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Unsteady non-Newtonian fluid flows with boundary conditions of friction type : the case of shear thickening fluids.

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Abstract

Motivated by applications to industrial processes like lubrication or extrusion / injection we consider in this paper non-stationary flow problems for general incompressible dilatant (shear thickening) fluids. The conservation of mass and momentum lead to a p-Laplacian unsteady Stokes system where the real parameter p is greater than 2. Such fluids undergo complex slip fluid-solid interface laws of friction type which may be described by a subdifferential boundary condition. Hence the fluid velocity and pressure satisfy a non-linear parabolic variational inequality and belong to Banach spaces depending on p. We prove the existence of a solution by using a fixed point technique combined with compactness and monotonicity arguments.

Keywords: unsteady shear thickening fluid flow, p-Laplacian, slip boundary condition of friction type, non-linear variational parabolic inequality, monotonicity methods
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1. Introduction

Many complex fluids involved in industrial processes like extrusion / injection or lubrification do not satisfy the classical linear constitutive law of Newtonian fluids and obey rather a power law depending on a real parameter p given as

$$\sigma = 2\mu\Big(\theta, \upsilon, \big|D(\upsilon)\big|\Big)\big|D(\upsilon)\big|^{p-2}D(\upsilon) - \pi \mathrm{Id}_{\mathbb{R}^3}$$

where μ is a given mapping, υ is the fluid velocity, π is the pressure, θ is the temperature and $D(\upsilon)$ and σ are the strain rate tensor and the stress tensor respectively. The case $p > 2$ models the behaviour of dilatant (or shear thickening) fluids like colloidal fluids and the case $p \in (1, 2)$ corresponds to pseudo-plastic (or shear thinning) fluids like molten polymers (see [2, 36, 26] for instance). When $p = 2$ the dependence of the mapping μ with respect to the temperature, the velocity and the modulus of the strain rate tensor still yields a non-linear constitutive law and we obtain a generalization of Newtonian fluids which fits the behaviour of fluids like oils (see [20] for instance).

If the fluid is incompressible the conservation of mass and momentum lead to the p-Laplacian Stokes system

$$\begin{cases} \dfrac{\partial \upsilon}{\partial t} - 2div\Big(\mu(\theta, \upsilon, |D(\upsilon)|)|D(\upsilon)|^{p-2}D(\upsilon)\Big) + \nabla\pi = f \quad \text{in } (0,T)\times\Omega \\ div(\upsilon) = 0 \qquad \text{in } (0,T)\times\Omega \end{cases}$$

where $(0, T)$ is the time interval ($T > 0$) and $\Omega \subset \mathbb{R}^3$ is the fluid flow domain.

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Such complex fluids exhibit also complex behaviour at the boundary. Indeed experimental studies have shown that threshold slip phenomena may occur (see for instance [3, 11, 29, 30, 35, 19, 27, 28, 33]). More precisely whenever the norm of the shear stress is less than the threshold, the fluid tangential velocity coincide with the wall velocity otherwise some slip occurs and the fluid relative tangential velocity is in the opposite direction of the shear stress but its modulus is an unknwon of the problem. From the mechanical point of view this kind of behaviour is reminiscent of Tresca's friction for solids ([10]) and from the mathematical point of view it can be described by a subdifferential boundary condition, namely

$$\upsilon_n = 0, \quad \upsilon_\tau - s \in -\partial\psi_{\overline{B}_{\mathbb{R}^3}(0,k)}(\sigma_\tau)$$

where k is a given positive threshold, $\partial\psi_{\overline{B}_{\mathbb{R}^3}(0,k)}$ is the subdifferential of the indicator function of the closed ball $\overline{B}_{\mathbb{R}^3}(0,k)$, s is the sliding velocity of the wall, υ_n, υ_τ and σ_τ are the normal component of the velocity, the tangential component of the velocity and the shear stress respectively.

The first existence results with this kind of boundary conditions have been obtained by H.Fujita ([12, 13]) for stationary Newtonian Stokes flows, paving the way to an abundant literature in the case of steady or unsteady Newtonian fluid flows (see for instance [14, 15, 31, 16, 17, 21, 32, 22, 23, 34, 6, 7, 8]). The case of non-Newtonian fluids satisfying a power law with $p \neq 2$ leads to a very different mathematical framework since both the velocity and the pressure are expected to belong to Banach spaces depending on p. The reader is referred to [24, 25] or [4, 9] and the reference therein for the case of a constant mapping μ with classical boundary conditions like Dirichlet, Neumann or Navier boundary conditions. The general case of a constitutive power law with Tresca's friction boundary conditions is considered in [5] for steady flows.

The aim of this paper is to extend the study to non-stationary flows. As a first step we start with the case $p \geq 2$ (i.e. shear thickening fluids or generalized Newtonian fluids) while the case $p \in (1, 2)$ will be considered in a forthcoming paper. Moreover we assume that the temperature θ is a given data. This simplified framework is fully justified when the convection term of the heat equation can be neglected: in such a case the heat equation is decoupled from the flow problem and can be solved independently, which allows to consider θ as a data in the study of the flow problem.

The paper is organized as follows. In Section 2 we introduce the mathematical framework and we derive the formulation of the problem as a non-linear parabolic variational inequality. In Section 3 we consider an auxiliary problem where the mapping μ depends on the modulus of the strain rate tensor $\left|D(\upsilon)\right|$ while its two other arguments are given data. By replacing the time-derivative of υ by a finite difference we build a sequence of approximate solutions and we establish its convergence to the unique solution of the auxiliary flow problem by using compactness and monotonicity arguments. This allows us to apply a fixed point technique combined with De Rham theorem to prove the existence of a solution to our original fluid flow problem in Section 4.

2. Description of the problem and variational formulation

We consider this fluid flow problem in a domain $\Omega \subset \mathbb{R}^3$ given by

$$\Omega = \left\{(x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : \ x' \in \omega, \ 0 < x_3 < h(x')\right\}$$

where ω is a non empty bounded domain of $\mathbb{R}^2$ with a Lipschitz continuous boundary and h is a Lipschitz continuous function which is bounded from above and from below by some positive real numbers.

We denote by $D(\upsilon) = \left(d_{ij}(\upsilon)\right)_{1\leq i,j\leq 3}$ the strain rate tensor given by

$$d_{ij}(\upsilon) = \frac{1}{2}\left(\frac{\partial \upsilon_i}{\partial x_j} + \frac{\partial \upsilon_j}{\partial x_i}\right) \quad 1 \leq i, j \leq 3,$$

where υ is the fluid velocity. We assume that the fluid is non-Newtonian and satisfies the following constitutive law depending on a parameter $p > 1$:

$$\sigma = 2\mu\Big(\theta, \upsilon, \left|D(\upsilon)\right|\Big)\left|D(\upsilon)\right|^{p-2}D(\upsilon) - \pi \mathrm{Id}_{\mathbb{R}^3} \tag{1}$$

where $\sigma = (\sigma_{ij})_{1\le i,j\le 3}$ is the stress tensor, θ is the temperature, π is the pressure and μ is a given mapping such that

$$(o,e,d) \mapsto \mu(o,e,d) \quad \text{is continuous on } \mathbb{R}\times\mathbb{R}^3\times\mathbb{R}_+, \tag{2}$$

$$d \mapsto \mu(.,.,d) \quad \text{is monotone increasing on } \mathbb{R}_+, \tag{3}$$

$$\text{there exists } (\mu_0,\mu_1)\in\mathbb{R}^2 \text{ such that } \ 0<\mu_0\le\mu(o,e,d)\le\mu_1 \quad \text{for all } (o,e,d)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}_+. \tag{4}$$

The fluid is assumed to be incompressible. The problem is then described by the conservation of mass and momentum (p-Laplacian Stokes system)

$$\begin{cases} \dfrac{\partial \upsilon}{\partial t} - 2div\Big(\mu(\theta,\upsilon,|D(\upsilon)|)|D(\upsilon)|^{p-2}D(\upsilon)\Big) + \nabla\pi = f \quad \text{in } (0,T)\times\Omega \\ div(\upsilon) = 0 \qquad \text{in } (0,T)\times\Omega \end{cases} \tag{5}$$

where $(0,T)$ is a non-trivial time interval (i.e. $T>0$) and f represents the vector of external forces.

We denote by $u\cdot v$ the Euclidean inner product of two vectors u and v in $\mathbb{R}^3$. We decompose the boundary of Ω as $\partial\Omega = \Gamma_0\cup\Gamma_L\cup\Gamma_1$ with

$$\Gamma_0 = \Big\{(x',x_3)\in\overline{\Omega}:\ x_3 = 0\Big\}, \qquad \Gamma_1 = \Big\{(x',x_3)\in\overline{\Omega}:\ x_3 = h(x')\Big\}$$

and Γ_L the lateral part of $\partial\Omega$.

Let us now describe the boundary conditions for the velocity. Let $g:\partial\Omega\to\mathbb{R}^3$ be such that

$$\int_{\partial\Omega} g\cdot n\, dY = 0, \quad g = 0 \text{ on } \Gamma_1, \quad g\neq 0 \text{ on } \Gamma_L, \quad g\cdot n = 0 \text{ on } \Gamma_0$$

with $n=(n_1,n_2,n_3)$ the unit outward normal vector to $\partial\Omega$.

In lubrication or extrusion / injection industrial processes the usual physical setting is given by a fixed wall for the upper part of the boundary of the fluid domain while the lower part is a moving device with a given sliding velocity. We assume that the velocity is given on $(0,T)\times(\Gamma_1\cup\Gamma_L)$ by the following non-homogeneous Dirichlet conditions

$$\upsilon = 0 \text{ on } (0,T)\times\Gamma_1, \quad \upsilon = g\xi \text{ on } (0,T)\times\Gamma_L \tag{6}$$

where ξ is a function depending only on the time variable such that

$$\xi(0) = 1. \tag{7}$$

We define the normal and tangential velocities on $\partial\Omega$ as $\upsilon_n = \upsilon\cdot n$ and $\upsilon_\tau = \upsilon - \upsilon_n n$ and the normal and tangential components of the stress vector are given by

$$\sigma_n = \sum_{i,j=1}^{3}\sigma_{ij}n_jn_i, \quad \sigma_\tau = \left(\sum_{j=1}^{3}\sigma_{ij}n_j - \sigma_n n_i\right)_{1\le i\le 3}.$$

We assume that the fluid is subjected to friction boundary conditions on Γ_0, i.e.

$$\upsilon\cdot n = 0 \qquad \text{on } (0,T)\times\Gamma_0 \quad \text{(slip condition)} \tag{8}$$

and υ_τ is unknown and satisfies Tresca's friction law [10]

$$\left.\begin{array}{l} |\sigma_\tau| < k \Rightarrow \upsilon_\tau = s \\ |\sigma_\tau| = k \Rightarrow \exists\lambda\ge 0 \quad \upsilon_\tau = s - \lambda\sigma_\tau \end{array}\right\} \qquad \text{on } (0,T)\times\Gamma_0 \quad \text{(Tresca's friction law)} \tag{9}$$

where s is the sliding velocity of the lower part of the boundary and k is the positive friction threshold.

We complete the description of the problem with the initial condition

$$\upsilon(0) = \upsilon_0 \text{ in } \Omega \tag{10}$$

and we assume that $\upsilon_0 \in \left(W^{1,p}(\Omega)\right)^3$ such that

$$div(\upsilon_0) = 0 \text{ in } \Omega, \quad \upsilon_0 = 0 \text{ on } \Gamma_1, \quad \upsilon_0 = g \text{ on } \Gamma_L, \quad \upsilon_0 \cdot n = 0 \text{ on } \Gamma_0. \tag{11}$$

Let us introduce now the functional framework. We denote here and throughout this paper by $\mathbf{X}$ the functional space X^3. We consider the following subspaces of $\mathbf{W}^{1,p}(\Omega)$:

$$V_0^p = \left\{\varphi \in \mathbf{W}^{1,p}(\Omega);\ \ \varphi = 0 \text{ on } \Gamma_1 \cup \Gamma_L \text{ and } \varphi \cdot n = 0 \text{ on } \Gamma_0\right\}$$

and

$$V_{0.div}^p = \left\{\varphi \in V_0^p;\ \ div(\varphi) = 0 \text{ in } \Omega\right\}, \quad V_{\Gamma_1}^p = \left\{\varphi \in \mathbf{W}^{1,p}(\Omega);\ \ \varphi = 0 \text{ on } \Gamma_1\right\}$$

endowed with the norm

$$\|\upsilon\|_{1.p} = \left(\int_\Omega |\nabla \upsilon|^p \, dx\right)^{\frac{1}{p}}.$$

By using Korn's inequality ([37]), there exists $C_{Korn} > 0$ such that

$$\left(\int_\Omega \left|D(u)\right|^p dx\right)^{1/p} = \|D(u)\|_{(L^p(\Omega))^{3\times 3}} \geq C_{Korn}\|u\|_{1.p} \quad \forall u \in V_{\Gamma_1}^p \tag{12}$$

and recalling that the mapping $z \mapsto z^p$ is monotone increasing on $\mathbb{R}^+$ we have

$$\left(\int_\Omega \left|D(u)\right|^p dx\right)^{1/p} = \|D(u)\|_{(L^p(\Omega))^{3\times 3}} \leq \|u\|_{1.p} \quad \forall u \in \mathbf{W}^{1,p}(\Omega). \tag{13}$$

Starting from the definition of the strain rate tensor σ (see (1)) we introduce the mapping $\mathcal{F} : \mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ given by

$$\mathcal{F}(\lambda_0, \lambda_1, \lambda_2) = 2\mu(\lambda_0, \lambda_1, |\lambda_2|)|\lambda_2|^{p-2}\lambda_2 \quad \text{if } \lambda_2 \neq 0_{\mathbb{R}^{3\times 3}}, \quad \mathcal{F}(\lambda_0, \lambda_1, \lambda_2) = 0_{\mathbb{R}^{3\times 3}} \quad \text{otherwise.} \tag{14}$$

With (4) we have immediately

$$\left|\mathcal{F}(\lambda_0, \lambda_1, \lambda_2)\right| \leq 2\mu_1|\lambda_2|^{p-1} \quad \forall(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3}. \tag{15}$$

Let $q > 1$ such that $q - p + 1 > 0$. Then for any $\theta \in L^{\tilde{q}}\left(0, T; L^{\tilde{p}}(\Omega)\right)$ with $\tilde{q} \geq 1$ and $\tilde{p} \geq 1$, and for any $u \in L^q\left(0, T; \mathbf{W}^{1,p}(\Omega)\right)$ we have

$$\mathcal{F}\left(\theta, u + \upsilon_0\xi, D(u + \upsilon_0\xi)\right) \in L^{\frac{q}{p-1}}\left(0, T; \left(L^{p'}(\Omega)\right)^{3\times 3}\right)$$

where $p' = \dfrac{p}{p-1}$ (resp. $q' = \dfrac{q}{q-1}$) is the conjugate number of p (resp. q).

