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ON THE REAL ABELIAN MAIN CONJECTURE IN THE NON SEMI-SIMPLE CASE

GEORGES GRAS

ABSTRACT. Let K/\mathbb{Q} be a real cyclic extension of degree divisible by p . We analyze the *statement* of the “Real Abelian Main Conjecture”, for the p -class group \mathcal{H}_K of K , in this non semi-simple case. The classical *algebraic* definition of the p -adic isotopic components $\mathcal{H}_{K,\varphi}^{\text{alg}}$, for irreducible p -adic characters φ , is inappropriate with respect to analytical formulas, because of capitulation of p -classes in the p -sub-extension of K/\mathbb{Q} . In the 1970’s we have given an *arithmetic* definition, $\mathcal{H}_{K,\varphi}^{\text{ar}}$, and formulated the conjecture, still unproven, $\#\mathcal{H}_{K,\varphi}^{\text{ar}} = \#(\mathcal{E}_K/\mathcal{E}_K^\circ \mathcal{F}_K)_{\varphi_0}$, in terms of units \mathcal{E}_K then \mathcal{E}_K° (generated by units of the strict subfields of K) and cyclotomic units \mathcal{F}_K , where φ_0 is the tame part of φ . We prove that the conjecture holds as soon as there exists a prime ℓ , totally inert in K , such that \mathcal{H}_K capitulates in $K(\mu_\ell)$, existence having been checked, in various circumstances, as a promising new tool.

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1. INTRODUCTION AND PRELIMINARY REMARKS

Let K/\mathbb{Q} be a real cyclic extension of Galois group $\mathcal{G}_K \simeq \gamma \oplus \Gamma$, γ of prime-to- p order d , $\Gamma \simeq \mathbb{Z}/p^e\mathbb{Z}$, $e \geq 1$, and let k be the subfield of K fixed by Γ . Let \mathcal{H}_K be the p -class group of K , as $\mathbb{Z}_p[\mathcal{G}_K]$ -module and let E_K (resp. F_K) be the group of units (resp. of Leopoldt’s cyclotomic units) of K and set

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$\mathcal{E}_K = E_K \otimes \mathbb{Z}_p$, $\mathcal{F}_K = F_K \otimes \mathbb{Z}_p$. We consider $\mathcal{E}_K^\circ = E_K^\circ \otimes \mathbb{Z}_p$, where E_K° is the subgroup of E_K generated by the units of the strict subfields of K .

The ‘‘Real Abelian Main Conjecture’’ (RAMC for short) writes:

$$(1.1) \quad \#\mathcal{H}_{K,\varphi}^{\text{ar}} = \#(\mathcal{E}_K/\mathcal{E}_K^\circ \mathcal{F}_K)_{\varphi_0}, \quad \varphi = \varphi_0 \varphi_p$$

(φ_p of p -power order, φ_0 , of prime-to- p order, is called the tame part of the irreducible p -adic character φ of K); RAMC claims analytic expressions of orders of suitable p -adic isotopic components $\mathcal{H}_{K,\varphi}^{\text{ar}}$ of the p -class group of K and was first stated in the papers [3] (1976), [4] (1977) (especially for the *non semi-simple case*); the semi-simple case $p \nmid [K : \mathbb{Q}]$, claiming $\#\mathcal{H}_{K,\varphi} = \#(\mathcal{E}_K/\mathcal{F}_K)_\varphi$, was the purpose of [5] (1977). These conjectures were presented at the meeting ‘‘Journées arithmétiques de Caen’’ (1976) as it is mentioned for instance in a paper of Solomon [19] (1990), in a survey of Ribet [20] (2008) and in Washington’s book [21, § 15.3] (1997). The non semi-simple RAMC is still unproven, contrary to some affirmations in the literature.

Main result (Theorem 4.4). The RAMC (1.1) holds for K as soon as there exists a prime number $\ell \equiv 1 \pmod{2p^N}$, totally inert in K , such that \mathcal{H}_K capitulates in the auxiliary p -sub-extension of $K(\mu_\ell)/K$.

The classical *algebraic* definition of the p -adic isotopic components, in the form $\mathcal{H}_{K,\varphi}^{\text{alg}} := \mathcal{H}_K \otimes_{\mathbb{Z}_p[\mathcal{G}_K]} \mathbb{Z}_p[\mu_{dp^e}]$ with obvious Galois action, is inappropriate with respect to analytic formulas, because of frequent capitulations of p -classes in the p -sub-extension K/k of K/\mathbb{Q} . In our previous works, we have given, for any rational character χ of K and $\varphi \mid \chi$, an *arithmetic* definition of these χ and φ -components, denoted $\mathcal{H}_{K,\chi}^{\text{ar}}$ and $\mathcal{H}_{K,\varphi}^{\text{ar}}$, giving the expected formulas $\#\mathcal{H}_{K,\chi}^{\text{ar}} = \prod_{\varphi \mid \chi} \#\mathcal{H}_{K,\varphi}^{\text{ar}}$ (Theorem 2.2). For convenience, we will refer to the english translation of [3] (1976) given in [7] (2021).

The RAMC has been proven in some semi-simple cases by means of an impressive series of articles (beginning with the geometrical methods of Ribet, Mazur–Wiles, Rubin, then ending by means of Kolyvagin’s Euler Systems from Thaine’s arithmetic approach), then in the non semi-simple case for *relative class groups* of imaginary abelian fields by Solomon and Greither (in the imaginary case, capitulation phenomena of ‘‘minus class groups’’ do not exist and the two definitions of the p -adic isotopic components coincide).

