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MOMENTS OF ORTHOGONAL POLYNOMIALS AND EXPONENTIAL GENERATING FUNCTIONS

IRA M. GESSEL AND JIANG ZENG

Dedicated to the memory of Richard Askey

ABSTRACT. Starting from the moment sequences of classical orthogonal polynomials we derive the orthogonality purely algebraically. We consider also the moments of ($q = 1$) classical orthogonal polynomials, and study those cases in which the exponential generating function has a nice form. In the opposite direction, we show that the generalized Dumont-Foata polynomials with six parameters are the moments of rescaled continuous dual Hahn polynomials. Finally we show that one of our methods can be applied to deal with the moments of Askey-Wilson polynomials.

1. INTRODUCTION

Many of the most important sequences in enumerative combinatorics—the factorials, derangement numbers, Bell numbers, Stirling polynomials, secant numbers, tangent numbers, Eulerian polynomials, Bernoulli numbers, and Catalan numbers—arise as moments of well-known orthogonal polynomials. With the exception of the Bernoulli and Catalan numbers, these orthogonal polynomials are all Sheffer type; see [41, 44]. One characteristic of these sequences is that their ordinary generating functions have simple continued fractions. For some recent work on the moments of classical orthogonal polynomials we refer the reader to [9, 14, 34, 10, 11, 24].

There is another sequence which appears in a number of enumerative applications, and which also has a simple continued fraction. The *Genocchi numbers* may be defined by

$$\sum_{n=0}^{\infty} G_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} G_{2n+2} \frac{x^{2n+2}}{(2n+2)!} = x \tan \frac{x}{2}.$$

So $G_n = 0$ when n is odd or $n = 0$, $G_2 = 1$, $G_4 = 1$, $G_6 = 3$, $G_8 = 17$, $G_{10} = 155$, and $G_{12} = 2073$.

A closely related sequence is the median Genocchi numbers H_{2n+1} , which first appeared in Seidel's work [38]. These numbers do not seem to have a simple exponential generating function and may be defined by $H_1 = 1$, and for $n \geq 1$,

$$H_{2n+1} = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} \binom{n}{2k-1} G_{2n+2-2k}. \quad (1.1)$$

So $H_1 = 1$, $H_3 = 1$, $H_5 = 2$, $H_7 = 8$, $H_9 = 56$, $H_{11} = 608$, $H_{13} = 9440$; see [40, 16, 18, 20].

A comprehensive discussion of the combinatorial properties of Genocchi numbers has been given by Viennot [40]. In particular, he showed that the Genocchi numbers and median Genocchi numbers H_{2n+1} have the S-fraction expansions

$$\begin{aligned} \sum_{n=0}^{\infty} G_{2n+2} t^{2n} &= S(t^2; 1^2, 1 \cdot 2, 2^2, 2 \cdot 3, 3^2, 3 \cdot 4, 4^2, \dots) \\ &= \frac{1}{1 - \frac{1^2 t^2}{1 - \frac{1 \cdot 2 t^2}{1 - \frac{2^2 t^2}{1 - \dots}}}} \end{aligned} \quad (1.2)$$

$$\begin{aligned} \sum_{n=0}^{\infty} H_{2n+1} t^{2n} &= S(t^2; 1^2, 1^2, 2^2, 2^2, 3^2, 3^2, \dots) \\ &= \frac{1}{1 - \frac{1^2 t^2}{1 - \frac{1^2 t^2}{1 - \frac{2^2 t^2}{1 - \dots}}}} \end{aligned} \quad (1.3)$$

Some recent papers (see [4, 21, 31]) have shown there are renewed interest on Genocchi numbers and median Genocchi numbers.

In his combinatorial approach to orthogonal polynomials [41, p. V-10], Viennot briefly alludes to the monic orthogonal polynomials whose moments are the Genocchi numbers (i.e., the n th moment is G_{2n+2}) and the median Genocchi numbers H_{2n+1} but he does not give any explicit formula for them.

Many years ago, one of the authors (IG) was learning about continued fractions and their connection to orthogonal polynomials, and saw in Viennot's work [41] the simple continued fraction for the Genocchi numbers. He wondered what the corresponding orthogonal polynomials were and wrote to Richard Askey, asking if he knew. Askey immediately wrote back to say that they were a special case of the continuous dual Hahn polynomials, and this paper grew out of an attempt to understand his reply.

As the Wilson polynomials are the most general ($q = 1$) classical orthogonal polynomials, we shall first consider the moments of the Wilson polynomials. In the general case, there are four parameters a, b, c, d and the ordinary generating function of these moments can be expressed as a hypergeometric series. We show that these moments have a simple exponential generating function when $a = 0$ or $a = 1/2$.

We then consider the continuous dual Hahn polynomials and the (continuous) Hahn polynomials along with their rescaled versions. In particular, we show that the moments of a rescaled version of the continuous dual Hahn polynomials are the generalized Dumont-Foata polynomials, which are a refinement of both Genocchi numbers and median Genocchi numbers. It would be possible to derive the moment generating function for Hahn polynomials from that for Wilson polynomials, but we will give a separate derivation and use our method for variety. Moreover, we show that the latter method can be applied to derive immediately a formula for the moments of the Askey-Wilson polynomials.

2. MOMENTS OF WILSON POLYNOMIALS

The monic Wilson polynomials $W_n(x)$ (see [2, 42, 26]) are defined by

$$W_n(x^2) = (-1)^n \frac{(a+b)_n (a+c)_n (a+d)_n}{(a+b+c+d+n-1)_n} \widetilde{W}(x^2; a, b, c, d) \quad (2.1)$$

where

$$\widetilde{W}(x^2; a, b, c, d) = {}_4F_3 \left(\begin{matrix} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix} ; 1 \right).$$

If a, b, c, d are positive or $a = \bar{b}$ and/or $c = \bar{d}$ and the real parts are positive, then the orthogonality reads as follows:

$$\begin{aligned} \frac{1}{2\pi} \int_0^\infty \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)} \right|^2 W_m(x^2) W_n(x^2) dx \\ = \frac{n! \Gamma(n+a+b) \cdots \Gamma(n+c+d)}{(n+a+b+c+d-1)_n \Gamma(2n+a+b+c+d)} \delta_{mn}. \end{aligned} \quad (2.2a)$$

Consider the moment sequence $w_n(a) := w_n(a, b, c, d)$ of the Wilson polynomials defined by

$$\begin{aligned} \frac{1}{2\pi} \int_0^\infty \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)} \right|^2 x^{2n} dx \\ = w_n(a) \frac{\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)}. \end{aligned} \quad (2.2b)$$

It follows that $w_0(a) = 1$ and for $n \geq 0$,

$$w_{n+1}(a) = \frac{(a+b)(a+c)(a+d)}{a+b+c+d} w_n(a+1) - a^2 w_n(a). \quad (2.3)$$

Proposition 1. *There holds*

$$\begin{aligned} \sum_{n \geq 0} w_n(a) t^n &= \sum_{n \geq 0} \frac{(a+b)_n (a+c)_n (a+d)_n t^n}{(a+b+c+d)_n \prod_{l=0}^n (1+(a+l)^2 t)} \\ &= \frac{1}{1+a^2 t} {}_4F_3 \left(\begin{matrix} a+b, a+c, a+d, 1 \\ a+b+c+d, a+1+i/\sqrt{t}, a+1-i/\sqrt{t} \end{matrix}; 1 \right). \end{aligned}$$

Proof. Set $F(a, t) = \sum_{n \geq 0} w_n(a) t^n$. The recurrence (2.3) implies that

$$F(a, t) = 1 + \frac{(a+b)(a+c)(a+d)}{(a+b+c+d)} t F(a+1, t) - a^2 t F(a, t). \quad (2.4)$$

Hence

$$F(a, t) = \frac{1}{1+a^2 t} + \frac{(a+b)(a+c)(a+d)t}{(a+b+c+d)(1+a^2 t)} F(a+1, t).$$

The formula follows then by iterating the above functional equation. \square

Here is the formal approach to the Wilson polynomials. We define a linear functional \mathcal{L} on even polynomials by $\mathcal{L}(x^{2n}) = w_n(a)$.

