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► **To cite this version:**

Pierre Clavier, Li Guo, Sylvie Paycha, Bin Zhang. Renormalisation via locality morphisms. *Revista Colombiana de Matemáticas*, 2019, 53 (Suppl), pp.113-141. 10.15446/recolma.v53nsupl.84010 . hal-04133424

**HAL Id: hal-04133424**

**<https://hal.science/hal-04133424v1>**

Submitted on 19 Jun 2023

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# RENORMALISATION VIA LOCALITY MORPHISMS

PIERRE CLAVIER, LI GUO, SYLVIE PAYCHA, AND BIN ZHANG

ABSTRACT. This is a survey on renormalisation in the locality setup highlighting the role that locality morphisms can play for renormalisation purposes. Having set up a general framework to build regularisation maps, we illustrate renormalisation by locality algebra homomorphisms on three examples, the renormalisation at poles of conical zeta functions, branched zeta functions and iterated integrals stemming from Kreimer's toy model.

## CONTENTS

Introduction	2
1. Partial versus locality structures	5
1.1. Partial semigroups	5
1.2. Relating various partial structures	8
1.3. Transitive partial structures	9
2. Locality algebras	9
2.1. Basic definitions	9
2.2. The locality algebra of meromorphic germs with linear poles	10
2.3. Locality algebra of lattice cones	11
2.4. Locality algebra of decorated rooted forests.	12
2.5. Locality algebra of meromorphic germs of symbols	12
3. Locality morphisms	13
3.1. Basic notions and examples	13
3.2. Locality morphisms on the algebra of meromorphic germs of symbols	14
3.3. Locality morphisms on lattice cones	14
3.4. Linear operators lifted to the algebra of rooted forests.	14
3.5. Operations lifted to the algebra of rooted forests	16
4. Renormalisation by locality morphisms	16
4.1. A renormalisation scheme	17
4.2. Renormalised conical zeta values	17
4.3. Branched zeta values	17
4.4. Kreimer's toy model	18
References	18

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2010 *Mathematics Subject Classification.* 08A55,16T99,81T15, 32A20, 52B20.

*Key words and phrases.* locality, renormalisation, partial algebra, operated algebra, Hopf algebra, Rota-Baxter algebra, symbols.

## INTRODUCTION

The study of certain models, might they stem from quantum field theory or be of a pure mathematical nature like that of measuring the volume of a convex cone, gives rise to *formal expressions*. Their evaluation can yield infinities instead of the desired numerical invariants, a situation which calls for renormalisation. Getting rid of the infinities first calls for a *regularisation* which turns the formal expressions into meaningful regularised expressions, typically described in terms of an algebra homomorphism

$$(1) \quad \phi^{\text{reg}} : A \longrightarrow M$$

defined on a certain algebra  $A$  (the algebra structure reflects the structure of the family of formal expressions) with values in a space  $M$  of meromorphic functions. Next, these meromorphic functions are processed to only keep the holomorphic parts which can then be evaluated at zero, giving rise to the *renormalised values* of the formal expressions.

A basic requirement is that the renormalised values obey a multiplicative property, a requirement reminiscent of the *locality principle* in quantum field theory. It states that an object is only directly influenced by its immediate surroundings which in practice translates to the independence and multiplicativity of measurements of independent events.

In the case of an univariate regularisation,  $M = \mathbb{C}[\varepsilon^{-1}, \varepsilon]$  and projecting onto the holomorphic part  $M_+ = \mathbb{C}[[\varepsilon]]$  (by means of the minimal subtraction scheme) gives rise to a Rota-Baxter operator  $\pi_+ : M \longrightarrow M_+$ . Yet the Rota-Baxter operator itself is not multiplicative so that a mere minimal subtraction scheme  $\pi_+ \circ \phi^{\text{reg}}$  does not preserve multiplicativity. However, if the space  $A$  carries a suitable Hopf algebra structure, an algebraic Birkhoff factorisation à la Connes and Kreimer [CK] implemented on the regularised map  $\phi^{\text{reg}} : A \longrightarrow M$  guarantees the multiplicativity of the renormalised map  $\phi_+^{\text{ren}} : A \longrightarrow M_+$ , that is, the conservation of products after renormalisation.

An alternative to univariate regularisation is multivariate regularisation, a setup which enables to encode locality as a guiding principle, thus opening the way to new opportunities, and new challenges. On the one hand, the multivariate minimal subtraction scheme, when available, is generally not an algebra homomorphism, and does not even give rise to a Rota-Baxter operator, a major obstacle to the implementation of an algebraic Birkhoff factorisation even when  $A$  carries a Hopf algebra structure. Yet on the other hand, the very multivariate nature of the scheme provides an easy book keeping device to preserve products for certain pairs of elements, we call *independent pairs*; in accordance with the locality principle in quantum field theory, multiplicativity holds for independent pairs of arguments. The latter property, together with the multiplicativity of the evaluation map at zero, allows the renormalisation to preserve products for independent pairs of elements.

An algebraic formulation of the locality principle in renormalisation was discussed in [CGPZ1]. There, we express locality as a symmetric binary relation, study locality versions of algebraic structures, and develop a machinery used to preserve locality during the renormalisation procedure. It turns out that the locality setup is important not only in renormalisation, but also crucial in exploring deeper structures as we can see in the example of lattice cones.

The key asset of a locality setup lies in the fact that the minimal subtraction scheme can be viewed as *locality algebra homomorphism*. For example, the algebra  $\mathcal{M}$  of multivariate meromorphic germs with linear poles carries a locality algebra structure  $(\mathcal{M}, \perp^Q, \cdot)$  and the corresponding projection  $\pi_+$  from the decomposition  $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-^Q$  to  $\mathcal{M}_+$  (which as we

stressed previously is not an algebra homomorphism) is a locality algebra homomorphism. Thus, for a regularisation, provided the source space  $A$  can be equipped with a *locality algebra*  $(A, \top_A)$  structure, and provided the regularised map  $\phi^{\text{reg}}$  takes its values in  $\mathcal{M}$  and it is a locality algebra homomorphism, then a *multivariate subtraction scheme* can be implemented on the regularised maps  $\phi^{\text{reg}} : (A, \top_A) \rightarrow (\mathcal{M}, \perp^Q)$ . In this locality setup, the renormalised map  $\pi_+ \circ \phi^{\text{reg}}$  still do not preserve products yet they preserve partial products, which is what one needs.

To sum up, we work in a setup which encompasses the principle of locality; locality detects pairs of independent elements of  $A$  and partial multiplicativity amounts to multiplicativity on pairs of independent elements. The purpose of this survey based on previous work by the authors [CGPZ1, CGPZ2, CGPZ3] and [GPZ1, GPZ2] is i) to demonstrate how to achieve a multivariate regularisation of a formal expression so as to build a locality algebra homomorphism (so a multivariate version of (1))

$$(2) \quad \phi^{\text{reg}} : (A, \top_A) \longrightarrow (\mathcal{M}, \perp^Q)$$

on a locality algebra  $(A, \top^A)$  with values in the locality algebra  $(\mathcal{M}, \perp^Q)$ , and ii) to renormalise the resulting regularised locality algebra homomorphism, describing the general theory and illustrating it by examples. Let us describe the contents of the paper in more detail.

Since there are different approaches to explore locality, in Section 1, much of which is borrowed from [Zh], we review and compare various partial structures with the locality structures introduced in [CGPZ1]. In particular, we view the locality setup as a symmetric version of the more general  $\mathcal{R}$ -setup which comprises partial semigroups (Definition 1.7) introduced in [Sch] and we relate  $\mathcal{R}$ -monoids to the selective category of Li-Bland and Weinstein [LW] with one object (Proposition 1.9). Thereafter, for the sake of simplicity, we choose to keep to the locality setup which turns out to be sufficient for the renormalisation purposes we have in mind.

In Section 2, we introduce locality algebras (Definition 2.1), a notion we first illustrate by the pivotal example of  $\mathbb{R}^\infty$  (Example 2.2) equipped with an inner product  $Q$  which induces an orthogonality relation  $\perp^Q$ , after which we discuss in Paragraph 2.2, the algebra  $\mathcal{M}$  of multivariate meromorphic germs with linear poles at zero, equipped with a locality relation induced by  $\perp^Q$ , which by a slight abuse of notation is denoted by the same symbol (see Proposition 2.3). Other relevant examples are the locality algebra of *lattice cones* in Paragraph 2.3 and the locality algebra of *properly decorated rooted forests* in Paragraph 2.4.

