Nonlinear Modes of Cantilever Beams at Extreme Amplitudes: Numerical Computation and Experiments

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ABSTRACT
A novel method for the numerical computation of the nonlinear normal modes (NNMs) of a highly flexible cantilever beam is presented. The flexible cantilever is modeled using a 2D finite element discretization of the geometrically exact beam model, wherein geometric nonlinearities relating to the rotation are kept entirely intact. The model is then solved using the proposed solution method, which is fully frequency domain-based and involves a novel combination of a harmonic balance (HBM) Fourier expansion with asymptotic numerical (ANM) continuation for periodic solutions. The NNMs are also calculated experimentally using a flexible cantilever specimen mounted to a shaker table. The experimental NNMs can be compared to their numerical counterparts in order to validate the frequency domain numerical technique.

Keywords: Nonlinear modes, geometric nonlinearity, geometrically exact beam, finite element analysis, continuation methods

INTRODUCTION

The scientific literature on the subject of cantilever beams is very dense across a variety of different studies. Modern studies on highly flexible beam structures often use the geometrically exact beam model, wherein any geometric nonlinearities related to the rotation of the structure are kept exact. Based on the work of Reissner [1] and Simo [2], existing methods for the simulation of flexible beams often apply a finite element discretization and are typically resolved using time domain numerical schemes [3-5].

In this paper, a novel method is introduced for the numerical simulation of highly flexible beams structures. This method proposes a finite element discretization of the geometrically exact beam model to be solved using a frequency domain numerical scheme as opposed to time domain formulations. The method of solving, rooted entirely in the frequency domain, illustrates the novelty of this approach. The discretized equations of motion are solving through a unique combination of the harmonic balance method (HBM) and asymptotic numerical method (ANM), the latter of which represents a continuation method based on pseudo-arclength parameterization and first mentioned in the work of Damil and Potier-Ferry [6]. This unique scheme for the continuation of periodic solutions has several consequential advantages. Most importantly in the present context, this method allows direct and efficient computation of the nonlinear normal modes (NNMs) of the system. The NNMs in this context are defined as the periodic solutions of the *free* and *undamped* dynamical system [7], and represent a visual characterization of the nonlinear mode shape dependence on the amplitude of vibration, also called the *backbone* curve. They are equivalent to invariant manifolds of the phase space [8, 9]. Several methods exist for their analytical calculation (see, for example, [10]); their numerical and experimental calculations are described here.

ANALYTICAL BACKGROUND

The analytical background, including the derivation of the geometrically exact model and the subsequent finite element discretization, is outlined in detail in previous work [11] and is therefore largely omitted for brevity. The 2D finite element model, based on a total Lagrangian nonlinear formulation and using Timoshenko beam elements, is written, for all time t:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{C}\dot{\boldsymbol{q}} + \boldsymbol{f}_{int}(\boldsymbol{q}) = \boldsymbol{f}_{ext} \tag{1}$$

where $\boldsymbol{M}$ and $\boldsymbol{C}$ represent the mass and damping matrices, respectively, of size $3n \times 3n$ for $n \times n$ degrees of freedom; $\boldsymbol{q}$ the vector of the discretized degrees of freedom, of size $3n$; $\boldsymbol{f}_{int}(\boldsymbol{q})$ the nonlinear internal force vector, which implicitly contains the stiffness matrix $\boldsymbol{K} = \partial \boldsymbol{f}_{int}/\partial \boldsymbol{q}|_{\boldsymbol{q}=0}$, of size $3n$; and $\boldsymbol{f}_{ext}$ the external force vector, also of size $3n$. The nonlinear modes, however, are defined as the free solutions of the underlying conservative system. In this case, equation (1) contains no damping $\boldsymbol{C}\dot{\boldsymbol{q}}$ or forcing $\boldsymbol{f}_{ext}$ terms. It is this autonomous system that is solved using the numerical scheme outlined in the next section in order to compute the nonlinear modes of the system.

NUMERICAL ANALYSIS

The numerical procedure for solving the finite element model described above is rooted in continuation of periodic solutions. A two-step, frequency-domain formulation for continuation of periodic solutions has been implemented into a custom solver coded in MATLAB. The custom solver, called **MANLAB**, is capable of tracing the solution(s) of periodic systems by combining the aforementioned two-step procedure: a harmonic balance method (HBM) modal expansion followed by asymptotic numerical method (ANM) continuation.