Hence we may define $\displaystyle\int_0^T \int_\Omega \mathcal{F}\left(\theta, u + \upsilon_0\xi, D(u + \upsilon_0\xi)\right) : D(\overline{\varphi})\, dxdt$ for all $\overline{\varphi} \in L^{\frac{q}{q-p+1}}\left(0, T; \mathbf{W}^{1,p}(\Omega)\right)$.

In order to deal with homogeneous boundary conditions on $(0, T) \times (\Gamma_1 \cup \Gamma_L)$, we set $\overline{\upsilon} = \upsilon - \upsilon_0\xi$. The variational formulation of the problem (5) with the boundary conditions (6), (8)-(9) and the initial condition (10) is given by

Problem (P) Let $f \in L^{q'}\Big(0,T;\mathbf{L}^2(\Omega)\Big)$, $k \in L^{q'}\Big(0,T;L^{p'}_+(\Gamma_0)\Big)$, μ satisfying (2)-(4), $\theta \in L^{\tilde{q}}\Big(0,T;L^{\tilde{p}}(\Omega)\Big)$ with $\tilde{q} \geq 1$ and $\tilde{p} \geq 1$, $s \in L^{q}\Big(0,T;\mathbf{L}^{p}(\Gamma_0)\Big)$, $\xi \in W^{1,q'}(0,T)$ satisfying (7) and $\upsilon_0 \in \mathbf{W}^{1,p}(\Omega)$ satisfying (11). Find $\overline{\upsilon} \in C\Big([0,T];\mathbf{L}^2(\Omega)\Big) \cap L^q\Big(0,T;V^p_{0.div}\Big)$ with $\dfrac{\partial \overline{\upsilon}}{\partial t} \in L^{q'}\Big(0,T;(V^p_{0.div})'\Big)$, and $\pi \in H^{-1}\Big(0,T;L^{p'}_0(\Omega)\Big)$, satisfying the following non-linear parabolic variational inequality

$$\begin{aligned}&\Big\langle \frac{\partial}{\partial t}(\overline{\upsilon},\tilde{\vartheta})_{\mathbf{L}^2(\Omega)},\zeta\Big\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} + \int_0^T\int_\Omega \mathcal{F}\Big(\theta,\overline{\upsilon}+\upsilon_0\xi,D(\overline{\upsilon}+\upsilon_0\xi)\Big) : D(\tilde{\vartheta})\zeta\,dxdt - \Big\langle \int_\Omega \pi\, div(\tilde{\vartheta}),\zeta\Big\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}\\ &+J(\overline{\upsilon}+\tilde{\vartheta}\zeta) - J(\overline{\upsilon}) \geq \int_0^T \Big(f - \frac{\partial \xi}{\partial t}\upsilon_0,\tilde{\vartheta}\Big)_{\mathbf{L}^2(\Omega)}\zeta\,dt \qquad \forall \tilde{\vartheta}\in V^p_0,\ \forall \zeta \in \mathcal{D}(0,T)\end{aligned} \tag{16}$$

and the initial condition

$$\overline{\upsilon}(0) = \upsilon_0 - \upsilon_0\xi(0) = 0 \qquad \text{in } \Omega \tag{17}$$

where

$$J : \begin{cases} L^q(0,T;V^p_0) & \to \mathbb{R} \\ \overline{\varphi} & \mapsto \displaystyle\int_0^T\int_{\Gamma_0} k|\overline{\varphi}-\tilde{s}|\,dx'dt, \qquad \tilde{s} = s - (\upsilon_0)_\tau\xi \end{cases}$$

and $\Big\langle .,. \Big\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}$ and $(.,.)_{\mathbf{L}^2(\Omega)}$ denote respectively the duality product between $\mathcal{D}(0,T)$ and $\mathcal{D}'(0,T)$ and the inner product in $\mathbf{L}^2(\Omega)$.

Then we observe first that $L^q\Big(0,T;V^p_{0.div}\Big) \subset L^{\frac{q}{q-p+1}}\Big(0,T;\mathbf{W}^{1,p}(\Omega)\Big)$ if and only if $q \geq \dfrac{q}{q-p+1}$ i.e. if and only if $q \geq p$. It allows us to consider the operator $\mathcal{A} : L^q\Big(0,T;V^p_{0.div}\Big) \to \Big(L^q\Big(0,T;V^p_{0.div}\Big)\Big)'$ defined by

$$\Big[\mathcal{A}(u),\overline{\varphi}\Big] = \int_0^T\int_\Omega \mathcal{F}\Big(\theta,u+\upsilon_0\xi,D(u+\upsilon_0\xi)\Big) : D(\overline{\varphi})\,dxdt \qquad \forall \overline{\varphi} \in L^q\Big(0,T;V^p_{0.div}\Big), \quad \forall u \in L^q\Big(0,T;V^p_{0.div}\Big)$$

where $\Big[\cdot,\cdot\Big]$ denotes the duality product between $L^q\Big(0,T;V^p_{0.div}\Big)$ and its dual $\Big(L^q\Big(0,T;V^p_{0.div}\Big)\Big)'$.

With (15) $\mathcal{A}$ is clearly a bounded operator. Nevertheless if $q > p$ we can not establish the coercivity of $\mathcal{A}$. Indeed for any $(u,\hat{\upsilon}) \in \Big(L^q\Big(0,T;V^p_{0.div}\Big)\Big)^2$ we have

$$\begin{aligned}\Big[\mathcal{A}(u),u-\hat{\upsilon}\Big] &= 2\int_0^T\int_\Omega \mu\Big(\theta,u+\upsilon_0\xi,|D(u+\upsilon_0\xi)|\Big)\Big|D(u+\upsilon_0\xi)\Big|^p\,dxdt\\ &- \int_0^T\int_\Omega \mathcal{F}\Big(\theta,u+\upsilon_0\xi,D(u+\upsilon_0\xi)\Big) : D(\upsilon_0\xi+\hat{\upsilon})\,dxdt.\end{aligned}$$

Recalling that $L^q\Big(0,T;V^p_{0.div}\Big) \subset L^{\frac{q}{q-p+1}}\Big(0,T;V^p_{0.div}\Big)$ we obtain with (4) and (12)-(13)

$$\begin{aligned}\Big[\mathcal{A}(u),u-\hat{\upsilon}\Big] &\geq 2\mu_0\int_0^T\int_\Omega \Big|D(u+\upsilon_0\xi)\Big|^p\,dxdt - 2\mu_1\int_0^T\int_\Omega \Big|D(u+\upsilon_0\xi)\Big|^{p-1}\Big|D(\upsilon_0\xi+\hat{\upsilon})\Big|\,dxdt\\ &\geq 2(C_{Korn})^p\mu_0\|u+\upsilon_0\xi\|^p_{L^p(0,T;V^p_{\Gamma_1})} - 2\mu_1\|u+\upsilon_0\xi\|^{p-1}_{L^q(0,T;V^p_{\Gamma_1})}\|\upsilon_0\xi+\hat{\upsilon}\|_{L^{\frac{q}{q-p+1}}(0,T;V^p_{\Gamma_1})}.\end{aligned}$$

Since $L^p\Big(0,T;V^p_{0.div}\Big) \subset L^p\Big(0,T;V^p_{\Gamma_1}\Big)$ with the same norm, we have

$$\begin{aligned}\Big[\mathcal{A}(u),u-\hat{\upsilon}\Big] \geq\ & 2(C_{Korn})^p\mu_0\Big|\|u\|_{L^p(0,T;V^p_{0.div})} - \|\upsilon_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}\Big|^p\\ & -2\mu_1\Big(\|u\|_{L^q(0,T;V^p_{0.div})} + \|\upsilon_0\xi\|_{L^q(0,T;V^p_{\Gamma_1})}\Big)^{p-1}\|\upsilon_0\xi+\hat{\upsilon}\|_{L^{\frac{q}{q-p+1}}(0,T;V^p_{\Gamma_1})}\end{aligned}$$

So, whenever $\|u\|_{L^q(0,T;V^p_{0.div})} \neq 0$, we get

$$\begin{aligned}\frac{\left[\mathcal{A}(u), u-\hat{v}\right]}{\|u\|_{L^q(0,T;V^p_{0.div})}} \geq \quad & 2(C_{Korn})^p \mu_0 \frac{\|u\|^p_{L^p(0,T;V^p_{0.div})}}{\|u\|_{L^q(0,T;V^p_{0.div})}} \left|1 - \frac{\|v_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{\|u\|_{L^p(0,T;V^p_{0.div})}}\right|^p \\ & -2\mu_1 \|u\|^{p-2}_{L^q(0,T;V^p_{0.div})} \left(1 + \frac{\|v_0\xi\|_{L^q(0,T;V^p_{\Gamma_1})}}{\|u\|_{L^q(0,T;V^p_{0.div})}}\right)^{p-1} \|v_0\xi + \hat{v}\|_{L^{\frac{q}{q-p+1}}(0,T;V^p_{\Gamma_1})}\end{aligned} \tag{18}$$

But if $q > p$ the embedding of $L^q\Big(0,T;V^p_{0.div}\Big)$ in $L^p\Big(0,T;V^p_{0.div}\Big)$ is continuous i.e. there exists $C_q > 0$ such that

$$\|u\|_{L^p(0,T;V^p_{0.div})} \leq C_q \|u\|_{L^q(0,T;V^p_{0.div})} \quad \forall u \in L^q\Big(0,T;V^p_{0.div}\Big).$$

It follows that

$$\frac{\|u\|^p_{L^p(0,T;V^p_{0.div})}}{\|u\|_{L^q(0,T;V^p_{0.div})}} \leq C_q \|u\|^{p-1}_{L^p(0,T;V^p_{0.div})} \quad \forall u \in L^q\Big(0,T;V^p_{0.div}\Big)$$

and we can not infer from (18) that $\mathcal{A}$ is coercive.

On the other hand, when $q = p \geq 2$ we have

$$\begin{aligned}\frac{\left[\mathcal{A}(u), u-\hat{v}\right]}{\|u\|_{L^p(0,T;V^p_{0.div})}} \geq \quad & 2(C_{Korn})^p \mu_0 \|u\|^{p-1}_{L^p(0,T;V^p_{0.div})} \left|1 - \frac{\|v_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{\|u\|_{L^p(0,T;V^p_{0.div})}}\right|^p \\ & -2\mu_1 \|u\|^{p-2}_{L^p(0,T;V^p_{0.div})} \left(1 + \frac{\|v_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{\|u\|_{L^p(0,T;V^p_{0.div})}}\right)^{p-1} \|v_0\xi + \hat{v}\|_{L^p(0,T;V^p_{\Gamma_1})}\end{aligned}$$

and recalling that $J(u) \geq 0$ for all $u \in L^p\Big(0,T;V^p_{0.div}\Big)$ we obtain

$$\lim_{\|u\|_{L^p(0,T;V^p_{0.div})} \to +\infty} \frac{[\mathcal{A}u, u-\hat{v}] + J(u)}{\|u\|_{L^p(0,T;V^p_{0.div})}} = +\infty$$

which proves the coercivity of the operator $\mathcal{A}$.

Hence we will assume for now on that $p = q \geq 2$. In order to prove that problem (P) admits at least a solution (with $p = q \geq 2$) we will apply a fixed point method with respect to the second argument of the mapping μ. As a first step we consider in the next section the simplified case when the viscosity μ depends only on a given temperature field θ, a given velocity field $\mathbf{u}$ and the modulus of the strain rate tensor $\left|D(v)\right| = \left|D(\overline{v} + v_0\xi)\right|$.

3. Auxiliary flow problem

Let $p \geq 2$ and

$$\mathcal{K}^p = L^p\Big(0,T;V^p_{0.div}\Big).$$

Since $p \geq 2$ we have $(\mathcal{K}^p)' = L^{p'}\Big(0,T;(V^p_{0.div})'\Big)$ and $\mathcal{K}^p \subset (\mathcal{K}^p)'$. Moreover let $H = \Big\{\psi \in \mathbf{L}^2(\Omega);\ \ div(\psi) = 0 \text{ in } \Omega,\ \psi \cdot n = 0 \text{ on } \partial\Omega\Big\}$ endowed with the norm of $\mathbf{L}^2(\Omega)$ and $\mathcal{H} = L^2\Big(0,T;H\Big)$. Recalling that the embedding of $V^p_{0.div}$ into H is dense and continuous (see Theorem 2.8, Chapter I in [18] for instance), we know that any function $\overline{\varphi} \in \mathcal{K}^p$ such that $\dfrac{\partial\overline{\varphi}}{\partial t} \in (\mathcal{K}^p)'$, by possibly modifying $\overline{\varphi}$ on a negligible subset of $[0,T]$, belongs to $C\Big([0,T];H\Big)$ (see [24]). Hence we may define

$$\phi^p = \left\{\overline{\varphi} \in \mathcal{K}^p;\ \frac{\partial\overline{\varphi}}{\partial t} \in (\mathcal{K}^p)',\ \overline{\varphi}(0) = 0\right\}.$$

For the sake of notational simplicity we will denote from now on by $\overline{\varphi}'$ the time derivative of any function $\overline{\varphi}$ (in the distribution sense).

Now let $\mathbf{u} \in L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big)$ be given. We define the operator $\mathcal{A}_{\mathbf{u}} : \mathcal{K}^p \to (\mathcal{K}^p)'$ by

$$\Big[\mathcal{A}_{\mathbf{u}}(u),\overline{\varphi}\Big] = \int_0^T \int_\Omega \mathcal{F}\Big(\theta,\mathbf{u}+\upsilon_0\xi, D(u+\upsilon_0\xi)\Big) : D(\overline{\varphi})\,dxdt \qquad \forall \overline{\varphi} \in \mathcal{K}^p, \quad \forall u \in \mathcal{K}^p$$

where we recall that $\Big[\cdot,\cdot\Big]$ denotes the duality product between $\mathcal{K}^p = L^p\Big(0,T;V^p_{0.div}\Big)$ and its dual $(\mathcal{K}^p)' = L^{p'}\Big(0,T;(V^p_{0.div})'\Big)$.

