



HAL
open science

Fiberwise criteria for Twisted Forms of Algebraic Structures

Philippe Gille, Arturo Pianzola

► **To cite this version:**

Philippe Gille, Arturo Pianzola. Fiberwise criteria for Twisted Forms of Algebraic Structures. 2023.
hal-04135047v2

HAL Id: hal-04135047

<https://hal.science/hal-04135047v2>

Preprint submitted on 14 Sep 2023 (v2), last revised 19 Oct 2023 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Fiberwise Criteria for Twisted Forms of Algebraic Structures

P. Gille¹ and A. Pianzola^{2,3}

In memoriam Professor Georgia Benkart

¹*UMR 5208 du CNRS - Institut Camille Jordan - Université Claude Bernard Lyon 1, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne cedex - France.*

²*Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1, Canada.*

³*Centro de Altos Estudios en Ciencia Exactas, Avenida de Mayo 866, (1084) Buenos Aires, Argentina.*

Abstract

We provide a criterion for certain algebraic objects over Jacobson schemes to be forms of each other based on their behaviour at closed fibres. This criterion permits to answer a question that I. Burban had asked the authors.

Keywords: Reductive group scheme, torsor, Lie algebras.

MSC 2000: 14L30, 17B67, 11E72, 14E20.

1 Introduction

Let \mathbf{G} be a group scheme over a base scheme S . The concept of \mathbf{G} -torsors over S is to be found in many areas of mathematics and mathematical physics; they are useful tools to frame problems in a language that is partial to powerful methods from algebraic geometry. \mathbf{G} acts on itself by right multiplication. A \mathbf{G} -torsor is a scheme \mathbf{X} with a right action of \mathbf{G} that “locally” looks like \mathbf{G} with this action. By locally we mean that there exists a faithfully flat and locally finitely presented scheme morphism $T \rightarrow S$ such that \mathbf{X}_T and \mathbf{G}_T are isomorphic as T -schemes with the corresponding induced \mathbf{G}_T -action.

The treatise [SGA3] shows how a deep understanding of \mathbf{G} can be had through the study of its fibres $\mathbf{G}_s := \mathbf{G} \times_S \text{Spec}(\kappa(s))$ where $\kappa(s)$ is the residue field of $s \in S$. It thus seems natural, given a right action of \mathbf{G} on an S -scheme \mathbf{X} , to study situations under which the fibres \mathbf{X}_s being \mathbf{G}_s -torsors yield information about \mathbf{X} itself being a \mathbf{G} -torsor. In practice the fibres of closed points are more tractable. The existence of “enough” closed points brings up the natural working hypothesis that S be a Jacobson scheme. It is under this assumption that we are able to create a fiberwise criterion for torsors under certain reductive group schemes.

This work grew up out of the desire to answer a question that I. Burban posed to the authors a couple of years ago [Bu]. His question and the (positive) answer that can be given using our main result is given in the last section.

Acknowledgments. The second author wishes to sincerely thank CNRS for facilitating an invited researcher visit to the Camille Jordan Institute.

2 Preliminaries on group schemes and smoothness

Throughout this paper S is a (base) scheme with structure sheaf \mathcal{O}_S . By an S -*functor* (resp. *monoid*, *group*) we mean a contravariant functor \mathbf{F} from the category of schemes over S to the category of sets (resp. monoids, groups). When \mathbf{F} is representable, i.e. a scheme, we say that \mathbf{F} is an S -*scheme* (resp. *monoid scheme*, *group scheme*). The use of bold face characters in the text is used to emphasize the functorial nature of the object under consideration. In particular, if X is an S -scheme then its functor of points

$$T \mapsto \text{Hom}_{S\text{-Sch}}(T, X)$$

will be denoted by \mathbf{X} .

If \mathbf{F} is an S -functor and T a scheme over S , we denote by \mathbf{F}_T the T -functor obtained by base change. If $T = \text{Spec}(R)$ we denote $\mathbf{F}(T)$ and \mathbf{F}_T by $\mathbf{F}(R)$ and \mathbf{F}_R respectively. If \mathbf{F} is an S -scheme, then $\mathbf{F}_T = \mathbf{F} \times_S T$ and, as it is customary, we denote $\mathbf{F} \times_{\text{Spec}(R)} T$ by $\mathbf{F} \times_R T$ and $\mathbf{F} \times_S \text{Spec}(R)$ by $\mathbf{F} \times_S R$.

We denote as usual by \mathbf{O}_S the affine S -ring scheme $S[t] = \text{Spec}(\mathbb{Z}[t]) \times_{\mathbb{Z}} S$. Thus $\mathbf{O}_S(T) = \mathcal{O}_T(T)$. Recall that an \mathbf{O}_S -*module* is an abelian S -group \mathbf{M} together with an $\mathbf{O}_S(T)$ -module structure on $\mathbf{M}(T)$ that is functorial on T . The concept of an \mathbf{O}_S -algebra is defined similarly. Base change is defined and denoted as it is for functors.

We mainly use the terminology and notation of Grothendieck–Dieudonné [EGA-I], which for the most part agrees with that of Demazure–Grothendieck used in [SGA3, Exp. I and II]. Below we briefly review those concepts and results that are relevant to this paper.

2.1 Groups attached to quasi-coherent modules

Let \mathcal{E} be a quasi-coherent module over S . We denote its dual by \mathcal{E}^\vee . For each morphism $f : T \rightarrow S$ we let $\mathcal{E}_T = f^*(\mathcal{E})$ be the inverse image of \mathcal{E} under the morphism f , and define an abelian S -group $\mathbf{V}(\mathcal{E})$ by $\mathbf{V}(\mathcal{E})(T) = \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{E}_T, \mathcal{O}_T) = \Gamma(T, (\mathcal{E}_T)^\vee)$. \mathbf{V} is actually an S -scheme; it is represented by the affine scheme $\mathrm{Spec}(\mathbf{Sym}(\mathcal{E}))$ where $\mathbf{Sym}(\mathcal{E})$ is the symmetric \mathcal{O}_S -algebra of \mathcal{E} [EGA-I, 9.4.9]. If \mathcal{E} is of finite type (resp. of finite presentation), then $\mathbf{V}(\mathcal{E})$ is an S -scheme of finite type (resp. of finite presentation), *ibid*, 9.4.11.

The abelian S -group $\mathbf{W}(\mathcal{E})$ is defined by $\mathbf{W}(\mathcal{E})(T) = \Gamma(T, \mathcal{E}_T)$. Recall that if \mathcal{E} is locally free of finite type then $\mathbf{W}(\mathcal{E}) \simeq \mathbf{V}(\mathcal{E}^\vee)$. In particular $\mathbf{W}(\mathcal{E})$ is in this case an affine S -scheme. Note that the abelian S -groups $\mathbf{V}(\mathcal{E})$ and $\mathbf{W}(\mathcal{E})$ have natural \mathbf{O}_S -module structures.

