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EQUIDISTRIBUTIONS AROUND SPECIAL KINDS OF DESCENTS AND EXCEDANCES

BIN HAN, JIANXI MAO, AND JIANG ZENG

ABSTRACT. We consider a sequence of four variable polynomials by refining Stieltjes' continued fraction for Eulerian polynomials. Using combinatorial theory of Jacobi-type continued fractions and bijections we derive various combinatorial interpretations in terms of permutation statistics for these polynomials, which include special kinds of descents and excedances in a recent paper of Baril and Kirgizov. As a by-product, we derive several equidistribution results for permutation statistics, which enables us to confirm and strengthen a recent conjecture of Vajnovszki and also to obtain several companion permutation statistics for two bivariate statistics in a conjecture of Baril and Kirgizov.

1. INTRODUCTION

It is well-known [9, 19, 25] that the statistics "des" and "exc" are equidistributed over permutations of $[n] := \{1, \dots, n\}$, their common generating function being the Eulerian polynomials $A_n(t)$, i.e.,

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des } \sigma} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc } \sigma},$$

which satisfy the identity

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{r=0}^{\infty} t^r (r+1)^n.$$

Since MacMahon's pioneering work [16] various combinatorial variants and refinements of Eulerian polynomials have appeared, see [2, 8, 13, 17, 18, 24] for some recent papers.

In a recent paper [1] Baril and Kirgizov considered some special descents, excedances and cycles of permutations, that we recall in the following. For a permutation $\sigma := \sigma(1)\sigma(2)\cdots\sigma(n)$ of $1 \dots n$, an index $i \in [1, n-1]$ is called a

- *descent* (resp. *excedance*) if $\sigma(i) > \sigma(i+1)$ (resp. $\sigma(i) > i$);
- *descent of type 2* if i is a descent and $\sigma(j) < \sigma(i)$ for $j < i$;
- *pure excedance* if i is an excedance and $\sigma(j) \notin [i, \sigma(i)]$ for $j < i$;

and an index $i \in [2, n]$ is called a

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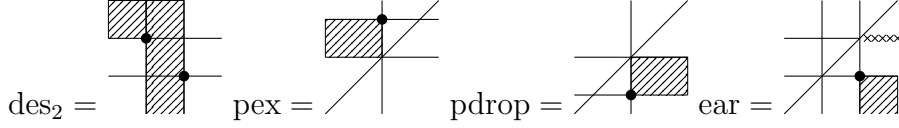


FIGURE 1. Illustration of the mesh patterns des_2 and pex and ear , where the cross line means that the value cannot be in the segment of the horizontal line

- *drop* if $i > \sigma(i)$;
- *pure drop* if i is a drop and $\sigma(j) \notin [\sigma(i), i]$ for $j > i$.

Let $\text{des } \sigma$ (resp. $\text{exc } \sigma$, $\text{drop } \sigma$, $\text{des}_2 \sigma$, $\text{pex } \sigma$ and $\text{pdrop } \sigma$) denote the number of descents (resp. excedances, drops, descents of type 2, pure excedances and pure drops) of σ . Identifying σ with the bijection $i \mapsto \sigma(i)$ on $[n]$ we can decompose σ into disjoint cycles $(i, \sigma(i), \dots, \sigma^\ell(i))$ with $\sigma^{\ell+1}(i) = i$ and $i \in [n]$. A cycle with $\ell = 1$ is called a *fixed point* of σ . Let $\text{cyc } \sigma$ (resp. $\text{fix } \sigma$) denote the number of cycles (resp. fixed points) of σ . The number of non trivial cycles of σ [23, A136394] is defined by

$$\text{pcyc } \sigma = \text{cyc } \sigma - \text{fix } \sigma. \quad (1.1)$$

For example, if $n = 8$ and $\sigma = 23146875$, the descent indexes of type 2 are $\{2, 6\}$; the pure excedance indexes are $\{1, 5\}$ and the pure drop indexes are $\{3, 8\}$. Thus $\text{des}_2 \sigma = 2$, $\text{pex } \sigma = 2$, and $\text{pdrop } \sigma = 2$. Factorizing σ as product of disjoint cycles $(123)(4)(568)(7)$, we derive $\text{cyc } \sigma = 4$, $\text{fix } \sigma = 2$, and $\text{pcyc } \sigma = 2$.

A *mesh pattern* of length k is a pair (τ, R) , where τ is a permutation of length k and R is a subset of $\llbracket 0, k \rrbracket \times \llbracket 0, k \rrbracket$ with $\llbracket 0, k \rrbracket = \{0, 1, \dots, k\}$. Let (i, j) denote the box whose corners have coordinates (i, j) , $(i, j + 1)$, $(i + 1, j + 1)$, and $(i + 1, j)$. Note that a descent of type 2 can be viewed as an occurrence of the mesh pattern $(21, L_1)$ where $L_1 = \{1\} \times [0, 2] \cup \{(0, 2)\}$. By abuse of notation, we use des_2 to denote the mesh pattern corresponding to an occurrence of descent of type 2 in Figure 1. Similarly, we use pex (resp. pdrop) to denote an occurrence of pure excedance in Figure 1 although pex (resp. pdrop) is not a mesh pattern. See [1, 14] for further information about mesh patterns.

Recently Baril and Kirgizov [1] proved the equidistribution of the statistics "des₂", "pex" and "pcyc" over \mathfrak{S}_n by bijections and conclude their paper with the following two conjectures on the equidistribution of two pairs of bistatistics.

Conjecture 1.1 (Baril and Kirgizov). The two bistatistics $(\text{des}_2, \text{cyc})$ and (pex, cyc) are equidistributed on \mathfrak{S}_n .

Conjecture 1.2 (Vajnovszki). The two bistatistics $(\text{des}_2, \text{des})$ and (pex, exc) are equidistributed on \mathfrak{S}_n .

In this paper we shall take a different approach to their problems through the combinatorial theory of J-continued fractions developed by Flajolet and Viennot in the 1980's [7, 12], see [2, 6, 13, 24] for recent developments of this theory. Recall that a J-type continued

fraction is a formal power series defined by

$$\sum_{n=0}^{\infty} a_n z^n = \frac{1}{1 - \gamma_0 z - \frac{\beta_1 z^2}{1 - \gamma_1 z - \frac{\beta_2 z^2}{\dots}}},$$

where $(\gamma_n)_{n \geq 0}$ and $(\beta_n)_{n \geq 1}$ are two sequences in some commutative ring.

Define the polynomials $A_n(t, \lambda, y, w)$ by the J-fraction

$$\sum_{n \geq 0} z^n A_n(t, \lambda, y, w) = \frac{1}{1 - wz - \frac{t\lambda y z^2}{1 - (w+t+1)z - \frac{t(\lambda+1)(y+1)z^2}{\dots}}} \quad (1.2)$$

with $\gamma_n = w + n(t+1)$ and $\beta_n = t(\lambda + n - 1)(y + n - 1)$.

It is known that $A_n(t, 1, 1, 1)$ equals the Eulerian polynomial $A_n(t)$, see [13, 24]. Recently Sokal and the third author [24] have generalized the J-fraction for Eulerian polynomials in infinitely many indeterminates, which are also generalizations of the polynomials $A_n(t, \lambda, y, w)$. The aim of this paper is to generalize the results in [1] by exploring the combinatorial interpretations of the polynomials $A_n(t, \lambda, y, w)$ in light of the aforementioned statistics. In particular, we confirm and strengthen Conjecture 1.2 (see Corollary 1.6) and obtain five equidistributed companions of the bivariate (pex, cyc) in Conjecture 1.1 (see Theorem 1.7).

1.1. Main results. For $\sigma \in \mathfrak{S}_n$, an index $i \in [n]$ is called (see [24]) a

- *cycle peak* (cpeak) if $\sigma^{-1}(i) < i < \sigma(i)$;
- *cycle valley* (cval) if $\sigma^{-1}(i) > i > \sigma(i)$;
- *cycle double rise* (cdrise) if $\sigma^{-1}(i) < i < \sigma(i)$;
- *cycle double fall* (cdfall) if $\sigma^{-1}(i) > i > \sigma(i)$;
- *fixed point* (fix) if $\sigma^{-1}(i) = i = \sigma(i)$.

Clearly every index i belongs to exactly one of these five types; we refer to this classification as the *cycle classification*. Next, an index $i \in [n]$ (or a value $\sigma(i)$) is called a

- *record* (rec) (or *left-to-right maximum*) if $\sigma(j) < \sigma(i)$ for all $j < i$ (The index 1 is always a record];
- *antirecord* (arec) (or *right-to-left minimum*) if $\sigma(j) > \sigma(i)$ for all $j > i$ (The index n is always an antirecord);
- *exclusive record* (erec) if it is a record and not also an antirecord;
- *exclusive antirecord* (earec) if it is an antirecord and not also a record.
- *exclusive antirecord cycle peak* (eareccpeak) if i is an exclusive antirecord and also a cycle peak.

The statistic eareccpeak was introduced in [24], in this paper we adopt the following concise notation instead

$$\text{ear} := \text{eareccpeak}. \quad (1.3)$$

An illustration of the pattern ear is given in Figure 1. Also, we shall denote the set of indexes of each type by capitalizing the first letter of type name. Hence Cpeak σ denotes the set of indexes of cycle peaks of σ . For example, if $\sigma = 23147865 = (123)(4)(6857)$, then Earec $\sigma = \{3, 8\}$ as $\sigma(3) = 1$ and $\sigma(8) = 5$ and Cpeak $\sigma = \{3, 7, 8\}$, so Ear $\sigma = \{3, 8\}$ and ear $\sigma = 2$.

Our first result provides three interpretations for the polynomials $A_n(t, \lambda, y, w)$ in (1.2).

Theorem 1.3. *We have*

$$A_n(t, \lambda, y, w) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc } \sigma} \lambda^{\text{pex } \sigma} y^{\text{ear } \sigma} w^{\text{fix } \sigma} \quad (1.4a)$$

$$= \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc } \sigma} \lambda^{\text{pcyc } \sigma} y^{\text{ear } \sigma} w^{\text{fix } \sigma} \quad (1.4b)$$

$$= \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc } \sigma} \lambda^{\text{pcyc } \sigma} y^{\text{pex } \sigma} w^{\text{fix } \sigma}. \quad (1.4c)$$

By (1.2), the polynomial $A_n(t, \lambda, y, w)$ is invariant under $\lambda \leftrightarrow y$. Hence, the above theorem implies immediately the following result.