We consider in this section the following auxiliary flow problem:

Problem $(P_{\mathbf{u}})$ Let $f \in L^{p'}\Big(0,T;\mathbf{L}^2(\Omega)\Big)$, $k \in L^{p'}\Big(0,T;L^{p'}_+(\Gamma_0)\Big)$, μ satisfying (2)-(4), $\theta \in L^{\tilde{q}}\Big(0,T;L^{\tilde{p}}(\Omega)\Big)$ with $\tilde{q} \geq 1$ and $\tilde{p} \geq 1$, $s \in L^p\Big(0,T;\mathbf{L}^p(\Gamma_0)\Big)$, $\xi \in W^{1,p'}(0,T)$ satisfying (7) and $\upsilon_0 \in \mathbf{W}^{1,p}(\Omega)$ satisfying (11). Find $\overline{\upsilon} \in C\Big([0,T];H\Big) \cap \mathcal{K}^p$ with $\overline{\upsilon}' \in (\mathcal{K}^p)'$ satisfying the following parabolic variational inequality

$$[\overline{\upsilon}', \overline{\varphi}-\overline{\upsilon}] + \Big[\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}), \overline{\varphi}-\overline{\upsilon}\Big] + J(\overline{\varphi}) - J(\overline{\upsilon}) \geq \Big[\overline{f}, \overline{\varphi}-\overline{\upsilon}\Big] \qquad \forall \overline{\varphi} \in L^p\Big(0,T;V^p_{0.div}\Big) \tag{19}$$

and the initial condition (17) where $\overline{f} \in (\mathcal{K}^p)' = L^{p'}\Big(0,T;(V^p_{0.div})'\Big)$ is defined as

$$\Big[\overline{f},\overline{\varphi}\Big] = \int_0^T \left(f - \upsilon_0\frac{\partial \xi}{\partial t}, \overline{\varphi}\right)_{\mathbf{L}^2(\Omega)} dt \qquad \forall \overline{\varphi} \in \mathcal{K}^p = L^p\Big(0,T;V^p_{0.div}\Big).$$

Let $\overline{\upsilon}$ be a solution of Problem $(P_{\mathbf{u}})$. Then for all $\overline{\varphi} \in \phi^p$ we have

$$\begin{aligned}
&[\overline{\varphi}', \overline{\varphi}-\overline{\upsilon}] + \Big[\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}), \overline{\varphi}-\overline{\upsilon}\Big] + J(\overline{\varphi}) - J(\overline{\upsilon}) \\
&= [\overline{\upsilon}', \overline{\varphi}-\overline{\upsilon}] + \Big[\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}), \overline{\varphi}-\overline{\upsilon}\Big] + J(\overline{\varphi}) - J(\overline{\upsilon}) + [\overline{\varphi}'-\overline{\upsilon}', \overline{\varphi}-\overline{\upsilon}] \\
&= [\overline{\upsilon}', \overline{\varphi}-\overline{\upsilon}] + \Big[\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}), \overline{\varphi}-\overline{\upsilon}\Big] + J(\overline{\varphi}) - J(\overline{\upsilon}) + \frac{1}{2}\underbrace{\|\overline{\varphi}(T)-\overline{\upsilon}(T)\|^2_{\mathbf{L}^2(\Omega)}}_{\geq 0} - \frac{1}{2}\|\underbrace{\overline{\varphi}(0)-\overline{\upsilon}(0)}_{=0}\|^2_{\mathbf{L}^2(\Omega)} \geq \Big[\overline{f}, \overline{\varphi}-\overline{\upsilon}\Big].
\end{aligned}$$

Hence any solution $\overline{\upsilon}$ of Problem $(P_{\mathbf{u}})$ is also a solution of this weaker variational inequality.

In order to prove the existence of a solution $\overline{\upsilon} \in \mathcal{K}^p$ of this weaker inequality, we introduce an approximation of the time derivative of the test-function $\overline{\varphi}$ and we will consider a sequence of elliptic variational inequalities.

Proposition 3.1. *Let* $\mathbf{u} \in L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big)$ *and* $\theta \in L^{\tilde{q}}\Big(0,T;L^{\tilde{p}}(\Omega)\Big)$ *with* $\tilde{q} \geq 1$ *and* $\tilde{p} \geq 1$. *Let* $f \in L^{p'}\Big(0,T;\mathbf{L}^2(\Omega)\Big)$, $k \in L^{p'}\Big(0,T;L^{p'}_+(\Gamma_0)\Big)$, μ *satisfying (2)-(4),* $s \in L^p\Big(0,T;\mathbf{L}^p(\Gamma_0)\Big)$, $\xi \in W^{1,p'}(0,T)$ *satisfying (7) and* $\upsilon_0 \in \mathbf{W}^{1,p}(\Omega)$ *satisfying (11). Then there exists* $\overline{\upsilon} \in \mathcal{K}^p$ *such that*

$$[\overline{\varphi}', \overline{\varphi}-\overline{\upsilon}] + \Big[\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}), \overline{\varphi}-\overline{\upsilon}\Big] + J(\overline{\varphi}) - J(\overline{\upsilon}) \geq \Big[\overline{f}, \overline{\varphi}-\overline{\upsilon}\Big] \qquad \forall \overline{\varphi} \in \phi^p. \tag{20}$$

Proof The main idea of the proof is to replace the time derivative in the first term of (20) by a finite time difference. More precisely we define the semi-group of contractions $\Big(S(h)\Big)_{h\geq 0}$ in $X = \mathcal{K}^p, \mathcal{H}, (\mathcal{K}^p)'$ by $S(h)\overline{\varphi} = \overline{\varphi}_{-h}$ i.e.

$$\overline{\varphi}_{-h}(t) = \Big(S(h)\overline{\varphi}\Big)(t) = \begin{cases} 0 & \text{if} \quad 0 < t < h \\ \overline{\varphi}(t-h) & \text{if} \quad h < t < T \end{cases}$$

for all $\overline{\varphi} \in X$. From standard results (see Chapter 2.7 in [24]) we infer that $-\Lambda$, with $\Lambda = \dfrac{d}{dt}$, is the infinitesimal generator of $\Big(S(h)\Big)_{h\geq 0}$ with domain $D(\Lambda, X)$ in X. Moreover, since $\mathcal{K}^p \subset \mathcal{H}$, we have $\phi^p = \mathcal{K}^p \cap D(\Lambda, (\mathcal{K}^p)')$.

Now we approach the variational parabolic inequality (20) by the following sequence of variational elliptic problems depending on the parameter $h > 0$:

Find $\overline{\upsilon}_h \in \mathcal{K}^p$ such that

$$\left[\frac{I - S(h)}{h}\overline{\upsilon}_h, \overline{\varphi} - \overline{\upsilon}_h\right] + \left[\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}_h), \overline{\varphi} - \overline{\upsilon}_h\right] + J(\overline{\varphi}) - J(\overline{\upsilon}_h) \geq \left[\overline{f}, \overline{\varphi} - \overline{\upsilon}_h\right] \qquad \forall \overline{\varphi} \in \mathcal{K}^p. \tag{21}$$

For all $h > 0$ we define the operator $\mathcal{B}_h : \mathcal{K}^p \to (\mathcal{K}^p)'$ by

$$\mathcal{B}_h(u) = \frac{I - S(h)}{h}u + \mathcal{A}_{\mathbf{u}}(u) \qquad \forall u \in \mathcal{K}^p.$$

For all $(u, v) \in (\mathcal{K}^p)^2$ we have

$$\left[\mathcal{B}_h(u) - \mathcal{B}_h(v), u - v\right] = \frac{1}{h}\int_0^T \int_\Omega \Big(\big(I - S(h)\big)(u - v)\Big) \cdot (u - v)\, dxdt + \left[\mathcal{A}_{\mathbf{u}}(u) - \mathcal{A}_{\mathbf{u}}(v), u - v\right].$$

With Lemma 1 in [5] we know that the mapping $\lambda_2 \mapsto \mathcal{F}(\cdot, \cdot, \lambda_2)$ is monotone in $\mathbb{R}^{3\times 3}$. It follows that

$$\left[\mathcal{A}_{\mathbf{u}}(u) - \mathcal{A}_{\mathbf{u}}(v), u - v\right] = \int_0^T \int_\Omega \Big(\mathcal{F}\Big(\theta, \mathbf{u} + \upsilon_0\xi, D(u + \upsilon_0\xi)\Big) - \mathcal{F}\Big(\theta, \mathbf{u} + \upsilon_0\xi, D(v + \upsilon_0\xi)\Big)\Big) : D(u - v)\, dxdt \geq 0 \tag{22}$$

and thus

$$\left[\mathcal{B}_h(u) - \mathcal{B}_h(v), u - v\right] \geq \frac{1}{h}\Big(\|u - v\|_{\mathcal{H}} - \|S(h)(u - v)\|_{\mathcal{H}}\Big)\|u - v\|_{\mathcal{H}} \geq 0.$$

Hence $\mathcal{B}_h$ is monotone. Using (2) and (4) we obtain immediately that it is bounded and hemicontinuous. Furthermore with the same computations as in Section 2 we get

$$\begin{aligned}\left[\mathcal{B}_h(u), u\right] + J(u) &= \left[\frac{I - S(h)}{h}u, u\right] + \left[\mathcal{A}_{\mathbf{u}}(u), u\right] + J(u) \geq \left[\mathcal{A}_{\mathbf{u}}(u), u\right] \\ &\geq 2(C_{Korn})^p\mu_0\Big|\|u\|_{L^p(0,T;V^p_{0,div})} - \|\upsilon_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}\Big|^p \\ &\quad -2\mu_1\Big(\|u\|_{L^p(0,T;V^p_{0,div})} + \|\upsilon_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}\Big)^{p-1}\|\upsilon_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}\end{aligned}$$

for all $u \in \mathcal{K}^p$ and we infer that $\mathcal{B}_h$ is coercive.

Finally for all $(u, v) \in (\mathcal{K}^p)^2$ we have

$$\Big|J(u) - J(v)\Big| \leq \int_0^T \int_{\Gamma_0} k|u - v|\, dx'dt \leq C_\gamma\|k\|_{L^{p'}(0,T;L^{p'}(\Gamma_0))}\|u - v\|_{\mathcal{K}^p}$$

where C_γ is the norm of the trace operator $\gamma : V^p_{\Gamma_1} \to \mathbf{L}^p(\partial\Omega)$. Thus J is Lipschitz continuous on $\mathcal{K}^p$. It follows that J is proper, convex and lower semi-continuous on $\mathcal{K}^p$ and we may conclude that (21) admits at least a solution $\overline{\upsilon}_h$ for all $h > 0$.

Moreover the sequence $(\overline{\upsilon}_h)_{h>0}$ is bounded in $\mathcal{K}^p$. Indeed with $\overline{\varphi} = 0$ in (21) we have

$$\left[\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}_h), \overline{\upsilon}_h\right] \leq \left[\frac{I - S(h)}{h}\overline{\upsilon}_h, \overline{\upsilon}_h\right] + \left[\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}_h), \overline{\upsilon}_h\right] + J(\overline{\upsilon}_h) \leq \left[\overline{f}, \overline{\upsilon}_h\right] + J(0)$$

which yields

$$2(C_{Korn})^p\mu_0\Big|\|\overline{\upsilon}_h\|_{\mathcal{K}^p} - \|\upsilon_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}\Big|^p - 2\mu_1\Big(\|\overline{\upsilon}_h\|_{\mathcal{K}^p} + \|\upsilon_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}\Big)^{p-1}\|\upsilon_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})} \leq \|\overline{f}\|_{(\mathcal{K}^p)'}\|\overline{\upsilon}_h\|_{\mathcal{K}^p} + J(0).$$

If $\|\overline{v}_h\|_{\mathcal{K}^p} \neq 0$ we get

$$2(C_{Korn})^p\mu_0\left|1-\frac{\|v_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{\|\overline{v}_h\|_{\mathcal{K}^p}}\right|^p-2\mu_1\left(1+\frac{\|v_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{\|\overline{v}_h\|_{\mathcal{K}^p}}\right)^{p-1}\frac{\|v_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{\|\overline{v}_h\|_{\mathcal{K}^p}}-\|\overline{f}\|_{(\mathcal{K}^p)'}\|\overline{v}_h\|^{1-p}_{\mathcal{K}^p}-\frac{J(0)}{\|\overline{v}_h\|^p_{\mathcal{K}^p}}\leq 0.$$

But the mapping

$$t\mapsto 2(C_{Korn})^p\mu_0\left|1-\frac{\|v_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{t}\right|^p-2\mu_1\left(1+\frac{\|v_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{t}\right)^{p-1}\frac{\|v_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{t}-\|\overline{f}\|_{(\mathcal{K}^p)'}t^{1-p}-\frac{J(0)}{t^p}$$

admits $2(C_{Korn})^p\mu_0 > 0$ as limit when t tends to $+\infty$. So there exists a real number $C > 0$, independent of h, such that

$$\|\overline{v}_h\|_{\mathcal{K}^p}\leq C.$$

With (15) we obtain that the operator $\mathcal{A}_{\mathbf{u}}$ is bounded and we infer that the sequence $\Big(\mathcal{A}_{\mathbf{u}}(\overline{v}_h)\Big)_{h>0}$ is also bounded in $(\mathcal{K}^p)'$. Thus, by possibly extracting a subsequence, still denoted $(\overline{v}_h)_{h>0}$, we have:

$$\overline{v}_h\rightharpoonup\overline{v}\quad\text{weakly in }\mathcal{K}^p \tag{23}$$

and

$$\mathcal{A}_{\mathbf{u}}(\overline{v}_h)\rightharpoonup\chi\quad\text{weakly in }(\mathcal{K}^p)'. \tag{24}$$

Starting from (21) we obtain

$$\begin{aligned}&\left[\frac{I-S(h)}{h}\overline{\varphi},\overline{\varphi}-\overline{v}_h\right]+\left[\mathcal{A}_{\mathbf{u}}(\overline{v}_h),\overline{\varphi}-\overline{v}_h\right]+J(\overline{\varphi})-J(\overline{v}_h)\\&\geq\left[\frac{I-S(h)}{h}\overline{v}_h,\overline{\varphi}-\overline{v}_h\right]+\left[\mathcal{A}_{\mathbf{u}}(\overline{v}_h),\overline{\varphi}-\overline{v}_h\right]+J(\overline{\varphi})-J(\overline{v}_h)\\&\geq\left[\overline{f},\overline{\varphi}-\overline{v}_h\right]\qquad\forall\overline{\varphi}\in\mathcal{K}^p.\end{aligned}$$

Now let $\overline{\varphi}\in\phi^p$. From (23)-(24) and recalling that $\phi^p=\mathcal{K}^p\cap D\Big(\Lambda,(\mathcal{K}^p)'\Big)$, we get

$$\begin{aligned}\limsup_{h\to0}\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}_h),\overline{v}_h\Big]&\leq\limsup_{h\to0}\left(\left[\frac{I-S(h)}{h}\overline{\varphi},\overline{\varphi}-\overline{v}_h\right]+\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}_h),\overline{\varphi}\Big]+J(\overline{\varphi})-J(\overline{v}_h)-\Big[\overline{f},\overline{\varphi}-\overline{v}_h\Big]\right)\\&\leq[\Lambda\overline{\varphi},\overline{\varphi}-\overline{v}]+[\chi,\overline{\varphi}]+J(\overline{\varphi})-\liminf_{h\to0}J(\overline{v}_h)-\Big[\overline{f},\overline{\varphi}-\overline{v}\Big].\end{aligned}$$

