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Abstract

A cycle with four blocks $C(k_1, k_2, k_3, k_4)$ is an oriented cycle formed of four blocks of lengths k_1, k_2, k_3 and k_4 respectively. We conjecture that for every positive integers k_1, k_2, k_3, k_4, there is an integer $g(k_1, k_2, k_3, k_4)$ such that every strongly connected digraph with chromatic number greater than $g(k_1, k_2, k_3, k_4)$ contains a subdivision of $C(k_1, k_2, k_3, k_4)$. As evidence, we prove this conjecture for $k_2 = k_3 = k_4 = 1$.

Key words: Chromatic number, four blocks cycle, subdivision

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1. Introduction

Throughout this paper, all graphs are considered to be simple, that is, there are no loops and no multiple edges. By giving an orientation to each edge of a graph G, the obtained oriented graph is called a digraph. Reciprocally, the graph obtained from a digraph D by ignoring the directions of its arcs is called the underlying graph of D, and denoted by $G(D)$ (a circuit of length 2 in D corresponds to one edge in $G(D)$). The chromatic number of a digraph D, denoted by $\chi(D)$, is the chromatic number of its underlying graph. A digraph D is said

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to be k-chromatic if $\chi(D) \geqslant k$. If x is a vertex of D, we denote by $N^+(x)$ (resp. $N^-(x)$) the set of vertices y such that (x, y) (resp. (y, x)) is an arc of D. The out-degree (resp. in-degree) of x is the cardinality of $N^+(x)$ (resp. $N^-(x)$).

An oriented path (resp. oriented cycle) is an orientation of a path (resp. cycle). The length of a path (resp. cycle) is the number of its arcs. An oriented path (resp. oriented cycle) is said to be directed if all its arcs have the same orientation. Given an oriented path P (resp. oriented cycle C), a block is a maximal directed subpath of P (resp. of C). We denote by $P(k_1, k_2, ..., k_n)$ (resp. $C(k_1, k_2, ..., k_n)$) the oriented path (resp. oriented cycle) formed of n blocks of lengths $k_1, k_2, ..., k_{n-1}$ and k_n respectively.

A digraph D is said to be acyclic if it contains no directed cycles. D is said to be strongly connected if for any two vertices x and y there is a directed path from x to y. Recall that a subdivision of a digraph D is a digraph D' obtained from D by replacing each arc (x, y) by an xy-dipath of length at least 1, all new paths being internally disjoint.

An important question to be asked is the following:

Problem 1. *Which are the graphs G such that every graph with sufficiently high chromatic number contains G as a subgraph?*

In this context, Erdös and Hajnal [7] proved that every graph with chromatic number at least k contains an odd cycle of length at least k. A counterpart of this theorem for even length was settled by Mihók and Schiermeyer [13]: every graph with chromatic number at least k contains an even cycle of length at least k. Further results on graphs with prescribed lengths of cycles have been obtained [9, 10, 12, 13, 15].

In their article, Cohen et al. [5] investigated a generalization of Problem 1 by considering the analogous problem for directed graphs:

Problem 2. *Which are the digraphs D such that every k-chromatic digraph*

contains D as a subdigraph?

A famous theorem by Erdös [6] states that there exist digraphs with arbitrarily large chromatic number and arbitrarily high girth. This implies that if D is a digraph containing an oriented cycle, there exist digraphs with arbitrarily high chromatic number with no subdigraph isomorphic to D. Thus the only possible candidates to answer Problem 2 are the oriented trees. Burr [4] conjectured that every $(2k-2)$-chromatic digraph contains every oriented tree T of order k, and he was able to prove that every $(k-1)^2$-chromatic digraph contains a copy of any oriented tree T of order k. The best known bound, due to Addario-Berry et al. [2], is in $(k/2)^2$. For special oriented trees, better bounds on the chromatic number are known. The most famous one, known as Gallai-Roy theorem, deals with directed paths:

Theorem 3. *(Gallai [8], Roy [14]) Every k-chromatic digraph contains a directed path of length $k-1$.*

However, for paths with two blocks, the best possible upper bound has been determined by Addario-Berry et al. as follows:

Theorem 4. *(Addario-Berry et al. [1]) Let k_1 and k_2 be positive integers such that $k_1+k_2 \geqslant 3$. Every (k_1+k_2+1)-chromatic digraph D contains any two blocks path $P(k_1,k_2)$.*

The following famous theorem of Bondy shows that the story does not stop here:

Theorem 5. *(Bondy [3]) Every strong digraph D contains a directed cycle of length at least $\chi(D)$.*

The strong connectivity assumption is indeed necessary, because there exists acyclic digraphs (transitive tournaments) with large chromatic number and no directed cycle.

