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# Fiberwise criteria for Twisted Forms of Algebraic Structures

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*In memoriam Professor Georgia Benkart*

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## Abstract

We provide a criterion for certain algebraic objects over Jacobson schemes to be forms of each other based on their behaviour at closed fibres. This criterion permits to answer a question that I. Burban had asked the authors.

*Keywords:* Reductive group scheme, torsor, Lie algebras.

*MSC 2000:* 14L30, 17B67, 11E72, 14E20.

## 1 Introduction

Let  $\mathbf{G}$  be a group scheme over a base scheme  $S$ . The concept of  $\mathbf{G}$ -torsors over  $S$  is to be found in many areas of mathematics and mathematical physics; they are powerful tools to frame problems in a language that is partial to powerful algebraic geometry methods.  $\mathbf{G}$  acts on itself by right multiplication. A  $\mathbf{G}$ -torsor is a scheme  $\mathbf{X}$  with a right action of  $\mathbf{G}$  that “locally” looks like  $\mathbf{G}$  with this action. By locally we

mean that there exists a faithfully flat and finitely presented scheme  $T$  over  $S$  such that  $\mathbf{X}_T$  and  $\mathbf{G}_T$  are isomorphic as  $T$ -schemes with a  $\mathbf{G}_T$ -action.

The treatise [SGA3] shows how a deep understanding of  $\mathbf{G}$  can be had through the study of its fibres  $\mathbf{G}_s$ ,  $s \in S$ . It thus seem natural, given a right action of  $\mathbf{G}$  on an  $S$ -scheme  $\mathbf{X}$ , to study situations under which the fibres  $\mathbf{X}_s$  being  $\mathbf{G}_s$ -torsors yield information about  $\mathbf{X}$  itself being a  $\mathbf{G}$ -torsor. In practice the fibres of closed points are more tractable. The existence of “enough” closed points brings up the natural working hypothesis that  $S$  be a Jacobson scheme. It is under this assumption that we are able to create a fiberwise criterion for torsors under certain reductive group schemes.

This work grew up out of the desire to answer a question that I. Burban posed to the authors a few years ago [Bu]. His question and the (positive) answer that can be given using our main result is given in the last section.

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## 2 Preliminaries on group schemes and smoothness

Throughout this paper  $S$  is a (base) scheme with structure sheaf  $\mathcal{O}_S$ . By an  $S$ -*functor* (resp. *monoid*, *group*) we mean a contravariant functor  $\mathbf{F}$  from the category of schemes over  $S$  to the category of sets (resp. monoid, groups). When  $\mathbf{F}$  is representable, i.e. a scheme, we say that  $\mathbf{F}$  is an  $S$ -*scheme* (resp. *monoid scheme*, *group scheme*). The use of bold face characters in the text is used to emphasize the functorial nature of the object under consideration. In particular, if  $X$  is an  $S$ -scheme then its functor of points

$$T \mapsto \text{Hom}_{S\text{-Sch}}(T, X)$$

will be denoted by  $\mathbf{X}$ .

If  $\mathbf{F}$  is an  $S$ -functor and  $T$  a scheme over  $S$ , we denote by  $\mathbf{F}_T$  the  $T$ -functor obtained by base change. If  $T = \text{Spec}(R)$  we denote  $\mathbf{F}(T)$  and  $\mathbf{F}_T$  by  $\mathbf{F}(R)$  and  $\mathbf{F}_R$  respectively. If  $\mathbf{F}$  is an  $S$ -scheme, then  $\mathbf{F}_T = \mathbf{F} \times_S T$  and, as it is customary, we denote  $\mathbf{F} \times_{\text{Spec}(R)} T$  by  $\mathbf{F} \times_R T$ .

We denote as usual by  $\mathbf{O}_S$  the affine  $S$ -ring scheme  $S[T] = \text{Spec}(\mathbb{Z}[T]) \times_{\mathbb{Z}} S$ . Thus  $\mathbf{O}_S(T) = \mathcal{O}_T(T)$ . Recall that an  $\mathbf{O}_S$ -*module* is an  $S$ -abelian group  $\mathbf{M}$  together with  $\mathbf{O}_S(T)$ -module structure on  $\mathbf{M}(T)$  that is functorial on  $T$ . The concept of  $\mathbf{O}_S$ -*algebra* is defined similarly.

We mainly use the terminology and notation of Grothendieck-Dieudonné [EGA-I], which agrees with that of Demazure-Grothendieck used in [SGA3, Exp. I and II]. Below we briefly review concepts/results that are relevant to this paper.

## 2.1 Groups attached to quasi-coherent modules

Let  $\mathcal{E}$  be a quasi-coherent module over  $S$ . We denote its dual by  $\mathcal{E}^\vee$ . For each morphism  $f : T \rightarrow S$  we let  $\mathcal{E}_T = f^*(\mathcal{E})$  be the inverse image of  $\mathcal{E}$  under the morphism  $f$ , and define an abelian  $S$ -group  $\mathbf{V}(\mathcal{E})$  by  $\mathbf{V}(\mathcal{E})(T) = \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{E}_T, \mathcal{O}_T) = \Gamma(T, (\mathcal{E}_T)^\vee)$ .  $\mathbf{V}$  is actually an  $S$ -scheme; it is represented by the affine scheme  $\mathrm{Spec}(\mathbf{Sym}(\mathcal{E}))$  where  $\mathbf{Sym}(\mathcal{E})$  is the  $\mathcal{O}_S$ -symmetric algebra of  $\mathcal{E}$  [EGA-I, 9.4.9]. If  $\mathcal{E}$  is of finite type (resp. of finite presentation), then  $\mathbf{V}(\mathcal{E})$  is an  $S$ -scheme of finite type (resp. of finite presentation), *ibid*, 9.4.11.

The abelian  $S$ -group  $\mathbf{W}(\mathcal{E})$  is defined by  $\mathbf{W}(\mathcal{E})(T) = \Gamma(T, \mathcal{E}_T)$ . Recall that if  $\mathcal{E}$  is locally free of finite type then  $\mathbf{W}(\mathcal{E}) \simeq \mathbf{V}(\mathcal{E}^\vee)$ . In particular  $\mathbf{W}(\mathcal{E})$  is in this case an affine  $S$ -scheme. Note that the  $S$ -abelian groups  $\mathbf{V}(\mathcal{E})$  and  $\mathbf{W}(\mathcal{E})$  have natural  $\mathbf{O}_S$ -module structures

To  $\mathcal{E}$  we attach the  $S$ -functors  $\mathbf{End}_{\mathbf{O}_S\text{-mod}}(\mathcal{E})$  and  $\mathbf{Aut}_{\mathbf{O}_S\text{-mod}}(\mathcal{E})$  whose functor of points are given by  $T \mapsto \mathrm{End}_{\mathcal{O}_T\text{-mod}}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_T)$  and  $T \mapsto \mathrm{Aut}_{\mathcal{O}_T\text{-mod}}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_T)$  respectively. The first one is naturally an  $S$ -monoid and the latter an  $S$ -group with “composition of endomorphisms” as multiplication. They are affine schemes whenever  $\mathcal{E}$  is locally free of finite rank.