Corollary 1.4. *The six bivariate statistics (pex, ear), (ear, pex), (ear, pcyc), (pcyc, ear), (pex, pcyc) and (pcyc, pex) are equidistributed on \mathfrak{S}_n .*

Now we consider three specializations of $A_n(t, \lambda, y, w)$. First let $B_n(t, \lambda, w) = A_n(t, \lambda, 1, w) = A_n(t, 1, \lambda, w)$, namely,

$$\sum_{n \geq 0} z^n B_n(t, w, \lambda) = \frac{1}{1 - wz - \frac{t\lambda z^2}{1 - (w+t+1)z - \frac{2t(\lambda+1)z^2}{\dots}}} \quad (1.5)$$

with $\gamma_n = w + n(t+1)$ and $\beta_n = nt(\lambda + n - 1)$.

Remark. By (1.4c) we recover the fix and cycle (p, q) -Eulerian polynomials [15, 17, 26]

$$A_n(x, p, 1, pq) = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{exc } \sigma} p^{\text{cyc } \sigma} q^{\text{fix } \sigma}. \quad (1.6)$$

To deal with descent statistics, we recall some linear statistics from [13]. For $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) \in \mathfrak{S}_n$ with convention $0-\infty$, i.e., $\sigma(\mathbf{0}) = \mathbf{0}$ and $\sigma(\mathbf{n}+1) = \mathbf{n}+1$, a value $\sigma(i)$ ($1 \leq i \leq n$) is called a

- *double ascent* (dasc) if $\sigma(i-1) < \sigma(i)$ and $\sigma(i) < \sigma(i+1)$;
- *double descent* (ddes) if $\sigma(i-1) > \sigma(i)$ and $\sigma(i) > \sigma(i+1)$;

- *peak* (peak) if $\sigma(i-1) < \sigma(i)$ and $\sigma(i) > \sigma(i+1)$;
- *valley* (valley) if $\sigma(i-1) > \sigma(i)$ and $\sigma(i) < \sigma(i+1)$.

A double ascent $\sigma(i)$ ($1 \leq i \leq n$) is called a *foremaximum* of σ if it is at the same time a record. Denote the number of foremaxima of σ by $\text{fmax } \sigma$. For example, if $\sigma = 34215876$, then $\text{dasc } \sigma = \text{ddes } \sigma = \text{peak } \sigma = \text{val } \sigma = 2$ and $\text{fmax } \sigma = 2$ as the foremaxima of σ are 3, 5.

Theorem 1.5. *We have*

$$B_n(t, \lambda, w) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc } \sigma} \lambda^{\text{pcyc } \sigma} w^{\text{fix } \sigma} \quad (1.7a)$$

$$= \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc } \sigma} \lambda^{\text{ear } \sigma} w^{\text{fix } \sigma} \quad (1.7b)$$

$$= \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc } \sigma} \lambda^{\text{pex } \sigma} w^{\text{fix } \sigma} \quad (1.7c)$$

$$= \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des } \sigma} \lambda^{\text{des}_2 \sigma} w^{\text{fmax } \sigma} \quad (1.7d)$$

and

$$\sum_{n \geq 0} B_n(t, w, \lambda) \frac{z^n}{n!} = e^{wz} \left(\frac{1-t}{e^{tz} - te^z} \right)^\lambda. \quad (1.7e)$$

The following corollary of Theorem 1.5 confirms and generalizes Conjecture 1.2.

Corollary 1.6. *The four bivariate statistics (exc, pcyc), (exc, ear), (des, des₂) and (exc, pex) are equidistributed over \mathfrak{S}_n .*

Remark. We will provide bijective proofs of Corollary 1.6 in Lemma 2.2 and Theorem 1.9.

Next let $C_n(y, \lambda) = A_n(1, \lambda, y, \lambda) = A_n(1, y, \lambda, \lambda)$. We obtain the following result directly from Theorem 1.5.

Theorem 1.7. *We have*

$$C_n(y, \lambda) = \sum_{\sigma \in \mathfrak{S}_n} y^{\text{pex } \sigma} \lambda^{\text{ear } \sigma + \text{fix } \sigma} = \sum_{\sigma \in \mathfrak{S}_n} y^{\text{ear } \sigma} \lambda^{\text{pex } \sigma + \text{fix } \sigma} \quad (1.8a)$$

$$= \sum_{\sigma \in \mathfrak{S}_n} y^{\text{pcyc } \sigma} \lambda^{\text{ear } \sigma + \text{fix } \sigma} = \sum_{\sigma \in \mathfrak{S}_n} y^{\text{ear } \sigma} \lambda^{\text{cyc } \sigma} \quad (1.8b)$$

$$= \sum_{\sigma \in \mathfrak{S}_n} y^{\text{pcyc } \sigma} \lambda^{\text{pex } \sigma + \text{fix } \sigma} = \sum_{\sigma \in \mathfrak{S}_n} y^{\text{pex } \sigma} \lambda^{\text{cyc } \sigma}. \quad (1.8c)$$

Finally let $D_n(t, \lambda, y) = A_n(t, \lambda, y, 0)$. From Theorem 1.3 we deduce

$$D_n(t, \lambda, y) = \sum_{\sigma \in \mathfrak{D}_n} t^{\text{exc } \sigma} \lambda^{\text{pex } \sigma} y^{\text{ear } \sigma} \quad (1.9a)$$

$$= \sum_{\sigma \in \mathfrak{D}_n} t^{\text{exc } \sigma} \lambda^{\text{cyc } \sigma} y^{\text{ear } \sigma} \quad (1.9b)$$

$$= \sum_{\sigma \in \mathfrak{D}_n} t^{\text{exc } \sigma} \lambda^{\text{cyc } \sigma} y^{\text{pex } \sigma}, \quad (1.9c)$$

where \mathfrak{D}_n is the set of derangements in \mathfrak{S}_n .

Let \mathfrak{D}_n^* the subset of \mathfrak{D}_n consisting of derangements without cycle double rise. Furthermore, for $k \in [n]$ define the set

$$\mathfrak{D}_n^*(k) = \{\sigma \in \mathfrak{D}_n \mid \text{exc}(\sigma) = k, \text{cdrise}(\sigma) = 0\}. \quad (1.10)$$

We show that the polynomials $D_n(t, \lambda, y)$ have a nice γ -positive formula, see [3, 13] for further informations.

Theorem 1.8. *We have*

$$D_n(t, \lambda, y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,k}(\lambda, y) t^k (1+t)^{n-2k}, \quad (1.11)$$

where the gamma coefficient $\gamma_{n,k}(\lambda, y)$ has the following interpretations

$$\gamma_{n,k}(\lambda, y) = \sum_{\sigma \in \mathfrak{D}_n^*(k)} \lambda^{\text{pex } \sigma} y^{\text{ear } \sigma} \quad (1.12a)$$

$$= \sum_{\sigma \in \mathfrak{D}_n^*(k)} \lambda^{\text{cyc } \sigma} y^{\text{ear } \sigma} \quad (1.12b)$$

$$= \sum_{\sigma \in \mathfrak{D}_n^*(k)} \lambda^{\text{cyc } \sigma} y^{\text{pex } \sigma}. \quad (1.12c)$$

Remark. For $\sigma \in \mathfrak{D}_n^*(k)$, the mapping $\sigma \mapsto \sigma^{-1}$ is a bijection from $\mathfrak{D}_n^*(k)$ to $\mathfrak{D}_n^{**}(k)$ with

$$\mathfrak{D}_n^{**}(k) = \{\sigma \in \mathfrak{D}_n \mid \text{drop}(\sigma) = k, \text{cdfall}(\sigma) = 0\}. \quad (1.13)$$

Thus, when $y = 1$ both (1.12b) and (1.12c) reduce to [22, Theorem 11].

The rest of this paper is organized as follows: we first prove the main results in Section 2. In Section 3, we will construct two bijections on \mathfrak{S}_n to prove the equality between (1.7c) and (1.7d), namely we have the following result.

Theorem 1.9. *There are bijections $\Phi_1 : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ and $\Phi_2 : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ such that*

$$(\text{des}, \text{des}_2) \sigma = (\text{exc}, \text{ear}) \Phi_1(\sigma); \quad (1.14a)$$

$$(\text{des}, \text{des}_2, \text{fmax}) \sigma = (\text{exc}, \text{pex}, \text{fix}) \Phi_2(\sigma). \quad (1.14b)$$

Remark. Note that Φ_2 gives a bijective proof of Conjecture 1.2.

The proof of Theorem 1.3 relies on the fact the polynomial $A_n(t, \lambda, y, w)$ can be obtained by specializing the two master polynomials Q_n and \widehat{Q}_n in [24], see (1.18) and (1.22). Actually, we shall derive Theorem 1.3 from the *first and second master J-fractions* for

permutations in [24, Theorems 2.9 and 2.14], and a dual form of the second master J-fraction, see Proposition 1.13. For reader's convenience we shall recall the two master J-fractions for permutations in the next section.

As the polynomials Q_n and \widehat{Q}_n are originally defined using cyclic statistics of permutations, it is then suggested in [24] to seek for interpretations using linear statistics for these master polynomials. In Section 3, we shall give two such interpretations for the polynomials Q_n and as an application, we give a group action proof for a gamma-expansion formula Eq.(1.12a) (see Theorem 1.8). Finally we conclude the paper with some open questions.

1.2. Two master J-fractions for permutations. We recall the two master J-fractions for permutations in [24]. First we associate to each permutation $\sigma \in \mathfrak{S}_n$ a pictorial representation (Figure 5) by placing vertices $1, 2, \dots, n$ along horizontal axis and then draw an arc from i to $\sigma(i)$ above (resp. below) the horizontal axis in case $\sigma(i) > i$ (resp. $\sigma(i) < i$), if $\sigma(i) = i$ we do not draw any arc. Of course, the arrows on the arc are redundant, because the arrow on an arc above (resp. below) the axis always point to the right (resp. left). We then say that a quadruplet $i < j < k < l$ forms an

- *upper crossing* (ucross) if $k = \sigma(i)$ and $l = \sigma(j)$;
- *lower crossing* (lcross) if $i = \sigma(k)$ and $j = \sigma(l)$;
- *upper nesting* (unest) if $l = \sigma(i)$ and $k = \sigma(j)$;
- *lower nesting* (lnest) if $i = \sigma(l)$ and $j = \sigma(k)$.