Since any directed cycle of length at least k can be seen as a subdivision of the directed cycle C_k of length k, Cohen et al. [5] conjectured that Bondy's theorem can be extended to all oriented cycles:

Conjecture 6. *(Cohen et al. [5]) For every oriented cycle C, there exists a constant $f(C)$ such that every strong digraph with chromatic number at least $f(C)$ contains a subdivision of C.*

Cohen et al. [5] noticed that the strongly connected connectivity assumption is also necessary in Conjecture 6. This follows from proving the existence of acyclic digraphs with large chromatic number and no subdivisions of C for any oriented cycle C:

Theorem 7. *(Cohen et al. [5]) For any positive integers b, c, there exists an acyclic digraph D with $\chi(D) \geqslant c$ in which all oriented cycles have more than b blocks.*

In fact, Cohen et al. [5] proved Conjecture 6 in their article for cycles with two blocks. In particular, they showed that the chromatic number of strong digraphs with no subdivisions of two blocks cycles $C(k_1, k_2)$ is bounded from above by $O((k_1 + k_2)^4)$. More recently, Kim et al. [11] improved this upper bound to $O((k_1 + k_2)^2)$, and asked whether this upper bound can be improved to $O(k_1 + k_2)$, which is remained to be open.

Furthermore, Cohen et al. confirmed the above conjecture for the case of cycles with four blocks as follows:

Theorem 8. *(Cohen et al. [5]) Let D be a strongly connected digraph with no subdivisions of four blocks cycles $C(1, 1, 1, 1)$, then $\chi(D) \leqslant 24$.*

In this paper, we are interested in the chromatic number of digraphs containing no subdivisions of cycles with four blocks $C(k_1, k_2, k_3, k_4)$. In particular, we conjecture the following:

Conjecture 9. *For every positive integers k_1, k_2, k_3 and k_4, there is an integer $g(k_1, k_2, k_3, k_4)$ such that every strongly connected digraph with no subdivisions of $C(k_1, k_2, k_3, k_4)$ has chromatic number at most $g(k_1, k_2, k_3, k_4)$.*

We confirm this conjecture when $k_2 = k_3 = k_4 = 1$ (for arbitrary k_1) using

the simple notion of a maximal tree and the technique of digraphs decomposing. Precisely, we prove that $g(k_1, 1, 1, 1) = 8^3.k_1$.

2. Main Result

We introduce some basic definitions and terminologies that will be elementary to state our main theorem.

A tree is a connected graph containing no cycles. An oriented tree is an orientation of a tree. An out-tree is an oriented tree in which all vertices have in-degree at most 1. This implies that an out-tree has exactly one vertex of in-degree 0, called the source. Given a digraph D having a spanning out-tree T with source r, the level of a vertex x with respect to T, denoted by $l_T(x)$, is the length of the unique (r, x)-directed path in T. For a nonnegative integer i, we define $L_i(T) := \{x \in V(T) | l_T(x) = i\}$. For a vertex x of D, the ancestors of x are the vertices that belong to $T[r, x]$. If y is an ancestor of x with respect to T, we write $y \leqslant_T x$. In this case, the vertices x and y are said to be comparable with respect to the order $(V(T), \leqslant_T)$ induced by T. Conversely, we denote by $S(x)$ the set of vertices v of G such that x is an ancestor of v. Furthermore, T_x denotes the subtree of T rooted at x and induced by $S(x)$. For two vertices x_1 and x_2 of D, the least common ancestor z of x_1 and x_2 is the common ancestor of x_1 and x_2 having the highest level in T. Note that the latter notion is well-defined since r is a common ancestor of all vertices. An arc (x, y) of D is said to be forward with respect to T if $l_T(x) < l_T(y)$, otherwise it is called a backward arc. We say that T is a maximal out-tree of D if for every backward arc (x, y) of D, y is an ancestor of x.