Lemma 2. *For $k \geq 0$,*

$$\mathcal{L}(x^{2n}(a+ix)_k (a-ix)_k) = \frac{(a+b)_k (a+c)_k (a+d)_k}{(a+b+c+d)_k} w_n(a+k). \quad (2.5)$$

Proof. We prove this by induction on k . It is clear for $k = 0$. For $k \geq 0$, we have

$$\begin{aligned} &\mathcal{L}(x^{2n}(a+ix)_{k+1} (a-ix)_{k+1}) \\ &= \mathcal{L}(x^{2n}(a+ix)_k (a-ix)_k ((a+k)^2 + x^2)) \\ &= \frac{(a+b)_k (a+c)_k (a+d)_k}{(a+b+c+d)_k} ((a+k)^2 w_n(a+k) + w_{n+1}(a+k)). \end{aligned}$$

By (2.3) the formula is valid for $k+1$. \square

The Wilson polynomials have the orthogonality relation [26, (9.1.2)]

$$\begin{aligned} &\mathcal{L}(W_n(x^2) W_m(x^2)) \\ &= n! \frac{(a+b)_n (a+c)_n (a+d)_n (b+c)_n (b+d)_n (c+d)_n}{(a+b+c+d+n-1)_n (a+b+c+d)_{2n}} \delta_{mn} \quad (m, n \geq 0). \end{aligned} \quad (2.6)$$

We can verify the orthogonality directly using Lemma 2. Indeed, by induction on $n \geq 0$,

$$\begin{aligned} &\mathcal{L}((a+ix)_m (a-ix)_m (b+ix)_n (b-ix)_n) \\ &= \frac{(a+b)_{m+n} (a+c)_m (a+d)_m (b+c)_n (b+d)_n}{(a+b+c+d)_{m+n}}. \end{aligned} \quad (2.7)$$

We evaluate

$$\begin{aligned}
& \mathcal{L} \left(\tilde{W}_n(x^2; a, b, c, d) (b + ix)_m (b - ix)_m \right) \\
&= \frac{(a + b)_m (b + c)_m (b + d)_m}{(a + b + c + d)_m} {}_3F_2 \left(\begin{matrix} -n, n + a + b + c + d - 1, a + b + m \\ a + b, a + b + c + d + m \end{matrix}; 1 \right) \\
&= \frac{(a + b)_m (b + c)_m (b + d)_m}{(a + b + c + d)_m} \frac{(1 - n - c - d)_n (-m)_n}{(a + b)_n (1 - n - a - b - c - d - m)_n} \\
&\quad \text{(by the Pfaff-Saalschütz theorem)} \\
&= \frac{(a + b)_m (b + c)_m (b + d)_m (c + d)_n (-m)_n}{(a + b + c + d)_{m+n} (a + b)_n}.
\end{aligned}$$

This is 0 for $m < n$, which implies orthogonality.

The orthogonality (2.6) implies the three-term recurrence relation [26, (9.1.5)]

$$xW_n(x) = W_{n+1}(x) + b_n W_n(x) + \lambda_n W_{n-1}(x) \quad (2.8)$$

with $\lambda_n = A_{n-1}C_n$, $b_n = A_n + C_n - a^2$, where

$$\begin{cases} A_n = \frac{(n + a + b + c + d - 1)(n + a + b)(n + a + c)(n + a + d)}{(2n + a + b + c + d - 1)(2n + a + b + c + d)}, \\ C_n = \frac{n(n + b + c - 1)(n + b + d - 1)(n + c + d - 1)}{(2n + a + b + c + d - 2)(2n + a + b + c + d - 1)}. \end{cases}$$

Indeed, from (2.6) we derive that

$$\lambda_n = \frac{\mathcal{L}(x^2 W_{n-1}(x^2) W_n(x^2))}{\mathcal{L}(W_{n-1}(x^2)^2)} = \frac{\mathcal{L}(W_n(x^2)^2)}{\mathcal{L}(W_{n-1}(x^2)^2)} = A_{n-1}C_n.$$

Extracting the coefficient of x^n in (2.8) we have

$$b_n = [x^{n-1}]W_n(x) - [x^n]W_{n+1}(x) \quad (2.9)$$

where $[x^k]W_n(x)$ is the coefficient of x^k in $W_n(x)$. As

$$(a + i\sqrt{x})_k (a - i\sqrt{x})_k = \prod_{l=0}^{k-1} ((a + l)^2 + x),$$

we derive from (2.1) that

$$[x^{n-1}]W_n(x) = -\frac{n(a + b + n - 1)(a + c + n - 1)(a + d + n - 1)}{a + b + c + d + 2n - 2} + \sum_{l=0}^{n-1} (a + l)^2,$$

which yields $b_n = A_n + C_n - a^2$ by (2.9).

It is known [8] that the recurrence (2.8) is equivalent to the following J-fraction expansion of the moments $w_n(a)$, where the J-fraction $J(t; a_1, b_1, a_2, b_2, \dots)$ is defined

to be

$$\frac{1}{1 - a_1 t - \frac{b_1 t^2}{1 - a_2 t - \frac{b_2 t^2}{1 - \dots}}}$$

Proposition 3. *We have expansion*

$$\sum_{n=0}^{\infty} w_n(a) t^n = J(t; A_0 + C_0 - a^2, A_0 C_1, \dots, A_n + C_n - a^2, A_n C_{n+1}, \dots). \quad (2.10)$$

Recall the following contraction formulae [41] transforming S-fraction to J-fraction:

$$S(t; \alpha_1, \dots, \alpha_n, \dots) = J(t; \gamma_0, \beta_1, \gamma_0, \beta_1, \dots, \gamma_n, \beta_{n+1}, \dots) \quad (2.11a)$$

$$= 1 + \gamma_0 t J(t; \gamma'_0, \beta'_1, \dots, \gamma'_n, \beta'_n, \dots) \quad (2.11b)$$

with $\gamma_0 = \alpha_1, \gamma'_0 = \alpha_1 + \alpha_2$ and for $n \geq 1$

$$\begin{aligned} \gamma_n &= \alpha_{2n} + \alpha_{2n+1}, & \beta_n &= \alpha_{2n-1} \alpha_{2n}; \\ \gamma'_n &= \alpha_{2n-1} + \alpha_{2n}, & \beta'_n &= \alpha_{2n} \alpha_{2n+1}. \end{aligned}$$

Thus, when $a = 0$ we can transform (2.10) to the S-fraction.

Corollary 4. *We have*

$$\begin{aligned} \sum_{n=0}^{\infty} w_n(0) t^n &= S \left(t; \frac{bcd}{b+c+d}, \frac{(b+c)(c+d)(b+d)}{(b+c+d)(b+c+d+1)}, \dots, \right. \\ &\quad \frac{(b+n)(c+n)(d+n)(b+c+d+n-1)}{(b+c+d+2n-1)(b+c+d+2n)}, \\ &\quad \left. \frac{(n+1)(b+c+n)(c+d+n)(b+d+n)}{(b+c+d+2n)(b+c+d+2n+1)}, \dots \right). \quad (2.12) \end{aligned}$$

3. EXPONENTIAL GENERATING FUNCTIONS

To derive exponential generating functions from ordinary generating functions we use the following lemma, which we will also apply to other orthogonal polynomials.

