The stochastic Jacobi flow

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by

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Summary. The problem of conditioning on the occupation field was investigated for the Brownian motion in 1998 independently by Aldous [4] and Warren and Yor [34] and recently for the loop soup at intensity 1/2 by Werner [35], Sabot and Tarrès [30], and Lupu, Sabot and Tarrès [22]. We consider this problem in the case of the Brownian loop soup on the real line, and show that it is connected with a flow version of Jacobi processes, called Jacobi flow. We give a pathwise construction of this flow simultaneously for all parameters by means of a common Brownian motion, via the perturbed reflecting Brownian motion. The Jacobi flow is related to Fleming–Viot processes, as established by Bertoin and Le Gall [9] and Dawson and Li [11]. This relation allows us to interpret Perkins' disintegration theorem between Feller continuous state branching-processes and Fleming–Viot processes as a decomposition of Gaussian measures. Our approach gives a unified framework for the problems of disintegrating on the real line. The connection with Bass–Burdzy flows which was drawn in Warren [33] and Lupu, Sabot and Tarrès [23] is shown to be valid in the general case.

Keywords. Jacobi flow, perturbed reflecting Brownian motion, loop soup, local time, Fleming–Viot processes, Perkins' disintegration theorem, Brownian burglar.

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1. Introduction

A Fleming–Viot process is a measure-valued branching process which models the evolution of a population of constant size 1. In [8], Bertoin and Le Gall construct a generalized version of the process through a flow of bridges, which are in correspondance with exchangeable coalescents. We can understand this flow as follows (it is the flow $\widehat{B}$ in the notation of [8]). One represents the population at any time as the interval $[0,1]$, each point of the interval (the label) representing a particle. Only a finite number of particles at time 0 will beget descendants at time $t>0$. Partitioning the population at time t into families will yield a finite partition of the interval $[0,1]$, the i-th interval from the left representing the descendants at time t of the ancestor at time 0 with the i-th lowest label. One can then construct at each time t a piecewise constant bridge from 0 to 1.

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The bridge jumps at the label of a particle at time 0 which has descendants at time t, the size of the jump being the size of its offspring, and stays constant otherwise. One can proceed similarly between times s and t (and not only 0 and t), and obtain a collection of bridges which naturally satisfies a flow property. Incorporating immigration in the model translates into adding an extra jump at 0 to the bridge, see [13]. In the case where the coalescents are so-called Λ-coalescents, Dawson and Li [11] consider the flow $\mathcal{Y}$ such that the bridges of [8], [13] are the maps $v \in [0,1] \to \mathcal{Y}_{s,t}(v)$, with $\mathcal{Y}_{s,t}(0)$ representing the size of the population descending from immigrants which arrived at rate δ between time s and t. They construct it as a solution of a stochastic differential equation (SDE) driven by a white noise and a Poisson random measure.

We consider the flow $\mathcal{Y}$ in the case where the SDE appearing in [11] is only driven by the white noise. The flow lines of $\mathcal{Y}$ are the solutions of the SDE

$$\mathrm{d}Y_t = 2\sqrt{Y_t(1-Y_t)}\mathrm{d}\gamma_t + (\delta(1-Y_t) - \delta' Y_t)\,\mathrm{d}t, \tag{1.1}$$

where γ denotes a one-dimensional Brownian motion, and $\delta, \delta' \in \mathbb{R}$. In the case $\delta = \delta' = 0$, the dual of $\mathcal{Y}$ corresponds to the classical Kingman coalescent. Following [34] we call the solution (Y_t) of (1.1) a Jacobi(δ, δ') process. These processes arise as the ratio of Gamma processes, see Proposition 8 in [34] in the case of nonnegative parameters, and Theorem 4 in [24]. A Jacobi(δ, δ') process is called in population genetics a Wright–Fisher diffusion with mutation rates (δ, δ').

The Jacobi flow $\mathcal{Y}$ is the flow version of the Jacobi processes Jacobi(δ, δ'), see Definitions 4.2 and 4.3.

One of the goals of this paper is to give a pathwise construction, from a one-dimensional two-sided Brownian motion, of Jacobi(δ, δ') flows simultaneously for all $\delta, \delta' \in \mathbb{R}$. This is done by means of the two-sided perturbed reflecting Brownian motion defined as follows. Let $(B_t,\, t \geqslant 0)$ and $(B'_t,\, t \geqslant 0)$ be two independent standard one-dimensional Brownian motions. Denote by $\mathfrak{L}$ and $\mathfrak{L}'$ their associated local time processes at position zero. The two-sided Brownian motion $(B_t,\, t \in \mathbb{R})$ is defined as

$$B_t := \begin{cases} B'_{-t} & t \leqslant 0, \\ B_t & t \geqslant 0. \end{cases}$$

We also let $\mathfrak{L}_t := -\mathfrak{L}'_{-t}$ for $t \leqslant 0$. For $\mu > 0$, we define the two-sided perturbed reflecting Brownian motion (PRBM) or μ-process by

$$X_t := \begin{cases} |B'_{-t}| + \mu\mathfrak{L}'_{-t}, & \text{if } t \leqslant 0, \\ |B_t| - \mu\mathfrak{L}_t, & \text{if } t \geqslant 0, \end{cases} \tag{1.2}$$

which can simply be written as $X_t := |B_t| - \mu\mathfrak{L}_t$, $t \in \mathbb{R}$. [The case $\mu = 1$ is special: on the positive half-line, it is distributed as a Brownian motion while in the negative half-line, it is a time-reversed three-dimensional Bessel process.] For general properties of PRBM, see [19, 36] and the references therein.

The promised pathwise construction of Jacobi(δ, δ') flows is as follows:

Theorem I (Theorem 4.6). *Let $\delta, \delta' \in \mathbb{R}$. Let $\mathcal{Y}$ be defined as in (4.11). Then $\mathcal{Y}$ is a* Jacobi(δ, δ') *flow.*

We will see that the flow $\mathcal{Y}$ constructed in (4.11) is a measurable function of the PRBM (X_t). Furthermore, $\mathcal{Y}$ satisfies an SDE driven by a specific martingale measure, see (4.12). This last result can be interpreted as a "flow version" of the well-known Perkins' disintegration theorem where we disintegrate, with respect to a flow line, an associated flow of squared Bessel processes; see Section 2 for the definition and construction of this associated flow of squared Bessel processes. In particular, Theorem 4.6 shows that Perkins' disintegration theorem between Feller continuous state branching-processes and Fleming–Viot processes amounts to a decomposition of Gaussian measures.

Another goal of the paper is to describe the "contour function" of Jacobi flows. We show that this contour function is a version of the PRBM conditioned on its occupation field. As such, it gives a unified framework for related problems on disintegrating a standard Brownian motion [4, 34, 33, 7, 17] and inverting the Ray–Knight identity of the Gaussian free field on the real line [23]. The connection with Bass–Burdzy flows which was drawn in these papers is shown to be valid in the general case. It provides a construction of the PRBM, or equivalently ([21]) of the one-dimensional loop soup, conditioned on its occupation field. The corresponding result for the loop soup at intensity $\frac{1}{2}$ on a metric graph is given in [1].

The problem of conditioning a Brownian motion on its occupation field has been treated by Aldous [4] and Warren and Yor [34], via different approaches. Aldous used the tree structure of the Brownian excursion to show that the genealogy of the conditioned Brownian motion is a time-changed Kingman coalescent (which is the dual of a Fleming–Viot process). Later, Berestycki and Berestycki [7] found an analogous result with excursion theory. Warren and Yor [34] solved the question by constructing the conditioned Brownian motion in time (rather that in space, as in the case of Aldous [4]). If $|B|$ is a reflecting Brownian motion, they define a process Z, called Brownian burglar, by

$$Z_t := \int_0^{|B|_{A_t}} \frac{\mathrm{d}r}{L^{|B|}(\tau_1^0(|B|), r)}, \qquad A_t := \inf\left\{ s > 0 \,:\, \int_0^s \frac{\mathrm{d}r}{L^{|B|}(\tau_1^0(|B|), |B_r|)^2} > t \right\}$$

where $L^{|B|}$ and $\tau(|B|)$ are naturally the local time and the inverse local time of $|B|$. They showed that Z is independent of the occupation field $(L^{|B|}(\tau_1^0(|B|), r),\, r \geqslant 0)$. In these works, the link with the Fleming–Viot process has been suggested but not made explicit. In [34], the authors state that "*the results [...] can be seen as describing a contour process for the Fleming–Viot process*". Recently, Gufler, Kersting and Wakolbinger [17] gave a rigorous connection between these two models by constructing the Brownian excursion via an enriched version of the lookdown process, hence giving a precise meaning to the statement of [34].

In our case, we will recover the connection with Fleming–Viot processes by constructing Z through our version of Perkins' disintegration theorem. The idea, which originates in a paper of Tóth and Werner [31], is to define the burglar via its local time flow. Actually, this approach will allow us to solve the analogous problem for the whole class of PRBM, giving rise to a family of "burglars".

To construct the burglars, we define

$$L(t,\, r) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\infty}^{t} \mathbf{1}_{\{r \leqslant X_s \leqslant r+\varepsilon\}}\, \mathrm{d}s, \qquad t \in \mathbb{R},\, r \in \mathbb{R},$$

as the local time of the continuous semimartingale X at time t and position r; we work with a bicontinuous version of local times $L(\cdot,\cdot)$, as in ([29], Theorem VI.1.7). Let

$$\tau_t^r := \inf\{s \in \mathbb{R} \,:\, L(s,r) > t\} \tag{1.3}$$

be the inverse local time of X. Let

$$t_0^* := \sup\left\{t \leqslant 0 \,:\, L(\tau_1^0, X_t) = 0\right\}, \qquad t^* := \inf\left\{t \geqslant 0 \,:\, L(\tau_1^0, X_t) = 0\right\},$$

with the convention $\inf \varnothing = +\infty$ and $\sup \varnothing = -\infty$. Define for $t \in \mathbb{R}$,

$$Z_t^{(1)} := \int_0^{X_{A_t^{(1)}}} \frac{\mathrm{d}r}{L(\tau_1^0, r)}, \qquad A_t^{(1)} := \inf\left\{s \in \mathbb{R} \,:\, \int_0^s \frac{\mathrm{d}r}{L(\tau_1^0, X_s)^2} > t\right\}. \tag{1.4}$$

The process $Z^{(1)}$ is the burglar associated with the process X between times t_0^* and t^*. Similarly, let for $t \geqslant 0$,

$$Z_t^{(2)} := \int_0^{X_{A_t^{(2)}}} \frac{\mathrm{d}r}{L(\tau_1^0, r)}, \qquad A_t^{(2)} := \inf\left\{s \geqslant 0 \,:\, \int_{\tau_1^0 - s}^{\tau_1^0} \frac{\mathrm{d}r}{L(\tau_1^0, X_s)^2} > t\right\}. \tag{1.5}$$

The burglar $Z^{(2)}$ is associated with the process X between times t^* and τ_1^0. A precise statement of the following result will be given in Section 5.1.

Theorem II (Theorem 5.3). *The processes $Z^{(1)}$ and $Z^{(2)}$ are independent of $(L(\tau_1^0, r),\, r \in \mathbb{R})$. The local time flows of the processes $Z^{(1)}$ and $Z^{(2)}$ are the left part and the right part of a flow which is a* Jacobi$(\delta, 0)$ *flow in the positive time-axis and a* Jacobi$(\delta, 2)$ *flow in the negative time-axis.*

The above result gives a description of the conditional law of a PRBM given its occupation field up to τ_0^1 (and from t_0^*). It suffices to take the processes $Z^{(1)}$ and $Z^{(2)}$ independently of the occupation field, and invert the transformations (1.4) and (1.5). We can also obtain a burglar by disintegrating the positive part of a PRBM with respect to its occupation field, see Section 5.4. This burglar can be interpreted as the contour function of the Jacobi$(\delta, 0)$ flow. Each of its excursion away from 0 being associated with a continuous tree in the manner of Aldous [4] and Duquesne and Le Gall [12], the burglar can be viewed as the contour function of a Fleming–Viot forest.

We present now an interesting connection with the Bass–Burdzy flow. This flow was introduced by Bass and Burdzy in [6]. The Bass–Burdzy flow with parameters (β_1, β_2), for $\beta_1,\, \beta_2 \in \mathbb{R}$, is the collection of processes $(\mathcal{R}_t(x),\, t \geqslant 0,\, x \in \mathbb{R})$ which are the strong solutions of

$$\mathcal{R}_t(x) = x + \gamma_t + \beta_1 \int_0^t \mathbf{1}_{\{\mathcal{R}_s(x) < 0\}} \mathrm{d}s + \beta_2 \int_0^t \mathbf{1}_{\{\mathcal{R}_s(x) > 0\}} \mathrm{d}s$$

where γ is a standard Brownian motion common for all processes $\mathcal{R}_\cdot(x)$. In words, $\mathcal{R}_t(x)$ is a Brownian motion with drift β_1 when negative and β_2 when positive. Following [18], for $t \geqslant 0$, denote by $\mathcal{R}_t^{-1}(0)$ the real x such that $\mathcal{R}_t(x) = 0$. When $\beta_1 = 0$ and $\beta_2 = 1$, the process $\mathcal{R}_t^{-1}(0)$ is shown in Warren [33] to be a time-change of the Brownian burglar of Warren and Yor [34]. In a recent paper [23], Lupu, Sabot and Tarrès showed that in the case $\beta_1 = -\frac{1}{2}, \beta_2 = \frac{1}{2}$, the process $\mathcal{R}_t^{-1}(0)$ is the scaling limit of a self-interacting process

involved in the inversion of the Ray–Knight identity, see the next paragraph. We show that in general, one can recover all the burglars $Z^{(2)}$ associated with the PRBM.

Theorem III (Theorem 5.12 and Proposition 5.13). *Taking* $\beta_1 = \frac{\delta}{2} - 1$ *and* $\beta_2 = \frac{\delta}{2}$, *the process* $\mathcal{R}_t^{-1}(0)$ *is a time-change of the process* $Z^{(2)}$ *defined in* (1.5).

To prove the theorem, we use a new approach based on a renewal argument. The process $Z^{(2)}$ has a kind of Markov property. If one appropriately scales the process after time t, the scaled process will be independent of the past and with always the same distribution, see Theorem 5.6. We use this property to show that the images of a point on the real line by these transformations form actually a Lévy process up to hitting 0, and then deduce that it must be a flow line of a Bass–Burdzy flow for some parameters. Finally, we use some random variable whose distribution is easily identified to compute the parameters. This renewal property breaks down when looking at $Z^{(1)}$. Roughly speaking, one needs to remember where the infimum was before doing the scaling. Still, we believe that a similar connection should hold with some perturbed Bass–Burdzy flow.

Yet another, non-trivial, application of our construction concerns the inversion of the Ray–Knight identity on the line and its connection with loop soups. Consider the PRBM X. By excursion theory, the excursions of X above its infimum process forms a Poisson point process of Brownian excursions rooted at various points of the real line. By exploring the real line in the upwards direction, one can consider the excursions that one encounters along the way as Brownian loops rooted at their minimum. As shown by Lupu [21], the collection of loops has the law of a Brownian loop soup on the real line with intensity $\frac{1}{\mu}$. As a result, conditioning a PRBM on its occupation field may be seen as conditioning a loop soup on its occupation field.

A similar problem was studied recently by Lupu, Sabot and Tarrès [22] in the case $\mu = 2$, equivalently $\delta = 1$, which is the case of the loop soup at intensity $\frac{1}{2}$, related to the Gaussian free field (the local time is BESQ^1, hence the square of the Gaussian Free Field, which is simply the Brownian motion on the real line). In their setting (reformulated in terms of loop soups), the authors add a Brownian motion up to a fixed local time at zero to a loop soup on $\mathbb{R}\backslash\{0\}$. Conditionally on the occupation field, they manage to reconstruct the Brownian motion path up to a random time. The reconstruction process is actually in terms of the burglar $Z^{(2)}$, stopped when it reaches local time 1 at some position (which happens in a finite time almost surely).

We solve the problem for any intensity of the loop soup. Specifically, take a loop soup in the positive half-line, and add Brownian excursions up to local time 1. We get the positive part of a PRBM. A space-time transformation of this PRBM gives a burglar Z^+ similar to $Z^{(2)}$ in (1.5), which we have interpreted as the contour function of the Jacobi$(\delta, 0)$ flow; see (5.12) for the definition of Z^+. One then gets a reconstruction of the Brownian excursions added to the loop soup in terms of the Jacobi flow, or in terms of a Bass–Burdzy flow driven by a reflecting Brownian motion with drift. More precisely, we have

Theorem IV (Theorems 5.15 and 5.16). *Let Z^+ be as in (5.12) and the occupation field f as in (5.11). Then the process Z^+ is independent of f and is the burglar associated with the positive part of the PRBM.*

Finally, we may wonder whether it is possible to find the law of a loop soup conditioned on its occupation field on a cable graph introduced by Lupu in [20]. In general, the loop soup loses its Markovian properties. But in the case of intensity $\frac{1}{2}$, the relation of the loop soup with the Gaussian free field indicates that a certain Markov property should hold. It is the topic of [1], where the description of the conditioned loop soup uses the framework of this paper. In the discrete case, the analogous problem was answered by Werner [35] (description that we could label in "space", via the link with the random current model) and Sabot, Tarrès [30], Lupu, Sabot and Tarrès [22] (description in "time", via a self-interacting process).

The paper is organized as follows. Flows of BESQ processes are studied in Section 2. Section 3 collects all the transformations that will be needed to state the disintegration theorems and construct the burglars. Section 4 contains the disintegration result stated for the BESQ$^{\delta,\delta'}$ flows and as a corollary the construction of the Jacobi flows from the PRBM. Theorem II on the conditioning of a PRBM is proved in Section 5.1. In Section 5.3, we will state the Markov property for $Z^{(2)}$ and prove the link with the Bass–Burdzy flow given in Theorem III. Section 5.4 connects the contour function of the Jacobi$(\delta, 0)$ flow to the Bass–Burdzy flow driven by a reflecting Brownian motion with drift. Section A studies properties of the BESQ and Jacobi flows with emphasis on perfect flow properties, and Section B contains a Girsanov theorem for Jacobi flows.

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2. The BESQ flows

We give the definition of the BESQ flows, then embed them in the PRBM. For any bounded Borel function $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ with compact support, let

$$W(g) := \int_{-\infty}^{\infty} g\big(L(t, X_t), X_t\big)\mathrm{sgn}(B_t)\mathrm{d}B_t. \tag{2.1}$$

As shown in [2], the stochastic integral is well defined and W is a white noise on $\mathbb{R}_+ \times \mathbb{R}$. Define for any $x \geqslant r$ in $\mathbb{R}$ and any $a \geqslant 0$,

$$S_{r,x}(a) := L(\tau_a^r, x), \qquad S^*_{r,x}(a) := L(\tau_a^{-r}, -x). \tag{2.2}$$

Theorem 2.1 ([2], Theorem 5.1)**.** *Let W be the white noise defined via (2.1). For any $r \in \mathbb{R}$ and any $a \geqslant 0$, $S_{r,r+\cdot}(a)$ and $S^*_{r,r+\cdot}(a)$ are the pathwise unique solutions, which are*

strong[4]*, of the following SDEs:*

$$S_{r,r+h}(a) = a + 2\int_r^{r+h} W([0, S_{r,s}(a)], \mathrm{d}s) + \frac{2}{\mu}h,\ h \geqslant 0, \tag{2.3}$$

$$S^*_{r,r+h}(a) = a - 2\int_r^{r+h} W^*([0, S^*_{r,s}(a)], \mathrm{d}s) + (2 - \frac{2}{\mu})h,\ h \in [0, T_0(S^*_{r,r+\cdot}(a))], \tag{2.4}$$

where W^* *is the image of* W *under the map* $(a, s) \mapsto (a, -s)$ *and* $T_0(S^*_{r,r+\cdot}(a)) := \inf\{h \geqslant 0 : S^*_{r,r+h}(a) = 0\}$.

Remark. If $I_t := \inf_{s\leqslant t} X_s$ denotes the infimum process of X, we can rewrite $T_0(S^*_{r,r+\cdot}(a))$ in (2.4) as $-I_{\tau_a^{-r}} - r$.

Ray–Knight theorems are usually statements on marginal distributions. Recall that the squared Bessel process of dimension $\delta \in \mathbb{R}$ started at $x \geqslant 0$, denoted BESQ^δ_x, is the pathwise unique solution of

$$S_t = x + 2\int_0^t \sqrt{|S_s|}\,\mathrm{d}\gamma_s + \delta t, \qquad t \geqslant 0,$$

where, as before, γ is a standard Brownian motion. The BESQ^δ hits zero at a positive time if and only if $\delta < 2$. It is absorbed at 0 when $\delta = 0$ and is reflecting at 0 when $\delta \in (0, 2)$. When $\delta < 0$, after hitting 0, it behaves as a $\mathrm{BESQ}^{-\delta}_0$ in the negative half-line, see e.g. [15]. See Le Gall and Yor [19] and Yor ([36], Chapter 9) for references on Ray–Knight theorems.

Ray–Knight theorems show that the flows of squared Bessel processes are embedded in the PRBM. The setting of Dawson and Li [11] includes the construction of such flows. We impose some further regularity conditions in order to give it the structure of a flow in the sense of [5, 31].

Definition 2.2. *Let* $\delta > 0$. *We call* BESQ^δ *flow (or non-killed* BESQ^δ *flow) a collection* $\mathcal{S}$ *of continuous processes* $(\mathcal{S}_{r,x}(a),\ x \geqslant r)_{r\in\mathbb{R}, a\geqslant 0}$ *such that:*

1) for each $(r, a) \in \mathbb{R}\times\mathbb{R}_+$, *the process* $(\mathcal{S}_{r,x}(a),\ x \geqslant r)$ *is almost surely the strong solution of the following SDE*

$$\mathcal{S}_{r,x}(a) = a + 2\int_r^x \mathcal{W}([0, \mathcal{S}_{r,s}(a)], \mathrm{d}s) + \delta(x - r) \tag{2.5}$$

where $\mathcal{W}$ *is a white noise on* $\mathbb{R}_+ \times \mathbb{R}$.

2) Almost surely,

(i) for all $r \in \mathbb{R}$ *and* $a \geqslant 0$, $\mathcal{S}_{r,r}(a) = a$,

(ii) for all $r \leqslant x$, $a \mapsto \mathcal{S}_{r,x}(a)$ *is càdlàg,*

(iii) for all $r' \leqslant r$ *and all* $a', a \geqslant 0$, *if* $\mathcal{S}_{r',r}(a') > a$ *(resp.* $\mathcal{S}_{r',r}(a') < a$*), then* $\mathcal{S}_{r',x}(a') \geqslant \mathcal{S}_{r,x}(a)$ *(resp.* $\mathcal{S}_{r',x}(a') \leqslant \mathcal{S}_{r,x}(a)$*) for all* $x \geqslant r$.