See Figure 2 and Figure 3. We also need a refined version of the above statistics. The basic idea is that, rather than counting the *total* numbers of quadruplets $i < j < k < l$ that form upper (resp. lower) crossings or nestings, we should instead count the number of upper (resp. lower) crossings or nestings that use a particular vertex j (resp. k) in second (resp. third) position, and then attribute weights to the vertex j (resp. k) depending on those values. More precisely, we define

$$\text{ucross}(j, \sigma) = \#\{i < j < k < l: k = \sigma(i) \text{ and } l = \sigma(j)\} \quad (1.15a)$$

$$\text{unest}(j, \sigma) = \#\{i < j < k < l: k = \sigma(j) \text{ and } l = \sigma(i)\} \quad (1.15b)$$

$$\text{lcross}(k, \sigma) = \#\{i < j < k < l: i = \sigma(k) \text{ and } j = \sigma(l)\} \quad (1.15c)$$

$$\text{lnest}(k, \sigma) = \#\{i < j < k < l: i = \sigma(l) \text{ and } j = \sigma(k)\} \quad (1.15d)$$

We also consider the degenerate cases with $j = k$, by saying that a triplet $i < j < l$ forms an

- *upper pseudo-nesting* (upsnest) if $l = \sigma(i)$ and $j = \sigma(j)$;
- *lower pseudo-nesting* (lpsnest) if $i = \sigma(l)$ and $j = \sigma(j)$.

See Figure 4. Note that $\text{upsnest}(\sigma) = \text{lpsnest}(\sigma)$ for all σ (see [24]). We therefore write these two statistics simply as

$$\text{lev}(\sigma) = \text{upsnest}(\sigma) = \text{lpsnest}(\sigma).$$

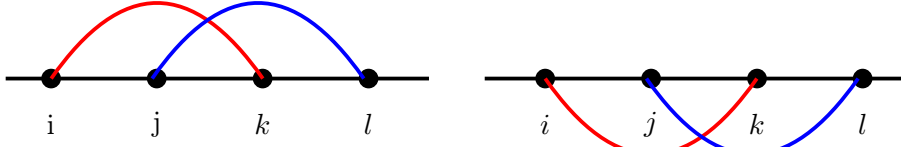


FIGURE 2. Upper crossing and lower crossing

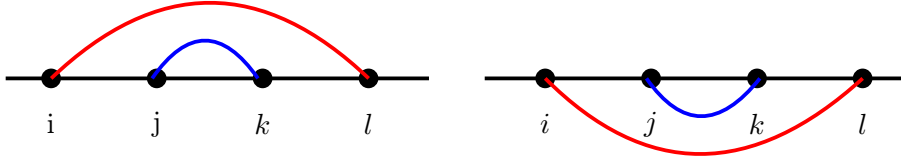


FIGURE 3. Upper nesting and lower nesting

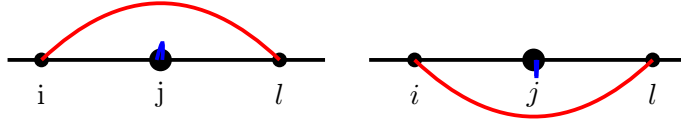


FIGURE 4. Upper pseudo-nesting and lower pseudo-nesting of a fixed point

The refined level of a fixed point j ($\sigma(j) = j$) is defined by

$$\text{lev}(j, \sigma) = \#\{i < j < l : l = \sigma(i)\} = \#\{i < j < l : i = \sigma(l)\}. \quad (1.16)$$

And we obviously have

$$\text{ucross}(\sigma) = \sum_{j \in \text{cval}} \text{ucross}(j, \sigma) \quad (1.17)$$

and analogously for the other four statistics lcross , unest , lnest and lev .

We now introduce five infinite families of indeterminates $\mathbf{a} = (a_{\ell, \ell'})_{\ell, \ell' \geq 0}$, $\mathbf{b} = (b_{\ell, \ell'})_{\ell, \ell' \geq 0}$, $\mathbf{c} = (c_{\ell, \ell'})_{\ell, \ell' \geq 0}$, $\mathbf{d} = (d_{\ell, \ell'})_{\ell, \ell' \geq 0}$, $\mathbf{e} = (e_{\ell})_{\ell \geq 0}$ and define the polynomial $Q_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e})$ by

$$\begin{aligned} Q_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) = & \sum_{\sigma \in \mathfrak{S}_n} \prod_{i \in \text{Cval}} a_{\text{ucross}(i, \sigma), \text{unest}(i, \sigma)} \prod_{i \in \text{Cpeak}} b_{\text{lcross}(i, \sigma), \text{lnest}(i, \sigma)} \times \\ & \prod_{i \in \text{Cdfall}} c_{\text{lcross}(i, \sigma), \text{lnest}(i, \sigma)} \prod_{i \in \text{Cdrise}} d_{\text{ucross}(i, \sigma), \text{unest}(i, \sigma)} \prod_{i \in \text{Fix}} e_{\text{lev}(i, \sigma)}. \end{aligned} \quad (1.18)$$

The following is the first master J-fraction for permutations in [24, Theorem 2.9].

Theorem 1.10. [24] *The ordinary generating function of the polynomials $Q_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e})$ has the J-type continued fraction*

$$\sum_{n=0}^{\infty} Q_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) z^n =$$

$$\frac{1}{1 - e_0 z - \frac{a_{00} b_{00} z^2}{1 - (c_{00} + d_{00} + e_1) z - \frac{(a_{01} + a_{10})(b_{01} + b_{10}) z^2}{1 - (c_{01} + c_{10} + d_{01} + d_{10} + e_2) z - \frac{(a_{02} + a_{11} + a_{20})(b_{02} + b_{11} + b_{20}) z^2}{1 - \dots}}}}
 \quad (1.19)$$

with coefficients

$$\gamma_n = c_{n-1}^* + d_{n-1}^* + e_n \quad (1.20a)$$

$$\beta_n = a_{n-1}^* b_{n-1}^* \quad (1.20b)$$

where

$$a_{n-1}^* \stackrel{\text{def}}{=} \sum_{\ell=0}^{n-1} a_{\ell, n-1-\ell} \quad (1.21)$$

and likewise for $\mathbf{b}, \mathbf{c}, \mathbf{d}$.

We again introduce five infinite families of indeterminates: $\mathbf{a} = (a_\ell)_{\ell \geq 0}$, $\mathbf{b} = (b_{\ell, \ell'})_{\ell, \ell' \geq 0}$, $\mathbf{c} = (c_{\ell, \ell'})_{\ell, \ell' \geq 0}$, $\mathbf{d} = (d_{\ell, \ell'})_{\ell, \ell' \geq 0}$, $\mathbf{e} = (e_\ell)_{\ell \geq 0}$; please note that \mathbf{a} now has one index rather than two. We then define the polynomial $\widehat{Q}_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \lambda)$ by

$$\begin{aligned}
 \widehat{Q}_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \lambda) = & \\
 & \sum_{\sigma \in \mathfrak{S}_n} \lambda^{\text{cyc}(\sigma)} \prod_{i \in \text{Cval}} a_{\text{ucross}(i, \sigma) + \text{unest}(i, \sigma)} \prod_{i \in \text{Cpeak}} b_{\text{lcross}(i, \sigma), \text{lnest}(i, \sigma)} \times \\
 & \prod_{i \in \text{Cdfall}} c_{\text{lcross}(i, \sigma), \text{lnest}(i, \sigma)} \prod_{i \in \text{Cdrise}} d_{\text{ucross}(i, \sigma) + \text{unest}(i, \sigma), \text{unest}(\sigma^{-1}(i), \sigma)} \prod_{i \in \text{Fix}} e_{\text{lev}(i, \sigma)}. \quad (1.22)
 \end{aligned}$$

Note that here, in contrast to Theorem 1.10, \widehat{Q}_n depends on $\text{ucross}(i, \sigma)$ and $\text{unest}(i, \sigma)$ only via their sum; and note also the somewhat bizarre appearance of $\text{unest}(\sigma^{-1}(i), \sigma)$ as the second index on \mathbf{d} . The following is the second master J-fraction for permutations in [24, Theorem 2.14].

Theorem 1.11. [24] *The ordinary generating function of the polynomials $\widehat{Q}_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \lambda)$ has the J-type continued fraction*

$$\sum_{n=0}^{\infty} \widehat{Q}_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \lambda) z^n = \frac{1}{1 - \lambda e_0 z - \frac{\lambda a_0 b_{00} z^2}{1 - (c_{00} + d_{00} + \lambda e_1) z - \frac{(\lambda + 1) a_1 (b_{01} + b_{10}) z^2}{1 - (c_{01} + c_{10} + d_{10} + d_{11} + \lambda e_2) z - \frac{(\lambda + 2) a_2 (b_{02} + b_{11} + b_{20}) z^2}{1 - \dots}}}}$$

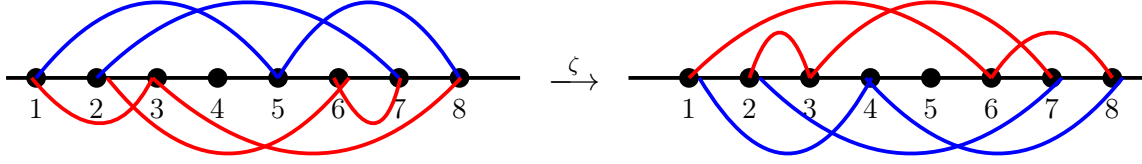


FIGURE 5. Pictorial representation of $\sigma = 57148263$ (left) and $\zeta(\sigma) = 63715824$.

