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► **To cite this version:**

Philippe Gille, Arturo Pianzola. Fiberwise criteria for Twisted Forms of Algebraic Structures. 2023.  
hal-04135047v2

**HAL Id: hal-04135047**

**<https://hal.science/hal-04135047v2>**

Preprint submitted on 14 Sep 2023 (v2), last revised 19 Oct 2023 (v3)

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# Fiberwise Criteria for Twisted Forms of Algebraic Structures

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*In memoriam Professor Georgia Benkart*

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## Abstract

We provide a criterion for certain algebraic objects over Jacobson schemes to be forms of each other based on their behaviour at closed fibres. This criterion permits to answer a question that I. Burban had asked the authors.

*Keywords:* Reductive group scheme, torsor, Lie algebras.

*MSC 2000:* 14L30, 17B67, 11E72, 14E20.

## 1 Introduction

Let  $\mathbf{G}$  be a group scheme over a base scheme  $S$ . The concept of  $\mathbf{G}$ -torsors over  $S$  is to be found in many areas of mathematics and mathematical physics; they are useful tools to frame problems in a language that is partial to powerful methods from algebraic geometry.  $\mathbf{G}$  acts on itself by right multiplication. A  $\mathbf{G}$ -torsor is a scheme  $\mathbf{X}$  with a right action of  $\mathbf{G}$  that “locally” looks like  $\mathbf{G}$  with this action. By locally we mean that there exists a faithfully flat and locally finitely presented scheme morphism  $T \rightarrow S$  such that  $\mathbf{X}_T$  and  $\mathbf{G}_T$  are isomorphic as  $T$ -schemes with the corresponding induced  $\mathbf{G}_T$ -action.

The treatise [SGA3] shows how a deep understanding of  $\mathbf{G}$  can be had through the study of its fibres  $\mathbf{G}_s := \mathbf{G} \times_S \text{Spec}(\kappa(s))$  where  $\kappa(s)$  is the residue field of  $s \in S$ . It thus seems natural, given a right action of  $\mathbf{G}$  on an  $S$ -scheme  $\mathbf{X}$ , to study situations under which the fibres  $\mathbf{X}_s$  being  $\mathbf{G}_s$ -torsors yield information about  $\mathbf{X}$  itself being a  $\mathbf{G}$ -torsor. In practice the fibres of closed points are more tractable. The existence of “enough” closed points brings up the natural working hypothesis that  $S$  be a Jacobson scheme. It is under this assumption that we are able to create a fiberwise criterion for torsors under certain reductive group schemes.

This work grew up out of the desire to answer a question that I. Burban posed to the authors a couple of years ago [Bu]. His question and the (positive) answer that can be given using our main result is given in the last section.

**Acknowledgments.** The second author wishes to sincerely thank CNRS for facilitating an invited researcher visit to the Camille Jordan Institute.

## 2 Preliminaries on group schemes and smoothness

Throughout this paper  $S$  is a (base) scheme with structure sheaf  $\mathcal{O}_S$ . By an  $S$ -*functor* (resp. *monoid*, *group*) we mean a contravariant functor  $\mathbf{F}$  from the category of schemes over  $S$  to the category of sets (resp. monoids, groups). When  $\mathbf{F}$  is representable, i.e. a scheme, we say that  $\mathbf{F}$  is an  $S$ -*scheme* (resp. *monoid scheme*, *group scheme*). The use of bold face characters in the text is used to emphasize the functorial nature of the object under consideration. In particular, if  $X$  is an  $S$ -scheme then its functor of points

$$T \mapsto \text{Hom}_{S\text{-Sch}}(T, X)$$

will be denoted by  $\mathbf{X}$ .

If  $\mathbf{F}$  is an  $S$ -functor and  $T$  a scheme over  $S$ , we denote by  $\mathbf{F}_T$  the  $T$ -functor obtained by base change. If  $T = \text{Spec}(R)$  we denote  $\mathbf{F}(T)$  and  $\mathbf{F}_T$  by  $\mathbf{F}(R)$  and  $\mathbf{F}_R$  respectively. If  $\mathbf{F}$  is an  $S$ -scheme, then  $\mathbf{F}_T = \mathbf{F} \times_S T$  and, as it is customary, we denote  $\mathbf{F} \times_{\text{Spec}(R)} T$  by  $\mathbf{F} \times_R T$  and  $\mathbf{F} \times_S \text{Spec}(R)$  by  $\mathbf{F} \times_S R$ .

We denote as usual by  $\mathbf{O}_S$  the affine  $S$ -ring scheme  $S[t] = \text{Spec}(\mathbb{Z}[t]) \times_{\mathbb{Z}} S$ . Thus  $\mathbf{O}_S(T) = \mathcal{O}_T(T)$ . Recall that an  $\mathbf{O}_S$ -*module* is an abelian  $S$ -group  $\mathbf{M}$  together with an  $\mathbf{O}_S(T)$ -module structure on  $\mathbf{M}(T)$  that is functorial on  $T$ . The concept of an  $\mathbf{O}_S$ -algebra is defined similarly. Base change is defined and denoted as it is for functors.

We mainly use the terminology and notation of Grothendieck–Dieudonné [EGA-I], which for the most part agrees with that of Demazure–Grothendieck used in [SGA3, Exp. I and II]. Below we briefly review those concepts and results that are relevant to this paper.

## 2.1 Groups attached to quasi-coherent modules

Let  $\mathcal{E}$  be a quasi-coherent module over  $S$ . We denote its dual by  $\mathcal{E}^\vee$ . For each morphism  $f : T \rightarrow S$  we let  $\mathcal{E}_T = f^*(\mathcal{E})$  be the inverse image of  $\mathcal{E}$  under the morphism  $f$ , and define an abelian  $S$ -group  $\mathbf{V}(\mathcal{E})$  by  $\mathbf{V}(\mathcal{E})(T) = \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{E}_T, \mathcal{O}_T) = \Gamma(T, (\mathcal{E}_T)^\vee)$ .  $\mathbf{V}$  is actually an  $S$ -scheme; it is represented by the affine scheme  $\mathrm{Spec}(\mathbf{Sym}(\mathcal{E}))$  where  $\mathbf{Sym}(\mathcal{E})$  is the symmetric  $\mathcal{O}_S$ -algebra of  $\mathcal{E}$  [EGA-I, 9.4.9]. If  $\mathcal{E}$  is of finite type (resp. of finite presentation), then  $\mathbf{V}(\mathcal{E})$  is an  $S$ -scheme of finite type (resp. of finite presentation), *ibid*, 9.4.11.

The abelian  $S$ -group  $\mathbf{W}(\mathcal{E})$  is defined by  $\mathbf{W}(\mathcal{E})(T) = \Gamma(T, \mathcal{E}_T)$ . Recall that if  $\mathcal{E}$  is locally free of finite type then  $\mathbf{W}(\mathcal{E}) \simeq \mathbf{V}(\mathcal{E}^\vee)$ . In particular  $\mathbf{W}(\mathcal{E})$  is in this case an affine  $S$ -scheme. Note that the abelian  $S$ -groups  $\mathbf{V}(\mathcal{E})$  and  $\mathbf{W}(\mathcal{E})$  have natural  $\mathbf{O}_S$ -module structures.