Similarly to an  $\mathbf{O}_S$ -module  $\mathbf{L}$  we attach an  $S$ -monoid  $\mathbf{Aut}_{\mathbf{O}_S\text{-mod}}(\mathbf{L})$  and an  $S$ -group  $\mathbf{Aut}_{\mathbf{O}_S\text{-mod}}(\mathbf{L})$ . For example  $\mathbf{Aut}_{\mathbf{O}_S\text{-mod}}(\mathbf{L})(T) = \mathrm{Aut}_{\mathcal{O}_T\text{-mod}}(\mathbf{L}_T)$ .

## 2.2 Group schemes and Lie algebras

Throughout  $\mathbf{G}$  will denote an  $S$ -group scheme and we denote by  $\epsilon \in \mathbf{G}(S)$  its unit section. We refer to [SGA3, I and II], [DG] and [LLR, §1] for details in what follows.

If  $T$  is a scheme we denote by  $T[\epsilon]$  the corresponding scheme of dual number [SGA3, II.2]. The functor  $T \rightarrow \mathbf{Lie}(\mathbf{G})(T) := \ker(\mathbf{G}(T[\epsilon]) \rightarrow \mathbf{G}(T))$  is an  $\mathbf{O}_S$ -module isomorphic to  $\mathbf{V}(\omega_{\mathbf{G}/S}^1)$  where  $\omega_{\mathbf{G}/S}^1 = \epsilon^*(\Omega_{\mathbf{G}/S}^1)$  *ibid*. Prop. 3.3 and 3.6. Furthermore, the  $\mathbf{O}_S(T)$ -module  $\mathbf{Lie}(\mathbf{G})(T)$  has a natural Lie algebra structure. It is thus an  $\mathbf{O}_S$ -Lie algebra. Recall that for all scheme morphisms  $T \rightarrow S$  we have an  $\mathbf{O}_T$ -Lie algebra isomorphism

$$(2.1) \quad \mathbf{Lie}(\mathbf{G}) \times_S T = \mathbf{Lie}(\mathbf{G}_T).$$

The  $\mathcal{O}_T(T)$ -Lie algebra  $\mathbf{Lie}(\mathbf{G})(T)$  is denoted by  $\mathrm{Lie}(\mathbf{G})$ . From the above isomorphism we have

$$(2.2) \quad \mathbf{Lie}(\mathbf{G})(T) = \mathrm{Lie}(\mathbf{G}_T).$$

We denote by  $\mathcal{L}ie(\mathbf{G})$  the vector group sheaf (fibration vectorielle) of sections of the affine scheme  $\mathbf{Lie}(\mathbf{G}) \rightarrow S$ . In other words,  $\mathcal{L}ie(\mathbf{G})$  is the  $\mathcal{O}_S$ -module  $(\omega_{\mathbf{G}/S}^1)^\vee = \mathrm{Hom}_{\mathcal{O}_S}(\omega_{\mathbf{G}/S}^1, \mathcal{O}_S)$ . Note that  $\mathcal{L}ie(\mathbf{G})$  is naturally an  $\mathcal{O}_S$ -Lie algebra.

**Remark 2.1.** In general  $\mathcal{L}ie(\mathbf{G})$  does not determine  $\mathbf{Lie}(\mathbf{G})$ , but it does if  $\omega_{\mathbf{G}/S}^1$  is locally free of finite type, in particular if  $\mathbf{G}$  is smooth. In this case  $\mathbf{Lie}(\mathbf{G}) = \mathbf{W}(\mathcal{L}ie(\mathbf{G}))$ . See [SGA3, II Lemma 4.11.7]. If  $S = \text{Spec}(R)$  the  $R$ -Lie algebra  $\mathbf{Lie}(\mathbf{G}) = \mathcal{L}ie(\mathbf{G})(S)$  is a locally free  $R$ -module of finite type. For any ring extension  $S/R$  we have  $\mathbf{Lie}(\mathbf{G}_S) = \mathbf{Lie}(\mathbf{G})(S) = \mathbf{Lie}(\mathbf{G}) \otimes_R S$ .

**Lemma 2.2.** *Let  $\mathcal{L}$  be an  $\mathcal{O}_S$ -algebra which is locally free of finite type (as an  $\mathcal{O}_S$ -module). Let  $\mathbf{L} = \mathbf{W}(\mathcal{L})$ . Then.*

- (1) *There is a natural isomorphism of affine  $S$ -group schemes  $\mathbf{Aut}_{\mathcal{O}_S\text{-alg}}(\mathcal{L}) \simeq \mathbf{Aut}_{\mathbf{O}_S\text{-alg}}(\mathbf{L})$ .*
- (2) *There is a natural  $\mathbf{O}_S$ -Lie algebra isomorphism  $\mathbf{Der}_{\mathbf{O}_S\text{-alg}}(\mathbf{L}) \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_S\text{-alg}}(\mathbf{L}))$ .*

*Proof.* (1) Since  $\mathbf{GL}(\mathcal{L}) := \mathbf{Aut}_{\mathbf{O}_S\text{-mod}}(\mathcal{L})$  is locally isomorphic to  $\mathbf{GL}_{d,S}$  for some  $d$ , it is an affine  $S$ -group scheme. Thus  $\mathbf{Aut}_{\mathcal{O}_S\text{-alg}}(\mathcal{L})$ , being a closed subgroup of  $\mathbf{GL}(\mathcal{L})$ , is also affine. By [SGA3, I Prop. 4.6.2]  $\mathbf{Aut}_{\mathcal{O}_S\text{-alg}}(\mathcal{L}) \simeq \mathbf{Aut}_{\mathbf{O}_S\text{-alg}}(\mathbf{L})$ .