(1.23)

with coefficients

$$\gamma_n = \sum_{\ell=0}^{n-1} c_{\ell, n-1-\ell} + \sum_{\ell=0}^{n-1} d_{n-1, \ell} + \lambda e_n \quad (1.24a)$$

$$\beta_n = (\lambda + n - 1) a_{n-1} \sum_{\ell=0}^{n-1} b_{\ell, n-1-\ell} \quad (1.24b)$$

For $\sigma \in \mathfrak{S}_n$, we define the *complementation* σ^c and *reversal* σ^r of σ by

$$\sigma^c(i) = n + 1 - \sigma(i) \quad \text{and} \quad \sigma^r(i) = \sigma(n + 1 - i) \quad \text{for } i \in [n]. \quad (1.25)$$

Let $\zeta : \sigma \mapsto \tau$ be the transformation by reversal combined with complementation, i.e.,

$$\zeta(\sigma) = (\sigma^r)^c. \quad (1.26)$$

By the pictorial representation of σ we can visualize the operation ζ by a geometric interpretation, i.e., rotate a graphical representation of a permutation by 180 degrees, see Figure 5.

Lemma 1.12. For $\sigma \in \mathfrak{S}_n$, we have

$$(\text{drop}, \text{pdrop}, \text{fix})\sigma = (\text{exc}, \text{pex}, \text{fix})\zeta(\sigma). \quad (1.27)$$

Proof. This is obvious by the geometric interpretation of ζ . \square

We derive a dual version of Theorem 1.11 from (1.22).

Proposition 1.13 (Dual form of Theorem 1.11). We have

$$\begin{aligned} \widehat{Q}_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \lambda) = & \sum_{\sigma \in \mathfrak{S}_n} \lambda^{\text{cyc}(\sigma)} \prod_{i \in \text{Cval}} b_{\text{ucross}(i, \sigma), \text{unest}(i, \sigma)} \prod_{i \in \text{Cpeak}} a_{\text{lcross}(i, \sigma) + \text{lnest}(i, \sigma)} \times \\ & \prod_{i \in \text{Cdfall}} d_{\text{lcross}(i, \sigma) + \text{lnest}(i, \sigma), \text{lnest}(\sigma^{-1}(i), \sigma)} \prod_{i \in \text{Cdrise}} c_{\text{ucross}(i, \sigma), \text{unest}(i, \sigma)} \prod_{i \in \text{Fix}} e_{\text{lev}(i, \sigma)}. \end{aligned} \quad (1.28)$$

Proof. For $\sigma \in \mathfrak{S}_n$ let $\tau = \zeta(\sigma)$ be the reversal combined with complementation of σ . For $i \in [n]$ let $i^c = n + 1 - i$, then the following properties are obvious (Figure 5)

- $i \in \text{Cval}(\tau) \Leftrightarrow i^c \in \text{Cpeak}(\sigma)$;
- $i \in \text{Cdrise}(\tau) \Leftrightarrow i^c \in \text{Cdfall}(\sigma)$;
- $i \in \text{Fix}(\tau) \Leftrightarrow i^c \in \text{Fix}(\sigma)$;

and

- $\text{ucross}(i, \tau) = \text{lcross}(i^c, \sigma)$ and $\text{lcross}(i, \tau) = \text{ucross}(i^c, \sigma)$;
- $\text{unest}(i, \tau) = \text{lrest}(i^c, \sigma)$ and $\text{lrest}(i, \tau) = \text{unest}(i^c, \sigma)$;
- $\text{lev}(i, \tau) = \text{lev}(i^c, \sigma)$;
- $\text{unest}(\tau^{-1}(i), \tau) = \text{lrest}(\sigma^{-1}(i^c), \sigma)$ and $\text{lrest}(\tau^{-1}(i), \tau) = \text{unest}(\sigma^{-1}(i^c), \sigma)$.

Moreover, the geometric interpretation of ζ implies straightforwardly that $\text{cyc}(\tau) = \text{cyc}(\sigma)$. Thus we derive Eq.(1.28) from Eq. (1.22). \square

2. PROOF OF THE MAIN RESULTS

2.1. Proof of Theorem 1.3.

Lemma 2.1. *For $\sigma \in \mathfrak{S}_n$, we have*

- a) $\text{exc } \sigma = \text{cval } \sigma + \text{cdrise } \sigma$;
- b) $i \in [n]$ is a pure excedance of σ if and only if i is a cval and $\text{ucross}(i, \sigma) = 0$;
- c) $i \in [n]$ is an eareccpeak (ear) of σ if and only if i is a cpeak and $\text{lrest}(i, \sigma) = 0$.

Proof. Let $\sigma \in \mathfrak{S}_n$ and $i \in [n]$.

- a) If i is an excedance, i.e., $\sigma(i) > i$, then $\sigma^{-1}(i) > i$ or $\sigma^{-1}(i) < i$, namely i is either a double rise or a cycle valley. Inversely, if i is a double rise or a cycle valley, then $\sigma(i) > i$.
- b) By definition, $i \in [n]$ is a pure excedance of σ if and only if $\sigma(i) > i$, $\sigma^{-1}(i) > i$ and $\forall j < i$ we have $\sigma(j) \notin [i, \sigma(i)]$, that means i is a cval and $\text{ucross}(i, \sigma) = 0$.
- c) By definition, $i \in [n]$ is an eareccpeak of σ if and only if there are $j < i$ and $k < i$ such that $\sigma(j) = i$ and $\sigma(i) = k$ and $\forall l > i$, $\sigma(l) > \sigma(i)$, that is, i is an exclusive anitirecord and cycle peak with $\text{lrest}(i, \sigma) = 0$.

\square

Now we are ready to prove Theorem 1.3. Our idea is to first specialize the J-fractions in Theorems 1.10 and 1.11 to obtain Eq. (1.2) and then apply the corresponding combinatorial interpretations.

- a) For Eq. (1.4a) we obtain the J-fraction (1.2) by taking the following substitutions in (1.19),

$$\begin{cases} \mathbf{a}_{\ell, \ell'} &= t (\ell > 0), & \mathbf{a}_{0, \ell'} &= \lambda t; \\ \mathbf{b}_{\ell, \ell'} &= 1 (\ell' > 0), & \mathbf{b}_{\ell, 0} &= y; \\ \mathbf{c}_{\ell, \ell'} &= 1, & \mathbf{d}_{\ell, \ell'} &= t, & \mathbf{e}_{\ell} &= w. \end{cases} \quad (2.1)$$

Then it is easy to see that Eq. (1.18) reduces to the combinatorial interpretation (1.4a) by Lemma 2.1.

- b) For (1.4b) we obtain the J-fraction (1.2) by taking the following substitutions in (1.23),

$$\begin{cases} a_\ell &= t, & b_{\ell,\ell'} &= 1 (\ell' > 0), & b_{\ell,0} &= y; \\ c_{\ell,\ell'} &= 1, & d_{\ell,\ell'} &= t, & e_\ell &= w/\lambda, \end{cases} \quad (2.2)$$

Then Eq. (1.22) reduces to the combinatorial interpretation (1.4b) by Lemma 2.1.

- c) For (1.4c) we first obtain the J-fraction (1.2) by taking the following substitutions in (1.23),

$$\begin{cases} a_\ell &= 1, & b_{\ell,\ell'} &= t (\ell > 0), & b_{0,\ell'} &= ty; \\ c_{\ell,\ell'} &= t, & d_{\ell,\ell'} &= 1, & e_\ell &= w/\lambda, \end{cases} \quad (2.3)$$

Then Eq. (1.28) reduces to the combinatorial interpretation (1.4c) by Lemma 2.1 and Eq. (1.1). □

2.2. Proof of Theorem 1.5. For $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) \in \mathfrak{S}_n$ with convention $\infty-0$, i.e., $\sigma(0) = \infty$ and $\sigma(n+1) = 0$, an index $i < n$ is an ascent if $\sigma(i) < \sigma(i+1)$ and an ascent of type 2 if $\sigma(i)$ is also a left-to-right minimum. We denote the number of ascents (resp. ascents of type 2 and left-to-right minima) of σ by $\text{asc } \sigma$ (resp. $\text{asc}_2 \sigma$ and lrmin). A double descent $\sigma(i)$ ($i \in [n]$), i.e., $\sigma(i-1) > \sigma(i) > \sigma(i+1)$, is called a *foreminimum* of σ if it is also a left-to-right minimum. Denote the number of foreminima of σ by $\text{fmin } \sigma$. It is routine to verify

$$(\text{des}_2, \text{des}, \text{fmax}, \text{rec}) \sigma = (\text{asc}_2, \text{asc}, \text{fmin}, \text{lrmin}) \sigma^c, \quad (2.4)$$

where σ^c is the complementation of σ (see (1.25)).

Lemma 2.2. *There is a bijection $\varphi : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ such that*

$$(\text{des}_2, \text{des}, \text{fmax}, \text{rec}) \varphi(\sigma) = (\text{pcyc}, \text{exc}, \text{fix}, \text{cyc}) \sigma. \quad (2.5)$$

Proof. We recall a variation of Foata's transformation *fondamentale* $\phi : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$, see [9] and [25, p.23]. Starting from a permutation $\sigma \in \mathfrak{S}_n$ we factorize it into disjoint cycles, say $\sigma = C_1 C_2 \dots C_k$, where each cycle C_i is written as a sequence $C_i = (a_i, \sigma(a_i), \dots, \sigma^{l-1}(a_i))$ with $\sigma^l(a_i) = a_i$ for some $l \in [n]$ and $i \in [k]$. We say that the factorization is *standard* if

- the letter a_i is the smallest of the cycle C_i ,
- the sequence $a_1 > \dots > a_k$ is decreasing.

The permutation $\phi(\sigma)$ is obtained by dropping the parentheses in the standard factorization of σ . Note that $\text{asc}_2 \phi(\sigma) = \text{pcyc}(\sigma)$. Indeed, an index i is an ascent of type 2 of $\tau := \phi(\sigma)$ if and only if $\tau(i)$ is the smallest element of a pure cycle of σ . Thus the transformation ϕ has the following property:

$$(\text{asc}_2, \text{asc}, \text{fmin}, \text{lrmin}) \phi(\sigma) = (\text{pcyc}, \text{exc}, \text{fix}, \text{cyc}) \sigma. \quad (2.6)$$

Let $\varphi(\sigma) = (\phi(\sigma))^c$. We obtain (2.5) by combining (2.6) and (2.4). \square

Remark. Since $\text{des}_2 = \text{rec} - \text{fmax}$ and $\text{pcyc} = \text{cyc} - \text{fix}$, the equality $\text{rec}(\varphi(\sigma)) = \text{cyc}(\sigma)$ is redundant in (2.5).

