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# The dual Derrida–Retaux conjecture

by

Xinxing Chen<sup>1</sup>, Yueyun Hu<sup>2</sup>, and Zhan Shi<sup>3</sup>

*Dedicated to the memory of Professor Francis Comets*

**Summary.** We consider a recursive system  $(X_n)$  which was introduced by Collet et al. [10]) as a spin glass model, and later by Derrida, Hakim, and Vannimenus [13] and by Derrida and Retaux [14] as a simplified hierarchical renormalization model. The system  $(X_n)$  is expected to possess highly nontrivial universalities at or near criticality. In the nearly supercritical regime, Derrida and Retaux [14] conjectured that the free energy of the system decays exponentially with exponent  $(p - p_c)^{-\frac{1}{2}}$  as  $p \downarrow p_c$ . We study the nearly subcritical regime ( $p \uparrow p_c$ ) and aim at a dual version of the Derrida–Retaux conjecture; our main result states that as  $n \rightarrow \infty$ , both  $\mathbf{E}(X_n)$  and  $\mathbf{P}(X_n \neq 0)$  decay exponentially with exponent  $(p_c - p)^{\frac{1}{2} + o(1)}$ , where  $o(1) \rightarrow 0$  as  $p \uparrow p_c$ .

**Keywords.** Recursive system, subcritical regime, Derrida–Retaux conjecture.

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## 1 Introduction

Fix an integer  $m \geq 2$ . Let  $X_0^* > 0$  be a random variable taking values in  $\{1, 2, \dots\}$  with  $\mathbf{P}(X_0^* \geq 2) > 0$ . For any random variable  $X$ , we write  $\mathcal{L}_X$  for its law. Let  $X_0$  be a random variable with law

$$\mathcal{L}_{X_0} = (1 - p) \delta_0 + p \mathcal{L}_{X_0^*},$$

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where  $p \in [0, 1]$ , and  $\delta_0$  denotes the Dirac measure at 0.

Consider the following recurrence relation: for all  $n \geq 0$ ,

$$(1.1) \quad X_{n+1} = (X_{n,1} + \cdots + X_{n,m} - 1)^+,$$

where  $X_{n,i}$ ,  $i \geq 1$ , are independent copies of  $X_n$ , and  $x^+ := \max\{x, 0\}$  for all  $x \in \mathbb{R}$ .

Note that the law of each random variable  $X_n$  is completely determined by the law of  $X_0$ , and that  $X_n$  is stochastically non-decreasing in  $p$ . By (1.1), we have

$$\mathbf{E}(X_{n+1}) \leq \mathbf{E}(X_{n,1} + \cdots + X_{n,m}) = m \mathbf{E}(X_n);$$

therefore

$$F_\infty(p) := \lim_{n \rightarrow \infty} \downarrow \frac{\mathbf{E}(X_n)}{m^n} \in [0, \infty)$$

exists (and is called the free energy). The following phase transition was established by Collet et al. [10]:

**Theorem A ([10]).** *Assume  $\mathbf{E}(X_0^* m^{X_0^*}) < \infty$ . Let*

$$(1.2) \quad p_c = p_c(X_0^*) := \frac{1}{1 + \mathbf{E}\{[(m-1)X_0^* - 1]m^{X_0^*}\}} \in (0, 1).$$

- (i) *If  $p > p_c$ , then  $F_\infty(p) > 0$ .*
- (ii) *If  $p \leq p_c$ , then  $\mathbf{E}(X_n) \rightarrow 0$ .*

Alternatively we may define  $p_c$  through the free energy  $F_\infty$  by letting  $p_c := \inf\{p : F_\infty(p) > 0\}$ . Then the assumption  $\mathbf{E}(X_0^* m^{X_0^*}) < \infty$  is equivalent to saying that  $p_c > 0$ . We also remark that under the assumption of Theorem A,

$$(1.3) \quad p = p_c \Leftrightarrow \mathbf{E}(m^{X_0}) = (m-1)\mathbf{E}(X_0 m^{X_0}).$$

More generally, it was proved in Collet et al. [10] that for any  $n \geq 0$ ,

$$(1.4) \quad p = p_c \Leftrightarrow \mathbf{E}(m^{X_n}) = (m-1)\mathbf{E}(X_n m^{X_n}).$$

In the language of Collet et al. [10], the identities in (1.3) and (1.4) mean that  $X_0$  lies on the critical manifold.

We say that the system is subcritical if  $p < p_c$ , critical if  $p = p_c$ , and supercritical if  $p > p_c$ .

The recursion (1.1) was introduced by Collet et al. [10] as a simplified version of a spin glass model, and later by Derrida, Hakim, and Vannimenus [13] and Derrida and Retaux [14] as a simplified hierarchical renormalization model, in order to understand the depinning transition of a line in presence of strong disorder, see Giacomin, Lacoïn, and Toninelli [16], also Berger, Giacomin and Lacoïn [3] for an infinite order transition of a copolymer model. It was studied from the point of view of iterations of random functions (Li and Rogers [21], Jordan [20]), and appeared as a special case in the family of max-type recursive models analyzed in the seminal paper of Aldous and Bandyopadhyay [1]. The recursion (1.1) is also connected to a parking scheme recently investigated by Goldschmidt and Przykucki [17], Curien and Hénard [12], Contat and Curien [11], and Aldous et al. [2]. See [19] for an extension to the case when  $m$  is random, and [18, 5] for an exactly solvable version in continuous time.

We call a Derrida–Retaux system the process  $(X_n)_{n \geq 0}$  defined in (1.1). In fact, our motivation of studying (1.1) comes from Derrida and Retaux [14] who conjectured that  $(X_n)_{n \geq 0}$  possesses highly nontrivial universalities at or near criticality; these universality properties are believed to hold for a general class of pinning and hierarchical renormalization models, though none of them has been completely proved so far.

In the nearly supercritical regime, the Derrida–Retaux conjecture (see [14]) says that if  $p_c > 0$  (and under some additional integrability conditions on  $X_0^*$ ), then

$$(1.5) \quad F_\infty(p) = \exp \left( - \frac{C_1 + o(1)}{(p - p_c)^{1/2}} \right), \quad p \downarrow p_c,$$

for some constant  $C_1 \in (0, \infty)$  depending on the law of  $X_0^*$ .

A partial answer to the conjecture (1.5) was given in [4]: Assuming  $\mathbf{E}[X_0^{*3} m^{X_0^*}] < \infty$ , we have

$$F_\infty(p) = \exp \left( - \frac{1}{(p - p_c)^{\frac{1}{2} + o(1)}} \right), \quad p \downarrow p_c.$$

In words,  $\frac{1}{2}$  was shown to be the correct exponent, in the exponential scale, for the free energy in the nearly supercritical case.

For the critical regime  $p = p_c$ , we refer to [6] for a list of open questions concerning the behaviors of  $X_n$ , and to [7] for some recent progress. To insist on the criticality, we denote by  $(Y_n)$  the Derrida–Retaux system defined by the recurrence relation (1.1), exactly in the

same way as for  $(X_n)$ , but with

$$(1.6) \quad \mathcal{L}_{Y_0} = (1 - p_c) \delta_0 + p_c \mathcal{L}_{X_0^*}.$$

It was shown in [7] that assuming the condition

$$(1.7) \quad \mathbf{E}(s^{X_0^*}) < \infty, \quad \text{for some } s > m,$$

we have

$$\mathbf{E}(Y_n) = n^{-2+o(1)} \quad \text{and} \quad \mathbf{P}(Y_n \geq 1) = n^{-2+o(1)}, \quad n \rightarrow \infty.$$

In this paper, we study the subcritical regime. Assume from now on that  $p \in (0, p_c)$ . There should be a kind of dual phenomenon for  $p \uparrow p_c$  in the sense that a certain transition should also be expected as in the Derrida–Retaux conjecture for the supercritical regime, and with the same exponent  $\frac{1}{2}$ . More precisely, under some additional integrability conditions on  $X_0^*$ , we expect the existence of a constant  $\kappa(p) \in (0, \infty)$  for all  $p < p_c$  such that

$$(1.8) \quad \mathbf{E}(X_n) = e^{-(\kappa(p)+o(1))n}, \quad n \rightarrow \infty,$$

where the exponent  $\kappa(p)$  would satisfy

$$(1.9) \quad \kappa(p) = (p_c - p)^{\frac{1}{2}+o(1)}, \quad p \uparrow p_c.$$

We call (1.8) and (1.9) the dual Derrida–Retaux conjecture. Roughly saying, the critical manifold, characterised by the identity on the right-hand-side of (1.3), says that if the initial distribution lies on the critical manifold, then the system always lies on the critical manifold. If the initial distribution does not exactly lie on the critical manifold, but only in a neighbourhood, with distance  $\varepsilon$  (by Collet et al. [10], this is equivalent to saying that  $|p - p_c|$  is of order  $\varepsilon$ ), then for a long time, of order  $\varepsilon^{-\frac{1}{2}}$ , the system lies in the  $\varepsilon$ -neighbourhood of the critical manifold before drifting away definitely. This phenomenon does not depend on the sign of  $p - p_c$ ; in other words, it is common for both supercritical and subcritical regimes. The quantity of time  $\varepsilon^{-\frac{1}{2}}$  leads to the Derrida–Retaux conjecture in the supercritical case, and to the dual conjecture in the subcritical case. This is why we would also expect to see the exponent  $\frac{1}{2}$  in the dual conjecture for the subcritical regime, as in the Derrida–Retaux conjecture for the nearly supercritical regime.