(2) Since  $\mathbf{L}$  is a good  $\mathbf{O}_S$ -module [SGA3, II Def. 4.4 and Ex. 4.4.2], we have an  $\mathbf{O}_S$ -module isomorphisms  $\eta : \mathbf{End}_{\mathbf{O}_S\text{-mod}}(\mathbf{L}) \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_S\text{-mod}}(\mathbf{L}))$  [SGA3, II Prop. 4.5]. By [SGA3, I Prop.4.6.2] this is in fact a morphism of  $S$ -schemes. We claim that the restriction of  $\eta$  to  $\mathbf{Der}_{\mathbf{O}_S\text{-alg}}(\mathbf{L})$  is our isomorphism. The proof reduces to the case when  $S$  is affine, which can be found in [DG, II §4 2.3].  $\square$

### 2.3 Connected component of the identity, forms and type of a reductive group scheme.

Let  $\mathbf{G}$  be an  $S$ -group scheme which is locally of finite presentation. We consider the  $S$ -subgroup  $\mathbf{G}^\circ$  of  $\mathbf{G}$  [SGA3, VI<sub>B</sub>.3.1] defined by

$$\mathbf{G}^\circ(\mathbf{T}) = \left\{ u \in \mathbf{G}(\mathbf{T}) \mid \forall s \in S, u_s(\mathbf{T}_s) \subset \mathbf{G}_s^\circ \right\}.$$

If  $\mathbf{G}$  is smooth along the unit section,  $\mathbf{G}^\circ$  is representable by a smooth  $S$ -group scheme called *the connected component of the identity of  $\mathbf{G}$*  and furthermore the fibre  $(\mathbf{G}^\circ)_s$  is the connected component of the identity of the  $\kappa(s)$ -algebraic group  $\mathbf{G}_s$  [SGA3, VI<sub>B</sub>.4.1].

Let  $\mathbf{G}$  be an  $S$  group scheme. An  $S$ -group scheme  $\mathbf{G}'$  is called a (*twisted*) *form of  $\mathbf{G}$* , if there exist a faithfully flat extension  $\mathbf{T} \rightarrow S$  such that  $\mathbf{G}_{\mathbf{T}}$  and  $\mathbf{G}'_{\mathbf{T}}$  are isomorphic  $\mathbf{T}$ -group schemes. In a similar fashion one defines forms of  $\mathbf{O}_S$  or  $\mathcal{O}_S$ -algebras.

Let  $\mathbf{G}$  be a reductive  $K$ -group scheme, where  $K$  is an algebraically closed field. If  $\mathbf{T}$  is a maximal torus of  $\mathbf{G}$ , then  $\mathbf{T}$  is split and defines a root datum which is up to

isomorphism independent of the choice of  $\mathbf{T}$ . It is called the *type of  $\mathbf{G}$*  and is denoted by  $\mathcal{R}(\mathbf{G})$ .

Let  $\mathbf{G}$  be a reductive  $S$ -group scheme. If  $s \in S$  the *type of  $\mathbf{G}$  at  $s$*  is the type of the reductive  $\kappa(s)$ -group  $\mathbf{G}_{\bar{s}} := \mathbf{G}_s \times_{\kappa(s)} \kappa(s)$ . Because  $\mathbf{G}$  possesses maximal tori locally for the étale topology one knows that the type function  $s \mapsto \text{type of } \mathbf{G}_{\bar{s}}$  is locally constant [SGA3, XXII 2.8]. We say that  $\mathbf{G}$  is of *constant type* if the type function is constant. This is the case, for example, if  $S$  is connected.

## 2.4 Specific properties in characteristic zero

We assume in this section that  $S$  is a  $\mathbb{Q}$ -scheme. Let  $\mathbf{G}$  be a semisimple adjoint  $S$ -group scheme of constant type. We will denote the  $\mathbf{O}_S$ -Lie algebra  $\mathbf{Lie}(\mathbf{G})$  by  $\mathbf{L}$ , and the  $\mathcal{O}_S$ -Lie algebra  $\mathcal{L}ie(\mathbf{G})$  by  $\mathcal{L}$ . Recall that  $\mathcal{L}$  is locally free of finite type.

Since  $\mathbf{G}$  and its simply connected cover  $\mathbf{G}^{\text{sc}}$  have the same Lie algebra, we have isomorphisms of affine  $S$ -group schemes [SGA3, XXIV 3.6 and 7.3.1]

$$(2.3) \quad \mathbf{Aut}(\mathbf{G}) \simeq \mathbf{Aut}(\mathbf{G}^{\text{sc}}) \simeq \mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathcal{L}) \simeq \mathbf{Aut}_{\mathbf{O}_S\text{-Lie}}(\mathbf{L})$$

where the last isomorphism is given by Remark 2.1 and Lemma 2.2.(1). In particular, the  $S$ -scheme  $\mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathcal{L})$  is smooth and affine. By (2.3) and [SGA3, XXIV 1.3 and 1.8] we have an isomorphism

$$(2.4) \quad \mathbf{G} \simeq \mathbf{Aut}_{\mathbf{O}_S\text{-Lie}}(\mathbf{L})^\circ.$$

This yields an  $\mathbf{O}_S$ -Lie algebra isomorphism

$$(2.5) \quad \mathbf{L} \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_S\text{-Lie}}(\mathbf{L})) = \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_S\text{-Lie}}(\mathbf{L})^\circ).$$

Recall [SGA3, II Theo. 4.7 and Prop. 4.8] the adjoint representation  $\text{Ad} : \mathbf{G} \rightarrow \mathbf{Aut}_{\mathbf{O}_S\text{-mod}}(\mathbf{L})$  and the induced  $\mathbf{O}_S$ -Lie algebra homomorphism  $\text{ad} : \mathbf{L} \rightarrow \mathbf{End}_{\mathbf{O}_S\text{-mod}}(\mathbf{L})$ .

**Lemma 2.3.** *Under the identification  $\mathbf{L} \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathbf{O}_S\text{-Lie}}(\mathbf{L}))$  of (2.5) the map  $\text{ad}$  induces an  $\mathbf{O}_S$ -Lie algebra isomorphism between  $\mathbf{L}$  and  $\mathbf{Der}_{\mathbf{O}_S\text{-Lie}}(\mathbf{L})$ .*

*Proof.* All the morphisms under consideration are  $S$ -scheme morphisms, so the question is local on  $S$  and we may assume that  $S = \text{Spec}(R)$  for some  $\mathbb{Q}$ -ring  $R$ . Consider the  $R$ -Lie algebra  $\mathbf{L} = \mathbf{L}(R)$ . Since by Remark 2.1  $\mathbf{L} = \mathbf{W}(\mathbf{L})$  where  $\mathbf{W}(\mathbf{L})(T) = \mathbf{L} \otimes_R \mathcal{O}_T(T)$ , it will suffice to show that for all ring extensions  $S/R$  the  $S$ -Lie algebra homomorphism  $d = \text{ad}_S : \mathbf{L} \otimes_R S \rightarrow \mathbf{Der}_{S\text{-Lie}}(\mathbf{L} \otimes_R S)$  is an isomorphism. By replacing  $\mathbf{G}$  by  $\mathbf{G}_S$  we may assume that  $R = S$ . By [SGA3, II 4.7.2] the map  $d$  is nothing but the adjoint representation  $d(x) = \text{ad}_L(x)$ . Since  $\mathbf{G}$  is of constant type