For example, for $\sigma = 23146875 \in \mathfrak{S}_8$, then $\text{stan}(\sigma) = (7)(568)(4)(123)$ and $\phi(\sigma) = 75684123$ and $\varphi(\sigma) = 24315876$, it is easy to see that

$$\begin{aligned} (\text{des}_2, \text{des}, \text{fmax}, \text{rec})\varphi(\sigma) &= (\text{asc}_2, \text{asc}, \text{fmin}, \text{lrn}) \phi(\sigma) \\ &= (\text{pcyc}, \text{exc}, \text{fix}, \text{cyc}) \sigma = (2, 4, 2, 4). \end{aligned}$$

Proof of Theorem 1.5. The identities (1.7a), (1.7b) and (1.7c) follow directly from Theorem 1.3. By Lemma 2.2 we derive Eq. (1.7d). Finally, the equivalence of (1.5), (1.7e) and (1.7a) follows from [26, Theorem 1] and [26, Theorem 3]. \square

2.3. Proof of Theorem 1.8. As $D_n(t, \lambda, y) = A_n(t, \lambda, y, 0)$, it follows from (1.2) that

$$\sum_{n \geq 0} z^n D_n(t, \lambda, y) = \frac{1}{1 - 0z - \frac{t\lambda y z^2}{1 - (t+1)z - \frac{t(\lambda+1)(y+1)z^2}{\dots}}} \quad (2.7)$$

with $\gamma_n = n(t+1)$ and $\beta_n = t(\lambda+n-1)(y+n-1)$.

Clearly, Theorem 1.8 is proved if we show that for each combinatorial interpretation of $\gamma_{n,k}(\lambda, y)$ in (1.12a)-(1.12c) the corresponding generating function has the following J-fraction expansion

$$\sum_{n \geq 0} z^n \sum_{k \geq 0} \gamma_{n,k}(\lambda, y) t^k = \frac{1}{1 - 0z - \frac{t\lambda y z^2}{1 - 1 \cdot z - \frac{t(\lambda+1)(y+1)z^2}{\dots}}} \quad (2.8)$$

with $\gamma_n = n$ and $\beta_n = t(\lambda+n-1)(y+n-1)$.

a) For Eq.(1.12a), taking the following substitutions in in two sides of (1.19),

$$\begin{cases} \mathbf{a}_{\ell, \ell'} &= t (\ell > 0), & \mathbf{a}_{0, \ell'} &= \lambda t; \\ \mathbf{b}_{\ell, \ell'} &= 1 (\ell' > 0), & \mathbf{b}_{\ell, 0} &= y; \\ \mathbf{c}_{\ell, \ell'} &= 1, & \mathbf{d}_{\ell, \ell'} &= 0, & \mathbf{e}_\ell &= 0, \end{cases} \quad (2.9)$$

we obtain

$$\sum_{n \geq 0} z^n \sum_{\sigma \in \mathcal{D}_n^*} t^{\text{exc } \sigma} \lambda^{\text{pex } \sigma} y^{\text{ear } \sigma} = \frac{1}{1 - 0z - \frac{t\lambda y z^2}{1 - 1 \cdot z - \frac{t(\lambda+1)(y+1)z^2}{\dots}}} \quad (2.10)$$

with $\gamma_n = n$ and $\beta_n = t(\lambda+n-1)(y+n-1)$.

b) For Eq.(1.12b), taking the following substitutions in two sides of (1.23),

$$\begin{cases} \mathbf{a}_\ell &= t, & \mathbf{b}_{\ell,\ell'} = 1 (\ell' > 0), & \mathbf{b}_{\ell,0} = y; \\ \mathbf{c}_{\ell,\ell'} &= 1, & \mathbf{d}_{\ell,\ell'} = 0, & \mathbf{e}_\ell = 0, \end{cases} \quad (2.11)$$

we obtain the same J-fraction for $\sum_{\sigma \in \mathcal{D}_n^*} t^{\text{exc } \sigma} \lambda^{\text{cyc } \sigma} y^{\text{ear } \sigma}$.

c) For Eq.(1.12c), we use the dual form (1.28) for the combinatorial interpretation of $\widehat{Q}_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \lambda)$. Taking the following substitutions in two sides of (1.23)

$$\begin{cases} \mathbf{a}_\ell &= 1, & \mathbf{b}_{\ell,\ell'} = t (\ell' > 0), & \mathbf{b}_{\ell,0} = ty; \\ \mathbf{c}_{\ell,\ell'} &= 0, & \mathbf{d}_{\ell,\ell'} = 1, & \mathbf{e}_\ell = 0, \end{cases} \quad (2.12)$$

we obtain the same J-fraction for $\sum_{\sigma \in \mathcal{D}_n^*} t^{\text{exc } \sigma} \lambda^{\text{cyc } \sigma} y^{\text{pex } \sigma}$.

□

2.4. Proof of Theorem 1.9. Given a permutation $\sigma = \sigma(1) \cdots \sigma(n)$, if i is a descent, i.e., $\sigma(i) > \sigma(i+1)$, the letters $\sigma(i)$ and $\sigma(i+1)$ are called *descent top* and *descent bottom*, respectively. For entry $i \in \{1, \dots, n\}$, we define the refined patterns, see [22],

$$(31-2)(i, \sigma) = \#\{j : 1 < j < \sigma^{-1}(i) \text{ and } \sigma(j) < i < \sigma(j-1)\}; \quad (2.13a)$$

$$(2-31)(i, \sigma) = \#\{j : \sigma^{-1}(i) < j < n \text{ and } \sigma(j+1) < i < \sigma(j)\}; \quad (2.13b)$$

and also the refined cycle statistics:

$$\text{icross}(i, \sigma) = \begin{cases} \text{ucross}(i, \sigma) & \text{if } i \in \text{Cval } \sigma \cup \text{Cdrise } \sigma; \\ \text{lcross}(i, \sigma) & \text{if } i \in \text{Cpeak } \sigma \cup \text{Fix } \sigma; \\ \text{lcross}(i, \sigma) + 1 & \text{if } i \in \text{Cdfall } \sigma. \end{cases} \quad (2.13c)$$

$$\text{cross}(i, \sigma) = \begin{cases} \text{ucross}(i, \sigma) & \text{if } i \in \text{Cval } \sigma \cup \text{Fix } \sigma; \\ \text{ucross}(i, \sigma) + 1 & \text{if } i \in \text{Cdrise } \sigma; \\ \text{lcross}(i, \sigma) & \text{if } i \in \text{Cpeak } \sigma \cup \text{Cdfall } \sigma. \end{cases} \quad (2.13d)$$

$$\text{nest}(i, \sigma) = \begin{cases} \text{unest}(i, \sigma) & \text{if } i < \sigma(i); \\ \text{lnest}(i, \sigma) & \text{if } i > \sigma(i); \\ \text{lev}(i, \sigma) & \text{if } i = \sigma(i). \end{cases} \quad (2.13e)$$

Note that $\text{ucross}(j, \sigma)$ and $\text{unest}(j, \sigma)$ can be nonzero only when j is a cycle valley or a cycle double rise, while $\text{lcross}(k, \sigma)$ and $\text{lnest}(k, \sigma)$ can be nonzero only when k is a cycle peak or a cycle double fall. Thus $\text{icross}(i, \sigma) = \text{cross}(i, \sigma) = 0$ if $i \in \text{Fix } \sigma$.

2.4.1. *The bijection Φ_1 .* Given a permutation $\sigma = \sigma(1) \cdots \sigma(n)$, we proceed as follows:

- Determine the sets of descent bottoms F and descent tops F' , and their complements G and G' , respectively;
- let f and g be the increasing arrangements of F and G , respectively;

- construct the biword $\binom{f}{f'}$: for each $j \in f$ starting from the **largest** (right), the entry in f' below j is the $((31 - 2)(j, \sigma) + 1)$ th largest entry of F' that is larger than j and not yet chosen.
- construct the biword $\binom{g}{g'}$: for each j in g starting from the **smallest** (left), the entry in g' below j is the $((31 - 2)(j, \sigma) + 1)$ th smallest entry of G' that is not larger than j and not yet chosen;
- Form the biword $w = \binom{f \ g}{f' \ g'}$ by concatenating the biwords $\binom{f}{f'}$, and $\binom{g}{g'}$. Sorting the columns so that the top row is in increasing order, we obtain the permutation $\tau = \Phi_1(\sigma)$ as the bottom row of the rearranged biword.

Example 2.3. For $\sigma = 4 \ 7 \ 1 \ 8 \ 6 \ 3 \ 2 \ 5$ we have

$\sigma =$	4	7	1	8	6	3	2	5	Φ_1	$\tau =$	8	3	6	1	5	7	2	4
$(31 - 2)(i, \sigma)$	0	0	0	0	1	1	1	2		nest(i, τ)	0	1	1	0	2	0	1	0
$(2 - 31)(i, \sigma)$	2	1	0	0	0	0	0	0		icross(i, τ)	0	0	0	0	0	1	0	2

(1) Determine the descent bottoms and descent tops of σ and their complements:

$$F = \{1, 2, 3, 6\}, \quad F' = \{3, 6, 7, 8\},$$

$$G = \{4, 5, 7, 8\}, \quad G' = \{1, 2, 4, 5\}.$$

(2) Compute the statistics $(31 - 2)(i, \sigma)$ and $(2 - 31)(i, \sigma)$ for $i = 1, \dots, 8$, see the above table (left).

(3) Form the biwords, for clarity we write the non-zero $(31 - 2)(i, \sigma)$ numbers in f and g , respectively, as subscripts of their corresponding letters:

$$\binom{f}{f'} = \binom{1 \ 2_1 \ 3_1 \ 6_1}{8 \ 3 \ 6 \ 7}, \quad \binom{g}{g'} = \binom{4 \ 5_2 \ 7 \ 8}{1 \ 5 \ 2 \ 4}.$$

(4) By concatenation

$$\tau = \binom{f \ g}{f' \ g'} = \binom{1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8}{8 \ 3 \ 6 \ 1 \ 5 \ 7 \ 2 \ 4}.$$

Remark 2.4. We can order f' and g' in the similar way as in [4]. Recall that an *inversion top number* (resp. *inversion bottom number*) of a letter $x := \sigma(i)$ in the word σ is the number of occurrences of inversions of form (i, j) (resp. (j, i)). Then f' (resp. g') is the permutation of descent tops (resp. nondescent tops) in σ such that the inversion bottom (resp. top) number of each letter x in f' (resp. g') that below i in f (resp. g) is $(31 - 2)(i, \sigma)$.