Section 3 is dedicated to the main protagonists of this paper, namely *locality morphisms* (Definition 3.2) of locality algebras, so maps between locality algebras which, as well as preserving the locality relation and locality vector space structure, further preserve the related partial product.

Amongst these are *locality projections*  $\pi_+^Q : \mathcal{M} \rightarrow \mathcal{M}_+$  onto the space  $\mathcal{M}_+$  of holomorphic germs at zero built from the inner product  $Q$ . Such projections arise from the decomposition  $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-^Q$  (Eq. (12)) induced by  $Q$ , and their locality as morphisms is a consequence of the fact that  $\mathcal{M}_+$  (resp.  $\mathcal{M}_-^Q$ ) is a locality subalgebra (resp. locality ideal) of  $\mathcal{M}$ , (Proposition 2.4).

Composed with the evaluation at zero  $\text{ev}_0$  these projections yield useful *renormalisation schemes* discussed in Paragraph 4.1:

$$(3) \quad \text{ev}_0 \circ \pi_+^Q : \mathcal{M} \longrightarrow \mathbb{C},$$

which can be viewed as a *multivariate minimal subtraction scheme*.

With this multivariate minimal subtraction scheme, a renormalisation process is reduced to two steps:

- (i) to construct the regularised map  $\phi^{\text{reg}} : (A, \top_A) \longrightarrow (\mathcal{M}, \perp^Q)$ ;
- (ii) to implement renormalisation schemes of the type (3) to the regularised map  $\phi^{\text{reg}} : (A, \top_A) \longrightarrow (\mathcal{M}, \perp^Q)$  in order to build the *renormalised map*  $\phi^{\text{ren}} := \text{ev}_0 \circ \pi_+^Q \circ \phi^{\text{reg}} : A \rightarrow \mathbb{C}$ .

Various locality maps built in Section 3 are interpreted in Section 4 as regularisation maps  $\phi^{\text{reg}} : A \longrightarrow \mathcal{M}$  which need to be renormalised, all of which stem from formal sums and integrals as multivariate regularisations.

We first illustrate (in Paragraphs 3.3 and 4.2) this multivariate approach with *conical zeta functions* (resp. *branched zeta functions*), which to a lattice cone (resp. a decorated rooted forest), assign a renormalised value of the *regularised conical zeta function* (resp. *regularised branched zeta function*) at poles. The (partial) multiplicativity of the maps encoded in their very construction in our multivariate locality setup, ensures their multiplicativity on orthogonal lattice cones (resp. independent decorated rooted forests).

In [CGPZ1, GPZ1], *conical zeta functions* (Paragraph 4.2), which generalise multiple zeta functions were built using exponential sums on lattice cones. The exponential sum  $S$ , resp. integral  $I$  on a lattice cone correspond to the discrete, resp. continuous Laplace transformation of the characteristic function of the lattice cone (Proposition 3.7). One easily checks that Laplace transforms of characteristic functions of smooth cones define meromorphic maps with linear poles; the fact that  $S$  and  $I$  take their values in  $\mathcal{M}$  for any convex lattice cone, then follows from their additivity on disjoint unions combined with the fact that any convex lattice cone can be subdivided in smooth lattice cones. Both maps define locality algebra homomorphisms on the locality algebra of lattice cones for a locality relation induced by the orthogonality relation  $\perp^Q$  on  $\mathbb{R}^\infty$ . Their multiplicativity on orthogonal lattice cones follows from the usual homomorphism property of the exponential map on these cones.

A second example which provides an alternative generalisation of multiple zeta functions, is given by *branched zeta functions* [CGPZ2] (discussed in Paragraph 4.3) associated with rooted forests (Paragraph 2.4). These are built by means of a *branching procedure* which strongly relies on the universal properties of properly decorated rooted forests (see Proposition 3.11). Such a branching procedure lifts a map  $\phi$  defined on the decoration set to what we call a *branched map*  $\widehat{\phi}$  on the algebra of decorated forests (see (17)). Applied to a summation map  $\phi = \mathfrak{S}_\lambda$  on the locality algebra  $\Omega$  of meromorphic germs of symbols, this branching procedure gives rise to a branched sum  $\widehat{\mathfrak{S}}_\lambda$  acting on the algebra of properly decorated rooted forests by meromorphic family of symbols on  $\mathbb{R}_{\geq 0}$ . The universal property underlying the construction ensures the multiplicativity on independent forests. Combining this with the locality morphism given by the Hadamard finite part at infinity (15)-a linear form on polyhomogeneous pseudodifferential symbols which coincides on

smoothing symbols with the limit at infinity- extended to  $\Omega$ , gives rise to *branched regularised zeta functions*  $\zeta^{\text{reg},\lambda}$  defined on the locality algebra of properly  $\Omega$ -decorated rooted forests.

In Paragraph 3.5 we describe similar constructions based on the universal properties of properly decorated rooted forests [CGPZ3], which yield a third example (Paragraph 4.4), namely  $\mathcal{M}$ -valued maps stemming from iterated integrals arising in Kreimer's toy model [K].

To sum up, the locality setup combined with the multivariate regularisation provides a way to preserve (partial) multiplicativity while renormalising, in accordance with the locality principle in physics.

## 1. PARTIAL VERSUS LOCALITY STRUCTURES

We review and compare various partial product structures with the locality structures introduced in [CGPZ1]; although the concept of algebraic locality structures is to our knowledge new in the context of renormalisation, partial products have been used in other contexts, hence the need to relate the two concepts, partial and locality products. This section is based on [Zh].

**1.1. Partial semigroups.** We start with a generalisation of the notion of a locality set introduced in [CGPZ1], by dropping the symmetry property of the relation required in [CGPZ1]:

**Definition 1.1.** (i) An  $\mathcal{R}$ -set is a couple  $(X, \top)$  with  $X$  a set and  $\top \subset X \times X$  a binary relation on  $X$ . We also write  $X \times_{\top} X$  for  $\top$ .

(ii) Let  $(X, \top)$  be an  $\mathcal{R}$ -set and  $U \subset X$ . We write  ${}^{\top}U$  (resp.  $U^{\top}$ ) the **left polar set** (resp. **right polar set**) of  $U$ ; defined by

$$(4) \quad {}^{\top}U := \{x \in X \mid (x, u) \in \top \forall u \in U\}$$

(resp.

$$(5) \quad U^{\top} := \{x \in X \mid (u, x) \in \top \forall u \in U\}.$$

If  $\top$  is a symmetric binary relation, we call, as in [CGPZ1], the couple  $(X, \top)$  a **locality set**, in which case  ${}^{\top}U = U^{\top}$ .

Let **RS** (resp. **LS**) denote the category of  $\mathcal{R}$ -sets (resp. locality sets) whose morphisms are maps  $\phi : (X, \top_X) \rightarrow (Y, \top_Y)$  such that  $(\phi \times \phi)(\top_X) \subset \top_Y$ , called  **$\mathcal{R}$ -maps** (resp. **locality maps**).

We equip an  $\mathcal{R}$ -set with four distinct, however related, partial product structures, the first one is a generalisation (dropping the symmetry condition) taken from [Zh] of the locality relation introduced in [CGPZ1]:

**Definition 1.2.** An  $\mathcal{R}$ -semigroup is an  $\mathcal{R}$ -set  $(X, \top)$  together with a partial product map

$$\begin{aligned} \mu : \top &\mapsto X \\ (x, y) &\mapsto xy \end{aligned}$$

which we denote by  $(X, \top, \mu)$ , such that:

(i) For any subset  $U \subset X$ ,

$$(6) \quad \mu(({}^{\top}U \times {}^{\top}U) \cap \top) \subseteq {}^{\top}U,$$

(ii) For any subset  $U \subset X$ ,

$$(7) \quad \mu((U^\top \times U^\top) \cap \top) \subseteq U^\top.$$

(iii) For any  $a, b, c$  in  $X$  such that any couple lies in  $\top$  we have  $(ab)c = a(bc)$ .

If  $\top$  is a symmetric binary relation, condition (6) coincides with (7) and we call  $(X, \top, \mu)$  a **locality semigroup**.

