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Fiberwise Criteria for Twisted Forms of Algebraic Structures

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In memoriam Professor Georgia Benkart

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Abstract

We provide a criterion for certain algebraic objects over Jacobson schemes to be forms of each other based on their behaviour at closed fibres. This criterion permits to answer a question related to current and affine Kac-Moody Lie algebras that I. Burban had asked the authors.

Keywords: Reductive group scheme, torsor, Lie algebras.

MSC 2000: 14L30, 17B67, 11E72, 14E20.

1 Introduction

Let \mathbf{G} be a group scheme over a base scheme S . The concept of \mathbf{G} -torsors over S is to be found in many areas of mathematics and mathematical physics; they are useful tools to frame problems in a language that is partial to powerful methods from algebraic geometry. \mathbf{G} acts on itself by right multiplication. A \mathbf{G} -torsor is a scheme \mathbf{X} with a right action of \mathbf{G} that “locally” looks like \mathbf{G} with this action. By locally we mean that there exists a faithfully flat and locally finitely presented scheme morphism

$T \rightarrow S$ such that \mathbf{X}_T and \mathbf{G}_T are isomorphic as T -schemes with the corresponding induced \mathbf{G}_T -action.

The treatise [SGA3] shows how a deep understanding of \mathbf{G} can be had through the study of its fibres $\mathbf{G}_s := \mathbf{G} \times_S \text{Spec}(\kappa(s))$ where $\kappa(s)$ is the residue field of $s \in S$. It thus seems natural, given a right action of \mathbf{G} on an S -scheme \mathbf{X} , to study situations under which the fibres \mathbf{X}_s being \mathbf{G}_s -torsors yield information about \mathbf{X} itself being a \mathbf{G} -torsor. In practice the fibres of closed points are more tractable. The existence of “enough” closed points brings up the natural working hypothesis that S be a Jacobson scheme. It is under this assumption that we are able to create a fiberwise criterion for torsors under certain reductive group schemes.

This work grew up out of the desire to answer a question that I. Burban posed to the authors a couple of years ago [Bu]. His question and the (positive) answer that can be given using our main result is given in the last section.

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2 Preliminaries on group schemes and smoothness

Throughout this paper S is a (base) scheme with structure sheaf \mathcal{O}_S . By an S -*functor* (resp. *monoid*, *group*) we mean a contravariant functor \mathbf{F} from the category of schemes over S to the category of sets (resp. monoids, groups). When \mathbf{F} is representable, i.e. a scheme, we say that \mathbf{F} is an S -*scheme* (resp. *monoid scheme*, *group scheme*). The use of bold face characters in the text is used to emphasize the functorial nature of the object under consideration. In particular, if X is an S -scheme then its functor of points

$$T \mapsto \text{Hom}_{S\text{-Sch}}(T, X)$$

will be denoted by \mathbf{X} .

If \mathbf{F} is an S -functor and T a scheme over S , we denote by \mathbf{F}_T the T -functor obtained by base change. If $T = \text{Spec}(R)$ we denote $\mathbf{F}(T)$ and \mathbf{F}_T by $\mathbf{F}(R)$ and \mathbf{F}_R respectively. If \mathbf{F} is an S -scheme, then $\mathbf{F}_T = \mathbf{F} \times_S T$ and, as it is customary, we denote $\mathbf{F} \times_{\text{Spec}(R)} T$ by $\mathbf{F} \times_R T$ and $\mathbf{F} \times_S \text{Spec}(R)$ by $\mathbf{F} \times_S R$.

We denote as usual by \mathbf{O}_S the affine S -ring scheme $S[t] = \text{Spec}(\mathbb{Z}[t]) \times_{\mathbb{Z}} S$. Thus $\mathbf{O}_S(T) = \mathcal{O}_T(T)$. Recall that an \mathbf{O}_S -*module* is an abelian S -group \mathbf{M} together with an $\mathbf{O}_S(T)$ -module structure on $\mathbf{M}(T)$ that is functorial on T . The concept of an \mathbf{O}_S -algebra is defined similarly. Base change is defined and denoted as it is for functors.

We mainly use the terminology and notation of Grothendieck–Dieudonné [EGA-I], which for the most part agrees with that of Demazure–Grothendieck used in [SGA3, Exp. I and II]. Below we briefly review those concepts and results that are relevant to this paper.

2.1 Groups attached to quasi-coherent modules

Let \mathcal{E} be a quasi-coherent module over S . We denote its dual by \mathcal{E}^\vee . For each morphism $f : T \rightarrow S$ we let $\mathcal{E}_T = f^*(\mathcal{E})$ be the inverse image of \mathcal{E} under the morphism f .¹ Define an abelian S -group $\mathbf{V}(\mathcal{E})$ by $\mathbf{V}(\mathcal{E})(T) = \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{E}_T, \mathcal{O}_T) = \Gamma(T, (\mathcal{E}_T)^\vee)$. \mathbf{V} is actually an S -scheme; it is represented by the affine scheme $\mathrm{Spec}(\mathbf{Sym}(\mathcal{E}))$ where $\mathbf{Sym}(\mathcal{E})$ is the symmetric \mathcal{O}_S -algebra of \mathcal{E} [EGA-I, 9.4.9]. If \mathcal{E} is of finite type (resp. of finite presentation), then $\mathbf{V}(\mathcal{E})$ is an S -scheme of finite type (resp. of finite presentation), *ibid*, 9.4.11.

The abelian S -group $\mathbf{W}(\mathcal{E})$ is defined by $\mathbf{W}(\mathcal{E})(T) = \Gamma(T, \mathcal{E}_T)$. Recall that if \mathcal{E} is locally free of finite type then $\mathbf{W}(\mathcal{E}) \simeq \mathbf{V}(\mathcal{E}^\vee)$. In particular $\mathbf{W}(\mathcal{E})$ is in this case an affine S -scheme. Note that the abelian S -groups $\mathbf{V}(\mathcal{E})$ and $\mathbf{W}(\mathcal{E})$ have natural \mathbf{O}_S -module structures.

Remark 2.1. We can view \mathbf{V} (resp. \mathbf{W}) as a contravariant (resp. covariant) functor from the category of quasi-coherent \mathcal{O}_S -modules to the category of \mathbf{O}_S -modules. These functors are full and faithful [SGA3, I Prop. 4.6.2].

To an \mathbf{O}_S -module \mathbf{L} we attach the \mathbf{O}_S -module $\mathbf{End}_{\mathbf{O}_S}(\mathbf{L})$ defined by $T \mapsto \mathrm{End}_{\mathbf{O}_T}(\mathbf{L}_T)$, and the S -group $\mathbf{Aut}_{\mathbf{O}_S}(\mathbf{L})$ defined by $T \mapsto \mathrm{Aut}_{\mathbf{O}_T}(\mathbf{L}_T)$. Of particular interest to us will be the case when \mathbf{L} is an \mathbf{O}_S -algebra. We then define an \mathbf{O}_S -submodule $\mathbf{Der}(\mathbf{L})$ of $\mathbf{End}_{\mathbf{O}_S}(\mathbf{L})$ by $T \mapsto \mathrm{Der}_{\mathbf{O}_T}(\mathbf{L}_T)$. Finally, we define the S -group $\mathbf{Aut}_{\mathbf{O}_S\text{-alg}}(\mathbf{L})$ via $T \mapsto \mathrm{Aut}_{\mathbf{O}_T\text{-alg}}(\mathbf{L}_T)$.