Since the functional J is lower semi-continuous and convex for the strong topology, it is also lower semi-continuous for the weak topology thus

$$J(\overline{v})\leq\liminf_{h\to0}J(\overline{v}_h)$$

and we obtain

$$\limsup_{h\to0}\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}_h),\overline{v}_h\Big]\leq[\Lambda\overline{\varphi},\overline{\varphi}-\overline{v}]+[\chi,\overline{\varphi}]+J(\overline{\varphi})-J(\overline{v})-\Big[\overline{f},\overline{\varphi}-\overline{v}\Big]$$

i.e.

$$\limsup_{h\to0}\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}_h),\overline{v}_h-\overline{v}\Big]\leq[\Lambda\overline{\varphi},\overline{\varphi}-\overline{v}]+J(\overline{\varphi})-J(\overline{v})+\Big[\chi-\overline{f},\overline{\varphi}-\overline{v}\Big]\qquad\forall\overline{\varphi}\in\phi^p.$$

Since $S(h)\Big(\mathcal{K}^p\Big)\subset\mathcal{K}^p$ for all $h\geq0$, we infer that the operator Λ and $\mathcal{K}^p$ are compatible which means that for all $\overline{v}\in\mathcal{K}^p$, there exists a regularizing sequence $(\overline{v}_j)_{j\in\mathbb{N}}\in\phi^p$ satisfying

$$\lim_{j\to+\infty}\overline{v}_j=\overline{v}\quad\text{in }\mathcal{K}^p\quad\text{and}\quad\limsup_{j\to+\infty}[\Lambda\overline{v}_j,\overline{v}_j-\overline{v}]\leq0 \tag{25}$$

(see Chapter 2, Theorem 9.1 in [24]). It follows that

$$\limsup_{h\to 0}\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}_h),\overline{v}_h-\overline{v}\Big]\le[\Lambda\overline{v}_j,\overline{v}_j-\overline{v}]+J(\overline{v}_j)-J(\overline{v})+\Big[\chi-\overline{f},\overline{v}_j-\overline{v}\Big]\quad\forall\, j\in\mathbb{N}.$$

Owing (25) we obtain

$$\limsup_{h\to 0}\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}_h),\overline{v}_h-\overline{v}\Big]\le 0. \tag{26}$$

With (22) we already know that $\mathcal{A}_{\mathbf{u}}$ is monotone and with (2) and (15) we obtain that $\mathcal{A}_{\mathbf{u}}$ is hemicontinuous and bounded. We infer that $\mathcal{A}_{\mathbf{u}}$ is pseudo-monotone and with (26) we get

$$\liminf_{h\to 0}\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}_h),\overline{v}_h-\overline{\varphi}\Big]\ge\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}),\overline{v}-\overline{\varphi}\Big]\qquad\forall\overline{\varphi}\in\mathcal{K}^p. \tag{27}$$

Hence

$$\begin{aligned}\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}),\overline{v}-\overline{\varphi}\Big]&\le\liminf_{h\to 0}\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}_h),\overline{v}_h-\overline{\varphi}\Big]\\&\le\liminf_{h\to 0}\left(\Big[\frac{I-S(h)}{h}\overline{\varphi},\overline{\varphi}-\overline{v}_h\Big]+J(\overline{\varphi})-J(\overline{v}_h)-\Big[\overline{f},\overline{\varphi}-\overline{v}_h\Big]\right)\\&\le\limsup_{h\to 0}\left(\Big[\frac{I-S(h)}{h}\overline{\varphi},\overline{\varphi}-\overline{v}_h\Big]+J(\overline{\varphi})-J(\overline{v}_h)-\Big[\overline{f},\overline{\varphi}-\overline{v}_h\Big]\right)\quad\forall\overline{\varphi}\in\mathcal{K}^p.\end{aligned}$$

Thus

$$\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}),\overline{v}-\overline{\varphi}\Big]\le[\Lambda\overline{\varphi},\overline{\varphi}-\overline{v}]+J(\overline{\varphi})-J(\overline{v})-\Big[\overline{f},\overline{\varphi}-\overline{v}\Big]$$

i.e.

$$[\Lambda\overline{\varphi},\overline{\varphi}-\overline{v}]+\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}),\overline{\varphi}-\overline{v}\Big]+J(\overline{\varphi})-J(\overline{v})\ge\Big[\overline{f},\overline{\varphi}-\overline{v}\Big]\qquad\forall\overline{\varphi}\in\phi^p$$

and we may conclude that $\overline{v}$ satisfies (20). □

Remark 3.1. *We may improve the convergence result (24) by showing that $\chi=\mathcal{A}_{\mathbf{u}}(\overline{v})$ i.e.*

$$\mathcal{A}_{\mathbf{u}}(\overline{v}_h)\rightharpoonup\mathcal{A}_{\mathbf{u}}(\overline{v})\quad\textit{weakly in }(\mathcal{K}^p)'. \tag{28}$$

Indeed with (27) and (24)

$$\liminf_{h\to 0}\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}_h),\overline{v}_h\Big]-[\chi,\overline{\varphi}]\ge\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}),\overline{v}-\overline{\varphi}\Big]\qquad\forall\overline{\varphi}\in\mathcal{K}^p. \tag{29}$$

With $\overline{\varphi}=\overline{v}$ we get

$$\liminf_{h\to 0}\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}_h),\overline{v}_h\Big]\ge[\chi,\overline{v}]$$

and thus with (26) and (24)

$$\liminf_{h\to 0}\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}_h),\overline{v}_h\Big]=[\chi,\overline{v}]=\limsup_{h\to 0}\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}_h),\overline{v}_h\Big].$$

Going back to (29) with $\overline{\varphi}=\overline{v}\pm\overline{w}$ with $\overline{w}\in\mathcal{K}^p$ we obtain

$$[\chi,\overline{w}]=\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}),\overline{w}\Big]\qquad\forall\overline{w}\in\mathcal{K}^p$$

which yields the announced result.

Now we can prove that problem $(P_{\mathbf{u}})$ admits an unique solution.

Theorem 3.1. *Let* $\mathbf{u} \in L^p\big(0,T;\mathbf{L}^p(\Omega)\big)$ *and* $\theta \in L^{\tilde{q}}\big(0,T;L^{\tilde{p}}(\Omega)\big)$ *with* $\tilde{q} \geq 1$ *and* $\tilde{p} \geq 1$. *Let* $f \in L^{p'}\big(0,T;\mathbf{L}^2(\Omega)\big)$, $k \in L^{p'}\big(0,T;L^{p'}_{+}(\Gamma_0)\big)$, μ *satisfying (2)-(4),* $s \in L^p\big(0,T;\mathbf{L}^p(\Gamma_0)\big)$, $\xi \in W^{1,p'}(0,T)$ *satisfying (7) and* $\upsilon_0 \in \boldsymbol{W}^{1,p}(\Omega)$ *satisfying (11). Then problem* $(P_{\mathbf{u}})$ *admits an unique solution.*

Proof Let $h > 0$ and $\overline{\upsilon}_h \in \mathcal{K}^p$ be a solution of (21). Let $\tilde{\psi} \in V^p_{0.div}$ and $\zeta \in L^p(0,T)$. With $\overline{\varphi} = \overline{\upsilon}_h + \tilde{\psi}\zeta \in \mathcal{K}^p$ we obtain

$$\begin{aligned}\Big[\frac{I-S(h)}{h}\overline{\upsilon}_h, \tilde{\psi}\zeta\Big] &\geq \Big[\overline{f}, \tilde{\psi}\zeta\Big] - \Big[\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}_h), \tilde{\psi}\zeta\Big] - J(\overline{\upsilon}_h + \tilde{\psi}\zeta) + J(\overline{\upsilon}_h)\\ &\geq -(\|\overline{f}\|_{(\mathcal{K}^p)'} + \|\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}_h)\|_{(\mathcal{K}^p)'} + C_\gamma\|k\|_{L^{p'}(0,T;L^{p'}(\Gamma_0))})\|\tilde{\psi}\zeta\|_{\mathcal{K}^p}.\end{aligned}$$

With $\overline{\varphi} = \overline{\upsilon}_h - \tilde{\psi}\zeta \in \mathcal{K}^p$ we have

$$\begin{aligned}\Big[\frac{I-S(h)}{h}\overline{\upsilon}_h, \tilde{\psi}\zeta\Big] &\leq \Big[\overline{f}, \tilde{\psi}\zeta\Big] - \Big[\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}_h), \tilde{\psi}\zeta\Big] + J(\overline{\upsilon}_h - \tilde{\psi}\zeta) - J(\overline{\upsilon}_h)\\ &\leq (\|\overline{f}\|_{(\mathcal{K}^p)'} + \|\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}_h)\|_{(\mathcal{K}^p)'} + C_\gamma\|k\|_{L^{p'}(0,T;L^{p'}(\Gamma_0))})\|\tilde{\psi}\zeta\|_{\mathcal{K}^p}.\end{aligned}$$

Hence

$$\Big|\Big[\frac{I-S(h)}{h}\overline{\upsilon}_h, \tilde{\psi}\zeta\Big]\Big| \leq \Big(\|\overline{f}\|_{(\mathcal{K}^p)'} + \|\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}_h)\|_{(\mathcal{K}^p)'} + C_\gamma\|k\|_{L^{p'}(0,T;L^{p'}(\Gamma_0))}\Big)\|\tilde{\psi}\zeta\|_{\mathcal{K}^p}$$

and recalling that $\big(\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}_h)\big)_{h>0}$ is bounded in $(\mathcal{K}^p)'$ we get

$$\Big|\Big[\frac{I-S(h)}{h}\overline{\upsilon}_h, \tilde{\psi}\zeta\Big]\Big| \leq C\|\tilde{\psi}\zeta\|_{\mathcal{K}^p} \tag{30}$$

where C is independent of h.

On the other hand,

$$\begin{aligned}\Big[\frac{I-S(h)}{h}\overline{\upsilon}_h, \tilde{\psi}\zeta\Big] &= \frac{1}{h}\Big(\int_0^T\int_\Omega\big(\overline{\upsilon}_h(t)\cdot\tilde{\psi}\big)\zeta(t)\,dxdt - \int_h^T\int_\Omega\big(\overline{\upsilon}_h(t-h)\cdot\tilde{\psi}\big)\zeta(t)\,dxdt\Big)\\ &= \frac{1}{h}\Big(\int_0^{T-h}\int_\Omega\big(\overline{\upsilon}_h(t')\cdot\tilde{\psi}\big)\big(\zeta(t') - \zeta(t'+h)\big)\,dxdt' + \int_{T-h}^T\int_\Omega\big(\overline{\upsilon}_h(t')\cdot\tilde{\psi}\big)\zeta(t')\,dxdt'\Big).\end{aligned}$$

Let us assume now that $\zeta \in C^\infty\big([0,T]\big)$ such that $\zeta(t) = 0$ for all $t \in [b,T]$ with $b \in (0,T)$. Then for all $h \in (0, T-b)$ we have

$$\Big[\frac{I-S(h)}{h}\overline{\upsilon}_h, \tilde{\psi}\zeta\Big] = \frac{1}{h}\Big(\int_0^{T-h}\int_\Omega\big(\overline{\upsilon}_h(t')\cdot\tilde{\psi}\big)\big(\zeta(t') - \zeta(t'+h)\big)\,dxdt'\Big).$$

From Taylor's formula we have

$$\zeta(t'+h) - \zeta(t') = h\zeta'(t') + \frac{h^2}{2}\zeta''(t'+c_{t'}h) \qquad \text{with } 0 \leq c_{t'} \leq 1$$

for all $t' \in [0, T-h]$. Thus

$$\begin{aligned}\Big[\frac{I-S(h)}{h}\overline{\upsilon}_h, \tilde{\psi}\zeta\Big] &= -\int_0^{T-h}\int_\Omega\big(\overline{\upsilon}_h(t')\cdot\tilde{\psi}\big)\zeta'(t')\,dxdt' - \frac{h}{2}\int_0^{T-h}\int_\Omega\big(\overline{\upsilon}_h(t')\cdot\tilde{\psi}\big)\zeta''(t'+c_{t'}h)\,dxdt'\\ &= -\int_0^{T}\int_\Omega\big(\overline{\upsilon}_h(t')\cdot\tilde{\psi}\big)\zeta'(t')\,dxdt' - \frac{h}{2}\int_0^{T-h}\int_\Omega\big(\overline{\upsilon}_h(t')\cdot\tilde{\psi}\big)\zeta''(t'+c_{t'}h)\,dxdt'.\end{aligned} \tag{31}$$

But

$$\Big|\frac{h}{2}\int_0^{T-h}\int_\Omega\big(\overline{\upsilon}_h(t')\cdot\tilde{\psi}\big)\zeta''(t'+c_{t'}h)\,dxdt'\Big| \leq \frac{h}{2}\|\zeta''\|_{C([0,T])}\int_0^T\int_\Omega|\overline{\upsilon}_h(t')\|\tilde{\psi}|\,dxdt'.$$

Since the embedding of $\mathcal{K}^p$ into $L^2\big(0,T;\mathbf{L}^2(\Omega)\big)$ is continuous and the sequence $(\overline{v}_h)_{h>0}$ is bounded in $\mathcal{K}^p$ we have

$$\left|\frac{h}{2}\int_0^{T-h}\int_\Omega\big(\overline{v}_h(t')\cdot\tilde{\psi}\big)\zeta''(t'+c_{t'}h)\,dxdt'\right| \le C_*h\|\tilde{\psi}\|_{\mathbf{L}^2(\Omega)}\|\zeta''\|_{C([0,T])} \tag{32}$$

where C_* is independent of h. With (30) and (32) we obtain

$$\left|\int_0^{T}\int_\Omega\big(\overline{v}_h(t')\cdot\tilde{\psi}\big)\zeta'(t')\,dxdt'\right| \le C\|\tilde{\psi}\zeta\|_{\mathcal{K}^p} + C_*h\|\tilde{\psi}\|_{\mathbf{L}^2(\Omega)}\|\zeta''\|_{C([0,T])}.$$

Then we consider a converging subsequence of $(\overline{v}_h)_{h>0}$, still denoted $(\overline{v}_h)_{h>0}$, satisfying (23)-(24). As h tends to zero we get

$$\left|\int_0^{T}\int_\Omega\big(\overline{v}(t')\cdot\tilde{\psi}\big)\zeta'(t')\,dxdt'\right| \le C\|\tilde{\psi}\zeta\|_{\mathcal{K}^p}$$

which implies that $\overline{v}'$ belongs to $(\mathcal{K}^p)'$ and

$$\|\overline{v}'\|_{(\mathcal{K}^p)'} \le C.$$

On the other hand, for all $\overline{\varphi}\in\phi^p$ we have

$$\begin{aligned}&[\overline{v}',\overline{\varphi}-\overline{v}]+\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}),\overline{\varphi}-\overline{v}\Big]+J(\overline{\varphi})-J(\overline{v})-\Big[\overline{f},\overline{\varphi}-\overline{v}\Big]\\ &=[\overline{\varphi}',\overline{\varphi}-\overline{v}]+\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}),\overline{\varphi}-\overline{v}\Big]+J(\overline{\varphi})-J(\overline{v})-\Big[\overline{f},\overline{\varphi}-\overline{v}\Big]-\frac{1}{2}\|\overline{\varphi}(T)-\overline{v}(T)\|^2_{\mathbf{L}^2(\Omega)}+\frac{1}{2}\|\underbrace{\overline{\varphi}(0)-\overline{v}(0)}_{\ge 0}\|^2_{\mathbf{L}^2(\Omega)}\end{aligned}$$

and with (20)

$$[\overline{v}',\overline{\varphi}-\overline{v}]+\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}),\overline{\varphi}-\overline{v}\Big]+J(\overline{\varphi})-J(\overline{v})-\Big[\overline{f},\overline{\varphi}-\overline{v}\Big]\ge-\frac{1}{2}\|\overline{\varphi}(T)-\overline{v}(T)\|^2_{\mathbf{L}^2(\Omega)}.$$

If $\overline{\varphi}(T)=\overline{v}(T)$ we may conclude

$$[\overline{v}',\overline{\varphi}-\overline{v}]+\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}),\overline{\varphi}-\overline{v}\Big]+J(\overline{\varphi})-J(\overline{v})\ge\Big[\overline{f},\overline{\varphi}-\overline{v}\Big].$$

Otherwise, let $\overline{\varphi}\in\phi^p$. For all $\epsilon\in(0,T/2)$ we define $\overline{\varphi}_\epsilon=\overline{\varphi}\mathcal{X}_\epsilon+\overline{v}(1-\mathcal{X}_\epsilon)$ with $\mathcal{X}_\epsilon\in C^\infty\big([0,T];\mathbb{R}\big)$ such that

$$\mathcal{X}_\epsilon(t)=\begin{cases}1 & \text{if}\quad t\in[0,T-\epsilon]\\ 0 & \text{if } t\in\left[T-\dfrac{\epsilon}{2},T\right]\end{cases}$$

and $\mathcal{X}_\epsilon(t)\in[0,1]$ if $t\in\left[T-\epsilon,T-\dfrac{\epsilon}{2}\right]$.