Lemma 2.5. Let $\Phi_1(\sigma) = \tau$ for $\sigma \in \mathfrak{S}_n$. For $i \in \{1, \dots, n\}$,

$$\text{nest}(i, \tau) = (31 - 2)(i, \sigma); \tag{2.14a}$$

$$\text{icross}(i, \tau) = (2 - 31)(i, \sigma). \tag{2.14b}$$

Lemma 2.6. The mapping Φ_1 is a bijection on \mathfrak{S}_n .

Proof. We construct the inverse Φ_1^{-1} similarly as in [4]. Let $\Phi_1(\sigma) = \tau$. Form two biwords $\binom{f}{f'}$ and $\binom{g}{g'}$, where f (resp. f', g, g') is the set of excedance positions (resp. excedance values, nonexcedance positions, nonexcedance values) of τ , with f and g ordered increasingly and $\binom{i}{j}$ is a column if $\tau(i) = j$. We get σ by constructing the descent blocks of σ , see [4].

For $i = 1, \dots, n$, we say that i is an opener (resp. closer, insider, outsider) of τ if i is a letter in the word fg' (resp. $f'g, ff, gg'$).

If 1 is an outsider, then put 1 as a block. And if 1 is an opener, then put $(\infty, 1)$ as an uncomplete block. For $i \geq 2$, we proceed as follows:

- If i is an opener, then put the uncomplete block (∞, i) to the left of $(\text{nest}(i, \tau) + 1)$ th uncomplete block from the left.
- If i is an outsider, then put (i) as a block to the left of $(\text{nest}(i, \tau) + 1)$ th uncomplete block from the left.
- If i is an insider, then put i into $(\text{nest}(i, \tau) + 1)$ th uncomplete block from the left and to the right of ∞ .
- if i is a closer, then replace the ∞ of $(\text{nest}(i, \tau) + 1)$ th uncomplete block by i , then the block is complete.

After the entry n has been constructed, the blocks are all complete and remove all parenthesis. Reading the entries from left to right, we immediately get the permutation σ . Let $\Phi'_1(\tau) = \sigma$. Then we prove $\sigma = \sigma$, then $\Phi^{-1} = \Phi'_1$. The set of descent bottoms (resp. descent tops, nondescent bottoms, nondescent tops) in σ transfers to the set of excedance position (resp. excedance value, nonexcedance position, nonexcedance value) in τ by the operation Φ_1 , and by Φ'_1 it transfer to set of descent top (resp. descent bottom, nondescent top, nondescent bottom) in σ . That is the set of descent top, descent bottom, nondescent top and nondescent bottom in σ are the same as those in σ . Similarly we get $(31 - 2)(j, \sigma) = (31 - 2)(j, \sigma)$ and $(2 - 31)(j, \sigma) = (2 - 31)(j, \sigma)$ for $j \in [n]$.

Suppose $\sigma(j) = \sigma(j)$ for $j = 1, 2, \dots, i-1$, and $\sigma(i) \neq \sigma(i)$. Assume $\sigma(i) = k < l = \sigma(i)$. If $\sigma(i-1)$ is a descent top, then k, l are descent bottoms. Then for l in σ and σ , since $\sigma(i-1) > k > l$, we have $(31 - 2)(l, \sigma) > (31 - 2)(l, \sigma)$, contradictory to $(31 - 2)(l, \sigma) = (31 - 2)(l, \sigma)$. If $\sigma(i-1)$ is a nondescent top, then k, l are nondescent bottoms. Suppose $\sigma(j) = k$, then $\sigma(j-1) < k$ since k is a nondescent bottom. And since $l > k$, then there exist two consecutive index $i < x, x+1 < j$ such that $\sigma(x) > k > \sigma(x+1)$. Then we have $(31 - 2)(k, \sigma) < (31 - 2)(k, \sigma)$, contradictory to $(31 - 2)(k, \sigma) = (31 - 2)(k, \sigma)$. We complete the proof. \square

Example 2.7. We illustrate Φ_1^{-1} on $\tau = 8\ 3\ 6\ 1\ 5\ 7\ 2\ 4$.

$$\begin{aligned}
& (\infty, 1) \longrightarrow (\infty, 1)(\infty, 2) \\
& \longrightarrow (\infty, 1)(\infty, 3, 2) \longrightarrow (4)(\infty, 1)(\infty, 3, 2) \\
& \longrightarrow (4)(\infty, 1)(\infty, 3, 2)(5) \longrightarrow (4)(\infty, 1)(\infty, 6, 3, 2)(5) \\
& \longrightarrow (4)(7, 1)(\infty, 6, 3, 2)(5) \longrightarrow (4)(7, 1)(8, 6, 3, 2)(5).
\end{aligned}$$

Then we have $\Phi_1^{-1}(\tau) = \sigma = 4\ 7\ 1\ 8\ 6\ 3\ 2\ 5$.

Lemma 2.8. *Let $\Phi_1(\sigma) = \tau$ for $\sigma \in \mathfrak{S}_n$. Then*

$$(\text{cpeak}, \text{cval}, \text{cdrise}, \text{cdfall} + \text{fix}) \tau = (\text{peak}, \text{val}, \text{ddes}, \text{dasc}) \sigma, \tag{2.15}$$

$$(\text{des}, \text{des}_2) \sigma = (\text{exc}, \text{ear}) \Phi_1(\sigma). \tag{2.16}$$

Proof. We just prove (2.15). If j is a peak of σ , then j is a descent top and nondescent bottom, so j is in f' and g . Since in $\binom{f}{f'}$, $\tau(k) = j > k$, and in $\binom{g}{g'}$, $\tau(j) = l < j$, then j is a cycle peak in τ . If j is a valley in σ , then j is a descent bottom of σ and nondescent top of σ , so j is in f and g' , so j is a cycle valley of τ . Similarly we have if j is a double descent of σ , then j is in f and f' , so j is a cycle double rise in τ . And if j is a double ascent of σ , then j is in g and g' , so j is a cycle double fall or a fixed point in τ . Let $\Phi_1(\sigma) = \tau$. It is not difficult to check that $\text{des} = \text{peak} + \text{ddes} = \text{val} + \text{ddes}$ and by Lemma 2.1, $\text{exc} = \text{cval} + \text{cdrise}$. So by Lemma 2.5 we have $\text{exc}(\tau) = \text{des}(\sigma)$. If i is a left-to-right maximum of σ , then $(31 - 2)(i, \sigma) = 0$. By definition of descent of type 2, $i \in \text{Des}_2(\sigma)$ if and only if i is a descent and left to right maximum, then i is a peak and $(31 - 2)(i, \sigma) = 0$. By Lemma 2.5 we have $i \in \text{Cpeak}(\tau)$ and $\text{lnest}(i, \tau) = 0$, that is i is an eareccpeak. If $i \notin \text{Des}_2(\sigma)$, then either i is not a peak or $(31 - 2)(i, \sigma) \neq 0$. Then we have $i \notin \text{Cpeak}(\tau)$ or $\text{lnest}(i, \tau) \neq 0$, that is i is not an eareccpeak. So we have $\text{ear}(\tau) = \text{des}_2(\sigma)$. \square

2.4.2. *The bijection Φ_2 .* We first recall the bijection Φ_{SZ} of Shin and Zeng [21], which is a variation of the bijection Φ in [4]. Given a permutation $\sigma = \sigma(1) \cdots \sigma(n)$, we proceed as follows:

- Determine the sets of descent tops F and descent bottoms F' , and their complements G and G' , respectively;
- let f and g be the increasing permutations of F and G , respectively;
- construct the biword $\binom{f}{f'}$: for each j in the first row f starting from the **smallest** (left), the entry in f' that below j is the $((31 - 2)(j, \sigma) + 1)$ th largest entry of f' that is smaller than j and not yet chosen.
- construct the biword $\binom{g}{g'}$: for each j in the first row g starting from the **largest** (right), the entry below j in g' is the $((31 - 2)(j, \sigma) + 1)$ th smallest entry of g' that is not smaller than j and not yet chosen.
- Rearranging the columns so that the top row is in increasing order, we obtain the permutation $\tau = \Phi_{SZ}(\sigma)$ as the bottom row of the rearranged biword.

Example 2.9. For $\sigma = 4\ 7\ 1\ 8\ 6\ 3\ 2\ 5$ with

$$\begin{array}{c|c|c|c|c|c|c|c|c} \sigma = & 4 & 7 & 1 & 8 & 6 & 3 & 2 & 5 \\ \hline (31 - 2)(i, \sigma) & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ \hline (2 - 31)(i, \sigma) & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \xrightarrow{\Phi_{SZ}} \begin{array}{c|c|c|c|c|c|c|c|c} \tau = & 5 & 7 & 1 & 4 & 8 & 2 & 6 & 3 \\ \hline \text{cross}(i, \tau) & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline \text{nest}(i, \tau) & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \end{array}.$$

We have

$$\binom{f}{f'} = \begin{pmatrix} 3_1 & 6_1 & 7 & 8 \\ 1 & 2 & 6 & 3 \end{pmatrix}, \quad \binom{g}{g'} = \begin{pmatrix} 1 & 2_1 & 4 & 5_2 \\ 5 & 7 & 4 & 8 \end{pmatrix}.$$

Hence

$$\tau = \begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 1 & 4 & 8 & 2 & 6 & 3 \end{pmatrix}.$$

Similar to Lemma 2.5 and Lemma 2.6 we have the following result in [21].

Lemma 2.10. *The mapping $\Phi_{SZ} : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ is a bijection. For $\sigma \in \mathfrak{S}_n$, if $\Phi_{SZ}(\sigma) = \tau$, then for $i \in [n]$,*

$$(31-2)(i, \sigma) = \text{cross}(i, \tau), \quad (2.17a)$$

$$(2-31)(i, \sigma) = \text{nest}(i, \tau). \quad (2.17b)$$

By Lemma 2.1, if $i \in \text{Pex}(\sigma)$, then $i \in \text{Cval}(\sigma)$ and $\text{ucross}(i, \sigma) = 0$. And by definition of "pdrop", we see that if $i \in \text{Pdrop}(\sigma)$, then $i \in \text{Cpeak}(\sigma)$ and $\text{lcross}(i, \sigma) = 0$. Recall that Fix (resp. Valley, Peak, Ddes, Dasc, Fmax) is the set valued statistic, namely Fix σ denotes the set of fixed points of σ .