Example 2.2. Assume $S = \mathrm{Spec}(R)$. Let \mathcal{L} be a quasi-coherent \mathcal{O}_S -module, and let L be the corresponding R -module. Let $\mathbf{L} = \mathbf{W}(\mathcal{L})$. Recall that for an S -scheme T , $\mathbf{L}(T)$ is the $\mathcal{O}_T(T)$ -module of global section of the \mathcal{O}_T -module $\mathcal{L} \otimes_{\mathcal{O}_S} \mathcal{O}_T$. There is a natural $\mathcal{O}_T(T)$ -module morphism

$$L \otimes_R \mathcal{O}_T(T) \rightarrow \mathbf{L}(T).$$

This map is an isomorphism in two important cases:

- (i) $T = \mathrm{Spec}(R')$ is affine.
- (ii) \mathcal{L} is locally free of finite type (i.e., L is projective of finite type).

In these two cases, we have

$$\mathbf{End}_{\mathbf{O}_S}(\mathbf{L})(T) = \mathrm{End}_{\mathcal{O}_T(T)\text{-alg}}(L \otimes_R \mathcal{O}(T)).$$

If furthermore \mathcal{L} is an \mathcal{O}_S -algebra, then L is an R -algebra and

$$\mathbf{Der}_{\mathbf{O}_S}(\mathbf{L})(T) = \mathrm{Der}_{\mathcal{O}_T(T)}(L \otimes_R \mathcal{O}(T))$$

¹ We will also use the [SGA3] notation $\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_T$ for the inverse image.

while

$$\mathbf{Aut}_{\mathbf{O}_S\text{-alg}}(\mathbf{L})(T) = \mathbf{Aut}_{\mathcal{O}_T(T)\text{-alg}}(L \otimes_R \mathcal{O}(T)).$$

Lemma 2.3. *Let \mathcal{L} be an \mathcal{O}_S -Lie algebra. Assume that as an \mathcal{O}_S -module \mathcal{L} is locally free of finite rank. Then.*

(1) *The natural \mathbf{O}_S -module morphism $\mathbf{W}(\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{L})) \rightarrow \mathbf{End}_{\mathbf{O}_S}(\mathbf{W}(\mathcal{L}))$ is an isomorphism.*

(2) *By restriction the map of (1) induces an \mathbf{O}_S -module isomorphism*

$$\mathbf{W}(\mathcal{D}er_{\mathcal{O}_S}(\mathcal{L})) \simeq \mathbf{Der}_{\mathbf{O}_S}(\mathbf{W}(\mathcal{L}))$$

(3) *If we denote $\mathbf{W}(\mathcal{L})$ by \mathbf{L} , then $\mathbf{End}_{\mathbf{O}_S}(\mathbf{L})$, $\mathbf{Aut}_{\mathbf{O}_S}(\mathbf{L})$ and $\mathbf{Aut}_{\mathbf{O}_S\text{-alg}}(\mathbf{L})$ are representable by affine S -schemes of finite presentation.*

Proof. (1) That the map is an isomorphisms boils down to the following fact. Let R be a ring and M and R -module. If R'/R is a ring extension the natural R' -module homomorphism $\mathbf{End}_R(M) \otimes_R R' \rightarrow \mathbf{End}_{R'}(M \otimes_R R')$ is an isomorphism whenever M is projective of finite type.

(2) This follows from the definitions.

(3) Assume that $S = \text{Spec}(R)$. Then \mathcal{L} corresponds to an R -module L which is projective of finite rank. By definition $\mathbf{L}(T) = L \otimes_R \mathcal{O}_T(T)$. The \mathcal{O}_S -module $\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{L})$ corresponds to the R -module $\mathbf{End}_R(L) \simeq L^* \otimes_R L$. Because the R -module map $\mathbf{End}_{R\text{-mod}}(L) \otimes_R R' \rightarrow \mathbf{End}_{R'\text{-mod}}(L \otimes_R R')$ is an isomorphism for all R'/R , it follows that $\mathbf{End}_{\mathbf{O}_S}(\mathbf{L})$ is represented by the affine R -scheme of finite presentation $\mathbf{W}(\mathcal{L}^\vee \otimes_{\mathcal{O}_S} \mathcal{L})$.

It is clear that $\mathbf{Aut}_{\mathbf{O}_S}(\mathbf{L})$ is an affine S -scheme of finite presentation since it is locally isomorphic to $\mathbf{GL}_{n,S}$ for some n . Finally $\mathbf{Aut}_{\mathbf{O}_S\text{-alg}}(\mathbf{L})$ is a closed subscheme of $\mathbf{Aut}_{\mathbf{O}_S}(\mathbf{L})$, hence also affine. It is of finite presentation since L is locally free of finite rank.

From the foregoing considerations it follows that our two functors are affine S -schemes which are locally of finite presentation. Since their structure morphisms are affine, they are quasi-compact and separated, hence of finite presentation. \square

2.2 Group schemes and Lie algebras

Throughout \mathbf{G} will denote an S -group scheme and we denote by $e \in \mathbf{G}(S)$ its unit section. We refer to [SGA3, I and II], [DG] and [LLR, §1] for details in what follows.

If T is a scheme we denote by $T[\epsilon]$ the corresponding scheme of dual number [SGA3, II.2]. The functor $T \rightarrow \mathbf{Lie}(\mathbf{G})(T) := \ker(\mathbf{G}(T[\epsilon]) \rightarrow \mathbf{G}(T))$ is an \mathbf{O}_S -module isomorphic to $\mathbf{V}(\omega_{\mathbf{G}/S}^1)$ where $\omega_{\mathbf{G}/S}^1 = e^*(\Omega_{\mathbf{G}/S}^1)$ *ibid.* Prop. 3.3 and 3.6.

Furthermore, the $\mathbf{O}_S(\mathbf{T})$ -module $\mathbf{Lie}(\mathbf{G})(\mathbf{T})$ has a natural Lie algebra structure. It is thus an \mathbf{O}_S -Lie algebra. Recall that for all scheme morphisms $\mathbf{T} \rightarrow \mathbf{S}$ we have a natural $\mathbf{O}_{\mathbf{T}}$ -Lie algebra isomorphism

$$(2.1) \quad \mathbf{Lie}(\mathbf{G}) \times_S \mathbf{T} \simeq \mathbf{Lie}(\mathbf{G}_{\mathbf{T}}).$$

We denote by $\mathcal{L}ie(\mathbf{G})$ the vector group sheaf (fibration vectorielle) of sections of the affine scheme $\mathbf{Lie}(\mathbf{G}) \rightarrow \mathbf{S}$. In other words, $\mathcal{L}ie(\mathbf{G})$ is the \mathcal{O}_S -module $(\omega_{\mathbf{G}/\mathbf{S}}^1)^\vee = \mathrm{Hom}_{\mathcal{O}_S}(\omega_{\mathbf{G}/\mathbf{S}}^1, \mathcal{O}_S)$. Note that $\mathcal{L}ie(\mathbf{G})$ is naturally an \mathcal{O}_S -Lie algebra.