Let $\varepsilon : \mathbb{Q}[t] \rightarrow \mathbb{Q}[t]$ be the linear transformation defined by

$$\varepsilon \left(\sum_{n=0}^{\infty} u_n t^n \right) = \sum_{n=0}^{\infty} u_n \frac{t^n}{n!}. \quad (3.1)$$

Lemma 5. *For any nonnegative integers m and n we have*

$$\varepsilon \left(\frac{t^m}{(1 - \alpha t)(1 - (\alpha + 1)t) \cdots (1 - (\alpha + m)t)} \right) = e^{\alpha t} \frac{(e^t - 1)^m}{m!}.$$

Remark 1. *We shall give two proofs. The first one uses a partial fraction expansion, while the second one doesn't require explicitly doing the partial fraction expansion.*

First proof. Expanding the left side by partial fractions we get

$$\frac{t^m}{\prod_{k=0}^m (1 - (\alpha + k)t)} = \frac{1}{m!} \sum_{k=0}^m \binom{m}{k} \frac{(-1)^k}{1 - (\alpha + m - k)t}.$$

The result follows from the fact that $\varepsilon(\frac{1}{1-\beta t}) = e^{\beta t}$ and the binomial theorem. \square

Second proof. Expanding $e^{\alpha t}(e^t - 1)^m/m!$ by the binomial theorem we obtain a linear combination of terms $e^{(\alpha+k)t}$, $k = 0, \dots, m$, and $\varepsilon^{-1}(e^{(\alpha+k)t}) = 1/(1 - (\alpha + k)t)$. So $\varepsilon^{-1}(e^{\alpha t}(e^t - 1)^m/m!)$ must be a proper rational function of the form

$$\frac{N(t)}{(1 - \alpha t)(1 - (\alpha + 1)t) \cdots (1 - (\alpha + m)t)},$$

where the degree of $N(t)$ is at most m . But since the first nonzero coefficient of $e^{\alpha t}(e^t - 1)^m/m!$ is $t^m/m!$, $N(t)$ must be t^m . \square

Remark 2. *We may also reduce the first proof to $\alpha = 0$ and apply the formula*

$$\varepsilon^{-1} \left(e^{\alpha t} \sum_{n=0}^{\infty} u_n \frac{t^n}{n!} \right) = \frac{1}{1 - \alpha t} \sum_{n=0}^{\infty} u_n \left(\frac{t}{1 - \alpha t} \right)^n. \quad (3.2)$$

If $\alpha = -m/2$ then we have

$$\varepsilon \left(\frac{t^m}{(1 + \frac{m}{2}t) \cdots (1 - \frac{m}{2}t)} \right) = e^{-\frac{m}{2}t} \frac{(e^t - 1)^m}{m!} = \frac{(2 \sinh \frac{t}{2})^m}{m!}. \quad (3.3a)$$

If $m = 2n$, this is

$$\varepsilon \left(\frac{t^{2n}}{(1 - t^2)(1 - 2^2 t^2) \cdots (1 - n^2 t^2)} \right) = \frac{(2 \sinh \frac{t}{2})^{2n}}{(2n)!}, \quad (3.3b)$$

and if $m = 2n + 1$ this is

$$\varepsilon \left(\frac{t^{2n+1}}{(1 - (\frac{1}{2})^2 t^2)(1 - (\frac{3}{2})^2 t^2) \cdots (1 - (n + \frac{1}{2})^2 t^2)} \right) = \frac{(2 \sinh \frac{t}{2})^{2n+1}}{(2n + 1)!}. \quad (3.3c)$$

If $a = 0$ or $a = \frac{1}{2}$ then there is a nice exponential generating function for the moments of the Wilson polynomials.

Theorem 6. *We have*

$$\sum_{n=0}^{\infty} w_n(0) \frac{t^{2n}}{(2n)!} = {}_3F_2 \left(\begin{matrix} b, c, d \\ b + c + d, \frac{1}{2} \end{matrix}; \sin^2 \frac{t}{2} \right), \quad (3.4)$$

$$\sum_{n=0}^{\infty} w_n(1/2) \frac{t^{2n+1}}{(2n+1)!} = 2 \sin \frac{t}{2} {}_3F_2 \left(\begin{matrix} b + \frac{1}{2}, c + \frac{1}{2}, d + \frac{1}{2} \\ b + c + d + \frac{1}{2}, \frac{3}{2} \end{matrix}; \sin^2 \frac{t}{2} \right). \quad (3.5)$$

Proof. Returning to the moments of the Wilson polynomials, for $a = 0$ we have

$$\sum_{n=0}^{\infty} (-1)^n w_n(0) t^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{(b)_n (c)_n (d)_n t^{2n}}{(b+c+d)_n \prod_{l=0}^n (1-l^2 t^2)}.$$

So by (3.3b),

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n w_n(0) \frac{t^{2n}}{(2n)!} &= \sum_{n=0}^{\infty} (-1)^n \frac{(b)_n (c)_n (d)_n (2 \sinh \frac{t}{2})^{2n}}{(b+c+d)_n (2n)!} \\ &= {}_3F_2 \left(\begin{matrix} b, c, d \\ b+c+d, \frac{1}{2} \end{matrix}; -\sinh^2 \frac{t}{2} \right). \end{aligned}$$

Replacing t with it , we get (3.4).

Similarly, for $a = 1/2$ we have

$$\sum_{n=0}^{\infty} w_n(1/2) t^n = \sum_{n=0}^{\infty} \frac{(b+1/2)_n (c+1/2)_n (d+1/2)_n (-1)^n t^{2n}}{(b+c+d+1/2)_n \prod_{l=0}^n (1-(l+1/2)^2 t^{2n})}.$$

Hence

$$\sum_{n=0}^{\infty} (-1)^n w_n(1/2) t^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(b+1/2)_n (c+1/2)_n (d+1/2)_n t^{2n+1}}{(b+c+d+1/2)_n \prod_{l=0}^n (1-(l+1/2)^2 t^{2n})}.$$

Therefore by (3.3c),

$$\sum_{n=0}^{\infty} w_n(1/2) \frac{(-1)^n t^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{(b+1/2)_n (c+1/2)_n (d+1/2)_n (2 \sinh \frac{t}{2})^{2n+1}}{(b+c+d+1/2)_n (2n+1)!}.$$

Replacing t with it , we may write this as (3.5). \square

We can also get the exponential generating functions (3.4) and (3.5) as follows. We have

$$\cosh \left(2x \arcsin \frac{z}{2} \right) = \sum_{n=0}^{\infty} x^2 (x^2 + 1^2) \cdots (x^2 + (n-1)^2) \frac{z^{2n}}{(2n)!}.$$

So, by Lemma 2 with $a = 0$, we have

$$\mathcal{L} \left(\cosh \left(2x \arcsin \frac{z}{2} \right) \right) = \sum_{n=0}^{\infty} \frac{(b)_n (c)_n (d)_n}{(b+c+d)_n} \frac{z^{2n}}{(2n)!}.$$

Setting $z = 2 \sin \frac{t}{2}$, we get

$$\begin{aligned} \mathcal{L}(\cosh(xt)) &= \sum_{n=0}^{\infty} \mathcal{L}(x^{2n}) \frac{t^{2n}}{(2n)!} \\ &= {}_3F_2 \left(\begin{matrix} b, c, d \\ b+c+d, \frac{1}{2} \end{matrix}; \sin^2 \frac{t}{2} \right). \end{aligned} \quad (3.6)$$

Similarly,

$$\begin{aligned} \sinh \left(2x \arcsin \frac{z}{2} \right) \\ = x \sum_{n=0}^{\infty} \left(x^2 + \left(\frac{1}{2} \right)^2 \right) \left(x^2 + \left(\frac{3}{2} \right)^2 \right) \cdots \left(x^2 + \left(n - \frac{1}{2} \right)^2 \right) \frac{z^{2n+1}}{(2n+1)!}. \end{aligned}$$