Finally, in the Iwasawa theory context, non semi-simple by nature, a specific Main Conjecture has been proved, in terms of p -adic L -functions, and is also at the origin of many articles (to avoid considerable bibliography, we refer to that given in [7]; a main overview on the story, precise proofs and classical references are given in Washington’s book [21, Chapters 6, 8, 13, 15, Notes]). The fundamental difference, regarding finite p -extensions, is that capitulation kernels are hidden in statements using pseudo-isomorphisms of $\mathbb{Z}_p[[T]]$ -modules, whence only giving results for the projective limit of the p -class groups in the \mathbb{Z}_p -extensions and, in general, no precise information is available in the finite layers (it’s quite clear in a numerical setting that any possible structure occurs in the first layers, up to the algebraic regularity predicted by Iwasawa’s theory; see for instance the numerical computations given in [13, 17]). Nevertheless, in the real case, Greenberg’s conjecture [10] (saying that $\lambda = \mu = 0$) makes it somewhat unnecessary. Indeed, if k is totally real and $K = k_\infty = \bigcup_{n \geq 0} k_n$ the cyclotomic \mathbb{Z}_p -extension of k , then $\lambda = \mu = 0$ is equivalent to the ‘‘stability’’ of the class groups \mathcal{H}_{k_n} in the tower, from some layer $n_0 \gg 0$, in which case, the arithmetic norms $\mathbf{N}_{k_{n+1}/k_n} : \mathcal{H}_{k_{n+1}} \rightarrow \mathcal{H}_{k_n}$ are isomorphisms, the transfer

maps $\mathbf{J}_{k_\infty/k_n} : \mathcal{H}_{k_n} \rightarrow \mathcal{H}_{k_\infty}$ being never injective since any \mathcal{H}_{k_n} capitulates in k_∞ (see, e.g., [9, Theorem 1.2, Remark 1.3] and Grandet–Jaulent [2, Théorème, p. 214] proving that $\mathcal{H}_{k_n} \simeq \text{Ker}(\mathbf{J}_{k_\infty/k_n}) \oplus \mathbf{J}_{k_\infty/k_n}(\mathcal{H}_{k_n})$ for $n \gg 0$).

Thus, in the first layers, the $\#\mathcal{H}_{k_n}$'s are increasing in some random way and the RAMC makes sense in these layers, up to the stabilization from k_{n_0} , where, for all p -adic character φ_n of k_n , $\varphi_n = \varphi_{n,0}\varphi_{n,p}$, the relation $\#\mathcal{H}_{k_n,\varphi_n}^{\text{ar}} = \#(\mathcal{E}_{k_n}/\mathcal{E}_{k_n}^\circ \mathcal{F}_{k_n})_{\varphi_{n,0}}$ holds for all $n \geq 0$ and becomes constant from n_0 , if RAMC is true. As we have explained, *the case of even p -adic characters in a non semi-simple context*, was less understood because of a problematic definition of the isotopic components and also of cyclotomic units; the more conceptual case of Iwasawa's theory has been privileged and generalized in many directions in the framework of Euler's systems.

2. ABELIAN EXTENSIONS AND CHARACTERS

Let \mathcal{G} be the Galois group of the maximal abelian extension \mathbb{Q}^{ab} of \mathbb{Q} and let K be any subfield of finite degree of \mathbb{Q}^{ab} ; put $\mathcal{G}_K = \text{Gal}(K/\mathbb{Q})$.

2.1. Abelian characters. Let Ψ be the set of irreducible characters of \mathcal{G} , of degree 1 and finite order, with values in an algebraic closure $\overline{\mathbb{Q}_p}$. We define the set of irreducible p -adic characters Φ , for a prime $p \geq 2$, the set \mathcal{X} of irreducible rational characters and the subsets of irreducible characters Ψ_K , Φ_K , \mathcal{X}_K , of K . The notation $\psi \mid \varphi \mid \chi$ (for $\psi \in \Psi$, $\varphi \in \Phi$, $\chi \in \mathcal{X}$) means that φ is a term of χ and ψ a term of φ . The set \mathcal{X} has the following elementary property to be considered as the “Main Theorem” for rational components [14, 15]:

Theorem 2.1. *Let K/\mathbb{Q} be an abelian extension and let $(A_\chi)_{\chi \in \mathcal{X}_K}$, $(A'_\chi)_{\chi \in \mathcal{X}_K}$ be two families of positive numbers, indexed by the set \mathcal{X}_K of irreducible rational characters of K . If for all subfields k of K , one has $\prod_{\chi \in \mathcal{X}_k} A'_\chi = \prod_{\chi \in \mathcal{X}_k} A_\chi$, then $A'_\chi = A_\chi$ for all $\chi \in \mathcal{X}_K$.*

The interest of this property is that analytic formulas (giving for instance orders A_K of some finite p -adic invariants \mathcal{A}_K of abelian fields K) may be *canonically* decomposed under the form $A_K = \prod_{\chi \in \mathcal{X}_K} A_\chi$, to be compared with algebraic formula $\#\mathcal{A}_K = \prod_{\chi \in \mathcal{X}_K} \#\mathcal{A}_\chi$ for suitable canonical $\mathbb{Z}_p[\mathcal{G}_K]$ -modules \mathcal{A}_χ , so that $\#\mathcal{A}_\chi = A_\chi$ for all χ ; the RAMC being the same statement, replacing rational characters χ by p -adic ones φ , under the existence of natural relations $\#\mathcal{A}_\chi = \prod_{\varphi \mid \chi} \#\mathcal{A}_\varphi$ and $A_\chi = \prod_{\varphi \mid \chi} A_\varphi$. Unfortunately, classical algebraic definitions of the \mathcal{A}_φ 's does not fulfill these equalities in the non semi-simple case $\#\mathcal{G}_K \equiv 0 \pmod{p}$ as we have shown in [8, 9].

Noting that the fixed field K_χ of $\text{Ker}(\chi)$ is cyclic, there is no restriction to assume K/\mathbb{Q} real cyclic in all the sequel, then such that $K = K_\chi$.

2.2. Algebraic versus Arithmetic isotopic components. Let K/\mathbb{Q} be a real cyclic extension of Galois group $\mathcal{G}_K \simeq \gamma \oplus \Gamma$, γ of prime-to- p order d , $\Gamma \simeq \mathbb{Z}/p^e\mathbb{Z}$, $e \geq 1$. Let $k := K^\Gamma$ and $K_0 := K^\gamma$.

2.2.1. Characters of K . Let χ be the rational character defining K . The field of values of $\psi \mid \chi$ is $\mathbb{Q}(\mu_{dp^e})$, direct compositum $\mathbb{Q}(\mu_d)\mathbb{Q}(\mu_{p^e})$; thus $\psi = \psi_0 \cdot \psi_p$, ψ_0 of order d , ψ_p of order p^e and $\chi = \chi_0 \cdot \chi_p$, $\chi_0 \in \mathcal{X}_k$ above ψ_0 , $\chi_p \in \mathcal{X}_{K_0}$ above ψ_p . Similarly, in the direct compositum $\mathbb{Q}_p(\mu_d)\mathbb{Q}_p(\mu_{p^e})$, irreducible p -adic characters $\varphi \mid \chi$ are of the form $\varphi_0 \cdot \varphi_p$, $\varphi_p = \chi_p$ since $\text{Gal}(\mathbb{Q}_p(\mu_{p^e})/\mathbb{Q}_p) \simeq \text{Gal}(\mathbb{Q}(\mu_{p^e})/\mathbb{Q})$.