Remark 2.4. In general $\mathcal{L}ie(\mathbf{G})$ does not determine $\mathbf{Lie}(\mathbf{G})$, but it does if $\omega_{\mathbf{G}/\mathbf{S}}^1$ is locally free of finite type, in particular if \mathbf{G} is smooth. In this case $\mathbf{Lie}(\mathbf{G}) = \mathbf{W}(\mathcal{L}ie(\mathbf{G}))$. See [SGA3, II Lemma 4.11.7]. If $\mathbf{S} = \mathrm{Spec}(R)$ the R -Lie algebra $\mathbf{Lie}(\mathbf{G}) = \mathcal{L}ie(\mathbf{G})(\mathbf{S})$ is a projective R -module of finite type. For any ring extension R'/R we have $\mathbf{Lie}(\mathbf{G}_{R'}) = \mathbf{Lie}(\mathbf{G})(R') = \mathbf{Lie}(\mathbf{G}) \otimes_R R'$.

Lemma 2.5. *Let \mathcal{L} be an \mathcal{O}_S -algebra which is locally free of finite type (as an \mathcal{O}_S -module). Let $\mathbf{L} = \mathbf{W}(\mathcal{L})$. There is a natural \mathbf{O}_S -Lie algebra isomorphism $\mathbf{Der}_{\mathbf{O}_S}(\mathbf{L}) \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_S\text{-alg}}(\mathbf{L}))$.*

Proof. $\mathbf{Aut}_{\mathbf{O}_S}(\mathbf{L})$ is an \mathbf{S} -group scheme by Lemma 2.3(3). Since \mathbf{L} is a good \mathbf{O}_S -module [SGA3, II Def. 4.4 and Ex. 4.4.2], we have an \mathbf{O}_S -module isomorphisms $\eta : \mathbf{End}_{\mathbf{O}_S}(\mathbf{L}) \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_S}(\mathbf{L}))$ [SGA3, II Prop. 4.5]. We claim that the restriction of η to $\mathbf{Der}_{\mathbf{O}_S\text{-alg}}(\mathbf{L})$ is our isomorphism. The proof reduces to the case when \mathbf{S} is affine, which can be found in [DG, II §4 2.3]. \square

2.3 Connected component of the identity, forms and type of a reductive group scheme.

Let \mathbf{G} be an \mathbf{S} -group scheme which is locally of finite presentation. We consider the \mathbf{S} -subgroup (functor) \mathbf{G}° of \mathbf{G} [SGA3, VI_B.3.1] defined by

$$\mathbf{G}^\circ(\mathbf{T}) = \left\{ u \in \mathbf{G}(\mathbf{T}) \mid \forall s \in \mathbf{S}, u_s(\mathbf{T}_s) \subset \mathbf{G}_s^\circ \right\}.$$

where \mathbf{G}_s° is the connected component of the identity of the $\kappa(s)$ -algebraic group \mathbf{G}_s . If \mathbf{G} is smooth along the unit section, \mathbf{G}° is representable by a smooth \mathbf{S} -group scheme called *the connected component of the identity of \mathbf{G}* [SGA3, VI_B.4.1]. Furthermore, the fibre $(\mathbf{G}^\circ)_s$ is naturally isomorphic to \mathbf{G}_s° .

Let \mathbf{G} be a reductive S -group scheme, that is a smooth and affine S -group whose geometric fibres are *connected* reductive algebraic groups. An S -group scheme \mathbf{G}' is called a (*twisted*) *form of \mathbf{G}* , if there exist a faithfully flat and locally presented extension $T \rightarrow S$ such that \mathbf{G}_T and \mathbf{G}'_T are isomorphic T -group schemes. In a similar fashion one defines forms of \mathbf{O}_S -algebras.

Let \mathbf{G} be a reductive K -group scheme, where K is an algebraically closed field. If \mathbf{T} is a maximal torus of \mathbf{G} , then \mathbf{T} is split and defines a root datum which is up to isomorphism independent of the choice of \mathbf{T} . It is called the *type of \mathbf{G}* and is denoted by $\mathcal{R}(\mathbf{G})$.

Let \mathbf{G} be a reductive S -group scheme. If $s \in S$, the *type of \mathbf{G} at s* is the type of the reductive $\overline{\kappa(s)}$ -group $\mathbf{G}_{\overline{s}} := \mathbf{G}_s \times_{\kappa(s)} \overline{\kappa(s)}$. Because \mathbf{G} possesses maximal tori locally for the étale topology, one knows that the type function $s \mapsto$ type of $\mathbf{G}_{\overline{s}}$ is locally constant [SGA3, XXII 2.8]. We say that \mathbf{G} is of *constant type* if the type function is constant. This is the case, for example, if S is connected.

2.4 Specific properties in characteristic zero

We assume in this section that S is a \mathbb{Q} -scheme. Let \mathbf{G} be a semisimple adjoint S -group scheme of constant type. We will denote the \mathbf{O}_S -Lie algebra $\mathbf{Lie}(\mathbf{G})$ by \mathbf{L} , and the \mathcal{O}_S -Lie algebra $\mathcal{L}ie(\mathbf{G})$ by \mathcal{L} . Recall (see Remark 2.4) that \mathcal{L} is locally free of finite type and that $\mathbf{W}(\mathcal{L}) = \mathbf{L}$. Since \mathbf{G} and its simply connected cover \mathbf{G}^{sc} have the same Lie algebra, we have isomorphisms of affine S -group schemes [SGA3, XXIV 3.6 and 7.3.1]

$$(2.2) \quad \mathbf{Aut}(\mathbf{G}) \simeq \mathbf{Aut}(\mathbf{G}^{\text{sc}}) \simeq \mathbf{Aut}_{\mathbf{O}_S\text{-Lie}}(\mathbf{L}).^2$$

In particular, the S -scheme $\mathbf{Aut}_{\mathbf{O}_S\text{-Lie}}(\mathbf{L})$ is smooth and affine. By (2.2) and [SGA3, XXIV 1.3 and 1.8] we have an isomorphism

$$(2.3) \quad \mathbf{G} \simeq \mathbf{Aut}_{\mathbf{O}_S\text{-Lie}}(\mathbf{L})^\circ.$$

This yields an \mathbf{O}_S -Lie algebra isomorphism

$$(2.4) \quad \mathbf{L} \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_S\text{-Lie}}(\mathbf{L})) = \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_S\text{-Lie}}(\mathbf{L})^\circ).$$

Recall [SGA3, II Theo. 4.7 and Prop. 4.8] the adjoint representation $\text{Ad} : \mathbf{G} \rightarrow \mathbf{Aut}_{\mathbf{O}_S\text{-mod}}(\mathbf{L})$ and the induced a \mathbf{O}_S -Lie algebra homomorphism $\text{ad} : \mathbf{L} \rightarrow \mathbf{End}_{\mathbf{O}_S\text{-mod}}(\mathbf{L})$.