Example 2.1. Assume $S = \mathrm{Spec}(R)$. Let \mathcal{L} be a quasi-coherent \mathcal{O}_S -module and denote by L the corresponding R -module. We denote $\mathbf{W}(\mathcal{L})$ by $\mathbf{W}(L)$. For all S -scheme T by definition $\mathbf{W}(L)(T)$ is the $\mathcal{O}_T(T)$ -module $L \otimes_R \mathcal{O}_T(T)$. Similarly if \mathcal{L} is an \mathcal{O}_S -algebra.

Remark 2.2. We can view \mathbf{V} (resp. \mathbf{W}) as a contravariant (resp. covariant) functor from the category of quasi-coherent \mathcal{O}_S -modules to the category of \mathbf{O}_S -modules. These functors are full and faithful [SGA3, I Prop. 4.6.2]

To \mathcal{E} we attach the S -functor $\mathbf{End}_{\mathcal{O}_S\text{-mod}}(\mathcal{E})$ whose functor of points is given by $T \mapsto \mathrm{End}_{\mathcal{O}_T\text{-mod}}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_T)$. This can be viewed as an abelian S -group or an \mathbf{O}_S -module. We also have the S -group $\mathbf{Aut}_{\mathcal{O}_S\text{-mod}}(\mathcal{E})$ given by $T \mapsto \mathrm{Aut}_{\mathcal{O}_T\text{-mod}}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_T)$. If in addition \mathcal{E} is an \mathcal{O}_S -algebra, one defines the S -group $\mathbf{Aut}_{\mathcal{O}_S\text{-alg}}(\mathcal{E})$ in the obvious way. It is an S -subgroup of $\mathbf{Aut}_{\mathcal{O}_S\text{-mod}}(\mathcal{E})$.

Similarly to an \mathbf{O}_S -module \mathbf{L} we attach an abelian S -group (in fact an \mathbf{O}_S -module) $\mathbf{End}_{\mathbf{O}_S\text{-mod}}(\mathbf{L})$ via $T \mapsto \mathrm{End}_{\mathbf{O}_T\text{-mod}}(\mathbf{L}_T)$. Finally if \mathbf{L} is an \mathbf{O}_S -algebra, the S -group $\mathbf{Aut}_{\mathbf{O}_S\text{-alg}}(\mathbf{L})$ is given by the functor of points $T \mapsto \mathrm{Aut}_{\mathbf{O}_T\text{-alg}}(\mathbf{L}_T)$.

Lemma 2.3. *Let \mathcal{L} be an \mathcal{O}_S -algebra. Assume that as an \mathcal{O}_S -module \mathcal{L} is locally free of finite rank. Then.*

(1) *The natural maps*

$$\mathbf{End}_{\mathcal{O}_S\text{-mod}}(\mathcal{L}) \rightarrow \mathbf{End}_{\mathbf{O}_S\text{-mod}}(\mathbf{W}(\mathcal{L}))$$

and

$$\mathbf{Aut}_{\mathcal{O}_S\text{-alg}}(\mathcal{L}) \rightarrow \mathbf{Aut}_{\mathbf{O}_S\text{-alg}}(\mathbf{W}(\mathcal{L}))$$

are S -functor isomorphisms.

(2) *$\mathbf{End}_{\mathcal{O}_S\text{-mod}}(\mathcal{L})$ and $\mathbf{Aut}_{\mathcal{O}_S\text{-alg}}(\mathcal{L})$ are representable by affine S -schemes of finite presentation.*

Proof. (1) From their definition we see that the maps under consideration are functorial. That they are bijective on points follows from Remark 2.2.

(2) Assume that $S = \text{Spec}(R)$. Then \mathcal{L} corresponds to an R -module L which is projective of finite rank. The \mathcal{O}_S -module $\mathcal{E}nd_{\mathcal{O}_S\text{-mod}}(\mathcal{L})$ corresponds to the R -module $\text{End}_{R\text{-mod}}(L) \simeq L^* \otimes_R L$. Because the R -module map $\text{End}_{R\text{-mod}}(L) \otimes_R R' \rightarrow \text{End}_{R'\text{-mod}}(L \otimes_R R')$ is an isomorphism for all R'/R , it follows that $\mathbf{End}_{\mathcal{O}_S\text{-mod}}(\mathcal{L})$ is represented by the affine R -scheme of finite presentation $\mathbf{W}(L^* \otimes_R L)$.

It is clear that $\mathbf{Aut}_{\mathcal{O}_S\text{-alg}}(\mathcal{L})$ is a closed subscheme of $\mathbf{End}_{\mathcal{O}_S\text{-mod}}(\mathcal{L})$, hence also affine, which is of finite presentation since L is locally free of finite rank.

From the foregoing considerations it follows that our two functors are affine S -schemes which are locally of finite presentation. Since their structure morphisms are affine, they are quasi-compact and separated, hence of finite presentation. \square

Remark 2.4. Let \mathcal{L} be a quasi-coherent \mathcal{O}_S -module, and consider the corresponding \mathcal{O}_S -module $\mathcal{E}nd_{\mathcal{O}_S\text{-mod}}(\mathcal{L})$. There is a natural \mathbf{O}_S -module morphism

$$\mathbf{W}(\mathcal{E}nd_{\mathcal{O}_S\text{-mod}}(\mathcal{L})) \rightarrow \mathbf{End}_{\mathbf{O}_S\text{-mod}}(\mathbf{W}(\mathcal{L})).$$

This morphism *need not* be an isomorphism. It is if \mathcal{L} is locally free of finite type.

2.2 Group schemes and Lie algebras

Throughout \mathbf{G} will denote an S -group scheme and we denote by $e \in \mathbf{G}(S)$ its unit section. We refer to [SGA3, I and II], [DG] and [LLR, §1] for details in what follows.

If T is a scheme we denote by $T[\epsilon]$ the corresponding scheme of dual number [SGA3, II.2]. The functor $T \rightarrow \mathbf{Lie}(\mathbf{G})(T) := \ker(\mathbf{G}(T[\epsilon]) \rightarrow \mathbf{G}(T))$ is an \mathbf{O}_S -module isomorphic to $\mathbf{V}(\omega_{\mathbf{G}/S}^1)$ where $\omega_{\mathbf{G}/S}^1 = e^*(\Omega_{\mathbf{G}/S}^1)$ *ibid.* Prop. 3.3 and 3.6. Furthermore, the $\mathbf{O}_S(T)$ -module $\mathbf{Lie}(\mathbf{G})(T)$ has a natural Lie algebra structure. It is thus an \mathbf{O}_S -Lie algebra. Recall that for all scheme morphisms $T \rightarrow S$ we have a natural \mathbf{O}_T -Lie algebra isomorphism

$$(2.1) \quad \mathbf{Lie}(\mathbf{G}) \times_S T \simeq \mathbf{Lie}(\mathbf{G}_T).$$

The $\mathcal{O}_S(S)$ -Lie algebra $\mathbf{Lie}(\mathbf{G})(S)$ is denoted by $\mathbf{Lie}(\mathbf{G})$. From the above isomorphism we have

$$(2.2) \quad \mathbf{Lie}(\mathbf{G})(T) = \mathbf{Lie}(\mathbf{G}_T).$$

We denote by $\mathcal{L}ie(\mathbf{G})$ the vector group sheaf (fibration vectorielle) of sections of the affine scheme $\mathbf{Lie}(\mathbf{G}) \rightarrow S$. In other words, $\mathcal{L}ie(\mathbf{G})$ is the \mathcal{O}_S -module $(\omega_{\mathbf{G}/S}^1)^\vee = \text{Hom}_{\mathcal{O}_S}(\omega_{\mathbf{G}/S}^1, \mathcal{O}_S)$. Note that $\mathcal{L}ie(\mathbf{G})$ is naturally an \mathcal{O}_S -Lie algebra.