Whence $\Psi_K = \Psi_k \cdot \Psi_{K_0}$, $\Phi_K = \Phi_k \cdot \Phi_{K_0}$, $\mathcal{X}_K = \mathcal{X}_k \cdot \mathcal{X}_{K_0}$. In the writing $\varphi = \varphi_0 \cdot \varphi_p$, φ_0 will be called the tame part of φ . Any $\rho \in \mathcal{X}_K$ corresponds to K_ρ , cyclic of degree $d'p^{e'}$, $d' \mid d$ and $e' \leq e$; so, any result for χ and $\varphi \mid \chi$, holds in the same way for ρ and its p -adic characters. Thus, in general, we will state results only for χ .

2.2.2. Definitions of the isotopic components. The non semi-simple context is problematic for the definition of isotopic p -adic components in the form $\mathcal{H}_{K,\varphi}$, $\varphi \mid \chi$, since $\mathcal{H}_K = \bigoplus_{\rho \in \mathcal{X}_K} \bigoplus_{\varphi \mid \rho} \mathcal{H}_{K,\varphi}$ does not make sense in general. Let's recall the definitions and explain how the phenomenon of capitulation gives rise to difficulties about the classical algebraic definition.

For $\mathcal{G}_K = \gamma \oplus \Gamma$ cyclic of order dp^e , $e \geq 1$, and for all $\varphi \mid \chi$, the first classical definition is the following, where P_χ is the dp^e th cyclotomic polynomial, σ_χ a generator of \mathcal{G}_K and $P_\varphi \mid P_\chi$ the local cyclotomic polynomial associated to the action $\tau \in \mathcal{G}_K \mapsto \psi(\tau)$, $\psi \mid \varphi$ and $P_\varphi(\psi(\tau)) = 0$:

$$(2.1) \quad \widehat{\mathcal{H}}_{K,\varphi}^{\text{alg}} := \mathcal{H}_K \otimes_{\mathbb{Z}_p[\mathcal{G}_K]} \mathbb{Z}_p[\mu_{dp^e}] \simeq \mathcal{H}_K / P_\varphi(\sigma_\chi) \cdot \mathcal{H}_K,$$

as $\mathbb{Z}_p[\mu_{dp^e}]$ -module (see, e.g., Solomon [19, II, §1], Greither [11, Definition, p. 451] or Mazigh [16, Introduction]). Another algebraic definition, giving $\mathbb{Z}_p[\mu_{dp^e}]$ -modules, is that of the kernels of the actions of $P_\chi(\sigma_\chi)$ and $P_\varphi(\sigma_\chi)$; it defines what we will call the notions of *algebraic* χ and φ -objects:

$$(2.2) \quad \begin{cases} \mathcal{H}_{K,\chi}^{\text{alg}} := \{x \in \mathcal{H}_K, P_\chi(\sigma_\chi) \cdot x = 1\} \\ \mathcal{H}_{K,\varphi}^{\text{alg}} := \{x \in \mathcal{H}_K, P_\varphi(\sigma_\chi) \cdot x = 1\}. \end{cases}$$

From such an algebraic definition of the χ -objects $\mathcal{H}_{K,\chi}^{\text{alg}}$ in (2.2), we have proved in [7, §3.2.4, Theorem 3.7, Definition 3.11], the interpretation:

$$\mathcal{H}_{K,\chi}^{\text{alg}} = \{x \in \mathcal{H}_K, \nu_{K/\kappa}(x) = 1, \forall \kappa \subsetneq K\},$$

where $\nu_{K/\kappa} = \sum_{\sigma \in \text{Gal}(K/\kappa)} \sigma$ is the algebraic norm in K/κ .

This gives rise to our *arithmetic* definitions of χ and φ -objects, replacing $\nu_{K/\kappa}$ by the arithmetic norm $\mathbf{N}_{K/\kappa}$, noting that $\nu_{K/\kappa} = \mathbf{J}_{K/\kappa} \circ \mathbf{N}_{K/\kappa}$, where $\mathbf{J}_{K/\kappa}$ is the transfer map (or extension of classes deduced of that of ideals):

$$(2.3) \quad \begin{cases} \mathcal{H}_{K,\chi}^{\text{ar}} := \{x \in \mathcal{H}_K, \mathbf{N}_{K/\kappa}(x) = 1, \forall \kappa \subsetneq K\}, \\ \mathcal{H}_{K,\varphi}^{\text{ar}} := \{x \in \mathcal{H}_K, \mathbf{N}_{K/\kappa}(x) = 1, \forall \kappa \subsetneq K \ \& \ P_\varphi(\sigma_\chi) \cdot x = 1\} \end{cases}$$

The above definitions of χ -objects lead to an unexpected semi-simplicity, in accordance with analytic formulas and the non semi-simple RAMC [7, Theorem 4.5]:

Theorem 2.2. *Let K/\mathbb{Q} , $K = K_\chi$, $\chi =: \chi_0 \chi_p$, be a real cyclic extension of Galois group $\mathcal{G}_K \simeq \gamma \oplus \Gamma$, with γ of prime-to- p order d , and $\Gamma \simeq \mathbb{Z}/p^e \mathbb{Z}$, $e \geq 1$. Let $\varphi \mid \chi$, $\varphi =: \varphi_0 \cdot \varphi_p$, $\varphi_0 \mid \chi_0$ (see §2.2.1) and put $e_{\varphi_0} := \frac{1}{d} \sum_{\tau \in \gamma} \varphi_0(\tau^{-1}) \tau$. If the $\mathbb{Z}_p[\mathcal{G}_K]$ -module $\mathcal{A}_{K,\chi}^{\text{ar}}$ is a χ -object, then $\mathcal{A}_{K,\chi}^{\text{ar}} = \bigoplus_{\varphi \mid \chi} \mathcal{A}_{K,\varphi}^{\text{ar}}$, where $\mathcal{A}_{K,\varphi}^{\text{ar}} := (\mathcal{A}_{K,\chi}^{\text{ar}})^{e_{\varphi_0}}$, also denoted $(\mathcal{A}_{K,\chi}^{\text{ar}})_{\varphi_0}$ (cf. definitions (2.3)).*