² La “définition évidente” of the S -group scheme $\underline{\mathcal{A}ut}_{\mathbf{O}_S\text{-alg-de-Lie}}(\mathcal{L})$ of 7.3.1(iii) is precisely our $\mathbf{Aut}_{\mathbf{O}_S\text{-Lie}}(\mathbf{L})$, so that the asertion is that the natural map $\mathbf{Aut}(\mathbf{G}^{\text{sc}}) \rightarrow \mathbf{Aut}_{\mathbf{O}_S\text{-Lie}}(\mathbf{L})$ is an isomorphism.

Lemma 2.6. *Under the identification $\mathbf{L} \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L}))$ of (2.4) the map ad induces an \mathcal{O}_S -Lie algebra isomorphism between \mathbf{L} and $\mathbf{Der}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L})$.*

Proof. All the morphisms under consideration are S -scheme morphisms, so the question is local on S and we may assume that $S = \text{Spec}(R)$ for some \mathbb{Q} -ring R . Consider the R -Lie algebra $L = \mathbf{L}(R)$. This is a projective R -module of finite type. Since $\mathbf{L}(T) = L \otimes_R \mathcal{O}_T(T)$ and $\mathbf{Der}_{\mathcal{O}_S}(\mathbf{L})(T) = \text{Der}_{\mathcal{O}_T(T)}(L \otimes_R \mathcal{O}_T(T))$ for all S -schemes T (see Example 2.2), it will suffice to show that for all ring extensions R'/R the R' -Lie algebra homomorphism $d = \text{ad}'_R : L \otimes_R R' \rightarrow \text{Der}_{R'}(L \otimes_R R')$ is an isomorphism. By replacing \mathbf{G} by $\mathbf{G}_{R'}$ we may assume that $R = R'$. By [SGA3, II 4.7.2] the map d is nothing but the adjoint representation $d(x) = \text{ad}_L(x)$. Since \mathbf{G} is of constant type there exists a unique Chevalley group \mathbf{G}_0 such that \mathbf{G} is a twisted form of $\mathbf{G}_0 \times_{\mathbb{Z}} R$. Let $\mathfrak{g} = \mathbf{Lie}(\mathbf{G}_0)(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then \mathfrak{g} is a finite dimensional split semisimple Lie algebra over \mathbb{Q} and L is a twisted form of $\mathfrak{g} \otimes_{\mathbb{Q}} R$, that is, there exists an fppf extension (in fact étale cover) R' of R such that the R' -Lie algebras $L \otimes_R R'$ and $(\mathfrak{g} \otimes_{\mathbb{Q}} R) \otimes_R R' \simeq \mathfrak{g} \otimes_{\mathbb{Q}} R'$ are isomorphic. Fix one such isomorphism

$$(2.5) \quad \psi : L \otimes_R R' \rightarrow \mathfrak{g} \otimes_{\mathbb{Q}} R'.$$

If $d(x) = 0$, then x belongs to the centre of L and therefore $\psi(x \otimes 1)$ belongs to the centre of $\mathfrak{g} \otimes_{\mathbb{Q}} R'$, which is trivial. Since R'/R is faithfully flat $x = 0$ so that d is injective and we henceforth identify L with an R -submodule of $\text{Der}_R(L)$. To show that d is surjective, that is that every derivation of L is inner, we reason as follows. If $L = \mathfrak{g} \otimes_{\mathbb{Q}} R$ every derivation is inner by Whitehead's Lemma (see [P1, Example 4.9]).³ For L arbitrary, we appeal to the isomorphism (2.5). By the split case we have $(\text{Der}_R(L)/L) \otimes_R R' = 0$. Thus $L = \text{Der}_R(L)$ as desired. \square

For answering Burban's question we will use the following variation of the previous result.

Lemma 2.7. *Let \mathcal{L} be an \mathcal{O}_S -Lie algebra, and consider the corresponding adjoint representation $\text{ad} : \mathcal{L} \rightarrow \mathcal{D}er(\mathcal{L})$. Assume that \mathcal{L} and $\mathcal{D}er(\mathcal{L})$ are locally free of finite type (as \mathcal{O}_S -modules), and that the fibres \mathcal{L}_s are $\kappa(s)$ -semisimple Lie algebras (necessarily of finite dimension) for all $s \in S$. Then ad is an \mathcal{O}_S -Lie algebra isomorphism.*

Proof. Along standard lines we may assume that $S = \text{Spec}(R)$ with R a local ring. In this case \mathcal{L} corresponds to a free R -module L of finite rank, and $\mathcal{D}er_{\mathcal{O}_S}(\mathcal{L})$ corresponds to the R -module $\text{Der}_R(L)$, which is also free of finite rank by assumption. Since every derivation of $L \otimes_R \kappa(s)$ is inner the Lemma follows by Nakayama considerations. \square

³ Note that our L is denoted by \mathcal{L} in [P1].

2.5 Semicontinuity property for Lie algebras

For convenience we recall the following well-known fact.

Lemma 2.8. *Let \mathcal{F} be a quasi-coherent \mathcal{O}_S -module of finite presentation. Then the function $s \mapsto \dim_{\kappa(s)} (\mathcal{F} \otimes_{\mathcal{O}_S} \kappa(s))$ is upper semi-continuous.*

Proof. The statement is local and therefore reduces then to the case of a ring R and of an R -module M of finite presentation [Stacks, Tag 01PC]. If R is noetherian then M is coherent and the result can be found in [Ha, 12.7.2]. For lack of a reference we provide a proof in the general case.

We consider an exact sequence $R^m \xrightarrow{A} R^n \rightarrow M \rightarrow 0$ where A is a matrix of size (m, n) with entries in R . Let $\mathfrak{q} \in \text{Spec}(R)$ and $r = \dim_{\kappa(\mathfrak{q})}(M \otimes_R \kappa(\mathfrak{q}))$. Since the sequence $\kappa(\mathfrak{q})^m \xrightarrow{A_{\kappa(\mathfrak{q})}} \kappa(\mathfrak{q})^n \rightarrow M_{\kappa(\mathfrak{q})} \rightarrow 0$ is exact, there exists a minor B of A of size $(n - r, n - r)$ such that $\det(B_{\kappa(\mathfrak{q})}) \neq 0 \in \kappa(\mathfrak{q})$. Consider the basic open set U of $\text{Spec}(R)$ consisting of prime ideals of R that do not contain $f = \det(B) \in R$. From $\det B_{\kappa(\mathfrak{q})} \neq 0$ it follows that $\mathfrak{q} \in U$. It is clear that for all $\mathfrak{p} \in U$ the image of $A_{\kappa(\mathfrak{p})}$ is a $\kappa(\mathfrak{p})$ -space of dimension at least $n - r$. Thus

$$\dim_{\kappa(\mathfrak{p})} (M \otimes_R \kappa(\mathfrak{p})) \leq r = \dim_{\kappa(\mathfrak{q})} (M \otimes_R \kappa(\mathfrak{q}))$$

for all $\mathfrak{p} \in U$, so that our function is thus upper semi-continuous as desired. \square

Lemma 2.9. *Let \mathbf{G} be an S -group scheme that is locally of finite presentation. Then $\omega_{\mathbf{G}/S}^1$ is an \mathcal{O}_S -module of finite presentation and the map $s \mapsto \dim_{\kappa(s)} \text{Lie}(\mathbf{G}_s)$ is upper semi-continuous.*

Proof. That $\omega_{\mathbf{G}/S}^1$ is of finite presentation follows from [Stacks, Tag 01V2, 01V3].