there exist a unique Chevalley group  $\mathbf{G}_0$  such that  $\mathbf{G}$  is a twisted form of  $\mathbf{G}_0 \times_{\mathbb{Z}} R$ . Let  $\mathfrak{g} = \mathbf{Lie}(\mathbf{G}_0)(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ . Then  $\mathfrak{g}$  is a finite dimensional split semisimple Lie algebra over  $\mathbb{Q}$  and  $L$  is a twisted form of  $\mathfrak{g} \otimes_{\mathbb{Q}} R$ , that is, there exists an fppf extension (in fact étale cover)  $S$  of  $R$  such that the  $S$ -Lie algebras  $L \otimes_R S$  and  $(\mathfrak{g} \otimes_{\mathbb{Q}} R) \otimes_R S \simeq \mathfrak{g} \otimes_{\mathbb{Q}} S$  are isomorphic. Fix one such isomorphism

$$(2.6) \quad \psi : L \otimes_R S \rightarrow \mathfrak{g} \otimes_{\mathbb{Q}} S.$$

If  $d(x) = 0$ , then  $x$  belongs to the centre of  $L$  and therefore  $\psi(x \otimes 1)$  belongs to the centre of  $\mathfrak{g} \otimes_{\mathbb{Q}} S$ , which is trivial. Since  $S/R$  is faithfully flat  $x = 0$  so that  $d$  is injective and we henceforth identify  $L$  with an  $R$ -submodule of  $\mathrm{Der}(L)$ . That  $d$  is surjective, that is that every derivation of  $L$  is inner, we reason as follows. If  $L = \mathfrak{g} \otimes_{\mathbb{Q}} R$  every derivation is inner by Whitehead's Lemma (see [P1, Example 4.9]).<sup>1</sup> For  $L$  arbitrary, we appeal to the isomorphism (2.6). By the split case we have  $(\mathrm{Der}(L)/L) \otimes_R S = 0$ . Thus  $L = \mathrm{Der}(L)$  as desired.  $\square$

## 2.5 Semicontinuity property for Lie algebras

For convenience we recall the following well-known fact.

**Lemma 2.4.** *Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_S$ -module of finite presentation. Then the function  $s \mapsto \dim_{\kappa(s)} (\mathcal{F} \otimes_{\mathcal{O}_S} \kappa(s))$  is upper semi-continuous.*

*Proof.* The statement is local and therefore reduces then to the case of a ring  $R$  and of an  $R$ -module  $M$  of finite presentation [Stacks, Tag 01PC]. If  $R$  is noetherian then  $M$  is coherent and the result can be found in [Ha, 12.7.2]. For lack of a reference we provide a proof in the general case. Let  $\mathfrak{p}, \mathfrak{q}$  be prime ideals of  $R$  such that  $\mathfrak{p} \subset \mathfrak{q}$ . We want to show that  $\dim_{\kappa(\mathfrak{p})} (M \otimes_R \kappa(\mathfrak{p})) \leq \dim_{\kappa(\mathfrak{q})} (M \otimes_R \kappa(\mathfrak{q}))$ . Replacing  $R$  by  $R_{\mathfrak{q}}$ , we can assume that  $R$  is local of maximal ideal  $\mathfrak{q}$ .

We consider an exact sequence  $R^m \xrightarrow{A} R^n \rightarrow M \rightarrow 0$  where  $A$  is a matrix of size  $(m, n)$  with entries in  $R$ . Let  $r = \dim_{\kappa(\mathfrak{q})} (M \otimes_R \kappa(\mathfrak{q}))$ , so that the rank of  $A_{\kappa(\mathfrak{q})}$  is  $n - r$ . There thus exists a minor  $B$  of  $A$  of size  $(n - r, n - r)$  such that  $\det(B) \in R$  is not zero when passing to  $\kappa(\mathfrak{q})$ , hence is invertible in  $R$ . It follows that  $\det(B_{\kappa(\mathfrak{p})})$  is invertible so that the rank of  $A_{\kappa(\mathfrak{p})}$  is  $\geq n - r$ . We conclude that  $\dim_{\kappa(\mathfrak{p})} (M \otimes_R \kappa(\mathfrak{p})) \geq r$  as desired.  $\square$

**Lemma 2.5.** *Let  $\mathbf{G}$  be an  $S$ -group scheme that is locally of finite presentation. Then  $\omega_{\mathbf{G}/S}^1$  is an  $\mathcal{O}_S$ -module of finite presentation and the map  $s \mapsto \dim_{\kappa(s)} \mathrm{Lie}(\mathbf{G}_s)$  is upper semi-continuous.*

<sup>1</sup> Note that our  $L$  is denoted by  $\mathcal{L}$  in [P1].

*Proof.* That  $\omega_{\mathbf{G}/S}^1$  is of finite presentation follows from [Stacks, Tag 01V2, 01V3].

By (2.2) we have a  $\kappa(s)$ –Lie algebra isomorphism  $\mathrm{Lie}(\mathbf{G}_s) \simeq \mathbf{Lie}(\mathbf{G})(\kappa(s))$ . From the isomorphism  $\mathbf{Lie}(\mathbf{G}) \simeq \mathbf{V}(\omega_{\mathbf{G}/S}^1)$ , we see that the dimension of  $\mathbf{Lie}(\mathbf{G})(\kappa(s))$  at a point  $s \in S$  is the dimension of the  $\kappa(s)$ –space  $\mathrm{Hom}_{\kappa(s)}(\omega_{\mathbf{G}/S}^1 \otimes_{\mathcal{O}_S} \kappa(s), \kappa(s))$ , which is the dimension of  $\omega_{\mathbf{G}/S}^1 \otimes_{\mathcal{O}_S} \kappa(s)$  since  $\omega_{\mathbf{G}/S}^1$  is an  $\mathcal{O}_S$ –module of finite type. Furthermore, since  $\omega_{\mathbf{G}/S}^1$  is in fact of finite presentation, Lemma 2.4 implies that the map  $s \mapsto \dim_{\kappa(s)}(\mathbf{Lie}(\mathbf{G})(\kappa(s)))$  is upper semi-continuous.  $\square$

## 3 Group schemes over Jacobson schemes

### 3.1 Jacobson schemes and smoothness

We refer the reader to [EGAIV, §10] for general results about Jacobson schemes. Let us just recall that a scheme  $S$  is *Jacobson* if for each open cover  $S = \bigcup_{i \in I} \mathrm{Spec}(R_i)$ , then each  $R_i$  is a Jacobson ring, i.e. if every radical ideal  $I$  is the intersection of the maximal ideals containing it [Stacks, Tag 00G4].