**Example 2.1.** Assume  $S = \mathrm{Spec}(R)$ . Let  $\mathcal{L}$  be a quasi-coherent  $\mathcal{O}_S$ -module and denote by  $L$  the corresponding  $R$ -module. We denote  $\mathbf{W}(\mathcal{L})$  by  $\mathbf{W}(L)$ . For all  $S$ -scheme  $T$  by definition  $\mathbf{W}(L)(T)$  is the  $\mathcal{O}_T(T)$ -module  $L \otimes_R \mathcal{O}_T(T)$ . Similarly if  $\mathcal{L}$  is an  $\mathcal{O}_S$ -algebra.

**Remark 2.2.** We can view  $\mathbf{V}$  (resp.  $\mathbf{W}$ ) as a contravariant (resp. covariant) functor from the category of quasi-coherent  $\mathcal{O}_S$ -modules to the category of  $\mathbf{O}_S$ -modules. These functors are full and faithful [SGA3, I Prop. 4.6.2]

To  $\mathcal{E}$  we attach the  $S$ -functor  $\mathbf{End}_{\mathcal{O}_S\text{-mod}}(\mathcal{E})$  whose functor of points is given by  $T \mapsto \mathrm{End}_{\mathcal{O}_T\text{-mod}}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_T)$ . This can be viewed as an abelian  $S$ -group or an  $\mathbf{O}_S$ -module. We also have the  $S$ -group  $\mathbf{Aut}_{\mathcal{O}_S\text{-mod}}(\mathcal{E})$  given by  $T \mapsto \mathrm{Aut}_{\mathcal{O}_T\text{-mod}}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_T)$ . If in addition  $\mathcal{E}$  is an  $\mathcal{O}_S$ -algebra, one defines the  $S$ -group  $\mathbf{Aut}_{\mathcal{O}_S\text{-alg}}(\mathcal{E})$  in the obvious way. It is an  $S$ -subgroup of  $\mathbf{Aut}_{\mathcal{O}_S\text{-mod}}(\mathcal{E})$ .

Similarly to an  $\mathbf{O}_S$ -module  $\mathbf{L}$  we attach an abelian  $S$ -group (in fact an  $\mathbf{O}_S$ -module)  $\mathbf{End}_{\mathbf{O}_S\text{-mod}}(\mathbf{L})$  via  $T \mapsto \mathrm{End}_{\mathbf{O}_T\text{-mod}}(\mathbf{L}_T)$ . Finally if  $\mathbf{L}$  is an  $\mathbf{O}_S$ -algebra, the  $S$ -group  $\mathbf{Aut}_{\mathbf{O}_S\text{-alg}}(\mathbf{L})$  is given by the functor of points  $T \mapsto \mathrm{Aut}_{\mathbf{O}_T\text{-alg}}(\mathbf{L}_T)$ .

**Lemma 2.3.** *Let  $\mathcal{L}$  be an  $\mathcal{O}_S$ -algebra. Assume that as an  $\mathcal{O}_S$ -module  $\mathcal{L}$  is locally free of finite rank. Then.*

(1) *The natural maps*

$$\mathbf{End}_{\mathcal{O}_S\text{-mod}}(\mathcal{L}) \rightarrow \mathbf{End}_{\mathbf{O}_S\text{-mod}}(\mathbf{W}(\mathcal{L}))$$

*and*

$$\mathbf{Aut}_{\mathcal{O}_S\text{-alg}}(\mathcal{L}) \rightarrow \mathbf{Aut}_{\mathbf{O}_S\text{-alg}}(\mathbf{W}(\mathcal{L}))$$

*are  $S$ -functor isomorphisms.*

(2)  *$\mathbf{End}_{\mathcal{O}_S\text{-mod}}(\mathcal{L})$  and  $\mathbf{Aut}_{\mathcal{O}_S\text{-alg}}(\mathcal{L})$  are representable by affine  $S$ -schemes of finite presentation.*

*Proof.* (1) From their definition we see that the maps under consideration are functorial. That they are bijective on points follows from Remark 2.2.

(2) Assume that  $S = \text{Spec}(R)$ . Then  $\mathcal{L}$  corresponds to an  $R$ -module  $L$  which is projective of finite rank. The  $\mathcal{O}_S$ -module  $\mathcal{E}nd_{\mathcal{O}_S\text{-mod}}(\mathcal{L})$  corresponds to the  $R$ -module  $\text{End}_{R\text{-mod}}(L) \simeq L^* \otimes_R L$ . Because the  $R$ -module map  $\text{End}_{R\text{-mod}}(L) \otimes_R R' \rightarrow \text{End}_{R'\text{-mod}}(L \otimes_R R')$  is an isomorphism for all  $R'/R$ , it follows that  $\mathbf{End}_{\mathcal{O}_S\text{-mod}}(\mathcal{L})$  is represented by the affine  $R$ -scheme of finite presentation  $\mathbf{W}(L^* \otimes_R L)$ .

It is clear that  $\mathbf{Aut}_{\mathcal{O}_S\text{-alg}}(\mathcal{L})$  is a closed subscheme of  $\mathbf{End}_{\mathcal{O}_S\text{-mod}}(\mathcal{L})$ , hence also affine, which is of finite presentation since  $L$  is locally free of finite rank.

From the foregoing considerations it follows that our two functors are affine  $S$ -schemes which are locally of finite presentation. Since their structure morphisms are affine, they are quasi-compact and separated, hence of finite presentation.  $\square$

**Remark 2.4.** Let  $\mathcal{L}$  be a quasi-coherent  $\mathcal{O}_S$ -module, and consider the corresponding  $\mathcal{O}_S$ -module  $\mathcal{E}nd_{\mathcal{O}_S\text{-mod}}(\mathcal{L})$ . There is a natural  $\mathbf{O}_S$ -module morphism

$$\mathbf{W}(\mathcal{E}nd_{\mathcal{O}_S\text{-mod}}(\mathcal{L})) \rightarrow \mathbf{End}_{\mathbf{O}_S\text{-mod}}(\mathbf{W}(\mathcal{L})).$$

This morphism *need not* be an isomorphism. It is if  $\mathcal{L}$  is locally free of finite type.

## 2.2 Group schemes and Lie algebras

Throughout  $\mathbf{G}$  will denote an  $S$ -group scheme and we denote by  $e \in \mathbf{G}(S)$  its unit section. We refer to [SGA3, I and II], [DG] and [LLR, §1] for details in what follows.