By construction $\overline{\varphi}_\epsilon\in\phi^p$ and $\overline{\varphi}_\epsilon(T)=\overline{v}(T)$. Thus

$$[\overline{v}',\overline{\varphi}_\epsilon-\overline{v}]+\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}),\overline{\varphi}_\epsilon-\overline{v}\Big]+J(\overline{\varphi}_\epsilon)-J(\overline{v})\ge\Big[\overline{f},\overline{\varphi}_\epsilon-\overline{v}\Big]\qquad\forall\epsilon\in\left(0,\frac{T}{2}\right). \tag{33}$$

On the other hand, we have

$$\|\overline{\varphi}_\epsilon-\overline{\varphi}\|^p_{\mathcal{K}^p}=\int_0^T|1-\mathcal{X}_\epsilon|^p\|\overline{v}-\overline{\varphi}\|^p_{V^p_{0,div}}\,dt\qquad\forall\epsilon\in\left(0,\frac{T}{2}\right)$$

and Lebesgue's theorem implies that $\|\overline{\varphi}_\epsilon-\overline{\varphi}\|^p_{\mathcal{K}^p}$ tends to zero as ϵ tends to zero. We can pass to the limit in (33) as ϵ tends to zero and we get

$$[\overline{v}',\overline{\varphi}-\overline{v}]+\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}),\overline{\varphi}-\overline{v}\Big]+J(\overline{\varphi})-J(\overline{v})\ge\Big[\overline{f},\overline{\varphi}-\overline{v}\Big]\qquad\forall\overline{\varphi}\in\phi^p.$$

Finally, from (25) we infer that ϕ^p is dense in $\mathcal{K}^p$, and we obtain

$$[\overline{v}', \overline{\varphi} - \overline{v}] + \left[\mathcal{A}_{\mathbf{u}}(\overline{v}), \overline{\varphi} - \overline{v}\right] + J(\overline{\varphi}) - J(\overline{v}) \geq \left[\overline{f}, \overline{\varphi} - \overline{v}\right] \qquad \forall \overline{\varphi} \in \mathcal{K}^p. \tag{34}$$

Let us prove now that $\overline{v}(0) = 0$. Let $\tilde{\psi} \in \mathbf{W}^{1,p}_{0.div} = \left\{w \in \mathbf{W}^{1,p}_0(\Omega);\ div(w) = 0 \text{ in } \Omega\right\}$ and $\zeta \in L^p(0,T)$. With $\overline{\varphi} = \overline{v}_h \pm \tilde{\psi}\zeta \in \mathcal{K}^p$ in (21) we have $J(\overline{v}_h \pm \tilde{\psi}\zeta) = J(\overline{v}_h)$ and

$$\left[\frac{I - S(h)}{h}\overline{v}_h, \tilde{\psi}\zeta\right] + \left[\mathcal{A}_{\mathbf{u}}(\overline{v}_h), \tilde{\psi}\zeta\right] = \left[\overline{f}, \tilde{\psi}\zeta\right].$$

Let us assume now that $\zeta \in C^\infty\Big([0,T]\Big)$ such that $\zeta = 0$ on $[b,T]$ with $b \in (0,T)$ and $\zeta(0) = 1$. With (31) we have

$$\begin{aligned} &-\int_0^T \int_\Omega \Big(\overline{v}_h(t') \cdot \tilde{\psi}\Big)\zeta'(t')\, dxdt' - \frac{h}{2}\int_0^{T-h} \int_\Omega \Big(\overline{v}_h(t') \cdot \tilde{\psi}\Big)\zeta''(t' + c_{t'}h)\, dxdt' \\ &+\Big[\mathcal{A}_{\mathbf{u}}(\overline{v}_h), \tilde{\psi}\zeta\Big] = \Big[\overline{f}, \tilde{\psi}\zeta\Big] \qquad \forall h \in (0, T-b). \end{aligned}$$

We pass to the limit when h tends to zero and using (28) and (32) we obtain

$$-\int_0^T \int_\Omega \Big(\overline{v}(t') \cdot \tilde{\psi}\Big)\zeta'(t')\, dxdt' + \Big[\mathcal{A}_{\mathbf{u}}(\overline{v}), \tilde{\psi}\zeta\Big] = \Big[\overline{f}, \tilde{\psi}\zeta\Big].$$

With (34) and $\overline{\varphi} = \overline{v} \pm \tilde{\psi}\zeta \in \mathcal{K}^p$ we have also

$$[\overline{v}', \tilde{\psi}\zeta] + \Big[\mathcal{A}_{\mathbf{u}}(\overline{v}), \tilde{\psi}\zeta\Big] = \Big[\overline{f}, \tilde{\psi}\zeta\Big].$$

Hence

$$\begin{aligned} 0 &= \int_0^T \int_\Omega \Big(\overline{v}(t') \cdot \tilde{\psi}\Big)\zeta'(t')\, dxdt' + [\overline{v}', \tilde{\psi}\zeta] = \int_0^T \int_\Omega \Big(\overline{v}(t') \cdot \tilde{\psi}\Big)\zeta'(t')\, dxdt' + \int_0^T \Big\langle \overline{v}'(t'), \tilde{\psi}\zeta(t')\Big\rangle_{(V^p_{0.div})', V^p_{0.div}} dt' \\ &= \int_\Omega \Big(\overline{v}(T) \cdot \tilde{\psi}\Big)\zeta(T)\, dx - \int_\Omega \Big(\overline{v}(0) \cdot \tilde{\psi}\Big)\zeta(0)\, dx = -\int_\Omega \Big(\overline{v}(0) \cdot \tilde{\psi}\Big)\, dx \qquad \forall \tilde{\psi} \in \mathbf{W}^{1,p}_{0.div}. \end{aligned}$$

Since $\mathbf{W}^{1,p}_{0.div}$ is dense in H (see for instance Theorem 2.9 in [1] and Theorem 2.8 Chapter 1 in [18]) and $\overline{v}(0) \in H$ we get

$$\int_\Omega |\overline{v}(0)|^2\, dx = \|\overline{v}(0)\|^2_{\mathbf{L}^2(\Omega)} = 0$$

and thus $\overline{v}(0) = 0$. Hence problem $(P_{\mathbf{u}})$ admits at least a solution.

Let us prove now the uniqueness of the solution by a contradiction argument. So let us assume that problem $(P_{\mathbf{u}})$ admits two solutions $\overline{v}_1$ and $\overline{v}_2$. We have

$$[\overline{v}_1', \overline{\varphi} - \overline{v}_1] + \Big[\mathcal{A}_{\mathbf{u}}(\overline{v}_1), \overline{\varphi} - \overline{v}_1\Big] + J(\overline{\varphi}) - J(\overline{v}_1) \geq \Big[\overline{f}, \overline{\varphi} - \overline{v}_1\Big] \qquad \forall \overline{\varphi} \in \mathcal{K}^p = L^p\Big(0, T; V^p_{0.div}\Big)$$

and

$$[\overline{v}_2', \overline{\varphi} - \overline{v}_2] + \Big[\mathcal{A}_{\mathbf{u}}(\overline{v}_2), \overline{\varphi} - \overline{v}_2\Big] + J(\overline{\varphi}) - J(\overline{v}_2) \geq \Big[\overline{f}, \overline{\varphi} - \overline{v}_2\Big] \qquad \forall \overline{\varphi} \in \mathcal{K}^p = L^p\Big(0, T; V^p_{0.div}\Big).$$

Let $t \in (0,T]$. We may choose $\overline{\varphi} = \overline{v}_2\mathbf{1}_{[0,t]} + \overline{v}_1\Big(1 - \mathbf{1}_{[0,t])}\Big)$ in the first inequality and $\overline{\varphi} = \overline{v}_1\mathbf{1}_{[0,t]} + \overline{v}_2\Big(1 - \mathbf{1}_{[0,t]}\Big)$ in the second one where $\mathbf{1}_{[0,t]}$ denotes the indicator function of the interval $[0,t]$. By adding the two inequalities, we get

$$[\overline{v}_1' - \overline{v}_2', (\overline{v}_1 - \overline{v}_2)\mathbf{1}_{[0,t]}] + \Big[\mathcal{A}_{\mathbf{u}}(\overline{v}_1) - \mathcal{A}_{\mathbf{u}}(\overline{v}_2), (\overline{v}_1 - \overline{v}_2)\mathbf{1}_{[0,t]}\Big] \leq 0$$

i.e.

$$\int_0^t \int_\Omega \Big(\mathcal{F}\Big(\theta, \mathbf{u}+\upsilon_0\xi, D(\overline{\upsilon}_1+\upsilon_0\xi)\Big) - \mathcal{F}\Big(\theta, \mathbf{u}+\upsilon_0\xi, D(\overline{\upsilon}_2+\upsilon_0\xi)\Big)\Big) : D(\overline{\upsilon}_1-\overline{\upsilon}_2)\Big)\, dxdt'$$
$$+\frac{1}{2}\|\overline{\upsilon}_1(t)-\overline{\upsilon}_2(t)\|^2_{\mathbf{L}^2(\Omega)} \leq \frac{1}{2}\|\overline{\upsilon}_1(0)-\overline{\upsilon}_2(0)\|^2_{\mathbf{L}^2(\Omega)}.$$

Recalling that the mapping $\lambda_2 \mapsto \mathcal{F}(\cdot,\cdot,\lambda_2)$ is monotone on $\mathbb{R}^{3\times 3}$ we obtain

$$0 \leq \int_0^t \int_\Omega \Big(\mathcal{F}\Big(\theta, \mathbf{u}+\upsilon_0\xi, D(\overline{\upsilon}_1+\upsilon_0\xi)\Big) - \mathcal{F}\Big(\theta, \mathbf{u}+\upsilon_0\xi, D(\overline{\upsilon}_2+\upsilon_0\xi)\Big)\Big) : D(\overline{\upsilon}_1-\overline{\upsilon}_2)\Big)\, dxdt'.$$

Hence

$$\|\overline{\upsilon}_1(t)-\overline{\upsilon}_2(t)\|^2_{\mathbf{L}^2(\Omega)} \leq \|\overline{\upsilon}_1(0)-\overline{\upsilon}_2(0)\|^2_{\mathbf{L}^2(\Omega)} = 0 \quad \forall t \in (0,T]$$

which proves the uniqueness of the solution of problem $(P_{\mathbf{u}})$. □

4. Existence result for problem (P)

As already explained at the end of Section 2 we will prove now the existence of a solution to problem (P) by using a fixed point argument. More precisely we consider the mapping $\Lambda : L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big) \to L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big)$ which is defined by $\Lambda(\mathbf{u}) = \overline{\upsilon}$ where $\overline{\upsilon}$ is the unique solution of problem $(P_{\mathbf{u}})$ for all $\mathbf{u} \in L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big)$. By using Schauder's fixed point theorem we will prove that Λ admits a fixed point and we will use De Rham's theorem to establish the existence of the pressure π.

Theorem 4.1. *Let $f \in L^{p'}\Big(0,T;\mathbf{L}^2(\Omega)\Big)$, $k \in L^{p'}\Big(0,T;L^{p'}_+(\Gamma_0)\Big)$, μ satisfying (2)-(4), $\theta \in L^{\tilde{q}}\Big(0,T;L^{\tilde{p}}(\Omega)\Big)$ with $\tilde{q} \geq 1$ and $\tilde{p} \geq 1$, $s \in L^p\Big(0,T;\mathbf{L}^p(\Gamma_0)\Big)$, $\xi \in W^{1,p'}(0,T)$ satisfying (7) and $\upsilon_0 \in \boldsymbol{W}^{1,p}(\Omega)$ satisfying (11). Then problem (P) admits a solution i.e. there exist $\overline{\upsilon} \in C\Big([0,T];\mathbf{L}^2(\Omega)\Big) \cap L^p(0,T;V^p_{0.div})$ with $\dfrac{\partial \overline{\upsilon}}{\partial t} \in L^{p'}\Big(0,T;(V^p_{0.div})'\Big)$ and $\pi \in H^{-1}\Big(0,T;L^{p'}_0(\Omega)\Big)$ such that*

$$\Big\langle \frac{\partial}{\partial t}(\overline{\upsilon},\tilde{\vartheta})_{\mathbf{L}^2(\Omega)}, \zeta \Big\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} + \int_0^T \int_\Omega \mathcal{F}\Big(\theta, \overline{\upsilon}+\upsilon_0\xi, D(\overline{\upsilon}+\upsilon_0\xi)\Big) : D(\tilde{\vartheta})\zeta\, dxdt - \Big\langle \int_\Omega \pi\, div(\tilde{\vartheta}), \zeta \Big\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}$$
$$+J(\overline{\upsilon}+\tilde{\vartheta}\zeta) - J(\overline{\upsilon}) \geq \int_0^T \left(f - \frac{\partial \xi}{\partial t}\upsilon_0, \tilde{\vartheta}\right)_{\mathbf{L}^2(\Omega)} \zeta\, dt \qquad \forall \tilde{\vartheta} \in V^p_0,\ \forall \zeta \in \mathcal{D}(0,T)$$

and the initial condition

$$\overline{\upsilon}(0) = \upsilon_0 - \upsilon_0\xi(0) = 0 \qquad \text{in } \Omega.$$

Proof As a first step we prove that Λ satisfies the assumptions of Schauder's fixed point theorem.

