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A dynamic Euler–Bernoulli beam equation frictionally damped on an elastic foundation

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Abstract

This paper deals with a dynamic Euler–Bernoulli beam equation. The beam relies on a foundation composed of a continuous distribution of linear elastic springs. In addition to this time dependent uniformly distributed force, the model includes a continuous distribution of Coulomb frictional dampers, formalized by a partial differential inclusion. Under appropriate regularity assumptions on the initial data, the existence of a weak solution is obtained as a limit of a sequence of solutions associated with some physically relevant regularized problems.

Keywords: Euler–Bernoulli beam equation, Coulomb friction law, existence result

2008 MSC: 35B45, 35D30, 49J40, 74H20, 74M10

1. Introduction

This paper aims to give a mathematical result on existence for beams laid on foundations subjected to the action of a time dependent uniformly distributed force. The behavior of beams plays a crucial role in high-speed railway track design. The railway companies wish to improve rolling speed of trains in order to meet higher solicitations of this mean of transport in the travelling and freights domains all around the world. The factors inducing the appearance of defects of railway track should be identified to preserve the

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track quality required by traffic safety and passenger comfort. A considerable engineering and mathematical literature is devoted to study the railway track oscillation amplitudes. These oscillations lead to some undesirable deformations and they increase maintenance costs required in order to return the railway track to its initial position. Some investigations focus on the identification of a critical velocity of the load for which the oscillation amplitudes become large enough to cause non elastic displacement of the railway track (see [6]). The railway track efficiency depends on the limitation of these dynamic amplifications coming from the ability of the ballast to dissipate the energy transmitted by the passage of moving loads (see [9, 12]). The ballast is widely used as a constituent of railway tracks having some mechanical properties like flexibility in construction and maintenance. This granular material achieves the transmission to the platform of static and dynamic efforts induced by the running of trains and it is essentially composed by stones of different sizes and shapes interacting through surfaces and being in persistent frictional contact. On the one hand, some viscous foundation models are investigated in [4, 2, 3]. On the other hand, a foundation model with a non-smooth character nearer to the frictional dissipative character of the ballast was introduced in [18] and later on developed in [5, 16]. More precisely, we consider in this work a Winkler–Coulomb foundation which consists to associate in parallel a linear elastic Winkler foundation with a reaction per unit length that, at each cross section of the beam satisfies Coulomb’s friction law. The dissipation comes from an instantaneous reaction per unit length $r(t, x)$ at cross section x and time instant t depending on the sign of the transverse velocity of that cross section, namely we have

$$r(t, x) \in -f \operatorname{sign}\left(\frac{\partial u}{\partial t}(t, x)\right),$$

where f and $\frac{\partial u}{\partial t}(t, x)$ are the maximum force per unit length supporting the system of frictional dissipators of the foundation and $\frac{\partial u}{\partial t}(t, x)$ and the transverse velocity of the cross section, respectively. The transverse velocity of the cross section satisfies

$$\operatorname{sign}\left(\frac{\partial u}{\partial t}(t, x)\right) \begin{cases} = \frac{\frac{\partial u}{\partial t}(t, x)}{|\frac{\partial u}{\partial t}(t, x)|} & \text{if } \frac{\partial u}{\partial t}(t, x) \neq 0, \\ \in [-1, 1] & \text{if } \frac{\partial u}{\partial t}(t, x) = 0. \end{cases}$$

The reaction force r is an algebraic inclusion reflecting the multi-valued character of Coulomb friction law for vanishing velocity (see [7, 17]). This

reaction is quite different from the one provided by a continuous distribution of the linear viscous dampers, $r(t, x) = c \frac{\partial u}{\partial t}(t, x)$, where c denotes the viscous damping coefficient per unit length. The reaction force opposes the velocity in the both cases. However, the frictional reaction is independent of the magnitude of the velocity $\frac{\partial u}{\partial t}(t, x) \neq 0$ and limited to the interval $[-f, +f]$ while the reaction is proportional to the local velocity itself for the viscous damping.

We consider here a horizontal infinite linear elastic Euler–Bernoulli beam having a cross sectional area A , a central moment of inertia I , a mass density ρ and a Young’s modulus E (see Figure 1). The beam is connected to a fixed foundation bed by a system of linear elastic springs, with stiffness per unit length denoted by k , associated in parallel with a continuous distribution of viscous dampers, with damping coefficient c , and frictional dampers, with a maximum force per unit length f . Let $u(t, x)$ be the transverse displacement of the beam at time t of the material point of space coordinate x . A force F , depending on t and x , acts on the beam moving from left to right. The mathematical problem is formulated as follows:

$$\begin{aligned} \mu \frac{\partial^2 u}{\partial t^2}(t, x) + EI \frac{\partial^4 u}{\partial x^4}(t, x) + c \frac{\partial u}{\partial t}(t, x) + ku(t, x) \\ \in F(t, x) - f \operatorname{sign}\left(\frac{\partial u}{\partial t}(t, x)\right), \end{aligned} \quad (1.1)$$

with Cauchy initial data

$$u(0, \cdot) = u_0 \quad \text{and} \quad \frac{\partial u}{\partial t}(0, \cdot) = v_0, \quad (1.2)$$

where $\frac{\partial(\cdot)}{\partial t}$ and $\frac{\partial(\cdot)}{\partial x}$ denote the partial derivatives with respect to t and x , respectively, $\mu \stackrel{\text{def}}{=} \rho A$ is the mass of the beam per unit length of the beam. Some numerical experiments are reported and analyzed for problem (1.1)–(1.2) in [16] with a concentrated moving force $F(t, x) = F_0 \delta(x - vt)$, where $F_0 > 0$, δ and v are the Dirac delta function and a constant velocity, respectively. **Notice that the estimates obtained here do not seem enough to prove existence and uniqueness results for the full problem (i.e. with the Dirac delta function).**

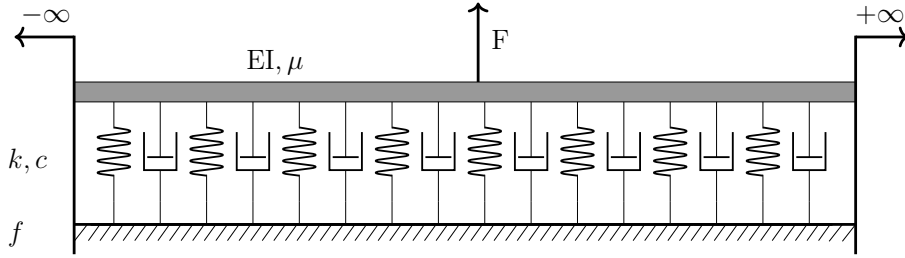


Figure 1: An infinite Euler–Bernoulli beam on a frictionally damped foundation under the action