[4]By strong solution, we mean that for instance $S_{r,r+h}(a)$ is measurable with respect to $\sigma\{W(\cdot, [r, s]) : r \leqslant s \leqslant r + h\}$. This terminology will be used elsewhere with the same remark.

Definition 2.3. *For $-\infty < \delta < 2$, we call killed* BESQ$^\delta$ *flow the flow solution of (2.5), where the process is absorbed when hitting* 0*, and which satisfies the same regularity conditions.*

For short, we call general BESQ$^\delta$ *flow a killed* BESQ$^\delta$ *when $\delta \leqslant 0$, a (non-killed)* BESQ$^\delta$ *flow when $\delta \geqslant 2$ and either a killed or non-killed* BESQ$^\delta$ *flow when $\delta \in (0,2)$.*

Dawson and Li [11] showed (in a more general setting) that equation (2.5) possesses a pathwise unique solution. We can show that the same is true in the case $\delta < 0$, with arguments similar to the ones of the proof of Theorem 4.6.

We check that the BESQ flows are naturally embedded in the PRBM X, which will give the existence of these flows for free. In agreement with (2.2), we define $S_{r,x}(a) = L(\tau_a^r, x)$ for any $r, x \in \mathbb{R}$. We call the collection of processes $\mathcal{L}_X := (S_{r,x}(a),\ -\infty < r, x < \infty,\ a \geqslant 0)$ the *local time flow* of X. The flow $S := (S_{r,x}(a),\ x \geqslant r)_{r\in\mathbb{R}, a\geqslant 0}$ is called the *forward local time flow of* X, while $S^* := (S_{-r,-x}(a),\ x \geqslant r)_{r\in\mathbb{R}, a\geqslant 0}$ is called the *backward local time flow of* X. Both flows are dual as we will see in equation (2.6).

Proposition 2.4. *Let $\mu > 0$.*

The forward flow $(S_{r,x}(a),\ x \geqslant r)_{r\in\mathbb{R}, a\geqslant 0}$ is a BESQ$^{2/\mu}$ *flow.*

If $\mu \in (0,1]$, the backward flow $(S_{-r,-x}(a),\ x \geqslant r)_{r\in\mathbb{R}, a\geqslant 0}$ is a killed BESQ$^{2-2/\mu}$ *flow.*

Proof. The finite-dimensional distributions coincide by Theorem 2.1 so we only have to check the regularity conditions (i), (ii) and (iii) in Definition 2.2. We have $S_{r,r}(a) = a$ indeed. Statement (ii) is a consequence of the continuity of the local times, and the observation that $a \to \tau_a^r$ is càdlàg by construction. We prove now (iii). We have $S_{r',r}(a') = L(\tau_{a'}^{r'}, r)$. By definition of τ_a^r, $S_{r',r}(a') > a$ is equivalent to $\tau_{a'}^{r'} > \tau_a^r$. Therefore, $S_{r',x}(a') = L(\tau_{a'}^{r'}, x) \geqslant L(\tau_a^r, x) = S_{r,x}(a)$. □

Conversely, we can recover X from its local time flow. It is the content of the following proposition.

Proposition 2.5. *The process X is a measurable function of its local time flow $\mathcal{L}_X = (S_{r,x}(a),\ -\infty < r, x < \infty,\ a \geqslant 0)$.*

Proof. By the occupation times formula, for any $a \geqslant 0$ and b, $\tau_a^b = \int_{\mathbb{R}} (S_{b,r}(a) - S_{0,r}(0))\mathrm{d}r$ hence $(\tau_a^b,\ a \geqslant 0,\ b \in \mathbb{R})$ is measurable with respect to $\mathcal{L}_X$. Therefore $L(t,x), t \in \mathbb{R}, x \in \mathbb{R}$, is measurable with respect to $\mathcal{L}_X$, which again by the occupation times formula yields that X is measurable as well (for any $s < t$, $\int_s^t X_u \mathrm{d}u = \int_{\mathbb{R}} x(L(t,x) - L(s,x))\mathrm{d}x$). □

We show now that any BESQ$^\delta$ flow can be constructed from a countable number of flow lines. It allows us to identify BESQ$^\delta$ flows with flows embedded in a PRBM.

Proposition 2.6. *Let $\mathcal{S}$ be a general* BESQ$^\delta$ *flow. Let $(r_n, a_n)_n$ be a dense countable set in $\mathbb{R} \times \mathbb{R}_+$. Almost surely, for any $x \geqslant r$ and $a \geqslant 0$,*

$$\mathcal{S}_{r,x}(a) = \inf_{\{n\,:\, r_n \leqslant r,\, \mathcal{S}_{r_n,r}(a_n) > a\}} \mathcal{S}_{r_n,x}(a_n).$$

Proof. We first check that almost surely, for any $r \in \mathbb{R}$, and any $0 \leqslant a < a'$, one can find some (r_n, a_n) such that $r_n \leqslant r$ and $\mathcal{S}_{r_n,r}(a_n) \in (a, a')$. From Theorem 2.1, one can reason on the local times of the PRBM (notice that one can make this identification because we only look at a countable number of flow lines). The previous claim property follows since it suffices to take $a < b_1 < b_2 < a'$, and (r_n, a_n) such that $L(\tau_r^{b_1}, r_n) < a_n < L(\tau_r^{b_2}, r_n)$. We would then have $L(\tau_{r_n}^{a_n}, r) \geqslant L(\tau_r^{b_1}, r) > a$ and $L(\tau_{r_n}^{a_n}, r) \leqslant L(\tau_r^{b_2}, r) < a'$.

Let us go back to the proof of the proposition. Fix $x \geqslant r$ and $a \geqslant 0$. We prove that $\mathcal{S}_{r,x}(a) = \inf_{\{n\,:\, r_n \leqslant r,\, \mathcal{S}_{r_n,r}(a_n) > a\}} \mathcal{S}_{r_n,x}(a_n)$. Notice that the set over which the infimum is taken is not empty. For any (a_n, r_n) such that $\mathcal{S}_{r_n,r}(a_n) > a$, we have $\mathcal{S}_{r_n,x}(a_n) \geqslant \mathcal{S}_{r,x}(a)$ by (iii) of Definition 2.2, which proves one inequality. If $a' > a$, we take some (a_n, r_n) such that $\mathcal{S}_{r_n,r}(a_n) \in (a, a')$. Then $\mathcal{S}_{r,x}(a) \leqslant \mathcal{S}_{r_n,x}(a_n) \leqslant \mathcal{S}_{r_n,x}(a')$. We then conclude by (ii) of Definition 2.2. □

Proposition 2.7. *Let $\delta \in \mathbb{R}$. Let $\mathcal{S}$ be a general* BESQ$^\delta$ *flow. Define its dual $\mathcal{S}^*$ by, for $r \leqslant x$,*

$$\mathcal{S}^*_{r,x}(a) := \inf\{b \geqslant 0 \,:\, \mathcal{S}_{-x,-r}(b) > a\}.$$

Then $\mathcal{S}^*$ is a BESQ$^{2-\delta}$ *flow (in the case $\delta \in (0,2)$, it is killed if $\mathcal{S}$ is not killed, and it is not killed if $\mathcal{S}$ is killed). Moreover, $(\mathcal{S}^*)^* = \mathcal{S}$.*

This proposition gives the dual of a Feller CSBP with immigration. We refer to Foucart, Ma and Mallein [14] for dual processes of CSBPs.

Proof. It is a direct consequence of Proposition 2.4 and the following claim: almost surely, for all $r, x \in \mathbb{R}$ and $a \geqslant 0$,

$$S_{r,x}(a) = \inf\{b \geqslant 0 \,:\, S_{x,r}(b) > a\}. \tag{2.6}$$

Let us prove this claim. Let $b \geqslant 0$ be such that $S_{x,r}(b) > a$. By definition, it means that $L(\tau_b^x, r) > a$, and since $L(\tau_a^r, r) = a$, we get $\tau_b^x > \tau_a^r$ so that $b = L(\tau_b^x, x) \geqslant L(\tau_a^r, x)$ which is $S_{r,x}(a)$ by definition. On the other hand, let $b \geqslant 0$ such that $S_{x,r}(b) \leqslant a$, i.e., $L(\tau_b^x, r) \leqslant a$. Take $s \geqslant 0$ such that $L(s, r) > a$, hence $s > \tau_b^x$. It implies that $L(s, x) \geqslant L(\tau_b^x, x) = b$ and $L(\tau_a^r, x) \geqslant b$ by making $s \downarrow \tau_a^r$. □

3. Space-time transformations

In this section, we consider a fixed positive continuous function $f : I \to (0, \infty)$ where I is an interval (not necessarily open nor bounded).

3.1. Transformations of flows.

The results in this section hold in a deterministic setting. They provide the transformations that will be used later in our disintegration results.

We introduce some general notation. Let $c \in I$ and define

$$\eta_{f,c}(x) := \int_c^x \frac{\mathrm{d}r}{f(r)}, \qquad x \in I.$$

Definition 3.1. *We denote by $\Psi(\cdot, f, c)$ the transformation with domain $D_f := \{(a, x) : a \in [0, f(x)], x \in I\}$ defined as*

$$\Psi(\cdot, f, c) : \begin{cases} D_f & \mapsto & [0,1] \times \eta_{f,c}(I) \\ (a, x) & \mapsto & \left(\frac{a}{f(x)}, \eta_{f,c}(x)\right). \end{cases}$$

If $g : I \to \mathbb{R}_+$ is a continuous function such that $g \leqslant f$, we denote by $\Psi(g, f, c)$ the function whose graph is the image of the graph of g by $\Psi(\cdot, f, c)$. That is

$$\Psi(g, f, c) : \begin{cases} \eta_{f,c}(I) & \mapsto & [0,1] \\ x & \mapsto & \frac{g \circ \eta_{f,c}^{-1}(x)}{f \circ \eta_{f,c}^{-1}(x)}. \end{cases}$$

When $c = 0$, we will simply write η_f for $\eta_{f,0}$ and $\Psi(\cdot, f)$ for $\Psi(\cdot, f, 0)$.

In this section, we will not need a precise definition of flows. Let $g : I \to \mathbb{R}_+$ be a continuous function. We will simply call g-flow a collection $\mathcal{L} := \{\mathcal{S}_{r,\cdot}(a),\ r \in I,\ a \in [0, g(r)]\}$ of continuous functions $\mathcal{S}_{r,\cdot}(a) : I \mapsto \mathbb{R}_+$ such that for every $r, x \in I$ and $a \in [0, g(r)]$, $\mathcal{S}_{r,r}(a) = a$ and $\mathcal{S}_{r,x}(a) \leqslant g(x)$. We call it flow only because we will deal exclusively with the BESQ and Jacobi flows. More generally, we give the following definition.

Definition 3.2. *We call flow in the domain D_f any flow which is a g-flow for some continuous nonnegative function $g \leqslant f$.*

Remark. We also allow the case $f \equiv \infty$, meaning that $\mathcal{L}$ is a flow in the domain $\mathbb{R}_+ \times \mathbb{R}$.

If $\mathcal{L}$ is a g-flow, the *forward flow of* $\mathcal{L}$ is the collection of functions $\mathcal{S} := (\mathcal{S}_{r,x}(a),\ x \in I \cap [r, \infty))_{r \in I, a \in [0, g(r)]}$ and the *backward flow* of $\mathcal{L}$ is the collection of functions $\mathcal{S}^* := (\mathcal{S}_{-r,-x}(a),\ x \in (-I) \cap [r, +\infty)_{r \in -I, a \in [0, g(-r)]}$ where $-I := \{-x,\ x \in I\}$. We already mentioned the Jacobi flow and the BESQ flow, which are in our terminology forward flows. The first one is a forward flow in the domain $[0,1] \times \mathbb{R}$ (here $g(r) \equiv 1$). The second one is a forward flow in the domain $\mathbb{R}_+ \times \mathbb{R}$ (formally taking $g(r) \equiv \infty$).

Definition 3.3. *Let $g \leqslant f$ be a continuous nonnegative function and $c \in I$. Consider a g-flow $\mathcal{L}$. We denote by $\Psi(\mathcal{L}, f, c)$ the flow such that for any (v, r) in the image of $D_g := \{(a, x) : a \in [0, g(x)], x \in I\}$ by $\Psi(\cdot, f, c)$, its flow line passing through (v, r) is the image of the flow line in $\mathcal{L}$ passing through the preimage of (v, r). It is defined through the following equation:*

$$\left(\Psi(\mathcal{L}, f, c)\right)_{r,x}(v) = \frac{\mathcal{S}_{\eta_{f,c}^{-1}(r), \eta_{f,c}^{-1}(x)}(v f \circ \eta_{f,c}^{-1}(r))}{f \circ \eta_{f,c}^{-1}(x)}$$

for all $r, x \in \eta_{f,c}(I)$, $v \in [0, \Psi(g, f, c)(r)]$. When $c = 0$, we will only write $\Psi(\mathcal{L}, f)$ for $\Psi(\mathcal{L}, f, 0)$.

Remark. The image flow $\Psi(\mathcal{L}, f, c)$ is a $\Psi(g, f, c)$-flow.

If we only look at the forward flow $\mathcal{S}$ (for example in (4.10)), we will still write $\Psi(\mathcal{S}, f, c)$ for the forward flow obtained as in Definition 3.3, restricted to $r \leqslant x$.

3.2. **Transformations of processes.** Let J be an interval of $\mathbb{R}$ containing 0 and $\mathcal{X} = (\mathcal{X}_t,\, t \in J)$ be some real-valued continuous process such that $\mathcal{X}_t \in I$ for all $t \in J$. We suppose that $\mathcal{X}$ admits a version of bicontinuous local times $L_{\mathcal{X}}(t,x)$, $t \in J$, $x \in I$, defined as the densities of the occupation times: for any Borel nonnegative function h and any $t \in J$,

$$\int_{J\cap(-\infty,t]} h(\mathcal{X}_s)\mathrm{d}s = \int_I h(x) L_{\mathcal{X}}(t,x)\mathrm{d}x.$$

We let $g(x) := L_{\mathcal{X}}(\infty, x)$ for $x \in I$ denote the total local time of $\mathcal{X}$ at position x and we suppose that $g(x) \leqslant f(x)$ on I. Define $\mathcal{L}_{\mathcal{X}} = (\mathcal{S}_{r,x}(a))_{r,x,a}$ as, for any x and r in I and $0 \leqslant a \leqslant g(r)$,

$$\mathcal{S}_{r,x}(a) := L_{\mathcal{X}}(\tau_a^r(\mathcal{X}), x),$$

where

$$\tau_a^r(\mathcal{X}) := \inf\{t \in J \,:\, L_{\mathcal{X}}(t,r) > a\}.$$

We used the convention that $\inf \varnothing = \infty$, so that when $a = g(r)$, $\tau_a^r(\mathcal{X}) = \infty$ and $\mathcal{S}_{r,x}(a) = g(x)$ for all x. Then $\mathcal{L}_{\mathcal{X}}$ is a g-flow, which (in agreement with the previous section) we will refer to as the *local time flow of* $\mathcal{X}$. Its forward, resp. backward flow is called *forward*, resp. *backward local time flow of* $\mathcal{X}$. By Definition 3.2, $\mathcal{L}_{\mathcal{X}}$ is a flow in the domain D_f.

Proposition 3.4. *We set*

$$C_f(t) := \int_0^t \frac{\mathrm{d}s}{f(\mathcal{X}_s)^2}, \qquad t \in J.$$

For $c \in I$, we let $\Upsilon(\mathcal{X}, f, c)$ be the process defined as

$$\Upsilon(\mathcal{X}, f, c)_t := \eta_{f,c}\left(\mathcal{X}_{C_f^{-1}(t)}\right) = \int_c^{\mathcal{X}_{C_f^{-1}(t)}} \frac{\mathrm{d}r}{f(r)}, \qquad t \in C_f(J). \tag{3.1}$$

Then, the process $\Upsilon(\mathcal{X}, f, c)$ possesses bicontinuous local times (given by (3.2)) and its local time flow is $\Psi(\mathcal{L}_{\mathcal{X}}, f, c)$.

For sake of brevity, we will write in the rest of this section $\mathcal{Z} := \Upsilon(\mathcal{X}, f, c)$.

Remarks. (i) The definition of Υ does not depend on the choice of the interval I on which is defined f.
(ii) Observe that $|C_f(t)| < \infty$ for all $t \in J$ and $t \to C_f(t)$ is strictly increasing so that $\mathcal{Z}$ is well defined and is a continuous process. Moreover, for all $t \in C_f(J)$, $\mathcal{Z}_t \in \eta_{f,c}(I)$.
(iii) In the course of the proof, we will prove that the local time $L_{\mathcal{Z}}(t,x)$ of $\mathcal{Z}$ at time $t \in C_f(J)$ and position $x \in \eta_{f,c}(I)$ is given by

$$L_{\mathcal{Z}}(t,x) = \frac{L_{\mathcal{X}}(C_f^{-1}(t), \eta_{f,c}^{-1}(x))}{f \circ \eta_{f,c}^{-1}(x)}. \tag{3.2}$$

In particular, the total local time of $\mathcal{Z}$ at position x is $\Psi(g, f, c)(x) \leqslant 1$ for all $x \in \eta_{f,c}(I)$.
(iv) Explicitly, the proposition means that

$$L_{\mathcal{Z}}(\tau_v^{\hat{r}}(\mathcal{Z}), \hat{x}) = \Psi(\mathcal{L}_{\mathcal{X}}, f, c)_{\hat{r},\hat{x}}(v)$$

for all $\hat{r}, \hat{x} \in \eta_{f,c}(I)$ and $v \in [0, \Psi(g, f, c)(\hat{r})]$ where $\tau_v^{\hat{r}}(\mathcal{Z}) := \inf\{t \in C_f(J) \,:\, L_{\mathcal{Z}}(t, \hat{r}) > v\}$.

Proof of the proposition. For simplicity, we suppose that $c = 0$. First we prove (3.2) by following the proof of Lemma 2 of [34]. Let $h : \mathbb{R} \to \mathbb{R}_+$ be a Borel function. Let $t_1 := \inf J$ and $t_2 := \sup J$. Using change of variables, we have for $t \in (t_1, t_2)$,

$$\begin{aligned}\int_{C_f(t_1)}^{C_f(t)} h(\mathcal{Z}_u)\mathrm{d}u &= \int_{t_1}^{t} h(\mathcal{Z}_{C_f(s)})\mathrm{d}C_f(s)\\ &= \int_{t_1}^{t} h\circ\eta_f(\mathcal{X}_s)\frac{\mathrm{d}s}{f(\mathcal{X}_s)^2}\\ &= \int_{I_f} h\circ\eta_f(x)\frac{L_{\mathcal{X}}(t,x)}{f(x)^2}\mathrm{d}x\end{aligned}$$

by the occupation times formula. Let $\widehat{L}(C_f(t), \eta_f(x)) := \frac{L_{\mathcal{X}}(t,x)}{f(x)}$. We get by the change of variables $z = \eta_f(x)$,

$$\int_{C_f(t_1)}^{C_f(t)} h(\mathcal{Z}_u)\mathrm{d}u = \int_{-\infty}^{\infty} h(z)\widehat{L}(C_f(t), z)\mathrm{d}z.$$

Therefore $\mathcal{Z}$ has local time given by $\widehat{L}$. It proves (3.2). Let $r, x \in I$ and $0 \leqslant v \leqslant \frac{g(r)}{f(r)}$. Set $\hat{r} := \eta_f(r)$, $\hat{x} := \eta_f(x)$, $\hat{\tau}_v^{\hat{r}} := C_f(\hat{t})$ where $\hat{t} := \tau^r_{vf(r)}(\mathcal{X})$. Then $\widehat{L}(\hat{\tau}_v^{\hat{r}}, \hat{r}) = v$, and since $v \to \hat{\tau}_v^{\hat{r}}$ is right-continuous, we deduce that $\hat{\tau}$ is the inverse local time of $\mathcal{Z}$. Moreover, by definition,

$$\widehat{L}(\hat{\tau}_v^{\hat{r}}, \hat{x}) = \frac{L_{\mathcal{X}}(\tau^r_{vf(r)}(\mathcal{X}), x)}{f(x)} = \frac{\mathcal{S}_{r,x}(vf(r))}{f(x)}.$$

By Definition 3.3, it is $\Psi(\mathcal{L}_{\mathcal{X}}, f)_{\hat{r},\hat{x}}(v)$ indeed. □

The transformation Υ behaves well under composition. It is the content of the following lemma. For a process X, we let $X_{s+\cdot}$ denote the process X_{s+t}, $t \geqslant 0$ (as long as X_{s+t} is well-defined).

Lemma 3.5. *Let $s \in J$ and $c' \in I$. We suppose that $f - L_{\mathcal{X}}(s,\cdot)$ is positive on an interval containing c' and $\{\mathcal{X}_{s+\cdot}\}$. Then, $\Upsilon\left(\mathcal{Z}_{C_f(s)+\cdot}, 1 - L_{\mathcal{Z}}(C_f(s),\cdot), \eta_{f,c}(c')\right)$ is well-defined and equals $\Upsilon\left(\mathcal{X}_{s+\cdot}, f - L_{\mathcal{X}}(s,\cdot), c'\right)$. In particular, the values at time 0 of the two processes are identical, i.e.*

$$\int_{\eta_{f,c}(c')}^{\mathcal{Z}_{C_f(s)}} \frac{\mathrm{d}r}{1 - L_{\mathcal{Z}}(C_f(s), r)} = \int_{c'}^{\mathcal{X}_s} \frac{\mathrm{d}r}{f(r) - L_{\mathcal{X}}(s, r)}. \tag{3.3}$$

Remark. By saying that $\Upsilon\left(\mathcal{Z}_{C_f(s)+\cdot}, 1 - L_{\mathcal{Z}}(C_f(s),\cdot), \eta_{f,c}(c')\right)$ is well-defined, we mean that

(i) the total local time of $\mathcal{Z}_{C_f(s)+\cdot}$ at position u is smaller than $1 - L_{\mathcal{Z}}(C_f(s), u)$ (which is clear since the total local time of $\mathcal{Z}$ is smaller than 1)
(ii) $1 - L_{\mathcal{Z}}(C_f(s),\cdot) > 0$ on an interval containing $\eta_{f,c}(c')$ and $\{\mathcal{Z}_{C_f(s)+\cdot}\}$.