Let $\mathbf{u} \in L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big)$. With $\overline{\varphi} = 0$ in (19) we obtain

$$[\overline{\upsilon}', \overline{\upsilon}] + \Big[\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}), \overline{\upsilon}\Big] + J(\overline{\upsilon}) \leq \Big[\overline{f}, \overline{\upsilon}\Big] + J(0).$$

But

$$[\overline{\upsilon}', \overline{\upsilon}] = \frac{1}{2}\|\overline{\upsilon}(T)\|^2_{\mathbf{L}^2(\Omega)} - \frac{1}{2}\| \underbrace{\overline{\upsilon}(0)}_{=0} \|^2_{\mathbf{L}^2(\Omega)} \geq 0.$$

Hence

$$\Big[\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}), \overline{\upsilon}\Big] \leq [\overline{\upsilon}', \overline{\upsilon}] + \Big[\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}), \overline{\upsilon}\Big] + J(\overline{\upsilon}) \leq \Big[\overline{f}, \overline{\upsilon}\Big] + J(0)$$

and with the same kind of computations as in Proposition 3.1 we get

$$2(C_{Korn})^p\mu_0\left|1-\frac{\|\upsilon_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{\|\overline{\upsilon}\|_{\mathcal{K}^p}}\right|^p-2\mu_1\left(1+\frac{\|\upsilon_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{\|\overline{\upsilon}\|_{\mathcal{K}^p}}\right)^{p-1}\frac{\|\upsilon_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{\|\overline{\upsilon}\|_{\mathcal{K}^p}}-\|\overline{f}\|_{(\mathcal{K}^p)'}\|\overline{\upsilon}\|^{1-p}_{\mathcal{K}^p}-\frac{J(0)}{\|\overline{\upsilon}\|^p_{\mathcal{K}^p}}\leq 0$$

if $\|\overline{\upsilon}\|_{\mathcal{K}^p}\neq 0$. Since the mapping

$$t\mapsto 2(C_{Korn})^p\mu_0\left|1-\frac{\|\upsilon_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{t}\right|^p-2\mu_1\left(1+\frac{\|\upsilon_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{t}\right)^{p-1}\frac{\|\upsilon_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{t}-\|\overline{f}\|_{(\mathcal{K}^p)'}t^{1-p}-\frac{J(0)}{t^p}$$

admits $2(C_{Korn})^p\mu_0>0$ as limit when t tends to $+\infty$, there exists a real number $C>0$, independent of $\mathbf{u}$, such that

$$\|\Lambda(\mathbf{u})\|_{\mathcal{K}^p}=\|\overline{\upsilon}\|_{\mathcal{K}^p}\leq C\qquad\forall\mathbf{u}\in L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big).\tag{35}$$

Hence

$$\|\Lambda(\mathbf{u})\|_{L^p(0,T;\mathbf{L}^p(\Omega))}\leq C_pC\qquad\forall\mathbf{u}\in L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big)$$

where C_p is the norm of the identity mapping from $\mathcal{K}^p=L^p\Big(0,T;V^p_{0.div}\Big)$ into $L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big)$.

Let us consider now $\tilde{\psi}\in V^p_{0.div}$ and $\zeta\in L^p(0,T)$. With $\overline{\varphi}=\overline{\upsilon}\pm\tilde{\psi}\zeta\in\mathcal{K}^p$ and with the same computations as in Theorem 3.1 we obtain

$$\left|[\overline{\upsilon}',\tilde{\psi}\zeta]\right|\leq\Big(\|\overline{f}\|_{(\mathcal{K}^p)'}+\|\mathcal{A}_{\mathbf{u}}(\overline{\upsilon})\|_{(\mathcal{K}^p)'}+C_\gamma\|k\|_{L^{p'}(0,T;L^{p'}(\Gamma_0))}\Big)\|\tilde{\psi}\zeta\|_{\mathcal{K}^p}$$

where we recall that C_γ is the norm of the trace operator $\gamma:V^p_{\Gamma_1}\to\mathbf{L}^p(\partial\Omega)$. But with (15) we have

$$\|\mathcal{A}_{\mathbf{u}}(\overline{\upsilon})\|_{(\mathcal{K}^p)'}\leq 2\mu_1\|\overline{\upsilon}+\upsilon_0\xi\|^{p-1}_{L^p(0,T;V^p_{\Gamma_1})}\leq 2\mu_1\Big(C+\|\upsilon_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}\Big)^{p-1}.$$

It follows that

$$\|\Big(\Lambda(\mathbf{u})\Big)'\|_{(\mathcal{K}^p)'}=\|\overline{\upsilon}'\|_{(\mathcal{K}^p)'}\leq\|\overline{f}\|_{(\mathcal{K}^p)'}+2\mu_1\Big(C+\|\upsilon_0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}\Big)^{p-1}+C_\gamma\|k\|_{L^{p'}(0,T;L^{p'}(\Gamma_0))}\tag{36}$$

for all $\mathbf{u}\in L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big)$.

By using Aubin's lemma we infer that the closed ball $\mathcal{B}=B_{L^p(0,T;\mathbf{L}^p(\Omega))}(0,C_pC)$ satisfies $\Lambda(\mathcal{B})\subset\mathcal{B}$ and that $\Lambda(\mathcal{B})$ is relatively compact in $L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big)$.

It remains to prove that the mapping Λ is continuous. Indeed, let $(\mathbf{u}_n)_{n\geq 0}$ be a sequence of $\mathcal{B}$ which converges strongly to $\mathbf{u}$. Let us prove that

$$\Lambda(\mathbf{u}_n)=\overline{\upsilon}_n\to\overline{\upsilon}=\Lambda(\mathbf{u})\quad\text{strongly in }L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big).$$

The sequence $(\overline{\upsilon}_n)_{n\geq 0}$ satisfies (35) and (36) so it admits strongly converging subsequences in $L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big)$. We consider such a subsequence, still denoted $(\overline{\upsilon}_n)_{n\geq 0}$.

For all $n\geq 0$ let us choose $\overline{\varphi}=\overline{\upsilon}$ as test-function in Problem $(P_{\mathbf{u}_n})$ and $\overline{\varphi}=\overline{\upsilon}_n$ as test-function in Problem $(P_{\mathbf{u}})$. By adding the two inequalities we obtain

$$[\overline{\upsilon}_n'-\overline{\upsilon}',\overline{\upsilon}_n-\overline{\upsilon}]+\Big[\mathcal{A}_{\mathbf{u}_n}(\overline{\upsilon}_n)-\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}),\overline{\upsilon}_n-\overline{\upsilon}\Big]\leq 0.$$

Since $\overline{\upsilon}_n(0)=\overline{\upsilon}(0)=0$ we have

$$[\overline{\upsilon}_n'-\overline{\upsilon}',\overline{\upsilon}_n-\overline{\upsilon}]=\frac{1}{2}\|\overline{\upsilon}_n(T)-\overline{\upsilon}(T)\|^2_{\mathbf{L}^2(\Omega)}$$

and thus

$$\frac{1}{2}\|\overline{\upsilon}_n(T)-\overline{\upsilon}(T)\|^2_{\mathbf{L}^2(\Omega)}+\Big[\mathcal{A}_{\mathbf{u}_n}(\overline{\upsilon}_n)-\mathcal{A}_{\mathbf{u}_n}(\overline{\upsilon}),\overline{\upsilon}_n-\overline{\upsilon}\Big]\leq\Big[\mathcal{A}_{\mathbf{u}}(\overline{\upsilon})-\mathcal{A}_{\mathbf{u}_n}(\overline{\upsilon}),\overline{\upsilon}_n-\overline{\upsilon}\Big].$$

We decompose the second term of the left hand side as

$$\Big[\mathcal{A}_{\mathbf{u}_n}(\overline{\upsilon}_n)-\mathcal{A}_{\mathbf{u}_n}(\overline{\upsilon}),\overline{\upsilon}_n-\overline{\upsilon}\Big]=\int_0^T\int_\Omega\Big(\mathcal{F}\Big(\theta,\mathbf{u}_n+\upsilon_0\xi,D(\overline{\upsilon}_n+\upsilon_0\xi)\Big)-\mathcal{F}\Big(\theta,\mathbf{u}_n+\upsilon_0\xi,D(\overline{\upsilon}+\upsilon_0\xi)\Big)\Big):D(\overline{\upsilon}_n-\overline{\upsilon})\,dxdt$$

so

$$\begin{aligned}&\Big[\mathcal{A}_{\mathbf{u}_n}(\overline{\upsilon}_n)-\mathcal{A}_{\mathbf{u}_n}(\overline{\upsilon}),\overline{\upsilon}_n-\overline{\upsilon}\Big]\\&=\mu_0\int_0^T\int_\Omega\Big(|D(\overline{\upsilon}_n+\upsilon_0\xi)|^{p-2}D(\overline{\upsilon}_n+\upsilon_0\xi)-|D(\overline{\upsilon}+\upsilon_0\xi)|^{p-2}D(\overline{\upsilon}+\upsilon_0\xi)\Big):D(\overline{\upsilon}_n-\overline{\upsilon})\,dxdt\\&+\int_0^T\int_\Omega\Big(\overline{\mathcal{F}}\Big(\theta,\mathbf{u}_n+\upsilon_0\xi,D(\overline{\upsilon}_n+\upsilon_0\xi)\Big)-\overline{\mathcal{F}}\Big(\theta,\mathbf{u}_n+\upsilon_0\xi,D(\overline{\upsilon}+\upsilon_0\xi)\Big)\Big):D(\overline{\upsilon}_n-\overline{\upsilon})\,dxdt\end{aligned}$$

where

$$\overline{\mathcal{F}}(\lambda_0,\lambda_1,\lambda_2)=2\overline{\mu}\Big(\lambda_0,\lambda_1,|\lambda_2|\Big)|\lambda_2|^{p-2}\lambda_2\quad\text{if }\lambda_2\neq 0_{\mathbb{R}^{3\times3}},\quad\overline{\mathcal{F}}(\lambda_0,\lambda_1,\lambda_2)=0_{\mathbb{R}^{3\times3}}\quad\text{otherwise}$$

and $\overline{\mu}=\mu-\dfrac{\mu_0}{2}$. Since $\overline{\mu}$ satisfies

$$\begin{aligned}&d\mapsto\overline{\mu}(\cdot,\cdot,d)\quad\text{is monotone increasing on }\mathbb{R}_+,\\&0<\frac{\mu_0}{2}\leq\overline{\mu}(o,e,d)\leq\mu_1-\frac{\mu_0}{2}\text{ for all }(o,e,d)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}_+,\end{aligned}$$

we infer with the same arguments as in Lemma 1 in [5] that $\lambda_2\mapsto\overline{\mathcal{F}}(\cdot,\cdot,\lambda_2)$ is monotone on $\mathbb{R}^{3\times3}$. Hence

$$\int_0^T\int_\Omega\Big(\overline{\mathcal{F}}\Big(\theta,\mathbf{u}_n+\upsilon_0\xi,D(\overline{\upsilon}_n+\upsilon_0\xi)\Big)-\overline{\mathcal{F}}\Big(\theta,\mathbf{u}_n+\upsilon_0\xi,D(\overline{\upsilon}+\upsilon_0\xi)\Big)\Big):D(\overline{\upsilon}_n-\overline{\upsilon})\,dxdt\geq 0$$

and

$$\begin{aligned}&\Big[\mathcal{A}_{\mathbf{u}_n}(\overline{\upsilon}_n)-\mathcal{A}_{\mathbf{u}_n}(\overline{\upsilon}),\overline{\upsilon}_n-\overline{\upsilon}\Big]\\&\geq\mu_0\int_0^T\int_\Omega\Big(|D(\overline{\upsilon}_n+\upsilon_0\xi)|^{p-2}D(\overline{\upsilon}_n+\upsilon_0\xi)-|D(\overline{\upsilon}+\upsilon_0\xi)|^{p-2}D(\overline{\upsilon}+\upsilon_0\xi)\Big):D(\overline{\upsilon}_n-\overline{\upsilon})\,dxdt.\end{aligned}$$

But for all $(\lambda,\lambda')\in\mathbb{R}^{3\times3}\times\mathbb{R}^{3\times3}$ we have

$$\Big(|\lambda|^{p-2}\lambda-|\lambda'|^{p-2}\lambda'\Big):(\lambda-\lambda')\geq\frac{1}{2^{p-1}}|\lambda-\lambda'|^p.\tag{37}$$

Indeed if $\lambda=\lambda'=0_{\mathbb{R}^{3\times3}}$ the result is obvious. Otherwise we get

$$\begin{aligned}&\Big(|\lambda|^{p-2}\lambda-|\lambda'|^{p-2}\lambda'\Big):(\lambda-\lambda')\geq\frac{1}{2}\Big(\Big(|\lambda|^{p-2}+|\lambda'|^{p-2}\Big)|\lambda-\lambda'|^2+\Big(|\lambda|^2-|\lambda'|^2\Big)\Big(|\lambda|^{p-2}-|\lambda'|^{p-2}\Big)\Big)\\&\geq\frac{1}{2}\Big(|\lambda|^{p-2}+|\lambda'|^{p-2}\Big)|\lambda-\lambda'|^2.\end{aligned}$$

Since $|\lambda|+|\lambda'|\neq 0$ we obtain

$$\begin{aligned}&\Big(|\lambda|^{p-2}\lambda-|\lambda'|^{p-2}\lambda'\Big):(\lambda-\lambda')\geq\frac{1}{2}\frac{|\lambda|^{p-2}+|\lambda'|^{p-2}}{\Big(|\lambda|+|\lambda'|\Big)^{p-2}}\Big(|\lambda|+|\lambda'|\Big)^{p-2}|\lambda-\lambda'|^2\\&\geq\frac{1}{2}\frac{|\lambda|^{p-2}+|\lambda'|^{p-2}}{\Big(|\lambda|+|\lambda'|\Big)^{p-2}}|\lambda-\lambda'|^p.\end{aligned}$$

Observing that

$$\left(\frac{|\lambda|+|\lambda'|}{2}\right)^{p-2} \leq \Big(\max\big(|\lambda|,|\lambda'|\big)\Big)^{p-2} \leq |\lambda|^{p-2}+|\lambda'|^{p-2}$$

we finally get

$$\Big(|\lambda|^{p-2}\lambda - |\lambda'|^{p-2}\lambda'\Big) : (\lambda-\lambda') \geq \frac{1}{2^{p-1}}|\lambda-\lambda'|^p.$$