Let $T > 0$. The definition of the subdifferential $\text{sign}\left(\frac{\partial u}{\partial t}(t, x)\right)$ leads to the weak formulation associated to (1.1)–(1.2) given by

$$\left\{ \begin{array}{l} \text{find } u \in \cap_{r=1}^2 W^{r, \infty}([0, T]; H^{4-2r}(\mathbb{R})) \text{ such that for all } w \in L^1([0, T] \times \mathbb{R}) \\ \cap L^2([0, T] \times \mathbb{R}) \text{ and for all } \theta \in L^\infty([0, T] \times \mathbb{R}), \text{ we have} \\ \bullet \int_{[0, T] \times \mathbb{R}} \left(\left(\mu \frac{\partial^2 u}{\partial t^2} + ku + c \frac{\partial u}{\partial t} + EI \frac{\partial^4 u}{\partial x^4} - F \right) \left(w - \frac{\partial u}{\partial t} \right) \theta \right) (t, x) dx dt \\ + \int_{[0, T] \times \mathbb{R}} f \left(|w(t, x)| - \left| \frac{\partial u}{\partial t}(t, x) \right| \right) dx dt \geq 0, \\ \bullet u(0, \cdot) = u_0 \quad \text{and} \quad \frac{\partial u}{\partial t}(0, \cdot) = v_0. \end{array} \right.$$

Notice the $W^{r, \infty}$ smoothness of solutions is classical for these kind problems (see [11, 13]). Observe also that the smoothness of u in this formulation has to be strengthened since the integrals are not all well-defined under the above regularity assumptions. This will be detailed in Theorem 1.1. Nevertheless, we can justify the choice of the test functions $w \in L^1([0, T] \times \mathbb{R}) \cap L^2([0, T] \times \mathbb{R})$ as follows. First, in order to define the integral $\int_{[0, T] \times \mathbb{R}} \mu \frac{\partial^2 u}{\partial t^2}(t, x) w(t, x) dx dt$, and due to the smoothness $\frac{\partial^2 u}{\partial t^2} \in L^\infty(0, T; L^2(\mathbb{R}))$, $w \in L^2([0, T] \times \mathbb{R})$ is required. Next, by equation (1.1), the term $\frac{\partial^4 u}{\partial x^4}$ belongs to $L^\infty([0, T] \times \mathbb{R}) + L^2([0, T], L^2(\mathbb{R}))$. Hence the smoothness $w \in L^2([0, T] \times \mathbb{R})$ has to be strengthened in $w \in L^1([0, T] \times \mathbb{R}) \cap L^2([0, T] \times \mathbb{R})$ in order to define the integral $\int_{[0, T] \times \mathbb{R}} \mu \frac{\partial^4 u}{\partial x^4}(t, x) w(t, x) dx dt$.

We assume that $\mu > 0$, $E > 0$, $I > 0$, $f > 0$. For an existence result, taking essentially in account the highest order derivatives, $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^4 u}{\partial x^4}$, we

do not assume any condition on $c \in \mathbb{R}$ and $k \in \mathbb{R}$. While for **the uniqueness result**, we restrict to the natural physical range $c \geq 0$, $k > 0$. See [5, 16] for further explanations. Before proceeding with the proofs, observe that the problem (1.1)–(1.2) can formally be written as

$$U'(t) \in \mathcal{A}U(t) + \mathcal{F}(t, U(t)), \quad (1.3)$$

with

$$\mathcal{A}(u, v) \stackrel{\text{def}}{=} \left(v, -\frac{1}{\mu}(\text{EI}u^{(4)} + cv + ku) \right), \quad (1.4)$$

and

$$\mathcal{F}(t, u, v) \stackrel{\text{def}}{=} \left(0, \frac{1}{\mu}(\text{F}(t) - f \text{sign}(v)) \right), \quad (1.5)$$

where $U(t) \stackrel{\text{def}}{=} (u(t), v(t))^{\text{T}}$. Relying on this formulation together with a suitable regularization of the sign function, we will obtain by means of the Hille–Yosida theorem an existence result for the weak formulation of the problem

Theorem 1.1. *Let $T > 0$, $\mu > 0$, $\text{E} > 0$, $\text{I} > 0$, $f > 0$ and let k, c be real numbers. Let also $\text{F} \in C^1([0, T]; L^2(\mathbb{R}))$, $(u_0, v_0) \in H^4(\mathbb{R}) \times H^2(\mathbb{R})$ be given. Then the weak problem associated to (1.1)–(1.2) admits at least one solution u with the following regularity:*

$$u \in \cap_{r=1}^2 \text{W}^{r, \infty}([0, T]; H^{4-2r}(\mathbb{R})) \cap_{0 < \varepsilon \leq 2} C^1([0, T]; H_{\text{loc}}^{2-\varepsilon}(\mathbb{R})), \quad (1.6a)$$

$$\frac{\partial u}{\partial t} \in L^1([0, T]; L^1(\mathbb{R})) \cap L^2([0, T]; L^2(\mathbb{R})), \quad (1.6b)$$

$$\frac{\partial^4 u}{\partial x^4} \in L^\infty([0, T] \times \mathbb{R}) + L^2([0, T]; L^2(\mathbb{R})). \quad (1.6c)$$

Moreover, there exists $\mathcal{N} \subset [0, T]$ a set of zero measure such that for all $w \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and all $t \in]0, T[\setminus \mathcal{N}$, we have

$$\begin{aligned} & \int_{\mathbb{R}} \left(\mu \frac{\partial^2 u}{\partial t^2} + ku + c \frac{\partial u}{\partial t} + \text{EI} \frac{\partial^4 u}{\partial x^4} - \text{F} \right)(t, x) \left(w(x) - \frac{\partial u}{\partial t}(t, x) \right) dx \\ & + \int_{\mathbb{R}} f \left(|w(x)| - \left| \frac{\partial u}{\partial t}(t, x) \right| \right) dx \geq 0. \end{aligned} \quad (1.7)$$

Under the assumptions that $c \geq 0$ and $k > 0$, the weak solution u associated to problem (1.1)–(1.2) is unique.

The paper is organized as follows. A general abstract existence result, in the spirit of Hille–Yosida theorem, is first given in Section 2. In Section 3, we study the steady state solution of a Euler–Bernoulli beam subjected to the action of a time dependent uniformly distributed force, on a foundation composed of a continuous distribution of linear elastic springs associated in parallel with an approximate/abstract distribution of Coulomb frictional dampers. This allows to use the general theorem obtained in Section 2. We also give some a priori estimates. Then, we apply those results to \mathcal{A} and a sequence $\{\mathcal{F}_n\}_{n \in \mathbb{N}^*}$ approximating operator \mathcal{F} (see Section 4). It provides a sequence of functions $\{u_n\}_{n \in \mathbb{N}^*}$ converging to some function $u \in \cap_{r=1}^2 W^{r,\infty}([0, T]; H^{4-2r}(\mathbb{R})) \cap_{0 < \varepsilon \leq 2} C^1([0, T]; H_{\text{loc}}^{2-\varepsilon}(\mathbb{R}))$ in a weak sense. In the last part of Section 4, we strengthen those convergences by means of a $W^{1,1}(0, T, L^1(\mathbb{R}))$ estimate (Lemmas 4.2–(iv) and 4.3). As a result, we get the dual properties (1.6b)–(1.6c). We finally prove that the function u satisfies the initial problem.