If  $T$  is a scheme we denote by  $T[\epsilon]$  the corresponding scheme of dual number [SGA3, II.2]. The functor  $T \rightarrow \mathbf{Lie}(\mathbf{G})(T) := \ker(\mathbf{G}(T[\epsilon]) \rightarrow \mathbf{G}(T))$  is an  $\mathbf{O}_S$ -module isomorphic to  $\mathbf{V}(\omega_{\mathbf{G}/S}^1)$  where  $\omega_{\mathbf{G}/S}^1 = e^*(\Omega_{\mathbf{G}/S}^1)$  *ibid.* Prop. 3.3 and 3.6. Furthermore, the  $\mathbf{O}_S(T)$ -module  $\mathbf{Lie}(\mathbf{G})(T)$  has a natural Lie algebra structure. It is thus an  $\mathbf{O}_S$ -Lie algebra. Recall that for all scheme morphisms  $T \rightarrow S$  we have a natural  $\mathbf{O}_T$ -Lie algebra isomorphism

$$(2.1) \quad \mathbf{Lie}(\mathbf{G}) \times_S T \simeq \mathbf{Lie}(\mathbf{G}_T).$$

The  $\mathcal{O}_S(S)$ -Lie algebra  $\mathbf{Lie}(\mathbf{G})(S)$  is denoted by  $\text{Lie}(\mathbf{G})$ . From the above isomorphism we have

$$(2.2) \quad \mathbf{Lie}(\mathbf{G})(T) = \text{Lie}(\mathbf{G}_T).$$

We denote by  $\mathcal{L}ie(\mathbf{G})$  the vector group sheaf (fibration vectorielle) of sections of the affine scheme  $\mathbf{Lie}(\mathbf{G}) \rightarrow S$ . In other words,  $\mathcal{L}ie(\mathbf{G})$  is the  $\mathcal{O}_S$ -module  $(\omega_{\mathbf{G}/S}^1)^\vee = \text{Hom}_{\mathcal{O}_S}(\omega_{\mathbf{G}/S}^1, \mathcal{O}_S)$ . Note that  $\mathcal{L}ie(\mathbf{G})$  is naturally an  $\mathcal{O}_S$ -Lie algebra.

**Remark 2.5.** In general  $\mathcal{L}ie(\mathbf{G})$  does not determine  $\mathbf{Lie}(\mathbf{G})$ , but it does if  $\omega_{\mathbf{G}/S}^1$  is locally free of finite type, in particular if  $\mathbf{G}$  is smooth. In this case  $\mathbf{Lie}(\mathbf{G}) = \mathbf{W}(\mathcal{L}ie(\mathbf{G}))$ . See [SGA3, II Lemma 4.11.7]. If  $S = \text{Spec}(R)$  the  $R$ -Lie algebra  $\mathbf{Lie}(\mathbf{G}) = \mathcal{L}ie(\mathbf{G})(S)$  is a locally free  $R$ -module of finite type. For any ring extension  $R'/R$  we have  $\mathbf{Lie}(\mathbf{G}_{R'}) = \mathbf{Lie}(\mathbf{G})(R') = \mathbf{Lie}(\mathbf{G}) \otimes_R R'$ .

**Lemma 2.6.** *Let  $\mathcal{L}$  be an  $\mathcal{O}_S$ -algebra which is locally free of finite type (as an  $\mathcal{O}_S$ -module). Let  $\mathbf{L} = \mathbf{W}(\mathcal{L})$ . There is a natural  $\mathbf{O}_S$ -Lie algebra isomorphism  $\mathbf{Der}_{\mathbf{O}_S\text{-alg}}(\mathbf{L}) \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_S\text{-alg}}(\mathbf{L}))$ .*

*Proof.* Since  $\mathbf{L}$  is a good  $\mathbf{O}_S$ -module [SGA3, II Def. 4.4 and Ex. 4.4.2], we have an  $\mathbf{O}_S$ -module isomorphisms  $\eta : \mathbf{End}_{\mathbf{O}_S\text{-mod}}(\mathbf{L}) \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_S\text{-mod}}(\mathbf{L}))$  [SGA3, II Prop. 4.5]. Since  $\mathcal{L}$  is locally free of finite type we can appeal to Lemma 2.3(1) to conclude that this is in fact an isomorphism of  $S$ -schemes. We claim that the restriction of  $\eta$  to  $\mathbf{Der}_{\mathbf{O}_S\text{-alg}}(\mathbf{L})$  is our isomorphism. The proof reduces to the case when  $S$  is affine, which can be found in [DG, II §4 2.3].  $\square$

### 2.3 Connected component of the identity, forms and type of a reductive group scheme.

Let  $\mathbf{G}$  be an  $S$ -group scheme which is locally of finite presentation. We consider the  $S$ -subgroup (functor)  $\mathbf{G}^\circ$  of  $\mathbf{G}$  [SGA3, VI<sub>B</sub>.3.1] defined by

$$\mathbf{G}^\circ(\mathbf{T}) = \left\{ u \in \mathbf{G}(\mathbf{T}) \mid \forall s \in S, u_s(\mathbf{T}_s) \subset \mathbf{G}_s^\circ \right\}.$$

where  $\mathbf{G}_s^\circ$  is the connected component of the identity of the  $\kappa(s)$ -algebraic group  $\mathbf{G}_s$ . If  $\mathbf{G}$  is smooth along the unit section,  $\mathbf{G}^\circ$  is representable by a smooth  $S$ -group scheme called *the connected component of the identity of  $\mathbf{G}$*  [SGA3, VI<sub>B</sub>.4.1]. Furthermore, the fibre  $(\mathbf{G}^\circ)_s$  is naturally isomorphic to  $\mathbf{G}_s^\circ$ .

Let  $\mathbf{G}$  be a reductive  $S$ -group scheme, that is a smooth and affine  $S$ -group whose geometric fibres are *connected* reductive algebraic groups. An  $S$ -group scheme  $\mathbf{G}'$  is called a (*twisted*) *form of  $\mathbf{G}$* , if there exist a faithfully flat and locally presented extension  $\mathbf{T} \rightarrow S$  such that  $\mathbf{G}_{\mathbf{T}}$  and  $\mathbf{G}'_{\mathbf{T}}$  are isomorphic  $\mathbf{T}$ -group schemes. In a similar fashion one defines forms of  $\mathbf{O}_S$ -algebras.

Let  $\mathbf{G}$  be a reductive  $K$ -group scheme, where  $K$  is an algebraically closed field. If  $\mathbf{T}$  is a maximal torus of  $\mathbf{G}$ , then  $\mathbf{T}$  is split and defines a root datum which is up to isomorphism independent of the choice of  $\mathbf{T}$ . It is called the *type of  $\mathbf{G}$*  and is denoted by  $\mathcal{R}(\mathbf{G})$ .

Let  $\mathbf{G}$  be a reductive  $\mathbb{S}$ -group scheme. If  $s \in \mathbb{S}$ , the *type of  $\mathbf{G}$  at  $s$*  is the type of the reductive  $\overline{\kappa(s)}$ -group  $\mathbf{G}_{\overline{s}} := \mathbf{G}_s \times_{\kappa(s)} \overline{\kappa(s)}$ . Because  $\mathbf{G}$  possesses maximal tori locally for the étale topology, one knows that the type function  $s \mapsto \text{type of } \mathbf{G}_{\overline{s}}$  is locally constant [SGA3, XXII 2.8]. We say that  $\mathbf{G}$  is of *constant type* if the type function is constant. This is the case, for example, if  $\mathbb{S}$  is connected.