Remark 2.5. In general $\mathcal{L}ie(\mathbf{G})$ does not determine $\mathbf{Lie}(\mathbf{G})$, but it does if $\omega_{\mathbf{G}/S}^1$ is locally free of finite type, in particular if \mathbf{G} is smooth. In this case $\mathbf{Lie}(\mathbf{G}) = \mathbf{W}(\mathcal{L}ie(\mathbf{G}))$. See [SGA3, II Lemma 4.11.7]. If $S = \text{Spec}(R)$ the R -Lie algebra $\mathbf{Lie}(\mathbf{G}) = \mathcal{L}ie(\mathbf{G})(S)$ is a locally free R -module of finite type. For any ring extension R'/R we have $\mathbf{Lie}(\mathbf{G}_{R'}) = \mathbf{Lie}(\mathbf{G})(R') = \mathbf{Lie}(\mathbf{G}) \otimes_R R'$.

Lemma 2.6. *Let \mathcal{L} be an \mathcal{O}_S -algebra which is locally free of finite type (as an \mathcal{O}_S -module). Let $\mathbf{L} = \mathbf{W}(\mathcal{L})$. There is a natural \mathbf{O}_S -Lie algebra isomorphism $\mathbf{Der}_{\mathbf{O}_S\text{-alg}}(\mathbf{L}) \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_S\text{-alg}}(\mathbf{L}))$.*

Proof. Since \mathbf{L} is a good \mathbf{O}_S -module [SGA3, II Def. 4.4 and Ex. 4.4.2], we have an \mathbf{O}_S -module isomorphisms $\eta : \mathbf{End}_{\mathbf{O}_S\text{-mod}}(\mathbf{L}) \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_S\text{-mod}}(\mathbf{L}))$ [SGA3, II Prop. 4.5]. Since \mathcal{L} is locally free of finite type we can appeal to Lemma 2.3(1) to conclude that this is in fact an isomorphism of S -schemes. We claim that the restriction of η to $\mathbf{Der}_{\mathbf{O}_S\text{-alg}}(\mathbf{L})$ is our isomorphism. The proof reduces to the case when S is affine, which can be found in [DG, II §4 2.3]. \square

2.3 Connected component of the identity, forms and type of a reductive group scheme.

Let \mathbf{G} be an S -group scheme which is locally of finite presentation. We consider the S -subgroup (functor) \mathbf{G}° of \mathbf{G} [SGA3, VI_B.3.1] defined by

$$\mathbf{G}^\circ(\mathbf{T}) = \left\{ u \in \mathbf{G}(\mathbf{T}) \mid \forall s \in S, u_s(\mathbf{T}_s) \subset \mathbf{G}_s^\circ \right\}.$$

where \mathbf{G}_s° is the connected component of the identity of the $\kappa(s)$ -algebraic group \mathbf{G}_s . If \mathbf{G} is smooth along the unit section, \mathbf{G}° is representable by a smooth S -group scheme called *the connected component of the identity of \mathbf{G}* [SGA3, VI_B.4.1]. Furthermore, the fibre $(\mathbf{G}^\circ)_s$ is naturally isomorphic to \mathbf{G}_s° .

Let \mathbf{G} be a reductive S -group scheme, that is a smooth and affine S -group whose geometric fibres are *connected* reductive algebraic groups. An S -group scheme \mathbf{G}' is called a (*twisted*) *form of \mathbf{G}* , if there exist a faithfully flat and locally presented extension $\mathbf{T} \rightarrow S$ such that $\mathbf{G}_{\mathbf{T}}$ and $\mathbf{G}'_{\mathbf{T}}$ are isomorphic \mathbf{T} -group schemes. In a similar fashion one defines forms of \mathbf{O}_S -algebras.

Let \mathbf{G} be a reductive K -group scheme, where K is an algebraically closed field. If \mathbf{T} is a maximal torus of \mathbf{G} , then \mathbf{T} is split and defines a root datum which is up to isomorphism independent of the choice of \mathbf{T} . It is called the *type of \mathbf{G}* and is denoted by $\mathcal{R}(\mathbf{G})$.

Let \mathbf{G} be a reductive \mathbb{S} -group scheme. If $s \in \mathbb{S}$, the *type of \mathbf{G} at s* is the type of the reductive $\overline{\kappa(s)}$ -group $\mathbf{G}_{\overline{s}} := \mathbf{G}_s \times_{\kappa(s)} \overline{\kappa(s)}$. Because \mathbf{G} possesses maximal tori locally for the étale topology, one knows that the type function $s \mapsto \text{type of } \mathbf{G}_{\overline{s}}$ is locally constant [SGA3, XXII 2.8]. We say that \mathbf{G} is of *constant type* if the type function is constant. This is the case, for example, if \mathbb{S} is connected.

2.4 Specific properties in characteristic zero

We assume in this section that \mathbb{S} is a \mathbb{Q} -scheme. Let \mathbf{G} be a semisimple adjoint \mathbb{S} -group scheme of constant type. We will denote the $\mathbf{O}_{\mathbb{S}}$ -Lie algebra $\mathbf{Lie}(\mathbf{G})$ by \mathbf{L} , and the $\mathcal{O}_{\mathbb{S}}$ -Lie algebra $\mathcal{L}ie(\mathbf{G})$ by \mathcal{L} . Recall (see Remark 2.5) that \mathcal{L} is locally free of finite type and that $\mathbf{W}(\mathcal{L}) = \mathbf{L}$. Since \mathbf{G} and its simply connected cover \mathbf{G}^{sc} have the same Lie algebra, we have isomorphisms of affine \mathbb{S} -group schemes [SGA3, XXIV 3.6 and 7.3.1]

$$(2.3) \quad \mathbf{Aut}(\mathbf{G}) \simeq \mathbf{Aut}(\mathbf{G}^{\text{sc}}) \simeq \mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathbf{L}).^1$$

In particular, the \mathbb{S} -scheme $\mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathbf{L})$ is smooth and affine. By (2.3) and [SGA3, XXIV 1.3 and 1.8] we have an isomorphism

$$(2.4) \quad \mathbf{G} \simeq \mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathbf{L})^\circ.$$

This yields an $\mathbf{O}_{\mathbb{S}}$ -Lie algebra isomorphism

$$(2.5) \quad \mathbf{L} \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathbf{L})) = \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathbf{L})^\circ).$$

Recall [SGA3, II Theo. 4.7 and Prop. 4.8] the adjoint representation $\text{Ad} : \mathbf{G} \rightarrow \mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-mod}}(\mathbf{L})$ and the induced $\mathbf{O}_{\mathbb{S}}$ -Lie algebra homomorphism $\text{ad} : \mathbf{L} \rightarrow \mathbf{End}_{\mathbf{O}_{\mathbb{S}}\text{-mod}}(\mathbf{L})$.