2. A preliminary existence result

Under suitable regularity assumptions, we give an existence and uniqueness result for the abstract problem (1.3) with a Cauchy initial data, and an equality in place of the inclusion. To this aim, it is convenient to denote by \mathcal{E} a Banach space. Since the discontinuous function \mathcal{F} in (1.5) cannot be easily handled within the natural functional frame of the Hille–Yosida theory, we first replace it by a general Lipschitz continuous function $\mathcal{G} : [0, T] \times \mathcal{E} \rightarrow \mathcal{E}$. This function \mathcal{G} will be used later on as an approximation of \mathcal{F} .

Proposition 2.1. *Let Λ be a strongly continuous semigroup defined on \mathcal{E} . Let $T > 0$ and $x \in \mathcal{E}$ and assume that $\mathcal{G} : [0, T] \times \mathcal{E} \rightarrow \mathcal{E}$ is Lipschitz continuous. The following problem has a unique solution:*

$$\begin{cases} \text{Find } U \in C^0([0, T]; \mathcal{E}) \text{ such that for all } t \in [0, T], \\ U(t) = \Lambda(t)x + \int_0^t \Lambda(t-s)\mathcal{G}(s, U(s)) \, ds. \end{cases}$$

Proof. It is enough to show that the operator

$$\begin{aligned} \Phi : C^0([0, T]; \mathcal{E}) &\rightarrow C^0([0, T]; \mathcal{E}) \\ U &\mapsto \Phi(U) \text{ with } \Phi(U) = \Lambda(t)x + \int_0^t \Lambda(t-s)\mathcal{G}(s, U(s)) \, ds \end{aligned}$$

is a contraction. For $h \in \mathbb{R}$, define the norm $\|\cdot\|_h$ on $C^0([0, T]; \mathcal{E})$ by $\|U\|_h \stackrel{\text{def}}{=} \sup_{0 \leq t \leq T} (e^{-ht} \|U(t)\|_{\mathcal{E}})$. Since Λ is a strongly continuous semigroup, there exists $M > 0$ and $\omega > 0$ such that, for any $(u, v) \in C^0([0, T]; \mathcal{E})^2$ and any $t \in [0, T]$

$$\|\Phi(U) - \Phi(V)\|_{\mathcal{E}} \leq M \int_0^t e^{\omega(t-s)} \|\mathcal{G}(s, U(s)) - \mathcal{G}(s, V(s))\|_{\mathcal{E}} ds. \quad (2.1)$$

Recall that \mathcal{G} is Lipschitz continuous. Hence, (2.1) implies that

$$\begin{aligned} \|\Phi(U) - \Phi(V)\|_h &\leq M \sup_{0 \leq t \leq T} \left(\int_0^t e^{\omega(t-s)} e^{-h(t-s)} e^{-hs} \|U(s) - V(s)\|_{\mathcal{E}} ds \right) \\ &\leq \frac{M}{h - \omega} \|U - V\|_h (1 - e^{(\omega-h)t}). \end{aligned}$$

Choosing $h = \omega + 2M$, we get

$$\|\Phi(U) - \Phi(V)\|_h \leq \frac{1}{2} \|U - V\|_h,$$

which concludes the proof. \square

As a corollary, we get

Corollary 2.2. *Assume that \mathcal{E} is reflexive. Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{E}$ be the infinitesimal generator of a strong continuous semigroup. Let $T > 0$ and $x \in \mathcal{D}(\mathcal{A})$ be given. Assume that $\mathcal{G} : [0, T] \times \mathcal{E} \rightarrow \mathcal{E}$ is a Lipschitz continuous function. Then the following problem admits exactly one solution*

$$(P) \quad \begin{cases} \text{Find } U \in C^0([0, T]; \mathcal{D}(\mathcal{A})) \cap C^1([0, T]; \mathcal{E}) \text{ such that for all } t \in [0, T], \\ U'(t) = \mathcal{A}U(t) + \mathcal{G}(t, U(t)) \text{ with } U(0) = x. \end{cases}$$

Proof. Since \mathcal{E} is reflexive, the Corollary follows from Propositions 2.1 and 4.3.9 and Lemma 4.1.1 of [1]. Note that the statements given in [1] hold true in the context of a function \mathcal{G} independent of the time, but can readily be extended to a function $\mathcal{G} : [0, T] \times \mathcal{E} \rightarrow \mathcal{E}$. \square

3. The case of the beam operator

We now apply the previous result to a more specific case, namely to the case of the beam operator \mathcal{A} in equation (1.1). Since we will use in the next section an approximation on the sign function, we are less definite for \mathcal{G} and only write $\mathcal{G}(t, u, v) = (0, \alpha(t) + \beta(v))$, see equation (3.2) below.

Let $\mathcal{E} \stackrel{\text{def}}{=} H^2(\mathbb{R}) \times L^2(\mathbb{R})$, endowed with the inner product

$$\langle (u_1, v_1), (u_2, v_2) \rangle_{\mathcal{E}} \stackrel{\text{def}}{=} \int_{\mathbb{R}} \left(u_1 u_2 + \frac{\text{EI}}{\mu} u_1'' u_2'' \right) (x) dx + \int_{\mathbb{R}} v_1 v_2(x) dx,$$

where $(\cdot)''$ denotes the second derivative with respect to x . From now on, notation \mathcal{A} denotes the beam operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{E} \rightarrow \mathcal{E}$ defined by

$$\mathcal{A}(u, v) \stackrel{\text{def}}{=} \left(v, -\frac{1}{\mu} \left(\text{EI} u^{(4)} + cv + ku \right) \right), \quad (3.1)$$

with $\mathcal{D}(\mathcal{A}) = H^4(\mathbb{R}) \times H^2(\mathbb{R})$. Let $\alpha \in C^1([0, T], L^2(\mathbb{R}))$ and let $\beta : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded, Lipschitz continuous function with $\beta(0) = 0$. Define

$$\mathcal{G} : C^0([0, T]; H^2(\mathbb{R}) \times L^2(\mathbb{R})) \rightarrow C^0([0, T]; H^2(\mathbb{R}) \times L^2(\mathbb{R}))$$

by

$$\mathcal{G}(t, u, v) \stackrel{\text{def}}{=} (0, \alpha(t) + \beta(v)). \quad (3.2)$$

Due to our assumptions on α and β , the function \mathcal{G} is Lipschitz continuous on $[0, T] \times \mathcal{E}$ ($T > 0$).

In order to apply Corollary 2.2 to problem (P) with \mathcal{A} and \mathcal{G} given by (3.1) and (3.2), respectively, we prove the following result.

Theorem 3.1. *With the above notations, for $\lambda > 0$ large enough, we have*

$$(\mathcal{A} - \lambda \text{Id}) \text{ is dissipative}, \quad (3.3a)$$

$$\text{Im}(\mathcal{A} - \lambda \text{Id}) = \mathcal{E}, \quad (3.3b)$$

where Id and $\text{Im}(\mathcal{A} - \lambda \text{Id})$ denote the identity of $\mathcal{D}(\mathcal{A})$ and the image of $(\mathcal{A} - \lambda \text{Id})$, respectively.