By Remark 2.4 we have a $\kappa(s)$ -Lie algebra isomorphism $\text{Lie}(\mathbf{G}_s) \simeq \mathbf{Lie}(\mathbf{G})(\kappa(s))$. From the isomorphism $\mathbf{Lie}(\mathbf{G}) \simeq \mathbf{V}(\omega_{\mathbf{G}/S}^1)$, we see that the dimension of $\mathbf{Lie}(\mathbf{G})(\kappa(s))$ at a point $s \in S$ is the dimension of the $\kappa(s)$ -space $\text{Hom}_{\kappa(s)} (\omega_{\mathbf{G}/S}^1 \otimes_{\mathcal{O}_S} \kappa(s), \kappa(s))$, which is the dimension of $\omega_{\mathbf{G}/S}^1 \otimes_{\mathcal{O}_S} \kappa(s)$ since $\omega_{\mathbf{G}/S}^1$ is an \mathcal{O}_S -module of finite type. Furthermore, since $\omega_{\mathbf{G}/S}^1$ is in fact of finite presentation, Lemma 2.8 implies that the map $s \mapsto \dim_{\kappa(s)} (\mathbf{Lie}(\mathbf{G})(\kappa(s)))$ is upper semi-continuous. \square

3 Group schemes over Jacobson schemes

3.1 Jacobson schemes and smoothness

We refer the reader to [EGAIV, §10] for general results about Jacobson schemes. Let X be a topological space. We recall that a subset X_0 of X is called *very dense* if

for every closed subset $Z \subset X$ we have $Z = \overline{Z \cap X_0}$. Let S be a scheme. View S as a topological space and let $S_0 \subset S$ the set of closed points. The scheme S is said to be *Jacobson* if S_0 is a very dense subset of S .

Let us recall for the sake of completeness that:

- (i) Very dense subsets of a topological space are dense.
- (ii) If $S = \text{Spec}(R)$, then S is Jacobson if and only if R is a Jacobson ring, that is every prime ideal of R is an intersection of maximal ideals.
- (iii) If $S = \bigcup_{i \in I} \text{Spec}(R_i)$, the S is Jacobson if and only if every R_i is a Jacobson ring.
- (iv) Every radical ideal I is the intersection of the maximal ideals containing it [Stacks, Tag 00G4].⁴

Proposition 3.1. *Let $f : X \rightarrow S$ be a scheme morphisms which is locally of finite presentation. Assume that $x \in X$ and $s = f(x)$ are such that:*

- (i) f is flat at x .
- (ii) f_s is smooth.

Then f is smooth on an open neighborhood of x .

Proof. By assumption there exists affine open subschemes $\text{Spec}(A) = U \subset X$ and $\text{Spec}(B) = V \subset S$ such that $x \in U$, $f(U) \subset V$ and A is a finitely presented B -algebra. Let $\mathfrak{q} \in \text{Spec}(A)$ and $\mathfrak{p} \in \text{Spec}(B)$ be the points corresponding to x and $f(x)$ respectively.

By (i) $A_{\mathfrak{q}}$ is flat over $B_{\mathfrak{p}}$, while by (ii) $A \otimes_B \kappa(\mathfrak{p})$ is smooth over $\kappa(\mathfrak{p})$. It follows from [Stacks, Tag 00TF] that the restriction of f to U is smooth at \mathfrak{q} , hence (by definition of smoothness) smooth also in a neighbourhood of \mathfrak{q} . \square

Corollary 3.2. *Let $f : X \rightarrow S$ be as above. Assume that (i) and (ii) hold for all closed points of X . If S is Jacobson, then f is smooth.*

Proof. By [EGAIV, Cor. 10.4.7] X is Jacobson. Let X_0 be the set of closed points of X viewed as a topological space. By the Proposition there exists an open $U \subset X$ containing X_0 in which f is smooth. Let $Z = X \setminus U$. If $Z \neq \emptyset$, then $Z \cap X_0 \neq \emptyset$ because X_0 is very dense. This contradicts $X_0 \subset U$. Thus $Z = \emptyset$ so that $U = X$ as desired. \square

⁴ We will not use these properties in what follows.

Theorem 3.3. *Let S be an integral Jacobson scheme, and \mathbf{G} be an S -group scheme of finite presentation. Assume that for all closed point $b \in S$ the fibres \mathbf{G}_b are smooth and of the same dimension d .⁵ Then.*

- (1) *For all $s \in S$ the algebraic $\kappa(s)$ -group \mathbf{G}_s is smooth of dimension d .*
- (2) *The S -functor \mathbf{G}° is representable by a smooth S -group scheme of relative dimension d which is open in \mathbf{G} .*
- (3) *If for each closed point b of S the fibre \mathbf{G}_b is connected, then \mathbf{G} is smooth and has connected geometric fibres (i.e., $\mathbf{G}_{\bar{\eta}}$ is connected).*

Proof. (1) We denote by $\eta : \text{Spec}(F) \rightarrow S$ the generic point of S . Chevalley's generic flatness theorem [G-W, 10.85] shows that there exists an open dense subset U of S such that $\mathbf{G} \times_S U$ is flat over U . We know that U is also Jacobson [EGAIV, Prop. 10.3.3]. Corollary 3.2 shows that $\mathbf{G} \times_S U$ is smooth over U . By [Stacks, Tag 05F7], up to shrinking U if necessary, the dimension function of the fibres of $\mathbf{G} \times_S U \rightarrow U$ is constant. Since by assumption this dimension has value d on the closed points of U , we get that $\mathbf{G} \times_S U$ is smooth of relative dimension d .⁶ In particular the algebraic F -group \mathbf{G}_F is smooth of dimension d .

Let $s \in S$ and let $c = \dim(\mathbf{G}_s)$. According to Chevalley's semicontinuity theorem [SGA3, VI_B.4.1]

$$F_c = \{x \in S : \dim(\mathbf{G}_x) \geq c\} \subset S$$

is a closed set. Since $F_c \neq \emptyset$ and S is Jacobson, it follows that there exists a closed point b in F_c . We thus have

$$(3.1) \quad \dim(\mathbf{G}_s) \leq \dim(\mathbf{G}_b) = d.$$

On the other hand since s belongs to $\overline{\{\eta\}} \subset F_\eta$, we have

$$(3.2) \quad \dim(\mathbf{G}_F) \leq \dim(\mathbf{G}_s).$$

Since \mathbf{G}_F is of dimension d it follows from (3.1) and (3.2) that $\dim(\mathbf{G}_s) = d$.