Lemma 2.7. *Under the identification $\mathbf{L} \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathbf{L}))$ of (2.5) the map ad induces an $\mathbf{O}_{\mathbb{S}}$ -Lie algebra isomorphism between \mathbf{L} and $\mathbf{Der}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathbf{L})$.*

Proof. All the morphisms under consideration are \mathbb{S} -scheme morphisms, so the question is local on \mathbb{S} and we may assume that $\mathbb{S} = \text{Spec}(R)$ for some \mathbb{Q} -ring R . Consider the R -Lie algebra $L = \mathbf{L}(R)$. Since $\mathbf{L} = \mathbf{W}(\mathbf{L})$ where $\mathbf{W}(\mathbf{L})(T) = \mathbf{L} \otimes_R \mathcal{O}_T(T)$, it will suffice to show that for all ring extensions R'/R the R' -Lie algebra homomorphism $d = \text{ad}'_R : \mathbf{L} \otimes_R R' \rightarrow \mathbf{Der}_{R'\text{-Lie}}(\mathbf{L} \otimes_R R')$ is an isomorphism. By replacing \mathbf{G} by \mathbf{G}'_R we may assume that $R = R'$. By [SGA3, II 4.7.2] the map d is nothing

¹ La “définition évidente” of the \mathbb{S} -group scheme $\underline{\mathcal{A}ut}_{\mathbf{O}_{\mathbb{S}}\text{-alg-de-Lie}}(\mathcal{L})$ of 7.3.1(iii) is precisely our $\mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathcal{L})$, so that the assertion is that the natural map $\mathbf{Aut}(\mathbf{G}^{\text{sc}}) \rightarrow \mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathcal{L})$ is an isomorphism. Finally $\mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathcal{L}) \simeq \mathbf{Aut}_{\mathbf{O}_{\mathbb{S}}\text{-Lie}}(\mathbf{L})$ by Lemma 2.3.

but the adjoint representation $d(x) = \text{ad}_L(x)$. Since \mathbf{G} is of constant type there exists a unique Chevalley group \mathbf{G}_0 such that \mathbf{G} is a twisted form of $\mathbf{G}_0 \times_{\mathbb{Z}} R$. Let $\mathfrak{g} = \mathbf{Lie}(\mathbf{G}_0)(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then \mathfrak{g} is a finite dimensional split semisimple Lie algebra over \mathbb{Q} and L is a twisted form of $\mathfrak{g} \otimes_{\mathbb{Q}} R$, that is, there exists an fppf extension (in fact étale cover) R' of R such that the R' -Lie algebras $L \otimes_R R'$ and $(\mathfrak{g} \otimes_{\mathbb{Q}} R) \otimes_R R' \simeq \mathfrak{g} \otimes_{\mathbb{Q}} R'$ are isomorphic. Fix one such isomorphism

$$(2.6) \quad \psi : L \otimes_R R' \rightarrow \mathfrak{g} \otimes_{\mathbb{Q}} R'.$$

If $d(x) = 0$, then x belongs to the centre of L and therefore $\psi(x \otimes 1)$ belongs to the centre of $\mathfrak{g} \otimes_{\mathbb{Q}} R'$, which is trivial. Since R'/R is faithfully flat $x = 0$ so that d is injective and we henceforth identify L with an R -submodule of $\text{Der}(L)$. To show that d is surjective, that is that every derivation of L is inner, we reason as follows. If $L = \mathfrak{g} \otimes_{\mathbb{Q}} R$ every derivation is inner by Whitehead's Lemma (see [P1, Example 4.9]).² For L arbitrary, we appeal to the isomorphism (2.6). By the split case we have $(\text{Der}(L)/L) \otimes_R R' = 0$. Thus $L = \text{Der}(L)$ as desired. \square

2.5 Semicontinuity property for Lie algebras

For convenience we recall the following well-known fact.

Lemma 2.8. *Let \mathcal{F} be a quasi-coherent \mathcal{O}_S -module of finite presentation. Then the function $s \mapsto \dim_{\kappa(s)} (\mathcal{F} \otimes_{\mathcal{O}_S} \kappa(s))$ is upper semi-continuous.*

Proof. The statement is local and therefore reduces then to the case of a ring R and of an R -module M of finite presentation [Stacks, Tag 01PC]. If R is noetherian then M is coherent and the result can be found in [Ha, 12.7.2]. For lack of a reference we provide a proof in the general case.

We consider an exact sequence $R^m \xrightarrow{A} R^n \rightarrow M \rightarrow 0$ where A is a matrix of size (m, n) with entries in R . Let $\mathfrak{q} \in \text{Spec}(R)$ and $r = \dim_{\kappa(\mathfrak{q})}(M \otimes_R \kappa(\mathfrak{q}))$. Since the sequence $\kappa(\mathfrak{q})^m \xrightarrow{A_{\kappa(\mathfrak{q})}} \kappa(\mathfrak{q})^n \rightarrow M_{\kappa(\mathfrak{q})} \rightarrow 0$ is exact, there exists a minor B of A of size $(n - r, n - r)$ such that $\det(B_{\kappa(\mathfrak{q})}) \neq 0 \in \kappa(\mathfrak{q})$. Consider the basic open set U of $\text{Spec}(R)$ consisting of prime ideals of R that do not contain $f = \det(B) \in R$. From $\det B_{\kappa(\mathfrak{q})} \neq 0$ it follows that $\mathfrak{q} \in U$. It is clear that for all $\mathfrak{p} \in U$ the image of $A_{\kappa(\mathfrak{p})}$ is a $\kappa(\mathfrak{p})$ -space of dimension at least $n - r$. Thus

$$\dim_{\kappa(\mathfrak{p})} (M \otimes_R \kappa(\mathfrak{p})) \leq r = \dim_{\kappa(\mathfrak{q})} (M \otimes_R \kappa(\mathfrak{q}))$$

for all $\mathfrak{p} \in U$, so that our function is thus upper semi-continuous as desired. \square

² Note that our L is denoted by \mathcal{L} in [P1].