**Proposition 3.1.** *Let  $f : X \rightarrow S$  be a scheme morphism which is locally of finite presentation. Assume that  $x \in X$  is such that:*

- (i)  $f$  is flat at  $x$ .
- (ii)  $f_{\kappa(x)}$  is smooth.

*Then  $f$  is smooth on an open neighborhood of  $x$ .*

*Proof.* By assumption there exists affine open subschemes  $\mathrm{Spec}(A) = U \subset X$  and  $\mathrm{Spec}(B) = V \subset S$  such that  $f(U) \subset V$  and  $A$  is a finitely presented  $B$ –algebra. Let  $\mathfrak{q} \in \mathrm{Spec}(A)$  and  $\mathfrak{p} \in \mathrm{Spec}(B)$  be the points corresponding to  $x$  and  $f(x)$  respectively.

By (i)  $A_{\mathfrak{q}}$  is flat over  $B_{\mathfrak{p}}$ , while by (ii)  $A \otimes_B \kappa(\mathfrak{p})$  is smooth over  $\kappa(\mathfrak{p})$ . It follows from [Stacks, Tag 00TF] that the restriction of  $f$  to  $U$  is smooth at  $\mathfrak{q}$ , hence (by definition of smoothness) smooth also in a neighbourhood of  $\mathfrak{q}$ .  $\square$

**Corollary 3.2.** *Let  $f : X \rightarrow S$  be as above. Assume that (i) and (ii) hold for all closed points of  $X$ . If  $S$  is Jacobson then  $f$  is smooth.*

*Proof.* By [EGAIV, Cor. 10.4.7]  $X$  is Jacobson. Let  $X_0$  be the set of closed points of  $X$ . It will suffice to show that each point  $x \in X_0$  has an open neighbourhood in which  $f$  is smooth. Indeed, if this is the case, then there exists an open  $U \subset X$  containing  $X_0$  in which  $f$  is smooth. Let  $Z = X \setminus U$ . If  $Z \neq \emptyset$ , then  $Z \cap X_0 \neq \emptyset$  by [EGAIV, Prop. 10.1.2 (a)]. This contradicts  $X_0 \subset U$ . Thus  $Z = \emptyset$  so that  $U = X$  as desired.



Since by the Proposition every  $x \in X_0$  has an open neighbourhood in which  $f$  is smooth the Corollary follows.  $\square$

**Proposition 3.3.** *Let  $S$  be an integral Jacobson scheme, and  $\mathbf{G}$  be an  $S$ -group scheme of finite presentation. Assume that for all closed point  $b \in S$  the fibres  $\mathbf{G}_b$  are smooth and of the same dimension  $d$ .<sup>2</sup>*

- (1) *For all  $s \in S$  the algebraic  $\kappa(s)$ -group  $\mathbf{G}_s$  is smooth of dimension  $d$ .*
- (2) *The  $S$ -functor  $\mathbf{G}^\circ$  is representable by a smooth  $S$ -group scheme of relative dimension  $d$  which is open in  $\mathbf{G}$ .*
- (3) *If for each closed point  $b \in S$  the fibre  $\mathbf{G}_b$  is connected then  $\mathbf{G}$  is smooth and has connected geometric fibres.*

*Proof.* (1) We denote by  $\eta : \text{Spec}(F) \rightarrow S$  the generic point of  $S$ . Chevalley's generic flatness theorem [G-W, 10.85] shows that there exists an open dense subset  $U$  of  $S$  such that  $\mathbf{G} \times_S U$  is flat over  $U$ . We know that  $U$  is also Jacobson [EGAIV, Prop. 10.3.3]. Corollary 3.2 shows that  $\mathbf{G} \times_S U$  is smooth over  $U$ . By [Stacks, Tag 05F7], up to shrinking  $U$  if necessary, the dimension function of the fibres of  $\mathbf{G} \times_S U \rightarrow U$  is constant. Since by assumption this dimension has value  $d$  on the closed points of  $U$ , we get that  $\mathbf{G} \times_S U$  is smooth of relative dimension  $d$ .<sup>3</sup> In particular the algebraic  $F$ -group  $\mathbf{G}_F$  is smooth of dimension  $d$ .

Let  $s \in S$  and let  $c = \dim(\mathbf{G}_s)$ . According to Chevalley's semicontinuity theorem [SGA3, VI<sub>B</sub>.4.1]

$$F_c = \{x \in S : \dim(\mathbf{G}_x) \geq c\} \subset S$$

is a closed set. Since  $F_c \cap \overline{\{s\}}$  contains  $s$  and is closed

$$(3.1) \quad \overline{\{s\}} \subset F_c.$$

Since  $S$  is Jacobson, it follows that there exists a closed point  $b$  in  $\overline{\{s\}}$ . We have

$$(3.2) \quad \dim(\mathbf{G}_s) \leq \dim(\mathbf{G}_b) = d.$$

On the other hand,  $s$  belongs in the adherence of  $\eta$ . The same semicontinuity reasoning used above shows that so that

$$(3.3) \quad \dim(\mathbf{G}_F) \leq \dim(\mathbf{G}_s).$$

Since  $\mathbf{G}_F$  is of dimension  $d$  it follows from (3.2) and (3.3) that  $\dim(\mathbf{G}_s) = d$ .

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<sup>2</sup> Since  $\mathbf{G}$  is of finite presentation  $\mathbf{G}_s$  is an algebraic  $\kappa(s)$ -group. Thus  $\mathbf{G}_s$  has a finite number of irreducible components. They all have the same (finite) dimension and this is also the dimension of  $\mathbf{G}_s$ .

<sup>3</sup> That is, all the fibres  $\mathbf{G}_s, s \in U$  are equidimensional and this dimension is  $d$ .

For establishing smoothness we use a variation of the previous argument by applying semicontinuity considerations to the Lie algebras. Lemma 2.5 yields the inequalities

$$\dim_F \operatorname{Lie}(\mathbf{G}_F) \leq \dim_{\kappa(s)} \operatorname{Lie}(\mathbf{G}_s) \leq \dim_{\kappa(b)} \operatorname{Lie}(\mathbf{G}_b).$$

Since  $\mathbf{G}$  is smooth of dimension  $d$  at  $\eta$  and at  $b$  we obtain that  $d = \dim_{\kappa(s)} \operatorname{Lie}(\mathbf{G}_s)$ . Thus  $\operatorname{Lie}(\mathbf{G}_s)$  and  $\mathbf{G}_s$  have the same dimension (namely  $d$ ). By the smoothness criterion [DG, II.5.2.1]  $\mathbf{G}_s$  is smooth. This completes the proof of (1).