Proof. Assume that $(u, v) \in \mathcal{D}(\mathcal{A})^2$. Then we have

$$\begin{aligned} \langle (\mathcal{A} - \lambda \text{Id})(u, v), (u, v) \rangle_{\mathcal{E}} &= -\lambda \int_{\mathbb{R}} \left(u^2 + \frac{\text{EI}}{\mu} (u'')^2 + v^2 \right) (x) dx \\ &+ \int_{\mathbb{R}} \left(uv + \frac{\text{EI}}{\mu} u'' v'' \right) (x) dx + \int_{\mathbb{R}} -\frac{1}{\mu} \left(\text{EI} u^{(4)} + cv + ku \right) (x) v(x) dx, \end{aligned}$$

which leads to

$$\begin{aligned}
\langle (\mathcal{A} - \lambda \text{Id})(u, v), (u, v) \rangle_{\mathcal{E}} &= -\lambda \int_{\mathbb{R}} \left(u^2 + \frac{\text{EI}}{\mu} (u'')^2 + v^2 \right) (x) \, dx \\
&+ \int_{\mathbb{R}} \left(uv + \frac{\text{EI}}{\mu} u'' v'' \right) (x) \, dx - \int_{\mathbb{R}} \frac{\text{EI}}{\mu} u'' v'' (x) \, dx - \frac{c}{\mu} \int_{\mathbb{R}} v^2 (x) \, dx \\
&- \frac{k}{\mu} \int_{\mathbb{R}} uv (x) \, dx \leq -\lambda \int_{\mathbb{R}} \left(u^2 + \frac{\text{EI}}{\mu} (u'')^2 + v^2 \right) (x) \, dx \\
&+ \left| 1 - \frac{k}{\mu} \right| \int_{\mathbb{R}} (u^2 + v^2) (x) \, dx + \left| \frac{c}{\mu} \right| \int_{\mathbb{R}} v^2 (x) \, dx \leq 0
\end{aligned}$$

for $\lambda > \left| 1 - \frac{k}{\mu} \right| + \left| \frac{c}{\mu} \right|$. Hence (3.3a) is proved.

We establish now that (3.3b) holds true. Let $(g_1, g_2) \in \text{H}^2(\mathbb{R}) \times \text{L}^2(\mathbb{R})$. We have to solve the following problem

$$\begin{cases} \lambda u - v = g_1, \\ \lambda v + \frac{1}{\mu} (\text{EI} u^{(4)} + cv + ku) = g_2. \end{cases} \quad (3.4)$$

It implies that

$$\frac{\text{EI}}{\mu} u^{(4)} + \left(\frac{k}{\mu} + \lambda \left(\lambda + \frac{c}{\mu} \right) \right) u = g_2 + \left(\lambda + \frac{c}{\mu} \right) g_1. \quad (3.5)$$

Let $\lambda > 0$ be large enough such that $\gamma \stackrel{\text{def}}{=} \frac{k}{\mu} + \lambda \left(\lambda + \frac{c}{\mu} \right) > 0$ and set $G \stackrel{\text{def}}{=} g_2 + \left(\lambda + \frac{c}{\mu} \right) g_1 \in \text{L}^2(\mathbb{R})$. According to the previous notations, identity (3.5) becomes

$$\frac{\text{EI}}{\mu} u^{(4)} + \gamma u = G. \quad (3.6)$$

We denote by ζ the dual variable to x . The Fourier transform of $v(x)$ is denoted by $\widehat{v}(\zeta)$, where the convention for the Fourier transform is given by:

$$\widehat{v}(\zeta) \stackrel{\text{def}}{=} \int_{\mathbb{R}} e^{-i\zeta x} v(x) \, dx.$$

Then it comes that (3.6) admits a solution $u \in \text{H}^4(\mathbb{R})$ given in Fourier variable by

$$\widehat{u}(\zeta) = \frac{\mu \widehat{G}(\zeta)}{\text{EI} \zeta^4 + \mu \gamma}.$$

Since $v = \lambda u - g_1$, we have $v \in \text{H}^2(\mathbb{R})$. Hence (3.3b) is proved. \square

We thus obtain the well-posedness of (P) and, under suitable assumptions on β , β -independent estimates on the solution U . This will be useful in section 4, when regularizing the function β :

Theorem 3.2. *Let $T > 0$. Let \mathcal{A} and \mathcal{G} be given by (3.1) and (3.2) respectively, with $\alpha \in C^1([0, T], L^2(\mathbb{R}))$ and $\beta : \mathbb{R} \rightarrow \mathbb{R}$ a bounded, Lipschitz continuous function such that $\beta(0) = 0$. Assume that $U_0 \in \mathcal{D}(\mathcal{A})$. Then the problem (P) admits exactly one solution $U = (u, v)$. Moreover, assuming that*

$$\forall y \in \mathbb{R} : y\beta(y) \leq 0 \quad \text{and} \quad \beta'(y) \leq 0. \quad (3.7)$$

then there exists $C_T > 0$, independent of β , such that

$$\sup_{t \in [0, T]} \|U(t)\|_{\mathcal{E}} + \int_{\mathbb{R}} |\beta(v)v| \, dx \leq C_T (\|U_0\|_{\mathcal{E}} + \|\alpha\|_{C^0([0, T], L^2(\mathbb{R}))}), \quad (3.8a)$$

$$\sup_{t \in [0, T]} \left\| \frac{dU}{dt}(t) \right\|_{\mathcal{E}} \leq C_T (\|U_0\|_{\mathcal{D}(\mathcal{A})} + \|\alpha\|_{C^1([0, T], L^2(\mathbb{R}))}). \quad (3.8b)$$

Proof. The existence and uniqueness results follow from Corollary 2.2 and the Hille–Yosida theorem, the verification is [left](#) to the reader.

We begin by proving (3.8a). To this aim, we assume that $\lambda_0 > 0$ is large enough. According to the notation $\mathcal{A}_{\lambda_0} \stackrel{\text{def}}{=} \mathcal{A} - \lambda_0 \text{Id}$, we get

$$\forall z \in \mathcal{D}(\mathcal{A}) : \langle \mathcal{A}_{\lambda_0} z, z \rangle_{\mathcal{E}} \leq 0.$$

Writing $U_{\lambda_0}(t) \stackrel{\text{def}}{=} e^{-\lambda_0 t} U(t)$, equations (P) provide

$$\frac{1}{2} \frac{d}{dt} \|U_{\lambda_0}\|_{\mathcal{E}}^2 \leq \langle \mathcal{A}_{\lambda_0} U_{\lambda_0}, U_{\lambda_0} \rangle_{\mathcal{E}}(t) + e^{-2\lambda_0 t} \int_{\mathbb{R}} (\alpha + \beta(v))v(t, x) \, dx.$$

Since $v\beta(v) \leq 0$, there exists $C_T > 0$ independent of β such that

$$\frac{1}{2} \frac{d}{dt} \|U_{\lambda_0}\|_{\mathcal{E}}^2 + e^{-2\lambda_0 t} \int_{\mathbb{R}} |\beta(v)v| \, dx \leq C_T (\|\alpha\|_{L^2(\mathbb{R})}^2(t) + \|U_{\lambda_0}\|_{\mathcal{E}}^2). \quad (3.9)$$

Now, (3.8a) follows from (3.9) and Grönwall's lemma.