For $\mathcal{A}_K = \mathcal{H}_K$, since K/k is totally ramified, arithmetic norms $\mathbf{N}_{L/K}$ are surjective (class field theory); so we have the following relations (not fulfilled by the $\mathcal{H}_{K,\rho}^{\text{alg}}$'s because of possible capitulations leading to non surjective $\nu_{K/\kappa}$'s; see §4.3 for some numerical examples):

Corollary 2.3. *We have $\#\mathcal{H}_K = \prod_{\rho \in \mathcal{X}_K} \#\mathcal{H}_{K,\rho}^{\text{ar}} = \prod_{\rho \in \mathcal{X}_K} \prod_{\varphi \mid \rho} \#\mathcal{H}_{K,\varphi}^{\text{ar}}$.*

2.2.3. *Practical characterization of the $\mathcal{H}_{K,\varphi}^{\text{ar}}$ ’s.* The computation of:

$$\mathcal{H}_{K,\varphi}^{\text{ar}} := \{x \in \mathcal{H}_K, \mathbf{N}_{K/\kappa}(x) = 1, \forall \kappa \subsetneq K \text{ \& } P_\varphi(\sigma_\chi) \cdot x = 1\},$$

may be obtained in two commutable steps, from Theorem 2.2:

(i) **Computation of $\mathcal{H}_{K,\chi}^{\text{ar}}$.** So, $\mathcal{H}_{K,\chi}^{\text{ar}} = \{x \in \mathcal{H}_K, \mathbf{N}_{K/\kappa}(x) = 1, \forall \kappa \subsetneq K\}$ is the group of “totally relative classes”. Let \overline{K} be the unique subfield of K such that $[K : \overline{K}] = p$; then the conditions $\mathbf{N}_{K/\kappa'}(x) = 1$ for the subfields κ' , such that $K^\gamma \subseteq \kappa' \subsetneq K$, is automatically fulfilled for the χ_0 -component of the kernel, $\mathcal{H}_{K/\overline{K}}^{\text{ar}}$, of $\mathbf{N}_{K/\overline{K}} : \mathcal{H}_K \rightarrow \mathcal{H}_{\overline{K}}$, thus giving for $\chi = \chi_0 \chi_p$:

$$(2.4) \quad \begin{cases} \mathcal{H}_{K/\overline{K}}^{\text{ar}} := \{x \in \mathcal{H}_K, \mathbf{N}_{K/\overline{K}}(x) = 1\}, \\ \mathcal{H}_{K,\chi}^{\text{ar}} := (\mathcal{H}_{K/\overline{K}}^{\text{ar}})^{e_{\chi_0}} =: \mathcal{H}_{K/\overline{K},\chi_0}^{\text{ar}}, \end{cases}$$

where $e_{\chi_0} \in \mathbb{Z}_p[\gamma]$ is the usual semi-simple idempotent.

(ii) **Computation of $\mathcal{H}_{K,\varphi}^{\text{ar}}$.** From Theorem (2.2) and the semi-simple decomposition $e_{\chi_0} = \sum_{\varphi_0 | \chi_0} e_{\varphi_0}$, one gets, for $\varphi = \varphi_0 \varphi_p$, $\varphi_0 | \chi_0$:

$$(2.5) \quad \begin{cases} \mathcal{H}_{K/\overline{K}}^{\text{ar}} := \{x \in \mathcal{H}_K, \mathbf{N}_{K/\overline{K}}(x) = 1\}, \\ \mathcal{H}_{K,\varphi}^{\text{ar}} = (\mathcal{H}_{K/\overline{K}}^{\text{ar}})^{e_{\varphi_0}} =: \mathcal{H}_{K/\overline{K},\varphi_0}^{\text{ar}}. \end{cases}$$

It is important to note that $\mathcal{H}_{K,\varphi}^{\text{ar}} = \{x \in \mathcal{H}_{K,\varphi_0}, \mathbf{N}_{K/\overline{K}}(x) = 1\}$.

(iii) **Analytic objects.** The $\mathbb{Z}_p[\mathcal{G}_K]$ -module $\mathcal{E}_K / \mathcal{E}_K^\circ \mathcal{F}_K$ is a χ -object since $\mathbf{N}_{K/\kappa}(\mathcal{E}_K) \subseteq \mathcal{E}_K^\circ$, for all $\kappa \subsetneq K$. Whence, from (ii):

$$(\mathcal{E}_K / \mathcal{E}_K^\circ \mathcal{F}_K)_\varphi = (\mathcal{E}_K / \mathcal{E}_K^\circ \mathcal{F}_K)^{e_{\varphi_0}} =: (\mathcal{E}_K / \mathcal{E}_K^\circ \mathcal{F}_K)_{\varphi_0}.$$

3. CLASS GROUPS OF REAL ABELIAN FIELDS – ANALYTIC FORMULAS

Denote by $\mathbf{E}_K := |E_K|$ the group of absolute value of the units of K , the Galois action being defined by $|\varepsilon|^\sigma = |\varepsilon^\sigma|$ for any unit ε and any $\sigma \in \mathcal{G}_K$.

3.1. **The Leopoldt cyclotomic units.** This aspect being very classical, we just recall the definitions (see Leopoldt [14, §8 (1)], [15], or Washington [21, Chap. 8]).

Definition 3.1. (i) Let $\rho \in \mathcal{X}_K \setminus \{1\}$ be of conductor f_ρ , let $\zeta_{2f_\rho} := \exp\left(\frac{i\pi}{f_\rho}\right)$, and put $\theta_\rho := \prod_{a \in A_\rho} (\zeta_{2f_\rho}^a - \zeta_{2f_\rho}^{-a})$, where $A_\rho \subset (\mathbb{Z}/f_\rho\mathbb{Z})^\times$ is a half-system of representatives corresponding to $\text{Gal}(\mathbb{Q}(\zeta_{2f_\rho} + \zeta_{2f_\rho}^{-1})/K_\rho)$.

(ii) Let \mathbf{F}_K be the intersection with \mathbf{E}_K of the multiplicative group generated by the conjugates of the $|\theta_\rho|$ ’s, for $\rho \in \mathcal{X}_K \setminus \{1\}$. This defines the group of cyclotomic units of K and we put $\mathcal{E}_K := \mathbf{E}_K \otimes \mathbb{Z}_p$, $\mathcal{F}_K := \mathbf{F}_K \otimes \mathbb{Z}_p$.

(iii) For the cyclic real field K of degree dp^e , let’s denote by \mathbf{E}_K° the subgroup of \mathbf{E}_K generated by the \mathbf{E}_κ ’s for all the subfields $\kappa \subsetneq K$.