Proof of Lemma 3.5. We first check that $\Upsilon\left(\mathcal{Z}_{C_f(s)+\cdot}, 1 - L_{\mathcal{Z}}(C_f(s),\cdot), \eta_{f,c}(c')\right)$ is well-defined, hence we check statement (ii) above since (i) is clear. By (3.2), for any $u \in \eta_{f,c}(I)$,

$$L_{\mathcal{Z}}(C_f(s), u) = \frac{L_{\mathcal{X}}(s, \eta_{f,c}^{-1}(u))}{f(\eta_{f,c}^{-1}(u))}.$$

We deduce that (ii) is satisfied since $\{\mathcal{Z}_{C_f(s)+\cdot}\} = \eta_{f,c}(\{\mathcal{X}_{s+\cdot}\})$. Let us prove the second statement of the lemma. Let $t \geqslant 0$ with $s + t \in J$. Let

$$t'' := \int_s^{s+t} \frac{\mathrm{d}u}{(f(\mathcal{X}_u) - L_{\mathcal{X}}(s, \mathcal{X}_u))^2}, \qquad c'' := \eta_{f,c}(c').$$

Substituting v for $C_f(u)$ in the above integral and using that $\mathcal{Z}_{C_f(u)} = \eta_{f,c}(\mathcal{X}_u)$ by (3.1), we get

$$t'' = \int_{C_f(s)}^{C_f(s+t)} \frac{\mathrm{d}v}{(1 - L_{\mathcal{Z}}(C_f(s), \mathcal{Z}_v))^2}. \tag{3.4}$$

Observe that by the change of variables $x = \eta_{f,c}^{-1}(r)$ in the integral,

$$\eta_{1-L_{\mathcal{Z}}(C_f(s),\cdot),c''}(u) = \int_{c''}^{u} \frac{\mathrm{d}r}{1 - L_{\mathcal{Z}}(C_f(s), r)} = \int_{c'}^{\eta_{f,c}^{-1}(u)} \frac{\mathrm{d}x}{f(x) - L_{\mathcal{X}}(s, x)}. \tag{3.5}$$

Let us consider $\Upsilon(\mathcal{Z}_{C_f(s)+\cdot}, 1 - L_{\mathcal{Z}}(C_f(s),\cdot), c'')$ at time t''. By definition and using equation (3.4),

$$\Upsilon(\mathcal{Z}_{C_f(s)+\cdot}, 1 - L_{\mathcal{Z}}(C_f(s),\cdot), c'')(t'') = \eta_{1-L_{\mathcal{Z}}(C_f(s),\cdot),c''}(\mathcal{Z}_{C_f(s+t)}).$$

Equation (3.5) with $u = \mathcal{Z}_{C_f(s+t)} = \eta_{f,c}(\mathcal{X}_{s+t})$ yields

$$\eta_{1-L_{\mathcal{Z}}(C_f(s),\cdot),c''}(\mathcal{Z}_{C_f(s+t)}) = \int_{c'}^{\mathcal{X}_{s+t}} \frac{\mathrm{d}x}{f(x) - L_{\mathcal{X}}(s, x)} = \eta_{f-L_{\mathcal{X}}(s,\cdot),c'}(\mathcal{X}_{s+t}).$$

By definition, it is the value of the process $\Upsilon(\mathcal{X}_{s+\cdot}, f - L_{\mathcal{X}}(s,\cdot), c')$ at time t''. The proof is complete. □

Notation 3.6. *When $c = \mathcal{X}_0$, we will write $\Upsilon(\mathcal{X}, f)$ for $\Upsilon(\mathcal{X}, f, \mathcal{X}_0)$.*

(i) Direct computations also show that $\Upsilon(\mathcal{X}, f)$ stays unchanged when replacing $\mathcal{X}$ by $\mathcal{X} + a$ and f by $f(\cdot - a)$ for $a \in \mathbb{R}$ arbitrary.

(ii) Since $\eta_{f,c}(\mathcal{X}_s) = \mathcal{Z}_{C_f(s)}$ by definition, Lemma 3.5 in the case that $c' = \mathcal{X}_s$ reads

$$\Upsilon\left(\mathcal{Z}_{C_f(s)+\cdot}, 1 - L_{\mathcal{Z}}(C_f(s),\cdot)\right) = \Upsilon\left(\mathcal{X}_{s+\cdot}, f - L_{\mathcal{X}}(s,\cdot)\right)$$

We finish this section by a continuity lemma on the transformation Υ that will be used later on.

Lemma 3.7. *We fix a couple $(\mathcal{X}, f)$ where $\mathcal{X} : J = [0, \infty) \to \mathbb{R}$, $f : I \to (0, \infty)$ are continuous and $\mathcal{X}_t \in I$ for all $t \geqslant 0$.*

We consider a sequence $(\mathcal{X}^n, f^n)_{n\geqslant 1}$ such that $\mathcal{X}^n : [0, j_n) \to \mathbb{R}$, $j_n \in \mathbb{R}_+ \cup \{+\infty\}$, $f^n : I_n \to (0, \infty)$ and $\mathcal{X}_t \in I_n$ for all $t \in [0, j_n)$. The sets I, I_n are intervals.

We make the following assumptions:

(i) for all n, $\mathcal{X}_0^n = \mathcal{X}_0 = 0$ and $\int_0^{j_n} \frac{\mathrm{d}s}{f^n(\mathcal{X}_s^n)^2} = \int_0^\infty \frac{\mathrm{d}s}{f(\mathcal{X}_s)^2} = \infty$;
(ii) $\lim_{n\to\infty} j_n = \infty$;
(iii) $\mathcal{X}^n$ converges uniformly to $\mathcal{X}$ on any compact set of $\mathbb{R}_+$;
(iv) for any compact set K of $\mathbb{R}_+$, there exists a compact set L in $\mathbb{R}$ such that
(a) for n large enough, $\mathcal{X}(K)$ and $\mathcal{X}^n(K)$ are contained in the set L which is itself contained in I and in I_n;
(b) f^n converges uniformly to f on L.

Then $\Upsilon(\mathcal{X}^n, f^n)$ converges to $\Upsilon(\mathcal{X}, f)$ (for the topology of uniform convergence on compact sets of $\mathbb{R}_+$).

See Appendix C for the proof.

4. Jacobi flows

In this section, we define Jacobi flows, and show how they arise by disintegration of the BESQ flows.

Let λ denote the Lebesgue measure.

Definition 4.1. *We introduce the covariance functional*

$$Q_h(A, \widehat{A}) = h\left(\lambda(A \cap \widehat{A}) - \lambda(A \cap [0,1])\lambda(\widehat{A} \cap [0,1])\right)$$

for A, $\widehat{A}$ Borel sets of $\mathbb{R}_+$, and $h \geqslant 0$. Let $(\mathcal{E}_s,\ -\infty < s < \infty)$ be a right-continuous filtration. We consider a collection of random variables $(\mathcal{M}(A\times[s,t]),\ s \leqslant t,\ A$ Borel set of $\mathbb{R}_+)$ such that for any $s \in \mathbb{R}$, A, $\widehat{A}$ Borel sets of $\mathbb{R}_+$, $(\mathcal{M}(A \times [s, s+h]))_{h\geqslant 0}$ and $(\mathcal{M}(\widehat{A} \times [s, s+h]))_{h\geqslant 0}$ are two continuous martingales with respect to the filtration $(\mathcal{E}_{s+h},\ h \geqslant 0)$ and

$$\langle \mathcal{M}(A \times [s, s+\cdot]), \mathcal{M}(\widehat{A} \times [s, s+\cdot])\rangle_h = Q_h(A, \widehat{A}). \tag{4.1}$$

From (4.1), we deduce that for fixed s, h, and $(A_n)_n$ disjoint Borel sets with $\lambda(\bigcup_n A_n) < \infty$, we have $\mathcal{M}(\bigcup_n A_n, [s, s+h]) = \sum_n \mathcal{M}(A_n, [s, s+h])$ in $L^2(\Omega)$. In other words, $\mathcal{M}$ is an $(\mathcal{E}_s)$-martingale measure with covariance functional Q on $\mathbb{R}_+ \times \mathbb{R}$ in the sense of [32].

The martingale measure $\mathcal{M}$ can be defined through a white noise $\mathcal{W}$ on $\mathbb{R}_+ \times \mathbb{R}$:

$$\mathcal{M}(A \times [s,t]) := \mathcal{W}(A \times [s,t]) - \lambda(A \cap [0,1])\mathcal{W}([0,1] \times [s,t]). \tag{4.2}$$

We now introduce the Jacobi flows.

Definition 4.2. *Let $\delta > 0$ and $\delta' \in \mathbb{R}$. Let $\mathcal{Y}$ be a collection of continuous processes $(\mathcal{Y}_{s,t}(v),\ t \geqslant s)_{s\in\mathbb{R},\, v\in[0,1]}$. We say that $\mathcal{Y}$ is a (non-killed)* Jacobi(δ, δ') *flow if it is solution of*

(4.3)

$$\mathcal{Y}_{s,t}(v) = v + 2\int_s^t \mathcal{M}\Big([0, \mathcal{Y}_{s,r}(v)], \mathrm{d}r\Big) + \int_s^t \delta(1 - \mathcal{Y}_{s,r}(v)) - \delta'\mathcal{Y}_{s,r}(v)\ \mathrm{d}r,\ t \in [0, T_1(\mathcal{Y}_{s,\cdot}(v))]$$

where $\mathcal{M}$ is a martingale measure on $\mathbb{R}_+\times\mathbb{R}$ with covariance functional Q and $T_1(\mathcal{Y}_{s,\cdot}(v)) := \inf\{t \geqslant s\ :\ \mathcal{Y}_{s,t}(v) = 1\}$ (it may be infinite). We require the regularity conditions: almost surely,

(i) for all $s \in \mathbb{R}$ and $v \in [0,1]$, $\mathcal{Y}_{s,s}(v) = v$,

(ii) for all $s \leqslant t$, $v \in [0,1) \mapsto \mathcal{Y}_{s,t}(v)$ is càdlàg,
(iii) for all $s' \leqslant s$ and all $v', v \in [0,1]$, if $\mathcal{Y}_{s',s}(v') > v$, then $\mathcal{Y}_{s',t}(v') \geqslant \mathcal{Y}_{s,t}(v)$ for all $t \geqslant s$,
(iv) for all $s \leqslant t \leqslant t'$, and $v \in [0,1]$, if $\mathcal{Y}_{s,t}(v) = 1$ then $\mathcal{Y}_{s,t'}(1) = 1$.

Definition 4.3. *Let $\delta < 2$ and $\delta' \in \mathbb{R}$. We say that $\mathcal{Y}$ is a killed Jacobi(δ, δ') flow if it is solution of (4.3) but is absorbed at 0 and satisfies the same regularity conditions. For $\delta \in \mathbb{R}$, we call general* Jacobi(δ, δ') *flow a Jacobi flow which is either killed or non-killed.*

Remark. Being non-killed means that the flow line is reflected at zero. With condition (iv), we arbitrarily decided to kill flow lines when they hit 1. We could also let them reflect at 1. Therefore, strictly speaking, our Jacobi flows are actually Jacobi flows killed at 1.

The existence of Jacobi flows will be obtained as a consequence of Theorem 4.6. As for the case of BESQ flows, the Jacobi flow can be constructed from a countable number of flow lines.

Definition 4.4. *Let $\delta, \delta' \in \mathbb{R}$ and $b \geqslant 0$. Let $\mathcal{W}$ be a white noise on $\mathbb{R}_+^2$. Consider the* BESQ$^\delta$ *flow $(\mathcal{S}_{r,x}(a), x \geqslant r)_{r \in \mathbb{R}, a \geqslant 0}$ driven by $\mathcal{W}$ as in Definition 2.2 or Definition 2.3. Let $f = (f(x),\, x \in [0, \mathfrak{d}_f))$ be the pathwise unique solution of*

$$f(x) = b + 2\int_0^x \mathcal{W}([0, f(s)], \mathrm{d}s) + (\delta + \delta')x \tag{4.4}$$

where the process is absorbed at 0 if it hits 0 at a positive time (we denote by $\mathfrak{d}_f$ this absorption time, possibly infinite). We call BESQ$_b^{\delta,\delta'}$ *flow driven by $\mathcal{W}$ the collection of continuous processes $\Theta = (\Theta_{r,x}(a),\, 0 \leqslant r \leqslant x < \mathfrak{d}_f),\, a \in [0, f(r)])$ where $\Theta_{r,\cdot}(a) = \mathcal{S}_{r,\cdot}(a)$ until it meets f, and is equal to f afterwards.*

The BESQ$_b^{\delta,\delta'}$ *flow is called killed when $\mathcal{S}$ is killed.*

When $b = 0$, by convention the flow will be defined for $x \geqslant r > 0$.

Remark. (i) To avoid trivial situations, we suppose either that $b > 0$ or $b = 0$ and $\delta + \delta' \geqslant 2$ (otherwise $\mathfrak{d}_f = 0$).
(ii) The flow lines may hit f only in the case $\delta' < 2$.

We suppose now that $b > 0$. We have $\mathfrak{d}_f < \infty$ if and only if $\delta + \delta' < 2$. We will use the transformation Ψ of Section 3.1 with f. We recall that $\eta_f(x) := \int_0^x \frac{\mathrm{d}r}{f(r)}$. In all cases, $\lim_{x \uparrow \mathfrak{d}_f} \eta_f(x) = +\infty$ (see Lemma 5.1). Define the martingale measure $\widetilde{\mathcal{W}}$ by, for any Borel set $A \subset \mathbb{R}_+$ with finite Lebesgue measure $\lambda(A)$,

$$\widetilde{\mathcal{W}}(A \times [0,t]) := \int_{r=0}^{\eta_f^{-1}(t)} \frac{1}{f(r)} \mathcal{W}\left(Af(r),\, \mathrm{d}r\right), \qquad t \geqslant 0, \tag{4.5}$$

where $Af(r) := \{af(r),\, a \in A\}$. If $(\mathscr{E}_x,\, x \geqslant 0)$ denotes the natural filtration of $\mathcal{W}$, with $\mathscr{E}_x := \sigma(\mathcal{W}(\cdot \times [0,r]), 0 \leqslant r \leqslant x)$, then the process $(\widetilde{\mathcal{W}}(A \times [0,t]))_{t \geqslant 0}$ is a continuous martingale with respect to the filtration $(\widetilde{\mathscr{E}}_t,\, t \geqslant 0) := (\mathscr{E}_{\eta_f^{-1}(t)},\, t \geqslant 0)$ whose quadratic

variation process is equal to $\lambda(A)t$. We deduce that $\widetilde{\mathcal{W}}$ is a white noise on $\mathbb{R}_+^2$ with respect to the filtration $(\widetilde{\mathscr{E}}_t,\, t \geqslant 0)$.

We define the martingale measure $\mathcal{M}^+$ on $\mathbb{R}_+^2$ by the identity for any Borel set $A \subset \mathbb{R}_+$:

$$\mathcal{M}^+(A \times [0,t]) := \widetilde{\mathcal{W}}\left(A \times [0,t]\right) - \lambda(A \cap [0,1])\, \widetilde{\mathcal{W}}\left([0,1] \times [0,t]\right), \qquad t \geqslant 0. \tag{4.6}$$

Notice that $(\mathcal{M}^+(A \times [0,t]))_{t,A}$ is a Gaussian process. It is actually an $(\widetilde{\mathscr{E}}_t)$-martingale measure with covariance functional Q as defined in Definition 4.1, from $s = 0$. From the definition of $\mathcal{M}^+$, we see that for any deterministic Borel set $A \subset \mathbb{R}_+$, and $t \geqslant 0$,

$$\mathcal{M}^+(A \times [0,t]) = \int_{r=0}^{\eta_f^{-1}(t)} \frac{1}{f(r)} \Big(\mathcal{W}\left(Af(r),\, \mathrm{d}r\right) - \lambda(A \cap [0,1])\, \mathcal{W}\left([0, f(r)],\, \mathrm{d}r\right) \Big). \tag{4.7}$$

We can rewrite it as

$$\begin{aligned} &\int_{r=0}^{\infty} \int_{u=0}^{\infty} g(u,r) \mathcal{M}^+(\mathrm{d}u, \mathrm{d}r) \\ &= \int_{r=0}^{\infty} \int_{u=0}^{\infty} \frac{1}{f(r)} g\Big(\frac{u}{f(r)}, \eta_f(r)\Big) \left(\mathcal{W}(\mathrm{d}u, \mathrm{d}r) - \mathbf{1}_{[0,f(r)]}(u) \frac{\mathrm{d}u}{f(r)} \mathcal{W}\left([0, f(r)],\, \mathrm{d}r\right) \right) \end{aligned} \tag{4.8}$$

for $g = \mathbf{1}_{A \times [0,t]}$. By definition of the stochastic integral, it is also true when $g = Z\mathbf{1}_{A \times (s,t]}$ where $0 \leqslant s \leqslant t$ and Z is a bounded $\widetilde{\mathscr{E}}_s$-measurable function. By linearity and density, (4.8) is true for any $(\widetilde{\mathscr{E}}_t)$-predictable square integrable function g with respect to the covariance of $\mathcal{M}^+$, i.e.

$$\mathbb{E}\left[\int_{t \geqslant 0} \int_{u \geqslant 0} g(u,t)^2 \mathrm{d}u\mathrm{d}t - \int_{t \geqslant 0} \left(\int_{u \in [0,1]} g(u,t) \mathrm{d}u \right)^2 \mathrm{d}t \right] < \infty.$$

Define

$$\gamma_x := \int_0^x \frac{1}{\sqrt{f(r)}} \mathcal{W}([0, f(r)], \mathrm{d}r), \qquad 0 \leqslant x < \mathfrak{d}_f. \tag{4.9}$$

The process γ is an $(\mathscr{E}_x)$-Brownian motion stopped at $\mathfrak{d}_f$ when $\mathfrak{d}_f < \infty$.

Proposition 4.5. *The martingale measure $\mathcal{M}^+$ is independent of the process f.*

Proof. The proof is an extension of that of Proposition 8 of [34] to the setting of martingale measures. Let γ be defined by (4.9). From (4.7), we see that $(\mathcal{M}^+(A \times [0, \eta_f(x)]),\, 0 \leqslant x < \mathfrak{d}_f)$ is a martingale which is orthogonal to γ. On the other hand, its increasing process is $x \to (\lambda(A) - \lambda(A \cap [0,1])^2)\eta_f(x)$. We deduce by Knight's theorem that $(\mathcal{M}^+(A \times [0,t]),\, t \geqslant 0)$ is a Brownian motion (with multiplicative constant) which is independent of γ. More generally, the multidimensional Knight's theorem implies that $(\mathcal{M}^+(A_1 \times [0,t]), \ldots, \mathcal{M}^+(A_n \times [0,t])\, t \geqslant 0)$ are independent of γ for any $n \geqslant 1$ and disjoint Borel sets $A_1, \ldots, A_n$ of $\mathbb{R}_+$. We deduce that $(\mathcal{M}^+(A \times [0,t]))_{t,A}$ is independent of γ. We observe that the process f is measurable with respect to the filtration of γ (it is the strong solution of $\mathrm{d}f(x) = 2\sqrt{f(x)}\mathrm{d}\gamma_x + (\delta + \delta')\mathrm{d}x$), therefore $(\mathcal{M}^+(A \times [0,t]))_{t,A}$ is independent of f. □

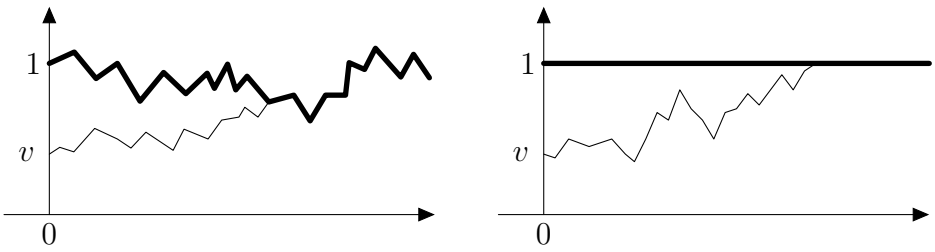

FIGURE 1. On the left: $\mathrm{BESQ}^{\delta,\delta'}$ flow. The process $x \to f(x)$ is traced in bold. A flow line $x \to S_{0,x}(v)$ is also represented. On the right: Jacobi(δ,δ') flow after a space-time transformation. The process $x \to f(x)$ is mapped to the constant function 1, and the image of the flow line at v is a flow line of the Jacobi flow.

In the notation of Section 3.1 with $I = [0, \mathfrak{d}_f)$ and $c = 0$, we define the flow $\mathcal{Y}^+ = (\mathcal{Y}_{s,t}(v), 0 \leqslant s \leqslant t, v \in [0,1])$ by

$$\mathcal{Y}^+ := \Psi(\Theta, f) \tag{4.10}$$

(we add the superscript in $\mathcal{Y}^+$ to stress that the flow is only defined for $0 \leqslant s \leqslant t$ whereas the flow $\mathcal{Y}$ is defined for all $-\infty < s \leqslant t$), see Figure 1. By definition,

$$\mathcal{Y}_{s,t}(v) = \frac{\Theta_{\eta_f^{-1}(s),\eta_f^{-1}(t)}\left(vf \circ \eta_f^{-1}(s)\right)}{f \circ \eta_f^{-1}(t)}, \qquad t \geqslant s \geqslant 0,\ v \in [0,1]. \tag{4.11}$$

Theorem 4.6. *Let $\delta, \delta' \in \mathbb{R}$. Let $b > 0$ and Θ be a $\mathrm{BESQ}_b^{\delta,\delta'}$ flow. Then, the flow $\mathcal{Y}^+$ is a* Jacobi(δ,δ') *flow, restricted to the positive time-axis. It is the $[0,1]$-valued pathwise unique solution, which is strong, of the following SDE. Let $v \in [0,1]$ and $s \geqslant 0$. Almost surely, for $t \geqslant s$,*

$$\mathcal{Y}_{s,t}(v) = v + 2\int_s^t \mathcal{M}^+([0, \mathcal{Y}_{s,r}(v)], \mathrm{d}r) + \int_s^t (\delta(1 - \mathcal{Y}_{s,r}(v)) - \delta'\mathcal{Y}_{s,r}(v))\,\mathrm{d}r \tag{4.12}$$

with the required absorption conditions[5]*. In particular, $\mathcal{Y}^+$ is independent of f.*

Remark. The theorem will be used only in the case $\delta' = 0$ in the rest of the paper. The general case will be used in a follow-up paper.