In order to prove (3.8b), we recall that $U \in C^0([0, T]; \mathcal{D}(\mathcal{A})) \cap C^1([0, T]; \mathcal{E})$. Hence, for $t > 0$ and $h \in \mathbb{R}$ such that $t \in [0, T]$ and $(t + h) \in [0, T]$, since β

is non increasing and $\mathcal{A} - \lambda_0 \text{Id}$ is dissipative, we have

$$\begin{aligned}
& \frac{1}{h^2} \left\langle \frac{dU}{dt}(t+h) - \frac{dU}{dt}(t), U(t+h) - U(t) \right\rangle_{\mathcal{E}} \\
&= \frac{1}{h^2} \langle \mathcal{A}(U(t+h) - U(t)), U(t+h) - U(t) \rangle_{\mathcal{E}} \\
&+ \frac{1}{h^2} (\beta(v(t+h)) - \beta(v(t)), v(t+h) - v(t))_{L^2(\mathbb{R})} \\
&+ \frac{1}{h^2} (\alpha(t+h) - \alpha(t), v(t+h) - v(t))_{L^2(\mathbb{R})} \\
&\leq (\lambda_0 + 1) \left\| \frac{U(t+h) - U(t)}{h} \right\|_{\mathcal{E}}^2 + \left\| \frac{\alpha(t+h) - \alpha(t)}{h} \right\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

Integrating on $[0, t]$ we get

$$\begin{aligned}
& \int_0^t \frac{1}{h^2} \left\langle \frac{dU}{dt}(s+h) - \frac{dU}{dt}(s), U(s+h) - U(s) \right\rangle_{\mathcal{E}} ds \\
&\leq C \int_0^t \left\| \frac{U(s+h) - U(s)}{h} \right\|_{\mathcal{E}}^2 ds + \int_0^t \left\| \frac{\alpha(s+h) - \alpha(s)}{h} \right\|_{L^2(\mathbb{R})}^2 ds.
\end{aligned} \tag{3.10}$$

By $U \in C^1([0, T]; \mathcal{E})$ and a density argument, we have

$$\begin{aligned}
& \int_0^t \frac{1}{h^2} \left\langle \frac{dU}{dt}(s+h) - \frac{dU}{dt}(s), U(s+h) - U(s) \right\rangle_{\mathcal{E}} ds \\
&= \frac{1}{2} \left\| \frac{U(t+h) - U(t)}{h} \right\|_{\mathcal{E}}^2 - \frac{1}{2} \left\| \frac{U(h) - U(0)}{h} \right\|_{\mathcal{E}}^2.
\end{aligned} \tag{3.11}$$

Using (3.10) and (3.11), we obtain

$$\begin{aligned}
& \left\| \frac{U(t+h) - U(t)}{h} \right\|_{\mathcal{E}}^2 \leq \left\| \frac{U(h) - U(0)}{h} \right\|_{\mathcal{E}}^2 \\
&+ C \int_0^t \left\| \frac{U(s+h) - U(s)}{h} \right\|_{\mathcal{E}}^2 ds + \int_0^t \left\| \frac{\alpha(s+h) - \alpha(s)}{h} \right\|_{L^2(\mathbb{R})}^2 ds.
\end{aligned}$$

Letting h tends to 0, we find

$$\left\| \frac{dU(t)}{dt} \right\|_{\mathcal{E}}^2 \leq \left\| \frac{dU}{dt}(0) \right\|_{\mathcal{E}}^2 + C \int_0^t \left\| \frac{dU}{dt}(s) \right\|_{\mathcal{E}}^2 ds + \int_0^t \|\alpha'(s)\|_{L^2(\mathbb{R})}^2 ds.$$

Observe that $U \in C^1([0, T]; \mathcal{E})$, then according to (P), we get

$$\frac{dU}{dt}(0) = \mathcal{A}U(0) + \mathcal{G}(0, U(0)).$$

Since $\mathcal{G}(0, 0) = \alpha(0)$ and \mathcal{G} is Lipschitz continuous on $[0, T] \times \mathcal{E}$, we may infer that

$$\left\| \frac{dU}{dt}(0) \right\|_{\mathcal{E}} \leq C(\|U_0\|_{\mathcal{D}(\mathcal{A})} + \|\alpha(0)\|_{L^2(\mathbb{R})}).$$

Then it follows that

$$\left\| \frac{dU}{dt}(t) \right\|_{\mathcal{E}} \leq C(\|U_0\|_{\mathcal{D}(\mathcal{A})} + \|\alpha\|_{C^1([0, T], L^2(\mathbb{R}))}) + C \int_0^t \left\| \frac{dU}{dt}(s) \right\|_{\mathcal{E}} ds$$

and the Grönwall's lemma leads to (3.8b). \square

4. The existence result for the beam equation

We now use the previous results with suitable functions α and β . The function α is simply defined as $\alpha \stackrel{\text{def}}{=} F/\mu$, with $F \in C^1([0, T], L^2(\mathbb{R}))$ a fixed function (see formula (1.5)). Next, we approximate the function $\varphi \stackrel{\text{def}}{=} \text{sign}$ in (1.1) by the function $\varphi_n : \mathbb{R} \rightarrow \mathbb{R}$ with $\varphi_n(z) \stackrel{\text{def}}{=} z/\sqrt{z^2 + \frac{1}{n}}$ ($n \in \mathbb{N}^*$) and set $\beta_n \stackrel{\text{def}}{=} -f\varphi_n/\mu$. This means that we replace the rigid response (1.1) by a very stiff response. Notice that $\lim_{n \rightarrow \infty} (z\varphi_n) = |\cdot|$ pointwise on \mathbb{R} and uniformly on any $\mathbb{R} \setminus [-\varepsilon, \varepsilon]$ ($\varepsilon > 0$).

We denote now by $U_n \stackrel{\text{def}}{=} (u_n, v_n)$ the solution considered in the Theorem 3.2 with $\alpha = F/\mu$, $\beta = \beta_n = -f\varphi_n/\mu$ and $U_0 \in \mathcal{D}(\mathcal{A})$ given. The family $\{U_n\}_{n \in \mathbb{N}^*}$ is bounded in $C^1([0, T]; \mathcal{E})$. More precisely, there exists a constant $C_T > 0$, independent of β_n (hence of $n \in \mathbb{N}^*$), such that

$$\|U_n\|_{C^0([0, T]; \mathcal{E})} + \left\| \frac{dU_n}{dt} \right\|_{C^0([0, T]; \mathcal{E})} + \int_{[0, T] \times \mathbb{R}} |v_n(x)\varphi_n(v_n(x))| dx dt \leq C_T. \quad (4.1)$$

It follows immediately that

Lemma 4.1. *There exists a constant $C_T > 0$, independent of $n \in \mathbb{N}^*$, such that*

$$\|u_n\|_{C^1(0, T; H^2(\mathbb{R}))} + \left\| \frac{\partial^2 u_n}{\partial t^2} \right\|_{C^0(0, T; L^2(\mathbb{R}))} \leq C_T. \quad (4.2)$$

Moreover, we have

$$\int_{[0, T] \times \mathbb{R}} \left| \left(\frac{\partial u_n}{\partial t} \varphi_n \left(\frac{\partial u_n}{\partial t} \right) \right) (t, x) \right| dx dt \leq C_T.$$

Proof. This follows from (4.1) with $v_n = \frac{\partial u_n}{\partial t}$ (see equation (P)) \square

Lemma 4.1 contains all the estimates that we need. Nevertheless, an additional argument, relying essentially on Fatou's lemma, is required to recover an inequality in the weak formulation of the problem ($n \rightarrow +\infty$). This argument is given in the proof of inequality (iv) below.