Lemma 2.9. *Let \mathbf{G} be an S -group scheme that is locally of finite presentation. Then $\omega_{\mathbf{G}/S}^1$ is an \mathcal{O}_S -module of finite presentation and the map $s \mapsto \dim_{\kappa(s)} \mathbf{Lie}(\mathbf{G}_s)$ is upper semi-continuous.*

Proof. That $\omega_{\mathbf{G}/S}^1$ is of finite presentation follows from [Stacks, Tag 01V2, 01V3].

By (2.2) we have a $\kappa(s)$ -Lie algebra isomorphism $\mathbf{Lie}(\mathbf{G}_s) \simeq \mathbf{Lie}(\mathbf{G})(\kappa(s))$. From the isomorphism $\mathbf{Lie}(\mathbf{G}) \simeq \mathbf{V}(\omega_{\mathbf{G}/S}^1)$, we see that the dimension of $\mathbf{Lie}(\mathbf{G})(\kappa(s))$ at a point $s \in S$ is the dimension of the $\kappa(s)$ -space $\mathrm{Hom}_{\kappa(s)}(\omega_{\mathbf{G}/S}^1 \otimes_{\mathcal{O}_S} \kappa(s), \kappa(s))$, which is the dimension of $\omega_{\mathbf{G}/S}^1 \otimes_{\mathcal{O}_S} \kappa(s)$ since $\omega_{\mathbf{G}/S}^1$ is an \mathcal{O}_S -module of finite type. Furthermore, since $\omega_{\mathbf{G}/S}^1$ is in fact of finite presentation, Lemma 2.8 implies that the map $s \mapsto \dim_{\kappa(s)}(\mathbf{Lie}(\mathbf{G})(\kappa(s)))$ is upper semi-continuous. \square

3 Group schemes over Jacobson schemes

3.1 Jacobson schemes and smoothness

We refer the reader to [EGAIV, §10] for general results about Jacobson schemes. Let X be a topological space. We recall that a subset X_0 of X is called *very dense* if for every closed subset $Z \subset X$ we have $Z = \overline{Z \cap X_0}$. Let S be a scheme. View S as a topological space and let $S_0 \subset S$ the set of closed points. The scheme S is said to be *Jacobson* if S_0 is a very dense subset of S .

Let us recall for the sake of completeness that:

- (i) Very dense subsets of a topological space are dense.
- (ii) If $S = \mathrm{Spec}(R)$, then S is Jacobson if and only if R is a Jacobson ring, that is every ideal of R is an intersection of maximal ideals.
- (iii) If $S = \bigcup_{i \in I} \mathrm{Spec}(R_i)$, the S is Jacobson if and only if every R_i is a Jacobson ring.
- (iv) Every radical ideal I is the intersection of the maximal ideals containing it [Stacks, Tag 00G4].³

Proposition 3.1. *Let $f : X \rightarrow S$ be a scheme morphisms which is locally of finite presentation. Assume that $x \in X$ is such that:*

- (i) f is flat at x .
- (ii) $f_{\kappa(x)}$ is smooth.

Then f is smooth on an open neighborhood of x .

³AP. We will not use these properties in what follows.

Proof. By assumption there exists affine open subschemes $\text{Spec}(A) = U \subset X$ and $\text{Spec}(B) = V \subset S$ such that $x \in U$, $f(U) \subset V$ and A is a finitely presented B -algebra. Let $\mathfrak{q} \in \text{Spec}(A)$ and $\mathfrak{p} \in \text{Spec}(B)$ be the points corresponding to x and $f(x)$ respectively.

By (i) $A_{\mathfrak{q}}$ is flat over $B_{\mathfrak{p}}$, while by (ii) $A \otimes_B \kappa(\mathfrak{p})$ is smooth over $\kappa(\mathfrak{p})$. It follows from [Stacks, Tag 00TF] that the restriction of f to U is smooth at \mathfrak{q} , hence (by definition of smoothness) smooth also in a neighbourhood of \mathfrak{q} . \square

Corollary 3.2. *Let $f : X \rightarrow S$ be as above. Assume that (i) and (ii) hold for all closed points of X . If S is Jacobson, then f is smooth.*

Proof. By [EGAIV, Cor. 10.4.7] X is Jacobson. Let X_0 be the set of closed points of X viewed as a topological space. By the Proposition there exists an open $U \subset X$ containing X_0 in which f is smooth. Let $Z = X \setminus U$. If $Z \neq \emptyset$, then $Z \cap X_0 \neq \emptyset$ because X_0 is very dense. This contradicts $X_0 \subset U$. Thus $Z = \emptyset$ so that $U = X$ as desired. \square

Proposition 3.3. *Let S be an integral Jacobson scheme, and \mathbf{G} be an S -group scheme of finite presentation. Assume that for all closed point $b \in S$ the fibres \mathbf{G}_b are smooth and of the same dimension d .⁴ Then.*

- (1) *For all $s \in S$ the algebraic $\kappa(s)$ -group \mathbf{G}_s is smooth of dimension d .*
- (2) *The S -functor \mathbf{G}° is representable by a smooth S -group scheme of relative dimension d which is open in \mathbf{G} .*
- (3) *If for each closed point b of S the fibre \mathbf{G}_b is connected, then \mathbf{G} is smooth and has connected geometric fibres (i.e., $\mathbf{G}_{\bar{b}}$ is connected).*

Proof. (1) We denote by $\eta : \text{Spec}(F) \rightarrow S$ the generic point of S . Chevalley's generic flatness theorem [G-W, 10.85] shows that there exists an open dense subset U of S such that $\mathbf{G} \times_S U$ is flat over U . We know that U is also Jacobson [EGAIV, Prop. 10.3.3]. Corollary 3.2 shows that $\mathbf{G} \times_S U$ is smooth over U . By [Stacks, Tag 05F7], up to shrinking U if necessary, the dimension function of the fibres of $\mathbf{G} \times_S U \rightarrow U$ is constant. Since by assumption this dimension has value d on the closed points of U , we get that $\mathbf{G} \times_S U$ is smooth of relative dimension d .⁵ In particular the algebraic F -group \mathbf{G}_F is smooth of dimension d .

Let $s \in S$ and let $c = \dim(\mathbf{G}_s)$. According to Chevalley's semicontinuity theorem [SGA3, VI_B.4.1]

$$F_c = \{x \in S : \dim(\mathbf{G}_x) \geq c\} \subset S$$

⁴ Since \mathbf{G} is of finite presentation \mathbf{G}_s is an algebraic $\kappa(s)$ -group. Thus \mathbf{G}_s has a finite number of irreducible components. They all have the same (finite) dimension and this is also the dimension of \mathbf{G}_s .