For establishing smoothness we use a variation of the previous argument by applying semicontinuity considerations to the Lie algebras. Lemma 2.9 yields the inequalities

$$\dim_F \text{Lie}(\mathbf{G}_F) \leq \dim_{\kappa(s)} \text{Lie}(\mathbf{G}_s) \leq \dim_{\kappa(b)} \text{Lie}(\mathbf{G}_b).$$

Since \mathbf{G} is smooth of dimension d at η and at b we obtain that $d = \dim_{\kappa(s)} \text{Lie}(\mathbf{G}_s)$. Thus $\text{Lie}(\mathbf{G}_s)$ and \mathbf{G}_s have the same dimension (namely d). By the smoothness criterion [DG, II.5.2.1] \mathbf{G}_s is smooth. This completes the proof of (1).

⁵ Since \mathbf{G} is of finite presentation \mathbf{G}_s is an algebraic $\kappa(s)$ -group. Thus \mathbf{G}_s has a finite number of irreducible components. They all have the same (finite) dimension and this is also the dimension of \mathbf{G}_s .

⁶ That is, all the fibres $\mathbf{G}_s, s \in U$, are equidimensional and this dimension is d .

(2) By (1) we see that assumption (ii) of [SGA3, VI_B Cor. 4.4] holds. It follows that the S -functor \mathbf{G}° is representable by a smooth S -group scheme which is open in \mathbf{G} .

(3) According to [Stacks, Tag 055I], the level set

$$E_n = \{s \in S \mid \mathbf{G}_s \text{ has } n \text{ geometrically connected components}\}$$

is a locally constructible subset of S for each $n \geq 1$. Since S is a Jacobson scheme, the set S_0 of closed points of the underlying topological space of S is very dense in S . Thus $S_0 \cap E_n$ is dense in E_n for each n [EGAIV, 10.1.2, (b')]. Our assumption implies that $E_n = \emptyset$ for $n \geq 2$ so that $E_1 = S$. That \mathbf{G} is smooth now follows from (2). \square

3.2 Forms

The following result generalizes to integral Jacobson schemes Theorem 1.5 of [PY], where the base is assumed to be a Dedekind scheme.

Theorem 3.4. *Let S be an integral Jacobson scheme and \mathbf{G}' an affine S -group scheme of finite presentation. Assume that \mathbf{G}'_b is reductive and of dimension d for all closed point $b \in S$. Then \mathbf{G}' is a reductive S -group scheme of constant type. In particular, there exists a unique Chevalley group \mathbf{G}_0 such that \mathbf{G}' is an S -form of $\mathbf{G} := \mathbf{G}_0 \times_{\mathbb{Z}} S$.*

Proof. Proposition 3.3.(3) shows that \mathbf{G}' is smooth of relative dimension d and has connected geometric fibres. According to [Co1, Prop. 3.1.9.(1)] \mathbf{G}' is reductive in a neighborhood of each closed point $b \in S$. Since S is a Jacobson scheme, it follows that \mathbf{G}' is reductive. Since the type function is locally constant, the connectedness of S implies that \mathbf{G}' has constant type t_0 (see §2.3). Let \mathbf{G}_0 be the Chevalley group scheme of type t_0 . According to [SGA3, XXIII 5.6] \mathbf{G}' is an S -form of \mathbf{G} . \square

Corollary 3.5. *Let \mathbf{G} and \mathbf{G}' be group schemes over an integral Jacobson scheme S . Assume that \mathbf{G} is reductive and that \mathbf{G}' is affine and of finite presentation. If \mathbf{G}'_b is a form of \mathbf{G}_b for each closed point $b \in S$, then \mathbf{G}' is an S -form of \mathbf{G} . In particular, \mathbf{G}' is a reductive S -group scheme \square*

3.3 Lie algebras

If \mathcal{L} is an \mathcal{O}_S -Lie algebra and $s \in S$, then the $\mathcal{O}_{\mathrm{Spec} \kappa(s)}$ -algebra $\mathcal{L} \otimes_{\mathcal{O}_S} \kappa(s)$ obtained by base change is simply a $\kappa(s)$ -Lie algebra that we will denote by \mathcal{L}_s .⁷ Let $\mathbf{L} = \mathbf{W}(\mathcal{L})$ and denote by a harmless abuse of notation the $\kappa(s)$ -algebra $\mathbf{L}(\kappa(s))$ by \mathbf{L}_s . Note that $\mathcal{L}_s = \mathbf{L}_s$.

The following result will allow us to answer Burban's question (see §4 below).

⁷ A $\mathcal{O}_{\mathrm{Spec} \kappa(s)}$ -Lie algebra "is the same" as a $\kappa(s)$ -Lie algebra.

Proposition 3.6. *Assume that S is an integral Jacobson \mathbb{Q} -scheme. Let \mathbf{G} be a semisimple adjoint S -group scheme and consider its \mathcal{O}_S -Lie algebra $\mathcal{L} = \mathcal{L}ie(\mathbf{G})$.*

Let \mathcal{L}' be a sheaf of \mathcal{O}_S -Lie algebras. We assume that as an \mathcal{O}_S -module \mathcal{L}' is locally free of rank d , and that for each closed point $b \in S$, \mathcal{L}'_b is a form of \mathcal{L}_b . Let $\mathbf{L}' = \mathbf{W}(\mathcal{L}')$.

(1) *The S -functor $\mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L}')$ is a smooth affine S -group scheme whose connected component of the identity \mathbf{G}' is an S -form of \mathbf{G} . In particular \mathbf{G}' is a semisimple adjoint S -group scheme.*

(2) *There is a natural isomorphism $\mathcal{L}ie(\mathbf{G}') \simeq \mathcal{L}'$.*

(3) *\mathcal{L}' is an S -form of \mathcal{L} .*

(3 bis) *\mathbf{L}' is an S -form of \mathbf{L} , where $\mathbf{L}' = \mathbf{W}(\mathcal{L}')$ and $\mathbf{L} = \mathbf{W}(\mathcal{L})$.*

Proof. (1) By Lemma 2.3 $\mathbf{H} := \mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L}')$ is an affine S -group scheme of finite presentation. We have $\mathbf{H}_s = \mathbf{Aut}(\mathbf{L}'_s)$ for all $s \in S$. These algebraic groups are smooth since $\kappa(s)$ is of characteristic 0.