Lemma 2.11. *We have $(\text{des}, \text{des}_2, \text{fmax})\sigma = (\text{drop}, \text{pdrop}, \text{fix})\Phi_{SZ}(\sigma)$.*

Proof. Let $\Phi_{SZ}(\sigma) = \tau$. Then similar to the proof of Lemma 2.8, we prove that

$$(\text{Cval}, \text{Cpeak}, \text{Cdfall}, \text{Cdrise} + \text{Fix}, \text{Fix})\tau = (\text{Val}, \text{Peak}, \text{Ddes}, \text{Dasc}, \text{Fmax})\sigma. \quad (2.18)$$

Note that if i is a foremaximum, then i is a double ascent and $(31-2)(i, \sigma) = 0$. Then by the bijection Φ_{SZ} , i is in g and g' . Since in g the entry is sorted from the largest, for all the entries $j > i$ in g , we have $\tau(j) \geq j > i$. We obtain that when i is sorted, i is not chosen and the smallest entry that are not chosen is i . By $(31-2)(i, \sigma) = 0$, we get i is a fixed point. It is not difficult to check that $\text{des} = \text{peak} + \text{ddes}$ and $\text{drop} = \text{cpeak} + \text{cdfall}$. By (2.18) we have $\text{drop}(\tau) = \text{des}(\sigma)$. If i is a descent of type 2, then i is a peak and $(31-2)(i, \sigma) = 0$. By (2.18) and Lemma 2.10 we have $i \in \text{Cpeak}(\tau)$ and $\text{lcross}(i, \tau) = 0$, that is i is a pure drop. If i is not a descent of type 2, then we have $i \notin \text{Cpeak}(\tau)$ or $\text{lcross}(i, \tau) \neq 0$, that is i is not a pure drop. So we have $\text{pdrop}(\tau) = \text{des}_2(\sigma)$. \square

Let $\Phi_2 = \zeta \circ \Phi_{SZ}$, where ζ is the *reversal and complementation* operation in (1.26). Combining Lemma 2.11 and Lemma 1.12 we obtain Eq. (1.14b). An illustration for the permutation τ in Example 2.9 is given in Figure 5.

3. INTERPRETATIONS USING LINEAR STATISTICS

The polynomials $Q_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e})$ being defined using cyclic statistics in (1.18), by the bijection Φ_{SZ} , Lemma 2.10 and Eq. (2.18), we derive the following interpretation using linear statistics.

Theorem 3.1. *(First linear version of Q_n)*

$$Q_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) =$$

$$\sum_{\sigma \in \mathfrak{S}_n} \prod_{i \in \text{Val}} \mathbf{a}_{(31-2)(i, \sigma), (2-31)(i, \sigma)} \prod_{i \in \text{Peak}} \mathbf{b}_{(31-2)(i, \sigma), (2-31)(i, \sigma)} \times$$

$$\prod_{i \in \text{Ddes}} c_{(31-2)(i,\sigma), (2-31)(i,\sigma)} \prod_{i \in \text{Dasc} \setminus \text{Fmax}} d_{(31-2)(i,\sigma)-1, (2-31)(i,\sigma)} \prod_{i \in \text{Fmax}} e_{(2-31)(i,\sigma)} \quad (3.1)$$

For $\sigma \in \mathfrak{S}_n$, a value $\sigma(i)$ is an *antirecord double ascent* (Arda) if it is an antirecord and at the same time a double ascent. Applying the bijection Φ_1 we obtain another interpretation of the polynomials Q_n .

Theorem 3.2. (*Second linear version of Q_n*)

$$Q_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) =$$

$$\sum_{\sigma \in \mathfrak{S}_n} \prod_{i \in \text{Val}} a_{(2-31)(i,\sigma), (31-2)(i,\sigma)} \prod_{i \in \text{Peak}} b_{(2-31)(i,\sigma), (31-2)(i,\sigma)} \times \\ \prod_{i \in \text{Dasc} \setminus \text{Arda}} c_{(2-31)(i,\sigma)-1, (31-2)(i,\sigma)} \prod_{i \in \text{Ddes}} d_{(2-31)(i,\sigma), (31-2)(i,\sigma)} \prod_{i \in \text{Arda}} e_{(31-2)(i,\sigma)}. \quad (3.2)$$

Proof. In view of Lemma 2.10 we just need to show that

$$\text{Arda } \sigma = \text{Fix } \Phi_1(\sigma).$$

If i is an antirecord, then $(2-31)(i,\sigma) = 0$. Thus if i is a Arda, then i is a double ascent and $(2-31)(i,\sigma) = 0$. Then by the bijection Φ_1 , i is in g and g' . Since in g the entry is sorted from the smallest, for all the entries $j < i$ in g , we have $\tau(j) \leq j < i$. We obtain that when i is sorted, i is not chosen and the largest entry that are not chosen is i . By $(2-31)(i,\sigma) = 0$, we get i is a fixed point. Suppose i is not a ARL, if i is not a double ascent, then i is either not in g or in g' . So i is not a fixed point. And if i is a double ascent, since $(2-31)(i,\sigma) \neq 0$ and i is the largest entry that are not chosen, then i is not a fixed point. \square

Remark. Using the reversal transformation $\sigma \mapsto \sigma^r$, we obtain a dual version of the above theorems for the boundary condition $\infty - 0$, i.e., $\sigma(0) = n + 1$ and $\sigma(n + 1) = 0$.

As an application of Theorem 3.1 we give a linear version for $A_n(t, \lambda, y, w)$ and its derangement counterpart $D_n(t, \lambda, y)$. This enables us to give a group action proof of (1.4a) in Theorem 1.8.

For $\sigma \in \mathfrak{S}_n$, recall that $\text{asc } \sigma$ is the number of ascents of σ (cf. (2.4)) and define the statistics *pure valley* (pval) and *pure peak* (ppeak) by

$$\text{pval } \sigma = |\{i \in [n] : i \in \text{Val } \sigma \text{ and } 31-2(i,\sigma) = 0\}|, \quad (3.3)$$

$$\text{ppeak } \sigma = |\{i \in [n] : i \in \text{Peak } \sigma \text{ and } 2-31(i,\sigma) = 0\}|. \quad (3.4)$$

Lemma 3.3. *We have*

$$A_n(t, \lambda, y, w) = \sum_{\sigma \in \mathfrak{S}_n} t^{(\text{asc} - \text{fmax}) \sigma} \lambda^{\text{pval } \sigma} y^{\text{ppeak } \sigma} w^{\text{fmax } \sigma}. \quad (3.5)$$

Proof. Applying the substitution (2.1) in Eq. (3.1) we obtain the right-hand side in (3.5) for $Q_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e})$, which is equal to $A_n(t, \lambda, y, w)$ by (1.4a). \square

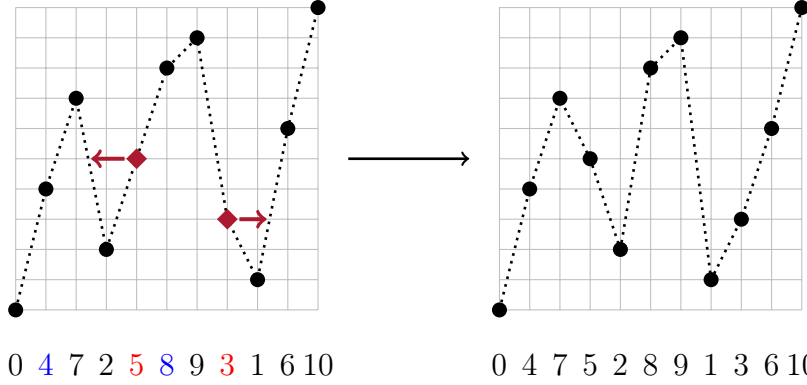


FIGURE 6. Valley-hopping on $\sigma = 472589316 \in \mathfrak{S}_9$ with $S = \{3, 4, 5\}$ yields $\varphi'_S(\sigma) = 475289136$

Let $\mathfrak{S}_{n,j} := \{\sigma \in \mathfrak{S}_n : \text{fmax}(\sigma) = j\}$, $\mathfrak{S}_{n,j}^* := \{\sigma \in \mathfrak{S}_{n,j} : \text{dd}(\sigma) = 0\}$ and

$$\mathfrak{S}_{n,j}^*(k) := \{\sigma \in \mathfrak{S}_{n,j}^* : \text{des}(\sigma) = k\}. \quad (3.6)$$

Theorem 3.4. *We have*

$$D_n(t, \lambda, y) = \sum_{\sigma \in \mathfrak{S}_{n,0}} t^{\text{asc} \sigma} \lambda^{\text{pval} \sigma} y^{\text{ppeak} \sigma} \quad (3.7)$$

and

$$D_n(t, \lambda, y) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_{n,k}(\lambda, y) t^k (1+t)^{n-2k}, \quad (3.8)$$

where the gamma coefficient $\gamma_{n,k}(\lambda, y)$ has the following combinatorial interpretation

$$\gamma_{n,k}(\lambda, y) = \sum_{\sigma \in \mathfrak{S}_{n,0}^*(k)} \lambda^{\text{pval} \sigma} y^{\text{ppeak} \sigma}. \quad (3.9)$$

Proof. As $D_n(t, \lambda, y) = A_n(t, \lambda, y, 0)$, letting $w = 0$ in (3.5) we obtain (3.7). Taking the following substitutions

$$\begin{cases} \mathbf{a}_{\ell, \ell'} = t \ (\ell > 0), & \mathbf{a}_{0, \ell'} = \lambda t; \\ \mathbf{b}_{\ell, \ell'} = 1 \ (\ell' > 0), & \mathbf{b}_{\ell, 0} = y; \\ \mathbf{c}_{\ell, \ell'} = 0, & \mathbf{d}_{\ell, \ell'} = t, \quad \mathbf{e}_{\ell} = 0, \end{cases} \quad (3.10)$$

in (1.18) and (3.1), respectively, we obtain

$$\sum_{\sigma \in \mathfrak{D}_n^*} t^{\text{exc} \sigma} \lambda^{\text{pex} \sigma} y^{\text{ear} \sigma} = \sum_{\sigma \in \mathfrak{S}_{n,0}^*} t^{\text{asc} \sigma} \lambda^{\text{pval} \sigma} y^{\text{ppeak} \sigma}. \quad (3.11)$$