Hence

$$\Big[\mathcal{A}_{\mathbf{u}_n}(\overline{\upsilon}_n) - \mathcal{A}_{\mathbf{u}_n}(\overline{\upsilon}), \overline{\upsilon}_n - \overline{\upsilon}\Big] \geq \frac{\mu_0}{2^{p-1}} \int_0^T \|D(\overline{\upsilon}_n - \overline{\upsilon})\|^p_{(L^p(\Omega))^{3\times 3}}\, dt.$$

On the other hand,

$$\begin{aligned}
&\Big[\mathcal{A}_{\mathbf{u}}(\overline{\upsilon}) - \mathcal{A}_{\mathbf{u}_n}(\overline{\upsilon}), \overline{\upsilon}_n - \overline{\upsilon}\Big] \\
&= \int_0^T \int_\Omega \Big(\overline{\mathcal{F}}\Big(\theta, \mathbf{u}+\upsilon_0\xi, D(\overline{\upsilon}+\upsilon_0\xi)\Big) - \overline{\mathcal{F}}\Big(\theta, \mathbf{u}_n+\upsilon_0\xi, D(\overline{\upsilon}+\upsilon_0\xi)\Big)\Big) : D(\overline{\upsilon}_n - \overline{\upsilon})\, dxdt \\
&\leq \Big\|\overline{\mathcal{F}}\Big(\theta, \mathbf{u}+\upsilon_0\xi, D(\overline{\upsilon}+\upsilon_0\xi)\Big) - \overline{\mathcal{F}}\Big(\theta, \mathbf{u}_n+\upsilon_0\xi, D(\overline{\upsilon}+\upsilon_0\xi)\Big)\Big\|_{L^{p'}(0,T;(L^{p'}(\Omega))^{3\times 3})} \|D(\overline{\upsilon}_n - \overline{\upsilon})\|_{L^p(0,T;(L^p(\Omega))^{3\times 3})}.
\end{aligned}$$

With Young's inequality

$$\begin{aligned}
&\|\overline{\mathcal{F}}\Big(\theta, \mathbf{u}+\upsilon_0\xi, D(\overline{\upsilon}+\upsilon_0\xi)\Big) - \overline{\mathcal{F}}\Big(\theta, \mathbf{u}_n+\upsilon_0\xi, D(\overline{\upsilon}+\upsilon_0\xi)\Big)\|_{L^{p'}(0,T;(L^{p'}(\Omega))^{3\times 3})} \|D(\overline{\upsilon}_n - \overline{\upsilon})\|_{L^p(0,T;(L^p(\Omega))^{3\times 3})} \\
&\leq C_\epsilon \Big\|\overline{\mathcal{F}}\Big(\theta, \mathbf{u}+\upsilon_0\xi, D(\overline{\upsilon}+\upsilon_0\xi)\Big) - \overline{\mathcal{F}}\Big(\theta, \mathbf{u}_n+\upsilon_0\xi, D(\overline{\upsilon}+\upsilon_0\xi)\Big)\Big\|^{p'}_{L^{p'}(0,T;(L^{p'}(\Omega))^{3\times 3})} + \epsilon \|D(\overline{\upsilon}_n - \overline{\upsilon})\|^p_{L^p(0,T;(L^p(\Omega))^{3\times 3})}
\end{aligned}$$

where $C_\epsilon = \dfrac{1}{p'}(p\epsilon)^{\frac{-1}{p-1}}$ and we choose $\epsilon = \dfrac{\mu_0}{2^p}$. We obtain

$$\begin{aligned}
&\frac{1}{2}\|\overline{\upsilon}_n(T) - \overline{\upsilon}(T)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\mu_0}{2^p}(C_p C_{Korn})^p \|\overline{\upsilon}_n - \overline{\upsilon}\|^p_{L^p(0,T;\mathbf{L}^p(\Omega))} \\
&\leq C_\epsilon \Big\|\overline{\mathcal{F}}\Big(\theta, \mathbf{u}+\upsilon_0\xi, D(\overline{\upsilon}+\upsilon_0\xi)\Big) - \overline{\mathcal{F}}\Big(\theta, \mathbf{u}_n+\upsilon_0\xi, D(\overline{\upsilon}+\upsilon_0\xi)\Big)\Big\|^{p'}_{L^{p'}(0,T;(L^{p'}(\Omega))^{3\times 3})}
\end{aligned}$$

where we recall that C_p is the norm of the identity mapping from $\mathcal{K}^p = L^p\Big(0,T;V^p_{0.div}\Big)$ into $L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big)$.

Since the sequence $(\mathbf{u}_n)_{n\geq 0}$ converges to $\mathbf{u}$ strongly in $L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big)$, there exists a subsequence, still denoted $(\mathbf{u}_n)_{n\geq 0}$, such that

$$\mathbf{u}_n \longrightarrow \mathbf{u} \quad \text{a.e. in } (0,T)\times\Omega.$$

By using the continuity and the uniform boundedness of the mapping $\overline{\mu}$, we infer from Lebesgue's theorem that

$$\overline{\mathcal{F}}\Big(\theta, \mathbf{u}_n+\upsilon_0\xi, D(\overline{\upsilon}+\upsilon_0\xi)\Big) \longrightarrow \overline{\mathcal{F}}\Big(\theta, \mathbf{u}+\upsilon_0\xi, D(\overline{\upsilon}+\upsilon_0\xi)\Big) \quad \text{strongly in } L^{p'}\Big(0,T;(L^{p'}(\Omega))^{3\times 3}\Big).$$

It follows that

$$\lim_{n\to+\infty} \|\overline{\upsilon}_n - \overline{\upsilon}\|_{L^p(0,T;\mathbf{L}^p(\Omega))} = 0.$$

So, any subsequence $(\overline{\upsilon}_n)_{n\geq 0}$ which is strongly convergent in $L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big)$ converges to $\overline{\upsilon} = \Lambda(\mathbf{u})$. Recalling that the whole sequence $(\overline{\upsilon}_n)_{n\geq 0}$ satisfies (35)-(36), we infer that the whole sequence $(\overline{\upsilon}_n)_{n\geq 0}$ converges to $\overline{\upsilon} = \Lambda(\mathbf{u})$ which proves the continuity of the mapping Λ.

With Schauder's fixed point theorem, we may conclude that Λ admits a fixed point $\mathbf{u} \in L^p\Big(0,T;\mathbf{L}^p(\Omega)\Big)$. Let us denote $\Lambda(\mathbf{u}) = \mathbf{u} = \overline{\upsilon}$. We obtain that $\overline{\upsilon} \in C\Big([0,T];\mathbf{L}^2(\Omega)\Big) \cap L^p\Big(0,T;V^p_{0.div}\Big)$ with $\overline{\upsilon}' \in L^{p'}\Big(0,T;(V^p_{0.div})'\Big)$, $\overline{\upsilon}(0) = 0$ and

$$[\overline{\upsilon}', \overline{\varphi} - \overline{\upsilon}] + \Big[\mathcal{A}_{\overline{\upsilon}}(\overline{\upsilon}), \overline{\varphi} - \overline{\upsilon}\Big] + J(\overline{\varphi}) - J(\overline{\upsilon}) \geq \Big[\overline{f}, \overline{\varphi} - \overline{\upsilon}\Big].$$

Hence $\overline{\upsilon}$ is a solution of Problem $(P_{\overline{\upsilon}})$.

In order to conclude the study of problem (P) it remains now to construct the pressure term. As usual the key tool is De Rham's theorem.

Reminding that $\overline{\upsilon} \in C\Big([0,T];\mathbf{L}^2(\Omega)\Big)$ we may define $F(t) \in (V_0^p)'$ for all $t \in [0,T]$ by

$$F(t)(\tilde{\vartheta}) = \left(\int_0^t \left(f - \upsilon_0 \frac{\partial \xi}{\partial t}\right) ds, \tilde{\vartheta}\right)_{\mathbf{L}^2(\Omega)} - \Big(\overline{\upsilon}(t), \tilde{\vartheta}\Big)_{\mathbf{L}^2(\Omega)} - \int_0^t \int_\Omega \mathcal{F}\Big(\theta, \overline{\upsilon} + \upsilon_0\xi, D(\overline{\upsilon} + \upsilon_0\xi)\Big) : D(\tilde{\vartheta})\, dxds \quad \forall \tilde{\vartheta} \in V_0^p.$$

The mapping F belongs to $C\Big([0,T];(V_0^p)'\Big)$. Moreover for all $\tilde{\vartheta} \in \mathbf{W}^{1,p}_{0.div}$ and for all $t \in [0,T]$ we may consider $\vartheta = \tilde{\vartheta}\mathbf{1}_{[[0,t]} \in L^p\Big(0,T;V^p_{0.div}\Big)$ and with $\overline{\varphi} = \overline{\upsilon} \pm \vartheta$ as a test function in Problem $(P_{\overline{\upsilon}})$ we obtain

$$\int_0^T \Big\langle \overline{\upsilon}', \vartheta \Big\rangle_{(V^p_{0.div})', V^p_{0.div}} dt + \int_0^T \int_\Omega \mathcal{F}\Big(\theta, \overline{\upsilon} + \upsilon_0\xi, D(\overline{\upsilon} + \upsilon_0\xi)\Big) : D(\vartheta)\, dxdt = \int_0^T \int_\Omega \left(f - \upsilon_0 \frac{\partial \xi}{\partial t}\right) \vartheta\, dxdt$$

which implies that $F(t)(\tilde{\vartheta}) = 0$. We infer that, for all $t \in [0,T]$, there exists an unique distribution $\tilde{\pi}(t) \in L_0^{p'}(\Omega)$ such that

$$F(t) = \nabla \tilde{\pi}(t)$$

(see for instance Lemma 2.7 in [1]). Since the gradient operator is an isomorphism from $L_0^{p'}(\Omega)$ into $\mathbf{W}^{-1,p'}(\Omega)$ (see Corollary 2.5 in [1]), we obtain that $\nabla\tilde{\pi} \in C\Big([0,T];\mathbf{W}^{-1,p'}(\Omega)\Big)$ and $\tilde{\pi} \in C\Big([0,T];L_0^{p'}(\Omega)\Big)$. Then, for all $t \in [0,T]$ and for all $\tilde{\vartheta} \in \mathbf{D}(\Omega) = (\mathcal{D}(\Omega))^3$, we have

$$\begin{aligned} F(t)(\tilde{\vartheta}) &= \int_0^t \int_\Omega \left(f - \upsilon_0 \frac{\partial \xi}{\partial t}\right) \tilde{\vartheta}\, dxds - \Big(\overline{\upsilon}(t), \tilde{\vartheta}\Big)_{\mathbf{L}^2(\Omega)} - \int_0^t \int_\Omega \mathcal{F}\Big(\theta, \overline{\upsilon} + \upsilon_0\xi, D(\overline{\upsilon} + \upsilon_0\xi)\Big) : D(\tilde{\vartheta})\, dxds \\ &= \Big\langle \nabla\tilde{\pi}(t), \tilde{\vartheta} \Big\rangle_{\mathbf{D}'(\Omega),\mathbf{D}(\Omega)} \quad \forall \tilde{\vartheta} \in \mathbf{D}(\Omega). \end{aligned}$$

Let us define π as the time derivative of $\tilde{\pi}$ in the distribution sense. We obtain $\pi = \dfrac{\partial \tilde{\pi}}{\partial t} \in \mathcal{D}'\Big(0,T;L_0^{p'}(\Omega)\Big)$ and

$$\int_\Omega \left(f - \upsilon_0 \frac{\partial \xi}{\partial t}\right) \tilde{\vartheta}\, dxds - \frac{\partial}{\partial t}\Big(\overline{\upsilon}, \tilde{\vartheta}\Big)_{\mathbf{L}^2(\Omega)} - \int_\Omega \mathcal{F}\Big(\theta, \overline{\upsilon} + \upsilon_0\xi, D(\overline{\upsilon} + \upsilon_0\xi)\Big) : D(\tilde{\vartheta})\, dx = \Big\langle \nabla\pi, \tilde{\vartheta} \Big\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \quad \forall \tilde{\vartheta} \in \mathbf{D}(\Omega)$$

in $\mathcal{D}'(0,T)$

Hence, for all $\zeta \in \mathcal{D}(0,T)$ and $\tilde{\vartheta} \in \mathbf{D}(\Omega)$, we have

$$\begin{aligned} &\left\langle \frac{\partial}{\partial t}(\overline{\upsilon}, \tilde{\vartheta})_{\mathbf{L}^2(\Omega)}, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} + \int_0^T \int_\Omega \mathcal{F}\Big(\theta, \overline{\upsilon} + \upsilon_0\xi, D(\overline{\upsilon} + \upsilon_0\xi)\Big) : D(\tilde{\vartheta})\zeta\, dxdt \\ &- \left\langle \int_\Omega \pi div(\tilde{\vartheta})\, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} = \int_0^T \int_\Omega \left(f - \upsilon_0 \frac{\partial \xi}{\partial t}\right) \tilde{\vartheta}\zeta\, dxdt. \end{aligned}$$

By density of $\mathbf{D}(\Omega)$ into $\mathbf{W}_0^{1,p}(\Omega)$ the same equality holds for all $\tilde{\vartheta} \in \mathbf{W}_0^{1,p}(\Omega)$. But, for all $\omega \in L_0^p(\Omega)$ we know that there exists $\tilde{\vartheta} \in \mathbf{W}_0^{1,p}(\Omega)$ such that

$$div(\tilde{\vartheta}) = \omega$$

and the mapping $P : L_0^p(\Omega) \to \mathbf{W}_0^{1,p}(\Omega)$ given by $\tilde{\vartheta} = P(\omega)$ is linear and continuous (see Corollary 3.1 in [1]). So for all $\omega \in L_0^p(\Omega)$ we have

$$\begin{aligned} &\left\langle \int_\Omega \pi\omega\, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} = -\int_0^T \Big(\overline{\upsilon}, P(\omega)\Big)_{\mathbf{L}^2(\Omega)} \zeta'\, dt \\ &+ \int_0^T \int_\Omega \mathcal{F}\Big(\theta, \overline{\upsilon} + \upsilon_0\xi, D(\overline{\upsilon} + \upsilon_0\xi)\Big) : D\Big(P(\omega)\Big)\zeta\, dxdt - \int_0^T \int_\Omega \left(f - \upsilon_0 \frac{\partial \xi}{\partial t}\right) P(\omega)\zeta\, dxdt \end{aligned}$$

and

$$\left|\left\langle \int_\Omega \pi\omega\, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}\right| \leq \|\overline{v}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\|P(\omega)\zeta'\|_{L^2(0,T;\mathbf{L}^2(\Omega))}$$
$$+2^{p-1}\mu_1 T^{\frac{1}{p}}\left(\|\overline{v}\|^{p-1}_{L^p(0,T;V^p_{0,div})} + \|D(\upsilon_0\xi)\|^{p-1}_{L^p(0,T;(L^p(\Omega)^{3\times 3})}\right)\|P(\omega)\zeta\|_{L^\infty(0,T;V_0^p)} + T^{\frac{1}{p}}\left\|f - \upsilon_0\frac{\partial\xi}{\partial t}\right\|_{L^{p'}(0,T;\mathbf{L}^2(\Omega))}\|P(\omega)\zeta\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}. \tag{38}$$