Proof. The regularity conditions (i), (iii) and (iv) of $\mathcal{Y}^+$ come from the properties of Θ. Let us check that (ii) is also satisfied. Let $u \in [0,1)$ and $s \leqslant t$. If $\mathcal{Y}_{s,t}(u) = 1$, then $\mathcal{Y}_{s,t}(v) = 1$ for all $v \geqslant u$ hence $v \to \mathcal{Y}_{s,t}(u)$ is càdlàg at u. Suppose then that $\mathcal{Y}_{s,t}(u) < 1$. There exists $v > u$ such that $\mathcal{Y}_{s,t}(v) < 1$. Indeed, the flow $(\mathcal{S}_{r,x}(a),\ x \geqslant r)_{r\in\mathbb{R},\, a\geqslant 0}$ verifies: almost surely, for all $r < x$ and $a \geqslant 0$, $\sup_{r'\in[r,x]} |\mathcal{S}_{r,r'}(b) - \mathcal{S}_{r,r'}(a)| \to 0$ as $b \downarrow a$. This statement follows from the identification $\mathcal{S}_{r,x}(a) = L(\tau_a^r, x)$ (see Section 2), the fact that $b \to \tau_b^r$ is càdlàg, and the bicontinuity of the local times. Since for $v > u$ close enough to

[5]The process is absorbed if it hits 1. If Θ is killed, then the process is also absorbed if it hits 0.

u, one has $\mathcal{Y}_{s,t}(v) < 1$, we can use the regularity of the flow $(\mathcal{S}_{r,x}(a),\ x \geqslant r)_{r\in\mathbb{R},\, a\geqslant 0}$ to see that $v \to \mathcal{Y}_{s,t}(v)$ is càdlàg at u.

We show now the SDE. To avoid too much notation, we prove it for $s = 0$. For brevity, we suppose that Θ is non-killed. The other case is similar, one just has to look at the times before the hitting time of 0. By (4.11), we have

$$\mathcal{Y}_{0,t}(v) = \frac{\Theta_{0,\eta_f^{-1}(t)}(bv)}{f \circ \eta_f^{-1}(t)}.$$

Equation (4.12) is equivalent to

$$\frac{\Theta_{0,x}(bv)}{f(x)} = v + 2\int_0^{\eta_f(x)} \mathcal{M}^+([0, \mathcal{Y}_{0,r}(v)], \mathrm{d}r) + \int_0^{\eta_f(x)} (\delta(1 - \mathcal{Y}_{0,r}(v)) - \delta'\mathcal{Y}_{0,r}(v))\, \mathrm{d}r. \tag{4.13}$$

Let $\xi_x(v) := \frac{\Theta_{0,x}(bv)}{f(x)}$. By (4.8) with $g(u, r) := \mathbf{1}_{[0,\mathcal{Y}_{0,r}(v)]\times[0,\eta_f(x)]}(u, r)$ (hence $g(\frac{u}{f(r)}, \eta_f(r)) = \mathbf{1}_{[0,\Theta_{0,r}(bv)]\times[0,x]}$), the first integral of the right-hand side of (4.13) is equal to

$$\int_0^x \frac{1}{f(r)} \left(\mathcal{W}([0, \Theta_{0,r}(bv)], \mathrm{d}r) - \xi_r(v)\mathcal{W}([0, f(r)], \mathrm{d}r)\right). \tag{4.14}$$

By a change of variables, the last term of the right-hand side in (4.13) is

$$\int_0^{\eta_f(x)} \delta(1 - \mathcal{Y}_{0,r}(v)) - \delta'\mathcal{Y}_{0,r}(v)\mathrm{d}r = \int_0^x \frac{\delta(1 - \xi_r(v)) - \delta'\xi_r(v)}{f(r)}\mathrm{d}r. \tag{4.15}$$

Let us verify (4.13). By (4.4), we have $\langle f, f\rangle_x = 4\int_0^x f(s)\mathrm{d}s$ and

$$\langle \Theta_{0,\cdot}(bv), f\rangle_x = 4\int_0^x \Theta_{0,s}(bv)\mathrm{d}s.$$

We deduce from Itô's formula that, as long as $\Theta_{0,x}(bv) < f(x)$, ($\mathrm{d} \equiv \mathrm{d}_x$ below)

$$\begin{aligned}
&\mathrm{d}\xi_x(v)\\
=\ & \frac{\mathrm{d}\Theta_{0,x}(bv)}{f(x)} - \frac{\Theta_{0,x}(bv)}{f(x)^2}\mathrm{d}f(x) + \Theta_{0,x}(bv)\frac{4f(x)}{f^3(x)}\mathrm{d}x - \frac{1}{f^2(x)}4\Theta_{0,x}(bv)\mathrm{d}x\\
=\ & \frac{\mathrm{d}\Theta_{0,x}(bv)}{f(x)} - \frac{\Theta_{0,x}(bv)}{f(x)^2}\mathrm{d}f(x)\\
=\ & \frac{1}{f(x)}\big(2\mathcal{W}([0, \Theta_{0,x}(bv)], \mathrm{d}x) + \delta\mathrm{d}x\big) - \frac{\xi_x(v)}{f(x)}\big(2\mathcal{W}([0, f(x)], \mathrm{d}x) + (\delta + \delta')\mathrm{d}x\big)\\
=\ & \frac{2}{f(x)}\left(\mathcal{W}([0, \Theta_{0,x}(bv)], \mathrm{d}x) - \xi_x(v)\mathcal{W}([0, f(x)], \mathrm{d}x)\right) + \frac{\delta(1 - \xi_x(v)) - \delta'\xi_x(v)}{f(x)}\mathrm{d}x.
\end{aligned}$$

Comparing with (4.14) and (4.15), we get (4.13) indeed.

Now we show that $\mathcal{Y}^+$ is the pathwise unique solution (by Yamada–Watanabe's theorem it will imply that it is a strong solution). This can be achieved by imitating the usual proof of the pathwise uniqueness of a one-dimensional SDE with non-Lipschitz coefficient. We give the details here for completeness. As before we take $s = 0$ for notational brevity. Define $a_0 := 1$ and $a_k := a_{k-1}e^{-k}$ for $k \geqslant 1$. Let ψ_k be a continuous function on $\mathbb{R}$ with

support in (a_k, a_{k-1}) such that $\int_{a_k}^{a_{k-1}} \psi_k(x)\mathrm{d}x = 1$ and $0 \leqslant \psi_k(x) \leqslant \frac{2}{kx}$ for $x \in (a_k, a_{k-1})$. Define

$$\phi_k(y) := \int_0^{|y|} \mathrm{d}x \int_0^x \psi_k(r)\mathrm{d}r, \qquad y \in \mathbb{R}.$$

Observe that ϕ_k is twice continuously differentiable and $|\phi_k'| \leqslant 1$. Moreover $\phi_k(y) \to |y|$ as $k \to \infty$. Suppose that $\mathcal{Y}^{(1)}_{0,t}(v)$ and $\mathcal{Y}^{(2)}_{0,t}(v)$, $t \geqslant 0$, are two solutions of (4.12) [with $s = 0$ there] with respect to the same martingale measure $\mathcal{M}^+$. Let $\Delta\mathcal{Y}_t := \mathcal{Y}^{(1)}_{0,t}(v) - \mathcal{Y}^{(2)}_{0,t}(v)$ for $t \geqslant 0$. By (4.6), we have $\int_0^t \mathcal{M}^+([0, \mathcal{Y}^{(i)}_{0,r}(v)], \mathrm{d}r) = \int_0^t \big(\widetilde{\mathcal{W}}([0, \mathcal{Y}^{(i)}_{0,r}(v)], \mathrm{d}r) - \mathcal{Y}^{(i)}_{0,r}(v)\widetilde{\mathcal{W}}([0,1], \mathrm{d}r)\big)$ for $i = 1, 2$. Then

$$\Delta\mathcal{Y}_t = 2\int_{r=0}^t \int_{s=0}^1 \big[(1_{\{s \leqslant \mathcal{Y}^{(1)}_{0,r}(v)\}} - \mathcal{Y}^{(1)}_{0,r}(v)) - (1_{\{s \leqslant \mathcal{Y}^{(2)}_{0,r}(v)\}} - \mathcal{Y}^{(2)}_{0,r}(v))\big]\widetilde{\mathcal{W}}(\mathrm{d}s, \mathrm{d}r) - (\delta + \delta')\int_0^t \Delta\mathcal{Y}_r \mathrm{d}r.$$

Since $0 \leqslant \mathcal{Y}^{(1)}_{0,r}(v), \mathcal{Y}^{(2)}_{0,r}(v) \leqslant 1$, $\mathrm{d}\langle\Delta\mathcal{Y}\rangle_r = 4\mathrm{d}r \int_{s=0}^1 \big[(1_{\{s \leqslant \mathcal{Y}^{(1)}_{0,r}(v)\}} - \mathcal{Y}^{(1)}_{0,r}(v)) - (1_{\{s \leqslant \mathcal{Y}^{(2)}_{0,r}(v)\}} - \mathcal{Y}^{(2)}_{0,r}(v))\big]^2 \mathrm{d}s$ and we observe that $\int_{s=0}^1 \big|(1_{\{s \leqslant \mathcal{Y}^{(1)}_{0,r}(v)\}} - \mathcal{Y}^{(1)}_{0,r}(v)) - (1_{\{s \leqslant \mathcal{Y}^{(2)}_{0,r}(v)\}} - \mathcal{Y}^{(2)}_{0,r}(v))\big|^2 \mathrm{d}s \leqslant 2|\Delta\mathcal{Y}_r|$. Applying Itô's formula to $\phi_k(\Delta\mathcal{Y}_t)$ gives that

$$\begin{aligned}
\mathbb{E}(\phi_k(\Delta\mathcal{Y}_t)) &= -(\delta + \delta')\,\mathbb{E}\int_0^t \phi_k'(\Delta\mathcal{Y}_r)\,\Delta\mathcal{Y}_r \mathrm{d}r + \frac{1}{2}\mathbb{E}\int_0^t \psi_k(|\Delta\mathcal{Y}_r|)\mathrm{d}\langle\Delta\mathcal{Y}\rangle_r \\
&\leqslant |\delta + \delta'|\,\mathbb{E}\int_0^t |\Delta\mathcal{Y}_r|\mathrm{d}r + \frac{8t}{k},
\end{aligned}$$

where in the last inequality we have used the facts that $|\phi_k'| \leqslant 1$ and $\psi_k(x)x \leqslant \frac{2}{k}$ for any $x \geqslant 0$. Letting $k \to \infty$ we deduce from Fatou's lemma that $\mathbb{E}(|\Delta\mathcal{Y}_t|) \leqslant |\delta + \delta'|\,\mathbb{E}\int_0^t |\Delta\mathcal{Y}_r|\mathrm{d}r$, yielding that $\Delta\mathcal{Y}_t = 0$ by Gronwall's inequality. This shows the pathwise uniqueness and completes the proof of the theorem. □

Let $\delta > 0$. Recall that X is a PRBM defined in (1.2) and S is its local time flow introduced in (2.2). Recall that S satisfies the SDEs (2.3) and (2.4) driven by W. The following corollary of Theorem 4.6 gives a pathwise construction of a Jacobi flow starting from a PRBM.

Corollary 4.7. *Let $b > 0$ and $\mu > 0$.*

1) *Let $f = (f(x),\, x \geqslant 0)$ defined by $f(x) = S_{0,x}(b) = L(\tau_b^0, x)$. Notice that $\mathfrak{d}_f = \inf\{x \geqslant 0 : L(\tau_b^0, x) = 0\}$. We have $\mathfrak{d}_f < \infty$ if and only if $\mu > 1$. Let $\Theta := (S_{r,x}(a),\, 0 \leqslant r \leqslant x < \mathfrak{d}_f,\, a \in [0, f(r)])$. The flow $\Psi(\Theta, f)$ is a* Jacobi$(2/\mu, 0)$ *flow in the positive time-axis, independent of f.*
2) *Let $f = (f(x),\, x \geqslant 0)$ defined by $f(x) = S_{0,-x}(b) = L(\tau_b^0, -x)$. Notice that $\mathfrak{d}_f = -\inf_{[0,\tau_b^0]} X$. Let $\Theta := (S_{-r,-x}(a),\, 0 \leqslant r \leqslant x < \mathfrak{d}_f,\, a \in [0, f(r)])$. The flow $\Psi(\Theta, f)$ is a killed* Jacobi$(2 - 2/\mu, 0)$ *flow in the positive time-axis, independent of f.*

Remark 4.8. We can rephrase the corollary in the following way. Let $b > 0$. Define the flow $Y = (Y_{s,t}(v), -\infty < s, t < \infty, v \in [0,1])$ by $Y := \Psi(\mathcal{L}_X^b, f)$ where $f(x) = S_{0,x}(b)$ and $\mathcal{L}_X^b$ is the local time flow of $(X_t,\, t \leqslant \tau_b^0)$. The forward flow of Y in the positive time-axis (i.e. $(Y_{s,t}(v), 0 \leqslant s \leqslant t,\, v \in [0,1])$) is a Jacobi$(2/\mu, 0)$ flow. Similarly, the backward

flow of Y in the negative time-axis (i.e. $(Y_{-s,-t}(v), 0 \leqslant s \leqslant t,\, v \in [0,1])$) is a killed Jacobi$(2-2/\mu, 0)$ flow. See Figure 2.

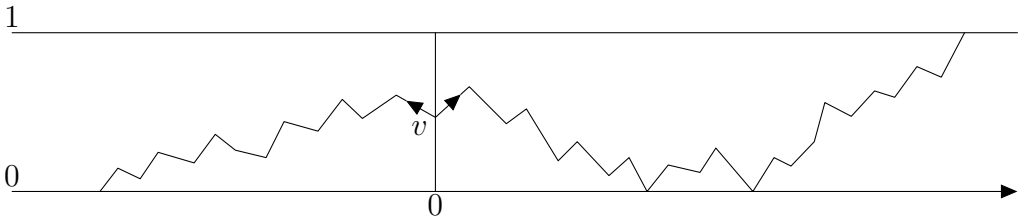

FIGURE 2. The flow Y. A forward flow line in the positive time-axis is a Jacobi$(2/\mu, 0)$ process. A backward flow line in the negative time-axis is a Jacobi$(2-2/\mu, 0)$ process absorbed at 0.

For later use, let for all $s \leqslant t \in \mathbb{R}$, and $v \in [0,1]$,

$$Y^*_{s,t}(v) := \inf\{u \in [0,1] \,:\, Y_{-t,-s}(u) > v\} \wedge 1.$$

Lemma 4.9. *For all $s, t \in \mathbb{R}$, and $v \in [0,1]$, $Y_{s,t}(v) = \inf\{u \in [0,1] \,:\, Y_{t,s}(u) > v\} \wedge 1$. In particular, for $s \leqslant t$ and $v \in [0,1]$, $Y^*_{s,t}(v) = Y_{-s,-t}(v)$.*

Proof. Let $s, t \in \mathbb{R}$. The case $v = 1$ is immediate so we can suppose $v < 1$. By definition of Y, we have

$$Y_{s,t}(v) = \frac{S_{\eta_f^{-1}(s),\eta_f^{-1}(t)}(vS_{0,\eta_f^{-1}(s)}(b))}{S_{0,\eta_f^{-1}(t)}(b)}.$$

By the duality (2.6) for S,

$$S_{\eta_f^{-1}(s),\eta_f^{-1}(t)}(c) = \inf\{a \geqslant 0 \,:\, S_{\eta_f^{-1}(t),\eta_f^{-1}(s)}(a) > c\}.$$

Taking $c = vS_{0,\eta_f^{-1}(s)}(b)$, we get

$$\begin{aligned} Y_{s,t}(v) &= \frac{1}{S_{0,\eta_f^{-1}(t)}(b)} \inf\left\{a \geqslant 0 \,:\, \frac{S_{\eta_f^{-1}(t),\eta_f^{-1}(s)}(a)}{S_{0,\eta_f^{-1}(s)}(b)} > v\right\} \\ &= \inf\left\{u \geqslant 0 \,:\, \frac{S_{\eta_f^{-1}(t),\eta_f^{-1}(s)}(uS_{0,\eta_f^{-1}(t)}(b))}{S_{0,\eta_f^{-1}(s)}(b)} > v\right\} \end{aligned}$$

which is $\inf\{u \in [0,1] \,:\, Y_{t,s}(u) > v\}$ indeed. □

5. Disintegration of the PRBM with respect to its occupation field

5.1. **The burglars.** Let $b > 0$. Consider the PRBM X defined in (1.2) up to τ_b^0. Recall from (2.2) that $S_{r,x}(a) = L(\tau_a^r, x)$. Set

$$t_0^* := \sup\left\{t \leqslant 0 \,:\, L(\tau_b^0, X_t) = 0\right\}, \qquad t^* := \inf\left\{t \geqslant 0 \,:\, L(\tau_b^0, X_t) = 0\right\}.$$

Notice that t^* is almost surely the hitting time of $\inf_{[0,\tau_b^0]} X$ and by the Ray–Knight theorems, $|t_0^*| < \infty$ if and only if $\mu > 1$ (if $\mu \leqslant 1$, $\{t \leqslant 0 \,:\, L(\tau_1^0, X_t) = 0\} = \varnothing$ hence $t_0^* = -\infty$).

We let $f(x) := S_{0,x}(b) = L(\tau_b^0, x)$ restricted to the interval I with boundaries

$$\inf I := \inf_{[0,\tau_b^0]} X = X_{t^*} \qquad \sup I := \inf\{x \geqslant 0 : L(\tau_b^0, x) = 0\} \tag{5.1}$$

($\sup I = X_{t_0^*}$ when $|t_0^*| < \infty$ and ∞ otherwise). We consider the processes

$$X_s^{(1)} := X_s,\ s \in (t_0^*, t^*), \qquad X_s^{(2)} := X_{\tau_b^0 - s},\ s \in [0, \tau_b^0 - t^*).$$

Recall the notation Υ in Section 3.2. We introduce the processes

$$Z^{(1)} := \Upsilon(X^{(1)}, f), \qquad Z^{(2)} := \Upsilon(X^{(2)}, f), \tag{5.2}$$

see Figure 3. In the case when $\mu = 1$, $Z^{(2)}$ defines a variant of the Brownian burglar introduced by Warren and Yor in [34] (its time-change $\widehat{Z}^{(2)}$ in Section 5.3 is distributed as the one appearing in [33], compare Theorem 5.12 and Proposition 5.13 with Theorem 1 in [33]).

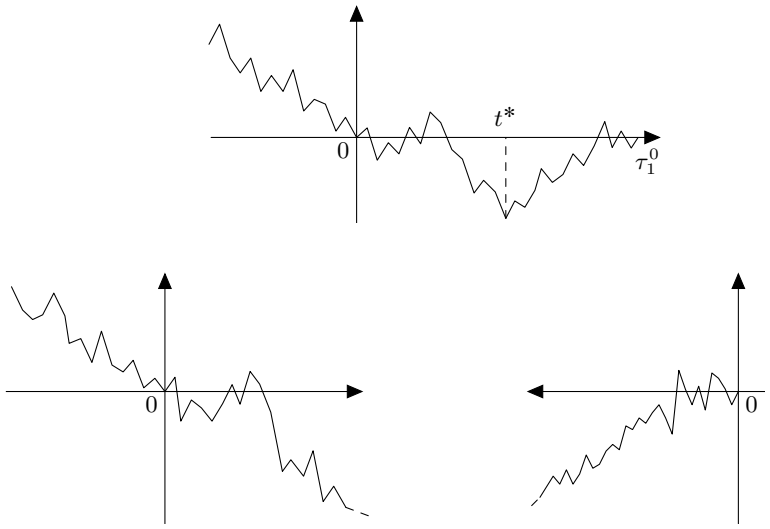

FIGURE 3. Illustration of the two burglars $Z^{(1)}$ and $Z^{(2)}$. Top: a PRBM with $\mu \in (0, 1]$ up to τ_1^0, with $t_0^* = -\infty$ and t^* being the time associated with the minimum. Bottom: the process before time t^* is mapped through a space-time transformation to the process $Z^{(1)}$ (left). That after time t^* is mapped to the process $Z^{(2)}$ (right).

Lemma 5.1. *Almost surely, the process $Z^{(1)}$ is defined on $\mathbb{R}$ with limits given by $\lim_{t\to-\infty} Z_t^{(1)} = +\infty$ and $\lim_{t\to\infty} Z_t^{(1)} = -\infty$; the process $Z^{(2)}$ is defined on $\mathbb{R}_+$ and $\lim_{t\to\infty} Z_t^{(2)} = -\infty$.*

Proof. We only prove it for $Z^{(1)}$. First we check that the transformation η_f maps the interval I defined in (5.1) onto $\mathbb{R}$. Consider $\sup \eta_f(I)$. When $\mu \leqslant 1$, $\sup I = \infty$ and $\lim_{y\to\infty} \eta_f(y) = \int_0^\infty \frac{\mathrm{d}r}{f(r)}$ which is infinite (Exercise X.3.20 in [29]). In the case $\mu > 1$, we need to show that $\int_0^{\sup I} \frac{\mathrm{d}r}{f(r)} = \infty$. By time reversal, $r \to f(\sup I - r)$ is a $\mathrm{BESQ}_0^{4-\delta}$ process, which, by scaling arguments, implies that the integral is infinite indeed. Consider

$\inf \eta_f(I)$. It is $\int_0^{\inf I} \frac{\mathrm{d}r}{f(r)}$ which is $-\infty$ by the same argument. Finally, we need to show that $\int_0^{t_0^*} \frac{\mathrm{d}s}{f(X_s^{(1)})^2} = -\infty$ and $\int_0^{t^*} \frac{\mathrm{d}s}{f(X_s^{(1)})^2} = \infty$. By the occupation time formula, $\int_0^{t_0^*} \frac{\mathrm{d}s}{f(X_s^{(1)})^2} = \int_{\inf I}^{\sup I} \frac{L(t_0^*,r)-L(0,r)}{f(r)^2}\mathrm{d}r = -\int_0^{\sup I} \frac{L(0,r)}{f(r)^2}\mathrm{d}r$. When $r < \sup I$ is close enough to $\sup I$, $L(0,r) = f(r)$, so that the integral is $-\infty$ indeed. Similarly,

$$\begin{aligned}\int_0^{t^*} \frac{\mathrm{d}s}{f(X_s^{(1)})^2} &= \int_{\inf I}^{\sup I} \frac{L(t^*,r)-L(0,r)}{f(r)^2}\mathrm{d}r \\ &\geqslant \int_{\inf I}^{0} \frac{L(t^*,r)-L(0,r)}{f(r)^2}\mathrm{d}r \\ &= \int_{\inf I}^{0} \frac{L(t^*,r)}{f(r)^2}\mathrm{d}r.\end{aligned}$$

On an event of probability arbitrarily close to 1 as $\varepsilon \to 0$, $L(t^*, \inf I + y)$ and $f(\inf I + y) - L(t^*, \inf I + y)$ on $[0, \varepsilon]$ are independent BESQ processes of respective dimensions δ and 2, starting at 0 (see Theorem 4.3 in [3]). Scaling arguments ensure again that the integral is infinite indeed. □

Using the notation Ψ of Section 3.1, define the process Ξ by

$$\Xi := \Psi(L(t^*, \cdot), f). \tag{5.3}$$

Recall from Remark 4.8 that Y denotes the flow $\Psi(\mathcal{L}_X^b, f)$. Its forward flow is a Jacobi$(2/\mu, 0)$ flow and its backward flow is a killed Jacobi$(2 - 2/\mu)$ flow. The following lemma shows that the process Ξ is measurable with respect to the flow Y.