3.2. **Arithmetic computation of $\#\mathcal{H}_{K,\chi}^{\text{ar}}$.** Since the RAMC is trivial for characters χ of p -power order ($\varphi = \chi$), we assume $d = [k : \mathbb{Q}] > 1$ and $\chi = \chi_0 \chi_p$, $\chi_0 \neq 1$. In that case, interpretation of Leopoldt’s formulas [14, 15] yields, in the spirit of Theorem 2.1 (see [3, pp. 71–75] for details and proofs, or [7, Section 7, §§7.1–7.3] for an overview):

Theorem 3.2. *Let $\mathcal{H}_{K,\chi}^{\text{ar}} := \{x \in \mathcal{H}_K, \mathbf{N}_{K/\kappa}(x) = 1, \text{ for all } \kappa \subsetneq K\}$. Then $\#\mathcal{H}_{K,\chi}^{\text{ar}} = (\mathcal{E}_K : \mathcal{E}_K^\circ \mathcal{F}_K)$, where \mathcal{E}_K° is the subgroup of \mathcal{E}_K generated by the \mathcal{E}_κ ’s for all the subfields $\kappa \subsetneq K$.*

Our purpose is to obtain $\#\mathcal{H}_{K,\varphi}^{\text{ar}} = \#(\mathcal{E}_K/\mathcal{E}_K^\circ \mathcal{F}_K)_{\varphi_0}$, for all $\varphi_0 \mid \chi_0$, which constitutes the RAMC. Such a result will be obtained if there exists an auxiliary cyclotomic extension $K(\mu_\ell)$, ℓ inert in K , in which \mathcal{H}_K capitulates, a trick showing that some natural properties “above K ” are logically overhead the arithmetic of the base field.

Remark 3.3. The analytic formula given by this theorem, which appears to have been less known, seems more convenient than formulas using other groups \mathcal{F}'_K of cyclotomic units. Indeed, compare with [11, Theorem 4.14] using instead *algebraic isotopic components* in a partial semi-simple statement saying that for all $\varphi_0 \in \Phi_k$, $\#(\mathcal{E}_K/\mathcal{F}'_K)_{\varphi_0} = \#\mathcal{H}_{K,\varphi_0}$, up to an explicit power of 2 and an index $(\mathbb{Z}[\mathcal{G}_K] : U)$ given in the framework of Sinnott’s group \mathcal{F}'_K of cyclotomic units larger than classical Leopoldt’s group of Definition 3.1.

3.3. Norm properties of Leopoldt’s cyclotomic units. Let $f > 1$ and let $m \mid f$, with $m > 1$, be any modulus; let $\mathbb{Q}(\zeta_m) \subseteq \mathbb{Q}(\zeta_f)$ be the corresponding cyclotomic fields with $\zeta_t := \exp\left(\frac{2i\pi}{t}\right)$, for all $t \geq 1$. Put $\eta_f := 1 - \zeta_f$, $\eta_m := 1 - \zeta_m$. It is well known that $\mathbf{N}_{\mathbb{Q}(\zeta_f)/\mathbb{Q}(\zeta_m)}(\eta_f) = \eta_m^\Omega$, with $\Omega = \prod_{\ell \mid f, \ell \nmid m} \left(1 - \left(\frac{\mathbb{Q}(\zeta_m)}{\ell}\right)^{-1}\right)$, where $\left(\frac{\mathbb{Q}(\zeta_m)}{\ell}\right) \in \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ denotes the Frobenius (or Artin) automorphism of the prime number $\ell \nmid m$ (see, e.g., [8, §4.2]). Applying this result to our context $L/K/\mathbb{Q}$ yields:

Proposition 3.4. *Let m be the conductor of K and let $L \subset K(\mu_\ell)$, ℓ being totally inert in K ; the conductor f of L is ℓm . Set $\eta_L := \mathbf{N}_{\mathbb{Q}^f/L}(\eta_f)$ and $\eta_K := \mathbf{N}_{\mathbb{Q}^m/K}(\eta_m)$; then, $\mathbf{N}_{L/K}(\eta_L) = \eta_K^\Omega$, with $\Omega = 1 - \left(\frac{K}{\ell}\right)^{-1}$, and Ωe_{φ_0} is invertible in $\mathbb{Z}_p[\mathcal{G}_K]e_{\varphi_0}$, for all $\varphi_0 \in \Phi_k \setminus \{1\}$. Then $\mathbf{N}_{L/K}(\mathcal{F}_{L,\varphi_0}) = \mathcal{F}_{K,\varphi_0}$ with the Definition 3.1 of cyclotomic units.*

Proof. Indeed, $\tau_{\ell,K} := \left(\frac{K}{\ell}\right)^{-1}$ generates \mathcal{G}_K since ℓ is inert in K/\mathbb{Q} ; write $\tau_{\ell,K} =: \tau\sigma$, τ of order d , σ of order p^e . So for $\psi_0 \mid \varphi_0 \neq 1$, $\psi_0(1-\tau) = 1 - \psi_0(\tau)$ is a unit of $\mathbb{Q}(\mu_d)$ if $\xi := \psi_0(\tau)$ is not of prime power order, otherwise, $(1-\xi)$ is a prime ideal above a prime $q \mid d$ and $1-\xi$ is a p -adic unit. Whence, $\psi_0(1-\tau_{\ell,K}) = 1 - \xi + \xi(1-\sigma)$; thus $1 - \tau_{\ell,K}$ is invertible in $\mathbb{Z}_p[\mathcal{G}_K]$ since $\psi_p(1-\sigma)$ is a uniformizing parameter at p . Since ℓ is totally inert in K/\mathbb{Q} , this applies to any subfield K' of K , of character $\chi' = \chi'_0\chi'_p$ with $\chi'_0 \neq 1$ and its p -adic characters $\varphi'_0 \neq 1$, with $L' = L \cap K'(\mu_\ell)$ and $\Omega' = 1 - \left(\frac{K'}{\ell}\right)^{-1}$.

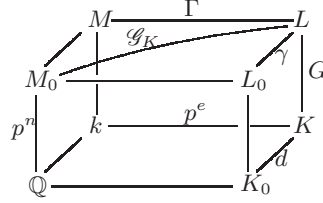
The module $\mathcal{F}_{K,\varphi_0}$, $\varphi_0 \mid \chi_0$, only depends on the cyclotomic units of the fields K' of characters $\chi' = \chi_0\chi_p^j$, $0 \leq j \leq e$, that is to say, $k \subseteq K' \subseteq K$.