Lemma 4.2. *Let $U_0 \in \mathcal{D}(\mathcal{A})$. Extracting if necessary subsequences, there exists $u \in \cap_{r=1}^2 W^{r,\infty}([0, T]; H^{4-2r}(\mathbb{R}))$ such that, for any $\epsilon > 0$ and any compact $\mathcal{K} \subset \mathbb{R}$, we have*

- (i) $u_n \xrightarrow[n \rightarrow \infty]{} u$ in $C^1([0, T]; H^{2-\epsilon}(\mathcal{K}))$,
- (ii) $u_n \xrightarrow[n \rightarrow \infty]{} u$ and $\frac{\partial u_n}{\partial t} \xrightarrow[n \rightarrow \infty]{} \frac{\partial u}{\partial t}$ weakly in any $L^2([0, T]; H^2(\mathbb{R}))$,
 $u_n \xrightarrow[n \rightarrow \infty]{} u$ and $\frac{\partial u_n}{\partial t} \xrightarrow[n \rightarrow \infty]{} \frac{\partial u}{\partial t}$ weakly-* in any $L^\infty([0, T]; H^2(\mathbb{R}))$,
- (iii) $\frac{\partial^2 u_n}{\partial t^2} \xrightarrow[n \rightarrow \infty]{} \frac{\partial^2 u}{\partial t^2}$ weakly in any $L^2([0, T]; L^2(\mathbb{R}))$,
 $\frac{\partial^2 u_n}{\partial t^2} \xrightarrow[n \rightarrow \infty]{} \frac{\partial^2 u}{\partial t^2}$ weakly-* in any $L^\infty([0, T]; L^2(\mathbb{R}))$,
- (iv) $\frac{\partial u}{\partial t} \in L^1([0, T] \times \mathbb{R}) \cap L^\infty([0, T]; L^2(\mathbb{R}))$. Moreover, for any measurable subset $A \subset \mathbb{R}$ and any $\theta \in L^\infty([0, T] \times \mathbb{R})$, we have

$$\begin{aligned} & \int_{[0, T] \times A} \left| \frac{\partial u}{\partial t}(t, x) \right| |\theta(t, x)| \, dx \, dt \\ & \leq \liminf_{n \rightarrow \infty} \int_{[0, T] \times A} \left| \frac{\partial u_n}{\partial t} \left(\varphi_n \left(\frac{\partial u_n}{\partial t} \right) \right) (t, x) \right| |\theta(t, x)| \, dx \, dt. \end{aligned}$$

Proof. Note that (i) follows from the fact that $\{u_n\}_{n \in \mathbb{N}^*}$ is bounded in $\cap_{r=1}^2 C^r([0, T]; H^{4-2r}(\mathbb{R}))$ (cf. Lemma 4.1), the compact embedding $H^2(\mathcal{K}) \hookrightarrow H^{2-\epsilon}(\mathcal{K})$, Aubin–Lions lemma and the diagonal argument. Properties (ii) and (iii) come from Lemma 4.1. We now prove property (iv). Notice that $\frac{\partial u}{\partial t} \in L^\infty([0, T], L^2(\mathbb{R}))$ is already known, see (ii). Next, according to (i), we may deduce that

$$\frac{\partial u_n}{\partial t}(t, x) \xrightarrow[n \rightarrow \infty]{} \frac{\partial u}{\partial t}(t, x) \text{ in } L^1([0, T] \times \mathcal{K}) \text{ for any compact } \mathcal{K} \subset \mathbb{R}.$$

Then, extracting if necessary subsequences, and by using a diagonal argument, we get

$$\frac{\partial u_n}{\partial t}(t, x) \xrightarrow{n \rightarrow \infty} \frac{\partial u}{\partial t}(t, x) \text{ almost everywhere.}$$

Assume that $(t, x) \in [0, T] \times \mathbb{R}$ is such that $\frac{\partial u}{\partial t}(t, x) \neq 0$ and let $a < \frac{\partial u}{\partial t}(t, x) < b$ and $ab > 0$. Since $z\varphi_n$ converges to $|\cdot|$ uniformly on $[a, b]$, we obtain

$$\left(\frac{\partial u_n}{\partial t} \varphi_n \left(\frac{\partial u_n}{\partial t} \right) \right) (t, x) \xrightarrow{n \rightarrow \infty} \left| \frac{\partial u}{\partial t}(t, x) \right|. \quad (4.3)$$

for almost every $(t, x) \in [0, T] \times \mathbb{R}$ such that $\frac{\partial u}{\partial t}(t, x) \neq 0$. Next, under the assumption that $(t, x) \in [0, T] \times \mathbb{R}$ satisfies $\frac{\partial u}{\partial t}(t, x) = 0$, we obtain

$$\left| \frac{\partial u}{\partial t}(t, x) \right| \leq \left| \left(\frac{\partial u_n}{\partial t} \varphi_n \left(\frac{\partial u_n}{\partial t} \right) \right) (t, x) \right|. \quad (4.4)$$

We may deduce from (4.3) and (4.4) that, for any measurable subset $A \subset \mathbb{R}$

$$\begin{aligned} & \int_{[0, T] \times A} \left| \frac{\partial u}{\partial t}(t, x) \right| |\theta(t, x)| \, dx \, dt \\ & \leq \int_{[0, T] \times A} \liminf_{n \rightarrow \infty} \left| \left(\frac{\partial u_n}{\partial t} \varphi_n \left(\frac{\partial u_n}{\partial t} \right) \right) (t, x) \right| |\theta(t, x)| \, dx \, dt, \end{aligned}$$

and by using Fatou's lemma, we find

$$\begin{aligned} & \int_{[0, T] \times A} \left| \frac{\partial u}{\partial t}(t, x) \right| |\theta(t, x)| \, dx \, dt \\ & \leq \liminf_{n \rightarrow \infty} \int_{[0, T] \times A} \left| \left(\frac{\partial u_n}{\partial t} \varphi_n \left(\frac{\partial u_n}{\partial t} \right) \right) (t, x) \right| |\theta(t, x)| \, dx \, dt. \end{aligned}$$

According to Lemma 4.1, we have

$$\int_{[0, T] \times \mathbb{R}} \left| \left(\frac{\partial u_n}{\partial t} \varphi_n \left(\frac{\partial u_n}{\partial t} \right) \right) (t, x) \right| |\theta(t, x)| \, dx \, dt \leq C_T.$$

Finally, taking $A = \mathbb{R}$, $\theta(t, x) = 1$, we get $\frac{\partial u}{\partial t} \in L^1([0, T] \times \mathbb{R})$, which proves (iv). \square

Some additional smoothness on u is given in (i) of the following lemma. Notice that property (iii) of (4.3) is (better than) the dual property of (iv) in (4.2).