Unfortunately, we have not been able to show the existence of  $\kappa(p)$  in (1.8). The main result of this paper is a weaker form of (1.8) and (1.9), which confirms that the exponent  $\frac{1}{2}$  does appear in the subcritical case:

**Theorem 1.1.** Assume (1.7). Let  $p = p_c - \varepsilon$  with  $\varepsilon \in (0, p_c)$ .

(i) There exists some positive constant  $C_2$ , independent of  $\varepsilon$ , such that

$$(1.10) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{E}(X_n) \leq -C_2 \varepsilon^{\frac{1}{2}}.$$

(ii) We have

$$(1.11) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{E}(X_n) \geq -\varepsilon^{\frac{1}{2} + o(1)},$$

where  $o(1)$  denotes some quantity which tends to 0 as  $\varepsilon \rightarrow 0$ .

We mention that the upper bound (1.10) holds for  $\mathbf{E}(m^{X_n}) - 1$  in place of  $\mathbf{E}(X_n)$ , see Proposition 2.1, whereas the lower bound (1.11) holds for  $\mathbf{P}(X_n \geq 1)$  in place of  $\mathbf{E}(X_n)$ , see Proposition 3.1.

**Remark 1.2.** In [4], the optimal condition for the validity of the usual Derrida–Retaux conjecture was proved to be  $\mathbf{E}((X_0^*)^3 m^{X_0^*}) < \infty$ . For the dual conjecture, we claim that (1.7) is optimal.

To see why (1.7) is also necessary for the validity of Theorem 1.1, we first remark that by definition, for all  $n \geq 1$ ,

$$X_n \geq \sum_{i=1}^{m^n} \mathbf{1}_{\{X_{0,i} \geq n+1\}},$$

where  $X_{0,i}$ ,  $i \geq 1$ , denote as before independent copies of  $X_0$ . Then

$$\mathbf{E}(X_n) \geq m^n \mathbf{P}(X_0 \geq n+1) = p m^n \mathbf{P}(X_0^* \geq n+1).$$

Assume that (1.10) holds for  $p = p_c - \varepsilon$  with  $\varepsilon \in (0, p_c)$ . We have, for some constant  $C_3 > 0$  and all sufficiently large  $n$ ,

$$\mathbf{E}(X_n) \leq e^{-C_3 n}.$$

It follows that for all sufficiently large  $n$ ,

$$\mathbf{P}(X_0^* \geq n+1) \leq \frac{m^{-n}}{p_c - \varepsilon} \mathbf{E}(X_n) \leq \frac{m^{-n}}{p_c - \varepsilon} e^{-C_3 n}.$$

Consequently, if (1.10) holds for  $\varepsilon \in (0, p_c)$ , then  $\mathbf{E}(s^{X_0^*}) < \infty$  as long as  $s \in (0, m e^{C_3})$ . This shows the optimality of (1.7).  $\square$

The proof of Theorem 1.1 relies on the aforementioned intuitive ideas, by using a coupling between the subcritical system  $(X_n)$  and the critical Derrida–Retaux system  $(Y_n)$  (see (1.6)). While the upper bound (1.10) in Theorem 1.1 follows from some precise estimates on the generating functions of  $X_n$ , the lower bound (1.11) relies on a coupling argument for the “survival probability”  $\mathbf{P}(X_n \geq 1)$  for subcritical systems and the Laplace transform of the number of open paths for critical systems. More precisely, let  $N_n$  (defined in (3.3)) denote the number of open paths. Then for  $p \in (0, p_c)$  and  $n \geq 1$ ,

$$\mathbf{P}(X_n \geq 1) \geq \mathbf{E}\left[\left(\frac{p}{p_c}\right)^{N_n} \mathbf{1}_{\{Y_n \geq 1\}}\right];$$

see Theorem 3.2. Under the conditional probability  $\mathbf{P}(\bullet | Y_n \geq 1)$ ,  $N_n$  is typically of order  $n^2$ , but a small deviation result (see (3.7)) says that for as  $n \rightarrow \infty$ ,

$$\mathbf{P}(1 \leq N_n \leq jn) \geq \exp\left(-\frac{n}{j^{1+o(1)}}\right),$$

with  $o(1)$  denoting an expression that does not depend on  $n$  and that converges to 0 when  $j \rightarrow \infty$ . Taking  $j$  to be the integer part of  $\varepsilon^{-1/2}$ , we obtain the lower bound (1.11).

The rest of the paper is organised as follows. The upper and lower bounds in Theorem 1.1 are proved in Sections 2 and 3, respectively. Some further remarks and questions are presented in Section 4.

Throughout the paper,  $C_i$ ,  $1 \leq i \leq 22$ , denote some positive constants whose values do not depend on  $p$ .

## 2 Upper bound

The proof of the upper bound in Theorem 1.1 is purely analytical, based on study of the moment generating function of the system. By the monotonicity in  $p$  of  $\mathbf{E}(X_n)$ , it suffices to prove the upper bound (1.10) in Theorem 1.1 for sufficiently small  $\varepsilon$ . The aim of this section is to prove the following result.

**Proposition 2.1.** *Under (1.7), there exist a positive constant  $C_2$  and a sufficiently small  $\varepsilon_0 \in (0, \frac{p_c}{2})$  such that for all  $p = p_c - \varepsilon$  with  $\varepsilon \in (0, \varepsilon_0)$ , there exists some  $s_0 > m$  such that*

$$(2.1) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log (\mathbf{E}(s_0^{X_n}) - 1) \leq -C_2 \varepsilon^{1/2}.$$

We note that  $s_0$  in (2.1) can be chosen such that  $s_0 - m$  is of order  $\varepsilon^{1/2}$  as  $\varepsilon \rightarrow 0$ .

Since  $\mathbf{E}(X_n) \leq \frac{1}{m-1}(\mathbf{E}(m^{X_n}) - 1) \leq \frac{1}{m-1}(\mathbf{E}(s_0^{X_n}) - 1)$ , the upper bound (1.10) in Theorem 1.1 will follow immediately from (2.1).

The rest of the section is devoted to the proof of Proposition 2.1. For the sake of clarity, the proof is divided into two parts. The first part collects some known estimates of the moment generating function for subcritical systems, followed by the second part which contains the proof of Proposition 2.1.

## 2.1 Preliminaries on the moment generating function

For any  $n \geq 0$ , we write

$$(2.2) \quad H_n(s) := \mathbf{E}(s^{X_n}).$$

In order to insist on the criticality, we write

$$G_n(s) := \mathbf{E}(s^{Y_n}),$$

where  $(Y_n)$  denotes as before the critical Derrida–Retaux system. We first list some known facts about *critical* systems.

**Fact 2.2. (The moment generating function at criticality).** *Assume  $\mathbf{E}(X_0^{*3} m^{X_0^*}) < \infty$ . We have*

$$(2.3) \quad \sup_{n \geq 0} G_n(m) \leq m^{\frac{1}{m-1}},$$

$$(2.4) \quad C_4 n^2 \leq \prod_{i=0}^{n-1} G_i(m)^{m-1} \leq C_5 n^2, \quad \forall n \geq 1,$$

for some positive constants  $C_4$  and  $C_5$ .

Fact 2.2 borrows from [10] in case  $m = 2$ . For general  $m \geq 2$ , see [6, Lemma 3] for (2.3) and [6, Propositions 1 and 2] for (2.4).