Let us denote by **RSg** (resp. **LSg**) the category of  $\mathcal{R}$ - (resp. locality) semigroups whose morphisms are  $\mathcal{R}$ -maps (resp. locality maps)

$$\phi : (X, \top_X, \mu_X) \longrightarrow (Y, \top_Y, \mu_Y),$$

which are partially multiplicative

$$(a, b) \in \top_X \implies \phi(\mu_X(a, b)) = \mu_Y(\phi(a), \phi(b)).$$

They are called  **$\mathcal{R}$ -morphisms** (resp. **locality morphisms**).

**Remark 1.3.** Note that a map between two locality semigroups is a locality morphism if and only if it is an  $\mathcal{R}$ -morphism.

**Remark 1.4.** It is easy to check that

- Eq. (6) is equivalent to

$$(8) \quad (x \top z \wedge y \top z \wedge x \top y) \implies (xy) \top z \quad \forall (x, y, z) \in X^3,$$

- Eq. (7) is equivalent to

$$(9) \quad (z \top x \wedge z \top y \wedge x \top y) \implies z \top (xy) \quad \forall (x, y, z) \in X^3.$$

The following definitions are taken from [Zh].

**Definition 1.5.** (see [Zh, Definition 3.1])

(i) A **strong  $\mathcal{R}$ -semigroup** is an  $\mathcal{R}$ -set  $(X, \top)$  together with a partial product map

$$\begin{aligned} \mu : \top &\mapsto X \\ (x, y) &\mapsto xy \end{aligned}$$

also denoted by  $(X, \top, \mu)$ , such that for any  $x, y, z \in X$  :

$$((x, y) \in \top \wedge (y, z) \in \top) \implies ((xy, z) \in \top \wedge (x, yz) \in \top \wedge (xy)z = x(yz)).$$

Let us denote by **SRSg** the category of strong  $\mathcal{R}$ -semigroups whose morphisms are  **$\mathcal{R}$ -morphisms**.

(ii) A **refined  $\mathcal{R}$ -semigroup** is an  $\mathcal{R}$ -set  $(X, \top)$  together with a partial product map

$$\begin{aligned} \mu : \top &\mapsto X \\ (x, y) &\mapsto xy \end{aligned}$$

such that:

- $(x, y) \in \top \implies ((y, z) \in \top \Leftrightarrow (xy, z) \in \top, \forall z \in X)$ ,
- $(y, z) \in \top \implies ((x, y) \in \top \Leftrightarrow (x, yz) \in \top, \forall x \in X)$ ,
- For any  $(x, y) \in \top$  and  $(y, z) \in \top$  we have  $(xy)z = x(yz)$ .

Let us denote by **RRSg** the category of refined  $\mathcal{R}$ -semigroups whose morphisms are  **$\mathcal{R}$ -morphisms**.

**Remark 1.6.** ([Zh, Proposition 3.3]) Every strong  $\mathcal{R}$ -semigroup is clearly an  $\mathcal{R}$ -semigroup, but the converse does not hold. See e.g. [Zh, Counterexample 3.4] and the subsequent paragraph.

The following definition is taken from [Sch]. See also [Zh, Definition 2.20].

**Definition 1.7.** A **partial semigroup** is an  $\mathcal{R}$ -set  $(X, \top)$  together with a partial product map

$$\begin{aligned} \mu : \top &\mapsto X \\ (x, y) &\mapsto xy \end{aligned}$$

such that for any  $x, y, z \in X$

$$(10) \quad ((x, y) \in \top \wedge (xy, z) \in \top) \Leftrightarrow ((y, z) \in \top \wedge (x, yz) \in \top)$$

in which case  $(xy)z = x(yz)$  also holds. Let us denote by **PSg** the category of partial semigroups whose morphisms are  **$\mathcal{R}$ -morphisms**.

The notion of partial semigroup relates to a particular instance of the **selective category** of Li-Bland and Weinstein introduced in [LW, Definition 2.1], whose definition we now recall.

**Definition 1.8.** A **selective category** is a category  $\mathcal{C}$  whose set of morphisms (resp. objects) we denote by  $\text{Mor}$  (resp.  $\text{Ob}$ ) together with a distinguished class  $\mathcal{S} \subset \text{Mor}$  of morphisms, called **suave**, and a class  $\top_{\mathcal{S}} \subset \mathcal{S} \times \mathcal{S}$  of pairs of suave morphisms called **congenial pairs**, such that:

- (i) Any identity morphism is suave so  $\text{Id}_x$  is suave for any  $x \in \text{Ob}$  which we write for short  $\text{Id} \in \mathcal{S}$ ;
- (ii) If  $f : X \rightarrow Y$  is suave,  $(\text{Id}_Y, f)$  and  $(f, \text{Id}_X)$  are congenial;
- (iii) If  $f$  is a suave isomorphism, its inverse  $f^{-1}$  is suave as well, and the pairs  $(f, f^{-1})$  and  $(f^{-1}, f)$  are both congenial;
- (iv) If  $f$  and  $g$  are suave and  $(f, g)$  is congenial, then  $f \circ g$  is suave, i.e., the composition is a map  $\circ : \top_{\mathcal{S}} \rightarrow \mathcal{S}$ ;
- (v) If  $f, g, h \in \mathcal{S}$ , then

$$((f, g) \in \top_{\mathcal{S}} \wedge (f \circ g, h) \in \top_{\mathcal{S}}) \iff ((g, h) \in \top_{\mathcal{S}} \wedge (f, g \circ h) \in \top_{\mathcal{S}}),$$

in which case  $(f, g, h)$  is called a **congenial triple**.

A **selective functor** between selective categories is one which takes congenial pairs to congenial pairs.

Recall that a category  $\mathcal{C} = (\text{Obj}(\mathcal{C}), \text{Mor}(\mathcal{C}))$  is **small** if  $\text{Obj}(\mathcal{C})$  and  $\text{Mor}(\mathcal{C})$  are sets and not proper classes.

**Proposition 1.9.** *A small selective category with one object reduces to a partial semigroup  $(S, \top \subset S \times S, m)$  built from a nonempty subset  $S \subset M$  of a monoid  $(M, \mu)$  with unit 1 such that,*

- (i)  $1 \in S$ ,  $1 \top S$  and  $S \top 1$ ;
- (ii)  $(S, \top)$  is stable under taking inverse (in  $M$ ) in the following sense: if  $s \in S$  is invertible in  $M$ , then its inverse  $s^{-1}$  is in  $S$  and  $(s, s^{-1}), (s^{-1}, s)$  are in  $\top$ .

*A selective morphism between selective categories with one object reduces to  $\mathcal{R}$ -morphisms of partial semigroups that preserve the identity (and hence inverses).*



*Proof.* With exactly one object, a small category  $\mathcal{C} = (\text{Obj}(\mathcal{C}), \text{Mor}(\mathcal{C}))$  boils down to a monoid  $M := \text{Mor}(\mathcal{C})$ , its distinguished class  $\mathcal{S}\text{Mor}$  of suave morphisms boils down to a subset  $S \subset M$ , and the class of congenial pairs of suave elements boils down to a subset  $\top \subset S \times S$ . Further conditions (i) – (iii) of a selective category boil down to the two conditions in the lemma, while conditions (iv) – (v) boil down to the condition that  $(S, \top)$  is a partial semigroup.

Finally a selective functor  $f : (S_1, \top_1) \rightarrow (S_2, \top_2)$  between selective categories  $(S_i, \top_i)$  with one object boils down to a  $\mathcal{R}$ -morphism of partial semigroups that preserve the identity.  $\square$

**Remark 1.10.** We need the category  $\mathcal{C}$  to be small, as even a category with only one object can be large. For example, take  $\mathcal{C}$  the category whose only object **Set** is the category of sets, and whose morphisms are the endofunctors of **Set**. In this example,  $\text{Mor}(\mathcal{C})$  has no monoid structure as it is not a set.

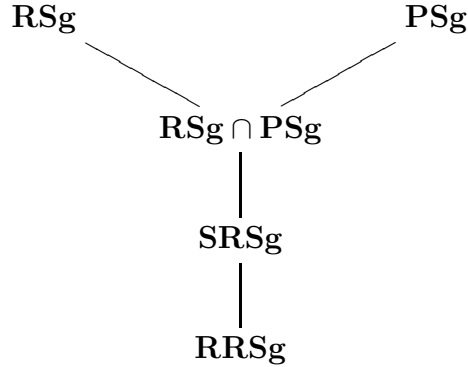
**1.2. Relating various partial structures.** We quote from [Zh] with the reference to the statements. We start with some general comparisons:

- $\text{RRSg} \subsetneq \text{SRSg}$  [Zh, Example 4.3].
- $\text{SRSg} \subsetneq \text{RSg}$  [Zh, Proposition 3.3 and Counterexample 3.4].
- $\text{SRSg} \subsetneq \text{PSg}$  [Zh, Example 3.6].
- $\text{SRSg} \subsetneq \text{RSg} \cap \text{PSg}$  [Zh, Proposition 3.7].

