

since  $(s_{d+1}M_B \circ_{\text{pre}} s_{d+1}F - s_{d+1}F \circ_{\text{multinec}} s_{d+1}M_A)^{\bar{i}}$  vanishes for  $\bar{i} \in \mathcal{T}_j$  if  $j < k$  or  $j = k$  and  $N(\bar{i}) < N(\bar{n})$ .

Then,  $(s_{d+1}M_B \circ_{\text{nec}} s_{d+1}M_B)^{\bar{n}} \circ (f[1])^{\otimes(N(\bar{n}))}$  is a boundary, and since  $(s_{d+1}M_B \circ_{\text{nec}} s_{d+1}M_B)^{\bar{n}}$  is a cycle and  $f$  is a quasi-isomorphism,  $(s_{d+1}M_B \circ_{\text{nec}} s_{d+1}M_B)^{\bar{n}}$  is itself a boundary, meaning that there

exists  $E \in \text{Multi}_d^{\bar{n}}(B)$  such that  $(s_{d+1}M_B \circ_{\text{nec}} s_{d+1}M_B)^{\bar{n}} = \partial(s_{d+1}E)$ . We thus define  $\mu_B^{\bar{i}} = M_B^{\bar{i}}$  for  $\bar{i} \in \mathcal{T}_j$  with  $j < k$  or  $j = k$  and  $N(\bar{i}) < N(\bar{n})$  and  $\mu_B^{\bar{n}} = M_B^{\bar{n}} - E$ . Then,  $(s_{d+1}\mu_B \circ_{\text{nec}} s_{d+1}\mu_B)^{\bar{i}} = 0$  for  $\bar{i} \in \mathcal{T}_j$  with  $j < k$  or  $j = k$  and  $N(\bar{i}) < N(\bar{n})$ .

Moreover, by (4.4) we have that  $(s_{d+1}M_B \circ_{\text{pre}} s_{d+1}F - s_{d+1}F \circ_{\text{multinec}} s_{d+1}M_A)^{\bar{n}}$  is a cycle so it can be written as

$$(s_{d+1}M_B \circ_{\text{pre}} s_{d+1}F - s_{d+1}F \circ_{\text{multinec}} s_{d+1}M_A)^{\bar{n}} = s_{d+1}E' \circ (F^1)^{\otimes N(\bar{n})} + \partial(s_{d+1}E'')$$

where  $E'$  is a cycle of  $\text{Multi}_d(B)$  and  $E'' \in \text{Multi}_d(A, B)$ .

We set  $\nu_B^{\bar{i}} = \mu_B^{\bar{i}}$  for  $\bar{i} \in \mathcal{T}_j$  with  $j < k$  or  $j = k$  and  $N(\bar{i}) < N(\bar{n})$ ,  $\nu_B^{\bar{n}} = \mu_B^{\bar{n}} - E'$  and  $G^{\bar{i}} = F^{\bar{i}}$  for  $i < n$ ,  $G^{\bar{n}} = F^{\bar{n}} - E''$ . Therefore, since we have only modified  $s_{d+1}\mu_B$  by adding a cycle, we have  $(s_{d+1}\nu_B \circ_{\text{nec}} s_{d+1}\nu_B)^{\bar{i}} = 0$  for  $\bar{i} \in \mathcal{T}_j$  with  $j < k$  or  $j = k$  and  $N(\bar{i}) < N(\bar{n})$ . It is straightforward to check that  $(s_{d+1}\nu_B \circ_{\text{pre}} s_{d+1}G - s_{d+1}G \circ_{\text{multinec}} s_{d+1}M_A)^{\bar{n}} = 0$ . This finishes the proof.  $\square$

### 4.3 Minimal models and quasi-isomorphisms of pre-Calabi-Yau algebras

We now remark that the previous results leads to the existence of minimal models for pre-Calabi-Yau algebras and prove that any quasi-isomorphism of  $d$ -pre-Calabi-Yau algebras admits a quasi-inverse.

**Definition 4.15.** A  $d$ -pre-Calabi-Yau algebra  $(B, s_{d+1}M_B)$  is **minimal** if  $M_B^1 = 0$ . Given a  $d$ -pre-Calabi-Yau algebra  $(A, s_{d+1}M_A)$ , a **minimal model of  $A$**  is a minimal  $d$ -pre-Calabi-Yau algebra  $(B, s_{d+1}M_B)$  together with a quasi-isomorphism  $P : A \rightarrow B$  of  $d$ -pre-Calabi-Yau algebras.