The next proposition shows an interesting structural property on digraphs having a spanning out-tree:

Proposition 10. *Given a digraph D having a spanning out-tree T, then D contains a maximal out-tree.*

Proof. Initially, set $T_0 := T$. If T_0 is maximal, there is nothing to do. Otherwise there is an arc (x, y) of D which is backward with respect to T_0 such that y is not an ancestor of x. Let T_1 be the out-tree obtained from T_0 by adding (x, y) to T_0, and deleting the arc of head y in T_0. We can easily see that the level of each vertex in T_1 is at least its level in T_0, and there exists a vertex (y) whose level has strictly increased. Since the level of a vertex cannot increase infinitely, we can see that after a finite number of repeating the above process we reach an out-tree which is maximal. □

An oriented cycle C is said to be antidirected if every vertex of C has either out-degree 2 or in-degree 2 in C. That is, it is a cycle in which all blocks have length equal 1. For a positive integer k, an antidirected cycle of length at least k is denoted by $\mathcal{A}_{\geqslant k}$. In their article, Cohen et al. [5] proved the following:

Theorem 11. *(Cohen et al.[5]) Let D be an oriented graph and k be an integer greater than 1. If $\chi(D) \geqslant 8k - 7$, then D contains an antidirected cycle $\mathcal{A}_{\geqslant 2k}$ of length at least $2k$.*

Given two digraphs D_1 and D_2, $D_1 \cup D_2$ is defined as the digraph whose vertex-set is $V(D_1) \cup V(D_2)$ and whose arc-set is $A(D_1) \cup A(D_2)$. The next lemma will be useful for the coming proofs:

Lemma 12. *$\chi(D_1 \cup D_2) \leqslant \chi(D_1) \times \chi(D_2)$ for any two digraphs D_1 and D_2.*

Proof. For $i \in \{1, 2\}$, let $\phi_i : V(D_i) \longrightarrow \{1, 2, ..., \chi(D_i)\}$ be a proper $\chi(D_i)$-coloring of D_i. Define ψ, the coloring of $V(D_1 \cup D_2)$, as follows:

$$\psi(x) = \begin{cases} (\phi_1(x), 1) & x \in V(D_1) \setminus V(D_2); \\ (\phi_1(x), \phi_2(x)) & x \in V(D_1) \cap V(D_2); \\ (1, \phi_2(x)) & x \in V(D_2) \setminus V(D_1). \end{cases}$$

We may easily verify that ψ is a proper coloring of $D_1 \cup D_2$ with color-set $\{1, 2, ..., \chi(D_1)\} \times \{1, 2, ..., \chi(D_2)\}$. Consequently, it follows that $\chi(D_1 \cup D_2) \leqslant \chi(D_1) \times \chi(D_2)$. □

Now we are ready to state our main theorem:

Theorem 13. *If D is a digraph having a spanning out-tree T with no subdivisions of $C(k,1,1,1)$, then the chromatic number of D is at most $8^3.k$.*

Proof. Due to Proposition 10, we may assume without loss of generality that T is a maximal out-tree in D. For $0 \leqslant i \leqslant k-1$, let $V_i := \cup_{\alpha \geqslant 0} L_{i+\alpha k}(T)$. Define D_i to be the subdigraph of D induced by V_i, and then partition the arcs of D_i as follows:

$$A_1 := \{(x,y) | l_T(x) < l_T(y) \text{ and } x \leqslant_T y\};$$

$$A_2 := \{(x,y) | l_T(x) > l_T(y) \text{ and } y \leqslant_T x\};$$

$$A_3 := A(D_i) \setminus (A_1 \cup A_2).$$

For $0 \leqslant i \leqslant k-1$ and $j = 1,2,3$, let D_i^j be the spanning subdigraph of D_i whose arc-set is A_j.

Claim 14. $\chi(D_i^1) \leqslant 8$ *for all* $i \in \{0,1,...,k-1\}$.