## 2.4 Specific properties in characteristic zero

We assume in this section that  $\mathbb{S}$  is a  $\mathbb{Q}$ -scheme. Let  $\mathbf{G}$  be a semisimple adjoint  $\mathbb{S}$ -group scheme of constant type. We will denote the  $\mathbf{O}_{\mathbb{S}}$ -Lie algebra  $\mathbf{Lie}(\mathbf{G})$  by  $\mathbf{L}$ , and the  $\mathcal{O}_{\mathbb{S}}$ -Lie algebra  $\mathcal{L}ie(\mathbf{G})$  by  $\mathcal{L}$ . Recall (see Remark 2.5) that  $\mathcal{L}$  is locally free of finite type and that  $\mathbf{W}(\mathcal{L}) = \mathbf{L}$ . Since  $\mathbf{G}$  and its simply connected cover  $\mathbf{G}^{\text{sc}}$  have the same Lie algebra, we have isomorphisms of affine  $\mathbb{S}$ -group schemes [SGA3, XXIV 3.6 and 7.3.1]

$$(2.3) \quad \mathbf{Aut}(\mathbf{G}) \simeq \mathbf{Aut}(\mathbf{G}^{\text{sc}}) \simeq \mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathbf{L}).^1$$

In particular, the  $\mathbb{S}$ -scheme  $\mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathbf{L})$  is smooth and affine. By (2.3) and [SGA3, XXIV 1.3 and 1.8] we have an isomorphism

$$(2.4) \quad \mathbf{G} \simeq \mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathbf{L})^\circ.$$

This yields an  $\mathbf{O}_{\mathbb{S}}$ -Lie algebra isomorphism

$$(2.5) \quad \mathbf{L} \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathbf{L})) = \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathbf{L})^\circ).$$

Recall [SGA3, II Theo. 4.7 and Prop. 4.8] the adjoint representation  $\text{Ad} : \mathbf{G} \rightarrow \mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-mod}}(\mathbf{L})$  and the induced  $\mathbf{O}_{\mathbb{S}}$ -Lie algebra homomorphism  $\text{ad} : \mathbf{L} \rightarrow \mathbf{End}_{\mathbf{O}_{\mathbb{S}}\text{-mod}}(\mathbf{L})$ .

**Lemma 2.7.** *Under the identification  $\mathbf{L} \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathbf{L}))$  of (2.5) the map  $\text{ad}$  induces an  $\mathbf{O}_{\mathbb{S}}$ -Lie algebra isomorphism between  $\mathbf{L}$  and  $\mathbf{Der}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathbf{L})$ .*

*Proof.* All the morphisms under consideration are  $\mathbb{S}$ -scheme morphisms, so the question is local on  $\mathbb{S}$  and we may assume that  $\mathbb{S} = \text{Spec}(R)$  for some  $\mathbb{Q}$ -ring  $R$ . Consider the  $R$ -Lie algebra  $L = \mathbf{L}(R)$ . Since  $\mathbf{L} = \mathbf{W}(\mathbf{L})$  where  $\mathbf{W}(\mathbf{L})(T) = L \otimes_R \mathcal{O}_T(T)$ , it will suffice to show that for all ring extensions  $R'/R$  the  $R'$ -Lie algebra homomorphism  $d = \text{ad}'_R : L \otimes_R R' \rightarrow \text{Der}_{R'\text{-Lie}}(L \otimes_R R')$  is an isomorphism. By replacing  $\mathbf{G}$  by  $\mathbf{G}'_R$  we may assume that  $R = R'$ . By [SGA3, II 4.7.2] the map  $d$  is nothing

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<sup>1</sup> La “définition évidente” of the  $\mathbb{S}$ -group scheme  $\underline{\mathcal{A}ut}_{\mathbf{O}_{\mathbb{S}}\text{-alg-de-Lie}}(\mathcal{L})$  of 7.3.1(iii) is precisely our  $\mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathcal{L})$ , so that the assertion is that the natural map  $\mathbf{Aut}(\mathbf{G}^{\text{sc}}) \rightarrow \mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathcal{L})$  is an isomorphism. Finally  $\mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathcal{L}) \simeq \mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathbf{L})$  by Lemma 2.3.

but the adjoint representation  $d(x) = \text{ad}_L(x)$ . Since  $\mathbf{G}$  is of constant type there exists a unique Chevalley group  $\mathbf{G}_0$  such that  $\mathbf{G}$  is a twisted form of  $\mathbf{G}_0 \times_{\mathbb{Z}} R$ . Let  $\mathfrak{g} = \mathbf{Lie}(\mathbf{G}_0)(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ . Then  $\mathfrak{g}$  is a finite dimensional split semisimple Lie algebra over  $\mathbb{Q}$  and  $L$  is a twisted form of  $\mathfrak{g} \otimes_{\mathbb{Q}} R$ , that is, there exists an fppf extension (in fact étale cover)  $R'$  of  $R$  such that the  $R'$ -Lie algebras  $L \otimes_R R'$  and  $(\mathfrak{g} \otimes_{\mathbb{Q}} R) \otimes_R R' \simeq \mathfrak{g} \otimes_{\mathbb{Q}} R'$  are isomorphic. Fix one such isomorphism

$$(2.6) \quad \psi : L \otimes_R R' \rightarrow \mathfrak{g} \otimes_{\mathbb{Q}} R'.$$

If  $d(x) = 0$ , then  $x$  belongs to the centre of  $L$  and therefore  $\psi(x \otimes 1)$  belongs to the centre of  $\mathfrak{g} \otimes_{\mathbb{Q}} R'$ , which is trivial. Since  $R'/R$  is faithfully flat  $x = 0$  so that  $d$  is injective and we henceforth identify  $L$  with an  $R$ -submodule of  $\text{Der}(L)$ . To show that  $d$  is surjective, that is that every derivation of  $L$  is inner, we reason as follows. If  $L = \mathfrak{g} \otimes_{\mathbb{Q}} R$  every derivation is inner by Whitehead's Lemma (see [P1, Example 4.9]).<sup>2</sup> For  $L$  arbitrary, we appeal to the isomorphism (2.6). By the split case we have  $(\text{Der}(L)/L) \otimes_R R' = 0$ . Thus  $L = \text{Der}(L)$  as desired.  $\square$

## 2.5 Semicontinuity property for Lie algebras

For convenience we recall the following well-known fact.

**Lemma 2.8.** *Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_S$ -module of finite presentation. Then the function  $s \mapsto \dim_{\kappa(s)} (\mathcal{F} \otimes_{\mathcal{O}_S} \kappa(s))$  is upper semi-continuous.*

*Proof.* The statement is local and therefore reduces then to the case of a ring  $R$  and of an  $R$ -module  $M$  of finite presentation [Stacks, Tag 01PC]. If  $R$  is noetherian then  $M$  is coherent and the result can be found in [Ha, 12.7.2]. For lack of a reference we provide a proof in the general case.

We consider an exact sequence  $R^m \xrightarrow{A} R^n \rightarrow M \rightarrow 0$  where  $A$  is a matrix of size  $(m, n)$  with entries in  $R$ . Let  $\mathfrak{q} \in \text{Spec}(R)$  and  $r = \dim_{\kappa(\mathfrak{q})}(M \otimes_R \kappa(\mathfrak{q}))$ . Since the sequence  $\kappa(\mathfrak{q})^m \xrightarrow{A_{\kappa(\mathfrak{q})}} \kappa(\mathfrak{q})^n \rightarrow M_{\kappa(\mathfrak{q})} \rightarrow 0$  is exact, there exists a minor  $B$  of  $A$  of size  $(n - r, n - r)$  such that  $\det(B_{\kappa(\mathfrak{q})}) \neq 0 \in \kappa(\mathfrak{q})$ . Consider the basic open set  $U$  of  $\text{Spec}(R)$  consisting of prime ideals of  $R$  that do not contain  $f = \det(B) \in R$ . From  $\det B_{\kappa(\mathfrak{q})} \neq 0$  it follows that  $\mathfrak{q} \in U$ . It is clear that for all  $\mathfrak{p} \in U$  the image of  $A_{\kappa(\mathfrak{p})}$  is a  $\kappa(\mathfrak{p})$ -space of dimension at least  $n - r$ . Thus

$$\dim_{\kappa(\mathfrak{p})} (M \otimes_R \kappa(\mathfrak{p})) \leq r = \dim_{\kappa(\mathfrak{q})} (M \otimes_R \kappa(\mathfrak{q}))$$

for all  $\mathfrak{p} \in U$ , so that our function is thus upper semi-continuous as desired.  $\square$

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<sup>2</sup> Note that our  $L$  is denoted by  $\mathcal{L}$  in [P1].