As for the case of  $A_\infty$ -algebras we deduce from the previous results that minimal models of pre-Calabi-Yau algebras always exist. Indeed, given a  $d$ -pre-Calabi-Yau algebra  $(A, s_{d+1}M_A)$ , let  $p : A \rightarrow H(A)$  be any quasi-isomorphism of dg vector spaces. By Theorem 4.14, there exists an  $d$ -pre-Calabi-Yau structure on  $H(A)$  and a morphism of  $d$ -pre-Calabi-Yau algebras  $P : A \rightarrow H(A)$  such that  $P^1 = p[1]$  is a quasi-isomorphism.

**Lemma 4.16.** Let  $(A, s_{d+1}M_A)$  be a  $d$ -pre-Calabi-Yau algebra. Given quasi-isomorphisms of dg vector spaces  $p : A \rightarrow H(A)$  and  $i : H(A) \rightarrow A$ , the  $d$ -pre-Calabi-Yau structures  $M_i$  and  $M_p$  on  $H(A)$  given in Theorem 4.13 and 4.14 are isomorphic. In particular, given quasi-isomorphisms  $p, p' : A \rightarrow H(A)$  (resp.  $i, i' : H(A) \rightarrow A$ ), the  $d$ -pre-Calabi-Yau structures  $M_p$  and  $M_{p'}$  (resp.  $M_i$  and  $M_{i'}$ ) on  $H(A)$  given in Theorem 4.14 (resp. in Theorem 4.13) are isomorphic.

*Proof.* Consider quasi-isomorphisms  $p : A \rightarrow H(A)$  and  $i : H(A) \rightarrow A$ . By Theorem 4.13, one can define a  $d$ -pre-Calabi-Yau structure  $M_p$  on  $H(A)$  as well as a  $d$ -pre-Calabi-Yau morphism  $P_A : A \rightarrow H(A)$  such that  $P^1 = p[1]$ . Moreover, by Theorem 4.14, one can construct a  $d$ -pre-Calabi-Yau structure  $M_i$  on  $H(A)$  and a  $d$ -pre-Calabi-Yau morphism  $I : H(A) \rightarrow A$  such that  $I^1 = i[1]$ . Those two constructions give isomorphic structures. Indeed, the composition  $P \circ I : (H(A), M_i) \rightarrow (H(A), M_p)$  is a  $d$ -pre-Calabi-Yau morphism such that  $(P \circ I)^1$  is an isomorphism since  $H(A)$  is minimal so  $P \circ I$  is an isomorphism of  $d$ -pre-Calabi-yau algebras by Lemma 4.9.  $\square$

**Definition 4.17.** Let  $(A, s_{d+1}M_A)$  and  $(B, s_{d+1}M_B)$  be  $d$ -pre-Calabi-Yau algebras. We consider quasi-isomorphisms of complexes  $i_A : H(A) \rightarrow A$  and  $p_B : B \rightarrow H(B)$ . Then, by Theorem 4.13  $H(A)$  is endowed with a  $d$ -pre-Calabi-Yau structure  $M_{i_A}$  and  $i_A$  extends to a  $d$ -pre-Calabi-Yau morphism  $I_A :$

$H(A) \rightarrow A$ . Moreover, by Theorem 4.14  $H(B)$  is endowed with a  $d$ -pre-Calabi-Yau structure  $M_{p_B}$  and  $p_B$  extends to a  $d$ -pre-Calabi-Yau morphism  $P_B : B \rightarrow H(B)$ . Given a  $d$ -pre-Calabi-Yau morphism  $F : A \rightarrow B$  we define a  $d$ -pre-Calabi-Yau morphism  $H(F) : H(A) \rightarrow H(B)$  as  $H(F) = P_B \circ F \circ I_A$ .

Using the previous results, we deduce that any quasi-isomorphism of  $d$ -pre-Calabi-Yau algebras admits a quasi-inverse.