In the first step, the system unknowns $\boldsymbol{u}(t)$ (the displacement of the cantilever beam, in the present case) are expanded as a Fourier series truncated to a number of harmonics H:

$$\boldsymbol{u}(t) = \boldsymbol{u}_0 + \sum_{k=1}^{H}[\boldsymbol{u}_k^c \cos(k\Omega t) + \boldsymbol{u}_k^s \sin(k\Omega t)] \tag{2}$$

where $\boldsymbol{u}_0$, $\boldsymbol{u}_k^c$ and $\boldsymbol{u}_k^s$ represent the Fourier coefficients of the unknowns $\boldsymbol{u}(t)$. It is known that solving via HBM proves difficult when H is large or if the system contains complex (i.e. non-polynomial) nonlinearities, as is the case with geometric nonlinearities. This difficulty is easily resolved, however, given that the ANM procedure necessitates a unique procedure called the *quadratic recast*, whereby non-polynomial nonlinearities are "recast" as polynomial nonlinearities of quadratic order or less through the introduction of so-called *auxiliary variables*. For example, $\cos(\theta)$ and $\sin(\theta)$ may be redefined as $c = \cos(\theta)$ and $s = \sin(\theta)$ where c and s represent solutions of linear differential equations [14], yielding the term $u_1\cos(\theta)$ as $u_1 c$ instead. The HBM is then easily applied to the larger system of simplified equations [12].

Finally, ANM continuation solves the system for $\boldsymbol{u}(t)$ as a function of a varying parameter, called λ. The ANM implements a pseudo-arclength parametrization to compute the branches of solution $\boldsymbol{u}(t)$, a method which is derived in detail in [13, 14]. An appropriate phase condition is applied to the system in order to initialize the solution on the backbone, leading to calculation of the NNMs as λ increases.

EXPERIMENTS

An experimental setup capable of reproducing the behavior of the flexible cantilever in extreme amplitude vibration is designed using a vibration table. A slender, highly flexible cantilever beam specimen (29.1 cm × 1.3 cm × 0.4 mm, stainless steel) is mounted to a shaker table forced with a periodic signal $\boldsymbol{F}(t) = \boldsymbol{F}_0\cos(\Omega t)$:

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \boldsymbol{C}\dot{\boldsymbol{u}}(t) + \boldsymbol{K}\boldsymbol{u}(t) + \boldsymbol{f}_{nl}[\boldsymbol{u}(t)] = \boldsymbol{F}_0\cos(\Omega t) \tag{3}$$

corresponding to the same parameters as defined above.

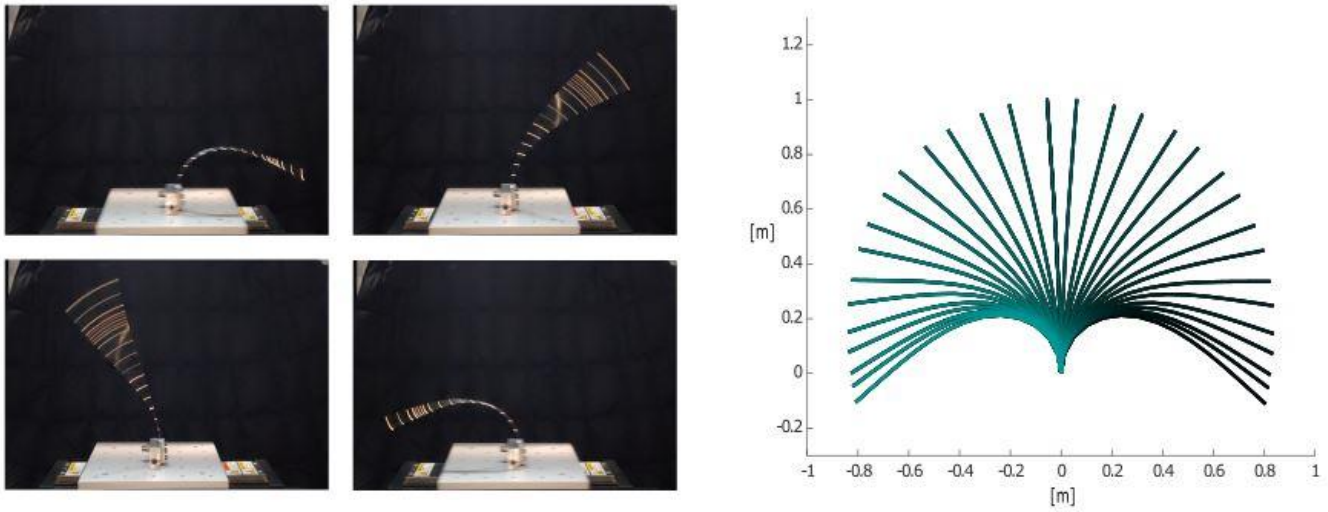
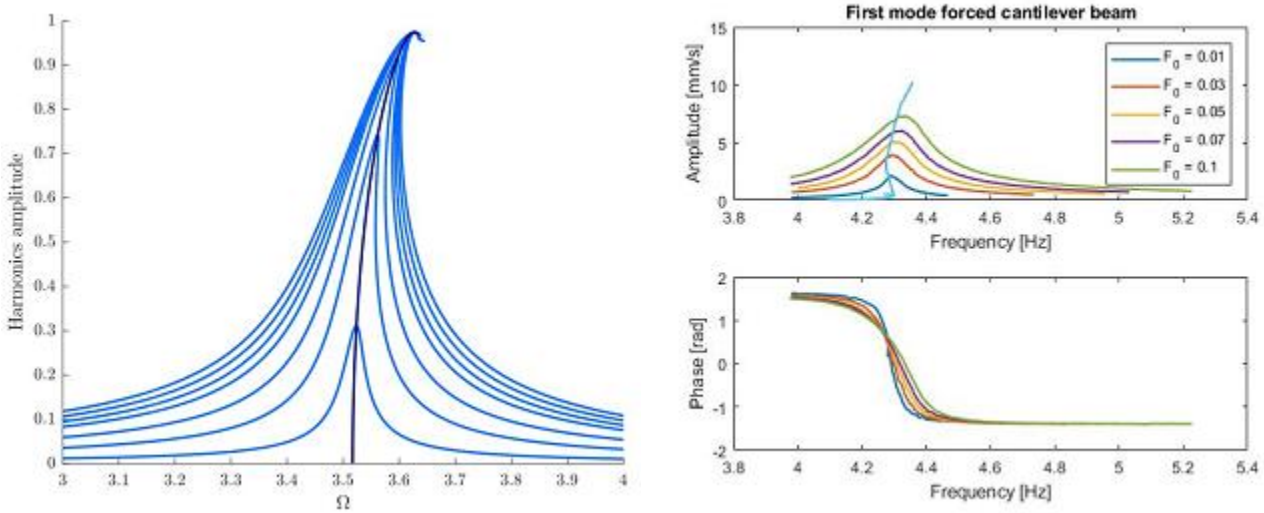