We consider from now on the *subcritical* regime  $p \in (0, p_c)$ . Plainly  $X_n$  is stochastically smaller than  $Y_n$  for all  $n \geq 0$ . In particular,  $H_n(s) \leq G_n(s)$  for any  $s \in [0, m]$ .



Define

$$(2.5) \quad \delta_n := H_n(m) - m(m-1)H'_n(m), \quad n \geq 0.$$

Note that  $\delta_n = \mathbf{E}[(1 - (m-1)X_n)m^{X_n}] \leq \mathbf{P}(X_n = 0) \leq 1$ . We shall see that  $\delta_n$  is in fact nonnegative.

Let  $\varepsilon := p_c - p > 0$ . From  $\mathbf{E}(m^{Y_0}) = (m-1)\mathbf{E}(Y_0 m^{Y_0})$  (see (1.4)), we deduce that

$$(2.6) \quad \delta_0 = C_6 \varepsilon,$$

where  $C_6 = \mathbf{E}[(m-1)X_0^* - 1]m^{X_0^*} + 1$ .

**Lemma 2.3. (The subcritical moment generating function).** *Suppose  $\mathbf{E}(X_0^{*3}m^{X_0^*}) < \infty$  and  $p \in (0, p_c)$ . Then for all  $n \geq 1$ ,*

$$(2.7) \quad \delta_n = \delta_0 \prod_{i=0}^{n-1} H_i(m)^{m-1} \in (0, 1],$$

$$(2.8) \quad \prod_{i=0}^{\infty} H_i(m)^{m-1} \leq \frac{1}{\delta_0},$$

$$(2.9) \quad \delta_n \leq C_5 n^2 \delta_0,$$

$$(2.10) \quad C_7 \mathbf{E}(X_n^3 m^{X_n}) \leq \prod_{i=0}^{n-1} H_i(m)^{m-1},$$

where  $C_5$  is as in (2.4), and  $C_7$  is some positive constant, independent of  $p$ .

*Proof.* The idea of the proof, based on the iteration of a suitable combination of the generating function  $H_n$  and its derivatives (up to the third derivative), goes back to [10, 14] and has already been explored in [6, 8]; in particular, (2.10) is essentially Lemma 1 of [8]. We give here the details for the sake of completeness. Let  $s \in [0, m]$  and  $n \geq 0$ . By (1.1),

$$H_{n+1}(s) = \frac{1}{s} H_n(s)^m + \left(1 - \frac{1}{s}\right) H_n(0)^m.$$

Taking the derivative with respect to  $s$  gives that

$$H'_{n+1}(s) = \frac{m}{s} H'_n(s) H_n(s)^{m-1} - \frac{1}{s^2} H_n(s)^m + \frac{1}{s^2} H_n(0)^m,$$

from which it follows that

$$(s-1)s H'_{n+1}(s) - H_{n+1}(s) = [m(s-1) H'_n(s) - H_n(s)] H_n(s)^{m-1},$$

Taking  $s = m$ , the identity reads:

$$\delta_{n+1} = \delta_n H_n(m)^{m-1},$$

which yields the equality in (2.7). Recall that  $\delta_0 > 0$  for  $p < p_c$ . Thus  $\delta_n > 0$ . Since we have already observed that  $\delta_n \leq 1$ , this yields (2.7). Letting  $n \rightarrow \infty$  gives (2.8), whereas (2.9) is a consequence of (2.4) because  $H_i(m) \leq G_i(m)$  for any  $i \geq 0$ .

To show (2.10), further differentiations lead to

$$\begin{aligned} (2.11) \quad H''_{n+1}(s) + \frac{2}{s} H'_{n+1}(s) &= \frac{m}{s} H''_n(s) H_n(s)^{m-1} + \frac{m(m-1)}{s} H'_n(s)^2 H_n(s)^{m-2}, \\ s H'''_{n+1}(s) + 3 H''_{n+1}(s) &= m H'''_n(s) H_n(s)^{m-1} + 3m(m-1) H'_n(s) H''_n(s) H_n(s)^{m-2} \\ (2.12) \quad &+ m(m-1)(m-2) H'_n(s)^3 H_n(s)^{m-3}. \end{aligned}$$

Define for all  $n \geq 0$ ,

$$\mathcal{D}_n(m) := (m-1)(m H'''_n(m) + 3 H''_n(m)) + (m-2)(H''_n(m) + \frac{2}{m} H'_n(m)).$$

Taking  $s = m$  in (2.11) and (2.12), we get that

$$\begin{aligned} \mathcal{D}_{n+1}(m) &= H_n(m)^{m-1} \mathcal{D}_n(m) - 3(m-1) \delta_n H''_n(m) H_n(m)^{m-2} \\ &\quad - \frac{m-2}{m} H'_n(m) H_n(m)^{m-3} [2H_n(m)^2 - m^2(m-1)^2 H'_n(m)^2 - m(m-1) H'_n(m) H_n(m)]. \end{aligned}$$

Note that  $H_n(m) > m(m-1) H'_n(m)$ , so the  $[\dots]$  term,  $2H_n(m)^2 - m^2(m-1)^2 H'_n(m)^2 - m(m-1) H'_n(m) H_n(m)$ , is greater than  $H_n(m)^2 - m(m-1) H'_n(m) H_n(m) = H_n(m) \delta_n$ , which is positive. Therefore, for all  $n \geq 0$ ,

$$\mathcal{D}_{n+1}(m) \leq H_n(m)^{m-1} \mathcal{D}_n(m).$$

Iterating the inequality gives that

$$\mathcal{D}_n(m) \leq \mathcal{D}_0(m) \prod_{i=0}^{n-1} H_i(m)^{m-1}, \quad \forall n \geq 1.$$

By definition,  $\mathcal{D}_n(m) \geq m(m-1)H_n'''(m)$ ; thus

$$m(m-1)H_n'''(m) \leq \mathcal{D}_0(m) \prod_{i=0}^{n-1} H_i(m)^{m-1}.$$

Note that for any  $k \geq 0$ ,  $k^3 m^k \leq \frac{9}{2}k(k-1)(k-2)m^k + 8m^2 \mathbf{1}_{\{k \leq 2\}}$ , it follows that

$$\mathbf{E}(X_n^3 m^{X_n}) \leq \frac{9}{2}m^3 H_n'''(m) + 8m^2 \leq \left(\frac{9m^2}{2(m-1)}\mathcal{D}_0(m) + 8m^2\right) \prod_{i=0}^{n-1} H_i(m)^{m-1}.$$

Since  $X_0$  is stochastically smaller than  $Y_0$ , we have  $\mathcal{D}_0(m) \leq (m-1)(mG_0'''(m) + 3G_0''(m)) + (m-2)(G_0''(m) + \frac{2}{m}G_0'(m)) =: \mathcal{D}_0^Y(m)$ . This yields (2.10) with  $C_7 := (\frac{9m^2}{2(m-1)}\mathcal{D}_0^Y(m) + 8m^2)^{-1}$ .  $\square$

**Fact 2.4.** ([7, Lemma 4.5]) *Let  $p \in (0, 1)$ . If there exist  $t > m$ ,  $\theta \in (0, 1)$  and an integer  $M \geq 0$  such that  $\frac{m}{t} [\mathbf{E}(t^{X_M})]^{m-1} \leq \theta$ , then*

$$(2.13) \quad \mathbf{E}(t^{X_n}) \leq 1 + (t-m)\theta^{n-M}, \quad \forall n \geq M.$$

**Fact 2.5.** ([8, Corollary 1]) *Under (1.7), there exist positive constants  $C_8$  and  $C_9$  such that for all sufficiently large integer  $n$ ,*<sup>4</sup>

$$(2.14) \quad \mathbf{E}(Y_n^2 s^{Y_n}) \leq C_8 n, \quad s := m + \frac{C_9}{n}.$$

*A fortiori, for all  $p \in (0, p_c]$ ,*

$$(2.15) \quad \mathbf{E}(X_n^2 s^{X_n}) \leq C_8 n, \quad \forall s \in [m, m + \frac{C_9}{n}].$$

## 2.2 Proof of Proposition 2.1

Write, as before,  $H_n(s) := \mathbf{E}(s^{X_n})$ ,  $n \geq 0$ , for the moment generating function of  $X_n$ .