There are examples of  $\mathcal{R}$ -semigroups that are not partial semigroups and vice-versa:

- $\text{RSg} \not\subseteq \text{PSg}$  [Zh, Example 3.8].
- $\text{PSg} \not\subseteq \text{RSg}$  [Zh, Example 3.10].

Note that the last two conditions mean that  $\text{RSg} \cap \text{PSg} \subsetneq \text{PSg}$  and  $\text{RSg} \cap \text{PSg} \subsetneq \text{RSg}$ . Thus in summary, we have strict inclusions shown by the following Hasse diagram.



Here are examples of locality sets with a partial product which fulfills the following equivalence relation:

$$(x \top y \wedge (xy) \top z) \iff (x \top y \wedge y \top z \wedge x \top z) \iff (y \top z \wedge x \top (yz)),$$

namely, conditions (8), (9) (which are equivalent for locality semigroups) are equivalent to (10). So they are both locality and partial semigroups.

**Example 1.11.** (i) The set  $\mathbb{N}$  of natural numbers equipped with the coprime relation  $n \top m \iff n \wedge m = 1$  and the usual product of real numbers is a partial semigroup since

$$a \wedge b = 1 \text{ and } a b \wedge c = 1 \iff a \wedge b = 1 \text{ and } a \wedge c = 1 \text{ and } b \wedge c = 1 \iff c \wedge b = 1 \text{ and } a \wedge b c = 1,$$

and a locality semigroup since

$$a \wedge c = 1 \text{ and } b \wedge c = 1 \implies ab \wedge c = 1.$$

(ii) The power set  $\mathcal{P}(X)$  of a set  $X$  equipped with the disjointness relation  $A \top B \Leftrightarrow A \cap B = \emptyset$  and the product law given by the union  $\cup$  is a partial semigroup and we have

$$\begin{aligned} A \cap B = \emptyset \wedge (A \cup B) \cap C = \emptyset &\iff A \cap B = \emptyset \wedge A \cap C = \emptyset \wedge B \cap C = \emptyset \\ &\iff B \cap C = \emptyset \wedge A \cap (B \cup C) = \emptyset. \end{aligned}$$

It is also a locality semigroup since

$$A \cap C = \emptyset \wedge B \cap C = \emptyset \implies (A \cup B) \cap C = \emptyset.$$

**1.3. Transitive partial structures.** Here is a useful property of partial structures.

**Definition 1.12.** A locality set  $(X, \top)$  is called **transitive** if the relation  $\top$  is transitive, namely if for any  $a, b, c \in X$

$$((a, b) \in \top \wedge (b, c) \in \top) \implies (a, c) \in \top.$$

A partial structure  $(X, \top, \mu)$  such that  $(X, \top)$  is transitive is called a **transitive partial structure**. We write **tLSg** (resp. **tSLSg**, **tRSg**, **tPSg**) for the category of transitive locality semigroups (resp. transitive strong locality semigroups, transitive refined locality semigroups, transitive partial semigroups).

**Remark 1.13.** Transitive partial structures are not relevant in the context of locality understood in the sense of quantum field theory, since we do not expect the event  $A$  to be independent of the event  $C$  under the assumption that the event  $A$  is independent of the event  $B$  and the event  $B$  is independent of the event  $C$ . In fact, a transitive locality structure  $\top$  is *almost* reflexive, in that for every event  $a$ , if there exists  $b$  such that  $b \top a$ , then  $a$  is independent of itself.

We saw that locality semigroups and partial semigroups are distinct structures. However, we have the following result:

**Proposition 1.14.** [Zh, Proposition 3.9] **tLSg**  $\subsetneq$  **tPSg**.

The statement of [Zh] involves a non-strict inclusion  $\subseteq$ , yet [Zh, Example 3.10] gives a transitive partial semigroup which is not a locality semigroup.

## 2. LOCALITY ALGEBRAS

Throughout the paper we choose to work in the framework of locality structures, partially for simplicity but more so due to the fact the the applications we have in view do not require the more general framework of  $\mathcal{R}$ -structures.

**2.1. Basic definitions.** We borrow the subsequent definitions from [CGPZ1]. Among them, locality algebras are fundamental objects in multivariate renormalisation.

**Definition 2.1.** (i) A **locality vector space** is a vector space  $V$  over a field  $K$  equipped with a locality relation  $\top$  which is compatible with the linear structure on  $V$  in the sense that, for any subset  $X$  of  $V$ ,  $X^\top$  is a linear subspace of  $V$ .

- (ii) A **locality monoid** is a locality semigroup  $(G, \top, m_G)$  together with a **unit element**  $1_G \in G$  given by the defining property

$$\{1_G\}^\top = G \quad \text{and} \quad m_G(x, 1_G) = m_G(1_G, x) = x \quad \text{for all } x \in G.$$

- (iii) A (**resp. unital**) **locality algebra**  $(A, \top, +, \cdot, m_A)$  (**resp.**  $(A, \top, +, \cdot, m_A, 1_A)$ ) over  $K$  is a locality vector space  $(A, +, \cdot, \top)$  over  $K$  together with a locality bilinear map

$$m_A : A \times_\top A \rightarrow A$$

such that  $(A, \top, m_A)$  is a locality semigroup (**resp.** a locality monoid with **unit**  $1_A \in A$ ).  $(A, \top)$  is called **commutative** if  $(A, \top, m_A)$  is a commutative locality semigroup.

- (iv) A **sub-locality algebra** of a locality algebra  $(A, \top, m_A)$  is a linear subspace  $B$  of  $A$  such that with respect to the locality condition  $\top_B := (B \times B) \cap \top$  of  $\top$  and the partial product  $m_B := m_A|_{\top_B}$  on  $B$ ,  $(B, \top_B, m_B)$  is a locality algebra.
- (v) A sub-locality algebra  $I$  of a locality commutative algebra  $(A, \top, m_A)$  is called a **locality ideal** of  $A$  if  $m_A(I^\top, I) \subseteq I$ ; i.e. if for any  $b \in I$  we have  $m_A(c, b) \in I$  for all  $c \in \{b\}^\top$ .

**Example 2.2.** A pivotal example is the locality vector space  $(\mathbb{R}^\infty, \perp^Q)$ , where  $\mathbb{R}^\infty = \bigcup_{k \geq 1} \mathbb{R}^k$  and  $Q = (Q_k(\cdot, \cdot))_{k \geq 1}$  is the inner product on  $\mathbb{R}^\infty$  defined by the inner products on  $\mathbb{R}^k$

$$Q_k(\cdot, \cdot) : \mathbb{R}^k \otimes \mathbb{R}^k \rightarrow \mathbb{R}, \quad k \geq 1,$$

such that  $Q_{k+1}|_{\mathbb{R}^k \otimes \mathbb{R}^k} = Q_k$ . The inner product induces a locality relation on  $\mathbb{R}^\infty$

$$u \perp^Q v \Leftrightarrow Q(u, v) = 0,$$

which makes  $(\mathbb{R}^\infty, \perp^Q)$  a locality vector space.

The inner product also induces a locality set structure on the set of subspaces of  $\mathbb{R}^\infty$ :

$$U \perp^Q V \Leftrightarrow Q(u, v) = 0, \forall u \in U, v \in V.$$

**2.2. The locality algebra of meromorphic germs with linear poles.** Recall that for the filtered Euclidean space  $(\mathbb{R}^\infty, Q)$  from the standard embeddings  $i_n : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ , the inner product  $Q$  induces an isomorphism

$$Q_n^* : \mathbb{R}^n \rightarrow (\mathbb{R}^n)^*.$$

So

$$j_{n+1} := (Q_n^*)^{-1} \circ i_n^* \circ Q_{n+1}^* : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n,$$

induce a direct system

$$j_{n+1}^* : \mathcal{M}(\mathbb{R}^n \otimes \mathbb{C}) \rightarrow \mathcal{M}(\mathbb{R}^{n+1} \otimes \mathbb{C}),$$

and we set

$$(11) \quad \mathcal{M} := \mathcal{M}(\mathbb{C}^\infty) := \varinjlim_n \mathcal{M}(\mathbb{C}^n) = \varinjlim_n \mathcal{M}(\mathbb{R}^n \otimes \mathbb{C})$$

to be the algebra of multivariate meromorphic germs with linear poles and real coefficients [GPZ1, GPZ2].