Lemma 5.2. *Almost surely, for all $r \in \mathbb{R}$,*

$$\Xi_r = \inf\{v \in [0,1] : Y_{r,x}(v) = 1 \text{ for some } x < r\}. \tag{5.4}$$

Proof. By definition, $\Xi_r = \frac{L(t^*, \eta_f^{-1}(r))}{f(\eta_f^{-1}(r))}$, with $\eta_f(s) := \int_0^s \frac{\mathrm{d}r}{f(r)}$ for $s \in I$. By Definition 3.3, $Y_{r,x}(v) = \frac{1}{f(\eta_f^{-1}(x))} L(t_v, \eta_f^{-1}(x))$, with $t_v := \tau^{\eta_f^{-1}(r)}_{vf(\eta_f^{-1}(r))}$. We have $t_v > t^*$ for any $1 \geqslant v > \Xi_r$ and $\eta_f^{-1}(x) \to X_{t^*}$ as $x \to -\infty$. Observe that $Y_{r,x}(v) = 1$ for all x such that $\eta_f^{-1}(x) \leqslant \inf_{[t_v, \tau_b^0]} X$. On the other hand, if $v < \Xi_r$, then $t_v < t^*$ and $Y_{r,x}(v) < 1$ for all $x < r$. □

The law of the process $(\Xi_r, r \in \mathbb{R})$, called the "primitive Eve" process in Bertoin and Le Gall [8], will be given in Proposition 5.10.

Let $L^{(1)}(t,r)$, resp. $L^{(2)}(t,r)$, denote the local time of $Z^{(1)}$, resp. $Z^{(2)}$, at time t and position r. Observe from (3.2) that $L^{(1)}(\infty, r) + L^{(2)}(\infty, r) = 1$ for all $r \in \mathbb{R}$. On the other hand, (5.3) implies that $\Xi_r = L^{(1)}(\infty, r)$.

Theorem 5.3. *(i) The process $Z^{(1)}$ possesses local time flow Y "seen from the left", meaning that $L^{(1)}(\tau_v^r(Z^{(1)}), x) = Y_{r,x}(v)$ for $v \in [0, \Xi_r]$ and $r, x \in \mathbb{R}$.*

(ii) The process $Z^{(2)}$ possesses local time flow Y "seen from the right", meaning that $L^{(2)}(\tau_{u-}^r(Z^{(2)}), x) = 1 - Y_{r,x}(1-u)$ for $u \in [0, 1 - \Xi_r]$ and $r, x \in \mathbb{R}$.

(iii) The processes $Z^{(1)}$ and $Z^{(2)}$ are independent of $(L(\tau_b^0, x),\ x \in \mathbb{R})$ (and their distributions do not depend on the value of $b > 0$).

Proof. (i) Note that for any $r \in I$, $L_{X^{(1)}}(t, r) = L(t, r)$ for all $t \leqslant t^*$, hence the local time flow of $X^{(1)}$ is the flow $\mathcal{S}^1 := \{S_{r,x}(a),\ r, x \in I,\ 0 \leqslant a \leqslant L(t^*, r)\}$. The image flow by $\Psi(\cdot, f)$ is the flow Y "seen from the left" by definition.

(ii) Similarly for any $r \in I$, $L_{X^{(2)}}(t, r) = f(r) - L(\tau_b^0 - t, r)$ for all $0 \leqslant t \leqslant \tau_b^0 - t^*$. We deduce that the local time flow of $X^{(2)}$ is the flow $\{f(x) - S_{r,x}((f(r) - a)-),\ r, x \in I,\ 0 \leqslant a \leqslant f(r) - L(t^*, r)\}$. The image flow by $\Psi(\cdot, f)$ is the flow Y "seen from the right".

(iii) Again $Z^{(1)}$ and $Z^{(2)}$ are measurable with respect to their local time flow Y (by a proof similar to Proposition 2.5). By Lemma 4.9, the flow Y is measurable with respect to its forward flow. The forward flow of Y is independent of $(L(\tau_b^0, x),\ x \in \mathbb{R})$ by Theorem 4.6. It completes the proof. □

5.2. **A Markov property for the process $Z^{(2)}$.** The following proposition gives a way of constructing $Z^{(2)}$ by means of a Brownian motion stopped at a hitting time and an independent squared Bessel process.

Proposition 5.4. *Let $g < 0$ be a constant. Take a Brownian motion up to the hitting time of g (call it $B = (B_t, t \in [0, T_g^B])$). Take h such that $h(x) = 0$ if $x \leqslant g$ and $h(g + r)$, $r \geqslant 0$ is an independent* BESQ_0^δ. *Let $f(x) := L_B(T_g^B, x) + h(x)$, $x \in \mathbb{R}$ where L_B is the local time of the Brownian motion B. Then $\Upsilon(B, f)$ has the law of $Z^{(2)}$ and is independent of f.*

Proof. We construct a probability measure $\widetilde{\mathbb{P}}$ under which $X^{(1)}$ and $X^{(2)}$ have simple descriptions. The process B will stand for the process $X^{(2)}$ while h will stand for the local time process of $X^{(1)}$ under $\widetilde{\mathbb{P}}$ conditioned on some event (which is $\{J(A) = g\}$ in the notation below).

Let $m > |g|$ be an arbitrary constant (the exact value of m plays no role). We consider a probability $\widetilde{\mathbb{P}}$ on the product space $\mathbb{R}_+ \times C(\mathbb{R}, \mathbb{R})$ such that, under $\widetilde{\mathbb{P}}$, (A, X) has the following distribution: A is gamma$(\frac{\delta}{2}, 2m)$ distributed (meaning it has density $\frac{(2m)^{-\frac{\delta}{2}}}{\Gamma(\frac{\delta}{2})} x^{\frac{\delta}{2}-1} e^{-x/2m}$ on $\mathbb{R}_+$) and conditionally on $A = b$, X has distribution

$$c_b |J(b)|^{\frac{\delta}{2}-1} \mathbf{1}_{\{J(b) > -m\}} \cdot \mathbb{P}$$

where $J(b) := \inf_{t \in [0, \tau_b^0]} X_t$ and $c_b = \Gamma(\frac{\delta}{2}) \left(\frac{b}{2}\right)^{1-\frac{\delta}{2}} e^{\frac{b}{2m}}$ is the renormalizing constant. Notice that $J(b)$ is measurable with respect to $L(\tau_b^0, x)$, $x \in \mathbb{R}$. Let $X^{(1)}$, $X^{(2)}$, $Z^{(1)}$, $Z^{(2)}$ be under $\widetilde{\mathbb{P}}$ the processes of Section 5.1 with $b = A$. From Theorem 5.3 (iii), and using that $J(A)$ is measurable with respect to $(L(\tau_A^0, x), x \in \mathbb{R})$, we deduce that under $\widetilde{\mathbb{P}}$, $Z^{(2)}$ has the same law as under $\mathbb{P}$, and is independent of $(L(\tau_A^0, x), x \in \mathbb{R})$.

Denote by

$$T_r := \inf\{t \in \mathbb{R} :\ X_t = r\}, \tag{5.5}$$

the hitting time of $r \in \mathbb{R}$.

Lemma 5.5. *(i) Under $\widetilde{\mathbb{P}}(\cdot|J(A)=g)$, the law of the process $X^{(1)}$ is the one of $(X_s,\, s\leqslant T_g)$ under $\mathbb{P}$.*

(ii) Under $\widetilde{\mathbb{P}}(\cdot|J(A)=g)$, the law of the process $X^{(2)}$ is that of a Brownian motion stopped when hitting g.

(iii) Under $\widetilde{\mathbb{P}}(\cdot|J(A)=g)$, the two processes $X^{(1)}$ and $X^{(2)}$ are independent.

Proof of Lemma 5.5. These results can be deduced from [3]. First, the law of $(X_s,\, s\leqslant 0)$ is the same under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$, since $(X_s,\, s\leqslant 0)$ and $(X_s,\, s\geqslant 0)$ are independent. We restrict our attention to $(X_s,\, s\in[0,\tau_A^0])$. By Proposition 9.1 p.123, Section 9.2 of [36] and Corollary 3.4 in [3], we observe that $(A, X_s,\, s\in[0,\tau_A^0])$ under $\widetilde{\mathbb{P}}$ has the distribution of $(L(T_{-m},0), X_s,\, s\in[0,\mathsf{g}_m])$ under $\mathbb{P}$, where $\mathsf{g}_m:=\sup\{t\leqslant T_{-m}: X_t=0\}$. The lemma is then Theorem 3.2 of [3]. □

We can now complete the proof of Proposition 5.4. Let $g\in(-m,0)$. We recall that $Z^{(2)}$ is independent of $(L(\tau_A^0,x), x\in\mathbb{R})$ under $\widetilde{\mathbb{P}}$ so that the law of $Z^{(2)}$ is identical under $\widetilde{\mathbb{P}}(\cdot|J(A)=g)$ for any $g\in(-m,0)$ (and is equal to the law of $Z^{(2)}$ under $\mathbb{P}$). By definition, $Z^{(2)}=\Upsilon(X^{(2)},f)$ where $f(x):=L(\tau_A^0,x)$. The statement of the proposition comes from the description of X under $\widetilde{\mathbb{P}}(\cdot|J(A)=g)$ in Lemma 5.5. This holds for any $g<0$ since m is arbitrary. □

We state now the Markov property for $Z^{(2)}$. It will be the key ingredient in the forthcoming connection with the Bass–Burdzy flow. Let as before $L^{(2)}(t,x)$ denote the local time of $Z^{(2)}$ at time t and position x. Recall Notation 3.6.

Theorem 5.6. *For any $t\geqslant 0$, $\Upsilon(Z^{(2)}_{t+\cdot}, 1-L^{(2)}(t,\cdot))$ is independent of $(Z^{(2)}_r,\, r\in[0,s])$ and distributed as $Z^{(2)}$. In particular, $(Z^{(2)}_t, (L^{(2)}(t,x))_{x\in\mathbb{R}})$ is a Markov process.*

Before proving the theorem, we show that for fixed $t\geqslant 0$, the process $\Upsilon(Z^{(2)}_{t+\cdot}, 1-L^{(2)}(t,\cdot))$ is indeed well-defined.

Lemma 5.7. *For any fixed $t\geqslant 0$, the map $s\mapsto\Upsilon(Z^{(2)}_{s+\cdot}, 1-L^{(2)}(s,\cdot))$ is a.s. well-defined on a neighborhood (in $\mathbb{R}_+$) of t and is continuous at t (in the space $C(\mathbb{R}_+,\mathbb{R})$ of continuous processes on $\mathbb{R}_+$ endowed with the usual topology of uniform convergence over all compacts).*

See Appendix C for the proof of Lemma 5.7.

Proof of the theorem. Take B and f as in Proposition 5.4, with, say, $g=-1$. Recall the notation $C_f(t):=\int_0^t\frac{\mathrm{d}s}{f(B_s)^2}$. Set $Z^{(2)}:=\Upsilon(B,f)$. Applying Lemma 3.5 to $\mathcal{X}=B$, $s\in J=[0,T^B_{-1}), I=(-1,\max_{0\leqslant t\leqslant T^B_{-1}}B_t]$ and $c'=B_s$, we have

$$\Upsilon(B_{s+\cdot}, f-L_B(s,\cdot))=\Upsilon(Z^{(2)}_{C_f(s)+\cdot}, 1-L^{(2)}(C_f(s),\cdot)).$$

On the other hand, by Proposition 5.4, the Markov property of the Brownian motion at time s and Notation 3.6 (i), conditionally on $\{s<T^B_{-1}\}$, the left-hand side is distributed as $Z^{(2)}$ and is independent of $f-L_B(s,\cdot)$ and of $(B_r,\, r\in[0,s])$.

It follows that, conditionally on $\{s < T^B_{-1}\}$, $\Upsilon(Z^{(2)}_{C_f(s)+\cdot}, 1 - L^{(2)}(C_f(s), \cdot))$ is independent of $\sigma\{f, B_r,\, r \in [0,s]\}$ and has the law of $Z^{(2)}$. Using the continuity of $s \to \Upsilon(Z^{(2)}_{s+\cdot}, 1 - L^{(2)}(s,\cdot))$ stated in Lemma 3.7, and since $C_f(s)$ and $(Z^{(2)}_t,\, t \leqslant C_f(s))$ are $\sigma\{f, B_r,\, r \in [0,s]\}$-measurable, we deduce that for all $t \geqslant 0$, $\Upsilon(Z^{(2)}_{t+\cdot}, 1 - L^{(2)}(t,\cdot))$ is independent of $(Z^{(2)}_r,\, r \in [0,t])$ and is distributed as $Z^{(2)}$. □

Remark 5.8. As a consequence of Theorem 5.6, if T is a stopping time with respect to the natural filtration of $Z^{(2)}$ such that almost surely $s \to \Upsilon(Z^{(2)}_{s+\cdot}, 1 - L^{(2)}(s,\cdot))$ is continuous at T in $C(\mathbb{R}_+, \mathbb{R})$, then $\Upsilon(Z^{(2)}_{T+\cdot}, 1 - L^{(2)}(T,\cdot))$ is independent of $(Z^{(2)}_r,\, r \in [0,T])$ and distributed as $Z^{(2)}$.

The next result gives an invariance principle for $Z^{(2)}$ at small times.

Theorem 5.9. *The process* $\left(\frac{1}{\sqrt{a}} Z^{(2)}_{at},\, t \geqslant 0\right)$ *converges in distribution as* $a \downarrow 0$ *to a standard Brownian motion.*

Proof. We use again the representation $Z^{(2)} = \Upsilon(B, f)$ from Proposition 5.4 with, say, $g = -1$. By definition of Υ in Section 3.2, $Z^{(2)}_t = \eta_f(B_{C_f^{-1}(t)})$ with $\eta_f(x) = \int_0^x \frac{\mathrm{d}r}{f(r)}$, $C_f(t) = \int_0^t \frac{\mathrm{d}s}{f(B_s)^2}$. Let $a > 0$. Define $\widetilde{f}(r) := f(r\sqrt{a})$, $\widetilde{B}_t := \frac{1}{\sqrt{a}} B_{at}$, $\widetilde{Z}^{(2)}_t := \frac{1}{\sqrt{a}} Z^{(2)}_{at}$. We check that $\eta_{\widetilde{f}}(\widetilde{B}_t) = \frac{1}{\sqrt{a}} \eta_f(B_{at})$ which is $\frac{1}{\sqrt{a}} Z^{(2)}_{C_f(at)}$. Moreover $C_{\widetilde{f}}(t) = \int_0^t \frac{\mathrm{d}s}{\widetilde{f}(\widetilde{B}_s)^2} = \frac{1}{a} C_f(at)$ so that $\eta_{\widetilde{f}}(\widetilde{B}_t) = \frac{1}{\sqrt{a}} Z^{(2)}_{aC_{\widetilde{f}}(t)}$. We proved that $\Upsilon(\widetilde{B}, \widetilde{f}) = \widetilde{Z}^{(2)}$. By scaling $\widetilde{B}$ is a standard Brownian motion stopped when hitting $-\frac{1}{\sqrt{a}}$. Moreover, $\widetilde{f}$ converges when $a \downarrow 0$ to the constant function $f(0)$, and $\widetilde{B}$ and $f(0)$ are asymptotically independent. Use Lemma 3.7 (together with Skorokhod's representation theorem to suppose that the convergence of $(\widetilde{B}, \widetilde{f})$ is almost sure instead of in distribution) to complete the proof. □

We end this subsection by describing the law of the process $(\Xi_r)_{r \in \mathbb{R}}$ defined in (5.4). Let $a, b > 0$. Denote by $\mathrm{beta}(a,b)$ the distribution with density $\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \mathbf{1}_{(0,1)}(x)$.

Proposition 5.10. *The random variable* Ξ_0 *is distributed as* $\mathrm{beta}(\delta/2, 1)$. *Conditionally on* Ξ_0, $(\Xi_{-t})_{t \geqslant 0}$ *and* $(\Xi_t)_{t \geqslant 0}$ *are independent* $\mathrm{Jacobi}(\delta, 2)$ *and* $\mathrm{Jacobi}(\delta, 0)$ *processes.*

Proof. Using the representation $Z^{(2)} = \Upsilon(B, f)$ in Proposition 5.4 with $g = -1$, we deduce from (3.2) that $L^{(2)}(t,x) = \frac{L^B(C_f^{-1}(t), \eta_f^{-1}(x))}{f(\eta_f^{-1}(x))}$, with $C_f(t) := \int_0^t \frac{\mathrm{d}s}{f(B_s)^2}$, $\eta_f(x) := \int_0^x \frac{\mathrm{d}r}{f(r)}$. Note that f is a $\mathrm{BESQ}_0^{2+\delta}$ process on $[-1, 0]$, and a BESQ^δ on $\mathbb{R}_+$. Since $\mathfrak{d}_f := \inf\{x > 0 : f(x) = 0\} = \inf\{x > \sup_{0 \leqslant s \leqslant T^B_{-1}} B_s : h(x) = 0\}$, where we recall that h is a BESQ^δ on $[-1, \infty)$ with $h(-1) = 0$, independent of B, $C_f^{-1}(t) \to T^B_{-1}$ as $t \to \infty$ and $L^{(2)}(\infty, x) = \frac{L^B(T^B_{-1}, \eta_f^{-1}(x))}{f(\eta_f^{-1}(x))}$ for $x \in \mathbb{R}$. By the classical Ray–Knight theorem, $(L^B(T^B_{-1}, y), y \geqslant -1)$ is a BESQ_0^2 on $[-1, 0]$, and BESQ^0 on $[0, \infty)$. Since $f(y) = L^B(T^B_{-1}, y) + h(y)$, we deduce that $L^{(2)}(\infty, 0) = \frac{L^B(T^B_{-1}, 0)}{f(0)}$ is distributed as $\mathrm{beta}(1, \delta/2)$. By Warren and Yor [34], $L^{(2)}(\infty, \cdot)$

is a Jacobi process of parameters $(0,\delta)$ on $\mathbb{R}_+$ starting from $L^{(2)}(\infty,0)$. Let $\varepsilon\in(0,1)$. Again by Warren and Yor [34], $(L^{(2)}(\infty,\eta_f(-1+\varepsilon)+x),0\leqslant x\leqslant -\eta_f(-1+\varepsilon))$ is a Jacobi process of parameters $(2,\delta)$. Recall that η_f is independent of $L^{(2)}(\infty,\cdot)$. Observe that $L^{(2)}(\infty,\eta_f(-1+\varepsilon))$ is still distributed as beta$(1,\delta/2)$ which is in fact the stationary distribution of a Jacobi process of parameters $(2,\delta)$. By time-reversal, $(L^{(2)}(\infty,-x),0\leqslant x\leqslant -\eta_f(-1+\varepsilon))$ is a Jacobi process of parameters $(2,\delta)$. Let $\varepsilon\to 0$ we get the proposition by using the fact that $\Xi_r=1-L^{(2)}(\infty,r)$ for all $r\in\mathbb{R}$. □

5.3. **Link with the Bass–Burdzy flow and proof of Theorem III.** Let β_1, $\beta_2\in\mathbb{R}$, $\sigma>0$ and γ be a standard Brownian motion. Following [6], one defines the Bass–Burdzy flow of parameters (β_1,β_2) and diffusivity σ as the collection of homeomorphisms of the real line $(\mathcal{R}_t,\,t\geqslant 0)$ such that for any $x\in\mathbb{R}$, the process $(\mathcal{R}_t(x),\,t\geqslant 0)$ is the strong solution of the SDE

$$\mathcal{R}_t(x):=x+\sigma\gamma_t+\beta_1\int_0^t\mathbf{1}_{\{\mathcal{R}_s(x)<0\}}\mathrm{d}s+\beta_2\int_0^t\mathbf{1}_{\{\mathcal{R}_s(x)>0\}}\mathrm{d}s.$$

When $\sigma=1$, we just call it Bass–Burdzy flow with parameters (β_1,β_2) as defined in the introduction.

We continue to look at the process $Z^{(2)}$ and show that it is linked to the Bass–Burdzy flow via a time-change, extending the result in [33] to all parameters of the PRBM. Using the same time-change as in [33], we set

$$\zeta(t):=\int_0^t\frac{\mathrm{d}s}{(1-L^{(2)}(s,Z^{(2)}_s))^2},\qquad \widehat{Z}^{(2)}_t:=Z^{(2)}_{\zeta^{-1}(t)}. \tag{5.6}$$

Lemma 5.11. *We have* $\lim_{t\to t_{\max}}\zeta(t)=\infty$ *where* $t_{\max}:=\inf\{s\geqslant 0\,:\,L^{(2)}(s,Z^{(2)}_s)=1\}\in(0,\infty]$ *(it is finite almost surely if* $\mu>1$ *and infinite otherwise). Consequently, if* $t_{\max}<\infty$, $\lim_{t\to\infty}\widehat{Z}^{(2)}_t=Z^{(2)}_{t_{\max}}$, *and if* $t_{\max}=\infty$, $\lim_{t\to\infty}\widehat{Z}^{(2)}_t=-\infty$.