Whence the norm relation between the φ_0 -components of the Leopoldt cyclotomic units for any $\varphi_0 \neq 1$; indeed, Leopoldt’s definition 3.1 writes $\theta_f = \zeta_{2f} - \zeta_{2f}^{-1} = -\zeta_{2f}^{-1}(1 - \zeta_{2f}^2) = -\zeta_{2f}^{-1}(1 - \zeta_f) = -\zeta_{2f}^{-1}\eta_f$ and norms are taken over real fields L and K . \square

4. THE p -LOCALIZATION OF THE CHEVALLEY–HERBRAND FORMULA

Let K/\mathbb{Q} be a cyclic extension of Galois group $\mathcal{G}_K = \gamma \oplus \Gamma$, γ of prime-to- p order d and $\Gamma \simeq \mathbb{Z}/p^e\mathbb{Z}$, $e \geq 1$; let $k = K^\Gamma$, $K_0 = K^\gamma$. Let M_0/\mathbb{Q} be a real cyclic extension of degree p^n , $n \geq 1$, linearly disjoint from K , and let $L = M_0K$; so any prime ideal of K , ramified in L/K , is totally ramified (this will be the principle for the proof of the RAMC, only using $M_0 \subset \mathbb{Q}(\mu_\ell)$, $\ell \equiv 1$

(mod $2p^N$), $N \geq n$, ℓ inert in K/\mathbb{Q} , under capitulation of \mathcal{H}_K in L).



Under the previous assumptions, the Chevalley–Herbrand formula for p -class groups [1, pp. 402-406] is the following, where $G := \text{Gal}(L/K)$:

$$\#\mathcal{H}_L^G = \#\mathcal{H}_K \times \frac{p^{n(r-1)}}{(E_K : E_K \cap \mathbf{N}_{L/K}(L^\times))},$$

$r \geq 1$ being the number of prime ideals of K ramified in L/K . If $r = 1$, then:

$$\#\mathcal{H}_L^G = \#\mathcal{H}_K,$$

which does not imply any group isomorphism $\mathcal{H}_L^G \simeq \mathcal{H}_K$; this subtlety of Chevalley–Herbrand formula will be the key for the proof which needs to compute the $\#\mathcal{H}_{L,\varphi_0}^G$'s, $\varphi_0 \in \Phi_k$, that we call the p -localizations of the formula; even if one could guess the result, the frequent capitulation (total or not) of \mathcal{H}_K in L , goes against a trivial proof.

4.1. The p -localizations of \mathcal{H}_L^G as $\mathbb{Z}_p[\mathcal{G}_K]$ -module. The p -localizations of the Chevalley–Herbrand formula were given, for the case $p \nmid [K : \mathbb{Q}]$ (i.e., $K_0 = \mathbb{Q}$), in the 1970/1980's (see [8, 9] for some story), but here we must give some improvements, regarding χ as character of $\text{Gal}(L/M_0) \simeq \mathcal{G}_K$, with $\chi_p \neq 1$. In other words, for $\varphi_0 \in \Phi_k$, we consider $\mathcal{H}_{L,\varphi_0}$ as $\mathbb{Z}_p[\mathcal{G}_K]$ -module for which $(\mathcal{H}_{L,\varphi_0})^G = (\mathcal{H}_L^G)^{e_{\varphi_0}}$ (denoted $\mathcal{H}_{L,\varphi_0}^G$), since $(\mathcal{H}_L^{e_{\varphi_0}})^G = (\mathcal{H}_L^G)^{e_{\varphi_0}}$ by commutativity (non semi-simple Γ -modules).

To get a formula for the orders of the φ_0 -components $\mathcal{H}_{L,\varphi_0}^G$, we follow the process given in Jaulent's Thesis [12, Chapitre III, p. 167].

Theorem 4.1. *Let K/\mathbb{Q} be a real cyclic extension of Galois group $\mathcal{G}_K = \gamma \oplus \Gamma$, γ of prime-to- p order d and $\Gamma \simeq \mathbb{Z}/p^e\mathbb{Z}$, $e \geq 1$. Let $k := K^\Gamma$.*

Let M_0/\mathbb{Q} be the real cyclic extension of degree p^n , $n \in [1, N]$, contained in $\mathbb{Q}(\mu_\ell)$, where $\ell \equiv 1 \pmod{2p^N}$; we assume that ℓ is totally inert in K . Put $L = M_0K$ and $G := \text{Gal}(L/K) =: \langle \sigma \rangle$. Let $\mathbf{j}_{L/K}$ be the extension of ideals $I_K \rightarrow I_L^G$ and $\mathbf{J}_{L/K}$ the corresponding transfer $\mathbf{H}_K \rightarrow \mathbf{H}_L^G$ of classes.

a) *We have the exact sequence of $\mathbb{Z}[\mathcal{G}_K \times G]$ -modules:*

$$\begin{aligned} 1 \longrightarrow \text{Ker}(\mathbf{J}_{L/K}) \longrightarrow \text{H}^1(G, E_L) \longrightarrow \text{Coker}(\mathbf{j}_{L/K}) \longrightarrow \\ \longrightarrow \text{Coker}(\mathbf{J}_{L/K}) \longrightarrow \text{H}^2(G, E_L) \longrightarrow E_K/E_K \cap \mathbf{N}(L^\times) \rightarrow 1; \end{aligned}$$

If \mathcal{H}_K capitulates in L , then:

b) $\#\mathcal{H}_{L,\varphi_0}^G = \#\mathcal{H}_{K,\varphi_0}$, for all $\varphi_0 \in \Phi_k \setminus \{1\}$;

c) $\mathcal{H}_{K,\varphi_0} \simeq \text{H}^1(G, \mathcal{E}_{L,\varphi_0}) = \mathcal{E}_{L,\varphi_0}^* / \mathcal{E}_{L,\varphi_0}^{1-\sigma}$ and $\mathcal{H}_{L,\varphi_0}^G \simeq \text{H}^2(G, \mathcal{E}_{L,\varphi_0}) = \mathcal{E}_{K,\varphi_0} / \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi_0})$, for all $\varphi_0 \in \Phi_k \setminus \{1\}$.