The locality structure on  $(\mathbb{R}^\infty, \perp^Q)$  induces a locality structure on  $\mathcal{M}$ . For  $f \in \mathcal{M}(\mathbb{C}^n)$ , let  $\text{Dep}(f)$  denote the **dependence space** of  $f$ , defined as the smallest subspace of  $(\mathbb{C}^n)^*$  spanned by the linear forms on which  $f$  depends in the sense of [CGPZ1, Definitions 2.9 and 2.13].

**Proposition 2.3.** [CGPZ1, Proposition 3.9] *Equipped with the locality relation*

$$f_1 \perp^Q f_2 \iff \text{Dep}(f_1) \perp^Q \text{Dep}(f_2),$$

and the ordinary product of functions restricted on the graph of the locality relation, the locality set  $(\mathcal{M}, \perp^Q)$  carries a locality algebra structure.

The inner product  $Q$  induces a decomposition of  $\mathcal{M}$  [GPZ2]

$$(12) \quad \mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-^Q,$$

where  $\mathcal{M}_+$  is the subspace of holomorphic germs and  $\mathcal{M}_-^Q$  is the subspace generated by polar germs.

**Proposition 2.4.** [CGPZ1, Proposition 3.19] *The subspace  $\mathcal{M}_+$  is a subalgebra and sub-locality algebra of  $\mathcal{M}$ . The subspace  $\mathcal{M}_-^Q$  is not a subalgebra but a locality ideal of  $\mathcal{M}$ .*

There is another locality structure on  $\mathcal{M}$  which is also compatible with the ordinary product of functions.

Let  $\{e_n \mid n \in \mathbb{N}\}$  denote a  $Q$ -orthonormal basis of  $\mathbb{R}^\infty$ . We call the **support of  $f \in \mathcal{M}$** , denoted  $\text{Supp}(f)$ , the smallest subset  $J \subset \mathbb{N}$  such that  $\text{Dep}(f)$  is contained in the subspace spanned by  $\{e_j^* \mid j \in J\}$ . We thus equip  $\mathcal{M}$  with the locality relation

$$f_1 \top_D^Q f_2 \iff \text{Supp}(f_1) \cap \text{Supp}(f_2) = \emptyset,$$

which makes  $\mathcal{M}$  a locality vector space.

**Remark 2.5.** Since the  $K$ -linear span of  $\text{Supp}(f)$  contains  $\text{Dep}(f)$ , for  $f_1, f_2 \in \mathcal{M}$  we have  $f_1 \top_D^Q f_2 \implies f_1 \perp^Q f_2$ . Yet  $(e_1^* + e_2^*) \perp^Q (e_1^* - e_2^*)$  whereas these two linear forms are not  $\top_D$  independent since  $\text{Supp}(e_1^* + e_2^*) = \{1, 2\} = \text{Supp}(e_1^* - e_2^*)$ .

**Proposition 2.6.** *The locality set  $(\mathcal{M}, \top_D^Q)$  equipped with the product of functions is a locality algebra.*

*Proof.* This follows from Remark 2.5 and Proposition 2.3. □

**2.3. Locality algebra of lattice cones.** In the filtered Euclidean lattice space  $(\mathbb{R}^\infty, \mathbb{Z}^\infty, Q)$  such that  $Q(u, v) \in \mathbb{Q}$  for  $u, v \in \mathbb{Z}^\infty$ , a lattice cone is a pair  $(C, \Lambda_C)$  where  $C$  is a polyhedral cone in some  $\mathbb{R}^k$  generated by elements in  $\mathbb{Z}^\infty$  and  $\Lambda_C$  is a lattice generated by elements in  $\mathbb{Q}^\infty$  in the linear subspace spanned by  $C$ . Let  $\mathcal{C}_k$  be the set of lattice cones in  $\mathbb{R}^k$  and

$$\mathcal{C} = \bigcup_{k \geq 1} \mathcal{C}_k$$

be the set of lattice cones in  $(\mathbb{R}^\infty, \mathbb{Z}^\infty)$  which is the direct limit under the standard embeddings. Let  $\mathbb{Q}\mathcal{C}_k$  and  $\mathbb{Q}\mathcal{C}$  be the linear spans of  $\mathcal{C}_k$  and  $\mathcal{C}$  over  $\mathbb{Q}$ .

For two convex lattice cones  $(C_i, \Lambda_i)$  we set

$$(13) \quad (C_1, \Lambda_1) \perp^Q (C_2, \Lambda_2) \iff \text{span}(C_1) \perp^Q \text{span}(C_2).$$

This defines a locality relation on  $\mathbb{Q}\mathcal{C}$ .

For convex cones  $C := \langle u_1, \dots, u_m \rangle$  and  $D := \langle v_1, \dots, v_n \rangle$  spanned by  $u_1, \dots, u_m$  and  $v_1, \dots, v_n$  respectively, their Minkowski sum is the convex cone

$$C \cdot D := \langle u_1, \dots, u_m, v_1, \dots, v_n \rangle.$$

This operation can be extended to a product in  $\mathbb{QC}$ :

$$(14) \quad (C, \Lambda_C) \cdot (D, \Lambda_D) := (C \cdot D, \Lambda_C + \Lambda_D),$$

where  $\Lambda_C + \Lambda_D$  is the abelian group generated by  $\Lambda_C$  and  $\Lambda_D$  in  $\mathbb{Q}^\infty$ . This product endows a monoid structure on  $\mathcal{C}$  with unit  $(\{0\}, \{0\})$ , which also restricts to a locality monoid structure on  $(\mathcal{C}, \perp^{\mathcal{Q}})$ .

**Proposition 2.7.** [CGPZ1, Lemma 3.18] *The locality set  $(\mathbb{QC}, \perp^{\mathcal{Q}})$  equipped with the Minkowski sum is a (graded) locality algebra.*

As with the case for meromorphic germs, there is another subset  $\top_D$  of  $\perp^{\mathcal{Q}}$  which also makes  $\mathbb{QC}$  into a locality algebra. Let  $\{e_n \mid n \in \mathbb{N}\}$  be an orthonormal basis of  $\mathbb{R}^\infty$ . For a lattice cone  $(C, \Lambda_C)$ , we denote by  $\text{Supp}(C, \Lambda_C)$  the smallest subset  $J$  such that  $\text{span}(C)$  is contained in the subspace spanned by  $\{e_j^* \mid j \in J\}$  and equip  $\mathbb{QC}$  with the locality relation

$$(C_1, \Lambda_{C_1}) \top_D^{\mathcal{Q}} (C_2, \Lambda_{C_2}) \iff \text{Supp}(C_1, \Lambda_{C_1}) \cap \text{Supp}(C_2, \Lambda_{C_2}) = \emptyset,$$

which makes  $\mathbb{QC}$  a locality vector space.

**Proposition 2.8.** *The locality set  $(\mathbb{QC}, \top_D)$  equipped with the Minkowski sum is a locality algebra.*

**2.4. Locality algebra of decorated rooted forests.** Let  $(\Omega, \top_\Omega)$  be a locality set. A **properly  $(\Omega, \top_\Omega)$ -decorated rooted forest** is a pair  $(F, d)$ , where  $F$  is a (non-planar) rooted forest and  $d : V(F) \rightarrow \Omega$  is a map from the set  $V(F)$  of vertices of  $F$  to  $\Omega$  such that

$$v \neq v' \Rightarrow d(v) \top_\Omega d(v').$$

Let  $\mathcal{F}_{\Omega, \top_\Omega}$  denote the set of properly  $(\Omega, \top_\Omega)$ -decorated rooted forests and by  $K\mathcal{F}_{\Omega, \top_\Omega}$  its linear span. The set  $\mathcal{F}_{\Omega, \top_\Omega}$  carries a natural locality relation  $\top_{\mathcal{F}_{\Omega, \top_\Omega}}$  from  $(\Omega, \top_\Omega)$ , and this locality relation induces a locality relation  $\top_{\mathcal{F}_{\Omega, \top_\Omega}}$  on  $K\mathcal{F}_{\Omega, \top_\Omega}$ .