By using the continuous embedding of $\mathbf{W}^{1,p}_0(\Omega)$ into $\mathbf{L}^2(\Omega)$ and $H^1_0(0,T)$ into $L^\infty(0,T)$ we infer from (38) that there exists a positive real number C^*, independent of ω and ζ, such that

$$\left|\left\langle \int_\Omega \pi\omega\, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}\right| \leq C^*\|\omega\zeta\|_{H^1(0,T;L^p(\Omega))} \qquad \forall\omega \in L^p_0(\Omega),\ \forall\zeta \in \mathcal{D}(0,T).$$

Then, for all $\omega^* \in L^p(\Omega)$, we may apply the previous inequality with

$$\omega = \omega^* - \frac{1}{meas(\Omega)}\int_\Omega \omega^*\, dx \quad \in L^p_0(\Omega).$$

Since $\pi \in \mathcal{D}'\Big(0,T;L^{p'}_0(\Omega)\Big)$ we have

$$\left\langle \int_\Omega \pi\omega\, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} = \left\langle \int_\Omega \pi\left(\omega^* - \frac{1}{meas(\Omega)}\int_\Omega \omega^*\, dx\right) dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} = \left\langle \int_\Omega \pi\omega^*\, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}$$

and with Hölder's inequality

$$\|\omega\|_{L^p(\Omega)} \leq 2\|\omega^*\|_{L^p(\Omega)}.$$

So, for all $\omega^* \in L^p(\Omega)$ and $\zeta \in \mathcal{D}(0,T)$ we obtain

$$\left|\left\langle \int_\Omega \pi\omega^*\, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}\right| \leq 2C^*\|\omega^*\zeta\|_{H^1(0,T;L^p(\Omega))}.$$

Finally the density of $\mathcal{D}(0,T)\otimes L^p(\Omega)$ into $H^1_0\Big(0,T;L^p(\Omega)\Big)$ allows us to conclude that $\pi \in H^{-1}\Big(0,T;L^{p'}_0(\Omega)\Big)$.

Now let $\tilde{\vartheta} \in \mathbf{D}(\Omega) = \Big(\mathcal{D}(\Omega)\Big)^3$ and $\zeta \in \mathcal{D}(0,T)$. We have

$$\int_0^T\int_\Omega\left(f - \upsilon_0\frac{\partial\xi}{\partial t}\right)\tilde{\vartheta}\zeta\, dxdt - \left\langle \frac{\partial}{\partial t}(\overline{v},\tilde{\vartheta})_{\mathbf{L}^2(\Omega)}, \zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}$$
$$= \int_0^T\int_\Omega \mathcal{F}\Big(\theta, \overline{v} + \upsilon_0\xi, D(\overline{v} + \upsilon_0\xi)\Big) : D(\tilde{\vartheta})\zeta\, dxdt - \left\langle \int_\Omega \pi div(\tilde{\vartheta})\, dx, \zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}.$$

i.e.

$$\int_0^T\int_\Omega\left(f - \upsilon_0\frac{\partial\xi}{\partial t}\right)\tilde{\vartheta}\zeta\, dxdt - \left\langle \frac{\partial}{\partial t}(\overline{v},\tilde{\vartheta})_{\mathbf{L}^2(\Omega)}, \zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}$$
$$= \left\langle \int_\Omega \sigma : \nabla\tilde{\vartheta}\, dx, \zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}$$

with $\sigma = \mathcal{F}\Big(\theta, \overline{v} + \upsilon_0\xi, D(\overline{v} + \upsilon_0\xi)\Big) - \pi\mathrm{Id}_{\mathbb{R}^3}$ (see (1)). It follows that

$$\left|\left\langle \int_\Omega \sigma : \nabla\tilde{\vartheta}\, dx, \zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}\right| \leq \left(\mathbf{C}C_\infty T^{\frac{1}{p}}\left\|f - \upsilon_0\frac{\partial\xi}{\partial t}\right\|_{L^{p'}(0,T;\mathbf{L}^2(\Omega))} + \mathbf{C}\|\overline{v}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\right)\|\tilde{\vartheta}\zeta\|_{H^1(0,T;\mathbf{L}^p(\Omega))}$$

where $\mathbf{C}$, C_∞ denote the norm of the identity mappings from $\mathbf{L}^p(\Omega)$ into $\mathbf{L}^2(\Omega)$ and from $H^1(0,T)$ into $L^\infty(0,T)$ respectively. By density of $\mathcal{D}(0,T)\otimes\mathbf{D}(\Omega)$ into $H_0^1\Big(0,T;\mathbf{L}^p(\Omega)\Big)$ we may conclude that $div(\sigma)\in H^{-1}\Big(0,T;\mathbf{L}^{p'}(\Omega)\Big)$ and we have

$$\int_0^T\int_\Omega\left(f-\upsilon_0\frac{\partial\xi}{\partial t}\right)\tilde\vartheta\zeta\,dxdt-\left\langle\frac{\partial}{\partial t}(\overline{\upsilon},\tilde\vartheta)_{\mathbf{L}^2(\Omega)},\zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}$$
$$=-\left\langle\int_\Omega div(\sigma)\tilde\vartheta\,dx,\zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}=-\int_0^T\int_\Omega div(\sigma)\tilde\vartheta\zeta\,dxdt\qquad\forall\tilde\vartheta\in\mathbf{D}(\Omega),\ \forall\zeta\in\mathcal{D}(0,T).$$

By density of $\mathbf{D}(\Omega)$ into $\mathbf{L}^p(\Omega)$, the same equality holds for all $\tilde\vartheta\in\mathbf{L}^p(\Omega)$.

Next we observe that $\sigma\in H^{-1}\Big(0,T;\mathcal{Y}^{p'}\Big)$ where

$$\mathcal{Y}^{p'}=\Big\{\tilde\sigma\in\Big(L^{p'}(\Omega)\Big)^{3\times3};\ div(\tilde\sigma)\in\mathbf{L}^{p'}(\Omega)\Big\}.$$

It follows that we may use Green's formula and we get

$$-\int_0^T\int_\Omega div(\sigma)\tilde\vartheta\zeta\,dxdt=\int_0^T\int_\Omega(\sigma:\nabla\tilde\vartheta)\zeta\,dxdt-\int_0^T\int_{\partial\Omega}\left(\sum_{i,j=1}^3\sigma_{ij}n_j\tilde\vartheta_i\right)\zeta\,dYdt$$

for all $\tilde\vartheta\in\mathbf{W}^{1,p}(\Omega)$ and for all $\zeta\in\mathcal{D}(0,T)$.

Therefore, for all $\tilde\vartheta\in V_0^p$ and for all $\zeta\in\mathcal{D}(0,T)$ we get

$$\begin{aligned}&\left\langle\frac{\partial}{\partial t}(\overline{\upsilon},\tilde\vartheta)_{\mathbf{L}^2(\Omega)},\zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}+\int_0^T\int_\Omega\mathcal{F}\Big(\theta,\overline{\upsilon}+\upsilon_0\xi,D(\overline{\upsilon}+\upsilon_0\xi)\Big):D(\tilde\vartheta)\zeta\,dxdt-\int_0^T\int_\Omega\pi div(\tilde\vartheta)\zeta\,dxdt\\&+J(\overline{\upsilon}+\tilde\vartheta\zeta)-J(\overline{\upsilon})\,dt=\int_0^T\int_\Omega\left(f-\upsilon_0\frac{\partial\xi}{\partial t}\right)\tilde\vartheta\zeta\,dxdt+A(\overline{\upsilon},\tilde\vartheta)\end{aligned}\tag{39}$$

where

$$A(\overline{\upsilon},\tilde\vartheta)=\int_0^T\int_{\Gamma_0}(\sigma_\tau\cdot\tilde\vartheta)\zeta\,dx'dt\ +J(\overline{\upsilon}+\tilde\vartheta\zeta)-J(\overline{\upsilon}).$$

But

$$\int_{\partial\Omega}\tilde\vartheta\cdot n\,dY=0\qquad\forall\tilde\vartheta\in V_0^p.$$

From Lemma 3.3 in [1], we infer that there exists $\hat\vartheta\in\mathbf{W}^{1,p}(\Omega)$ satisfying

$$div(\hat\vartheta)=0\quad\text{in }\Omega,\quad\hat\vartheta=\tilde\vartheta\quad\text{on }\partial\Omega.$$

Hence $\hat\vartheta\in V_{0.div}^p$ and with $\overline{\varphi}=\overline{\upsilon}+\hat\vartheta\zeta$ as a test function in Problem $(P_{\overline{\upsilon}})$ we obtain

$$\begin{aligned}&\left\langle\frac{\partial}{\partial t}(\overline{\upsilon},\hat\vartheta)_{\mathbf{L}^2(\Omega)},\zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}+\int_0^T\int_\Omega\mathcal{F}\Big(\theta,\overline{\upsilon}+\upsilon_0\xi,D(\overline{\upsilon}+\upsilon_0\xi)\Big):D(\hat\vartheta)\zeta\,dxdt\\&+J(\overline{\upsilon}+\hat\vartheta\zeta)-J(\overline{\upsilon})-\int_0^T\int_\Omega\left(f-\upsilon_0\frac{\partial\xi}{\partial t}\right)\hat\vartheta\zeta\,dxdt\geq0\end{aligned}$$

and thus $A(\overline{\upsilon},\hat\vartheta)\geq0$. By observing that $A(\overline{\upsilon},\hat\vartheta)=A(\overline{\upsilon},\tilde\vartheta)$ since $\hat\vartheta=\tilde\vartheta$ on $\partial\Omega$ we get $A(\overline{\upsilon},\tilde\vartheta)\geq0$ and (39) becomes

$$\begin{aligned}&\left\langle\frac{\partial}{\partial t}(\overline{\upsilon},\tilde\vartheta)_{\mathbf{L}^2(\Omega)},\zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}+\int_0^T\int_\Omega\mathcal{F}\Big(\theta,\overline{\upsilon}+\upsilon_0\xi,D(\overline{\upsilon}+\upsilon_0\xi)\Big):D(\tilde\vartheta)\zeta\,dxdt-\left\langle\int_\Omega\pi div(\tilde\vartheta)\,dx,\zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}\\&+J(\overline{\upsilon}+\tilde\vartheta\zeta)-J(\overline{\upsilon})\,dt\geq\int_0^T\left(f-\upsilon_0\frac{\partial\xi}{\partial t},\tilde\vartheta\right)_{\mathbf{L}^2(\Omega)}\zeta\,dt\qquad\forall\tilde\vartheta\in V_0^p,\ \forall\zeta\in\mathcal{D}(0,T).\end{aligned}$$

Hence $(\overline{\upsilon},\pi)$ is a solution of Problem (P).

Remark 4.1. *If the vicosity μ does not depend on the velocity, then problem (P) admits an unique solution. Indeed, with Theorem 4.1 we already know that problem (P) admits at least two solutions and it remains to prove uniqueness. Let $(\overline{\upsilon}_1, \pi_1)$ and $(\overline{\upsilon}_2, \pi_2)$ be two solutions of (P). Hence*

$$\int_0^T \left\langle \frac{\partial \overline{\upsilon}_1}{\partial t}, \overline{\varphi} - \overline{\upsilon}_1 \right\rangle_{(V^p_{0.div})', V^p_{0.div}} dt + \left[\mathcal{A}_{\upsilon_1}(\overline{\upsilon}_1), \overline{\varphi} - \overline{\upsilon}_1 \right] + J(\overline{\varphi}) - J(\overline{\upsilon}_1) \geq \int_0^T (\overline{f}, \overline{\varphi} - \overline{\upsilon}_1)_{\mathbf{L}^2(\Omega)}$$

and

$$\int_0^T \left\langle \frac{\partial \overline{\upsilon}_2}{\partial t}, \overline{\varphi} - \overline{\upsilon}_2 \right\rangle_{(V^p_{0.div})', V^p_{0.div}} dt + \left[\mathcal{A}_{\upsilon_2}(\overline{\upsilon}_2), \overline{\varphi} - \overline{\upsilon}_2 \right] + J(\overline{\varphi}) - J(\overline{\upsilon}_2) \geq \int_0^T (\overline{f}, \overline{\varphi} - \overline{\upsilon}_2)_{\mathbf{L}^2(\Omega)}$$

for all $\overline{\varphi} \in L^p(0, T; V^p_{0.div})$. Since μ does not depend on the velocity we have

$$\mathcal{F}(\lambda_0, \lambda_1, \lambda_2) = 2\mu(\lambda_0, \lambda_2)|\lambda_2|^{p-2}\lambda_2 \quad \text{if } \lambda_2 \neq 0_{\mathbb{R}^{3\times 3}}, \quad \mathcal{F}(\lambda_0, \lambda_1, \lambda_2) = 0_{\mathbb{R}^{3\times 3}} \quad \text{otherwise}$$

and $\mathcal{A}_{\upsilon_1} = \mathcal{A}_{\upsilon_2} = \mathcal{A}_{\mathbf{u}}$ for any $\mathbf{u} \in L^p\Big(0, T; \mathbf{L}^p(\Omega)\Big)$. With Theorem 3.1 we infer that $\upsilon_1 = \upsilon_2$.

It follows that

$$\left\langle \int_\Omega (\pi_1 - \pi_2)\, div(\tilde{\vartheta})\, dx, \zeta \right\rangle_{\mathcal{D}'(0,T), \mathcal{D}(0,T)} = 0 \quad \forall \tilde{\vartheta} \in \mathbf{W}^{1,p}_0(\Omega),\ \forall \zeta \in \mathcal{D}(0, T)$$

and with the same notations as in Theorem 4.1 we get

$$\begin{aligned} &\left\langle \int_\Omega (\pi_1 - \pi_2)\, \omega^*\, dx, \zeta \right\rangle_{\mathcal{D}'(0,T), \mathcal{D}(0,T)} = \left\langle \int_\Omega (\pi_1 - \pi_2)\, \omega\, dx, \zeta \right\rangle_{\mathcal{D}'(0,T), \mathcal{D}(0,T)} \\ &= \left\langle \int_\Omega (\pi_1 - \pi_2)\, div\Big(P(\omega)\Big)\, dx, \zeta \right\rangle_{\mathcal{D}'(0,T), \mathcal{D}(0,T)} = 0 \quad \forall \omega^* \in L^p(\Omega),\ \forall \zeta \in \mathcal{D}(0, T) \end{aligned}$$

which yields $\pi_1 = \pi_2$ and allows us to conclude.

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