Proof of Claim 14. Assume to the contrary that $\chi(D_i^1) \geqslant 9$. Then Theorem 11 implies that D_i^1 contains an antidirected cycle of length at least 4, say $\mathcal{A}_{2k}$ with $k \geqslant 2$. Set $\mathcal{A}_{2k} := x_1x_2...x_kx_{k+1}x_{k+2}...x_{2k} = S_1 \cup S_2$, where $S_1 = \{x_i \in V(\mathcal{A}_{2k}) | \, |N^+(x_i) \cap V(\mathcal{A}_{2k})| = 2\}$ and $S_2 = \{x_i \in V(\mathcal{A}_{2k}) | \, |N^-(x_i) \cap V(\mathcal{A}_{2k})| = 2\}$. Let $x_i \in S_1$ such that $l_T(x_i)$ is maximal. Assume that $N^+(x_i) \cap V(\mathcal{A}_{2k}) = \{x_\alpha, x_\beta\}$. Clearly, x_α and x_β belong to S_2. By the definition of S_2, there exists $1 \leqslant \alpha', \beta' \leqslant 2k$ other than i such that $(x_{\alpha'}, x_\alpha)$ and $(x_{\beta'}, x_\beta)$ belong to $A(D_i^1)$. Note that probably $\alpha' = \beta'$. By the definition of D_i^1, $x_{\alpha'}, x_{\beta'}$ and x_i are all comparable with respect to the order induced by T. Without loss of generality, assume that $l_T(x_{\beta'}) < l_T(x_{\alpha'})$ and $l_T(x_\alpha) \leqslant l_T(x_\beta)$. Thus by the maximality of $l_T(x_i)$ in S_1, we get $l_T(x_{\beta'}) < l_T(x_{\alpha'}) < l_T(x_i)$. Therefore, $T[x_i, x_\alpha] \cup (x_i, x_\beta) \cup T[x_{\beta'}, x_{\alpha'}] \cup (x_{\alpha'}, x_\alpha) \cup (x_{\beta'}, x_\beta)$ is a subdivision of $C(k,1,1,1)$, a contradiction. □

Claim 15. $\chi(D_i^2) \leqslant 16$ *for all* $i \in \{0,1,...,k-1\}$.

Proof of Claim 15. Assume to the contrary that $\chi(D_i^2) \geqslant 17$. Then Theorem 11 implies that D_i^2 contains an antidirected cycle of length at least 6, say $\mathcal{A}_{2k}$ with $k \geqslant 3$. Set $\mathcal{A}_{2k} := x_1x_2...x_kx_{k+1}x_{k+2}...x_{2k} = S_1 \cup S_2$, where $S_1 = \{x_i \in V(\mathcal{A}_{2k})| \, |N^+(x_i) \cap V(\mathcal{A}_{2k})| = 2\}$ and $S_2 = \{x_i \in V(\mathcal{A}_{2k})| \, |N^-(x_i) \cap V(\mathcal{A}_{2k})| = 2\}$. Let $x_i \in S_2$ such that $l_T(x_i)$ is maximal. Assume that $N^-(x_i) \cap V(\mathcal{A}_{2k}) = \{x_\alpha, x_\beta\}$. Clearly, x_α and x_β belong to S_1. By the definition of S_1, there exists $1 \leqslant \alpha', \beta' \leqslant 2k$ other than i such that $(x_\alpha, x_{\alpha'})$ and $(x_\beta, x_{\beta'})$ belong to $A(D_i^2)$. Note that probably $\alpha' \neq \beta'$, since otherwise we get $k = 2$, a contradiction to our assumption. By the definition of A_2, $x_{\alpha'}, x_{\beta'}$ and x_i are all comparable with respect to the order induced by T. By the maximality of $l_T(x_i)$ in S_2, we get $l_T(x_{\alpha'}) < l_T(x_i)$ and $l_T(x_{\beta'}) < l_T(x_i)$. Without loss of generality, we may assume that $l_T(x_{\alpha'}) > l_T(x_{\beta'})$. By the definition of D_i^2, $l_T(x_{\alpha'}) - l_T(x_{\beta'}) \geqslant k$. Thus the union of $(x_\beta, x_{\beta'}) \cup T[x_{\beta'}, x_{\alpha'}], (x_\beta, x_i), (x_\alpha, x_{\alpha'})$ and (x_α, x_i) is a subdivision of $C(k, 1, 1, 1)$, a contradiction. □

Claim 16. *$\chi(D_i^3) \leqslant 4$ for all $i \in \{0, 1, ..., k-1\}$.*

Proof of Claim 16. Assume to the contrary that $\chi(D_i^3) \geqslant 5$. Due to Theorem 4, D_i^3 contains a copy Q of $P(2, 2)$, which is the union of two directed paths Q_1 and Q_2 which are disjoint except in their initial vertex y_0, say $Q_1 = (y_0, y_1, y_2)$ and $Q_2 = (y_0, z_1, z_2)$.

Assume that y_2 and z_2 are not comparable with respect to the tree-order, and let x be the least common ancestor of y_2 and z_2. By the definition of D_i^3, x is distinct from y_1 and z_1.