Let $b \in S$ be a closed point. Since the $\kappa(b)$ -Lie algebra \mathcal{L}'_b is a twisted form of \mathcal{L}_b , the corresponding $\kappa(b)$ -algebraic group $\mathbf{Aut}(\mathbf{L}'_b)$ is a twisted form of $\mathbf{Aut}(\mathbf{L}_b)$. Similarly for their connected component of the identity. Thus $\mathbf{H}_b^\circ = \mathbf{Aut}(\mathbf{L}'_b)^\circ$ is a twisted form of $\mathbf{Aut}(\mathbf{L}_b)^\circ$. Since \mathbf{G} is semisimple adjoint $\mathbf{Aut}(\mathbf{L}_b)^\circ = \mathbf{G}_b$. Thus \mathbf{H}_b° is a twisted form of \mathbf{G}_b . In particular all the \mathbf{H}_b are smooth groups of the same dimension. Theorem 3.3.(2) then shows that the S -subfunctor \mathbf{H}° of \mathbf{H} is representable by a smooth affine S -group scheme \mathbf{G}' .

According to [Co1, prop. 3.1.9.(1)], \mathbf{G}' is reductive on a neighborhood of each closed point $b \in S$. Since S is a Jacobson scheme, it follows that \mathbf{G}' is reductive. Since S is connected the type of \mathbf{G}' is constant. Since \mathbf{G}' and \mathbf{G} have the same type at closed points, their (constant) types coincide. Thus \mathbf{G}' is a twisted form of \mathbf{G} [SGA3, XXIII 5.6]. In particular, \mathbf{G}' is semisimple adjoint.

(2) By Lemma 2.5

$$(3.3) \quad \mathbf{Der}_{\mathcal{O}_S}(\mathbf{L}') \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathcal{O}_S\text{-alg}}(\mathbf{L}')) = \mathbf{Lie}(\mathbf{H}^\circ) = \mathbf{Lie}(\mathbf{G}').$$

By Lemma 2.3(2) $\mathbf{Der}_{\mathcal{O}_S}(\mathbf{L}') = \mathbf{W}(\mathcal{D}er_{\mathcal{O}_S}(\mathcal{L}'))$, while by Remark 2.4 we get that $\mathbf{Lie}(\mathbf{G}') = \mathbf{W}(\mathcal{L}ie(\mathbf{G}'))$. Since \mathbf{W} is faithful we conclude from 3.3 that $\mathcal{D}er_{\mathcal{O}_S}(\mathcal{L}') \simeq \mathcal{L}ie(\mathbf{G}')$. In particular $\mathcal{D}er_{\mathcal{O}_S}(\mathbf{L}')$ is a locally free \mathcal{O}_S -module of finite type. We can now apply Lemma 2.7 to conclude that $\mathcal{L}' \simeq \mathcal{D}er_{\mathcal{O}_S}(\mathcal{L}')$.

(3) From (1) and Remark 2.4 we get that $\mathbf{Lie}(\mathbf{G}') = \mathbf{W}(\mathcal{L}ie(\mathbf{G}'))$ is a form of $\mathbf{Lie}(\mathbf{G}) = \mathbf{W}(\mathcal{L}ie(\mathbf{G}))$. It follows from the fact that \mathbf{W} is faithful that $\mathcal{L}ie(\mathbf{G}')$ is a form of $\mathcal{L}ie(\mathbf{G})$. Thus (3) follows from (2).

(3 bis.) Follows from (3) since \mathbf{W} commutes with base change. \square

We finish this section with a variation of the previous Proposition which is of independent interest.

Proposition 3.7. *Assume that S is an integral Jacobson \mathbb{Q} -scheme. Let \mathcal{L}' be an \mathcal{O}_S -Lie algebra which is locally free of rank d . Assume that for each closed point $b \in S$ the $\kappa(b)$ -Lie algebra \mathcal{L}'_b is semisimple.⁸ Then there exists a unique finite dimensional split semisimple Lie \mathbb{Q} -algebra L such that \mathcal{L}' is a form of $\mathcal{L} := L \otimes_{\mathbb{Q}} \mathcal{O}_S$.⁹*

Proof. (1) Let $\mathbf{L}' = \mathbf{W}(\mathcal{L}')$. By Lemma 2.3 $\mathbf{H} := \mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L}')$ is an affine S -group of finite presentation. For each closed point $b \in S$, the algebraic group $\mathbf{H}_b = \mathbf{Aut}_{\kappa(b)\text{-Lie}}(\mathbf{L}'_b)$ is smooth of dimension d . Theorem 3.3.(2) shows that the connected component \mathbf{G}' of \mathbf{H} is a smooth S -group scheme of relative dimension d . Since $\mathbf{G}'_b = \mathbf{H}_b^\circ = \mathbf{Aut}_{\kappa(b)\text{-Lie}}(\mathbf{L}'_b)^\circ$ is semisimple adjoint, we can appeal to Proposition 3.4 to conclude that \mathbf{G}' is adjoint semisimple, hence a form of the S -group $\mathbf{G} = \mathbf{G}_0 \times_{\mathbb{Z}} S$ corresponding to a (unique) semisimple adjoint Chevalley \mathbb{Z} -group scheme \mathbf{G}_0 . To conclude that \mathcal{L}' is a twisted form of $\mathcal{L} \simeq \mathcal{L}ie(\mathbf{G})$ we reason as in the last Proposition. \square

4 Burban's question

In order to formulate Burban's question we need to recall the concept of loop algebra of a simple Lie finite dimensional complex Lie algebra \mathfrak{g} . The relevance of these algebras is that they provide concrete realizations of the (derived modulo their centres) affine Kac-Moody Lie algebras. See [Kac].

Let $R = \mathbb{C}[t^{\pm 1}]$. Fix a positive integer d , and set $R_d = \mathbb{C}[t^{\pm \frac{1}{d}}]$. The natural map $R \rightarrow R_d$ is faithfully flat and finite étale. Let $\xi \in \mathbb{C}$ be a primitive d -th root of unity. Then the elements of $\Gamma = \mathbb{Z}/d\mathbb{Z}$ act as automorphisms of R_d over R via

$$\bar{n}t^{\frac{1}{d}} = \xi^n t^{\frac{1}{d}}.$$

for $n \in \mathbb{Z}$. This action makes R_d into a Galois extension of R with Galois group Γ .

Let σ be an automorphism of \mathfrak{g} of order d . For $i \in \mathbb{Z}$ consider the eigenspace

$$\mathfrak{g}_i = \{x \in \mathfrak{g} : \sigma(x) = \xi^i x\}$$

Then $\mathfrak{g} = \bigoplus_{0 \leq i < d} \mathfrak{g}_i$. Out of this data we define the corresponding *loop algebra*

$$(4.1) \quad L(\mathfrak{g}, \sigma) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \otimes t^{\frac{i}{d}} \subset \mathfrak{g} \otimes_{\mathbb{C}} R_d.$$

⁸ Necessarily of finite dimension d .

⁹ That is, there exists an fppf covering $T \rightarrow S$ such that the \mathcal{O}_T -Lie algebras $\mathcal{L}' \otimes_{\mathcal{O}_S} \mathcal{O}_T$ and $\mathcal{L} \otimes_{\mathcal{O}_S} \mathcal{O}_T$ are isomorphic.