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<sup>4</sup>Strictly speaking, we first consider a truncating version of  $(X_n)$ : For any integer  $L \geq 1$ , let  $Z_0 = X_0 \mathbf{1}_{\{X_0 \leq L\}}$  and  $(Z_n)$  be the associated Derrida–Retaux system. As explained in Section 3 of [8], we may apply [8, Corollary 1] to  $(Z_n)$  and get (2.14) for  $Z_M$  in place of  $X_M$ , and with positive constants  $C_8$  and  $C_9$  that are independent of  $L$ . Then we let  $L \rightarrow \infty$  and obtain (2.14).

**Lemma 2.6.** Suppose  $\mathbf{E}(X_0^{*3} m^{X_0^*}) < \infty$ . There exist constants  $C_{10} > 0$  and  $C_{11} > 0$ , such that for  $p = p_c - \varepsilon$  with  $\varepsilon \in (0, \frac{p_c}{2})$  and  $1 \leq n \leq C_{10} \varepsilon^{-\frac{1}{2}}$ ,

$$\prod_{i=0}^{n-1} H_i(m)^{m-1} \geq C_{11} n^2.$$

*Proof of Lemma 2.6.* Write  $b_n = \prod_{i=0}^{n-1} H_i(m)^{m-1}$ . Note that  $1 \leq H_i(m) \leq G_i(m) \leq m^{\frac{1}{m-1}}$  (see (2.3)). Let  $C_{12} > 0$  (whose value depends only on  $m$ ) be such that  $(1+x)^{\frac{m-1}{2}} - 1 \geq C_{12} x$  for  $x \in [0, m^{\frac{1}{m-1}} - 1]$ . Then

$$b_{n+1}^{\frac{1}{2}} - b_n^{\frac{1}{2}} = b_n^{\frac{1}{2}} (H_n(m)^{\frac{m-1}{2}} - 1) \geq C_{12} b_n^{\frac{1}{2}} (H_n(m) - 1).$$

For the term on the right-hand side, we note that  $b_n \geq C_7 \mathbf{E}(X_n^3 m^{X_n})$  by (2.10), and that  $H_n(m) - 1 \geq (1 - \frac{1}{m}) \mathbf{E}(m^{X_n} \mathbf{1}_{\{X_n \geq 1\}})$ . Since  $\mathbf{E}(X_n^3 m^{X_n}) [\mathbf{E}(m^{X_n} \mathbf{1}_{\{X_n \geq 1\}})]^2 \geq [\mathbf{E}(X_n m^{X_n})]^3$  (Hölder's inequality), it follows that

$$\begin{aligned} b_n^{\frac{1}{2}} (H_n(m) - 1) &\geq C_7^{\frac{1}{2}} (1 - \frac{1}{m}) [\mathbf{E}(X_n^3 m^{X_n})]^{\frac{1}{2}} \mathbf{E}(m^{X_n} \mathbf{1}_{\{X_n \geq 1\}}) \\ &\geq C_7^{\frac{1}{2}} (1 - \frac{1}{m}) [\mathbf{E}(X_n m^{X_n})]^{\frac{3}{2}}. \end{aligned}$$

Consequently, for all  $n \geq 1$  and with  $C_{13} := C_{12} C_7^{\frac{1}{2}} (1 - \frac{1}{m})$ ,

$$b_{n+1}^{\frac{1}{2}} - b_n^{\frac{1}{2}} \geq C_{13} [\mathbf{E}(X_n m^{X_n})]^{\frac{3}{2}}.$$

By definition,  $\delta_n = H_n(m) - m(m-1)H'_n(m)$ , thus  $\mathbf{E}(X_n m^{X_n}) = \frac{H_n(m) - \delta_n}{m-1} \geq \frac{1 - \delta_n}{m-1}$ . Since  $\delta_n \leq C_5 n^2 \delta_0 = C_5 C_6 n^2 \varepsilon$  (see (2.9) for the inequality, and (2.6) for the equality), we have  $\delta_n \leq \frac{1}{2}$  for  $n \leq (2C_5 C_6 \varepsilon)^{-\frac{1}{2}}$ . Therefore,

$$\mathbf{E}(X_n m^{X_n}) \geq \frac{1 - \frac{1}{2}}{m-1} = \frac{1}{2(m-1)}, \quad \forall 1 \leq n \leq (2C_5 C_6 \varepsilon)^{-\frac{1}{2}}.$$

As such, for all  $1 \leq n \leq (2C_5 C_6 \varepsilon)^{-1/2}$ , we have  $b_{n+1}^{1/2} - b_n^{1/2} \geq \frac{C_{13}}{[2(m-1)]^{3/2}}$ , which implies that  $b_{n+1}^{1/2} \geq \frac{C_{13}}{[2(m-1)]^{3/2}} n + 1$ . The lemma follows immediately.  $\square$

**Lemma 2.7.** Assume (1.7). For any sufficiently small  $\varepsilon_0 > 0$ , there exist positive constants  $C_{14}$ ,  $C_{15}$ ,  $C_{16}$  and  $C_{17}$  such that for any  $p = p_c - \varepsilon$  with  $\varepsilon \in (0, \varepsilon_0)$ , we can find some integer  $M \in [C_{14} \varepsilon^{-\frac{1}{2}}, C_{15} \varepsilon^{-\frac{1}{2}}]$  and some  $s_* \in (m, m + \frac{C_{16}}{M}]$  such that

$$\frac{m}{s_*} H_M(s_*)^{m-1} < 1 - C_{17} \varepsilon^{\frac{1}{2}}.$$

*Proof.* Let  $\varepsilon_0 \in (0, \frac{\varepsilon_c}{2})$  be small (how small will be determined later) and  $\varepsilon \in (0, \varepsilon_0)$ . By Lemma 2.6, we have<sup>5</sup>

$$(2.16) \quad \prod_{i=0}^{C_{10}\varepsilon^{-\frac{1}{2}}-1} H_i(m)^{m-1} \geq C_{11} (C_{10}\varepsilon^{-\frac{1}{2}} - 1)^2 \geq \frac{C_{18}}{\varepsilon},$$

where  $C_{18} := \frac{1}{4}C_{11}C_{10}^2$ ; here appears the first constraint on  $\varepsilon_0$ :  $\varepsilon_0$  is sufficiently small such that  $C_{10}\varepsilon_0^{-\frac{1}{2}} \geq 2$  and that  $C_{11} (C_{10}\varepsilon^{-\frac{1}{2}} - 1)^2 \geq \frac{C_{18}}{\varepsilon}$  for  $\varepsilon \in (0, \varepsilon_0)$ . By (2.8), we have  $\prod_{i=0}^{\infty} H_i(m)^{m-1} \leq \frac{1}{\delta_0}$  which is  $\frac{1}{C_6\varepsilon}$  (see (2.6)). Thus

$$\prod_{i=C_{10}\varepsilon^{-\frac{1}{2}}}^{\infty} H_i(m)^{m-1} \leq \frac{1}{C_6 C_{18}};$$

in other words,

$$(2.17) \quad \prod_{i=C_{10}\varepsilon^{-\frac{1}{2}}}^{\infty} H_i(m) \leq \left(\frac{1}{C_6 C_{18}}\right)^{\frac{1}{m-1}} =: C_{19}.$$

Let  $\lambda > 0$  be a small constant whose value will be given later. Let  $C_{15} := C_{10} e^{\frac{2C_{19}}{\lambda}}$ . We have

$$\prod_{i=C_{10}\varepsilon^{-\frac{1}{2}}}^{C_{15}\varepsilon^{-\frac{1}{2}}} \left(1 + \frac{\lambda}{i}\right) \geq \sum_{i=C_{10}\varepsilon^{-\frac{1}{2}}}^{C_{15}\varepsilon^{-\frac{1}{2}}} \frac{\lambda}{i} \geq \lambda \log \frac{C_{15}}{C_{10}} = 2 C_{19}.$$

Compared with (2.17), it follows that there exists  $M \in [C_{10}\varepsilon^{-\frac{1}{2}}, C_{15}\varepsilon^{-\frac{1}{2}}] \cap \mathbb{Z}$  such that

$$(2.18) \quad H_M(m) < 1 + \frac{\lambda}{M}.$$

On the other hand, since  $M \geq C_{10}\varepsilon^{-\frac{1}{2}}$ , we have

$$\prod_{i=0}^{M-1} H_i(m)^{m-1} \geq \prod_{i=0}^{C_{10}\varepsilon^{-\frac{1}{2}}-1} H_i(m)^{m-1} \geq \frac{C_{18}}{\varepsilon},$$

the second inequality being from (2.16). Since  $H_M(m) - m(m-1)H'_M(m) = \delta_M$ , which equals  $\delta_0 \prod_{i=0}^{M-1} H_i(m)^{m-1}$  (see (2.7)), and  $\delta_0 = C_6\varepsilon$  (see (2.6)), it follows that

$$H_M(m) - m(m-1)H'_M(m) \geq C_6\varepsilon \frac{C_{18}}{\varepsilon} = C_6 C_{18} = 2C_{20},$$

---

<sup>5</sup>For notational brevity, we write  $C_{10}\varepsilon^{-\frac{1}{2}}$  instead of  $\lfloor C_{10}\varepsilon^{-\frac{1}{2}} \rfloor$ . Similar simplifications apply elsewhere.

with  $C_{20} := \frac{C_6 C_{18}}{2}$ .