Assume that $(T[x, y_2] \cup T[x, z_2]) \cap Q = \{y_2, z_2\}$. Observe that $l(T[x, y_2]) < k$ and $l(T[x, z_2]) < k$, since otherwise $T[x, y_2] \cup T[x, z_2] \cup Q$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction. Consider v to be the least common ancestor of y_0 and y_1. Note that $(T[v, y_0] \cup T[v, y_1]) \cap (T[x, y_2] \cup T[x, z_2]) = \phi$, since otherwise we get $l(T[x, y_2]) \geqslant k$ or $l(T[x, z_2]) \geqslant k$, a contradiction. In what follows, if y_1 and z_1 are comparable with respect to the tree-order, we will assume without loss of generality that $y_1 \leqslant_T z_1$. This implies that $T[v, y_1] \cap Q = \{y_1\}$. Thus the union

of $T[v, y_1] \cup (y_1, y_2)$, $T[v, y_0] \cup Q_2$, $T[x, y_2]$ and $T[x, z_2]$ would be a subdivision of $C(k, 1, 1, 1)$, a contradiction. This implies that $(T[x, y_2] \cup T[x, z_2]) \cap Q \neq \{y_2, z_2\}$. Without loss of generality, we will assume that $T[x, y_2] \cap Q - y_2 \neq \phi$. Let v be the last vertex of $T[x, y_2]$ in $Q - y_2$. By our assumption, z_2 is not an ancestor of y_2, so v is different from z_2. Also by the definition of A_3, y_1 is not an ancestor of y_2, so v is different from y_1. Thus v is either z_1 or y_0. Assume that $v = z_1$. Let t be the least common ancestor of y_0 and z_2. Due to our assumption and to the definition of A_3, it is easy to see that y_2 and z_1 are not on $T[t, z_2]$. Observe that $T[t, z_2] \cap Q_1 - y_0 = \phi$, since otherwise y_1 belongs to $T[t, z_2]$ and hence $T[y_1, z_2] \cup (y_1, y_2) \cup (z_1, z_2) \cup T[z_1, y_2]$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction. Thus the union of $T[t, z_2]$, $T[t, y_0] \cup Q_1$, (z_1, z_2) and $T[z_1, y_2]$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction. Therefore, $v = y_0$.

Assume that z_1 and y_1 are not comparable with respect to the tree-order, and let t be the least common ancestor of z_1 and y_1. By the definition of A_3, y_0 is not an ancestor neither of y_1 nor of z_1, and so $(T[t, y_1] \cup T[t, z_1]) \cap (Q \cup T[y_0, y_2]) = \{y_1, z_1\}$. Thus the union of $T[y_0, y_2]$, (y_0, z_1), $T[t, y_1] \cup (y_1, y_2)$ and $T[t, z_1]$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction. This implies that $y_1 \leqslant_T z_1$ or vice versa. If $y_1 \leqslant_T z_1$, then $T[y_0, y_2] \cup (y_0, z_1) \cup (y_1, y_2) \cup T[y_1, z_1]$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction. Thus $z_1 \leqslant_T y_1$ and so the union of $T[x, z_2]$, $T[x, y_0] \cup T[y_0, y_2]$, (z_1, z_2) and $T[z_1, y_1] \cup (y_1, y_2)$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction. Therefore, y_2 and z_2 are comparable with respect to the tree-order.

Without loss of generality, we will assume that $y_2 \leqslant_T z_2$. By the definition of A_3, $T[y_2, z_2] \cap Q = \{y_2, z_2\}$. We suppose that y_0 is an ancestor of y_2. Again by the definition of A_3, $T[y_0, y_2] \cap Q = \{y_0, y_2\}$. Now we assume that z_1 and y_1 are not ancestors, and we consider t to be the least common ancestor of y_1 and z_1. It is clear that $(T[t, y_1] \cup T[t, z_1]) \cap (Q \cup T[y_0, y_2]) = \{y_1, z_1\}$. Hence the union of $T[y_0, y_2]$, (y_0, z_1), $T[x, y_1] \cup (y_1, y_2)$ and $T[x, z_1]$ is a subdivision of $C(k, 1, 1, 1)$. This means that y_1 and z_1 are comparable with respect to the tree-order. If $y_1 \leqslant_T z_1$, then we may easily check that $T[y_1, z_1] \cup (y_1, y_2) \cup T[y_0, y_2] \cup (y_0, z_1)$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction. Thus $z_1 \leqslant_T y_1$ and hence we obtain

a subdivision of $C(k,1,1,1)$ which is formed by the union of $T[y_0,y_2]\cup T[y_2,z_2]$, (y_0,y_1), (z_1,z_2) and $T[z_1,y_1]$, a contradiction.