Proof. Let I_K and I_L (resp. P_K and P_L) be the ideal groups (resp. the subgroups of principal ideals), of K and L , respectively. Consider the exact sequences of $\mathbb{Z}[\mathcal{G}_K \times G]$ -modules:

$$(4.1) \quad 1 \rightarrow E_L \rightarrow L^\times \rightarrow P_L \rightarrow 1, \quad 1 \rightarrow E_K \rightarrow K^\times \rightarrow P_K \rightarrow 1$$

$$(4.2) \quad 1 \rightarrow P_L \rightarrow I_L \rightarrow \mathbf{H}_L \rightarrow 1, \quad 1 \rightarrow P_K \rightarrow I_K \rightarrow \mathbf{H}_K \rightarrow 1.$$

Lemma 4.2. *We have the following properties:*

- (i) $\text{Coker}(\mathbf{j}'_{L/K} : P_K \rightarrow P_L^G) \simeq H^1(G, E_L)$;
- (ii) $H^1(G, P_L) \simeq \text{Ker}[H^2(G, E_L) \rightarrow H^2(G, L^\times)]$;
- (iii) $H^1(G, I_L) = 1$.

Proof. We have, from exact sequences (4.1):

$$\begin{aligned} 1 \rightarrow E_L^G = E_K \rightarrow L^\times G = K^\times \rightarrow P_L^G \rightarrow H^1(G, E_L) \rightarrow H^1(G, L^\times) \\ \rightarrow H^1(G, P_L) \rightarrow H^2(G, E_L) \rightarrow H^2(G, L^\times). \end{aligned}$$

Since $H^1(G, L^\times) = 1$ (Hilbert's Theorem 90), this yields (i) and (ii). The claim (iii) is classical since I_L is a $\mathbb{Z}[G]$ -module generated by the prime ideals of L on which the Galois action is canonical. \square

From exact sequences (4.2), we obtain the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_K & \longrightarrow & I_K & \longrightarrow & \mathbf{H}_K & \longrightarrow & 1 \\ & & \downarrow \mathbf{j}'_{L/K} & & \downarrow \mathbf{j}_{L/K} & & \downarrow \mathbf{J}_{L/K} & & \\ 1 & \longrightarrow & P_L^G & \longrightarrow & I_L^G & \longrightarrow & \mathbf{H}_L^G & \longrightarrow & H^1(G, P_L) \longrightarrow 1. \end{array}$$

The snake lemma gives the exact sequence:

$$\begin{aligned} 1 \rightarrow \text{Ker}(\mathbf{j}'_{L/K}) \rightarrow \text{Ker}(\mathbf{j}_{L/K}) \rightarrow \text{Ker}(\mathbf{J}_{L/K}) \rightarrow \text{Coker}(\mathbf{j}'_{L/K}) = H^1(G, E_L) \\ \rightarrow \text{Coker}(\mathbf{j}_{L/K}) \rightarrow \text{Coker}(\mathbf{J}_{L/K}) \rightarrow H^2(G, E_L) \xrightarrow{u} H^2(G, L^\times), \end{aligned}$$

since $\text{Ker}(\mathbf{j}'_{L/K}) = \text{Ker}(\mathbf{j}_{L/K}) = 1$ and $H^2(G, L^\times) \simeq K^\times / \mathbf{N}_{L/K}(L^\times)$, the above exact sequence becomes with $\text{Im}(u) = E_K/E_K \cap \mathbf{N}_{L/K}(L^\times)$:

$$\begin{aligned} 1 \rightarrow \text{Ker}(\mathbf{J}_{L/K}) \rightarrow H^1(G, E_L) \rightarrow \text{Coker}(\mathbf{j}_{L/K}) \rightarrow \\ \text{Coker}(\mathbf{J}_{L/K}) \rightarrow H^2(G, E_L) \rightarrow E_K/E_K \cap \mathbf{N}_{L/K}(L^\times) \rightarrow 1, \end{aligned}$$

proving (a). Whence the localized exact sequences of $\mathbb{Z}_p[\mathcal{G}_K]$ -modules:

$$(4.3) \quad \begin{cases} 1 \rightarrow \text{Ker}(\mathbf{J}_{L/K})_{\varphi_0} \rightarrow H^1(G, \mathcal{E}_{L, \varphi_0}) \rightarrow \text{Coker}(\mathbf{j}_{L/K})_{\varphi_0} \rightarrow \\ \text{Coker}(\mathbf{J}_{L/K})_{\varphi_0} \rightarrow H^2(G, \mathcal{E}_{L, \varphi_0}) \rightarrow \mathcal{E}_{K, \varphi_0} / \mathcal{E}_{K, \varphi_0} \cap \mathbf{N}_{L/K}(L^\times) \rightarrow 1; \end{cases}$$

since L/K is totally ramified at the unique prime (ℓ) of K (inert in K/\mathbb{Q}), $\text{Coker}(\mathbf{j}_{L/K})_{\varphi_0} = 1$, $(\mathcal{E}_K / \mathcal{E}_K \cap \mathbf{N}_{L/K}(L^\times))_{\varphi_0} = 1$ for $\varphi_0 \in \Phi_k \setminus \{1\}$, giving:

$$(4.4) \quad \begin{cases} 1 \longrightarrow \text{Ker}(\mathbf{J}_{L/K})_{\varphi_0} \longrightarrow H^1(G, \mathcal{E}_{L, \varphi_0}) \longrightarrow 1 \\ 1 \longrightarrow \text{Coker}(\mathbf{J}_{L/K})_{\varphi_0} \longrightarrow H^2(G, \mathcal{E}_{L, \varphi_0}) \longrightarrow 1. \end{cases}$$

The capitulation of $\mathcal{H}_{K, \varphi_0}$ in L yields the semi-simple p -localized formula:

$$(4.5) \quad \frac{\#\text{Coker}(\mathbf{J}_{L/K})_{\varphi_0}}{\#\text{Ker}(\mathbf{J}_{L/K})_{\varphi_0}} = \frac{\#\mathcal{H}_{L, \varphi_0}^G}{\#\mathcal{H}_{K, \varphi_0}} = \frac{\#H^2(G, \mathcal{E}_{L, \varphi_0})}{\#H^1(G, \mathcal{E}_{L, \varphi_0})},$$

where $h(\mathcal{E}_{L, \varphi_0}) := \frac{\#H^2(G, \mathcal{E}_{L, \varphi_0})}{\#H^1(G, \mathcal{E}_{L, \varphi_0})}$ is the Herbrand quotient of $\mathcal{E}_{L, \varphi_0}$. Since