By assumption (1.7), there exists some  $s_1 > m$ , independent of  $p$ , such that  $\mathbf{E}(s_1^{X_0}) < \infty$ . Since  $H_M - m(m-1)H'_M$  is  $C^\infty$  on  $[0, s_1)$ , we can apply the mean-value theorem to see that for all  $s \in (m, s_1)$ , there exists  $u \in [m, s]$  such that

$$\begin{aligned} & H_M(s) - m(m-1)H'_M(s) \\ &= H_M(m) - m(m-1)H'_M(m) + (s-m)[H'_M(u) - m(m-1)H''_M(u)] \\ (2.19) \quad & \geq 2C_{20} - (s-m)m(m-1)H''_M(u). \end{aligned}$$

Let  $C_{21} := \min\{\frac{C_{20}}{m(m-1)C_8}, C_9, \frac{s_1-m}{2}\}$  (where  $C_8$  and  $C_9$  are the positive constants in Fact 2.5). We apply (2.19) to  $s = s_* := m + \frac{C_{21}}{M} \in (m, s_1)$ ; then  $(s_* - m)m(m-1)H''_M(u) \leq \frac{C_{21}}{M} m(m-1)C_8 M$  (by (2.15)), so that

$$(2.20) \quad H_M(s_*) - m(m-1)H'_M(s_*) \geq 2C_{20} - C_8 C_{21} m(m-1) \geq C_{20}.$$

On the other hand, by the convexity of  $H_M(\cdot)$ ,

$$H'_M(s_*) \geq \frac{H_M(s_*) - H_M(m)}{s_* - m} = \frac{M}{C_{21}}(H_M(s_*) - H_M(m)) \geq \frac{M}{C_{21}}(H_M(s_*) - 1 - \frac{\lambda}{M}),$$

where the last inequality follows from (2.18). Going back to (2.20), we obtain that

$$H_M(s_*) \geq C_{20} + m(m-1) \frac{M}{C_{21}}(H_M(s_*) - 1 - \frac{\lambda}{M}).$$

Let  $C_{22} := \frac{m(m-1)}{C_{21}}$ . We impose a new constraint on  $\varepsilon_0$ :  $C_{10}\varepsilon_0^{-\frac{1}{2}} > \frac{1}{C_{22}}$ . Then  $C_{22}M > 1$  as long as  $\varepsilon \in (0, \varepsilon_0)$ . We get that

$$H_M(s_*) - 1 \leq \frac{1 - C_{20} + \lambda C_{22}}{C_{22}M - 1}.$$

Now, we choose  $\lambda := \frac{C_{20}}{2C_{22}}$ ; thus

$$\frac{m}{s_*} H_M(s_*)^{m-1} = \frac{H_M(s_*)^{m-1}}{1 + \frac{C_{21}}{mM}} \leq \frac{1}{1 + \frac{C_{21}}{mM}} \left(1 + \frac{1 - \frac{1}{2}C_{20}}{C_{22}M - 1}\right)^{m-1}.$$

We further require that  $\varepsilon_0$  is small enough so that  $\frac{1}{1 + \frac{C_{21}}{mM}}(1 + \frac{1 - \frac{1}{2}C_{20}}{C_{22}M - 1})^{m-1} \leq 1 - \frac{C_{20}C_{21}}{3mM}$  for all  $M \geq C_{10}\varepsilon^{-\frac{1}{2}}$  with  $\varepsilon \in (0, \varepsilon_0)$ . [This is possible because  $-\frac{C_{21}}{m} + \frac{(1 - \frac{1}{2}C_{20})(m-1)}{C_{22}} < -\frac{C_{20}C_{21}}{3m}$ .] Consequently, for  $\varepsilon \in (0, \varepsilon_0)$ ,

$$\frac{m}{s_*} H_M(s_*)^{m-1} \leq 1 - \frac{C_{20}C_{21}}{3mM} \leq 1 - C_{17} \varepsilon^{\frac{1}{2}},$$

where  $C_{17} := \frac{C_{20}C_{21}}{3mC_{10}}$ . This yields Lemma 2.7.  $\square$

We now have all the ingredients for the proof of Proposition 2.1.

*Proof of Proposition 2.1.* By Lemma 2.7, there exist  $\varepsilon_0 > 0$  and  $C_{17} > 0$  such that for  $p = p_c - \varepsilon$  with  $\varepsilon \in (0, \varepsilon_0)$ , there exist  $s_* > m$  and an integer  $M$  satisfying  $\frac{m}{s_*} [\mathbf{E}(s_*^{X_M})]^{m-1} < 1 - C_{17} \varepsilon^{\frac{1}{2}}$ . So we are entitled to apply Fact 2.4 to  $t := s_*$  and  $\theta := 1 - C_{17} \varepsilon^{\frac{1}{2}}$  to see that for all integer  $n \geq M$ ,

$$\mathbf{E}(s_*^{X_n}) \leq 1 + (s_* - m)(1 - C_{17} \varepsilon^{\frac{1}{2}})^{n-M} \leq 1 + (s_* - m)e^{-(n-M)C_{17} \varepsilon^{\frac{1}{2}}}.$$

Therefore,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log (\mathbf{E}(s_*^{X_n}) - 1) \leq -C_{17} \varepsilon^{\frac{1}{2}},$$

which yields the desired inequality (2.1).  $\square$

### 3 Lower bound

The lower bound in Theorem 1.1 will be a straightforward consequence of the following result.

**Proposition 3.1.** *Assume (1.7). We have, for  $p = p_c - \varepsilon$  and  $\varepsilon \in (0, p_c)$ ,*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}(X_n \geq 1) \geq -\varepsilon^{\frac{1}{2} + o(1)},$$

where  $o(1)$  goes to 0 as  $\varepsilon \rightarrow 0$ .

The rest of the section is devoted to the proof of Proposition 3.1.

While the proof of the upper bound in Theorem 1.1 presented in Section 2 was purely analytic, the proof of its lower bound is probabilistic. It requires a simple hierarchical representation of the system, together with the notion of open paths in the system. The main ingredient in the proof of Proposition 3.1 is a coupling inequality (Theorem 3.2 below), connecting  $\mathbf{P}(X_n \geq 1)$  (with  $p < p_c$ ) to the Laplace transform of the number of open paths when the system is critical. For the sake of clarity, the coupling inequality and the proof of Proposition 3.1 are presented in distinct parts.

### 3.1 A coupling for subcritical and critical systems

Recall (1.6):  $\mathcal{L}_{Y_0} = (1 - p_c) \delta_0 + p_c \mathcal{L}_{X_0^*}$ . We are going to construct a coupling for recursive systems  $(X_n)$  and  $(Y_n)$  in the same probability space such that  $X_n \leq Y_n$  a.s. for all  $n \geq 0$ , where  $\mathcal{L}_{X_0} = (1 - p) \delta_0 + p \mathcal{L}_{X_0^*}$  with  $p := p_c - \varepsilon$ ,  $\varepsilon \in (0, p_c)$ . We use a natural hierarchical representation of Derrida–Retaux systems, as in [10], [14], or [7].