So, by Lemma 2 with $a = \frac{1}{2}$, we have

$$\begin{aligned} \mathcal{L}(x^{-1} \sinh(xt)) &= \sum_{n=0}^{\infty} \mathcal{L}(x^{2n}) \frac{t^{2n+1}}{(2n+1)!} \\ &= 2 \sin \frac{t}{2} {}_3F_2 \left(\begin{matrix} b + \frac{1}{2}, c + \frac{1}{2}, d + \frac{1}{2} \\ b+c+d, \frac{1}{2} \end{matrix}; \sin^2 \frac{t}{2} \right). \end{aligned} \quad (3.7)$$

4. MOMENTS OF CONTINUOUS DUAL HAHN POLYNOMIALS

If we take the limit as $d \rightarrow \infty$ in the Wilson polynomials we get the *continuous dual Hahn polynomials* $p_n(x) := p_n(x; a, b, c)$ defined by

$$p_n(x^2) = (-1)^n (a+b)_n (a+c)_n {}_3F_2 \left(\begin{matrix} -n, a+ix, a-ix \\ a+b, a+c \end{matrix}; 1 \right). \quad (4.1)$$

The first two values of $p_n(x^2; a, b, c)$ are the following:

$$\begin{aligned} p_1(x^2) &= x^2 - (ab + bc + ca) \\ p_2(x^2) &= x^4 - [1 + 2(a+b+c) + 2(ab+ac+bc)]x^2 \\ &\quad + a^2b^2 + 2a^2bc + a^2c^2 + 2b^2ac + 2c^2ab + b^2c^2 \\ &\quad + a^2b + a^2c + b^2a + 4abc + ac^2 + b^2c + bc^2 + ab + ac + bc. \end{aligned}$$

When either a, b , and c are all positive or one is positive and the other two are complex conjugates with positive real parts, Wilson's result [42] reduces to

$$\begin{aligned} \frac{1}{2\pi} \int_0^\infty \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)} \right|^2 p_m(x^2)p_n(x^2) dx \\ = \Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+b+c)n! \delta_{mn}. \end{aligned} \quad (4.2)$$

The corresponding moments $\mu_n(a, b, c)$ are given by

$$\begin{aligned} \frac{1}{2\pi} \int_0^\infty \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)} \right|^2 x^{2n} dx \\ = \mu_n(a, b, c)\Gamma(a+b)\Gamma(a+c)\Gamma(b+c). \end{aligned} \quad (4.3)$$

De Branges [12] ([2, p. 152] and [1]) also proved that $\mu_0(a, b, c) = 1$. The counterpart of (2.3) reads as follows:

$$\mu_{n+1}(a, b, c) = (a+b)(a+c)\mu_n(a+1, b, c) - a^2\mu_n(a, b, c), \quad (4.4)$$

which is equivalent to the generating function

$$\begin{aligned} \sum_{n \geq 0} \mu_n(a, b, c)t^n &= \sum_{n \geq 0} \frac{(a+b)_n(a+c)_n t^n}{\prod_{l=0}^n (1+(a+l)^2 t)} \\ &= \frac{1}{1+a^2 t} {}_3F_2 \left(\begin{matrix} a+b, a+c, 1 \\ a+1+i/\sqrt{t}, a+1-i/\sqrt{t} \end{matrix}; 1 \right). \end{aligned} \quad (4.5)$$

We remark that the above recurrence is nothing but the definition of a sequence of polynomials introduced by Dumont and Foata [13, 7, 45] as an extension of Genocchi numbers. In this context the following result was conjectured by Gandhi [19] in 1970 and first proved by Carlitz [6], and Riordan and Stein [37]. Here we provide a direct proof starting from (4.3).

Proposition 7. *For $n \geq 0$ we have*

$$\mu_n(1, 1, 1) = G_{2n+4}. \quad (4.6)$$

Proof: Since $|\Gamma(ix)|^2 = \frac{\pi}{x \sinh(\pi x)}$ and $|\Gamma(1+ix)|^2 = \frac{\pi x}{\sinh(\pi x)}$, we have

$$\begin{aligned} \mu_n(1, 1, 1) &= \frac{1}{2\pi} \int_0^\infty x^{2n} \left| \frac{(\Gamma(1+ix))^3}{\Gamma(2ix)} \right|^2 dx \\ &= 2\pi \int_0^\infty x^{2n+4} \frac{\cosh(\pi x)}{\sinh^2(\pi x)} dx \\ &= -2 \int_0^\infty x^{2n+4} d(1/(\sinh(\pi x))). \end{aligned}$$

Integrating by parts yields

$$\mu_n(1, 1, 1) = 4(n+2) \int_0^\infty \frac{x^{2n+3} dx}{\sinh(\pi x)}.$$

Equation (4.6) follows then from the known integral expression of Bernoulli numbers [17, p. 39]:

$$|B_{2n}| = \frac{2n}{2^{2n} - 1} \int_0^\infty \frac{x^{2n-1} dx}{\sinh(\pi x)},$$

and the formula $G_{2n} = 2(2^{2n} - 1)|B_{2n}|$. \square

From the J-fraction (2.10) for the moments of Wilson polynomials we derive the following J-fraction

$$\sum_{n \geq 0} \mu_n(a, b, c) t^n = J(t; (ab + bc + ca), (a + b)(b + c)(c + a), \dots, \\ (a + n)(b + n) + (b + n)(c + n) + (c + n)(a + n) - n(n + 1), \\ (n + 1)(a + b + n)(b + c + n)(c + a + n), \dots). \quad (4.7)$$

By (2.12), when $a = 0$ we have the S-fraction

$$\sum_{n=0}^{\infty} \mu_n(0, b, c) t^n = S(t; bc, b + c, (b + 1)(c + 1), 2(b + c + 1), \dots), \quad (4.8)$$

as is well-known, and $b = c = 1$ gives the Genocchi numbers (see (1.2)). In other words, the Genocchi numbers G_{2n+2} are the moments of the continuous dual Hahn polynomials $p_n(x, 0, 1, 1)$.

As we recall, Carlitz [7] gave a cumbersome formula for the exponential generating function of $\mu_n(a, b, c)$. By (3.4) the corresponding generating functions for $a = 0$ is

$$\sum_{n=0}^{\infty} \mu_n(0, b, c) \frac{t^{2n}}{(2n)!} = {}_2F_1 \left(\begin{matrix} b, c \\ \frac{1}{2} \end{matrix}; \sin^2 \frac{t}{2} \right), \quad (4.9)$$

which is equivalent to Carlitz's [7, Equation (4.2)]. In particular, the exponential generating function of Genocchi numbers $G_{2n+2} = \mu_n(0, 1, 1)$ also has the hypergeometric series representations:

$$1 + \sum_{n=1}^{\infty} G_{2n+2} \frac{t^{2n}}{(2n)!} = {}_2F_1 \left(\begin{matrix} 1, 1 \\ \frac{1}{2} \end{matrix}; \sin^2 \frac{t}{2} \right) \\ = \sec^2 \frac{t}{2} {}_2F_1 \left(\begin{matrix} 1, -\frac{1}{2} \\ \frac{1}{2} \end{matrix}; -\tan^2 \frac{t}{2} \right),$$

where the last formula follows from Pfaff's transformation (see [2, p. 68]).