Figure 1: (Left) views of the vibration table used in experiments involving transverse vibration of slender, flexible cantilevers and (right) a numerical simulation of extreme amplitude vibration of a flexible cantilever.

Velocity measurements (captured via laser vibrometers) and the forcing of the table are routed through a dSpace MicroLabBox, which integrates a Phase-Lock Loop (PLL) control schematic [15]. The PLL controller governs the behavior of the system, permitting experimental calculation of the nonlinear normal modes (NNMS) based on the work of Peeters, Kerschen et al. [16]. It is hypothesized that the NNMS, being defined as free solutions of the underlying conservative system $\boldsymbol{u_n}(t)$:

$$\boldsymbol{M}\ddot{\boldsymbol{u}}_n(t) + \boldsymbol{K}\boldsymbol{u}_n(t) + \boldsymbol{f}_{nl}[\boldsymbol{u}_n(t)] = 0 \tag{4}$$

can be estimated experimentally by controlling the system such that a phase resonance is attained [15, 16]. The phase resonance occurs at a phase difference of $\pi/2$ between the relative position $\boldsymbol{u}_n(t)$ and the forcing $\boldsymbol{F}(t)$) [15], or π if measuring the relative velocity $\dot{\boldsymbol{u}}_n(t)$, a value that the PLL controller "locks" onto before increasing the forcing amplitude in order to trace the backbone curve of the system, equivalent to the NNMs.



Figures 2 and 3: Amplitude of the first harmonic of the transverse displacement and NNMs at the tip of a cantilever beam, numerical simulations and experiments. (Left) numerical forced response and NNMs of a dimensionless cantilever beam. (Right) experimental forced response and NNMs of the cantilever beam system.

CONCLUSION

The modeling of highly flexible structures is a current and competitive subject of research, with new methods extending the capabilities of numerical modeling with each passing year. A novel method for the simulation of highly flexible cantilevers in the frequency domain along with an experimental comparison with physical reality were described. The proposed numerical scheme, rooted fully in the frequency domain, offers several advantages when compared with time domain formulations: most importantly, the capability to calculate directly the nonlinear normal modes (NNMs) of the system. Based on a finite element discretization of the geometrically exact beam model, this method is able to simulate the deformation of a flexible cantilever even at extreme amplitudes of vibration. Finally, the numerical results are compared to experiments based on flexible cantilever specimens in order to validate the accuracy of the proposed frequency domain numerical scheme.

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