Let  $\mathbb{T}$  be a (reversed) infinite  $m$ -ary tree. For any vertex  $v$  of  $\mathbb{T}$ , let  $|v|$  denote the generation of  $v$  (so  $|v| = 0$  if the vertex  $v$  is in the initial generation). We define a family of random variables  $(X(v), Y(v), v \in \mathbb{T})$  as follows. Let  $Y(v)$ , for  $v \in \mathbb{T}$  with  $|v| = 0$ , be i.i.d. having the law of  $Y_0$ . Let  $Z_0$  be a Bernoulli random variable with  $\mathbf{P}(Z_0 = 1) = 1 - \frac{\varepsilon}{p_c}$  and  $\mathbf{P}(Z_0 = 0) = \frac{\varepsilon}{p_c}$ . Let  $Z(v)$ ,  $|v| = 0$ , be independent copies of  $Z_0$ , and independent of  $(Y(v), |v| = 0)$ . Define  $X(v) := Y(v)Z(v)$  for  $|v| = 0$ , so that  $X(v)$  is distributed as  $X_0$  for  $|v| = 0$ .

For any  $v \in \mathbb{T}$  with  $|v| \geq 1$ , we write  $v^{(1)}, \dots, v^{(m)}$  for the  $m$  parents of  $v$  in generation  $|v| - 1$ , and define recursively

$$(3.1) \quad X(v) := (X(v^{(1)}) + \dots + X(v^{(m)}) - 1)^+,$$

$$(3.2) \quad Y(v) := (Y(v^{(1)}) + \dots + Y(v^{(m)}) - 1)^+.$$

As such,  $X(v)$  and  $Y(v)$  are well-defined for all  $v \in \mathbb{T}$ .

For  $n \geq 0$ , let  $\mathbf{e}_n$  denote the first lexicographic vertex in the  $n$ -th generation of  $\mathbb{T}$ . Let  $\mathbb{T}_n$  denote the (reversed) subtree formed by all the ancestors (including  $\mathbf{e}_n$  itself) of  $\mathbf{e}_n$  in the first  $n$  generations. See Figure 1 below for an example. By definition,  $X(\mathbf{e}_n) \leq Y(\mathbf{e}_n)$  a.s. for all  $n \geq 0$ , and  $(X(\mathbf{e}_n), n \geq 0)$  (resp.  $(Y(\mathbf{e}_n), n \geq 0)$ ) has the same law as  $(X_n, n \geq 0)$  (resp.  $(Y_n, n \geq 0)$ ).

For  $v \in \mathbb{T}$  with  $|v| = 0$  and integer  $n \geq 0$ , we write  $v_n$  for the unique descendant of  $v$  in generation  $n$ , and call  $(v = v_0, v_1, v_2, \dots, v_n)$  the path in  $\mathbb{T}$  from  $v$  to  $v_n$  (or: leading to  $v_n$ ).<sup>6</sup>

For any  $u \in \mathbb{T}$ , let  $u_*$  be the unique child of  $u$ . Denote by  $\mathbf{bro}(u)$  the set of the “brothers” of  $u$ , i.e., the parents of  $u_*$  that are not  $u$ . Let

$$\xi(u) := \sum_{y \in \mathbf{bro}(u)} Y(y).$$

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<sup>6</sup>Degenerate case: when  $|v| = 0$ , the path from  $v$  to  $v$  is reduced to the singleton  $v$ .





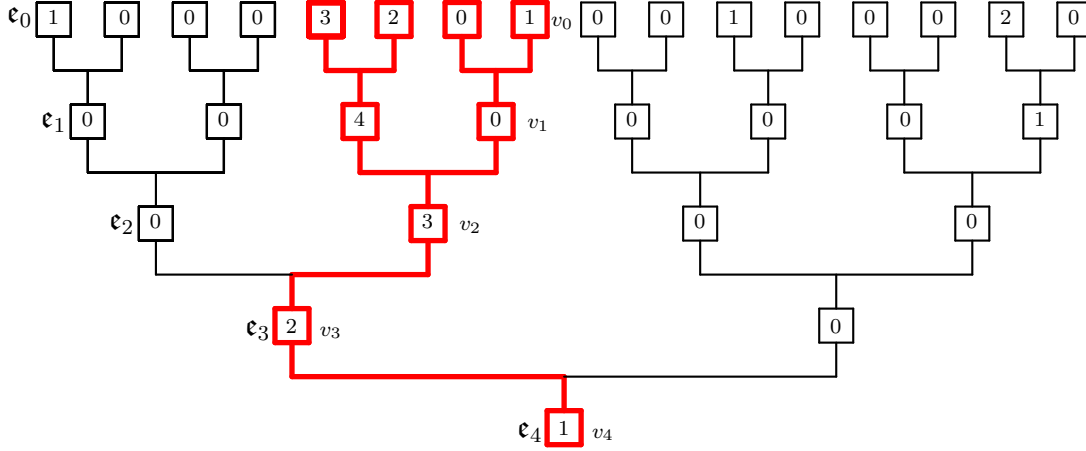


Figure 2: Open paths from the initial generation to  $\epsilon_4$  are marked in bold (and coloured in red), with  $N_4 = 4$  and  $Y_4 = 1$ .

*Proof of Theorem 3.2.* Let  $\mathcal{N}_n$  be the set of vertices  $v$  with  $|v| = 0$  such that  $v_n = \epsilon_n$  and that the path  $(v = v_0, v_1, v_2, \dots, v_n = \epsilon_n)$  is open. So  $N_n$  is exactly the cardinality of  $\mathcal{N}_n$ .

Since  $(Z(v), |v| = 0)$  and  $(Y(v), |v| = 0)$  are independent, and  $\mathbf{P}(Z(v) = 1) = \frac{p}{p_c}$  for any  $|v| = 0$ , it follows that

$$\mathbf{P}\left(\bigcap_{v \in \mathcal{N}_n} \{Z(v) = 1\} \mid \mathcal{N}_n, Y_n\right) = \left(\frac{p}{p_c}\right)^{N_n}.$$

By definition, if  $Z(v) = 1$  for all  $v \in \mathcal{N}_n$ , then  $X_n = Y_n$ . Consequently,

$$\mathbf{P}(X_n \geq 1) \geq \mathbf{P}\left(Y_n \geq 1, \bigcap_{v \in \mathcal{N}_n} \{Z(v) = 1\}\right) = \mathbf{E}\left[\left(\frac{p}{p_c}\right)^{N_n} \mathbf{1}_{\{Y_n \geq 1\}}\right],$$

as desired.  $\square$

The following fact, which relies solely on the critical system  $(Y_n)$ , will be useful in proving Proposition 3.1.

**Fact 3.3** ([7]). *Assume (1.7). We have*

$$(3.5) \quad \mathbf{P}(N_n \geq 1) = n^{-2+o(1)}, \quad n \rightarrow \infty.$$

*Fix any  $\lambda > 0$  and  $\varrho > 0$ . For all sufficiently large  $n$ ,*

$$(3.6) \quad \mathbf{P}(Y_n \geq \ell + 1, N_n \leq n^{2+\varrho}) \geq n^{-(2+\varrho)} m^{-\ell}, \quad \forall \ell \in [0, \lambda n].$$

See [7, Theorem 1.2 and Remark 2.4] for (3.5), whereas (3.6) is a straightforward consequence of [7, Proposition 5.1 and (5.13)].

## 3.2 Proof of Proposition 3.1

The main ingredient in the proof of Proposition 3.1 is the following deviation result for  $N_n$ , the number of open paths of the critical system  $(Y_n)$ .

**Lemma 3.4.** *Assume (1.7). We have, for  $j \rightarrow \infty$ ,*

$$(3.7) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}\left(Y_n \geq \frac{n}{j}, N_n \leq jn\right) \geq -\frac{1}{j^{1+o(1)}},$$

with  $o(1) \rightarrow 0$  as  $j \rightarrow \infty$ .