The J-fraction for $\mu_n(\frac{1}{2}, b, c)$ is

$$\sum_{n=0}^{\infty} \mu_n(\frac{1}{2}, b, c) t^n = J(t; bc + \frac{1}{2}(b + c), (b + c)(b + 1/2)(c + 1/2), \\ bc + \frac{5}{2}(b + c) + 2 \cdot 1^2, (b + c + 1)(b + 3/2)(c + 3/2), \\ bc + \frac{9}{2}(b + c) + 2 \cdot 2^2, (b + c + 2)(b + 5/2)(c + 5/2), \dots). \quad (4.10)$$

By (3.5) the corresponding generating functions for $a = \frac{1}{2}$ is

$$\sum_{n=0}^{\infty} \mu_n(\frac{1}{2}, b, c) \frac{t^{2n+1}}{(2n+1)!} = 2 \sin \frac{t}{2} {}_2F_1 \left(\begin{matrix} b + \frac{1}{2}, c + \frac{1}{2} \\ \frac{3}{2} \end{matrix}; \sin^2 \frac{t}{2} \right). \quad (4.11)$$

The following two nice special cases of (4.9) and (4.11) are known (see [17, p. 101]):

$$\frac{\cos(at/2)}{\cos t/2} = {}_2F_1 \left(\begin{matrix} (1-a)/2, (1+a)/2 \\ 1/2 \end{matrix}; \sin^2 \frac{t}{2} \right), \quad (4.12)$$

$$\frac{\sin(at/2)}{a \sin t/2} = {}_2F_1 \left(\begin{matrix} (1-a)/2, (1+a)/2 \\ 3/2 \end{matrix}; \sin^2 \frac{t}{2} \right). \quad (4.13)$$

To find the orthogonal polynomials whose moments are median Genocchi numbers we shall consider the *generalized Dumont-Foata polynomials* in 6 variables due to Dumont [15]:

$$\Gamma_{n+1}(\alpha, \bar{\alpha}) := \Gamma_{n+1}(\alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma, \bar{\gamma}),$$

which can be defined by the J-fraction

$$\begin{aligned} \sum_{n \geq 0} \Gamma_{n+1}(\alpha, \bar{\alpha}) t^n &= J(t; \alpha \bar{\beta} + \beta \bar{\gamma} + \gamma \bar{\alpha}, (\bar{\alpha} + \beta)(\bar{\beta} + \gamma)(\bar{\gamma} + \alpha), \dots, \\ &(\alpha + n)(\bar{\beta} + n) + (\beta + n)(\bar{\gamma} + n) + (\gamma + n)(\bar{\alpha} + n) - n(n+1), \\ &(n+1)(\bar{\alpha} + \beta + n)(\bar{\beta} + \gamma + n)(\bar{\gamma} + \alpha + n), \dots). \end{aligned} \quad (4.14)$$

Theorem 8. *The sequence $\{\Gamma_{n+1}(\alpha, \bar{\alpha})\}_n$ is the moment sequence of the rescaled continuous dual Hahn polynomials $Z_n(x) = p_n(x+d; a, b, c)$ with*

$$\begin{cases} a = \frac{1}{2}(\alpha + \bar{\alpha} + \beta - \bar{\beta} + \bar{\gamma} - \gamma), \\ b = \frac{1}{2}(\bar{\alpha} - \alpha + \beta + \bar{\beta} + \gamma - \bar{\gamma}), \\ c = \frac{1}{2}(\alpha - \bar{\alpha} - \beta + \bar{\beta} + \gamma + \bar{\gamma}), \\ d = \alpha \bar{\alpha} + \alpha(\beta - \bar{\beta}) - \bar{\alpha}(\gamma - \bar{\gamma}) - a^2. \end{cases} \quad (4.15)$$

More precisely, let $\psi: K[x] \rightarrow K$ be a linear functional such that $\psi(x^n) = \Gamma_{n+1}(\alpha, \bar{\alpha})$ for $n \geq 0$. Then

$$\psi(Z_m(x)Z_n(x)) = n! (\bar{\alpha} + \beta)_n (\alpha + \bar{\gamma})_n (\bar{\beta} + \gamma)_n \delta_{mn}. \quad (4.16)$$

where

$$\begin{aligned} Z_n(x) &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (\bar{\alpha} + \beta + k)_{n-k} (\alpha + \bar{\gamma} + k)_{n-k} \\ &\times \prod_{l=0}^{k-1} [x^2 + (\alpha + l)(\bar{\alpha} + l) + (\alpha + l)(\beta - \bar{\beta}) - (\bar{\alpha} + l)(\gamma - \bar{\gamma})]. \end{aligned} \quad (4.17)$$

Proof. It is known that the continuous dual Hahn polynomials $p_n(x)$ have the recurrence relation [26, (9.3.5)]:

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x) \quad (n \geq 0), \quad (4.18)$$

where

$$\begin{cases} b_n &= (n+a)(n+b) + (n+a)(n+c) + (n+b)(n+c) - n(n+1), \\ \lambda_n &= n(n-1+a+b)(n-1+a+c)(n-1+b+c). \end{cases} \quad (4.19)$$

Thus, the polynomials $Z(x) = p_n(x+d, a, b, c)$ with the substitution (4.15) have the recurrence relation

$$Z_{n+1}(x) = (x - \bar{b}_n)Z_n(x) - \bar{\lambda}_n Z_{n-1}(x) \quad (n \geq 0) \quad (4.20)$$

with

$$\begin{cases} \bar{b}_n &= (n+\alpha)(n+\bar{\beta}) + (n+\bar{\alpha})(n+\gamma) + (n+\beta)(n+\bar{\gamma}) - n(n+1), \\ \bar{\lambda}_n &= n(n-1+\bar{\alpha}+\beta)(n-1+\alpha+\bar{\gamma})(n-1+\bar{\beta}+\gamma). \end{cases} \quad (4.21)$$

This recurrence is equivalent to the J-fraction (4.14) for $\Gamma_n(\alpha, \bar{\alpha})$. \square

Proposition 9. *We have*

$$\begin{aligned} \Gamma_{n+1}(\alpha, \bar{\alpha}) &= (\alpha + \bar{\gamma})(\beta + \bar{\alpha})\Gamma_n(\alpha + 1, \bar{\alpha} + 1) \\ &\quad + [\alpha(\bar{\beta} - \beta) - \bar{\alpha}(\bar{\gamma} - \gamma) - \alpha\bar{\alpha}]\Gamma_n(\alpha, \bar{\alpha}) \end{aligned} \quad (4.22)$$

with $\Gamma_1(\alpha, \bar{\alpha}) = 1$, and

$$\begin{aligned} &\sum_{n \geq 0} \Gamma_{n+1}(\alpha, \bar{\alpha}) t^n \\ &= \sum_{n \geq 0} \frac{(\alpha + \bar{\gamma})_n (\beta + \bar{\alpha})_n t^n}{\prod_{k=0}^n (1 - [(\alpha + k)(\bar{\beta} - \beta) - (\bar{\alpha} + k)(\bar{\gamma} - \gamma) - (\alpha + k)(\bar{\alpha} + k)]t)}. \end{aligned} \quad (4.23)$$

Proof. Clearly, the recurrence (4.22) is equivalent to the generating function (4.23). So we just prove (4.23). It is known (see [8]) and easy to see that the moments of $p_n(x+d; a, b, c)$ are related to those of $p_n(x; a, b, c)$ as follows:

$$\Gamma_{n+1}(\alpha, \bar{\alpha}) = \sum_{k=0}^n \binom{n}{k} (-d)^{n-k} \mu_k(a, b, c). \quad (4.24)$$

Therefore

$$\begin{aligned} \sum_{n \geq 0} \Gamma_{n+1}(\alpha, \bar{\alpha}) t^n &= \sum_{n \geq 0} \sum_{k=0}^n \binom{n}{k} (-d)^{n-k} \mu_k(a, b, c) t^n \\ &= \sum_{k \geq 0} \mu_k(a, b, c) t^k \sum_{n \geq 0} \binom{n+k}{k} (-dt)^n \\ &= \sum_{k \geq 0} \mu_k(a, b, c) t^k (1+dt)^{-k-1}. \end{aligned}$$

Invoking (4.5) and (4.15) we derive the generating function. \square

Remark 3. Originally Dumont [15] defined the polynomials $\Gamma_n(\alpha, \bar{\alpha})$ combinatorially and conjectured the J -fraction in (4.14). Randrianarivony [36] and Zeng [45] proved Dumont's conjectured J -fraction by first establishing (4.23) and (4.22) from Dumont's combinatorial definition for $\Gamma_{n+1}(\alpha, \bar{\alpha})$. In 2010 Josuat-Vergès [23] gave a new proof of the J -fraction (4.14) starting from (4.22).