(2) By (1) we see that assumption (ii) of [SGA3, VI<sub>B</sub> Cor. 4.4] holds. It follows that the S-functor  $\mathbf{G}^\circ$  is representable by a smooth S-group scheme which is open in  $\mathbf{G}$ .

According to [Stacks, Tag 055I], the level set

$$E_n = \{s \in S \mid \mathbf{G}_s \text{ has } n \text{ geometrically connected components}\}$$

is a locally constructible subset of  $S$  for each  $n \geq 1$ . Since  $S$  is a Jacobson scheme, the set closed points of  $S_0$  is very dense in  $S$  so that  $S_0 \cap E_n$  is dense in  $E_n$  for each  $n$  [EGAIV, 10.1.2, (b')]. Our assumption implies that  $E_1 = S$ . That  $\mathbf{G}$  is smooth now follows from (2).  $\square$

## 3.2 Forms

**Proposition 3.4.** *Let  $S$  be an integral Jacobson scheme and  $\mathbf{G}$  an affine S-group scheme of finite presentation. Assume that  $\mathbf{G}_b$  is reductive and of dimension  $d$  for all closed point  $b \in S$ . Then  $\mathbf{G}$  is a reductive S-group scheme of constant type. In particular, there exists a unique Chevalley group  $\mathbf{G}_0$  such that  $\mathbf{G}$  is an S-form of  $\mathbf{G}_0 \times_{\mathbb{Z}} S$ .*

*Proof.* Proposition 3.3.(3) shows that  $\mathbf{G}$  is smooth of relative dimension  $d$  and has connected geometric fibres. According to [Co1, Prop. 3.1.9.(1)]  $\mathbf{G}$  is reductive in a neighborhood of each closed point  $b \in S$ . Since  $S$  is a Jacobson scheme, it follows that  $\mathbf{G}$  is reductive. Since the type function is locally constant, the connectedness of  $S$  implies that  $\mathbf{G}$  has constant type  $t_0$  (see §2.3). Let  $\mathbf{G}_0$  be the corresponding Chevalley group scheme. According to [SGA3, XXIII 5.6]  $\mathbf{G}$  is an S-form of  $\mathbf{G}_0 \times_{\mathbb{Z}} S$ .  $\square$

**Corollary 3.5.** *Let  $\mathbf{G}$  and  $\mathbf{G}'$  be group schemes over an integral Jacobson scheme  $S$ . Assume that  $\mathbf{G}$  is reductive and that  $\mathbf{G}'$  is affine and of finite presentation. If  $\mathbf{G}'_b$  is a form of  $\mathbf{G}_b$  for each closed point  $b \in S$ , then  $\mathbf{G}'$  is an S-form of  $\mathbf{G}$ . In particular,  $\mathbf{G}'$  is a reductive S-group scheme*  $\square$

### 3.3 Lie algebras

If  $\mathcal{L}$  is an  $\mathcal{O}_S$ -Lie algebra and  $s \in S$ , then the  $\mathcal{O}_{\text{Spec } \kappa(s)}$ -algebra  $\mathcal{L} \otimes_{\mathcal{O}_S} \kappa(s)$  obtained by base change is equivalent to a  $\kappa(s)$ -algebra that we will denote by  $\mathcal{L}_s$ . Along similar lines if  $\mathbf{L} = \mathbf{W}(\mathcal{L})$  we will denote the  $\kappa(s)$ -algebra  $\mathbf{L}(\kappa(s))$  by  $\mathbf{L}_s$ . Note that  $\mathcal{L}_s = \mathbf{L}_s$ .

**Proposition 3.6.** *We assume that  $S$  is an integral Jacobson  $\mathbb{Q}$ -scheme. Let  $\mathcal{L}$  be an  $\mathcal{O}_S$ -Lie algebra which is locally free of rank  $d$ , and let  $\mathbf{L} = \mathbf{W}(\mathcal{L})$  be its corresponding  $\mathcal{O}_S$ -Lie algebra. Assume that for each closed point  $b \in S$  the  $\kappa(b)$ -Lie algebra  $\mathbf{L}_b$  is semisimple.<sup>4</sup> Then there exists a unique finite dimensional split semisimple Lie  $\mathbb{Q}$ -algebra  $L_0$  such that  $\mathbf{L}$  is a form of  $\mathbf{L}_0 := L_0 \otimes_{\mathbb{Q}} S$ .<sup>5</sup>*

*Proof.* (1) The  $S$ -group scheme  $\mathbf{G} := \mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathcal{L})$  is a closed subgroup of  $\mathbf{GL}(\mathcal{L})$ , hence affine of finite presentation. Recall that by (2.3) we have  $\mathbf{G} \simeq \mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L})$ . For each closed point  $b \in S$ , the algebraic group  $\mathbf{G}_b = \mathbf{Aut}_{\kappa(b)\text{-Lie}}(\mathbf{L}_b)$  is smooth of dimension  $d$  and  $(\mathbf{G}_b)^\circ$  is semisimple adjoint. Proposition 3.3.(2) shows that  $\mathbf{G}^\circ$  is representable by an open subgroup scheme of  $\mathbf{G}$  which is smooth of relative dimension  $d$ . Appealing now to Proposition 3.4 yields that  $\mathbf{G}^\circ$  is adjoint semisimple and is a form of the  $S$ -group corresponding to a (unique) semisimple adjoint Chevalley group scheme  $\mathbf{G}_0^\circ$ . We denote by  $L_0$  the  $\mathbb{Q}$ -Lie algebra of  $\text{Lie}(\mathbf{G}_0^\circ) \otimes_{\mathbb{Z}} \mathbb{Q}$ . This yields the  $\mathcal{O}_S$ -Lie algebra that we denoted by  $\mathbf{L}_0$ . Observe that  $\text{Lie}(\mathbf{G})$  is a form of  $\mathbf{L}_0$  as  $\mathcal{O}_S$ -Lie algebras.

By Lemma 2.3 we have  $\mathbf{L} \simeq \mathbf{Der}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L})$ . Finally since  $\mathbf{G} \simeq \mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L})$  we can apply Lemma 2.2 to conclude that  $\text{Lie}(\mathbf{G}) \simeq \mathbf{Der}_{\mathcal{O}_S\text{-Lie}}(\mathbf{L})$ . It follows that  $\mathbf{L}$  is a form of  $\mathbf{L}_0$  as desired.  $\square$

The following variant of the heretofore presented ideas will allow us to answer Burban's question (see §4 below).