Proof. It suffices to treat the case $\mu>1$. Recall the construction of $Z^{(2)}$ in Proposition 5.4 and recall from (3.2) that $L^{(2)}(t,x)=\frac{L_B(C_f^{-1}(t),\eta_f^{-1}(x))}{f(\eta_f^{-1}(x))}$, with $C_f(t):=\int_0^t\frac{\mathrm{d}s}{f(B_s)^2}$, $\eta_f(x):=\int_0^x\frac{\mathrm{d}r}{f(r)}$ and $f(x)=L_B(T_g^B,x)+h(x)$ where $h(g+\cdot)$ is a BESQ_0^δ independent of B. Let $\widetilde{t}_{\max}:=C_f^{-1}(t_{\max})$ which is $\inf\{s\in[0,T_g^B]\,:\,L_B(s,B_s)=f(B_s)\}$. Using the occupation time formula, we can express $\zeta(t_{\max})$ as

$$\zeta(t_{\max})=\int_{\mathbb{R}}\frac{L_B(\widetilde{t}_{\max},y)}{f(y)(f(y)-L_B(\widetilde{t}_{\max},y))}\mathrm{d}y.$$

Let $y_0:=B_{\widetilde{t}_{\max}}$. We observe that y_0 is the last zero of h before $M:=\max_{[0,T_g^B]}B$ and $\widetilde{t}_{\max}$ is the last passage time of B at y_0 before T_g^B. Hence, for any $y\geqslant y_0$, $L_B(\widetilde{t}_{\max},y)=L_B(T_g^B,y)$ so that

$$\zeta(t_{\max})\geqslant\int_{y_0}^\infty\frac{L_B(T_g^B,y)}{f(y)(f(y)-L_B(T_g^B,y))}\mathrm{d}y=\int_{y_0}^\infty\frac{L_B(T_g^B,y)}{f(y)h(y)}\mathrm{d}y.$$

Notice that in a neighborhood of y_0, the numerator is positive and f is bounded from above while $\int_{y_0}^{y_0+\varepsilon} \frac{\mathrm{d}y}{h(y)} = \infty$ for all ε positive.[6] $\square$

The local time of $\widehat{Z}^{(2)}$ can be computed since

$$\begin{aligned}\int_0^t \mathbf{1}_{\{\widehat{Z}_s^{(2)}<x\}}\mathrm{d}s &= \int_0^{\zeta^{-1}(t)} \mathbf{1}_{\{Z_r^{(2)}<x\}} \frac{\mathrm{d}r}{(1-L^{(2)}(r,Z_r^{(2)}))^2} \\ &= \int_{-\infty}^x \mathrm{d}y \int_0^{\zeta^{-1}(t)} \frac{\mathrm{d}_r L^{(2)}(r,y)}{(1-L^{(2)}(r,y))^2} \\ &= \int_{-\infty}^x \mathrm{d}y \frac{L^{(2)}(\zeta^{-1}(t),y)}{1-L^{(2)}(\zeta^{-1}(t),y)}\end{aligned}$$

which proves, with natural notation, that $\widehat{L}^{(2)}(t,y) = \frac{L^{(2)}(\zeta^{-1}(t),y)}{1-L^{(2)}(\zeta^{-1}(t),y)}$. Consequently,

$$\zeta^{-1}(t) = \int_0^t (1-L^{(2)}(\zeta^{-1}(r),\widehat{Z}_r^{(2)}))^2\mathrm{d}r = \int_0^t \frac{\mathrm{d}r}{(1+\widehat{L}^{(2)}(r,\widehat{Z}_r^{(2)}))^2}.$$

It implies ζ^{-1} is adapted with respect to the natural filtration $\widehat{\mathcal{F}}^{(2)}$ of $\widehat{Z}^{(2)}$. We also introduce for $x \in \mathbb{R}$,

$$R_t(x) := \int_{Z^{(2)}_{\zeta^{-1}(t)}}^x \frac{\mathrm{d}y}{1-L^{(2)}(\zeta^{-1}(t),y)} = \int_{\widehat{Z}_t^{(2)}}^x \left(1+\widehat{L}^{(2)}(t,y)\right)\mathrm{d}y. \tag{5.7}$$

Observe that $x = \widehat{Z}_t^{(2)}$ if and only if $R_t(x) = 0$. Note that in the case $\mu > 1$, the flow $(R_t(x),\, x \in \mathbb{R})$ only gives the reconstruction of the burglar $Z^{(2)}_\cdot$ until the finite time $t_{\max}$.

Theorem 5.12. *The process $(R_t(x),\, x \in \mathbb{R})$ is a Bass–Burdzy flow for some parameters β_1 and β_2 and diffusivity σ. Moreover $\beta_2 = \beta_1 + 1$.*

Proof. First we note that a.s. for all $t \geqslant 0$,

$$H_t := R_t(x) - x - \int_0^t \mathbf{1}_{\{R_s(x)>0\}}\mathrm{d}s, \qquad x \in \mathbb{R}, \tag{5.8}$$

is well defined. In other words, the right-hand side of (5.8) does not depend on x. This can be seen as follows:

$$\int_0^t \mathbf{1}_{\{R_s(x)>0\}}\mathrm{d}s = \int_0^t \mathbf{1}_{\{\widehat{Z}_s^{(2)}<x\}}\mathrm{d}s = \int_{-\infty}^x \widehat{L}^{(2)}(t,y)\mathrm{d}y$$

showing that the derivative with respect to x of the right-hand side of (5.8) vanishes.

Clearly (H_t) is $(\widehat{\mathcal{F}}_t^{(2)})$-adapted. We are going to show that (H_t) is an $(\widehat{\mathcal{F}}_t^{(2)})$-Lévy process. As we shall see below, this boils down to showing the strong Markov property

[6] For example, we can express h in terms of the square of a Bessel meander whose law is absolutely continuous with respect to BESQ, see Section 3.6 in [36].

for R_t. We use equation (3.3) with $\mathcal{X} := Z^{(2)}_{\zeta^{-1}(s)+\cdot}$, $f(r) = 1 - L^{(2)}(\zeta^{-1}(s), r)$, $c = \widehat{Z}^{(2)}_s$, $c' = x$ and $\zeta^{-1}(t+s) - \zeta^{-1}(s)$ in lieu of s to get that

$$\int_{R_s(x)}^{\widetilde{Z}^{(2)}_{\tilde{t}}} \frac{\mathrm{d}r}{1 - \widetilde{L}^{(2)}(\tilde{t}, r)} = \int_x^{Z^{(2)}_{\zeta^{-1}(s+t)}} \frac{\mathrm{d}r}{1 - L^{(2)}(\zeta^{-1}(s+t), r)} \tag{5.9}$$

where we used the observation that $\eta_{f,c}(x) = R_s(x)$, the process $\widetilde{Z}^{(2)}$ is defined by

$$\widetilde{Z}^{(2)} := \Upsilon(Z^{(2)}_{\zeta^{-1}(s)+\cdot}, 1 - L^{(2)}(\zeta^{-1}(s), \cdot), Z^{(2)}_{\zeta^{-1}(s)}) \equiv \Upsilon(Z^{(2)}_{\zeta^{-1}(s)+\cdot}, 1 - L^{(2)}(\zeta^{-1}(s), \cdot)),$$

and $\widetilde{L}^{(2)}$ is the local time of the process $\widetilde{Z}^{(2)}$, $\tilde{t} = C_f(\zeta^{-1}(t+s) - \zeta^{-1}(s))$ with

$$C_f(v) := \int_0^v \frac{\mathrm{d}u}{(1 - L^{(2)}(\zeta^{-1}(s), Z^{(2)}_{\zeta^{-1}(s)+u}))^2}.$$

We get

$$\tilde{t} = \int_{\zeta^{-1}(s)}^{\zeta^{-1}(s+t)} \frac{\mathrm{d}u}{(1 - L^{(2)}(\zeta^{-1}(s), Z^{(2)}_u))^2}.$$

We can rewrite (5.9) as

$$\widetilde{R}_t(R_s(x)) = R_{t+s}(x)$$

with

$$\widetilde{R}_t(y) = \int_{\widetilde{Z}^{(2)}_{\tilde{t}}}^{y} \frac{\mathrm{d}r}{1 - \widetilde{L}^{(2)}(\widetilde{t}, r)}.$$

Let $\widetilde{\zeta}$ denote the ζ associated with $\widetilde{Z}^{(2)}$, i.e.,

$$\widetilde{\zeta}(v) := \int_0^v \frac{\mathrm{d}u}{(1 - \widetilde{L}^{(2)}(u, \widetilde{Z}^{(2)}_u))^2}.$$

Suppose for the time being that we know that

$$\widetilde{t} = \widetilde{\zeta}^{-1}(t). \tag{5.10}$$

By Remark 5.8 applied to the stopping time $\zeta^{-1}(s)$, $\widetilde{Z}^{(2)}$ is distributed as $Z^{(2)}$ and is independent of $\sigma\{Z^{(2)}_r, r \leqslant \zeta^{-1}(s)\}$, thus independent of $\widehat{\mathcal{F}}^{(2)}_s$. It implies that conditionally on $\widehat{\mathcal{F}}^{(2)}_s$ and $R_s(x) = y$, the process $(R_{t+s}(x))_{t\geqslant 0}$ has the law of $(R_t(y))_{t\geqslant 0}$. It follows that conditionally on $\widehat{\mathcal{F}}^{(2)}_s$ and $R_s(x) = y$, for all $t \geqslant 0$, $H_{t+s} - H_s = \widetilde{R}_t(y) - y - \int_0^t 1_{\{\widetilde{R}_s(y)>0\}} \mathrm{d}s = \widetilde{H}_t$, where and $\widetilde{H}$ is defined from $\widetilde{R}$ as H is from R. In other words, $H_{t+s} - H_s, t \geqslant 0$ is independent of $\widehat{\mathcal{F}}^{(2)}_s$ and has the same distribution as H. Then H is a (continuous) Lévy process hence of the form $\sigma\gamma_t + \beta_1 t$ with γ an $(\widehat{\mathcal{F}}^{(2)}_t)$-Brownian motion, which yields the theorem.

It remains to prove (5.10). We know from equation (3.2) that

$$\widetilde{L}^{(2)}(u, \widetilde{Z}^{(2)}_u) = \frac{L^{(2)}(C_f^{-1}(u), Z^{(2)}_{\zeta^{-1}(s)+C_f^{-1}(u)}) - L^{(2)}(\zeta^{-1}(s), Z^{(2)}_{\zeta^{-1}(s)+C_f^{-1}(u)})}{1 - L^{(2)}(\zeta^{-1}(s), Z^{(2)}_{\zeta^{-1}(s)+C_f^{-1}(u)})}.$$

Hence, by definition of $\widetilde{\zeta}$ and a change of variables,

$$\widetilde{\zeta}(\tilde{t}) = \int_0^{\tilde{t}} \frac{\mathrm{d}u}{(1-\widetilde{L}^{(2)}(u,\widetilde{Z}_u^{(2)}))^2} = \int_{\zeta^{-1}(s)}^{\zeta^{-1}(t+s)} \frac{\mathrm{d}u}{(1-L^{(2)}(u,Z_u^{(2)}))^2} = t$$

by definition of ζ, which completes the proof. □

We identify the parameters of the Bass–Burdzy flow in the following proposition.

Proposition 5.13. *The parameters of the Bass–Burdzy flow are* $\sigma = 1$, $\beta_1 = \frac{\delta}{2} - 1$ *and* $\beta_2 = \frac{\delta}{2}$.

Proof. We first show that $\sigma = 1$. Recall that $\widehat{Z}_t^{(2)} = Z^{(2)}_{\zeta^{-1}(t)}$. Since $\frac{1}{t}\zeta(t) \xrightarrow{t\to 0} 1$, one can deduce from Theorem 5.9 that $\left(\frac{1}{\sqrt{a}}\widehat{Z}^{(2)}_{at}, t \geqslant 0\right)$ converges in distribution as $a \downarrow 0$ to a standard Brownian motion. Therefore, the hitting time of x by $\widehat{Z}^{(2)}$, divided by x^2, converges in law when $x \downarrow 0$ to the hitting time of 1 by a standard Brownian motion. But the hitting time of x by $\widehat{Z}^{(2)}$ is the hitting time of 0 by $R_t(x)$. We deduce that necessarily $\sigma = 1$.

We now determine β_2 (or equivalently β_1 since we know that $\beta_2 = 1+\beta_1$). By definition of the Bass–Burdzy flow, if we start the flow $R_t(x)$ from $x^* = \sup_{t\geqslant 0} \widehat{Z}_t^{(2)}$, then $R_t(x^*) \geqslant 0$ for all $t \geqslant 0$ and $R_t(x^*) = x^* + \gamma_t + \beta_2 t$. Note that the unique zero is attained when $\widehat{Z}_t^{(2)}$ reaches its maximum x^*, that means

$$x^* = -\inf_{t\geqslant 0}(\gamma_t + \beta_2 t).$$

It implies that $\beta_2 > 0$ and x^* is exponentially distributed with parameter $2\beta_2$. On the other hand, observe that $x^* = \sup_{t\geqslant 0} Z_t^{(2)}$, so that, recalling from Section 5.1 that $\Xi_r = L^{(1)}(\infty, r) = 1 - L^{(2)}(\infty, r)$,

$$x^* := \inf\{r \geqslant 0 \,:\, \Xi_r = 1\}.$$

In other words, by Proposition 5.10, it is the hitting time of 0 by a Jacobi process with parameters $(0,\delta)$ with initial distribution beta$(1, \frac{\delta}{2})$. The proof will be complete once we prove that this hitting time is exponentially distributed with parameter δ. Let us prove it. Under some probability P^x, denote by V a Jacobi$(0,\delta)$ process starting from x, and let T_0 be its hitting time of 0. We can check that $M_t := \frac{V_t}{x} e^{\delta t}$ is a martingale. We can then define the probability measure Q^x with Radon–Nikodym derivative M_t with respect to P^x on the σ-algebra $\sigma(V_s,\, s \in [0,t])$. Under Q^x, the process V is a Jacobi$(4,\delta)$ process. We have $P^x(T_0 > t) = e^{-\delta t} E_{Q^x}\left[\frac{x}{V_t}\right]$. Taking x with the beta$(1,\frac{\delta}{2})$ distribution, and setting $P := \int_0^1 \frac{\delta}{2}(1-x)^{\frac{\delta}{2}-1} P^x \mathrm{d}x$, we get

$$\begin{aligned} P(T_0 > t) &= e^{-\delta t} \int_0^1 \frac{\delta}{2}(1-x)^{\frac{\delta}{2}-1} E_{Q^x}\left[\frac{x}{V_t}\right] \mathrm{d}x \\ &= e^{-\delta t} \int_0^1 \frac{\delta}{2}(1-x)^{\frac{\delta}{2}-1} x\, E_{Q^x}\left[\frac{1}{V_t}\right] \mathrm{d}x. \end{aligned}$$

Since beta$(2, \frac{\delta}{2})$ is the invariant distribution of a Jacobi$(4, \delta)$ process, one has

$$\int_0^1 \frac{\delta}{2}(1-x)^{\frac{\delta}{2}-1} x\, E_{Q^x}\left[\frac{1}{V_t}\right] \mathrm{d}x = \int_0^1 \frac{\delta}{2}(1-x)^{\frac{\delta}{2}-1} x\, E_{Q^x}\left[\frac{1}{V_0}\right] \mathrm{d}x = 1.$$

Hence $P(T_0 > t) = e^{-\delta t}$ so T_0 is indeed exponentially distributed with parameter δ. □

Remark. In light of Theorem III, Theorem 4.6 gives a flow version of the Ray–Knight theorems appearing in [18].

5.4. **Contour function of a Fleming–Viot forest.** Let $\delta > 0$ and $b > 0$. Let X be the PRBM associated with $\mu = \frac{2}{\delta}$, which we consider up to time τ_b^0 when the local time at position 0 hits b.

Notation 5.14. *We define for* $u \in (t_0^*, \tau_b^0)$ *and* $t \in (-\int_{t_0^*}^0 \mathbf{1}_{\{X_s>0\}}\mathrm{d}s, \int_0^{\tau_b^0} \mathbf{1}_{\{X_s>0\}}\mathrm{d}s]$,

$$A_u^+ := \int_0^u \mathbf{1}_{\{X_s>0\}}\mathrm{d}s, \qquad \alpha_t^+ := \inf\{u : A_u^+ > t\}, \qquad X_t^+ := X_{\alpha_t^+}.$$

The process X^+ is the process X looked above zero. The processes $(X_t^+,\, t \geqslant 0)$ and $(X_t^+,\, t \leqslant 0)$ are independent, the former being a reflecting Brownian motion, see [27]. We let as before

$$f(r) := L(\tau_b^0, r), \qquad r \in [0, \mathfrak{d}_f) \tag{5.11}$$

be the local time at position r at time τ_b^0, where $\mathfrak{d}_f$ denotes the hitting time of 0 by $L(\tau_b^0, r)$, $r \geqslant 0$ (which is $X_{t_0^*}$ if $|t_0^*| < \infty$ and ∞ otherwise). We want to reconstruct X^+ conditionally on f up to a certain random time. Recall the definition of the forward local time flow in Section 3.2. Let S^+ denote that of X^+, i.e. by definition

$$S^+_{r,x}(a) = L_{X^+}(\tau_a^r(X^+), x) = L(\tau_a^r, x), \qquad 0 \leqslant r \leqslant x < \mathfrak{d}_f,\ a \in [0, f(r)].$$

By Theorem 2.1 and Definition 4.4, S^+ is a $\mathrm{BESQ}_b^{\delta,0}$ flow. In the notation of Section 3.2, let $Z^+ := \Upsilon((X_t^+,\, t \in [0, \tau_b^0(X^+)]), f)$. Specifically,

$$Z^+_{C_f(t)} = \int_0^{X_t^+} \frac{\mathrm{d}r}{f(r)}, \qquad C_f(t) := \int_0^t \frac{\mathrm{d}s}{f(X_s^+)^2}, \qquad 0 \leqslant t \leqslant \tau_b^0(X^+). \tag{5.12}$$

The flow $Y^+ := \Psi(S^+, f)$ is a Jacobi$(\delta, 0)$ flow independent of f by Theorem 4.6. In particular, Z^+ can be thought of as the contour function of the Fleming–Viot forest embedded in Y^+, rooted at level 0. The following theorem is the analog of Theorem 5.3 (iii).

Theorem 5.15. *The process* Z^+ *is independent of* f.

Proof. The forward local time flow of Z^+ is composed of the flow lines of Y^+ located at the right of $Y^+_{0,\cdot}(0)$. The flow Y^+ is independent of f, and Z^+ is measurable with respect to its forward local time flow. □

Let us give now a description in time of the process Z^+. We introduce a variant of the Bass–Burdzy flow. Let $\gamma^{+,\delta}$ be a reflecting Brownian motion with drift $1-\frac{\delta}{2}$, see [16]. It is the absolute value of the unique strong solution of the SDE

$$\mathrm{d}U_t = \Big(1-\frac{\delta}{2}\Big)\mathrm{sgn}U_t\,\mathrm{d}t + \mathrm{d}B_t, \qquad U_0 = 0$$

where B is a standard Brownian motion. When $\delta = 2$, it is the usual reflecting Brownian motion. It is recurrent when $\delta \geqslant 2$, and transient when $\delta \in (0,2)$. It is proved in [16] that $\gamma^{+,\delta} := |U|$ is distributed as the process $(\sup_{s\in[0,t]}(B_s + (\frac{\delta}{2}-1)s) - (B_t + (\frac{\delta}{2}-1)t),\ t \geqslant 0)$.

One defines for any $x \geqslant 0$, the process $(\mathcal{R}_t^{+,\delta}(x),\ t \geqslant 0)$ which is adapted to the filtration of $\gamma^{+,\delta}$ and is the strong solution of the SDE [7]

$$\mathcal{R}_t^{+,\delta}(x) = x - \gamma_t^{+,\delta} + \int_0^t \mathbf{1}_{\{\mathcal{R}_s^{+,\delta}(x)>0\}}\mathrm{d}s.$$

Theorem 5.16 will construct such a solution. Let $(R_t^+(x),\ x \geqslant 0,\ t \geqslant 0)$ be the flow associated with Z^+ via (5.6) and (5.7), i.e. set

$$\zeta^+(t) := \int_0^t \frac{\mathrm{d}s}{(1-L^+(s,Z_s^+))^2}, \qquad \widehat{Z}_t^+ := Z^+_{(\zeta^+)^{-1}(t)} \tag{5.13}$$

where L^+ denotes the local times of Z^+, and for $x \geqslant 0$,

$$R_t^+(x) := \int_{\widehat{Z}_t^+}^x \frac{\mathrm{d}r}{1-L^+((\zeta^+)^{-1}(t),r)} = \int_{\widehat{Z}_t^+}^x \Big(1+\widehat{L}^+(t,y)\Big)\,\mathrm{d}y. \tag{5.14}$$

In the (second) equality, $\widehat{L}^+$ is the local time of $\widehat{Z}^+$. This equality is proved along the lines of (5.7). Notice that R^+ is measurable with respect to X^+.

With the notation $t^+_{\max} := \inf\{s \geqslant 0 : L^+(s,Z_s^+) = 1\} \in (0,\infty]$, we have the analog of Lemma 5.11, meaning that $\lim_{t\to t^+_{\max}} \zeta^+(t) = \infty$. The time $t^+_{\max}$ is finite almost surely if $\delta \in (0,2)$ and infinite otherwise. The proof follows the lines of the proof of Lemma 5.11. We need to replace there B by $(X_t^+,\ t \in [0,\tau_b^0(X^+)])$, T_g^B by $\tau_b^0(X^+)$, and $(h(x),\ x \geqslant 0)$ is the local time at x of $(X_t^+,\ t \leqslant 0)$, i.e. a BESQ_0^δ process.

Theorem 5.16. *The process $(R_t^+(x),\ x \geqslant 0,\ t \geqslant 0)$ has the law of $(\mathcal{R}_t^{+,\delta}(x),\ x \geqslant 0,\ t \geqslant 0)$. In particular, $(\widehat{Z}_t^+,\ t \geqslant 0)$ is distributed as $(\mathcal{R}_t^{+,\delta})^{-1}(0)$.*

Proof. We observe that $(X_t^+,\ t \in [0,\tau_b^0(X^+)], f)$ and $(X^+_{\tau_b^0(X^+)-t},\ t \in [0,\tau_b^0(X^+)], f)$ have the same distribution. Therefore, we will take without loss of generality $Z^+ = \Upsilon((X^+_{\tau_b^0(X^+)-t},\ t \in [0,\tau_b^0(X^+)]), f)$ (and take $\widehat{Z}^+$, $\widehat{L}^+$, R^+ as in (5.13), (5.14) associated with this Z^+).

As in the proof of Theorem 5.12, the process $R_t^+(x) - x - \int_0^t \mathbf{1}_{\{R_s^+(x)>0\}}\mathrm{d}s$ does not depend on x, since $\int_0^t \mathbf{1}_{\{R_s^+(x)>0\}}\mathrm{d}s = \int_0^t \mathbf{1}_{\{\widehat{Z}_s^+<x\}}\mathrm{d}s = \int_{-\infty}^x \widehat{L}^+(t,y)\mathrm{d}y$. Therefore, it suffices

[7] To obtain the existence and uniqueness of the (strong) solution, we may use Zvonkin's method: Let $h(x) := \int_0^x e^{2\max(y,0)}\mathrm{d}y, x \in \mathbb{R}$, be an increasing (and convex) function on $\mathbb{R}$. Applying Itô–Tanaka's formula to $h(\mathcal{R}_t^{+,\delta}(x))$, we see that $\eta_t := h(\mathcal{R}_t^{+,\delta}(x))$ satisfies the SDE: $\eta_t = h(x) + \int_0^t \sigma(\eta_s)\mathrm{d}\gamma_s^{+,\delta}$ with $\sigma(x) := -h'(h^{-1}(x)) = -1 - 2\max(x,0)$. In particular σ is a Lipschitz function, we may apply Theorem V.6 in Protter [28] to get the existence and uniqueness of the (strong) solution η and then that of $\mathcal{R}^{+,\delta}(x)$.

to show that $(R_t^+(0),\, t \geqslant 0)$ is distributed as $(-\gamma_t^{+,\delta},\, t \geqslant 0)$ (observe that by definition, $R_t^+(0) \leqslant 0$ for all t).