⁵ That is, all the fibres $\mathbf{G}_s, s \in U$ are equidimensional and this dimension is d .

is a closed set. Since $F_c \cap \overline{\{s\}}$ contains s and is closed

$$(3.1) \quad \overline{\{s\}} \subset F_c.$$

Since S is Jacobson, it follows that there exists a closed point b in $\overline{\{s\}}$. We have

$$(3.2) \quad \dim(\mathbf{G}_s) \leq \dim(\mathbf{G}_b) = d.$$

On the other hand s belongs to $\overline{\{\eta\}}$. The same semicontinuity reasoning used above shows that

$$(3.3) \quad \dim(\mathbf{G}_F) \leq \dim(\mathbf{G}_s).$$

Since \mathbf{G}_F is of dimension d it follows from (3.2) and (3.3) that $\dim(\mathbf{G}_s) = d$.

For establishing smoothness we use a variation of the previous argument by applying semicontinuity considerations to the Lie algebras. Lemma 2.9 yields the inequalities

$$\dim_F \operatorname{Lie}(\mathbf{G}_F) \leq \dim_{\kappa(s)} \operatorname{Lie}(\mathbf{G}_s) \leq \dim_{\kappa(b)} \operatorname{Lie}(\mathbf{G}_b).$$

Since \mathbf{G} is smooth of dimension d at η and at b we obtain that $d = \dim_{\kappa(s)} \operatorname{Lie}(\mathbf{G}_s)$. Thus $\operatorname{Lie}(\mathbf{G}_s)$ and \mathbf{G}_s have the same dimension (namely d). By the smoothness criterion [DG, II.5.2.1] \mathbf{G}_s is smooth. This completes the proof of (1).

(2) By (1) we see that assumption (ii) of [SGA3, VI_B Cor. 4.4] holds. It follows that the S -functor \mathbf{G}° is representable by a smooth S -group scheme which is open in \mathbf{G} .

(3) According to [Stacks, Tag 055I], the level set

$$E_n = \{s \in S \mid \mathbf{G}_s \text{ has } n \text{ geometrically connected components}\}$$

is a locally constructible subset of S for each $n \geq 1$. Since S is a Jacobson scheme, the set S_0 of closed points of the underlying topological space of S is very dense in S . Thus $S_0 \cap E_n$ is dense in E_n for each n [EGAIV, 10.1.2, (b')]. Our assumption implies that $E_n = \emptyset$ for $n \geq 2$ so that $E_1 = S$. That \mathbf{G} is smooth now follows from (2). \square

3.2 Forms

Proposition 3.4. *Let S be an integral Jacobson scheme and \mathbf{G} an affine S -group scheme of finite presentation. Assume that \mathbf{G}_b is reductive and of dimension d for all closed point $b \in S$. Then \mathbf{G} is a reductive S -group scheme of constant type. In particular, there exists a unique Chevalley group \mathbf{G}_0 such that \mathbf{G} is an S -form of $\mathbf{G}_0 \times_{\mathbb{Z}} S$.*

Proof. Proposition 3.3.(3) shows that \mathbf{G} is smooth of relative dimension d and has connected geometric fibres. According to [Co1, Prop. 3.1.9.(1)] \mathbf{G} is reductive in a neighborhood of each closed point $b \in S$. Since S is a Jacobson scheme, it follows that \mathbf{G} is reductive. Since the type function is locally constant, the connectedness of S implies that \mathbf{G} has constant type t_0 (see §2.3). Let \mathbf{G}_0 be the corresponding Chevalley group scheme. According to [SGA3, XXIII 5.6] \mathbf{G} is an S -form of $\mathbf{G}_0 \times_{\mathbb{Z}} S$. \square

Corollary 3.5. *Let \mathbf{G} and \mathbf{G}' be group schemes over an integral Jacobson scheme S . Assume that \mathbf{G} is reductive and that \mathbf{G}' is affine and of finite presentation. If \mathbf{G}'_b is a form of \mathbf{G}_b for each closed point $b \in S$, then \mathbf{G}' is an S -form of \mathbf{G} . In particular, \mathbf{G}' is a reductive S -group scheme* \square

3.3 Lie algebras

If \mathcal{L} is an \mathcal{O}_S -Lie algebra and $s \in S$, then the $\mathcal{O}_{\text{Spec } \kappa(s)}$ -algebra $\mathcal{L} \otimes_{\mathcal{O}_S} \kappa(s)$ obtained by base change is simply a $\kappa(s)$ -algebra that we will denote by \mathcal{L}_s .⁶ Let $\mathbf{L} = \mathbf{W}(\mathcal{L})$ and denote by a harmless abuse of notation the $\kappa(s)$ -algebra $\mathbf{L}(\kappa(s))$ by \mathbf{L}_s . Note that $\mathcal{L}_s = \mathbf{L}_s$.

Proposition 3.6. *Assume that S is an integral Jacobson \mathbb{Q} -scheme. Let \mathcal{L} be an \mathcal{O}_S -Lie algebra which is locally free of rank d , and let $\mathbf{L} = \mathbf{W}(\mathcal{L})$ be its corresponding \mathcal{O}_S -Lie algebra. Assume that for each closed point $b \in S$ the $\kappa(b)$ -Lie algebra \mathbf{L}_b is semisimple.⁷ Then there exists a unique finite dimensional split semisimple Lie \mathbb{Q} -algebra \mathbf{L}_0 such that \mathbf{L} is a form of $\mathbf{L}_0 := \mathbf{L}_0 \otimes_{\mathbb{Q}} \mathcal{O}_S$.⁸*

Proof. (1) By Lemma 2.3 $\mathbf{G} := \mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L})$ is an affine S -group of finite presentation. For each closed point $b \in S$, the algebraic group $\mathbf{G}_b = \mathbf{Aut}_{\kappa(b)\text{-Lie}}(\mathbf{L}_b)$ is smooth of dimension d and \mathbf{G}_b° is semisimple adjoint. Proposition 3.3.(2) shows that \mathbf{G}° is representable by an open subgroup scheme of \mathbf{G} which is smooth of relative dimension d . Appealing now to Proposition 3.4 yields that \mathbf{G}° is adjoint semisimple and is a form of the S -group corresponding to a (unique) semisimple adjoint Chevalley group scheme \mathbf{G}_0° . We denote by \mathbf{L}_0 the \mathbb{Q} -Lie algebra of $\text{Lie}(\mathbf{G}_0^\circ) \otimes_{\mathbb{Z}} \mathbb{Q}$. This yields the \mathcal{O}_S -Lie algebra that we denoted by \mathbf{L}_0 . Observe that $\mathbf{Lie}(\mathbf{G})$ is a form of \mathbf{L}_0 as an \mathcal{O}_S -Lie algebras.

By Lemma 2.7 we have $\mathbf{L} \simeq \mathbf{Der}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L})$. Finally since $\mathbf{G} \simeq \mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L})$ we can apply Lemma 2.6 to conclude that $\mathbf{Lie}(\mathbf{G}) \simeq \mathbf{Der}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L})$. It follows that \mathbf{L} is a form of \mathbf{L}_0 as desired. \square

⁶ A $\mathcal{O}_{\text{Spec } \kappa(s)}$ -algebra “is the same” as a $\kappa(s)$ -algebra.