Corollary 10. The median Genocchi numbers H_{2n+1} are the moments of the rescaled continuous dual Hahn polynomials $p_n(x - \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

Proof. Contracting the S -fraction (1.3) by (2.11a) we obtain the following J -fraction

$$\sum_{n=0}^{\infty} H_{2n+1} t^n = J(t; 2 \cdot 1^2, (1 \cdot 2)^2, 2 \cdot 2^2, (2 \cdot 3)^2, \dots, 2 \cdot n^2, (n \cdot (n+1))^2, \dots). \quad (4.25)$$

Comparing with (4.14) we see that $H_{2n+1} = \Gamma_{n+1}(1, 1, 1, 0, 1, 1)$. Hence $a = b = c = \frac{1}{2}$ and $d = -\frac{1}{4}$ by (4.15). \square

Our method does not produce an exponential generating function for the median Genocchi numbers here since the denominator in (4.23) with $\alpha = 1$ and $\bar{\alpha} = 0$ does not factorize nicely.

5. MOMENTS OF HAHN POLYNOMIALS

In this section we introduce another method for determining generating functions for moments of orthogonal polynomials. We apply it only to the Hahn polynomials, but it can also be used for the Wilson polynomials and Askey-Wilson polynomials (see Theorem 17).

Lemma 11. Let a_0, a_1, a_2, \dots be arbitrary. Then

$$\sum_{n=0}^{\infty} (x + a_0)(x + a_1) \cdots (x + a_{n-1}) \frac{t^n}{(1 + a_0 t) \cdots (1 + a_n t)} = \frac{1}{1 - xt}. \quad (5.1)$$

Proof. We have the indefinite sum

$$\begin{aligned} & \sum_{n=0}^m (x + a_0) \cdots (x + a_{n-1}) \frac{t^n}{(1 + a_0 t) \cdots (1 + a_n t)} \\ &= \frac{1}{1 - xt} \left[1 - (x + a_0) \cdots (x + a_m) \frac{t^{m+1}}{(1 + a_0 t) \cdots (1 + a_m t)} \right], \end{aligned} \quad (5.2)$$

which is easily proved by induction. The lemma follows by taking $m \rightarrow \infty$. \square

Remark 4. As pointed out by Knuth [25, equation (2.16)], a formula equivalent to Lemma 11 was discovered by François Nicole [33] in 1727. Nicole's formula was also used by Apéry to derive the formula

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}};$$

see van der Poorten [35, section 3]. The identity in the proof is equivalent to a special case of Newton's interpolation formula (see [32]). Moreover, equating coefficients of t^n in (5.1) gives the expansion of the polynomial x^n in the basis $\{(x + a_0) \cdots (x + a_k)\}_{0 \leq k \leq n-1}$, in which the coefficients can be computed by Newton's interpolation formula. We give such an example for the moments of Askey-Wilson polynomials at the end of this paper.

The following result is an immediate consequence of Lemma 11.

Proposition 12. Let L be a linear functional defined on polynomials in x and let a_0, a_1, \dots be complex numbers or indeterminates that do not involve x . Let

$$v_n = L((x + a_0)(x + a_1) \cdots (x + a_{n-1})).$$

Then

$$\sum_{n=0}^{\infty} L(x^n) t^n = \sum_{n=0}^{\infty} v_n \frac{t^n}{(1 + a_0 t) \cdots (1 + a_n t)}. \quad (5.3)$$

Remark 5. Substituting x by $x + T$ and a_j by $a_j - T$ ($j \in \mathbb{N}$) for any complex variable T in the above equation yields the formula for $L((x + T)^n)$.

The Hahn polynomials are defined in [26] by

$$Q_n(x; \alpha, \beta, N) = {}_3F_2 \left(\begin{matrix} -n, n + \alpha + \beta + 1, -x \\ \alpha + 1, -N \end{matrix} ; 1 \right). \quad (5.4)$$

(Some authors define $Q_n(x; \alpha, \beta, N)$ with $-N + 1$ replacing $-N$ in (5.4).) If N is a nonnegative integer then the polynomials $Q_n(x; \alpha, \beta, N)$ are defined only for $n = 0, 1, \dots, N$ and it is known that they are orthogonal with respect to the linear functional L_0 given by

$$L_0(p(x)) = \sum_{x=0}^N \binom{\alpha + x}{x} \binom{\beta + N - x}{N - x} p(x). \quad (5.5)$$

Applying Vandermonde's theorem we find that

$$L_0((x + \alpha + 1)_n) = \frac{(\alpha + \beta + 2)_N}{N!} \cdot \frac{(\alpha + 1)_n (\alpha + \beta + N + 2)_n}{(\alpha + \beta + 2)_n}.$$

This suggests that in studying moments of Hahn polynomials we reparametrize them by setting $\alpha = A - 1$, $\beta = C - A - 1$, and $N = B - C$ so that $\alpha + 1 = A$, $\alpha + \beta + N + 2 = B$, and $\alpha + \beta + 2 = C$.

Thus we define polynomials $R_n(x; A, B, C)$ by

$$\begin{aligned} R_n(x; A, B, C) &= Q_n(x; A-1, C-A-1, B-C) \\ &= {}_3F_2 \left(\begin{matrix} -n, n+C-1, -x \\ A, C-B \end{matrix} ; 1 \right) \end{aligned}$$

where A , B , and C are indeterminates. We will show that these polynomials are orthogonal with respect to the linear functional L on polynomials in x defined by

$$L((x+A)_m) = \frac{(A)_m(B)_m}{(C)_m}. \quad (5.6)$$

The polynomials $R_n(x; A, B, C)$ are closely related to the continuous Hahn polynomials, defined in [26] by

$$p_n(x; a, b, c, d) = i^n \frac{(a+c)_n(a+d)_n}{n!} {}_3F_2 \left(\begin{matrix} -n, n+a+b+c+d-1, a+ix \\ a+c, a+d \end{matrix} ; 1 \right).$$

Thus

$$R_n(x; A, B, C) = (-i)^n \frac{n!}{(A)_n(C-B)_n} p_n(ix; 0, B-A, A, C-B)$$

and

$$p_n(x; a, b, c, d) = i^n \frac{(a+c)_n(a+d)_n}{n!} R_n(-a-ix; a+c, b+c, a+b+c+d). \quad (5.7)$$