In Section 5.1, taking $b = 1$ in equation (5.2), we constructed the process $Z^{(2)}$ associated with

$$X_s^{(2)} := X_{\tau_1^0 - s},\ s \in [0, \tau_1^0 - t^*),$$

where t^* is the time such that $X_{t^*} = \inf_{[0,\tau_1^0]} X$. We proved in Proposition 5.13 that the time-changed process $\widehat{Z}^{(2)}$, defined in equation (5.6), is associated with the Bass–Burdzy flow $(R_t(x))_{t,x}$ with parameters $\beta_1 = \frac{\delta}{2} - 1$ and $\beta_2 = \frac{\delta}{2}$.

We define the process $X^{(2),+}$ as the process $X^{(2)}$ looked above 0 (which should be denoted by $(X^{(2)})^+$ in the notation 5.14). Specifically, $X_t^{(2),+} = X^+_{\tau_1^0(X^+)-t}$, $t \in [0, \tau_1^0(X^+) - \tau_\ell^0(X^+)]$ where $\ell := L(t^*, 0)$. We now introduce the process $Z^{(2),+} := \Upsilon(X^{(2),+}, f)$ where f is given by (5.11) with $b = 1$. We mention that $Z^{(2),+}$ is also the process $Z^{(2)}$ looked above 0. [We omit the details. Intuitively, each excursion of $Z^{(2),+}$ is obtained from the corresponding excursion of $X^{(2)}$ in $\mathbb{R}_+$ by the transformation $\Upsilon(\cdot, f)$.] Analogously to (5.13), we let

$$\widehat{Z}_t^{(2),+} := Z^{(2),+}_{(\zeta^{(2),+})^{-1}(t)},\ t < \int_0^\infty \mathbf{1}_{\{\widehat{Z}_s^{(2)} > 0\}} \mathrm{d}s =: T^+$$

where $\zeta^{(2),+}(t) := \int_0^t \frac{\mathrm{d}s}{(1 - L^{(2),+}(s, Z_s^{(2),+}))^2}$, $L^{(2),+}$ being the local times of $Z^{(2),+}$. Again, the process $\widehat{Z}^{(2),+}$ is the process $\widehat{Z}^{(2)}$ looked above 0.

We claim that for any fixed $M > 0$,

$$\mathbb{P}\big(T^+ \leqslant M \,|\, L(t^*, 0) = \ell\big) \to 0, \qquad \ell \to 0. \tag{5.15}$$

Let us prove (5.15). First, By Lemma 5.11, if $t_{\max} < \infty$ and $Z^{(2)}_{t_{\max}} > 0$, then $T^+ = \infty$. Suppose now that $t_{\max} = \infty$ or $t_{\max} < \infty$ and $Z^{(2)}_{t_{\max}} \leqslant 0$. By (5.6), we have

$$T^+ = \int_0^{t_{\max}} \frac{1_{\{Z_s^{(2)} > 0\}}}{(1 - L^{(2)}(s, Z_s^{(2)}))^2} \mathrm{d}s.$$

Recall from (3.2) that $L^{(2)}(t, x) = \frac{L_{X^{(2)}}(C_f^{-1}(t), \eta_f^{-1}(x))}{f(\eta_f^{-1}(x))}$, with $C_f(t) := \int_0^t \frac{\mathrm{d}s}{f(X_s^{(2)})^2}$, $\eta_f(x) := \int_0^x \frac{\mathrm{d}r}{f(r)}$. Therefore

$$T^+ = \int_0^{C_f^{-1}(t_{\max})} \frac{1_{\{X_r^{(2)} > 0\}}}{(f(X_r^{(2)}) - L_{X^{(2)}}(r, X_r^{(2)}))^2} \mathrm{d}r = \int_{\tau_1^0 - C_f^{-1}(t_{\max})}^{\tau_1^0} \frac{1_{\{X_s > 0\}}}{(L(s, X_s))^2} \mathrm{d}s.$$

If $t_{\max} = \infty$ (i.e. $\delta \geqslant 2$), $\tau_1^0 - C_f^{-1}(t_{\max}) = t^*$. If $t_{\max} < \infty$ and $Z^{(2)}_{t_{\max}} \leqslant 0$, the process $Z^{(2)}$ stays negative after $t_{\max}$, which is equivalent with saying that X is always negative between t^* and $\tau_1^0 - C_f^{-1}(t_{\max})$. Hence, in both cases, $T^+ = \int_{\tau_1^0 - C_f^{-1}(t_{\max})}^{\tau_1^0} \frac{1_{\{X_s > 0\}}}{(L(s, X_s))^2} \mathrm{d}s = \int_{t^*}^{\tau_1^0} \frac{1_{\{X_s > 0\}}}{(L(s, X_s))^2} \mathrm{d}s$. From the occupation times formula, we get that

$$T^+ = \int_0^\infty \left(\frac{1}{L(t^*, x)} - \frac{1}{f(x)} \right) \mathrm{d}x \geqslant \int_0^1 \left(\frac{1}{L(t^*, x)} - \frac{1}{f(x)} \right) \mathrm{d}x$$

(the upper boundary 1 is arbitrary). Using [26] Lemma 2.3, conditionally on $L(t^*,0)=\ell$, the processes $x\in\mathbb{R}_+\to L(t^*,x)$ and $x\in\mathbb{R}_+\to f(x)-L(t^*,x)$ are independent, with distribution respectively $\mathrm{BESQ}^{\delta}_{\ell}$ and $\mathrm{BESQ}^{0}_{1-\ell}$. The integral $\int_0^1\frac{1}{f(x)}\mathrm{d}x$ is tight as $\ell\to 0$ while the integral $\int_0^1\frac{1}{L(t^*,x)}\mathrm{d}x$ tends in law to ∞ by scaling. We deduce (5.15).

As in (5.14), we define for $x\geqslant 0$,

$$R_t^{(2),+}(x):=\int_{\widehat{Z}_t^{(2),+}}^{x}\frac{\mathrm{d}r}{1-L^{(2),+}((\zeta^{(2),+})^{-1}(t),r)}=\int_{\widehat{Z}_t^{(2),+}}^{x}\left(1+\widehat{L}^{(2),+}(t,y)\right)\mathrm{d}y.$$

By construction, $\widehat{Z}^{(2),+}$ is the process $(\widehat{Z}_t^{+},\,t\in[0,T^+))$, hence for all $x\geqslant 0$, the process $R^{(2),+}(x)$ coincides with $R^{+}(x)$ up to time T^+.

On the other hand, the process $R^{(2),+}(x)$ is deduced from $R(x)$ by a time-change: define

$$\tau_t^{(2),+}:=\inf\left\{s\geqslant 0\,:\,\int_0^s\mathbf{1}_{\{R_u(0)<0\}}\mathrm{d}u>t\right\}=\inf\left\{s\geqslant 0\,:\,\int_0^s\mathbf{1}_{\{\widehat{Z}_u^{(2)}>0\}}\mathrm{d}u>t\right\}.$$

Then we have for any $x\geqslant 0$ and $0\leqslant t<T^+$, [8]

$$R_t^{(2),+}(x)=R_{\tau_t^{(2),+}}(x). \tag{5.16}$$

Recall that we want to show that $(R_t^{+}(0),\,t\geqslant 0)$ is distributed as $(-\gamma_t^{+,\delta},\,t\geqslant 0)$. If we let γ be the Brownian motion driving the Bass–Burdzy flow $(R_t(x))_{t,x}$, we have

$$R_t(0)=\gamma_t+\frac{\delta}{2}t-\int_0^t\mathbf{1}_{\{R_s(0)<0\}}\mathrm{d}s.$$

By Tanaka's formula,

$$\min(R_t(0),0)=\int_0^t\mathbf{1}_{\{R_s(0)\leqslant 0\}}\mathrm{d}R_s(0)-\frac{1}{2}L_{R(0)}(t,0)$$

where $L_{R(0)}(t,0)$ is the local time of $R_\cdot(0)$ at time t at position 0. By (5.16), observing that $R_t^{(2),+}(0)\leqslant 0$,

$$R_t^{(2),+}(0)=\int_0^{\tau_t^{(2),+}}\mathbf{1}_{\{R_s(0)\leqslant 0\}}\mathrm{d}R_s(0)-\frac{1}{2}L_{R(0)}(\tau_t^{(2),+},0).$$

We have $L_{R(0)}(\tau_t^{(2),+},0)=L_{R^{(2),+}(0)}(t,0)$ which is the local time of $R_\cdot^{(2),+}(0)$ at time t at position 0. Using that $\int_0^{\tau_t^{(2),+}}\mathbf{1}_{\{R_s(0)\leqslant 0\}}\mathrm{d}s=\int_0^{\tau_t^{(2),+}}\mathbf{1}_{\{R_s(0)<0\}}\mathrm{d}s=t$, we get

$$R_t^{(2),+}(0)=\int_0^{\tau_t^{(2),+}}\mathbf{1}_{\{R_s(0)\leqslant 0\}}\mathrm{d}\gamma_s+(\frac{\delta}{2}-1)t-\frac{1}{2}L_{R^{(2),+}(0)}(t,0),\ 0\leqslant t<T^+.$$

[8] In fact, $\widehat{Z}_t^{(2),+}=Z^{(2),+}_{(\zeta^{(2),+})^{-1}(t)}=Z^{(2)}_{\alpha^{(2),+}\circ(\zeta^{(2),+})^{-1}(t)}$ where $\alpha^{(2),+}$ is the inverse of $\int_0^u\mathbf{1}_{\{Z_s^{(2)}>0\}}\mathrm{d}s$. We notice that $L^{(2),+}(s,x)=L^{(2)}(\alpha_s^{(2),+},x)$, $x\geqslant 0$. Hence $R_t^{(2),+}(x)=R_{\zeta^{(2)}\circ\alpha^{(2),+}\circ(\zeta^{(2),+})^{-1}(t)}(x)$. We check $\zeta^{(2)}\circ\alpha^{(2),+}\circ(\zeta^{(2),+})^{-1}(t)=\tau_t^{(2),+}$ by noticing that $\int_0^{\zeta^{(2)}\circ\alpha^{(2),+}\circ(\zeta^{(2),+})^{-1}(t)}\mathbf{1}_{\{\widehat{Z}_u^{(2)}>0\}}\mathrm{d}u=\int_0^{\alpha^{(2),+}\circ(\zeta^{(2),+})^{-1}(t)}\mathbf{1}_{\{Z_s^{(2)}>0\}}\frac{\mathrm{d}s}{(1-L^{(2)}(s,Z_s^{(2)}))^2}=\int_0^{(\zeta^{(2),+})^{-1}(t)}\frac{\mathrm{d}s}{(1-L^{(2),+}(s,Z_s^{(2),+}))^2}=t$.

By Dambis–Dubins–Schwarz theorem, there exists a standard Brownian motion $B^{(+)} = (B_t^{(+)},\, t \geqslant 0)$ such that

$$\int_0^{\tau_t^{(2),+}} \mathbf{1}_{\{R_s(0)<0\}} \mathrm{d}\gamma_s = B_t^{(+)},\ 0 \leqslant t < T^+.$$

By Skorokhod's lemma,

$$\frac{1}{2} L_{R^{(2),+}(0)}(t,0) = \sup_{s \leqslant t} \left(B_s^{(+)} + \left(\frac{\delta}{2} - 1\right) s \right),\ 0 \leqslant t < T^+.$$

Define

$$\gamma_t^{+,\delta} := \sup_{s \leqslant t} \left(B_s^{(+)} + \left(\frac{\delta}{2} - 1\right) s \right) - B_t^{(+)} - \left(\frac{\delta}{2} - 1\right) t,\ t \geqslant 0.$$

The process $-\gamma^{+,\delta}$ coincides with $R^{(2),+}(0)$ up to time T^+. We now prove that $R^+(0)$ is distributed as $-\gamma^{+,\delta}$. Fix $M > 0$ and a functional F measurable and bounded. Notice that $L(t^*, 0)$ is measurable with respect to the process $(X_s,\, s \in [0, \tau_1^0])$ looked below 0, which is independent of X^+, see [27]. Therefore R^+ and $L(t^*, 0)$ are independent, which implies that for all $\ell > 0$,

$$E[F(R_t^+(0),\, t \in [0, M])] = E[F(R_t^+(0),\, t \in [0, M]) \,|\, L(t^*, 0) \leqslant \ell].$$

We consider two cases. The first case is when $T^+ > M$. In this case, $R_t^+(0) = R_t^{(2),+}(0) = -\gamma_t^{+,\delta}$, for $t \in [0, M]$. The second case is when $T^+ \leqslant M$, which has probability going to 0 under $\mathbb{P}(\cdot \,|\, L(t^*, 0) \leqslant \ell)$ when $\ell \to 0$ by (5.15). We deduce that

$$E[F(R_t^+(0),\, t \in [0, M])] = E[F(-\gamma_t^{+,\delta},\, t \in [0, M])]$$

which is what we wanted to prove. □

Appendix A. Perfect flow property

Recall the definition of a BESQ$^\delta$ in Definition 2.2. We establish some flow properties for the BESQ$^\delta$ flow.

Proposition A.1. *Let $\mathcal{S}$ be either a non-killed* BESQ$^\delta$ *flow with $\delta > 0$ or a killed* BESQ$^\delta$ *flow with $\delta \leqslant 0$. The flow $\mathcal{S}$ satisfies the perfect flow property: almost surely, for every $r \leqslant x \leqslant y$, $\mathcal{S}_{r,y} = \mathcal{S}_{x,y} \circ \mathcal{S}_{r,x}$.*

Proof. The property is true when one of the inequalities in $r \leqslant x \leqslant y$ is an equality since $\mathcal{S}_{z,z}(a) = a$. Let $a \geqslant 0$ and $r < x < y$. We first treat the case $\delta > 0$. By Proposition 2.4, we can set $\mathcal{S} = (S_{r,x}(a),\, x \geqslant r)_{a,r \in \mathbb{R}}$. We want to show that $S_{r,y}(a) = S_{x,y} \circ S_{r,x}(a)$. Let $b := S_{r,x}(a)$. By definition of S, $L(\tau_a^r, x) = b$ hence $\tau_b^x \geqslant \tau_a^r$. If $t > \inf\{s > \tau_a^r \,:\, X_s > x\} =: \theta$, then $L(t, x) > b$ since a Brownian motion accumulates local time at any level that it visits. It implies that $\tau_b^x \leqslant \theta$. If $t \in (\tau_a^r, \theta)$, one has $L(t, x) = L(\tau_a^r, x) = b$. We conclude that $L(\tau_a^r, y) = L(\tau_b^x, y)$ indeed (and $\tau_b^x = \theta$). We now deal with the case $\delta \leqslant 0$. To be consistent with our setting, we will actually consider a BESQ$^{\delta'}$ flow with $\delta' \leqslant 0$, and rather take $\delta := 2 - \delta'$ which is now greater than 2. Then we can set $\mathcal{S} = (S_{-r,-x}(a))_{r \leqslant x, a \geqslant 0}$. We write $S_{-r,-x}(a) = L(\tau_a^{-r}, -x) =: b$ and see that $\tau_b^{-x} \geqslant \tau_a^{-r}$. Again, if $t > \inf\{s > \tau_a^{-r} \,:\, X_s < -x\}$, then $L(t, -x) > b$ because the PRBM does not

have point of monotonicity when $\delta \geqslant 2$, see [27]. We deduce that $L(\tau_a^{-r}, -y) = L(\tau_b^{-x}, y)$. □

When $\delta \in (0,2)$, the killed BESQ$^\delta$ flow loses this property. There will be exceptional times when a flow line starting at 0 will exit the boundary 0. This comes from the fact that in the case $\delta \in (0,2)$, the PRBM has points of monotonicity, see [27]. Still the flow satisfies a weaker form of the flow property stated in Proposition A.5. We need some preliminary results.

The following proposition shows that, for $\delta \in (0,2)$ the killed BESQ$^\delta$ flow is naturally embedded in its non-killed version.

Proposition A.2. *Let $\delta \in (0,2)$ and $\mathcal{S} = (\mathcal{S}_{r,x}(a),\, x \geqslant r)_{r\in\mathbb{R}, a\geqslant 0}$ be a non-killed* BESQ$^\delta$ *flow. Define a flow $\Pi(\mathcal{S}) := (\widetilde{\mathcal{S}}_{r,x}(a),\, x \geqslant r)_{r\in\mathbb{R}, a\geqslant 0}$ where for $r \in \mathbb{R}$ and $a \geqslant 0$, $\widetilde{\mathcal{S}}_{r,x}(a)$ is equal to $\mathcal{S}_{r,x}(a)$ for $x \geqslant r$ up to the time $\inf\{r' > r : \mathcal{S}_{r,r'}(a) = 0\}$, and is equal to 0 afterwards. Then $\Pi(\mathcal{S})$ is a killed* BESQ$^\delta$ *flow. We call $\Pi(\mathcal{S})$ the killed version of $\mathcal{S}$.*

Remark. Note that when $a = 0$, at times r when the flow line $(\mathcal{S}_{r,x}(0),\, x \geqslant r)$ starts an excursion away from 0, the flow line $(\widetilde{\mathcal{S}}_{r,x}(0),\, x \geqslant r)$ traces the same excursion then gets absorbed at 0 when coming back.

Proof. The flow lines of $\Pi(\mathcal{S})$ have the required distribution by construction so we need to check the regularity of $\Pi(\mathcal{S})$ imposed in Definition 2.2. Assumption (i) is clear. We prove (ii). The map $a \mapsto \widetilde{\mathcal{S}}_{r,x}(a)$ is nondecreasing. Indeed, if $\widetilde{\mathcal{S}}_{r,x}(a) > 0$, then $\widetilde{\mathcal{S}}_{r,x}(a) = \mathcal{S}_{r,x}(a) \leqslant \mathcal{S}_{r,x}(a') = \widetilde{\mathcal{S}}_{r,x}(a')$ for all $a' \geqslant a$. And obviously $\widetilde{\mathcal{S}}_{r,x}(a) \leqslant \widetilde{\mathcal{S}}_{r,x}(a')$ if $\widetilde{\mathcal{S}}_{r,x}(a) = 0$. We show the right-continuity. The case $\widetilde{\mathcal{S}}_{r,x}(a) > 0$ is immediate from the same arguments, using the right-continuity of $a \mapsto \mathcal{S}_{r,x}(a)$. Suppose now that $\widetilde{\mathcal{S}}_{r,x}(a) = 0$. We can suppose $x > r$. By construction, there exists $r' \in (r, x]$ such that $\mathcal{S}_{r,r'}(a) = 0$. Since $b \mapsto \mathcal{S}_{r,r'}(b)$ is piecewise constant, we would have $\mathcal{S}_{r,r'}(b) = 0$ for $b > a$ small enough, which implies $\mathcal{S}_{r',r}(b) = 0$ for the same set of b, hence $\widetilde{\mathcal{S}}_{r,r'}(b) = 0$. The right-continuity is therefore proved. Finally, we check condition (iii). We suppose $\widetilde{\mathcal{S}}_{r',r}(a') > a$. In particular, $\widetilde{\mathcal{S}}_{r',r}(a') > 0$ so $\widetilde{\mathcal{S}}_{r',r}(a') = \mathcal{S}_{r',r}(a')$ hence by construction $\widetilde{\mathcal{S}}_{r',x}(a') = \mathcal{S}_{r',x}(a') \geqslant \mathcal{S}_{r,x}(a) = \widetilde{\mathcal{S}}_{r,x}(a)$ for all x before $\widetilde{\mathcal{S}}_{r,\cdot}(a)$ hits 0 when it will get absorbed. The case $\widetilde{\mathcal{S}}_{r',r}(a') < a$ is dealt with similarly. □

Conversely, for $\delta \in (0,2)$, a non-killed BESQ$^\delta$ flow can also be constructed from its killed version. Recall the definition of the dual flow in Proposition 2.7.

Proposition A.3. *Let $\delta \in (0,2)$ and $\widetilde{\mathcal{S}}$ be a killed* BESQ$^\delta$ *flow. The dual of $\widetilde{\mathcal{S}}$ is a non-killed* BESQ$^{2-\delta}$ *flow, from which we can construct its killed version by Proposition A.2. Let $\mathcal{S}$ denote the dual of the latter. Then $\mathcal{S}$ is a non-killed* BESQ$^\delta$ *flow, and $\widetilde{\mathcal{S}}$ is the killed version of $\mathcal{S}$.*

Proof. We start with a lemma.

Lemma A.4. *Let $\delta \in (0,2)$. Let $\mathcal{S}$ be a non-killed* BESQ$^\delta$ *flow. Then the dual of $\mathcal{S}$ is the killed version of the dual of $\Pi(\mathcal{S})$, i.e. $\mathcal{S}^* = \Pi(\Pi(\mathcal{S})^*)$.*

Proof. Let $\mathcal{S}^*$, resp. $\Pi(\mathcal{S})^*$, denote the dual of $\mathcal{S}$, resp. $\Pi(\mathcal{S})$. Write $\widetilde{\mathcal{S}}$ for $\Pi(\mathcal{S})$. We have by definition, for any $r \leqslant x$,

$$\mathcal{S}^*_{r,x}(a) := \inf\{b \geqslant 0 : \mathcal{S}_{-x,-r}(b) > a\}, \qquad (\widetilde{\mathcal{S}})^*_{r,x}(a) := \inf\{b \geqslant 0 : \widetilde{\mathcal{S}}_{-x,-r}(b) > a\}.$$

Let $r < x$ and $a \geqslant 0$. We first show that $(\widetilde{\mathcal{S}})^*_{r,x}(a) \geqslant \mathcal{S}^*_{r,x}(a)$. Let $b > (\widetilde{\mathcal{S}})^*_{r,x}(a)$, hence $\widetilde{\mathcal{S}}_{-x,-r}(b) > a$. In particular, $\widetilde{\mathcal{S}}_{-x,-r}(b) > 0$ so that $\widetilde{\mathcal{S}}_{-x,-r}(b) = \mathcal{S}_{-x,-r}(b)$. Therefore $\mathcal{S}_{-x,-r}(b) > a$ hence the inequality.