**Proposition 3.7.** *Assume that  $S$  is an integral Jacobson  $\mathbb{Q}$ -scheme. Let  $\mathbf{G}$  be a semisimple adjoint  $S$ -group scheme and consider its  $\mathcal{O}_S$ -Lie algebra  $\mathcal{L} = \text{Lie}(\mathbf{G})$ .*

*Let  $\mathcal{L}'$  be a sheaf of  $\mathcal{O}_S$ -Lie algebras. We assume that  $\mathcal{L}'$  is locally free of rank  $d$ , and that for each closed point  $b \in S$ ,  $\mathcal{L}'_b$  is a form of  $\mathcal{L}_b$ .*

(1) *The  $S$ -functor  $\mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathcal{L}')$  is a smooth affine  $S$ -group scheme whose connected component of the identity  $\mathbf{G}'$  is a semisimple adjoint  $S$ -group scheme.*

(2) *There is a natural isomorphism  $\text{Lie}(\mathbf{G}') \simeq \mathcal{L}'$ .*

(3)  *$\mathbf{G}'$  is an  $S$ -form of  $\mathbf{G}$ .*

(4)  *$\mathcal{L}'$  is an  $S$ -form of  $\mathcal{L}$ .*

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<sup>4</sup> Necessarily of finite dimension  $d$ .

<sup>5</sup> By definition  $\mathbf{L}_0(\mathbb{T})$  is the  $\mathcal{O}_{\mathbb{T}}(\mathbb{T})$ -Lie algebra  $L_0 \otimes_{\mathbb{Q}} \mathcal{O}_{\mathbb{T}}(\mathbb{T})$ .

*Proof.* (1) As in the previous Proposition  $\mathbf{H} := \mathbf{Aut}_{\mathcal{O}_S\text{-Lie}}(\mathcal{L}')$  is an affine  $S$ -group scheme of finite presentation. For each closed point  $b \in S$ ,  $\mathbf{H}_b$  is smooth ( $\kappa(b)$  is of characteristic 0), hence of dimension  $d$ . Proposition 3.3.(1) shows that the  $S$ -subfunctor  $\mathbf{H}^\circ$  of  $\mathbf{H}$  is representable by a smooth affine  $S$ -group scheme  $\mathbf{G}'$  of dimension  $d$ .

According to [Co1, prop. 3.1.9.(1)],  $\mathbf{G}'$  is reductive on a neighborhood of each closed point  $b \in S$ . Since  $S$  is a Jacobson scheme, it follows that  $\mathbf{G}'$  is reductive. We consider now the center  $\mathbf{C}$  of  $\mathbf{G}'$  [SGA3, XXII 4.1.7] which is a multiplicative  $S$ -group scheme of finite type. According to [SGA3, IX 2.9] the subset  $U$  of points  $b$  of  $S$  where  $\mathbf{C}_b = 1$  is clopen. Since  $S$  is Jacobson,  $U$  contains a closed point  $b$  of  $S$ . Thus  $U \neq \emptyset$ , so that  $U = S$  since  $S$  is connected. It follows that  $\mathbf{C} = 1$ . Thus  $\mathbf{G}'$  is semisimple adjoint.

(2) Let  $\mathbf{L}' = \mathbf{W}(\mathcal{L}')$ . By Lemma 2.2 and (1)

$$\mathbf{Der}_{\mathcal{O}_S\text{-alg}}(\mathbf{L}') \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathcal{O}_S\text{-alg}}(\mathbf{L}')) \simeq \mathbf{Lie}(\mathbf{Aut}_{\mathcal{O}_S\text{-alg}}(\mathbf{L}')^\circ) = \mathbf{Lie}(\mathbf{G}').$$

On the other hand we have an isomorphism of  $\mathcal{O}_S$ -Lie algebras  $\mathbf{L}' \simeq \mathbf{Der}_{\mathcal{O}_S\text{-alg}}(\mathbf{L}')$  (Lemma 2.3), whence an isomorphism  $\mathbf{L}' \simeq \mathbf{Lie}(\mathbf{G}')$ . It follows that  $\mathbf{W}(\mathcal{L}') \simeq \mathbf{W}(\mathbf{Lie}(\mathbf{G}'))$ . Since  $\mathbf{W}$  is fully faithful  $\mathcal{L}'$  and  $\mathbf{Lie}(\mathbf{G}')$  are isomorphic  $\mathcal{O}_S$ -Lie algebras.

(3) Since  $S$  is connected the semisimple  $S$ -groups  $\mathbf{G}$  and  $\mathbf{G}'$  have constant type. Furthermore, the types are determined by their Lie algebras since the groups are adjoint. Given that by assumption the fibres of their Lie algebras at closed points are forms of each other, the types of  $\mathbf{G}$  and  $\mathbf{G}'$  coincide. We appeal again to [SGA3, XXIII 5.6] to conclude that  $\mathbf{G}'$  is a form of  $\mathbf{G}$ . Thus (3) holds.

(4) From (3) and Remark 2.1 we get that  $\mathbf{Lie}(\mathbf{G}') = \mathbf{W}(\mathbf{Lie}(\mathbf{G}'))$  is a form of  $\mathbf{Lie}(\mathbf{G}) = \mathbf{W}(\mathbf{Lie}(\mathbf{G}))$ . It follows that  $\mathbf{Lie}(\mathbf{G}')$  is a form of  $\mathbf{Lie}(\mathbf{G})$ . Thus (4) follows from (2).  $\square$

## 4 Burban's question

In order to formulate Burban's question we need to recall the concept of loop algebra of a simple Lie finite dimensional complex Lie algebra  $\mathfrak{g}$ .

Let  $R = \mathbb{C}[t^{\pm 1}]$ . Fix a positive integer  $d$ , and set  $S = k[t^{\pm \frac{1}{d}}]$ . The natural map  $R \rightarrow S$  is faithfully flat and finite étale. We put  $\xi = e^{\frac{i2\pi}{d}}$ ; then the elements of  $\Gamma = \mathbb{Z}/d\mathbb{Z}$  act as automorphisms of  $S$  over  $R$  via

$$\bar{e}_t^{\frac{1}{d}} = \xi^e t^{\frac{1}{d}}.$$

for  $e \in \mathbb{Z}$ . This action makes  $S$  into a Galois extension of  $R$  with Galois group  $\Gamma$ .

Let  $\sigma$  be an automorphism of  $\mathfrak{g}$  of order  $d$ . For  $i \in \mathbb{Z}$  consider the eigenspace

$$\mathfrak{g}_i = \{x \in \mathfrak{g} : \sigma(x) = \xi^i x\}$$

Then  $\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}_i$ , and in fact  $\mathfrak{g} = \bigoplus_{0 \leq i < d} \mathfrak{g}_i$ . Out of this data we define the corresponding *loop algebra*

$$(4.1) \quad L(\mathfrak{g}, \sigma) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \otimes t^{\frac{i}{d}} \subset \mathfrak{g} \otimes_{\mathbb{C}} R_d.$$

The simple but crucial observation is that  $L(\mathfrak{g}, \sigma)$  is stable under the action of  $R$ . Thus  $L(\mathfrak{g}, \sigma)$  is not only an infinite dimensional complex Lie algebra, but also an  $R$ -Lie algebra. As we shall see, it is the algebra structure over this ring that allows non-abelian cohomological considerations to enter into the picture.