Lemma 4.3. *With the notations of Lemma 4.2, by extracting if necessary subsequences, we have*

- (i) $\frac{\partial^4 u_n}{\partial x^4} \rightharpoonup \frac{\partial^4 u}{\partial x^4}$ weakly in $L^2([0, T] \times \mathcal{K})$ for any compact $\mathcal{K} \subset \mathbb{R}$,
- (ii) $\int_{[0, T] \times \mathcal{K}} \frac{\partial^4 u_n}{\partial x^4} \frac{\partial u_n}{\partial t} dx dt \xrightarrow{n \rightarrow \infty} \int_{[0, T] \times \mathcal{K}} \frac{\partial^4 u}{\partial x^4} \frac{\partial u}{\partial t} dx dt$ for any compact $\mathcal{K} \subset \mathbb{R}$,
- (iii) $\frac{\partial^4 u}{\partial x^4} \in L^\infty([0, T] \times \mathbb{R}) + L^2([0, T]; L^2(\mathbb{R}))$.

Proof. By using Lemma 4.2, (i), we have

$$\frac{\partial u_n}{\partial t} \xrightarrow{n \rightarrow \infty} \frac{\partial u}{\partial t} \text{ strongly in } L^2([0, T] \times \mathcal{K}). \quad (4.5)$$

Hence, property (ii) follows from (4.5) and (i). We now prove (i) and (iii). We write equation (P) as

$$\text{EI} \frac{\partial^4 u_n}{\partial x^4} = -\mu \frac{\partial^2 u_n}{\partial t^2} - c \frac{\partial u_n}{\partial t} - k u_n + F - f \varphi_n \left(\frac{\partial u_n}{\partial t} \right),$$

which we abridge in

$$\text{EI} \frac{\partial^4 u_n}{\partial x^4} = A_n - f \varphi_n \left(\frac{\partial u_n}{\partial t} \right),$$

where $A_n \stackrel{\text{def}}{=} -\mu \frac{\partial^2 u_n}{\partial t^2} - c \frac{\partial u_n}{\partial t} - k u_n + F$. Since $|f \varphi_n(\frac{\partial u_n}{\partial t})| \leq f$, extracting if necessary a subsequence, we find that $\{f \varphi_n(\frac{\partial u_n}{\partial t})\}_{n \in \mathbb{N}^*}$ converge weakly- $*$ in $L^\infty([0, T] \times \mathbb{R})$. Next, since $\{u_n\}_{n \in \mathbb{N}^*}$ is bounded in $\cap_{r=1}^2 W^{r, \infty}([0, T]; H^{4-2r}(\mathbb{R}))$ (see Lemma 4.1) and $F \in L^2([0, T] \times \mathbb{R})$, we get that $\{A_n\}_{n \in \mathbb{N}^*}$ converges weakly in $L^2([0, T] \times \mathbb{R})$. Property (i) and (iii) follow. \square

We finally prove the existence of a weak solution for the initial problem.

Proof of Theorem 1.1. Let $w \in C_c^2([0, T] \times \mathbb{R})$, $\theta \in L^\infty([0, T] \times \mathbb{R})$ such that $\theta \geq 0$, and let $\mathcal{K} \subset \mathbb{R}$ be a compact set. Then equation (P) provides

$$\begin{aligned} & \int_{[0, T] \times \mathcal{K}} \left(\mu \frac{\partial^2 u_n}{\partial t^2} + \text{EI} \frac{\partial^4 u_n}{\partial x^4} \right) (t, x) \left(w(t, x) - \frac{\partial u_n}{\partial t}(t, x) \right) \theta(t, x) dx dt \\ & + \int_{[0, T] \times \mathcal{K}} \left(c \frac{\partial u_n}{\partial t} + k u_n - F \right) (t, x) \left(w(t, x) - \frac{\partial u_n}{\partial t}(t, x) \right) \theta(t, x) dx dt \quad (4.6) \\ & = - \int_{[0, T] \times \mathcal{K}} f \varphi_n \left(\frac{\partial u_n}{\partial t} \right) (t, x) \left(w(t, x) - \frac{\partial u_n}{\partial t}(t, x) \right) \theta(t, x) dx dt. \end{aligned}$$

By using Lemmas 4.2 and 4.3, we easily get that the left hand side of (4.6) tends to the same quantity with u in place of u_n as n tends to $+\infty$. Concerning the right hand side of (4.6), we may notice that $|\varphi_n| \leq 1$. Hence, we find

$$\left| \int_{[0,T] \times \mathcal{K}} \varphi_n \left(\frac{\partial u_n}{\partial t} \right) w(t, x) \theta(t, x) \, dx \, dt \right| \leq \int_{[0,T] \times \mathcal{K}} |w(t, x)| \theta(t, x) \, dx \, dt.$$

Notice that Lemma 4.2 leads to

$$\begin{aligned} & \int_{[0,T] \times \mathcal{K}} \left| \frac{\partial u}{\partial t}(t, x) \right| \theta(t, x) \, dx \, dt \\ & \leq \liminf_{n \rightarrow \infty} \int_{[0,T] \times \mathcal{K}} \left[\frac{\partial u_n}{\partial t} \varphi_n \left(\frac{\partial u_n}{\partial t} \right) \right] \theta(t, x) \, dx \, dt. \end{aligned}$$

Finally, letting n tends to $+\infty$ in (4.6), we obtain

$$\begin{aligned} & \int_{[0,T] \times \mathcal{K}} \left(\mu \frac{\partial^2 u}{\partial t^2} + \text{EI} \frac{\partial^4 u}{\partial x^4} \right) (t, x) \left(w(t, x) - \frac{\partial u}{\partial t}(t, x) \right) \theta(t, x) \, dx \, dt \\ & + \int_{[0,T] \times \mathcal{K}} \left(c \frac{\partial u}{\partial t} + ku - \text{F} \right) (t, x) \left(w(t, x) - \frac{\partial u}{\partial t}(t, x) \right) \theta(t, x) \, dx \, dt \quad (4.7) \\ & \geq \int_{[0,T] \times \mathcal{K}} f \left(\left| \frac{\partial u}{\partial t}(t, x) \right| - |w(t, x)| \right) \theta(t, x) \, dx \, dt. \end{aligned}$$

Define the function ℓ by

$$\begin{aligned} \ell(t, x) & \stackrel{\text{def}}{=} \left(\mu \frac{\partial^2 u}{\partial t^2} + c \frac{\partial u}{\partial t} + ku - \text{F} \right) (t, x) \left(w(t, x) - \frac{\partial u}{\partial t}(t, x) \right) \theta(t, x) \\ & - f \left(\left| \frac{\partial u}{\partial t}(t, x) \right| - |w(t, x)| \right) \theta(t, x) - \text{EI} \left(\frac{\partial^4 u}{\partial x^4} \frac{\partial u}{\partial t} \right) (t, x) \theta(t, x). \end{aligned}$$