**Remark 2.9.** The symmetry of the concatenation of forests motivates the symmetry of the binary relation on the decoration set  $\Omega$ . Planar forests call for the more general non symmetric  $\mathcal{R}$  structure. Given a  $\mathcal{R}$ -set  $(\Omega, \top_\Omega)$ , one could also define the algebra of  $(\Omega, \top_\Omega)$ -**decorated planar forests** in a similar manner, yet taking care of preserving the order of the concatenation of vertices.

**Proposition 2.10.** [CGPZ2, Proposition 1.22] *The free module  $K\mathcal{F}_{\Omega, \top_\Omega}$  of properly  $(\Omega, \top_\Omega)$ -decorated rooted forests is a locality algebra for the concatenation product.*

**2.5. Locality algebra of meromorphic germs of symbols.** In analysis and geometry, the algebra of polyhomogeneous symbols plays an important role. We are in particular interested in the algebra  $\mathcal{S}(\mathbb{R}_{\geq 0})$  of polyhomogeneous symbols in  $\mathbb{R}_{\geq 0}$ .

For a polyhomogeneous symbol  $\sigma \in \mathcal{S}(\mathbb{R}_{\geq 0})$  with asymptotic expansion

$$\sigma \sim \sum_{j=0}^{\infty} a_j x^{\alpha-j},$$

the Hadamard finite part at infinity is defined by

$$(15) \quad \text{fp}_{+\infty} \sigma := \sum_{j=0}^{\infty} a_j \delta_{\alpha-j, 0},$$

(with  $\delta_{i,0}$  the Kronecker symbol). Unfortunately,  $\text{fp}_{+\infty}$  is not an algebra homomorphism on  $\mathfrak{S}(\mathbb{R}_{\geq 0})$ .

To make these structures compatible, we introduce the space  $\Omega$  of meromorphic germs of symbols on  $\mathbb{R}_{\geq 0}$  [CGPZ2], where germs are around 0 in the filtered Euclidean space  $(\mathbb{R}^\infty, Q)$ , and define the locality relation  $\perp^Q$  similar to that of meromorphic functions induced by the inner product  $Q$ . Then we have

**Proposition 2.11.** [CGPZ2, Proposition 4.15] *The triple  $(\Omega, \perp^Q, m_\Omega)$  is a commutative and unital locality algebra, with unit given by the constant function 1 and  $m_\Omega$  is the restriction of the pointwise function multiplication to the graph  $\perp^Q \subset \Omega \times \Omega$ .*

### 3. LOCALITY MORPHISMS

As indicated in the last section, we choose to work in the locality setup and not the  $\mathcal{R}$ -set framework to which many of the concepts below could be generalised.

**3.1. Basic notions and examples.** Recall that

**Definition 3.1.** A **locality map** from a locality set  $(X, \top_X)$  to a locality set  $(Y, \top_Y)$  is a map  $\phi : X \rightarrow Y$  such that  $(\phi \times \phi)(\top_X) \subseteq \top_Y$ . More generally, maps  $\phi, \psi : (X, \top_X) \rightarrow (Y, \top_Y)$  are called **independent** and denoted  $\phi \top \psi$  if  $(\phi \times \psi)(\top_X) \subseteq \top_Y$ .

So a locality map is a map independent of itself.

**Definition 3.2.** Let  $(U, \top_U)$  and  $(V, \top_V)$  be locality vector spaces. A linear map  $\phi : (U, \top_U) \rightarrow (V, \top_V)$  is called a **locality linear map** if it is a locality map.

**Definition 3.3.** A locality linear map  $f : (A, \top_A, \cdot_A) \rightarrow (B, \top_B, \cdot_B)$  between two (not necessarily unital) locality algebras is called a **locality algebra homomorphism** if

$$(16) \quad f(u \cdot_A v) = f(u) \cdot_B f(v) \quad \text{for all } (u, v) \in \top_A.$$

By the definition, the composition of locality morphisms is again a locality morphism, so we have the category **LA** of locality algebras over  $K$ .

Here are fundamental examples of locality morphisms on  $\mathcal{M}$ . The first one plays a central role in our multivariate minimal subtraction renormalisation scheme.

Since  $\mathcal{M}_-^Q$  is a locality ideal of  $\mathcal{M}$ , we have

**Proposition 3.4.** [CGPZ1, Proposition 3.19] **(The  $Q$ -orthogonal projection onto holomorphic germs).** *The projection  $\pi_+^Q : (\mathcal{M}, \perp^Q) \rightarrow (\mathcal{M}_+, \perp^Q)$  is a locality algebra homomorphism.*

Since  $\top_D^Q \subset \perp^Q$  and  $\text{Supp}(\pi_+^Q f) \subset \text{Supp}(f)$ , the projection  $\pi_+^Q$  is also a locality algebra homomorphism on  $(\mathcal{M}, \top_D^Q)$ .

**Remark 3.5.** We view the fact of going from the locality relation  $\perp^Q$  to the locality relation  $\top_D^Q$  with a smaller graph  $\top_D^Q \subset \perp^Q$ , as a reduction of the locality relation, which rigidifies the setup in a manner similar to that fact that the structure group of a principal bundle to a subgroup rigidifies the underlying geometric setup.

### 3.2. Locality morphisms on the algebra of meromorphic germs of symbols.

On the locality algebra  $(\Omega, \perp^{\mathcal{Q}})$  of meromorphic germs of symbols, we can define several important locality maps:

- the Hadamard finite part at infinity map  $\text{fp}_{+\infty} : \Omega \rightarrow \mathcal{M}$  ;
- locality maps:  $\mathfrak{S}_\lambda : \Omega \rightarrow \Omega$  with  $\lambda = 0, \pm 1$ .

These maps are constructed as follows. Though the Hadamard finite part at infinity map is not an algebra homomorphism on  $\mathfrak{S}(\mathbb{R}_{\geq 0})$ ; yet its extension to  $\Omega$  enjoys the following property.

**Proposition 3.6.** [CGPZ2, Proposition 4.17] *The Hadamard finite part at infinity map  $\text{fp}_{+\infty}$  extends to a locality algebra homomorphism*

$$\text{fp}_{+\infty} : (\Omega, \perp^{\mathcal{Q}}) \rightarrow (\mathcal{M}, \perp^{\mathcal{Q}}).$$

For  $\lambda = 1$  (resp.  $\lambda = -1$ ) we define

$$\mathfrak{S}_1(\sigma)(n) := \sum_{k=1}^n \sigma(k) \quad (\text{resp. } \mathfrak{S}_{-1}(\sigma)(n) := \sum_{k=1}^{n-1} \sigma(k)),$$

both maps can be interpolated by means of the Euler-MacLaurin formula [H, Eqn. (13.1.1)] to take values in  $\Omega$ ; and for  $\lambda = 0$  we define

$$\mathfrak{S}_0(\sigma)(x) := \mathfrak{J}(\sigma)(x) := \int_1^x \sigma(y) dy.$$

**3.3. Locality morphisms on lattice cones.** On a strongly convex lattice cone  $(C, \Lambda_C)$  with interior  $C^\circ$ , discrete (resp. continuous) Laplace transforms of characteristic functions lead to exponential sums (resp. integrals) and give rise to meromorphic functions

$$\sum_{\vec{n} \in C^\circ \cap \Lambda_C} e^{\langle \vec{e}, \vec{n} \rangle} \quad \left( \text{resp. } \int_C e^{\langle \vec{e}, \vec{x} \rangle} d\vec{x}_{\Lambda_C} \right).$$

These can be extended by linearity and subdivisions to any convex lattice cone, to build maps  $S^\circ$  and  $I$  from  $\mathbb{QC}$  to  $\mathcal{M}$ .

The idempotency  $(C, \Lambda_C) \cdot (C, \Lambda_C) = (C, \Lambda_C)$  for any lattice cone  $(C, \Lambda_C)$  implies that  $S^\circ$  and  $I$  are not algebra homomorphisms for the Minkowski sum  $\cdot$ , since otherwise they can only assume values  $t$  with  $t^2 = t$ , meaning  $t = 0$  or  $1$ . But in the locality setting, we have

**Proposition 3.7.** ([GPZ1, Proposition 3.7])  *$S^\circ$  and  $I$  are locality algebra homomorphisms from  $(\mathbb{QC}, \perp^{\mathcal{Q}})$  to  $(\mathcal{M}, \perp^{\mathcal{Q}})$ .*

Similarly,  $S^\circ$  and  $I$  are locality morphisms from  $(\mathbb{QC}, \top_D^{\mathcal{Q}})$  to  $(\mathcal{M}(\mathbb{C}^\infty), \top_D)$ , a useful property of these maps which shows the importance of locality algebra.