**Remark 3.5.** The correct rate in (3.7) should be  $-\frac{O(1)}{j}$ . Note that  $\mathbf{P}(Y_n \geq \frac{n}{j}, N_n \leq jn) \leq \mathbf{P}(Y_n \geq \frac{n}{j}) \leq m^{-\frac{n}{j}} \mathbf{E}(m^{Y_n})$ , and  $\mathbf{E}(m^{Y_n}) \rightarrow 1$  as  $n \rightarrow \infty$  (see [6, Theorem 3]), we obtain an upper bound for the probability term in (3.7):  $\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}(Y_n \geq \frac{n}{j}, N_n \leq jn) \leq -\frac{\log m}{j}$ . It would be interesting to study the small deviation probabilities of  $N_n$  conditioned on survival:  $\mathbf{P}(N_n \leq \alpha_n | Y_n \geq 1)$  for  $\alpha_n \rightarrow \infty$  and  $\alpha_n = o(n^2)$ .  $\square$

*Proof of Lemma 3.4.* The proof is based on an explicit construction of an event contained in  $\{Y_n \geq \frac{n}{j}, N_n \leq jn\}$ . Let  $\varrho \in (0, 1)$ . By (3.6), there exists  $j_0 \geq 1$  such that for all  $j \geq j_0$ ,

$$(3.8) \quad \mathbf{P}\left(Y_j \geq j+3, N_j \leq j^{2+\varrho}\right) \geq m^{-(j+2)} j^{-(2+\varrho)}.$$

Fix  $j \geq j_0$ . For all  $n > mj$ , let  $\ell = \ell(n, j) \geq 1$  be the smallest integer such that  $n \leq j + \ell + jm^\ell$ . Then  $n > j + \ell - 1 + jm^{\ell-1}$ , and  $\ell \sim \frac{\log n}{\log m}$  as  $n \rightarrow \infty$ . Let  $r = r(n, \ell) := \lceil \frac{n-j-\ell}{j} \rceil$  be the smallest integer satisfying  $r \geq \frac{n-j-\ell}{j}$ . Then  $m^{\ell-1} \leq r \leq m^\ell$ .

We use the hierarchical construction presented in Section 3.1. Recall (3.2), (3.3), and the fact that for any  $|v| = j$ ,  $(Y(v), N(v))$  is distributed as  $(Y_j, N_j)$ . By definition,

$$Y_{j+\ell} := Y(\mathbf{e}_{j+\ell}) \geq \sum_{|v|=j, v \in \mathbb{T}_{j+\ell}} Y(v) - \sum_{k=0}^{\ell-1} m^k \geq \sum_{v \in \mathcal{F}_r} Y(v) - \sum_{k=0}^{\ell-1} m^k,$$

where  $\{|v| = j, v \in \mathbb{T}_{j+\ell}\}$  is the set of ancestors of  $\mathbf{e}_{j+\ell}$  at generation  $j$  (so its cardinality equals  $m^\ell$ ), and  $\mathcal{F}_r$  denotes the first  $r$  ancestors (in the lexicographic order) in this set.

Let  $\mathcal{F}_r^c := \{|v| = j, v \in \mathbb{T}_{j+\ell}\} \setminus \mathcal{F}_r$ . On the event  $\cap_{v \in \mathcal{F}_r} \{Y(v) \geq j+3\}$ , we have  $Y_{j+\ell} \geq (j+3)r - \frac{m^\ell-1}{m-1} \geq (j+1)r$  as  $\frac{m^\ell-1}{m-1} \leq 2m^{\ell-1} \leq 2r$ .

On the other hand,  $N_{j+\ell} := N(\mathbf{e}_{j+\ell}) \leq \sum_{|v|=j, v \in \mathbb{T}_{j+\ell}} N(v)$ , so if  $N(v) \leq j^{2+\varrho}$  for all  $v \in \mathcal{F}_r$  and  $N(v) = 0$  for all  $v \in \mathcal{F}_r^c$ , then  $N_{j+\ell} \leq r j^{2+\varrho} \leq m^\ell j^{2+\varrho}$ . Consequently,

$$\begin{aligned} & \mathbf{P}(Y_{j+\ell} \geq (j+1)r, N_{j+\ell} \leq m^\ell j^{2+\varrho}) \\ & \geq \mathbf{P}\left(\bigcap_{v \in \mathcal{F}_r} \{Y(v) \geq j+3, N(v) \leq j^{2+\varrho}\}, \bigcap_{v \in \mathcal{F}_r^c} \{N(v) = 0\}\right) \\ & = \mathbf{P}(Y_j \geq j+3, N_j \leq j^{2+\varrho})^r \mathbf{P}(N_j = 0)^{m^\ell-r} \\ & \geq m^{-(j+2)r} j^{-(2+\varrho)r} \mathbf{P}(N_j = 0)^{m^\ell-r}, \end{aligned}$$

where the last inequality follows from (3.8). By (3.5),  $\mathbf{P}(N_j = 0) \geq e^{-j^{-2+\varrho}}$  for all  $j \geq j_0$  (we may eventually enlarge the value of  $j_0$  if needed), so  $\mathbf{P}(N_j = 0)^{m^\ell-r} \geq \mathbf{P}(N_j = 0)^{m^\ell} \geq e^{-m^\ell j^{-2+\varrho}}$ . Thus we have proved that

$$(3.9) \quad \mathbf{P}(Y_{j+\ell} \geq (j+1)r, N_{j+\ell} \leq m^\ell j^{2+\varrho}) \geq m^{-jr} j^{-(2+\varrho)r} e^{-m^\ell j^{-2+\varrho}}.$$

Now we deal with  $(Y_n, N_n)$ . At generation  $j+\ell$ , we have a family  $(Y(u), N(u))_{|u|=j+\ell}$  of  $m^{n-(j+\ell)}$  i.i.d. copies of  $(Y_{j+\ell}, N_{j+\ell})$ . For  $|u| = j+\ell$  with  $u \in \mathbb{T}_n$ , let  $[u, \mathbf{e}_n)$  be the path in  $\mathbb{T}_n$  from  $u$  to  $\mathbf{e}_n$  (including  $u$  but excluding  $\mathbf{e}_n$ ); let

$$A_u := \left\{ Y(u) \geq (j+1)r, N(u) \leq m^\ell j^{2+\varrho} \right\} \cap \bigcap_{w \in [u, \mathbf{e}_n), v \in \mathbf{bro}(w)} \{N(v) = 0\}.$$

On  $A_u$ , we have  $Y_n := Y(\mathbf{e}_n) = Y(u) - (n-j-\ell) \geq (j+1)r - (n-j-\ell) \geq r$  (the argument shows that  $Y(w) > 0$ , a fortiori  $N(w) > 0$ , for all  $w \in [u, \mathbf{e}_n]$ : this property is going to be used in the next paragraph), and  $N_n := N(\mathbf{e}_n) = N(u) \leq m^\ell j^{2+\varrho}$ . Therefore,

$$\mathbf{P}(Y_n \geq r, N_n \leq m^\ell j^{2+\varrho}) \geq \mathbf{P}\left(\bigcup_{u \in \mathbb{T}_n: |u|=j+\ell} A_u\right).$$

The events  $A_u$  have the same probability. We claim that they are also disjoint. Indeed, if  $A_u$  is realized for some  $u$ , then for all other  $u' \in \mathbb{T}_n$  with  $|u'| = j+\ell$ , there exists  $w \in [u, \mathbf{e}_n)$  and  $w' \in [u', \mathbf{e}_n)$  such that  $w' \in \mathbf{bro}(w)$ . Thus  $N(w') = 0$  by definition of  $A_u$ . This implies that  $A_{u'}$  is not realized (we have seen in the previous paragraph that on the event  $A_{u'}$ , we

would have  $N(w') > 0$  for all  $w' \in [u', \mathfrak{e}_n]$ . We have thus proved that the events  $A_u$  are disjoint. Consequently,

$$\mathbf{P}\left(\bigcup_{u \in \mathbb{T}_n: |u|=j+\ell} A_u\right) = m^{n-(j+\ell)} \mathbf{P}(A_{\mathfrak{e}_{j+\ell}}),$$

which implies that

$$\mathbf{P}(Y_n \geq r, N_n \leq m^\ell j^{2+\varrho}) \geq m^{n-(j+\ell)} \mathbf{P}(A_{\mathfrak{e}_{j+\ell}}).$$

We observe that

$$\mathbf{P}(A_{\mathfrak{e}_{j+\ell}}) = \mathbf{P}(Y_{j+\ell} \geq (j+1)r, N_{j+\ell} \leq m^\ell j^{2+\varrho}) \prod_{i=j+\ell}^{n-1} \mathbf{P}(N_i = 0)^{m^{-1}}.$$

Consider the two probability expressions on the right-hand side. The first probability expression is at least  $m^{-jr} j^{-(2+\varrho)r} e^{-m^\ell j^{-2+\varrho}}$  (see (3.9)), whereas  $\mathbf{P}(N_i = 0) \geq e^{-i^{-2+\varrho}}$  (see (3.5)). Therefore,

$$\mathbf{P}(Y_n \geq r, N_n \leq m^\ell j^{2+\varrho}) \geq m^{n-(j+\ell)-jr} j^{-(2+\varrho)r} e^{-m^\ell j^{-2+\varrho}} \prod_{i=j+\ell}^{n-1} e^{-(m-1)i^{-2+\varrho}}.$$

By the definition of  $r$ , we have  $n - (j + \ell) - jr \geq -j$ , and  $j^{-(2+\varrho)r} \geq e^{-(2+\varrho)m^\ell \log j} \geq e^{-(2+\varrho)\frac{nm \log j}{j}}$ . Note that  $m^\ell j^{-2+\varrho} \leq nmj^{-3+\varrho}$  and  $r \geq \frac{n}{2j}$  for all large  $n$ , we get that

$$\mathbf{P}\left(Y_n \geq \frac{n}{2j}, 1 \leq N_n \leq nmj^{1+\varrho}\right) \geq e^{-3m\frac{n \log j}{j}},$$

for all large  $n$ . This proves Lemma 3.4 as  $\varrho$  can be arbitrarily small.  $\square$

We have now all the ingredients for the proof of Proposition 3.1.