We suppose now that $\mathcal{S}^*_{r,s}(a) \neq 0$ for all $s \in (r, x)$, and we want to show that $\mathcal{S}^*_{r,x}(a) = (\widetilde{\mathcal{S}})^*_{r,x}(a)$. We only have to prove that $\mathcal{S}^*_{r,x}(a) \geqslant (\widetilde{\mathcal{S}})^*_{r,x}(a)$. Let $b > \mathcal{S}^*_{r,x}(a)$. Then $\mathcal{S}_{-x,-r}(b) > a$. If $\mathcal{S}_{-x,-s}(b) = 0$ for some $s \in (r, x)$, we would have by the perfect flow property $\mathcal{S}_{-s,-r}(0) > a$ hence $\mathcal{S}^*_{r,s}(a) = 0$, which is a contradiction. Therefore $\mathcal{S}_{-x,-s}(b) \neq 0$ for all $s \in (r, x)$, hence $\mathcal{S}_{-x,-r}(b) = \widetilde{\mathcal{S}}_{-x,-r}(b)$ so $\widetilde{\mathcal{S}}_{-x,-r}(b) > a$. We deduce that $\mathcal{S}^*_{r,x}(a) \geqslant (\widetilde{\mathcal{S}})^*_{r,x}(a)$.

Finally, we show that flow lines in $\mathcal{S}^*$ are absorbed at 0. So we suppose that $\mathcal{S}^*_{r,s}(a) = 0$ for some $s \in (r, x)$, and we want to show that $\mathcal{S}^*_{r,x}(a) = 0$. We have by definition, $\mathcal{S}_{-s,-r}(b) > a$ for all $b > 0$. Since $b \to \mathcal{S}_{-s,-r}(b)$ is piecewise constant, we must have $\mathcal{S}_{-s,-r}(0) > a$, therefore $\mathcal{S}_{-x,-r}(0) > a$ by the perfect flow property. It implies that $\mathcal{S}^*_{r,x}(a) = 0$. □

We go back to the proof of the proposition. We use the notation of the proposition. By definition, $\mathcal{S} = (\Pi((\widetilde{\mathcal{S}})^*))^*$. By the lemma (using $(\widetilde{\mathcal{S}})^*$, which is a non-killed BESQ$^{2-\delta}$ flow in place of $\mathcal{S}$ in the statement of the lemma), we have $\Pi(\mathcal{S}) = ((\widetilde{\mathcal{S}})^*)^* = \widetilde{\mathcal{S}}$. The fact that $\mathcal{S}$ is a non-killed BESQ$^\delta$ flow comes from Proposition 2.7. □

We can now state a flow property for a general BESQ$^\delta$ flow.

Proposition A.5. *Let $\delta \in \mathbb{R}$. A general* BESQ$^\delta$ *flow $\mathcal{S}$ satisfies the following properties almost surely:*

- *(Almost perfect flow property) If $r \leqslant x \leqslant y$, $a \geqslant 0$ and $\mathcal{S}_{r,x}(a) > 0$, then $\mathcal{S}_{x,y} \circ \mathcal{S}_{r,x}(a) = \mathcal{S}_{r,y}(a)$.*
- *(Coalescence) If $r', r < x$, $0 \leqslant a, a'$ and $\mathcal{S}_{r,x}(a) = \mathcal{S}_{r',x}(a')$, then $\mathcal{S}_{r,y}(a) = \mathcal{S}_{r',y}(a')$ for all $y \geqslant x$.*

Proof. These properties hold for non-killed BESQ$^\delta$ flows and killed BESQ$^\delta$ flows with $\delta \leqslant 0$ as a consequence of the perfect flow property, so we should only deal with $\delta \in (0, 2)$ and killed BESQ$^\delta$ flows. Write it $\widetilde{\mathcal{S}}$ and let rather $\mathcal{S}$ be the non-killed version of it (i.e. the flow such that $\Pi(\mathcal{S}) = \widetilde{\mathcal{S}}$). Let $r \leqslant x \leqslant y$, $a \geqslant 0$ and suppose $b := \widetilde{\mathcal{S}}_{r,x}(a) > 0$. From the definition of $\Pi(\mathcal{S})$, we have $\widetilde{\mathcal{S}}_{r,s}(a) = \mathcal{S}_{r,s}(a)$ for all s from r to the hitting time of 0, hence we can apply the perfect flow property of $\mathcal{S}$. After hitting 0, the flow line is absorbed. Let now $r, r' < x$, $0 \leqslant a, a'$ and suppose that $b := \widetilde{\mathcal{S}}_{r,x}(a) = \mathcal{S}_{r',x}(a')$. If $b > 0$, we can apply the almost perfect flow property to conclude that flow lines coalesce. If $b = 0$, both flow lines are absorbed at 0 (we use here the fact that $r, r' < x$ so that none are flow lines which are departing from 0 to trace an excursion). □

Remark. The coalescence property does not hold in full generality if we let $r = x$ for killed BESQ$^\delta$ flows when $\delta \in (0,2)$. Indeed, at times when a flow line starts at r an excursion away from 0, flow lines which are already absorbed at 0 will stay stuck at 0, while the flow line starting at r will leave 0.

Corollary A.6. *Let $\delta, \delta' \in \mathbb{R}$. A general* Jacobi$(\delta, \delta')$ *flow satisfies the following properties almost surely:*

- *(Almost perfect flow property) If $r \leqslant s \leqslant t$, $v \in [0,1]$ and $\mathcal{Y}_{r,s}(v) > 0$, then $\mathcal{Y}_{s,t} \circ \mathcal{Y}_{r,s}(v) = \mathcal{Y}_{r,t}(v)$.*
- *(Coalescence) If $r, r' < s$, $v, v' \in [0,1]$ and $\mathcal{Y}_{r,s}(v) = \mathcal{Y}_{r',s}(v)$, then $\mathcal{Y}_{r,t}(v) = \mathcal{Y}_{r',t}(v)$ for all $t \geqslant s$.*

Proof. Proposition A.5 stays true for BESQ$_b^{\delta,\delta'}$ flows by construction (for $a \in [0, f(r)]$ and $r, r' \in [0, \mathfrak{d}_f)$ there) then we use Theorem 4.6 to prove the corollary when $r, r' \geqslant 0$. It is true for all r, r' by stationarity. □

Appendix B. A Girsanov theorem for Jacobi flows

Let $\delta \in \mathbb{R}$. Consider a general Jacobi$(\delta, 0)$ flow $\mathcal{Y}$, driven by the martingale measure $\mathcal{M}$ given by Definition 4.1 under some measure $\mathbb{P}$. Fix $T > 0$, and restrict to the time interval $[0,T]$. Call $\mathcal{Y}^T$ the collection $(\mathcal{Y}_{s,t},\ 0 \leqslant s \leqslant t \leqslant T)$. Let $\delta' \in \mathbb{R}$. We want to relate the Jacobi(δ, δ') flow to the Jacobi$(\delta, 0)$ flow. We will use in this section the notation $\mathcal{M}_s(A) := \mathcal{M}(A \times [0,s])$ for $s \geqslant 0$ and A a Borel set of $[0,1]$.

For $\mathfrak{v} \in [0,1]$, let $\mathcal{F}_\mathfrak{v}^T$ be the σ-field generated by $(\mathcal{M}_s(A),\ s \in [0,T],\ A$ Borel set of $[0,\mathfrak{v}])$. Observe that $\mathcal{F}^T := (\mathcal{F}_\mathfrak{v}^T, \mathfrak{v} \in [0,1))$ forms a filtration.

We define

$$m_\mathfrak{v}^T := \mathrm{e}^{-\frac{\delta'}{2(1-\mathfrak{v})}\mathcal{M}_T([0,\mathfrak{v}]) - \frac{T}{8}\frac{\delta'^2}{1-\mathfrak{v}}\mathfrak{v}}, \qquad \mathfrak{v} \in [0,1).$$

Theorem B.1. *The process $(m_\mathfrak{v},\ \mathfrak{v} \in [0,1))$ is a $\mathcal{F}^T$-martingale. Introduce the measure $\mathbb{Q}$ on $\mathcal{F}_1^T$ such that*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_\mathfrak{v}^T} := m_\mathfrak{v}^T, \qquad \mathfrak{v} \in [0,1).$$

Under $\mathbb{Q}$, $\mathcal{M}$ has drift $-\frac{\delta'}{2}\mathbf{1}_{[0,1]\times[0,T]}(u,x)\mathrm{d}u\mathrm{d}x$, and $\mathcal{Y}^T$ is a general Jacobi(δ, δ') *flow on $[0,T]$.*

Proof. We use the representation in (4.6): $\mathcal{M}_T([0,u]) = \widetilde{\mathcal{W}}_T([0,u]) - u\widetilde{\mathcal{W}}_T([0,1])$ where $\widetilde{\mathcal{W}}$ is a white noise and we used the notation $\widetilde{\mathcal{W}}_s(A) := \widetilde{\mathcal{W}}(A \times [0,s])$ for $s \geqslant 0$ and Borel sets A. The random variable

$$\mathrm{e}^{-\frac{\delta'}{2(1-\mathfrak{v})}\widetilde{\mathcal{W}}_T([0,\mathfrak{v}]) - \frac{T}{8}\frac{\delta'^2}{(1-\mathfrak{v})^2}\mathfrak{v}} \tag{B.1}$$

has mean 1. Take this random variable as the Radon–Nikodym derivative of a new probability measure with respect to $\mathbb{P}$ on $\sigma(\widetilde{\mathcal{W}}_s(A), 0 \leqslant s \leqslant T, A$ Borel set of $[0,1])$.

This change of measure adds a drift $-\frac{\delta'}{2(1-\mathfrak{v})}\mathbf{1}_{[0,\mathfrak{v}]\times[0,T]}(u,x)\mathrm{d}u\mathrm{d}x$ to the white noise $\widetilde{\mathcal{W}}$. Therefore it adds a drift

$$-\frac{\delta'}{2}\Big(\frac{1}{(1-\mathfrak{v})}\mathbf{1}_{[0,\mathfrak{v}]\times[0,T]}(u,x)-\frac{\mathfrak{v}}{1-\mathfrak{v}}\Big)=-\frac{\delta'}{2}\Big(\mathbf{1}_{[0,\mathfrak{v}]\times[0,T]}(u,x)-\frac{\mathfrak{v}}{1-\mathfrak{v}}\mathbf{1}_{(\mathfrak{v},1]\times[0,T]}(u,x)\Big)$$

to the martingale measure $\mathcal{M}$. Use that $\mathcal{M}_T([0,\mathfrak{v}])=\widetilde{\mathcal{W}}_T([0,\mathfrak{v}])-\mathfrak{v}\widetilde{\mathcal{W}}_T([0,1])$, and average (B.1) over $\widetilde{\mathcal{W}}_T([0,1])$ which is independent of $\mathcal{M}$ to obtain that it is also the distribution of $\mathcal{M}$ under $m^T_{\mathfrak{v}}\cdot\mathbb{P}$. Notice that on $[0,\mathfrak{v}]\times[0,T]$, the drift is just $-\frac{\delta'}{2}$. We conclude that $m^T_{\mathfrak{v}}$ is the Radon-Nykodim derivative on $\mathcal{F}^T_{\mathfrak{v}}$ of the probability measure under which $\mathcal{M}$ has drift $-\frac{\delta'}{2}\mathbf{1}_{[0,1]\times[0,T]}(u,x)$. Hence m^T is a martingale, and $\mathcal{M}$ under $\mathbb{Q}$ has the required distribution (we could directly check that $m^T_{\mathfrak{v}}$ is a $\mathcal{F}^T_{\mathfrak{v}}$-martingale, by using the semimartingale decomposition of the bridge: $\mathcal{M}_T([0,u])=\sqrt{T}\widehat{B}_u-\int_0^u\frac{\mathrm{d}s}{1-s}\mathcal{M}_T([0,s])$ where $\widehat{B}$ is a standard $\mathcal{F}^T$-Brownian motion.).

Plug the drift into (4.3) to conclude that a flow line from $v\in[0,1]$ and $s\in[0,T]$ is a (possibly absorbed at 0) Jacobi(δ,δ') process up to the minimum between the hitting time of 1 and T. We set, under $\mathbb{Q}$, $\mathcal{Y}^T_{s,t}(1):=1$ for all $0\leqslant s\leqslant t\leqslant T$, and absorb at 1 any flow line which hits 1.

Let $\mathfrak{v}\in[0,1)$. From the absolute continuity of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{F}^T_{\mathfrak{v}}$, one concludes that the almost perfect flow property of Corollary A.6 holds as long as the flow lines do not touch $\mathfrak{v}$. Making $\mathfrak{v}$ tend to 1, this property is also true as long as they do not touch 1. With our convention, it is also true after the hitting time of 1. Hence $\mathcal{Y}^T$ possesses the almost perfect flow property also under $\mathbb{Q}$. We can similarly show that flow lines coalesce under $\mathbb{Q}$ in the sense of Corollary A.6.

Let us check the regularity conditions of Definition 4.3. Statement (i) is clear. We prove (ii). We already know that $v\in[0,1]\to\mathcal{Y}^T_{s,t}(v)$ is nondecreasing since flow lines coalesce. Again, for any $\mathfrak{v}\in(0,1)$, the right-continuity holds for any $0\leqslant s\leqslant t\leqslant T$ at any $v\in[0,1)$ such that the flow line $\mathcal{Y}_{s,\cdot}(v)$ did not hit $\mathfrak{v}$ on $[s,t]$. Then, it holds if the flow line did not hit 1. Since $\mathcal{Y}_{s,t}(v)=1$ if the flow line hit 1, we conclude that it is right-continuous at any $v\in[0,1)$. Condition (iii) is a consequence of the fact that flow lines coalesce and condition (iv) is satisfied by construction. □

Appendix C. Proofs of Lemmas 3.7 and 5.7

Proof of Lemma 3.7. Fix $M\geqslant 0$ and take a compact set L as in the statement of the lemma, for $K=[0,M]$. By assumption, there exists $\varepsilon\in(0,1)$ such that f^n and f are contained in $(\varepsilon,\varepsilon^{-1})$ on L for n large enough. Let $t\in[0,M]$. We have, since $\mathcal{X}([0,M])\cup\mathcal{X}^n([0,M])\subset L\subset I_n$ for n large enough,

$$|\eta_f(\mathcal{X}_t)-\eta_{f^n}(\mathcal{X}^n_t)|\leqslant\left|\int_0^{\mathcal{X}_t}\mathrm{d}r\left(\frac{1}{f(r)}-\frac{1}{f^n(r)}\right)\right|+\left|\int_{\mathcal{X}_t}^{\mathcal{X}^n_t}\frac{\mathrm{d}r}{f^n(r)}\right|.$$

The first term is bounded by $\|\frac{1}{f}-\frac{1}{f^n}\|_L\|\mathcal{X}\|_{[0,M]}$. The second term is bounded by $\frac{1}{\varepsilon}\|\mathcal{X}-\mathcal{X}^n\|_{[0,M]}$. Both terms go to 0 as $n\to\infty$. It shows the uniform convergence of $\eta_{f^n}\circ\mathcal{X}^n$ to $\eta_f\circ\mathcal{X}$ on $[0,M]$. Since $|f(\mathcal{X}_t)-f^n(\mathcal{X}^n_t)|\leqslant|f(\mathcal{X}_t)-f(\mathcal{X}^n_t)|+|f(\mathcal{X}^n_t)-f^n(\mathcal{X}^n_t)|$, that f is uniformly continuous on L, that $\mathcal{X}^n$ converges uniformly to $\mathcal{X}$ on $[0,M]$ and f^n converges

uniformly to f on L, we have that $f^n \circ \mathcal{X}^n$ converges uniformly to $f \circ \mathcal{X}$ on $[0, M]$. Let $C_{f_n}(t) := \int_0^t \frac{\mathrm{d}s}{f^n(\mathcal{X}^n_s)^2}$ for $t \in [0, j_n)$ and $C_f(t) := \int_0^t \frac{\mathrm{d}s}{f(\mathcal{X}_s)^2}$ for $t \geqslant 0$. Notice that the range of C_{f_n} and of C_f is $\mathbb{R}_+$ by our assumption (i). Dominated convergence implies that $C_{f^n}(t)$ converges to $C_f(t)$ for all $t \in [0, M]$. Fix $t \in (0, M)$ and set $x := C_f(t)$. Let $t_n := C^{-1}_{f_n}(x)$. Since $C_{f^n}(v) - C_{f^n}(u) \geqslant \varepsilon^2(v - u)$ for $v \geqslant u$ in $[0, M]$, we have $|C_{f^n}(t) - x| \geqslant \varepsilon^2 |t - t_n|$. We deduce that $t_n \to t$, i.e. $C^{-1}_{f^n}(x) \to C^{-1}_f(x)$. Since M is arbitrary, it proves that $C^{-1}_{f^n}$ converges to C^{-1}_f pointwise on $\mathbb{R}_+$. Since $C^{-1}_{f_n}, C^{-1}_f$ are increasing continuous functions on $\mathbb{R}_+$, it implies the uniform convergence on any compact set of $\mathbb{R}_+$. By composition of $\eta_{f^n} \circ \mathcal{X}^n$ and $C^{-1}_{f_n}$, we deduce that $\Upsilon(\mathcal{X}^n, f^n)$ converges to $\Upsilon(\mathcal{X}, f)$ uniformly on any compact. □

Proof of Lemma 5.7. First, for any $t \geqslant 0$, a.s., $1 - L^{(2)}(t, x) > 0$ for all $x \in \{Z^{(2)}_{t+u},\, u \geqslant 0\}$. Let us prove this statement. Because $Z^{(2)}$ accumulates some local time at each level that it crosses, if $L^{(2)}(t, x) = 1$, it implies that the process $Z^{(2)}$ will never visit x again. Hence necessarily, the only possibility to have $L^{(2)}(t, x) = 1$ for some $x \in \{Z^{(2)}_{t+u},\, u \geqslant 0\}$ is to have $x = Z^{(2)}_t$. So we need to show that for all $t \geqslant 0$, a.s., $L^{(2)}(t, Z^{(2)}_t) < 1$. Let $\varepsilon \in (0, 1)$. We have by the occupation times formula, for any $s < t$ and $M \geqslant 0$, $\int_s^t \mathbf{1}_{\{L^{(2)}(u, Z^{(2)}_u) \geqslant 1-\varepsilon,\, |Z^{(2)}_u| \leqslant M\}} \mathrm{d}u = \int_{x \in [-M, M]} \int_s^t \mathbf{1}_{\{L^{(2)}(u,x) \geqslant 1-\varepsilon\}} \mathrm{d}_u L^{(2)}(u, x) \mathrm{d}x \leqslant 2M\varepsilon(t - s)$. Taking expectation, we deduce that t is an accumulation of points s such that $\mathbb{P}(L^{(2)}(s, Z^{(2)}_s) \geqslant 1 - \varepsilon, |Z^{(2)}_s| \leqslant M) \leqslant 2M\varepsilon$, hence by continuity of $s \to L^{(2)}(s, Z^{(2)}_s)$ and of $Z^{(2)}$, $\mathbb{P}(L^{(2)}(t, Z^{(2)}_t) = 1, |Z^{(2)}_t| < M) \leqslant 2M\varepsilon$ which proves the claim by making $\varepsilon \to 0$ then $M \to \infty$. It implies that $\Upsilon(Z^{(2)}_{t+\cdot}, 1 - L^{(2)}(t, \cdot))$ is well-defined. Notice that by continuity of $Z^{(2)}$, it implies that $\Upsilon(Z^{(2)}_{s+\cdot}, 1 - L^{(2)}(s, \cdot))$ is well-defined on a neighborhood of t.

Let us go back to the proof of the continuity and fix $t \geqslant 0$. Let $t' < t < t''$ and $x'' := \inf\{x : L^{(2)}(t'', x) = 1\}$. We suppose that t' and t'' are close enough to t so that $x'' > \sup_{\{u \geqslant t'\}} Z^{(2)}_u$ (such t' and t'' exist a.s. from what we just proved). Let $(t_n)_n$ be a sequence which converges to t. We can suppose that $t_n \in (t', t'')$ for all n for simplicity. We apply Lemma 3.7 with $\mathcal{X}_u := Z^{(2)}_{t+u} - Z^{(2)}_t$, $f(x) := 1 - L^{(2)}(t, x + Z^{(2)}_t)$, $I = (-\infty, x'' - Z^{(2)}_t)$, and $\mathcal{X}^n$, f^n, I_n defined similarly by replacing t by t_n. Let us check the assumptions of Lemma 3.7. In the notation of that lemma, (i) and (ii) are verified since all processes start at 0, $j_n = \infty$ and $f(x) = f^n(x) = 1$ for $x \to -\infty$. Assumption (iii) comes from the uniform continuity of $Z^{(2)}$ on compact sets. Let K be a compact set of $\mathbb{R}_+$. Take $\ell_+ \in \mathbb{R}$ such that $\inf_{(t', t'')} Z^{(2)} + \ell_+ > \sup_{u \geqslant t'} Z^{(2)}_u$ and $\sup_{(t', t'')} Z^{(2)} + \ell_+ < x''$ (we can suppose that t and t'' are close enough so that ℓ_+ exists). Then for all $s \geqslant 0$, $Z^{(2)}_{t+s} - Z^{(2)}_t \leqslant \ell_+$. Indeed, $Z^{(2)}_{t+s} - Z^{(2)}_t \leqslant \sup_{u \geqslant t'} Z^{(2)}_u - \inf_{(t', t'')} Z^{(2)} \leqslant \ell_+$. The same lines replacing t by t_n imply that for all $s \geqslant 0$, $Z^{(2)}_{t_n+s} - Z^{(2)}_{t_n} \leqslant \ell_+$ for all n. Similarly, take $\ell_- \in \mathbb{R}$ such that $\sup_{(t', t'')} Z^{(2)} + \ell_- < \inf_{u \leqslant t'' + \sup K} Z^{(2)}_{t''+u}$. We have that for all $s \in K$, $Z^{(2)}_{t+s} - Z^{(2)}_t \geqslant \ell_-$ since $Z^{(2)}_{t+s} - Z^{(2)}_t \geqslant \inf_{u \leqslant t'' + \sup K} Z^{(2)}_u - \sup_{t', t''} Z^{(2)} \geqslant \ell_-$, similarly for t_n. The compact set $L = [\ell_-, \ell_+]$ satisfies assumption (iv). We deduce the convergence of

$\Upsilon(Z^{(2)}_{t_n+\cdot}, 1 - L^{(2)}(t_n, \cdot))$ towards $\Upsilon(Z^{(2)}_{t+\cdot}, 1 - L^{(2)}(t, \cdot))$ (on an event of probability 1 which does not depend on the choice of $(t_n)_n$). □

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