**Lemma 2.9.** *Let  $\mathbf{G}$  be an  $S$ -group scheme that is locally of finite presentation. Then  $\omega_{\mathbf{G}/S}^1$  is an  $\mathcal{O}_S$ -module of finite presentation and the map  $s \mapsto \dim_{\kappa(s)} \mathbf{Lie}(\mathbf{G}_s)$  is upper semi-continuous.*

*Proof.* That  $\omega_{\mathbf{G}/S}^1$  is of finite presentation follows from [Stacks, Tag 01V2, 01V3].

By (2.2) we have a  $\kappa(s)$ -Lie algebra isomorphism  $\mathbf{Lie}(\mathbf{G}_s) \simeq \mathbf{Lie}(\mathbf{G})(\kappa(s))$ . From the isomorphism  $\mathbf{Lie}(\mathbf{G}) \simeq \mathbf{V}(\omega_{\mathbf{G}/S}^1)$ , we see that the dimension of  $\mathbf{Lie}(\mathbf{G})(\kappa(s))$  at a point  $s \in S$  is the dimension of the  $\kappa(s)$ -space  $\mathrm{Hom}_{\kappa(s)}(\omega_{\mathbf{G}/S}^1 \otimes_{\mathcal{O}_S} \kappa(s), \kappa(s))$ , which is the dimension of  $\omega_{\mathbf{G}/S}^1 \otimes_{\mathcal{O}_S} \kappa(s)$  since  $\omega_{\mathbf{G}/S}^1$  is an  $\mathcal{O}_S$ -module of finite type. Furthermore, since  $\omega_{\mathbf{G}/S}^1$  is in fact of finite presentation, Lemma 2.8 implies that the map  $s \mapsto \dim_{\kappa(s)}(\mathbf{Lie}(\mathbf{G})(\kappa(s)))$  is upper semi-continuous.  $\square$

## 3 Group schemes over Jacobson schemes

### 3.1 Jacobson schemes and smoothness

We refer the reader to [EGAIV, §10] for general results about Jacobson schemes. Let  $X$  be a topological space. We recall that a subset  $X_0$  of  $X$  is called *very dense* if for every closed subset  $Z \subset X$  we have  $Z = \overline{Z \cap X_0}$ . Let  $S$  be a scheme. View  $S$  as a topological space and let  $S_0 \subset S$  the set of closed points. The scheme  $S$  is said to be *Jacobson* if  $S_0$  is a very dense subset of  $S$ .

Let us recall for the sake of completeness that:

- (i) Very dense subsets of a topological space are dense.
- (ii) If  $S = \mathrm{Spec}(R)$ , then  $S$  is Jacobson if and only if  $R$  is a Jacobson ring, that is every ideal of  $R$  is an intersection of maximal ideals.
- (iii) If  $S = \bigcup_{i \in I} \mathrm{Spec}(R_i)$ , the  $S$  is Jacobson if and only if every  $R_i$  is a Jacobson ring.

(iv) Every radical ideal  $I$  is the intersection of the maximal ideals containing it [Stacks, Tag 00G4].<sup>3</sup>

**Proposition 3.1.** *Let  $f : X \rightarrow S$  be a scheme morphism which is locally of finite presentation. Assume that  $x \in X$  is such that:*

- (i)  $f$  is flat at  $x$ .
- (ii)  $f_{\kappa(x)}$  is smooth.

*Then  $f$  is smooth on an open neighborhood of  $x$ .*

<sup>3</sup>AP. We will not use these properties in what follows.

*Proof.* By assumption there exists affine open subschemes  $\text{Spec}(A) = U \subset X$  and  $\text{Spec}(B) = V \subset S$  such that  $x \in U$ ,  $f(U) \subset V$  and  $A$  is a finitely presented  $B$ -algebra. Let  $\mathfrak{q} \in \text{Spec}(A)$  and  $\mathfrak{p} \in \text{Spec}(B)$  be the points corresponding to  $x$  and  $f(x)$  respectively.

By (i)  $A_{\mathfrak{q}}$  is flat over  $B_{\mathfrak{p}}$ , while by (ii)  $A \otimes_B \kappa(\mathfrak{p})$  is smooth over  $\kappa(\mathfrak{p})$ . It follows from [Stacks, Tag 00TF] that the restriction of  $f$  to  $U$  is smooth at  $\mathfrak{q}$ , hence (by definition of smoothness) smooth also in a neighbourhood of  $\mathfrak{q}$ .  $\square$

**Corollary 3.2.** *Let  $f : X \rightarrow S$  be as above. Assume that (i) and (ii) hold for all closed points of  $X$ . If  $S$  is Jacobson, then  $f$  is smooth.*

*Proof.* By [EGAIV, Cor. 10.4.7]  $X$  is Jacobson. Let  $X_0$  be the set of closed points of  $X$  viewed as a topological space. By the Proposition there exists an open  $U \subset X$  containing  $X_0$  in which  $f$  is smooth. Let  $Z = X \setminus U$ . If  $Z \neq \emptyset$ , then  $Z \cap X_0 \neq \emptyset$  because  $X_0$  is very dense. This contradicts  $X_0 \subset U$ . Thus  $Z = \emptyset$  so that  $U = X$  as desired.  $\square$

**Proposition 3.3.** *Let  $S$  be an integral Jacobson scheme, and  $\mathbf{G}$  be an  $S$ -group scheme of finite presentation. Assume that for all closed point  $b \in S$  the fibres  $\mathbf{G}_b$  are smooth and of the same dimension  $d$ .<sup>4</sup> Then.*

- (1) *For all  $s \in S$  the algebraic  $\kappa(s)$ -group  $\mathbf{G}_s$  is smooth of dimension  $d$ .*
- (2) *The  $S$ -functor  $\mathbf{G}^\circ$  is representable by a smooth  $S$ -group scheme of relative dimension  $d$  which is open in  $\mathbf{G}$ .*
- (3) *If for each closed point  $b$  of  $S$  the fibre  $\mathbf{G}_b$  is connected, then  $\mathbf{G}$  is smooth and has connected geometric fibres (i.e.,  $\mathbf{G}_{\bar{b}}$  is connected).*

*Proof.* (1) We denote by  $\eta : \text{Spec}(F) \rightarrow S$  the generic point of  $S$ . Chevalley's generic flatness theorem [G-W, 10.85] shows that there exists an open dense subset  $U$  of  $S$  such that  $\mathbf{G} \times_S U$  is flat over  $U$ . We know that  $U$  is also Jacobson [EGAIV, Prop. 10.3.3]. Corollary 3.2 shows that  $\mathbf{G} \times_S U$  is smooth over  $U$ . By [Stacks, Tag 05F7], up to shrinking  $U$  if necessary, the dimension function of the fibres of  $\mathbf{G} \times_S U \rightarrow U$  is constant. Since by assumption this dimension has value  $d$  on the closed points of  $U$ , we get that  $\mathbf{G} \times_S U$  is smooth of relative dimension  $d$ .<sup>5</sup> In particular the algebraic  $F$ -group  $\mathbf{G}_F$  is smooth of dimension  $d$ .