It follows that y_0 is not an ancestor of y_2. By the definition of A_3, y_1 is not an ancestor of y_2, so we consider x to be the least common ancestor of y_1 and y_2. It is clear that $T[x,y_2]\cap Q=\{y_2\}$, since otherwise we get either $y_0 \leqslant_T y_2$, or $z_1 \leqslant_T z_2$, a contradiction. Moreover, we may easily verify that $T[x,y_1]\cap Q - y_1 \subseteq \{z_1\}$. Assume that z_1 does not belong to $T[x,y_1]$, then the union of $T[x,y_2]\cup T[y_2,z_2]$, $T[x,y_1]$, Q_2 and (y_0,y_1) is a subdivision of $C(k,1,1,1)$, a contradiction. Thus z_1 belongs to $T[x,y_1]$, and so $T[x,y_2]\cup T[x,z_1]\cup Q_1\cup(y_0,z_1)$ forms a subdivision of $C(k,1,1,1)$, a contradiction. □

Therefore, Claims 14, 15, 16 together with Lemma 12 imply that $\chi(D_i) \leqslant 8^3$. As $V(D)$ is partitioned into $V(D_i)$ for $i=0,1,...k-1$, then D can be colored properly using $8^3.k$ colors by assigning 8^3 distinct colors to each D_i. This completes the proof. □

Note that the upper bound in the theorem above can easily be lowered. However, we made no attempt to it here for two reasons: Firstly, we would like to keep the proof as simple as possible; secondly using our method, there is no hope to get an optimal value for $\chi(D)$ and all what we look for is to prove that it is bounded.

Due to the fact that every strongly connected digraph contains a spanning out-tree, we can immediately deduce the following:

Corollary 17. *Let D be a strong digraph with no subdivisions of $C(k,1,1,1)$, then the chromatic number of D is at most $8^3.k$.*

References

[1] L. Addario-Berry, F. Havet, and S. Thomassé, Paths with two blocks in n-chromatic digraphs, *J. of Combinatorial Theory, Series B*, 97 (4), pages 620-626, 2007.

[2] L. Addario-Berry, F. Havet, C. L. Sales, B. A. Reed, and S. Thomassé, Oriented trees in digraphs, *Discrete Math*, 313 (8), pages 967-974, 2013.

[3] J. A. Bondy, Disconnected orientations and a conjecture of Las Vergnas, *J. London Math. Soc. (2)*, 14 (2), pages 277-282, 1976.

[4] S. A. Burr, Subtrees of directed graphs and hypergraphs, *Proceedings of the Eleventh Southeastern Conference on Combinatorics, Graph Theory and Computing*, Boca Raton, Congr. Numer, 28, pages 227-239, 1980.

[5] N. Cohen, F. Havet, W. Lochet, and N. Nisse, Subdivisions of oriented cycles in digraphs with large chromatic number, *J. Graph Theory*, 89, pages 439-456, 2018.

[6] P. Erdös, Graph Theory and probability, *Canad. J. Math.*, 11, pages 34-38, 1959.

[7] P. Erdös and A. Hajnal, On chromatic number of graphs and set-systems, *Acta Math. Acad. Sci. Hungar*, 17, pages 61-99, 1966.

[8] T. Gallai, On directed paths and circuits, *In Theory of Graphs (Proc. Colloq. Titany, 1966)*, pages 115-118, Academic Press, New York, 1968.

[9] A. Gyárfás, Graphs with k odd cycle lengths, *Discrete Math*, 103, pages 41-48, 1992.

[10] T. Kaiser, O. Rucký, and R. Skrekovski, Graphs with odd cycle lengths 5 and 7 are 3-colorable, *SIAM J. Discrete Math*, 25, pages 1069-1088, 2011.

[11] R. Kim, SJ. Kim, J. Ma, and B. Park, Cycles with two blocks in k-chromatic digraphs, *J. Graph Theory*, 00, pages 1-14, 2017.

[12] C. Löwenstein, D. Rautenbach, and I. Schiermeyer, Cycle length parities and the chromate number, *J. Graph Theory*, 64, pages 210-218, 2010.

[13] P. Mihók and I. Schiermeyer, Cycle lengths and chromatic number of graphs, *Discrete Math*, 286, pages 147-149, 2004.

[14] B. Roy, Nombre chromatique et plus longs chemins d'un graphe, *Rev. Française Informat. Recherche Opérationnelle*, 1 (5), pages 129-132, 1967.

[15] S. S. Wang, Structure and coloring of graphs with only small odd cycles, *SIAM J. Discrete Math*, 22, pages 1040-1072, 2008.