⁷ Necessarily of finite dimension d .

⁸ By definition $\mathbf{L}_0(\mathbb{T})$ is the $\mathcal{O}_{\mathbb{T}}(\mathbb{T})$ -Lie algebra $\mathbf{L}_0 \otimes_{\mathbb{Q}} \mathcal{O}_{\mathbb{T}}(\mathbb{T})$.

The following variant of the ideas presented heretofore will allow us to answer Burban's question (see §4 below).

Proposition 3.7. *Assume that S is an integral Jacobson \mathbb{Q} -scheme. Let \mathbf{G} be a semisimple adjoint S -group scheme and consider its \mathcal{O}_S -Lie algebra $\mathcal{L} = \mathcal{L}ie(\mathbf{G})$.*

Let \mathcal{L}' be a sheaf of \mathcal{O}_S -Lie algebras. We assume that as an \mathcal{O}_S -module \mathcal{L}' is locally free of rank d , and that for each closed point $b \in S$, \mathcal{L}'_b is a form of \mathcal{L}_b .

(1) *The S -functor $\mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathcal{L}')$ is a smooth affine S -group scheme whose connected component of the identity \mathbf{G}' is an S -form of \mathbf{G} . In particular \mathbf{G}' is a semisimple adjoint S -group scheme.*

(2) *There is a natural isomorphism $\mathcal{L}ie(\mathbf{G}') \simeq \mathcal{L}'$.*

(3) *\mathcal{L}' is an S -form of \mathcal{L} .*

(3 bis) *\mathbf{L}' is an S -form of \mathbf{L} , where $\mathbf{L}' = \mathbf{W}(\mathcal{L}')$ and $\mathbf{L} = \mathbf{W}(\mathcal{L})$.*

Proof. (1) Let $\mathbf{L}' = \mathbf{W}(\mathcal{L}')$. By Lemma 2.3 $\mathbf{H} := \mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathcal{L}') \simeq \mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L}')$ is an affine S -group scheme of finite presentation. Since $\mathbf{H}_T = \mathbf{Aut}_{\mathcal{O}_T\text{-Lie}}(\mathbf{L}'_T)$ for all S -scheme T , we have $\mathbf{H}_s = \mathbf{Aut}(\mathbf{L}'_s)$ for all $s \in S$, where $\mathbf{L}'_s := \mathbf{L}' \otimes_{\mathcal{O}_S} \kappa(s)$. These algebraic groups are smooth since $\kappa(s)$ is of characteristic 0.

Let $b \in S$ be a closed point. Since the $\kappa(b)$ -Lie algebra \mathcal{L}'_b is a twisted form of \mathcal{L}_b , the corresponding $\kappa(b)$ -algebraic group $\mathbf{Aut}(\mathcal{L}'_b)$ is a twisted form of $\mathbf{Aut}(\mathcal{L}_b)$. Similarly for their connected component of the identity. As we have seen that $\mathbf{Aut}(\mathcal{L}'_b) = \mathbf{H}_b$. On the other hand since \mathbf{G} is semisimple adjoint $\mathbf{Aut}(\mathcal{L}_b)^\circ = \mathbf{G}_b$. This yields that \mathbf{H}_b° is a twisted form of \mathbf{G}_b . In particular all the \mathbf{H}_b are smooth groups of the same dimension. Proposition 3.3.(2) then shows that the S -subfunctor \mathbf{H}° of \mathbf{H} is representable by a smooth affine S -group scheme \mathbf{G}' .

According to [Co1, prop. 3.1.9.(1)], \mathbf{G}' is reductive on a neighborhood of each closed point $b \in S$. Since S is a Jacobson scheme, it follows that \mathbf{G}' is reductive. Since S is connected the type of \mathbf{G}' is constant. Since \mathbf{G}' and \mathbf{G} have the same type at closed points, their (constant) types coincide. Thus \mathbf{G}' is a twisted form of \mathbf{G} [SGA3, XXIII 5.6]. In particular, \mathbf{G}' is semisimple adjoint.

(2) By Lemma 2.6

$$\mathbf{Der}_{\mathcal{O}_S\text{-alg}}(\mathbf{L}') \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathcal{O}_S\text{-alg}}(\mathbf{L}')) = \mathbf{Lie}(\mathbf{H}^\circ) = \mathbf{Lie}(\mathbf{G}').$$

On the other hand we have an isomorphism of \mathcal{O}_S -Lie algebras $\mathbf{L}' \simeq \mathbf{Der}_{\mathcal{O}_S\text{-alg}}(\mathbf{L}')$ (Lemma 2.7), whence an isomorphism $\mathbf{L}' \simeq \mathbf{Lie}(\mathbf{G}')$. It follows that $\mathbf{W}(\mathcal{L}') \simeq \mathbf{W}(\mathcal{L}ie(\mathbf{G}'))$. Since \mathbf{W} is full and faithful \mathcal{L}' and $\mathcal{L}ie(\mathbf{G}')$ are isomorphic \mathcal{O}_S -Lie algebras.

(3) From (2) and Remark 2.5 we get that $\mathbf{Lie}(\mathbf{G}') = \mathbf{W}(\mathcal{L}ie(\mathbf{G}'))$ is a form of $\mathbf{Lie}(\mathbf{G}) = \mathbf{W}(\mathcal{L}ie(\mathbf{G}))$. It follows that $\mathcal{L}ie(\mathbf{G}')$ is a form of $\mathcal{L}ie(\mathbf{G})$. Thus (3) follows from (2).

(3 bis.) Follows from (3) since \mathbf{W} commutes with base change. \square

4 Burban's question

In order to formulate Burban's question we need to recall the concept of loop algebra of a simple Lie finite dimensional complex Lie algebra \mathfrak{g} .

Let $R = \mathbb{C}[t^{\pm 1}]$. Fix a positive integer d , and set $R_d = \mathbb{C}[t^{\pm \frac{1}{d}}]$. The natural map $R \rightarrow R_d$ is faithfully flat and finite étale. Let $\xi \in \mathbb{C}$ be a primitive d -th root of unity. Then the elements of $\Gamma = \mathbb{Z}/d\mathbb{Z}$ act as automorphisms of R_d over R via

$$\bar{e}_{t^{\frac{1}{d}}} = \xi^e t^{\frac{1}{d}}.$$

for $e \in \mathbb{Z}$. This action makes R_d into a Galois extension of R with Galois group Γ .