Let  $s \in S$  and let  $c = \dim(\mathbf{G}_s)$ . According to Chevalley's semicontinuity theorem [SGA3, VI<sub>B</sub>.4.1]

$$F_c = \{x \in S : \dim(\mathbf{G}_x) \geq c\} \subset S$$

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<sup>4</sup> Since  $\mathbf{G}$  is of finite presentation  $\mathbf{G}_s$  is an algebraic  $\kappa(s)$ -group. Thus  $\mathbf{G}_s$  has a finite number of irreducible components. They all have the same (finite) dimension and this is also the dimension of  $\mathbf{G}_s$ .

<sup>5</sup> That is, all the fibres  $\mathbf{G}_s, s \in U$  are equidimensional and this dimension is  $d$ .

is a closed set. Since  $F_c \cap \overline{\{s\}}$  contains  $s$  and is closed

$$(3.1) \quad \overline{\{s\}} \subset F_c.$$

Since  $S$  is Jacobson, it follows that there exists a closed point  $b$  in  $\overline{\{s\}}$ . We have

$$(3.2) \quad \dim(\mathbf{G}_s) \leq \dim(\mathbf{G}_b) = d.$$

On the other hand  $s$  belongs to  $\overline{\{\eta\}}$ . The same semicontinuity reasoning used above shows that

$$(3.3) \quad \dim(\mathbf{G}_F) \leq \dim(\mathbf{G}_s).$$

Since  $\mathbf{G}_F$  is of dimension  $d$  it follows from (3.2) and (3.3) that  $\dim(\mathbf{G}_s) = d$ .

For establishing smoothness we use a variation of the previous argument by applying semicontinuity considerations to the Lie algebras. Lemma 2.9 yields the inequalities

$$\dim_F \operatorname{Lie}(\mathbf{G}_F) \leq \dim_{\kappa(s)} \operatorname{Lie}(\mathbf{G}_s) \leq \dim_{\kappa(b)} \operatorname{Lie}(\mathbf{G}_b).$$

Since  $\mathbf{G}$  is smooth of dimension  $d$  at  $\eta$  and at  $b$  we obtain that  $d = \dim_{\kappa(s)} \operatorname{Lie}(\mathbf{G}_s)$ . Thus  $\operatorname{Lie}(\mathbf{G}_s)$  and  $\mathbf{G}_s$  have the same dimension (namely  $d$ ). By the smoothness criterion [DG, II.5.2.1]  $\mathbf{G}_s$  is smooth. This completes the proof of (1).

(2) By (1) we see that assumption (ii) of [SGA3, VI<sub>B</sub> Cor. 4.4] holds. It follows that the  $S$ -functor  $\mathbf{G}^\circ$  is representable by a smooth  $S$ -group scheme which is open in  $\mathbf{G}$ .

(3) According to [Stacks, Tag 055I], the level set

$$E_n = \{s \in S \mid \mathbf{G}_s \text{ has } n \text{ geometrically connected components}\}$$

is a locally constructible subset of  $S$  for each  $n \geq 1$ . Since  $S$  is a Jacobson scheme, the set  $S_0$  of closed points of the underlying topological space of  $S$  is very dense in  $S$ . Thus  $S_0 \cap E_n$  is dense in  $E_n$  for each  $n$  [EGAIV, 10.1.2, (b')]. Our assumption implies that  $E_n = \emptyset$  for  $n \geq 2$  so that  $E_1 = S$ . That  $\mathbf{G}$  is smooth now follows from (2).  $\square$

## 3.2 Forms

**Proposition 3.4.** *Let  $S$  be an integral Jacobson scheme and  $\mathbf{G}$  an affine  $S$ -group scheme of finite presentation. Assume that  $\mathbf{G}_b$  is reductive and of dimension  $d$  for all closed point  $b \in S$ . Then  $\mathbf{G}$  is a reductive  $S$ -group scheme of constant type. In particular, there exists a unique Chevalley group  $\mathbf{G}_0$  such that  $\mathbf{G}$  is an  $S$ -form of  $\mathbf{G}_0 \times_{\mathbb{Z}} S$ .*

*Proof.* Proposition 3.3.(3) shows that  $\mathbf{G}$  is smooth of relative dimension  $d$  and has connected geometric fibres. According to [Co1, Prop. 3.1.9.(1)]  $\mathbf{G}$  is reductive in a neighborhood of each closed point  $b \in S$ . Since  $S$  is a Jacobson scheme, it follows that  $\mathbf{G}$  is reductive. Since the type function is locally constant, the connectedness of  $S$  implies that  $\mathbf{G}$  has constant type  $t_0$  (see §2.3). Let  $\mathbf{G}_0$  be the corresponding Chevalley group scheme. According to [SGA3, XXIII 5.6]  $\mathbf{G}$  is an  $S$ -form of  $\mathbf{G}_0 \times_{\mathbb{Z}} S$ .  $\square$

**Corollary 3.5.** *Let  $\mathbf{G}$  and  $\mathbf{G}'$  be group schemes over an integral Jacobson scheme  $S$ . Assume that  $\mathbf{G}$  is reductive and that  $\mathbf{G}'$  is affine and of finite presentation. If  $\mathbf{G}'_b$  is a form of  $\mathbf{G}_b$  for each closed point  $b \in S$ , then  $\mathbf{G}'$  is an  $S$ -form of  $\mathbf{G}$ . In particular,  $\mathbf{G}'$  is a reductive  $S$ -group scheme*  $\square$

### 3.3 Lie algebras

If  $\mathcal{L}$  is an  $\mathcal{O}_S$ -Lie algebra and  $s \in S$ , then the  $\mathcal{O}_{\text{Spec } \kappa(s)}$ -algebra  $\mathcal{L} \otimes_{\mathcal{O}_S} \kappa(s)$  obtained by base change is simply a  $\kappa(s)$ -algebra that we will denote by  $\mathcal{L}_s$ .<sup>6</sup> Let  $\mathbf{L} = \mathbf{W}(\mathcal{L})$  and denote by a harmless abuse of notation the  $\kappa(s)$ -algebra  $\mathbf{L}(\kappa(s))$  by  $\mathbf{L}_s$ . Note that  $\mathcal{L}_s = \mathbf{L}_s$ .