**3.4. Linear operators lifted to the algebra of rooted forests.** Let us briefly recall some definitions and results borrowed from [CGPZ3].

**Definition 3.8.** Let  $(\Omega, \top)$  be a locality set. A **locality  $(\Omega, \top)$ -operated set** or simply a **locality operated set** is a locality set  $(X, \top_X)$  together with a **partial action**  $\beta$  of  $\Omega$  on  $X$ : there is a subset  $\top_{\Omega, X} := \Omega \times_{\top} X \subseteq \Omega \times X$  and a map

$$\beta : \Omega \times_{\top} X \longrightarrow X, (\omega, x) \mapsto \beta^{\omega}(x)$$

satisfying the following compatibility conditions

(i) For

$$\Omega \times_{\top} X \times_{\top} X := \{(\omega, u, u') \in \Omega \times X \times X \mid (u, u') \in \top_X, (\omega, u), (\omega, u') \in \Omega \times_{\top} X\},$$

we have

$$\beta \times \text{Id}_X : \Omega \times_{\top} X \times_{\top} X \longrightarrow X \times_{\top} X.$$

In other words, If  $(\omega, u, u')$  is in  $\Omega \times_{\top} X \times_{\top} X$ , then  $(\beta^{\omega}(u), u')$  is in  $\top_X$ .

(ii) For

$$\Omega \times_{\top} \Omega \times_{\top} X := \{(\omega, \omega', u) \in \Omega \times \Omega \times X \mid (\omega, \omega') \in T_{\Omega}, (\omega, u), (\omega', u) \in \Omega \times_{\top} X\},$$

we have

$$\text{Id}_{\Omega} \times \beta : \Omega \times_{\top} \Omega \times_{\top} X \longrightarrow \Omega \times_{\top} X,$$

that is, if  $(\omega, \omega') \in T_{\Omega}, (\omega, u), (\omega', u) \in \Omega \times_{\top} X$ , then  $(\omega', \beta^{\omega}(u)) \in \Omega \times_{\top} X$ .

**Definition 3.9.** Let  $(\Omega, \top)$  be a locality set.

- (i) A **locality  $(\Omega, \top)$ -operated semigroup** is a quadruple  $(U, \top_U, \beta, m_U)$ , where  $(U, \top_U, m_U)$  is a locality semigroup and  $(U, \top_U, \beta)$  is a  $(\Omega, \top)$ -operated locality set such that if  $(\omega, u, u')$  is in  $\Omega \times_{\top} U \times_{\top} U$ , then  $(\omega, uu')$  is in  $\Omega \times_{\top} U$ ;
- (ii) A **locality  $(\Omega, \top)$ -operated monoid** is a quintuple  $(U, \top_U, \beta, m_U, 1_U)$ , where  $(U, \top_U, m_U, 1_U)$  is an locality monoid and  $(U, \top_U, \beta, m_U)$  is a  $(\Omega, \top)$ -operated locality semigroup such that  $\Omega \times 1_U \subset \Omega \times_{\top} U$ .
- (iii) A  **$(\Omega, \top)$ -operated locality nonunitary algebra** (resp.  **$(\Omega, \top)$ -operated locality unitary algebra**) is a quadruple  $(U, \top_U, \beta, m_U)$  (resp. quintuple  $(U, \top_U, \beta, m_U, 1_U)$ ) which is at the same time a locality algebra (resp. unitary algebra) and a locality  $(\Omega, \top)$ -operated semigroup (resp. monoid), satisfying the additional condition that for any  $\omega \in \Omega$ , the set  $\{\omega\}^{\top_{\Omega, U}} := \{u \in U \mid \omega \top_{\Omega, U} u\}$  is a subspace of  $U$  on which the action of  $\omega$  is linear. More precisely, the last condition means  
 let  $u_1, u_2 \in U$ . If  $u_1, u_2 \in \{\omega\}^{\top_{\Omega, U}}$  then for all  $k_1, k_2 \in K$ , we have  
 $k_1 u_1 + k_2 u_2 \in \{\omega\}^{\top_{\Omega, U}}$  and  $\beta^{\omega}(k_1 u_1 + k_2 u_2) = k_1 \beta^{\omega}(u_1) + k_2 \beta^{\omega}(u_2)$ .  
 (resp. this condition and  $\Omega \times 1_U \subset \Omega \times_{\top} U$ )

**Definition 3.10.** Given  $(\Omega, \top_{\Omega})$ -operated locality structures (sets, semigroups, monoids, nonunitary algebras, algebras)  $(U_i, \top_{U_i}, \beta_i), i = 1, 2$ , a **morphism of locality operated locality structures** is a locality map  $f : U_1 \rightarrow U_2$  such that  $f \circ \beta_1^{\omega} = \beta_2^{\omega} \circ f$  for all  $\omega \in \Omega$ .

The key property of  $K \mathcal{F}_{\Omega, \top_{\Omega}}$  is the following universal property.

**Proposition 3.11.**  $K \mathcal{F}_{\Omega, \top_{\Omega}}$  is an  $(\Omega, \top_{\Omega})$ -operated algebra, and it is the initial object in the category of commutative  $(\Omega, \top_{\Omega})$ -operated algebra.

Let  $(\Omega, \top_{\Omega})$  be a locality algebra. By the universal property of the initial object, a linear map  $\phi : \Omega \rightarrow \Omega$  such that  $\phi \top \text{Id}_{\Omega}$  induces a  $(\Omega, \top_{\Omega})$  locality operation on itself, and  $\phi$  lifts uniquely to a locality morphism of  $(\Omega, \top_{\Omega})$ -operated locality algebra for this action [CGPZ2, Corollary 1.24]

$$(17) \quad \widehat{\phi} : K \mathcal{F}_{\Omega, \top_{\Omega}} \longrightarrow \Omega.$$



This can be applied to the space  $(\Omega, \perp^Q)$  of multivariate meromorphic germs of symbols on  $\mathbb{R}_{\geq 0}$ . The interpolated summation map  $\mathfrak{S}_\lambda$  on  $\Omega$  (with  $\lambda = \pm 1$ ) give rise to branched maps

$$(18) \quad \widehat{\mathfrak{S}}_\lambda : (\mathbb{R}\mathcal{F}_{\Omega, \perp^Q}, \top_{\mathcal{F}_{\Omega, \perp^Q}}) \rightarrow (\Omega, \perp^Q)$$

**Proposition 3.12.** *The branched map  $\widehat{\mathfrak{S}}_\lambda$  is a locality algebra homomorphism.*

**3.5. Operations lifted to the algebra of rooted forests.** An operation  $\beta : \Omega \times U \rightarrow U$  of a locality set  $(\Omega, \top_\Omega)$  on a locality monoid  $(U, \top_U)$  induces a locality algebra morphism [CGPZ3, Proposition 2.6]

$$\Phi_\beta : (\mathbb{R}\mathcal{F}_{\Omega, \top_\Omega}, \top_{\mathcal{F}_{\Omega, \top_\Omega}}) \rightarrow (U, \top_U).$$

On the filtered Euclidean space  $(\mathbb{R}^\infty, Q)$  we have a direct system

$$j_{n+1}^* : (\mathbb{R}^n)^* \rightarrow (\mathbb{R}^{n+1})^*.$$

Let

$$\mathcal{L} = \varinjlim_n (\mathbb{R}^n)^*$$

be the direct limit of spaces of linear forms. To an element  $L \in \mathcal{L}$ , regarded as a linear function on  $\mathbb{R}^\infty \otimes \mathbb{C}$ , one assigns a homogeneous pseudodifferential symbol  $x \mapsto x^L$  on  $(0, +\infty)$  of order  $L$  i.e. for any  $z \in \mathbb{R}^\infty \otimes \mathbb{C}$ , it defines a smooth function on  $(0, +\infty)$  which is homogeneous in  $x$  of degree  $L(z)$ . Let

$$\Omega := \mathcal{L}; \quad U := \mathcal{M}[\mathcal{L}]$$

where  $\mathcal{M}[\mathcal{L}]$  is the group ring over  $\mathcal{M}$  generated by the additive monoid  $\mathcal{L}$ , equipped with the locality relation  $\perp^Q$  induced by that on  $\mathcal{M}$  and on  $\mathcal{L}$ :

$$\left( \sum_i f_i x^{L_i} \perp^Q \sum_j g_j x^{\ell_j} \right) \iff (\{f_i, L_i\}_i \perp^Q \{g_j, \ell_j\}_j).$$