Extracting the coefficient of t^k in (3.11) we deduce (3.8) from (1.4a). \square

3.1. Group action proof of (3.8). In the following we give a direct proof of (3.8) by applying the well-known *valley-hopping* action, see Foata and Strehl [10], Shapiro, Woan, and Getu [20] and Brändén [3]. Let $\sigma \in \mathfrak{S}_n$ with boundary condition $\sigma(0) = 0$ and $\sigma(n+1) = n+1$. Recall that for $x \in [n]$, see [10], the x -factorization of σ is defined by

$$\sigma = w_1 w_2 x w_3 w_4, \quad (3.12)$$

where w_2 (resp. w_3) is the maximal contiguous subword immediately to the left (resp. right) of x whose letters are all less than x . Note that w_1, \dots, w_4 may be empty. For instance, if x is a double ascent (resp. double descent), then $w_3 = \emptyset$ (resp. $w_2 = \emptyset$), and if x is a valley then $w_2 = w_3 = \emptyset$. Foata and Strehl [10] considered a mapping φ_x on permutations by exchanging w_2 and w_3 in (3.12):

$$\varphi_x(\sigma) = w_1 w_3 x w_2 w_4.$$

For instance, if $x = 3$ and $\sigma = 472589316 \in \mathfrak{S}_9$, then $w_1 = 472589, w_2 = \emptyset, w_3 = 1$ and $w_4 = 6$. Thus $\varphi_3(\sigma) = 472589136$. It is known (see [10]) that φ_x is an involution acting on \mathfrak{S}_n and that φ_x and φ_y commute for all $x, y \in [n]$. Brändén [3] introduced the modified mapping φ'_x by

$$\varphi'_x(\sigma) := \begin{cases} \varphi_x(\sigma), & \text{if } x \text{ is neither a peak nor foremaximum of } \sigma; \\ \sigma, & \text{if } x \text{ is a peak or foremaximum of } \sigma. \end{cases}$$

Note that the boundary condition matters, e.g., in the above example, if $\sigma(0) = 10$ instead, then 4 becomes a valley and will be fixed by φ'_4 . Also, we have $\varphi'_x(\sigma) = \sigma$ if x is a peak, valley or foremaximum, otherwise $\varphi'_x(\sigma)$ exchanges w_2 and w_3 in the x -factorization of σ , which is equivalent to moving x from a double ascent to a double descent or vice versa. Then φ'_x 's are involutions and commute. Hence, for any subset $S \subseteq [n]$ we can define the map $\varphi'_S : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ by

$$\varphi'_S(\sigma) = \prod_{x \in S} \varphi'_x(\sigma).$$

In other words, the group \mathbb{Z}_2^n acts on \mathfrak{S}_n via the mapping φ'_S with $S \subseteq [n]$. For example, let $\sigma = 472589316 \in \mathfrak{S}_9$, then $\text{Fmax}(\sigma) = \{4, 8\}$. If $S = \{3, 4, 5\}$, we have $\varphi'_S(\sigma) = 475289136$, see Fig. 6 for an illustration.

Recall that a permutation σ has a descent at i with $1 \leq i < n$ if $\sigma(i) > \sigma(i+1)$. We will say that a *run* of a permutation σ is a maximal interval of consecutive arguments of σ on which the values of σ are monotonic. If the values of σ increase on the interval then we speak of a rising run, else a decreasing run.

Lemma 3.5. *For $\sigma \in \mathfrak{S}_n$ the quintuple permutation statistic (peak, val, fmax, ppeak, pval) σ is invariant under the group action φ'_S with $S \subset [n]$, i.e.,*

$$(\text{peak, val, fmax, ppeak, pval}) \sigma = (\text{peak, val, fmax, ppeak, pval}) \varphi'_S(\sigma). \quad (3.13)$$

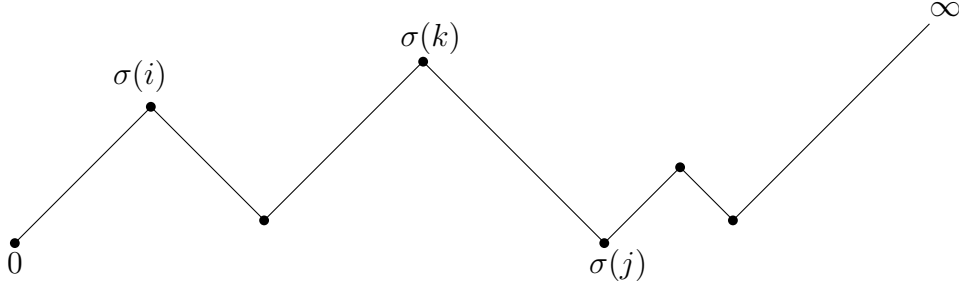


FIGURE 7. Valley-hopping on statistics ppeak

Proof. Clearly the triple statistic (peak, val, fmax) is invariant under the valley-hopping action, so it remains to verify the invariance for bi-statistic (ppeak, pval). For each $i \in [n]$, the statistic $2-31(i, \sigma)$ equals the number of maximal decreasing runs at the right of i , i.e., the sequences of values with consecutive arguments $k, k+1, \dots, j$ such that $\sigma^{-1}(i) < k < j$,

$$\sigma(k) > i > \sigma(j) \quad \text{with } \sigma(k) \in \text{Peak}(\sigma) \text{ and } \sigma(j) \in \text{Valley}(\sigma),$$

see Fig.7. Since the elements of $\text{Peak}(\sigma)$ and $\text{Valley}(\sigma)$ are fixed by the valley-hopping action, the number $2-31(i, \sigma)$ is invariant under the group action if $i \in \text{Peak}(\sigma)$. This proves the invariance of ppeak in (3.4). The case of pval is similar and omitted. \square

Lemma 3.6. *We have*

$$\sum_{\sigma \in \mathfrak{S}_{n,j}} \lambda^{\text{pval} \sigma} y^{\text{ppeak} \sigma} t^{\text{asc-fmax} \sigma} = \sum_{k=0}^{\lfloor (n-j)/2 \rfloor} \left(\sum_{\sigma \in \mathfrak{S}_{n,j}^*(k)} \lambda^{\text{pval} \sigma} y^{\text{ppeak} \sigma} \right) t^k (1+t)^{n-j-2k}. \quad (3.14)$$

Proof. For any permutation $\sigma \in \mathfrak{S}_n$, let $\text{Orb}(\sigma) = \{g(\sigma) : g \in \mathbb{Z}_2^n\}$ be the orbit of σ under the valley-hopping. The valley-hopping divides the set \mathfrak{S}_n into disjoint orbits. Moreover, for $\sigma \in \mathfrak{S}_n$, if x is a double descent of σ , then x is a double ascent of $\varphi'_x(\sigma)$. Hence, there is a unique permutation in each orbit which has no double descent. Now, let $\bar{\sigma}$ be such a unique element in $\text{Orb}(\sigma)$, then

$$\begin{aligned} \text{dasc}(\bar{\sigma}) &= n - \text{peak}(\bar{\sigma}) - \text{val}(\bar{\sigma}); \\ \text{des}(\bar{\sigma}) &= \text{peak}(\bar{\sigma}) = \text{val}(\bar{\sigma}). \end{aligned}$$

As $\text{asc} - \text{fmax} = \text{valley} + \text{dasc} - \text{fmax}$, we have

$$\sum_{\sigma' \in \text{Orb}(\sigma)} t^{(\text{asc-fmax}) \sigma'} = t^{\text{val}(\bar{\sigma})} (1+t)^{(\text{dasc-fmax})(\bar{\sigma})} = t^{\text{des}(\bar{\sigma})} (1+t)^{n-2 \text{des} \sigma - \text{fmax}(\bar{\sigma})}. \quad (3.15)$$

Therefore, by Lemma 3.5, we obtain (3.14). \square

Clearly (3.8) corresponds to the $j = 0$ case of (3.14).

4. CONCLUDING REMARKS

We notice that the bivariate statistics $(\text{des}_2, \text{fix})$ and (pex, fix) are not equidistributed on \mathfrak{S}_4 and the distribution of $(\text{des}_2, \text{pex})$ over \mathfrak{S}_6 is not symmetric. Let

$$P_n(x, y) = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}_2(\sigma)} y^{\text{ear}(\sigma)}. \tag{4.1}$$

We speculate that the polynomial $P_n(x, y)$ is invariant under $x \leftrightarrow y$.

Conjecture 4.1. The distribution of $(\text{des}_2, \text{ear})$ over permutations is symmetric.

By Theorem 1.7 and Eq. (1.2), we can reformulate Conjecture 1.1 as follows.

Conjecture 4.2.

$$\sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} y^{\text{des}_2 \sigma} \lambda^{\text{cyc} \sigma} z^n = \frac{1}{1 - \lambda z - \frac{\lambda y z^2}{1 - (\lambda + 2) z - \frac{(\lambda + 1)(y + 1) z^2}{\dots}}} \tag{4.2}$$

with $\gamma_n = \lambda + 2n$ and $\lambda_n = (\lambda + n - 1)(y + n - 1)$.

The Françon-Viennot bijection Φ_{FV} and Foata-Zeilberger bijection Φ_{FZ} are two fundamental bijections from permutations to *Laguerre histories* [11, 12]. The composition $\Phi_{FZ}^{-1} \circ \Phi_{FV}$ as a bijection Φ on \mathfrak{S}_n was first characterized in [4]. Since then some variations of this bijection appeared in [5, 13, 21, 22]. Our bijections Φ_1 and Φ_2 are similar to those in [5, 21]. Besides, the two equations (1.7b) and (1.7d) ask for a bijection Φ_3 on \mathfrak{S}_n satisfying

$$(\text{des}, \text{des}_2, \text{fmax}) \sigma = (\text{exc}, \text{ear}, \text{fix}) \Phi_3(\sigma). \tag{4.3}$$

Although a bijection via Laguerre histories could be given by combining Φ_{FV} and Φ_{FZ}^{-1} , a direct bijection similar to Φ_1 and Φ_2 would be interesting.

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