Lemma 13. *For nonnegative integers m and n we have*

$$L((x+A)_m(-x)_n) = \frac{(A)_{m+n}(B)_m(C-B)_n}{(C)_{m+n}}. \quad (5.8)$$

Proof. By Vandermonde's theorem we have

$$(-x)_n = \sum_{i=0}^n (-1)^i \binom{n}{i} (x+A+m)_i (A+m+i)_{n-i}.$$

Thus

$$\begin{aligned} L((x+A)_m(-x)_n) &= L \left(\sum_{i=0}^n (-1)^i \binom{n}{i} (x+A)_{m+i} (A+m+i)_{n-i} \right) \\ &= \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{(A)_{m+i}(B)_{m+i}}{(C)_{m+i}} (A+m+i)_{n-i} \\ &= \frac{(A)_{m+n}(B)_m(C-B)_n}{(C)_{m+n}} \end{aligned}$$

by Vandermonde's theorem again. \square

Proposition 14. *The polynomials $R_n(x; A, B, C)$ are orthogonal with respect to the linear functional L , and*

$$L(R_n(x; A, B, C)^2) = (-1)^n n! \frac{(B)_n (C - A)_n}{(A)_n (C - B)_n (C)_{n-1} (C + 2n - 1)}. \quad (5.9)$$

Proof. To show that the polynomials R_n are orthogonal, it suffices to show that for $0 \leq m < n$,

$$L(R_n(x; A, B, C)(x + A)_m) = 0.$$

We have

$$\begin{aligned} L(R_n(x; A, B, C)(x + A)_m) &= L\left(\sum_{k=0}^n \frac{(-n)_k (n + C - 1)_k}{k! (A)_k (C - B)_k} (-x)_k (x + A)_m\right) \\ &= \sum_{k=0}^n \frac{(-n)_k (n + C - 1)_k (A)_{m+k} (B)_m (C - B)_k}{k! (A)_k (C - B)_k (C)_{m+k}} \\ &= \frac{(A)_m (B)_m}{(C)_m} {}_3F_2\left(\begin{matrix} -n, n + C - 1, A + m \\ A, C + m \end{matrix}; 1\right) \\ &= \frac{(A)_m (B)_m (-m)_n (A - n - C + 1)_n}{(C)_m (A)_n (-m - n - C + 1)_n} \\ &= \frac{(A)_m (B)_m (C - A)_n (-m)_n}{(C)_{m+n} (A)_n}, \end{aligned} \quad (5.10)$$

where we have used the Pfaff-Saalschütz theorem to evaluate the ${}_3F_2$. So if $0 \leq m < n$ this is 0. Since $R_n(x; A, B, C)$ has leading coefficient $(C - 1)_{2n} / (A)_n (C - B)_n (C - 1)_n$, the case $m = n$ of (5.10) gives (5.9). \square

Proposition 15. *Let $M_n(A, B, C) = L(x^n)$. Then*

$$\sum_{n=0}^{\infty} M_n(A, B, C) t^n = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{(C)_n} \frac{t^n}{\prod_{l=0}^n (1 + (A + l)t)}, \quad (5.11)$$

and

$$\sum_{n=0}^{\infty} M_n(A, B, C) \frac{t^n}{n!} = e^{-At} {}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix}; 1 - e^{-t}\right). \quad (5.12)$$

Proof. Equation (5.11) follows from Proposition 12 and equation (5.6). By Lemma 5 we have

$$\varepsilon\left(\frac{t^n}{\prod_{l=0}^n (1 + (A + l)t)}\right) = e^{-(A+n)t} \frac{(e^t - 1)^n}{n!} = e^{-At} \frac{(1 - e^{-t})^n}{n!}$$

so applying ε to (5.11) gives (5.12). \square

A formula equivalent to (5.12) has been given by Dominici [14, Section 3.7].

Theorem 16. *The following S-fraction holds:*

$$\sum_{n=0}^{\infty} M_n(A, B, C)t^n = S\left(t; \frac{A(B-C)}{C}, \frac{B(C-A)}{C(C+1)}, \frac{(A+1)(B-C-1)C}{(C+1)(C+2)}, \frac{2(B+1)(C-A+1)}{(C+2)(C+3)}, \frac{(A+2)(B-C-2)(C+1)}{(C+3)(C+4)}, \frac{3(B+2)(C-A+2)}{(C+4)(C+5)}, \dots\right). \quad (5.13)$$

Proof. Let $F(t; A, B, C) = \sum_{n=0}^{\infty} M_n(A, B, C)t^n$. By (5.11) we have

$$F(t; A, B, C) = \frac{1}{1+At} G\left(\frac{t}{1+At}; A, B, C\right) \quad (5.14a)$$

with

$$G(t; A, B, C) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{(C)_n} \frac{t^n}{\prod_{l=0}^n (1+lt)}. \quad (5.14b)$$

We first expand $G(t; A, B, C)$ as an S-fraction

$$G(t; A, B, C) = S(t; C_1, C_2, \dots, C_n, \dots),$$

which can be written as a J-fraction

$$G(t; A, B, C) = J(t; C_1, C_1 C_2, C_2 + C_3, C_3 C_4, \dots), \quad (5.15)$$

where $C_n := C_n(A, B, C)$ ($n \geq 1$) are to be determined.

Rewriting (5.14b) as

$$G(t; A, B, C) = 1 + \frac{AB}{C} \cdot \frac{t}{1+t} \cdot G\left(\frac{t}{1+t}; A+1, B+1, C+1\right)$$

and using the J-fraction (5.15) to write

$$\begin{aligned} \frac{1}{1+t} \cdot G\left(\frac{t}{1+t}; A+1, B+1, C+1\right) \\ = J(t; C_1^+ - 1, C_1^+ C_2^+, C_2^+ + C_3^+ - 1, C_3^+ C_4^+, \dots) \end{aligned}$$

with $C_n^+ = C_n(A+1, B+1, C+1)$, we obtain

$$G(t; A, B, C) = 1 + \frac{AB}{C} t \cdot J(t; C_1^+ - 1, C_1^+ C_2^+, C_2^+ + C_3^+ - 1, C_3^+ C_4^+, \dots). \quad (5.16)$$

On the other hand, contracting the S-fraction by (2.11b) we have

$$G(t; A, B, C) = 1 + C_1 t \cdot J(t; C_1 + C_2, C_2 C_3, C_3 + C_4, C_4 C_5, \dots). \quad (5.17)$$

Comparing the above two J-fractions we derive

$$C_1 = \frac{AB}{C}, \quad C_1 + C_2 = C_1^+ - 1, \quad C_2 C_3 = C_1^+ C_2^+,$$

and for $n \geq 2$

$$\begin{aligned} C_{2n-1} + C_{2n} &= C_{2n-2}^+ + C_{2n-3}^+ - 1, \\ C_{2n} C_{2n+1} &= C_{2n-1}^+ C_{2n+2}^+. \end{aligned}$$

This yields immediately

$$\begin{cases} C_{2n-1} = \frac{(A+n-1)(B+n-1)(C+n-2)}{(C+2n-3)(C+2n-2)}; \\ C_{2n} = \frac{n(B-C-n+1)(A-C-n+1)}{(C+2n-2)(C+2n-1)}. \end{cases} \quad (5.18)$$

Next, by (5.14a) and (5.15) we have the J-fraction for $F(t; A, B, C)$:

$$F(t; A, B, C) = J(t; C_1 - A, C_1 C_2, C_2 + C_3 - A, C_3 C_4, \dots). \quad (5.19a)$$

It remains to determine a sequence α_n such that

$$F(t; A, B, C) = S(t; \alpha_1, \alpha_2, \dots, \alpha_n, \dots) \quad (5.19b)$$

$$= J(t; \alpha_1, \alpha_1 \alpha_2, \alpha_2 + \alpha_3, \alpha_3 \alpha_4, \dots). \quad (5.19c)$$

From (5.18), (5.19c) and (5.19a) we derive

$$\begin{aligned} \alpha_1 &= C_1 - A = \frac{A(B-C)}{C} \\ \alpha_2 &= \frac{C_1 C_2}{\alpha_1} = \frac{B(C-A)}{C(C+1)}. \end{aligned}$$

and for $n \geq 2$,

$$\begin{aligned} \alpha_{2n-1} &= C_{2n-1} + C_{2n-2} - A \\ &= \frac{(A+n)(B-C-n)(C+n-1)}{(C+n)(C+n+1)}, \\ \alpha_{2n} &= \frac{C_{2n} C_{2n-1}}{\alpha_{2n-1}} \\ &= \frac{n(B+n-1)(C-A+n-1)}{(C+2n-1)(C+2n-2)}. \end{aligned}$$

Plugging these values in (5.19b) yields the S-fraction in Theorem 16. \square

As an example of Proposition 15 let us consider the case $A = 1, B = 1, C = 2$. We have ${}_2F_1(1, 1; 2; z) = -z^{-1} \log(1-z)$, so

$$e^{-t} {}_2F_1\left(\begin{matrix} 1, 1 \\ 2 \end{matrix}; 1 - e^{-t}\right) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

where B_n is the n th Bernoulli number. So $M_n(1, 1, 2) = B_n$.