The simple but crucial observation is that $L(\mathfrak{g}, \sigma)$ is stable under the scalar action of R . Thus $L(\mathfrak{g}, \sigma)$ is not only an infinite dimensional complex Lie algebra, but also an R -Lie algebra. As we shall see, it is the algebra structure over this ring that allows non-abelian cohomological considerations to enter into the picture.

It is an easy linear algebra exercise to verify that we have a natural R_d -algebra isomorphism

$$(4.2) \quad L(\mathfrak{g}, \sigma) \otimes_R R_d \simeq \mathfrak{g} \otimes_{\mathbb{C}} R_d \simeq \mathfrak{g}_R \otimes_R R_d.$$

where $\mathfrak{g}_R := \mathfrak{g} \otimes_{\mathbb{C}} R$. This shows that the R -Lie algebra $L(\mathfrak{g}, \sigma)$ is a twisted form of $\mathfrak{g} \otimes_{\mathbb{C}} R$. It therefore corresponds to an $\mathbf{Aut}(\mathfrak{g}_R)$ -torsor \mathbf{X} over $\mathrm{Spec}(R)$. More precisely $\mathbf{X} = \mathbf{Isom}_{R\text{-Lie}}(\mathfrak{g}_R, L)$.¹⁰

We can now formulate Burban's questions (essentially verbatim except for some notation changes):

Question 1: Let L' be a Lie algebra over the ring $\mathbb{C}[t]$. Assume L' is free as a module and that for any complex number b the quotient Lie algebra $L'/(t-b)L'$ is isomorphic to \mathfrak{g} . Does it follow that L' is isomorphic to $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$ (as a Lie algebra over $\mathbb{C}[t]$)?

Question 2: Similarly, let L' be a Lie algebra over the ring $\mathbb{C}[t^{\pm 1}]$ which we assume is free as a module and such that for any $b \in \mathbb{C}^{\times}$ the quotient $L'/(t-b)L'$ is isomorphic to \mathfrak{g} . Does it follow that L' is isomorphic to a loop algebra (with respect to an automorphism of \mathfrak{g} of finite order)?

To answer these questions we take $S = \mathrm{Spec}(R)$ where $R = \mathbb{C}[t]$ or $\mathbb{C}[t^{\pm 1}]$, and where \mathcal{L}' and \mathcal{L} are the \mathcal{O}_S -Lie algebra corresponding to L' and $\mathfrak{g} \otimes_{\mathbb{C}} R$ respectively. By Proposition 3.6 we see that \mathcal{L}' corresponds to a torsor \mathbf{X}' over $\mathrm{Spec}(R)$ whose class is an element of $H^1(R, \mathbf{Aut}(\mathfrak{g}_R))$.

Recall (see [SGA3, XXIV 1.3 and 7.3.1]. See also §2.4) the split exact sequence of R -group schemes

$$(4.3) \quad 1 \rightarrow \mathbf{G} \rightarrow \mathbf{Aut}(L) \rightarrow \mathbf{Out}(L) \rightarrow 1$$

where \mathbf{G} is the split adjoint semisimple R -group scheme corresponding to \mathfrak{g} , and $\mathbf{Out}(L)$ is the constant R -group scheme corresponding to the finite (abstract) group $\mathbf{Out}(\mathfrak{g})$ of the symmetries of the Coxeter-Dynkin diagram of \mathfrak{g} . This allow us to compute the relevant H^1 and thus determine the nature of \mathbf{X}' , hence L' . According to [P3] or [CGP, Cor. 3.3], we have a bijection

$$H^1(R, \mathbf{Aut}(L)) \xrightarrow{\sim} H^1(R, \mathbf{Out}(L)).$$

¹⁰ See [DG, III §4] for the material on torsors used in this section.

Answer 1: If $R = \mathbb{C}[t]$ then $H^1(R, \mathbf{Out}(L)) = 1$ since R is simply connected. It follows that $H^1(R, \mathbf{Aut}(L)) = 1$ and therefore that $L' \simeq \mathfrak{g} \otimes_k R$.

Answer 2: If $R = \mathbb{C}[t^{\pm 1}]$ then $H^1(R, \mathbf{Out}(L))$ is the set of conjugacy classes of the (abstract) group $\mathbf{Out}(\mathfrak{g})$ which, in terms of forms of L , correspond to the loop algebras $L(\mathfrak{g}, \sigma)$ with $\sigma \in \mathbf{Out}(\mathfrak{g})$ (see [P2] for details, or more generally [P3]). Thus L' is a loop algebra.

References

- [Bu] I. Burban, Open correspondence, 2020.
- [CGP] V. Chernousov, P. Gille and A. Pianzola, *Three-point Lie algebras and Grothendieck's dessins d'enfants*, Mathematical Research Letters **23** (2016), 81-104.
- [Co1] B. Conrad, *Reductive group schemes*, in *Autour des schémas en groupes, vol. I*, Panoramas et Synthèses **42-43**, Soc. Math. France 2014.
- [DG] M. Demazure et P. Gabriel, *Groupes algébriques*, Masson (1970).
- [EGA-I] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique: I. Le langage des schémas*, Grundlehren der Mathematischen Wissenschaften **166** (2nd ed.). Berlin; New York: Springer-Verlag, 1971.
- [EGAIV] A. Grothendieck (avec la collaboration de J. Dieudonné), *Eléments de Géométrie Algébrique IV: Étude locale des schémas et des morphismes de schémas*, Publications mathématiques de l'I.H.É.S. no 20, 24, 28 and 32 (1964 - 1967).
- [G-W] U. Görtz and T. Wedhorn, *Algebraic Geometry I*, Viehweg+Teubner 2010.
- [Ha] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Springer.
- [Kac] V. Kac, *Infinite dimensional Lie algebras*, third edition, Cambridge University press, Cambridge, 1990.
- [LLR] Q. Liu, D. Lorenzini, M. Raynaud, *Néron models, Lie algebras, and reduction of curves of genus one*, Invent. Math. **157** (2004), 455-518.
- [P1] A. Pianzola, *Derivations of certain algebras defined by étale descent*. Math. Z. **264** (2010).

- [P2] A. Pianzola, *Affine Kac-Moody Lie algebras as torsors over the punctured line*, *Indagationes Mathematicae N.S.* **13**(2) (2002) 249-257.
- [P3] A. Pianzola, *Vanishing of H^1 for Dedekind rings and applications to loop algebras*, *C. R. Acad. Sci. Paris, Ser. I* **340** (2005), 633-638.
- [PY] G. Prasad and J-K. Yu, *On quasi-reductive group schemes, with an appendix by Brian Conrad*, *J. Algebraic Geom.* **15** (2006), 507-549.
- [SGA3] *Séminaire de Géométrie algébrique de l'I.H.E.S., 1963-1964, schémas en groupes, dirigé par M. Demazure et A. Grothendieck*, *Lecture Notes in Math.* 151-153. Springer (1970).
- [Stacks] Stacks project, <http://stacks.math.columbia.edu/>