Let σ be an automorphism of \mathfrak{g} of order d . For $i \in \mathbb{Z}$ consider the eigenspace

$$\mathfrak{g}_i = \{x \in \mathfrak{g} : \sigma(x) = \xi^i x\}$$

Then $\mathfrak{g} = \bigoplus_{0 \leq i < d} \mathfrak{g}_i$. Out of this data we define the corresponding *loop algebra*

$$(4.1) \quad L(\mathfrak{g}, \sigma) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \otimes t^{\frac{i}{d}} \subset \mathfrak{g} \otimes_{\mathbb{C}} R_d.$$

The simple but crucial observation is that $L(\mathfrak{g}, \sigma)$ is stable under the scalar action of R . Thus $L(\mathfrak{g}, \sigma)$ is not only an infinite dimensional complex Lie algebra, but also an R -Lie algebra. As we shall see, it is the algebra structure over this ring that allows non-abelian cohomological considerations to enter into the picture.

It is an easy linear algebra exercise to verify that we have a natural R_d -algebra isomorphism

$$(4.2) \quad L(\mathfrak{g}, \sigma) \otimes_R R_d \simeq \mathfrak{g} \otimes_{\mathbb{C}} R_d \simeq \mathfrak{g}_R \otimes_R R_d.$$

where $\mathfrak{g}_R := \mathfrak{g} \otimes_{\mathbb{C}} R$. This shows that the R -Lie algebra $L := L(\mathfrak{g}, \sigma)$ is a twisted form of $\mathfrak{g} \otimes_{\mathbb{C}} R$. It therefore corresponds to an $\mathbf{Aut}(\mathfrak{g}_R)$ -torsor \mathbf{X} over $\mathrm{Spec}(R)$. More precisely $\mathbf{X} = \mathbf{Isom}_{R\text{-Lie}}(\mathfrak{g}_R, L)$.⁹

We can now formulate Burban's questions (essentially verbatim except for some notation changes):

Question 1: Let L' be a Lie algebra over the ring $\mathbb{C}[t]$. Assume L' is free as a module and that for any complex number b the quotient Lie algebra $L'/(t-b)L'$ is

⁹ See [DG, III §4] for the material on torsors used in this section.

isomorphic to \mathfrak{g} . Does it follow that L' is isomorphic to $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$ (as a Lie algebra over $\mathbb{C}[t]$)?

Question 2: Similarly, let L' be a Lie algebra over the ring $\mathbb{C}[t^{\pm 1}]$ which we assume is free as a module and such that for any $b \in \mathbb{C}^{\times}$ the quotient $L'/(t-b)L'$ is isomorphic to \mathfrak{g} . Does it follow that L' is isomorphic to a loop algebra (with respect to an automorphism of \mathfrak{g} of finite order)?

To answer these questions we take $S = \text{Spec}(R)$ where $R = \mathbb{C}[t]$ or $\mathbb{C}[t^{\pm 1}]$, and where \mathcal{L}' and \mathcal{L} are the \mathcal{O}_S -Lie algebra corresponding to L' and $\mathfrak{g} \otimes_{\mathbb{C}} R$ respectively. By Proposition 3.7 we see that \mathcal{L}' corresponds to a torsor \mathbf{X}' over $\text{Spec}(R)$ whose class is an element of $H^1(R, \mathbf{Aut}(\mathfrak{g}_R))$.

Recall ([SGA3, XXIV 1.3 and 7.3.1]. See also §2.4) the split exact sequence of R -group schemes

$$(4.3) \quad 1 \rightarrow \mathbf{G} \rightarrow \mathbf{Aut}(L) \rightarrow \mathbf{Out}(L) \rightarrow 1$$

where \mathbf{G} is the split adjoint semisimple R -group scheme corresponding to \mathfrak{g} , and $\mathbf{Out}(L)$ is the constant R -group scheme corresponding to the finite (abstract) group $\text{Out}(\mathfrak{g})$ of the symmetries of the Coxeter-Dynkin diagram of \mathfrak{g} . This allow us to compute the relevant H^1 and thus determine the nature of \mathbf{X}' , hence L' . According to [CGP, Cor. 3.3], we have a bijection

$$H^1(R, \mathbf{Aut}(L)) \xrightarrow{\sim} H^1(R, \mathbf{Out}(L)).$$

Answer 1: If $R = \mathbb{C}[t]$ then $H^1(R, \mathbf{Out}(L)) = 1$ since R is simply connected. It follows that $H^1(R, \mathbf{Aut}(L)) = 1$ and therefore that $L' \simeq \mathfrak{g} \otimes_k R$.

Answer 2: If $R = \mathbb{C}[t^{\pm 1}]$ then furthermore $H^1(R, \mathbf{Out}(L))$ is the set of conjugacy classes of the (abstract) group $\text{Out}(\mathfrak{g})$ which, in terms of forms of L , correspond to the loop algebras $L(\mathfrak{g}, \sigma)$ with $\sigma \in \text{Out}(\mathfrak{g})$ (see [P2] for details, or more generally [P3]). Thus L' is a loop algebra.

References

- [Bu] I. Burban, Open correspondence, 2020.
- [CGP] V. Chernousov, P. Gille and A. Pianzola, *Three-point Lie algebras and Grothendieck's dessins d'enfants*, Mathematical Research Letters **23** (2016), 81-104.
- [Co1] B. Conrad, *Reductive group schemes*, in *Autour des schémas en groupes, vol. I*, Panoramas et Synthèses **42-43**, Soc. Math. France 2014.

- [DG] M. Demazure et P. Gabriel, *Groupes algébriques*, Masson (1970).
- [EGA-I] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique: I. Le langage des schémas*, Grundlehren der Mathematischen Wissenschaften **166** (2nd ed.). Berlin; New York: Springer-Verlag, 1971.
- [EGAIV] A. Grothendieck (avec la collaboration de J. Dieudonné), *Eléments de Géométrie Algébrique IV: Étude locale des schémas et des morphismes de schémas*, Publications mathématiques de l'I.H.É.S. no 20, 24, 28 and 32 (1964 - 1967).
- [G-W] U. Görtz and T. Wedhorn, *Algebraic Geometry I*, Vieweg+Teubner 2010.
- [Ha] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Springer.
- [LLR] Q. Liu, D. Lorenzini, M. Raynaud, *Néron models, Lie algebras, and reduction of curves of genus one*, Invent. Math. **157** (2004), 455-518.
- [P1] A. Pianzola, *Derivations of certain algebras defined by étale descent*. Math. Z. **264** (2010).
- [P2] A. Pianzola, *Affine Kac-Moody Lie algebras as torsors over the punctured line*, Indagationes Mathematicae N.S. **13**(2) (2002) 249-257.
- [P3] A. Pianzola, *Vanishing of H^1 for Dedekind rings and applications to loop algebras*, C. R. Acad. Sci. Paris, Ser. I **340** (2005), 633-638.
- [SGA3] *Séminaire de Géométrie algébrique de l'I.H.E.S., 1963-1964, schémas en groupes, dirigé par M. Demazure et A. Grothendieck*, Lecture Notes in Math. 151-153. Springer (1970).
- [Stacks] Stacks project, <http://stacks.math.columbia.edu/>