The orthogonal polynomials whose moments are Bernoulli numbers were considered by Touchard [39], but he was not able to find an explicit formula for them. An explicit formula (different from ours, but equivalent) was found by Wyman and Moser [43] and these polynomials were further studied by Carlitz [5]. One can show similarly that $M_n(1, 2, 3) = -2B_{n+1}$ and $M_n(1, 1, 3) = 2(B_n + B_{n+1})$. Krattenthaler [27, Section 2.7] proved a result equivalent to

$$\frac{e^t}{6} \sum_{n=0}^{\infty} M_n(2, 2, 4) \frac{t^n}{n!} = \sum_{n=0}^{\infty} B_{n+2} \frac{t^n}{n!}.$$

Fulmek and Krattenthaler [28, equation (5.14)] expressed the moments of general continuous Hahn polynomials (with some integrality and nonnegativity restrictions on the parameters) in terms of Bernoulli numbers, generalizing all of these formulas.

Chapoton [9] studied some special cases of Racah polynomials whose moments are the *median Bernoulli numbers* since they are, up to an easy power of 2, the main diagonal of Seidel's difference tableau of Bernoulli numbers [38, p. 181].

For the moments μ_n of the original Hahn polynomials

$$Q_n(x; \alpha, \beta, N) = R_n(x; \alpha + 1, \alpha + \beta + 2, \alpha + \beta + N + 2),$$

equation (5.12) gives

$$\sum_{n=0}^{\infty} \mu_n \frac{t^n}{n!} = e^{-(\alpha+1)t} {}_2F_1 \left(\begin{matrix} \alpha + 1, \alpha + \beta + N + 2 \\ \alpha + \beta + 2 \end{matrix}; 1 - e^{-t} \right).$$

Applying Pfaff's transformation gives

$$\sum_{n=0}^{\infty} \mu_n \frac{t^n}{n!} = {}_2F_1 \left(\begin{matrix} \alpha + 1, -N \\ \alpha + \beta + 2 \end{matrix}; 1 - e^t \right).$$

If N is a nonnegative integer, we may apply the terminating ${}_2F_1$ transformation

$${}_2F_1 \left(\begin{matrix} a, -N \\ b \end{matrix}; z \right) = \frac{(b-a)_N}{(b)_N} {}_2F_1 \left(\begin{matrix} a, -N \\ 1 + a - b - N \end{matrix}; 1 - z \right)$$

to obtain

$$\sum_{n=0}^{\infty} \mu_n \frac{t^n}{n!} = \frac{(\beta + 1)_N}{(\alpha + \beta + 2)_N} {}_2F_1 \left(\begin{matrix} \alpha + 1, -N \\ -\beta - N \end{matrix}; e^t \right).$$

Equating coefficients of $x^n/n!$ gives

$$\begin{aligned} \mu_n &= \frac{(\beta + 1)_N}{(\alpha + \beta + 2)_N} \sum_{k=0}^N \frac{(\alpha + 1)_k (-N)_k}{k! (-\beta - N)_k} k^n \\ &= \frac{N!}{(\alpha + \beta + 2)_N} \sum_{k=0}^n \binom{\alpha + k}{k} \binom{\beta + N - k}{N - k} k^n, \end{aligned}$$

so we see that, as expected, in this case the linear functional L is a constant multiple of the linear functional L_0 defined by (5.5).

6. CONCLUDING REMARKS

In what follows the standard q -notations (see [2, 22, 26]) will be used. The Askey-Wilson polynomials are defined by

$$p_n(x; a, b, c, d | q) := (ab, ac, ad; q)_n a^{-n} A_n(x) \quad (n \in \mathbb{N})$$

with

$$A_n(x) = {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q \right], \quad (6.1)$$

where $x = \cos \theta$, see [3, 22, 26].

In the last decade much work has been done to extend Viennot's results for moments of classical orthogonal polynomials to the moments of the Askey-Wilson polynomials; see [10, 11, 24, 30]. It would be interesting to see to what extent the methods of this paper can be q -generalized. We show that Proposition 12 works also for the moments of the Askey-Wilson polynomials.

For $0 < q < 1$, $\max\{|a|, |b|, |c|, |d|\} < 1$, $z = e^{i\theta}$ and $x = \cos \theta$, the linear functional $\mathcal{L}_q : \mathbb{C}[x] \mapsto \mathbb{C}$ associated to the orthogonal measure of the Askey-Wilson polynomials has the explicit integral form [3]:

$$\begin{aligned} \mathcal{L}_q(x^n) &= \frac{1}{2\pi} \frac{(ab, ac, ad, bc, bd, cd; q)_\infty}{(abcd; q)_\infty} \\ &\quad \times \int_0^\pi \frac{(\cos \theta)^n (e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_\infty}. \end{aligned}$$

Thus, the Askey-Wilson integral reads $\mathcal{L}_q(1) = 1$. As $(az, a/z; q)_n = (ae^{i\theta}, ae^{-i\theta}; q)_n$, the value $\mathcal{L}_q((az, a/z; q)_n)$ amounts to shifting a to aq^n in the integral, so

$$\mathcal{L}_q((az, a/z; q)_n) = \frac{(ab, ac, ad; q)_n}{(abcd; q)_n}. \quad (6.2)$$

The same argument yields the q -version of (2.7):

$$\mathcal{L}_q((az, a/z; q)_n (bz, b/z; q)_m) = \frac{(ab, q)_{m+n} (ac, ad; q)_n (bc, bd; q)_m}{(abcd; q)_{m+n}}. \quad (6.3)$$

As $x = (z + 1/z)/2$, choose $a_j = (q^{-j}/a + aq^j)/2$ for $j \in \mathbb{N}$ and $t \in \mathbb{C}$. Then

$$(x - a_0) \cdots (x - a_{n-1}) = (-1)^n (2a)^{-n} q^{-\binom{n}{2}} (az, a/z; q)_n.$$

Applying Proposition 12 to (6.2) we derive the generating function of the moments of Askey-Wilson polynomials.

Theorem 17. *We have*

$$\sum_{n=0}^{\infty} \mathcal{L}_q(x^n) u^n = \sum_{n=0}^{\infty} \frac{(ab, ac, ad; q)_n}{(abcd; q)_n} \frac{(-1)^n (2a)^{-n} q^{-\binom{n}{2}} u^n}{(1 - a_0 u) \dots (1 - a_n u)}. \quad (6.4)$$

It is interesting to note that computing the coefficient of u^n in the right side of (6.4) by partial fraction decomposition or using Newton's interpolation formula [30] yields the $t = 0$ case of the double sum formula for the Askey-Wilson moments in [11, Theorem 1.13]. The general case follows by applying the shifted version of Proposition 12, see Remark 5. Finally, we note that an algebraic proof of the orthogonality relation of the Askey-Wilson polynomials was given by Gasper and Rahman in [29, p. 190–191] using (6.3).

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