**Proposition 3.6.** *Assume that  $S$  is an integral Jacobson  $\mathbb{Q}$ -scheme. Let  $\mathcal{L}$  be an  $\mathcal{O}_S$ -Lie algebra which is locally free of rank  $d$ , and let  $\mathbf{L} = \mathbf{W}(\mathcal{L})$  be its corresponding  $\mathcal{O}_S$ -Lie algebra. Assume that for each closed point  $b \in S$  the  $\kappa(b)$ -Lie algebra  $\mathbf{L}_b$  is semisimple.<sup>7</sup> Then there exists a unique finite dimensional split semisimple Lie  $\mathbb{Q}$ -algebra  $\mathbf{L}_0$  such that  $\mathbf{L}$  is a form of  $\mathbf{L}_0 := \mathbf{L}_0 \otimes_{\mathbb{Q}} \mathcal{O}_S$ .<sup>8</sup>*

*Proof.* (1) By Lemma 2.3  $\mathbf{G} := \mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L})$  is an affine  $S$ -group of finite presentation. For each closed point  $b \in S$ , the algebraic group  $\mathbf{G}_b = \mathbf{Aut}_{\kappa(b)\text{-Lie}}(\mathbf{L}_b)$  is smooth of dimension  $d$  and  $\mathbf{G}_b^\circ$  is semisimple adjoint. Proposition 3.3.(2) shows that  $\mathbf{G}^\circ$  is representable by an open subgroup scheme of  $\mathbf{G}$  which is smooth of relative dimension  $d$ . Appealing now to Proposition 3.4 yields that  $\mathbf{G}^\circ$  is adjoint semisimple and is a form of the  $S$ -group corresponding to a (unique) semisimple adjoint Chevalley group scheme  $\mathbf{G}_0^\circ$ . We denote by  $\mathbf{L}_0$  the  $\mathbb{Q}$ -Lie algebra of  $\text{Lie}(\mathbf{G}_0^\circ) \otimes_{\mathbb{Z}} \mathbb{Q}$ . This yields the  $\mathcal{O}_S$ -Lie algebra that we denoted by  $\mathbf{L}_0$ . Observe that  $\mathbf{Lie}(\mathbf{G})$  is a form of  $\mathbf{L}_0$  as an  $\mathcal{O}_S$ -Lie algebras.

By Lemma 2.7 we have  $\mathbf{L} \simeq \mathbf{Der}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L})$ . Finally since  $\mathbf{G} \simeq \mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L})$  we can apply Lemma 2.6 to conclude that  $\mathbf{Lie}(\mathbf{G}) \simeq \mathbf{Der}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L})$ . It follows that  $\mathbf{L}$  is a form of  $\mathbf{L}_0$  as desired.  $\square$

<sup>6</sup> A  $\mathcal{O}_{\text{Spec } \kappa(s)}$ -algebra “is the same” as a  $\kappa(s)$ -algebra.

<sup>7</sup> Necessarily of finite dimension  $d$ .

<sup>8</sup> By definition  $\mathbf{L}_0(\mathbb{T})$  is the  $\mathcal{O}_{\mathbb{T}}(\mathbb{T})$ -Lie algebra  $\mathbf{L}_0 \otimes_{\mathbb{Q}} \mathcal{O}_{\mathbb{T}}(\mathbb{T})$ .

The following variant of the ideas presented heretofore will allow us to answer Burban's question (see §4 below).

**Proposition 3.7.** *Assume that  $S$  is an integral Jacobson  $\mathbb{Q}$ -scheme. Let  $\mathbf{G}$  be a semisimple adjoint  $S$ -group scheme and consider its  $\mathcal{O}_S$ -Lie algebra  $\mathcal{L} = \mathcal{L}ie(\mathbf{G})$ .*

*Let  $\mathcal{L}'$  be a sheaf of  $\mathcal{O}_S$ -Lie algebras. We assume that as an  $\mathcal{O}_S$ -module  $\mathcal{L}'$  is locally free of rank  $d$ , and that for each closed point  $b \in S$ ,  $\mathcal{L}'_b$  is a form of  $\mathcal{L}_b$ .*

(1) *The  $S$ -functor  $\mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathcal{L}')$  is a smooth affine  $S$ -group scheme whose connected component of the identity  $\mathbf{G}'$  is an  $S$ -form of  $\mathbf{G}$ . In particular  $\mathbf{G}'$  is a semisimple adjoint  $S$ -group scheme.*

(2) *There is a natural isomorphism  $\mathcal{L}ie(\mathbf{G}') \simeq \mathcal{L}'$ .*

(3)  *$\mathcal{L}'$  is an  $S$ -form of  $\mathcal{L}$ .*

(3 bis)  *$\mathbf{L}'$  is an  $S$ -form of  $\mathbf{L}$ , where  $\mathbf{L}' = \mathbf{W}(\mathcal{L}')$  and  $\mathbf{L} = \mathbf{W}(\mathcal{L})$ .*

*Proof.* (1) Let  $\mathbf{L}' = \mathbf{W}(\mathcal{L}')$ . By Lemma 2.3  $\mathbf{H} := \mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathcal{L}') \simeq \mathbf{Aut}_{\mathbf{O}_S\text{-Lie}}(\mathbf{L}')$  is an affine  $S$ -group scheme of finite presentation. Since  $\mathbf{H}_T = \mathbf{Aut}_{\mathbf{O}_T\text{-Lie}}(\mathbf{L}'_T)$  for all  $S$ -scheme  $T$ , we have  $\mathbf{H}_s = \mathbf{Aut}(\mathbf{L}'_s)$  for all  $s \in S$ , where  $\mathbf{L}'_s := \mathbf{L}' \otimes_{\mathcal{O}_S} \kappa(s)$ . These algebraic groups are smooth since  $\kappa(s)$  is of characteristic 0.

Let  $b \in S$  be a closed point. Since the  $\kappa(b)$ -Lie algebra  $\mathcal{L}'_b$  is a twisted form of  $\mathcal{L}_b$ , the corresponding  $\kappa(b)$ -algebraic group  $\mathbf{Aut}(\mathcal{L}'_b)$  is a twisted form of  $\mathbf{Aut}(\mathcal{L}_b)$ . Similarly for their connected component of the identity. As we have seen that  $\mathbf{Aut}(\mathcal{L}'_b) = \mathbf{H}_b$ . On the other hand since  $\mathbf{G}$  is semisimple adjoint  $\mathbf{Aut}(\mathcal{L}_b)^\circ = \mathbf{G}_b$ . This yields that  $\mathbf{H}_b^\circ$  is a twisted form of  $\mathbf{G}_b$ . In particular all the  $\mathbf{H}_b$  are smooth groups of the same dimension. Proposition 3.3.(2) then shows that the  $S$ -subfunctor  $\mathbf{H}^\circ$  of  $\mathbf{H}$  is representable by a smooth affine  $S$ -group scheme  $\mathbf{G}'$ .

According to [Co1, prop. 3.1.9.(1)],  $\mathbf{G}'$  is reductive on a neighborhood of each closed point  $b \in S$ . Since  $S$  is a Jacobson scheme, it follows that  $\mathbf{G}'$  is reductive. Since  $S$  is connected the type of  $\mathbf{G}'$  is constant. Since  $\mathbf{G}'$  and  $\mathbf{G}$  have the same type at closed points, their (constant) types coincide. Thus  $\mathbf{G}'$  is a twisted form of  $\mathbf{G}$  [SGA3, XXIII 5.6]. In particular,  $\mathbf{G}'$  is semisimple adjoint.

(2) By Lemma 2.6

$$\mathbf{Der}_{\mathbf{O}_S\text{-alg}}(\mathbf{L}') \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_S\text{-alg}}(\mathbf{L}')) = \mathbf{Lie}(\mathbf{H}^\circ) = \mathbf{Lie}(\mathbf{G}').$$

On the other hand we have an isomorphism of  $\mathbf{O}_S$ -Lie algebras  $\mathbf{L}' \simeq \mathbf{Der}_{\mathbf{O}_S\text{-alg}}(\mathbf{L}')$  (Lemma 2.7), whence an isomorphism  $\mathbf{L}' \simeq \mathbf{Lie}(\mathbf{G}')$ . It follows that  $\mathbf{W}(\mathcal{L}') \simeq \mathbf{W}(\mathcal{L}ie(\mathbf{G}'))$ . Since  $\mathbf{W}$  is full and faithful  $\mathcal{L}'$  and  $\mathcal{L}ie(\mathbf{G}')$  are isomorphic  $\mathcal{O}_S$ -Lie algebras.