It is an easy linear algebra exercise to verify that we have a natural  $R_d$ -algebra isomorphism

$$(4.2) \quad L(\mathfrak{g}, \sigma) \otimes_R R_d \simeq \mathfrak{g} \otimes_{\mathbb{C}} R_d \simeq \mathfrak{g}_R \otimes_R R_d.$$

where  $\mathfrak{g}_R := \mathfrak{g} \otimes_{\mathbb{C}} R$ . This shows that the  $R$ -Lie algebra  $L := L(\mathfrak{g}, \sigma)$  is a twisted form of  $\mathfrak{g} \otimes_{\mathbb{C}} R$ . It therefore corresponds to an  $\mathbf{Aut}(\mathfrak{g}_R)$  torsor  $\mathbf{X}$  over  $\mathrm{Spec}(R)$ . More precisely  $\mathbf{X} = \mathbf{Isom}_{R\text{-Lie}}(\mathfrak{g}_R, L)$ .<sup>6</sup>

We can now formulate Burban's questions (essentially verbatim except for some notation changes):

**Question 1:** Let  $L'$  be a Lie algebra over the ring  $\mathbb{C}[z]$ . Assume  $L'$  is free as a module and that for any complex number  $b$  the quotient Lie algebra  $L'/(z-b)L'$  is isomorphic to  $\mathfrak{g}$ . Does it follow that  $L'$  is isomorphic to  $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z]$  (as a Lie algebra over  $\mathbb{C}[z]$ )?

**Question 2:** Similarly, let  $L'$  be a Lie algebra over the ring  $\mathbb{C}[z, z^{-1}]$  which we assume is free as a module and such that for any  $b \in \mathbb{C}^\times$  the quotient  $L'/(z-b)L'$  is isomorphic to  $\mathfrak{g}$ . Does it follow that  $L'$  is isomorphic to a loop algebra (with respect to an automorphism of  $\mathfrak{g}$  of finite order)?

To answer these questions we take  $S = \mathrm{Spec}(R)$  where  $R = \mathbb{C}[t]$  or  $\mathbb{C}[t^{\pm 1}]$ , and where  $\mathcal{L}'$  and  $\mathcal{L}$  are the  $\mathcal{O}_S$ -Lie algebra corresponding to  $L'$  and  $\mathfrak{g} \otimes_{\mathbb{C}} R$  respectively. By Proposition 3.7 we see that  $\mathcal{L}'$  corresponds to a torsor  $\mathbf{X}'$  over  $\mathrm{Spec}(R)$  whose class is an element of  $H^1(R, \mathbf{Aut}(\mathfrak{g}))$ .

As we have observed there is a split exact sequence of  $R$ -group schemes

$$(4.3) \quad 1 \rightarrow \mathbf{G} \rightarrow \mathbf{Aut}(L) \rightarrow \mathbf{Out}(L) \rightarrow 1$$

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<sup>6</sup> See [DG, III §4] for the material on torsors used in this section.

where  $\mathbf{G}$  is the split adjoint semisimple  $R$ -group scheme corresponding to  $\mathfrak{g}$ , and  $\mathbf{Out}(L)$  is the constant  $R$ -group scheme corresponding to the finite (abstract) group  $\mathbf{Out}(\mathfrak{g})$  of the symmetries of the Coxeter-Dynkin diagram of  $\mathfrak{g}$ . This allow us to compute the relevant  $H^1$  and thus determine the nature of  $\mathbf{X}'$ , hence  $L'$ . According to [CGP, Cor. 3.3], we have a bijection

$$H^1(R, \mathbf{Aut}(L)) \xrightarrow{\sim} H^1(R, \mathbf{Out}(L)).$$

**Answer 1:** If  $R = \mathbb{C}[t]$  then  $H^1(R, \mathbf{Out}(L)) = 1$  since  $R$  is simply connected. It follows that  $H^1(R, \mathbf{Aut}(L)) = 1$  and therefore that  $L' \simeq \mathfrak{g} \otimes_k R$ .

**Answer 2:** If  $R = \mathbb{C}[t^{\pm 1}]$  then furthermore  $H^1(R, \mathbf{Out}(L))$  is the set of conjugacy classes of the (abstract) group  $\mathbf{Out}(\mathfrak{g})$  which, in terms of forms of  $L$ , correspond precisely to the loop algebras (see [P2] for details). Thus  $L'$  is a loop algebra.

## References

- [CGP] V. Chernousov, P. Gille and A. Pianzola, *Three-point Lie algebras and Grothendieck's dessins d'enfants*, Mathematical Research Letters **23** (2016), 81-104.
- [Bu] I. Burban, Open correspondence, 2020.
- [Co1] B. Conrad, *Reductive group schemes*, in *Autour des schémas en groupes, vol. I*, Panoramas et Synthèses **42-43**, Soc. Math. France 2014.
- [DG] M. Demazure et P. Gabriel, *Groupes algébriques*, Masson (1970).
- [EGA-I] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique: I. Le langage des schémas*, Grundlehren der Mathematischen Wissenschaften **166** (2nd ed.). Berlin; New York: Springer-Verlag, 1971.
- [EGAIV] A. Grothendieck (avec la collaboration de J. Dieudonné), *Eléments de Géométrie Algébrique IV: Étude locale des schémas et des morphismes de schémas*, Publications mathématiques de l'I.H.É.S. no 20, 24, 28 and 32 (1964 - 1967).
- [G-W] U. Görtz and T. Wedhorn, *Algebraic Geometry I*, Vieweg+Teubner 2010.
- [Ha] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Springer.
- [LLR] Q. Liu, D. Lorenzini, M. Raynaud, *Néron models, Lie algebras, and reduction of curves of genus one*, Invent. Math. **157** (2004), 455-518.

- [P1] A. Piazola, *Derivations of certain algebras defined by étale descent*. Math. Z. **264** (2010).
- [P2] A. Piazola, *Vanishing of  $H^1$  for Dedekind rings and applications to loop algebras*, C. R. Acad. Sci. Paris, Ser. I **340** (2005), 633-638.
- [SGA3] *Séminaire de Géométrie algébrique de l'I.H.E.S., 1963-1964, schémas en groupes, dirigé par M. Demazure et A. Grothendieck*, Lecture Notes in Math. 151-153. Springer (1970).
- [Stacks] Stacks project, <http://stacks.math.columbia.edu/>