*Proof of Proposition 3.1.* By Theorem 3.2, for  $p = p_c - \varepsilon$  with  $\varepsilon \in (0, p_c)$ ,

$$\begin{aligned} \mathbf{P}(X_n \geq 1) &\geq \mathbf{E}\left[(1 - p_c^{-1}\varepsilon)^{N_n} \mathbf{1}_{\{Y_n \geq 1, N_n \leq \varepsilon^{-1/2n}\}}\right] \\ &\geq (1 - p_c^{-1}\varepsilon)^{\varepsilon^{-1/2n}} \mathbf{P}(Y_n \geq 1, N_n \leq \varepsilon^{-1/2n}), \end{aligned}$$

which by (3.7) is larger than  $(1 - p_c^{-1}\varepsilon)^{\varepsilon^{-1/2n}} \exp(-n\varepsilon^{\frac{1}{2}+o(1)})$  for all large  $n$ , with  $o(1) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . This yields Proposition 3.1.  $\square$

## 4 Further remarks and questions

We present some comments and questions to indicate a few important differences between the dual Derrida–Retaux conjecture studied in this paper and the (usual) Derrida–Retaux conjecture.

### 4.1 Existence of the limit

It was explained in the introduction that part of the dual Derrida–Retaux conjecture was the existence of the constant

$$\kappa(p) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{E}(X_n),$$

for  $p \in (0, p_c)$ . We have not been able to prove the existence of the limit. For the (usual) Derrida–Retaux conjecture, the existence of the free energy

$$F_\infty(p) := \lim_{n \rightarrow \infty} \frac{\mathbf{E}(X_n)}{m^n},$$

is straightforward; indeed, we have seen in the introduction that  $n \mapsto \frac{\mathbf{E}(X_n)}{m^n}$  is non-increasing.

**Problem 4.1.** *Prove, under some suitable integrability assumption on the law of  $X_0^*$ , the existence of the limit*

$$\kappa(p) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{E}(X_n),$$

for all  $p \in (0, p_c)$ .

### 4.2 “Mixing time”

In the introduction, we explained the heuristics leading to the dual conjecture: if the initial distribution lies in an  $\varepsilon$ -neighbourhood of the critical manifold (meaning that  $|p - p_c|$  is of order  $\varepsilon$ ), then for a long time, of order  $\varepsilon^{-1/2}$ , the system lies in the  $\varepsilon$ -neighbourhood of the critical manifold before drifting away definitely. This phenomenon does not depend on the sign of  $p - p_c$ , and is common for both supercritical and subcritical regimes.

As such, the time (of order  $\varepsilon^{-1/2}$ ) during which the system lies in the  $\varepsilon$ -neighbourhood of the critical manifold before drifting away definitely plays a crucial role in both supercritical and subcritical regimes. In the supercritical regime, this time can be defined as the

smallest integer  $n$  such that  $\mathbf{E}(X_n)$  exceeds 3 (or any real number greater than  $m^{1/(m-1)}$ ); the heuristics described in the previous paragraph can be made rigorous in a weaker form, which led to a proof of a weaker version of the Derrida–Retaux conjecture in [4]. In the subcritical regime, however, it is not clear how to define rigorously a quantity playing the role of this particular time (a kind of “mixing time” necessary for a Markov chain to reach the stationary phase, except that here, it is the time necessary for the system to drift away), through the study of which one could prove the dual conjecture.

**Problem 4.2.** *Define and study an appropriate “mixing time” for the subcritical regime.*

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## References

- [1] Aldous, D.J. and Bandyopadhyay, A. (2005). A survey of max-type recursive distributional equations. *Ann. Appl. Probab.* **15**, 1047–1110.
- [2] Aldous, D., Contat, A., Curien, N. and Hénard, O. (2023). Parking on the infinite binary tree. *Probab. Theory Relat. Fields*. <https://doi.org/10.1007/s00440-023-01189-6>
- [3] Berger, Q., Giacomini, G. and Lacoïn, H. (2019). Disorder and critical phenomena: the  $\alpha = 0$  copolymer model. *Probab. Theory Related Fields* **174**, 787–819.
- [4] Chen, X., Dagard, V., Derrida, B., Hu, Y., Lifshits, M. and Shi, Z. (2021). The Derrida–Retaux conjecture on recursive models. *Ann. Probab.* **49**, 637–670.
- [5] Chen, X., Dagard, V., Derrida, B. and Shi, Z. (2020) The critical behaviors and the scaling functions of a coalescence equation. *J. Phys. A* **53**, 195202, 25 pp.
- [6] Chen, X., Derrida, B., Hu, Y., Lifshits, M. and Shi, Z. (2019). A max-type recursive model: some properties and open questions. In: *Sojourns in Probability Theory and Statistical Physics-III* (pp. 166–186). Springer, Singapore.
- [7] Chen, X., Hu, Y. and Shi, Z. (2022) The sustainability probability for the critical Derrida–Retaux model. *Probab. Theory Related Fields* **182**, 641–684.
- [8] Chen, X. and Shi, Z. (2021). The stable Derrida–Retaux system at criticality. In: *In and out of equilibrium, 3, Celebrating Vladas Sidoravicius*, 239–264, Progr. Probab. **77**, Birkhäuser/Springer, Cham.

- [9] Collet, P., Eckmann, J.P., Glaser, V. and Martin, A. (1984). A spin glass with random couplings. *J. Statist. Phys.* **36**, 89–106.
- [10] Collet, P., Eckmann, J.P., Glaser, V. and Martin, A. (1984). Study of the iterations of a mapping associated to a spin-glass model. *Commun. Math. Phys.* **94**, 353–370.
- [11] Contat, A. and Curien, N. (2021+). Parking on Cayley trees and frozen Erdős–Rényi graphs. [arXiv:2107.02116](https://arxiv.org/abs/2107.02116)
- [12] Curien, N. and Hénard, O. (2022). The phase transition for parking on Galton–Watson trees. *Discrete Anal.* Paper No. 1, 17 pp.
- [13] Derrida, B., Hakim, V. and Vannimenus, J. (1992). Effect of disorder on two-dimensional wetting. *J. Statist. Phys.* **66**, 1189–1213.
- [14] Derrida, B. and Retaux, M. (2014). The depinning transition in presence of disorder: a toy model. *J. Statist. Phys.* **156**, 268–290.
- [15] Derrida, B. and Shi, Z. (2020). Results and conjectures on a toy model of depinning. *Moscow Math. J.* **20**, 695–709.
- [16] Giacomin, G., Lacoïn, H. and Toninelli, F.L. (2010). Hierarchical pinning models, quadratic maps and quenched disorder. *Probab. Theory Related Fields* **147**, 185–216.
- [17] Goldschmidt, C. and Przykucki, M. (2019). Parking on a random tree. *Combin. Probab. Comput.* **28**, 23–45.
- [18] Hu, Y., Mallein, B. and Pain, M. (2020). An exactly solvable continuous-time Derrida–Retaux model. *Commun. Math. Phys.* **375**, 605–651.
- [19] Hu, Y. and Shi, Z. (2018). The free energy in the Derrida–Retaux recursive model. *J. Statist. Phys.* **172**, 718–741.
- [20] Jordan, J. (2002). Almost sure convergence for iterated functions of independent random variables. *Ann. Appl. Probab.* **12**, 985–1000.
- [21] Li, D. and Rogers, T.D. (1999). Asymptotic behavior for iterated functions of random variables. *Ann. Appl. Probab.* **9**, 1175–1201.
- [22] Tang, L.H. and Chaté, H. (2001). Rare-event induced binding transition of heteropolymers. *Phys. Rev. Letters* **86**(5), 830.