The map

$$\mathcal{J} : (L, f) \mapsto \left( x \mapsto \int_0^\infty \frac{f(y) y^{-L}}{y+x} dy \right),$$

defines an operation

$$(19) \quad \begin{aligned} \mathcal{J} : \mathcal{L} \times \mathcal{M}[\mathcal{L}] &\longrightarrow \mathcal{M}[\mathcal{L}] \\ (L, f) &\longmapsto \mathcal{J}(L, f) \end{aligned}$$

and can therefore be lifted to a map [CGPZ3, Eqs. (33)-(35)]

$$\mathcal{R} : (\mathbb{R}\mathcal{F}_{\mathcal{L}, \perp^Q}, \top_{\mathcal{F}_{\mathcal{L}, \perp^Q}}) \rightarrow (\mathcal{M}[\mathcal{L}], \perp^Q).$$

Composing the resulting map  $\mathcal{R}$  with the evaluation of the maps at  $x = 1$ , gives rise to a  $\mathcal{M}$ -valued locality morphism

$$\mathcal{R}_1 = ev_{x=1} \circ \mathcal{R} : (\mathbb{R}\mathcal{F}_{\mathcal{L}, \perp^Q}, \top_{\mathcal{F}_{\mathcal{L}, \perp^Q}}) \rightarrow (\mathcal{M}, \perp^Q).$$

#### 4. RENORMALISATION BY LOCALITY MORPHISMS

In this section we describe a general renormalisation scheme via multivariate regularisations, and implement it to renormalise formal sums on lattice cones, branched formal sums and branched formal integrals.

**4.1. A renormalisation scheme.** If a regularised theory is realised by a locality algebra morphism

$$(20) \quad \phi^{\text{reg}} : (\mathcal{A}, \top) \longrightarrow (\mathcal{M}, \perp^Q),$$

then by Proposition 2.4, the projection

$$\pi_+^Q : \mathcal{M} \rightarrow \mathcal{M}_+$$

is a locality homomorphism. Therefore we have a locality algebra homomorphism

$$\text{ev}_0 \circ \pi_+^Q \circ \phi^{\text{reg}} : (\mathcal{A}, \top) \longrightarrow \mathbb{C},$$

where  $\text{ev}_0$  is the evaluation at 0 and the locality relation on  $\mathbb{C}$  is  $\mathbb{C} \times \mathbb{C}$ . This locality algebra homomorphism  $\text{ev}_0 \circ \pi_+^Q \circ \phi^{\text{reg}}$  is taken as a renormalisation of  $\phi^{\text{reg}}$ . This gives a renormalisation scheme in this setting. When  $(\mathcal{A}, \top)$  is equipped with a suitable locality Hopf algebra structure. This renormalisation agrees with the one from the algebraic Birkhoff factorisation [CGPZ1, Theorem 5.9].

**4.2. Renormalised conical zeta values.** For a lattice cone  $(C, \Lambda)$  in  $(\mathbb{R}^\infty, \mathbb{Z}^\infty)$ , were they well-defined, the formal sums

$$\sum_{n \in C^\circ \cap \Lambda_C} 1 \quad \text{and} \quad \sum_{n \in C \cap \Lambda_C} 1,$$

would yield values characteristic of the lattice cone, but they are unfortunately divergent. In order to extract information from these divergent expressions, a univariate regularisation is shown to be less appropriate (see [GPZ3]) than multivariate regularisations, which appear as very natural:

$$S^o(C, \Lambda_C) := \sum_{n \in C^\circ \cap \Lambda_C} e^{\langle n, z \rangle} \quad \text{and} \quad S^c(C, \Lambda_C) := \sum_{n \in C \cap \Lambda_C} e^{\langle n, z \rangle}.$$

By subdivision techniques, we can extend  $S^o$  and  $S^c$  to linear maps from  $\mathbb{QC}$  to  $\mathcal{M}$ , which are locality algebra homomorphisms as discussed in Section 3.3. These are regularised maps for the formal expressions.

Therefore we have renormalised open conical zeta values for a lattice cone  $(C, \Lambda_C)$

$$\zeta^o(C, \Lambda_C) := (\text{ev}_0 \circ \pi_+^Q \circ S^o)(C, \Lambda_C)$$

and renormalised closed conical zeta values for a lattice cone  $(C, \Lambda_C)$

$$\zeta^c(C, \Lambda_C) := (\text{ev}_0 \circ \pi_+^Q \circ S^c)(C, \Lambda_C).$$

In fact, the function  $(\pi_+^Q \circ S^o)(C, \Lambda_C)$  or  $(\pi_+^Q \circ S^c)(C, \Lambda_C)$  contains important geometry information for lattice cones – they are building blocks of Euler-MacLaurin formula for lattice cones. Because of their geometric nature, these formal expression can easily be renormalised by means of locality morphisms.

**4.3. Branched zeta values.** In [CGPZ2], this multivariate renormalisation scheme was applied to renormalise a branched generalisation of multiple zeta values. Renormalised multiple zeta values are related to the renormalisation of the formal sum

$$\sum_{n_1 > \dots > n_k > 0} 1.$$

This formal sum is an iterated sum corresponding to a totally ordered structure, and can therefore be viewed as a sum over ladder trees. It generalises to branched sums on more general partially ordered structures such as rooted trees.

In order to renormalise such branched formal sums, we construct the regularisation maps from the locality morphisms in Section 3.4 and Section 3.2

$$(21) \quad \mathcal{Z}^\lambda = \underset{+\infty}{\text{fp}} \circ \widehat{\mathfrak{S}}_\lambda : (\mathbb{R}\mathcal{F}_{\Omega, \perp^Q}, \top_{\mathcal{F}_{\Omega, \perp^Q}}) \rightarrow (\mathcal{M}, \perp^Q).$$

Once the regularisation is chosen, a specific choice of meromorphic germs of symbols  $x \mapsto \sigma(s)(x) := \chi(x) x^{-s}$  on  $\mathbb{R}_{\geq 0}$  [CGPZ2, Definition 5.1], where  $\chi$  is an excision function around zero, leads to a generalisation of multiple zeta functions, namely **regularised branched zeta functions**

$$\zeta^{\text{reg}, \lambda} : \mathbb{R}\mathcal{F}_{\Omega, \perp^Q} \rightarrow \mathcal{M}.$$

Due to the locality of the morphisms involved in its construction,  $\zeta^{\text{reg}, \lambda}$  is a locality morphism of locality algebras. Composing on the left with the renormalised evaluation at zero  $\text{ev}_0 \circ \pi_+^Q$  leads to **renormalised branched zeta values**

$$\zeta^{\text{ren}, \lambda} : \mathbb{R}\mathcal{F}_{\Omega, \perp^Q} \rightarrow \mathbb{R}.$$

**4.4. Kreimer's toy model.** In [CGPZ3], this multivariate renormalisation scheme was applied to Kreimer's toy model [K] which recursively assigns formal iterated integrals to rooted forests induced by the formal grafting operator:

$$\beta_+(f)(x) = \int_0^\infty \frac{f(y)}{y+x} dy.$$

There are different ways to regularise these divergent integrals. We adapt the regularisation by universal property of rooted forests studied in Section 3.5:

$$\mathcal{R}_1 : \mathbb{R}\mathcal{F}_{\mathcal{L}, \perp^Q} \rightarrow \mathcal{M}.$$

Applying the renormalisation scheme to this locality map, we have the renormalised value to a properly decorated forest  $(F, d)$

$$(\text{ev}_0 \circ \pi_+^Q \circ \mathcal{R}_1)(F, d).$$

We refer the reader to [CGPZ3] for further details.

**Acknowledgements.** The authors acknowledge supports from the Natural Science Foundation of China (Grant Nos. 11521061 and 11771190) and the German Research Foundation (DFG project FOR 2402). They are grateful to the hospitalities of Sichuan University and University of Potsdam where parts of the work were completed.

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