**Theorem 4.18.** *Let  $(A, s_{d+1}M_A)$  and  $(B, s_{d+1}M_B)$  be  $d$ -pre-Calabi-Yau algebras and consider a  $d$ -pre-Calabi-Yau morphism  $F : (A, s_{d+1}M_A) \rightarrow (B, s_{d+1}M_B)$ . Consider quasi-isomorphisms  $i_A : H(A) \rightarrow A$  and  $p_B : B \rightarrow H(B)$  and  $H(F) : (H(A), M_{i_A}) \rightarrow (H(B), M_{p_B})$  given in Definition 4.17. Then, if  $F$  is a quasi-isomorphism, there exists a quasi-isomorphism  $G : (B, s_{d+1}M_B) \rightarrow (A, s_{d+1}M_A)$  of  $d$ -pre-Calabi-Yau algebras as well as quasi-isomorphisms  $i_B : H(B) \rightarrow B$ ,  $p_A : A \rightarrow H(A)$  such that  $p_A \circ i_A = \text{id}_{H(A)}$ ,  $p_B \circ i_B = \text{id}_{H(B)}$  and whose associated  $H(G)$  in the sense of Definition 4.17 is the inverse of  $H(F)$ .*

*Proof.* Since  $F$  is a quasi-isomorphism, it induces by Lemma 4.9 an isomorphism of  $d$ -pre-Calabi-Yau algebras  $H(F) : (H(A), M_{i_A}) \rightarrow (H(B), M_{p_B})$  which has an inverse denoted here by  $K : (H(B), M_{p_B}) \rightarrow (H(A), M_{i_A})$ . Consider  $i_B : H(B) \rightarrow B$ ,  $p_A : A \rightarrow H(A)$  such that  $p_A \circ i_A = \text{id}_{H(A)}$  and  $p_B \circ i_B = \text{id}_{H(B)}$ . By Theorem 4.13 and Theorem 4.14, those maps extend to  $d$ -pre-Calabi-Yau morphisms  $I_A : H(A) \rightarrow A$ ,  $P_A : A \rightarrow H(A)$ ,  $I_B : H(B) \rightarrow B$  and  $P_B : B \rightarrow H(B)$  such that  $(P_A \circ I_A)^1 = \text{id}_{H(A)[1]}$  and  $(P_B \circ I_B)^1 = \text{id}_{H(B)[1]}$ . Moreover, up to an automorphism of the  $d$ -pre-Calabi-Yau algebra  $H(A)$  (resp.  $H(B)$ ), it is possible to assume that  $P_A \circ I_A = \text{id}$  (resp.  $P_B \circ I_B = \text{id}$ ). Then  $G = I_A \circ K \circ P_B$  gives a quasi-inverse for  $F$  since by definition we have that  $H(G) = P_A \circ G \circ I_B = P_A \circ I_A \circ K \circ P_B \circ I_B = K$ .  $\square$

## References

- [1] Marion Boucrot, *Morphisms of pre-Calabi-Yau categories and morphisms of cyclic  $A_\infty$ -categories*, 2023. arXiv:2304.13661 [math.KT].  $\uparrow 3$
- [2] T. V. Kadeišvili, *On the theory of homology of fiber spaces*, Uspekhi Mat. Nauk **35** (1980), no. 3(213), 183–188. International Topology Conference (Moscow State Univ., Moscow, 1979). MR580645  $\uparrow 1$
- [3] Maxim Kontsevich, *Weak Calabi-Yau algebras* (2013), available at <https://math.berkeley.edu/~auroux/miami2013-notes/>. Notes taken from the talk at *Conference on Homological Mirror Symmetry*.  $\uparrow 1$
- [4] Maxim Kontsevich, Alex Takeda, and Yiannis Vlassopoulos, *Pre-Calabi-Yau algebras and topological quantum field theories*, arXiv, 2021. arXiv:2112.14667 [math.AG].  $\uparrow 2, 3, 12, 13$
- [5] Dan Petersen, *A closer look at Kadeishvili’s theorem*, High. Struct. **4** (2020), no. 2, 211–221. MR4133168  $\uparrow 1, 2, 4, 9, 10, 16$
- [6] Paul Seidel, *Fukaya  $A_\infty$ -structures associated to Lefschetz fibrations. II*, Algebra, geometry, and physics in the 21st century, 2017, pp. 295–364. MR3727564  $\uparrow 1$
- [7] James Dillon Stasheff, *Homotopy associativity of  $H$ -spaces. I, II*, Trans. Amer. Math. Soc. **108** (1963), 275–292; *ibid.* **108** (1963), 293–312. MR0158400  $\uparrow 4$
- [8] Masahiro Sugawara, *On the homotopy-commutativity of groups and loop spaces*, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **33** (1960/61), 257–269. MR120645  $\uparrow 4$
- [9] Thomas Tradler and Mahmoud Zeinalian, *Algebraic string operations*,  $K$ -Theory **38** (2007), no. 1, 59–82. MR2353864  $\uparrow 1$

MARION BOUCROT: Univ. Grenoble Alpes, CNRS, IF, 38000 Grenoble, France  
*E-mail adress* : marion.boucrot@univ-grenoble-alpes.fr