It belongs to $L^1([0, T] \times \mathbb{R})$, due to Lemmas 4.2 and 4.3. Taking $\mathcal{K} \stackrel{\text{def}}{=} \mathcal{K}_p \stackrel{\text{def}}{=} [-p, p]$ with $p \in \mathbb{N}$, we obtain by using the dominated convergence theorem that

$$\int_{[0,T] \times \mathcal{K}_p} \ell(t, x) \, dx \, dt \xrightarrow{p \rightarrow \infty} \int_{[0,T] \times \mathbb{R}} \ell(t, x) \, dx \, dt \quad (4.8)$$

In order to pass to the limit $p \rightarrow +\infty$ in $\int_{[0,T] \times \mathcal{K}_p} \frac{\partial^4 u}{\partial x^4}(t, x) w(t, x) \theta(t, x) \, dx \, dt$,

we write, for $p \in \mathbb{N}$ large enough such that $\text{supp}(w) \subset [0, T] \times \mathcal{K}_p$

$$\begin{aligned} \int_{[0, T] \times \mathcal{K}_p} \frac{\partial^4 u}{\partial x^4}(t, x) w(t, x) \theta(t, x) \, dx \, dt &= \int_{[0, T] \times \mathbb{R}} \frac{\partial^4 u}{\partial x^4}(t, x) w(t, x) \theta(t, x) \, dx \, dt \\ &\xrightarrow{p \rightarrow \infty} \int_{[0, T] \times \mathbb{R}} \frac{\partial^4 u}{\partial x^4}(t, x) w(t, x) \theta(t, x) \, dx \, dt. \end{aligned} \quad (4.9)$$

Thus according to (4.7)–(4.9), we obtain

$$\begin{aligned} &\int_{[0, T] \times \mathbb{R}} \left(\mu \frac{\partial^2 u}{\partial t^2} + \text{EI} \frac{\partial^4 u}{\partial x^4} \right)(t, x) \left(w(t, x) - \frac{\partial u}{\partial t}(t, x) \right) \theta(t, x) \, dx \, dt \\ &+ \int_{[0, T] \times \mathbb{R}} \left(c \frac{\partial u}{\partial t} + ku - F \right)(t, x) \left(w(t, x) - \frac{\partial u}{\partial t}(t, x) \right) \theta(t, x) \, dx \, dt \quad (4.10) \\ &\geq \int_{[0, T] \times \mathbb{R}} f \left(\left| \frac{\partial u}{\partial t}(t, x) \right| - |w(t, x)| \right) \theta(t, x) \, dx \, dt. \end{aligned}$$

Using Theorems 4.2 (ii), (iii), (iv), 4.3 (iii), and a density argument, this also holds true for any $w \in L^1([0, T] \times \mathbb{R}) \cap L^2([0, T] \times \mathbb{R})$. Last, equality $(u(0, \cdot), (\partial u / \partial t)(0, \cdot)) = (u_0, v_0)$ follows from 4.2-(i), which proves the existence result.

Taking $\theta \in L^\infty([0, T])$ such that $\theta \geq 0$, it follows from equation (4.10) that for any $w \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, there exists a set $\mathcal{N}_w \subset [0, T]$ with zero measure such that for any $t \in [0, T] \setminus \mathcal{N}_w$, we have

$$\begin{aligned} &\int_{\mathbb{R}} \left(\mu \frac{\partial^2 u}{\partial t^2} + \text{EI} \frac{\partial^4 u}{\partial x^4} \right)(t, x) \left(w(x) - \frac{\partial u}{\partial t}(t, x) \right) \, dx \\ &+ \int_{\mathbb{R}} \left(c \frac{\partial u}{\partial t} + ku - F \right)(t, x) \left(w(x) - \frac{\partial u}{\partial t}(t, x) \right) \, dx \quad (4.11) \\ &\geq \int_{\mathbb{R}} f \left(\left| \frac{\partial u}{\partial t}(t, x) \right| - |w(x)| \right) \, dx. \end{aligned}$$

Observe that $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ endowed with the $\|\cdot\|_{L^1(\mathbb{R})} + \|\cdot\|_{L^2(\mathbb{R})}$ norm is separable. This comes from the fact that $L^2([-q, q])$ is separable for any $q \in \mathbb{N}$. Let $\{w_i \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), i \in \mathbb{N}\}$ be a dense subset of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Define the zero measure set \mathcal{N} by $\mathcal{N} \stackrel{\text{def}}{=} \cup_{i \in \mathbb{N}} \mathcal{N}_{w_i}$. Then for any $t \in [0, T] \setminus \mathcal{N}$ and any $w = w_i, i \in \mathbb{N}$, inequality (4.11) holds true. By a density argument, we may infer that (1.7) holds for any $t \in [0, T] \setminus \mathcal{N}$ and $w \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Following the relevant physical assumptions for $c \geq 0$ and $k > 0$, uniqueness is established below by a standard approach. First, let us introduce u and v two solutions to (4.10), (1.2) and (1.6). Choosing $w = \frac{\partial u}{\partial t} \in L^1([0, T] \times \mathbb{R}) \cap L^2([0, T] \times \mathbb{R})$ (see (1.6b)) and $\theta = 1$ in (4.10), we obtain

$$\begin{aligned} & \int_{[0, \tau] \times \mathbb{R}} \left(\mu \frac{\partial^2 u}{\partial t^2} + \text{EI} \frac{\partial^4 u}{\partial x^4} \right) (t, x) \left(\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t} \right) (t, x) \, dx \, dt \\ & + \int_{[0, \tau] \times \mathbb{R}} \left(c \frac{\partial u}{\partial t} + ku - F \right) (t, x) \left(\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t} \right) (t, x) \, dx \, dt \quad (4.12) \\ & \geq \int_{[0, \tau] \times \mathbb{R}} f \left(\left| \frac{\partial u}{\partial t} (t, x) \right| - \left| \frac{\partial v}{\partial t} (t, x) \right| \right) \, dx \, dt, \end{aligned}$$

for all $\tau \in [0, T]$. We exchange u and v in (4.12) and add the resulting inequality to (4.12). We write $z \stackrel{\text{def}}{=} u - v$. Performing standard manipulations, this leads to

$$\begin{aligned} & \frac{\mu}{2} \left\| \frac{\partial z}{\partial t} (\tau, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\text{EI}}{2} \left\| \frac{\partial^2 z}{\partial x^2} (\tau, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ & + c \left\| \frac{\partial z}{\partial t} \right\|_{L^2([0, \tau] \times \mathbb{R})}^2 + \frac{k}{2} \|z(\tau, \cdot)\|_{L^2(\mathbb{R})}^2 \quad (4.13) \\ & \leq \frac{\mu}{2} \left\| \frac{\partial z}{\partial t} (0, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{k}{2} \|z(0, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\text{EI}}{2} \left\| \frac{\partial^2 z}{\partial x^2} (0, \cdot) \right\|_{L^2(\mathbb{R})}^2 \end{aligned}$$

Since $z(0, \cdot) = \frac{\partial z}{\partial t}(0, \cdot) = 0$, we get $z = 0$ a.e., which concludes the proof. \square

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