(3) From (2) and Remark 2.5 we get that  $\mathbf{Lie}(\mathbf{G}') = \mathbf{W}(\mathcal{L}ie(\mathbf{G}'))$  is a form of  $\mathbf{Lie}(\mathbf{G}) = \mathbf{W}(\mathcal{L}ie(\mathbf{G}))$ . It follows that  $\mathcal{L}ie(\mathbf{G}')$  is a form of  $\mathcal{L}ie(\mathbf{G})$ . Thus (3) follows from (2).

(3 bis.) Follows from (3) since  $\mathbf{W}$  commutes with base change.  $\square$

## 4 Burban's question

In order to formulate Burban's question we need to recall the concept of loop algebra of a simple Lie finite dimensional complex Lie algebra  $\mathfrak{g}$ .

Let  $R = \mathbb{C}[t^{\pm 1}]$ . Fix a positive integer  $d$ , and set  $R_d = \mathbb{C}[t^{\pm \frac{1}{d}}]$ . The natural map  $R \rightarrow R_d$  is faithfully flat and finite étale. Let  $\xi \in \mathbb{C}$  be a primitive  $d$ -th root of unity. Then the elements of  $\Gamma = \mathbb{Z}/d\mathbb{Z}$  act as automorphisms of  $R_d$  over  $R$  via

$$\bar{e}_{t^{\frac{1}{d}}} = \xi^e t^{\frac{1}{d}}.$$

for  $e \in \mathbb{Z}$ . This action makes  $R_d$  into a Galois extension of  $R$  with Galois group  $\Gamma$ .

Let  $\sigma$  be an automorphism of  $\mathfrak{g}$  of order  $d$ . For  $i \in \mathbb{Z}$  consider the eigenspace

$$\mathfrak{g}_i = \{x \in \mathfrak{g} : \sigma(x) = \xi^i x\}$$

Then  $\mathfrak{g} = \bigoplus_{0 \leq i < d} \mathfrak{g}_i$ . Out of this data we define the corresponding *loop algebra*

$$(4.1) \quad L(\mathfrak{g}, \sigma) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \otimes t^{\frac{i}{d}} \subset \mathfrak{g} \otimes_{\mathbb{C}} R_d.$$

The simple but crucial observation is that  $L(\mathfrak{g}, \sigma)$  is stable under the scalar action of  $R$ . Thus  $L(\mathfrak{g}, \sigma)$  is not only an infinite dimensional complex Lie algebra, but also an  $R$ -Lie algebra. As we shall see, it is the algebra structure over this ring that allows non-abelian cohomological considerations to enter into the picture.

It is an easy linear algebra exercise to verify that we have a natural  $R_d$ -algebra isomorphism

$$(4.2) \quad L(\mathfrak{g}, \sigma) \otimes_R R_d \simeq \mathfrak{g} \otimes_{\mathbb{C}} R_d \simeq \mathfrak{g}_R \otimes_R R_d.$$

where  $\mathfrak{g}_R := \mathfrak{g} \otimes_{\mathbb{C}} R$ . This shows that the  $R$ -Lie algebra  $L := L(\mathfrak{g}, \sigma)$  is a twisted form of  $\mathfrak{g} \otimes_{\mathbb{C}} R$ . It therefore corresponds to an  $\mathbf{Aut}(\mathfrak{g}_R)$ -torsor  $\mathbf{X}$  over  $\mathrm{Spec}(R)$ . More precisely  $\mathbf{X} = \mathbf{Isom}_{R\text{-Lie}}(\mathfrak{g}_R, L)$ .<sup>9</sup>

We can now formulate Burban's questions (essentially verbatim except for some notation changes):

**Question 1:** Let  $L'$  be a Lie algebra over the ring  $\mathbb{C}[t]$ . Assume  $L'$  is free as a module and that for any complex number  $b$  the quotient Lie algebra  $L'/(t-b)L'$  is

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<sup>9</sup> See [DG, III §4] for the material on torsors used in this section.

isomorphic to  $\mathfrak{g}$ . Does it follow that  $L'$  is isomorphic to  $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$  (as a Lie algebra over  $\mathbb{C}[t]$ )?

**Question 2:** Similarly, let  $L'$  be a Lie algebra over the ring  $\mathbb{C}[t^{\pm 1}]$  which we assume is free as a module and such that for any  $b \in \mathbb{C}^{\times}$  the quotient  $L'/(t-b)L'$  is isomorphic to  $\mathfrak{g}$ . Does it follow that  $L'$  is isomorphic to a loop algebra (with respect to an automorphism of  $\mathfrak{g}$  of finite order)?

To answer these questions we take  $S = \text{Spec}(R)$  where  $R = \mathbb{C}[t]$  or  $\mathbb{C}[t^{\pm 1}]$ , and where  $\mathcal{L}'$  and  $\mathcal{L}$  are the  $\mathcal{O}_S$ -Lie algebra corresponding to  $L'$  and  $\mathfrak{g} \otimes_{\mathbb{C}} R$  respectively. By Proposition 3.7 we see that  $\mathcal{L}'$  corresponds to a torsor  $\mathbf{X}'$  over  $\text{Spec}(R)$  whose class is an element of  $H^1(R, \mathbf{Aut}(\mathfrak{g}_R))$ .

Recall ([SGA3, XXIV 1.3 and 7.3.1]. See also §2.4) the split exact sequence of  $R$ -group schemes

$$(4.3) \quad 1 \rightarrow \mathbf{G} \rightarrow \mathbf{Aut}(L) \rightarrow \mathbf{Out}(L) \rightarrow 1$$

where  $\mathbf{G}$  is the split adjoint semisimple  $R$ -group scheme corresponding to  $\mathfrak{g}$ , and  $\mathbf{Out}(L)$  is the constant  $R$ -group scheme corresponding to the finite (abstract) group  $\text{Out}(\mathfrak{g})$  of the symmetries of the Coxeter-Dynkin diagram of  $\mathfrak{g}$ . This allow us to compute the relevant  $H^1$  and thus determine the nature of  $\mathbf{X}'$ , hence  $L'$ . According to [CGP, Cor. 3.3], we have a bijection

$$H^1(R, \mathbf{Aut}(L)) \xrightarrow{\sim} H^1(R, \mathbf{Out}(L)).$$

**Answer 1:** If  $R = \mathbb{C}[t]$  then  $H^1(R, \mathbf{Out}(L)) = 1$  since  $R$  is simply connected. It follows that  $H^1(R, \mathbf{Aut}(L)) = 1$  and therefore that  $L' \simeq \mathfrak{g} \otimes_k R$ .

**Answer 2:** If  $R = \mathbb{C}[t^{\pm 1}]$  then furthermore  $H^1(R, \mathbf{Out}(L))$  is the set of conjugacy classes of the (abstract) group  $\text{Out}(\mathfrak{g})$  which, in terms of forms of  $L$ , correspond to the loop algebras  $L(\mathfrak{g}, \sigma)$  with  $\sigma \in \text{Out}(\mathfrak{g})$  (see [P2] for details, or more generally [P3]). Thus  $L'$  is a loop algebra.

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