$(E_L \otimes \mathbb{Q}) \oplus \mathbb{Q}$ is the regular representation, there exists a ‘‘Minkowski unit’’ ε such that the $\mathbb{Z}[\gamma][\Gamma \times G]$ -module generated by ε is of prime-to- p index in E_L ; so, $\mathcal{E}_L \oplus \mathbb{Z}_p \simeq \mathbb{Z}_p[\gamma][\Gamma \times G]$ as γ -modules. Thus, $(\mathcal{E}_L \oplus \mathbb{Z}_p)_{\varphi_0} = \mathcal{E}_{L, \varphi_0} \simeq \mathbb{Z}_p[\mu_{d^*}][\Gamma \times G]$ for $\varphi_0 \neq 1$, proving that $h(\mathcal{E}_{L, \varphi_0}) = 1$. In particular, this yields $\#\mathcal{H}_{L, \varphi_0}^G = \#\mathcal{H}_{K, \varphi_0}$ (point (b)); then (c) comes from (4.4) giving the isomorphisms $\mathcal{H}_{L, \varphi_0}^G \simeq \mathcal{E}_{K, \varphi_0} / \mathbf{N}_{L/K}(\mathcal{E}_{L, \varphi_0})$ and $\mathcal{H}_{K, \varphi_0} \simeq \mathcal{E}_{L, \varphi_0}^* / \mathcal{E}_{L, \varphi_0}^{1-\sigma}$. \square

4.2. Proof of the RAMC under capitulation in $K(\mu_\ell)$. We can think about the main result involving Theorem 4.1 (b), (c), under capitulation of \mathcal{H}_K in $L \subset K(\mu_\ell)$, $\ell \equiv 1 \pmod{2p^N}$ totally inert in K ; so:

$$(4.6) \quad \#\mathcal{H}_{L,\varphi_0}^G = \#\mathcal{H}_{K,\varphi_0} \quad \& \quad \mathcal{H}_{L,\varphi_0}^G \simeq (\mathcal{E}_K/\mathbf{N}_{L/K}(\mathcal{E}_L))_{\varphi_0}, \quad \varphi_0 \in \Phi_k \setminus \{1\}.$$

Knowing only the relation $\#\mathcal{H}_{K,\chi}^{\text{ar}} = (\mathcal{E}_K : \mathcal{E}_K^\circ \mathcal{F}_K)$ (Theorem 3.2), we have to prove (Theorem 2.2 giving the semi-simple decomposition of an arithmetic χ -object into its φ -components), that:

$$\#\mathcal{H}_{K,\varphi}^{\text{ar}} = \#(\mathcal{E}_K/\mathcal{E}_K^\circ \mathcal{F}_K)_{\varphi_0}, \quad \forall \varphi = \varphi_0 \chi_p, \quad \varphi_0 \mid \chi_0;$$

in fact, we will prove $\#\mathcal{H}_{K,\varphi}^{\text{ar}} \leq \#(\mathcal{E}_K/\mathcal{E}_K^\circ \mathcal{F}_K)_{\varphi_0}$, but the above global formula $\#\mathcal{H}_{K,\chi}^{\text{ar}} = \#(\mathcal{E}_K/\mathcal{E}_K^\circ \mathcal{F}_K)$ in which:

$$\#\mathcal{H}_{K,\chi}^{\text{ar}} = \prod_{\varphi \mid \chi} \#\mathcal{H}_{K,\varphi}^{\text{ar}} \quad \text{and} \quad \#(\mathcal{E}_K/\mathcal{E}_K^\circ \mathcal{F}_K) = \prod_{\varphi_0 \mid \chi_0} \#(\mathcal{E}_K/\mathcal{E}_K^\circ \mathcal{F}_K)_{\varphi_0},$$

will imply equalities.

Lemma 4.3. *Let \overline{K} and \overline{L} be the subfields of K and L , respectively, such that $[K : \overline{K}] = [L : \overline{L}] = p$. If \mathcal{H}_K capitulates in L , then $\mathcal{H}_{\overline{K}}$ capitulates in \overline{L} .*

Proof. Recall that $\mathbf{N}_{L/K}(\mathcal{H}_L) = \mathcal{H}_K$ and $\mathbf{N}_{L/\overline{L}}(\mathcal{H}_L) = \mathcal{H}_{\overline{L}}$ since the extensions are totally ramified. The capitulation of \mathcal{H}_K in L is equivalent to $\nu_{L/K}(\mathcal{H}_L) = 1$ since $\nu_{L/K} = \mathbf{J}_{L/K} \circ \mathbf{N}_{L/K}$. If so, $1 = \mathbf{N}_{L/\overline{L}} \circ \nu_{L/K}(\mathcal{H}_L) = \nu_{\overline{L}/\overline{K}}(\mathbf{N}_{L/\overline{L}}(\mathcal{H}_L)) = \nu_{\overline{L}/\overline{K}}(\mathcal{H}_{\overline{L}})$ and $\mathcal{H}_{\overline{L}}$ capitulates in \overline{L} (analogous results for the subfields k_i of K regarding the extensions $k_i L^\Gamma \subset L$). \square

Theorem 4.4. *Let K/\mathbb{Q} be a real cyclic extension of Galois group $\mathcal{G}_K = \gamma \oplus \Gamma$, γ of prime-to- p order d and $\Gamma \simeq \mathbb{Z}/p^e \mathbb{Z}$, $e \geq 1$. Let M_0/\mathbb{Q} be the real cyclic extension of degree p^N , contained in $\mathbb{Q}(\mu_\ell)$, where $\ell \equiv 1 \pmod{2p^N}$, $N \geq 1$, and let $L = M_0 K$; we assume, moreover, that ℓ is totally inert in K/\mathbb{Q} .*

If \mathcal{H}_K capitulates in L , then $\#\mathcal{H}_{K,\varphi}^{\text{ar}} = \#(\mathcal{E}_K/\mathcal{E}_K^\circ \mathcal{F}_K)_{\varphi_0}$, for all $\varphi \mid \chi$, giving the RAMC for K .

Proof. For short, put $\mathbf{N} = \mathbf{N}_{L/K} = \mathbf{N}_{\overline{L}/\overline{K}}$, $\overline{\mathbf{N}} = \mathbf{N}_{L/\overline{L}} = \mathbf{N}_{K/\overline{K}}$. An index $(\mathcal{A} : \mathcal{B})_{\varphi_0}$ means $\#(\mathcal{A}/\mathcal{B})_{\varphi_0} = \#(\mathcal{A}_{\varphi_0}/\mathcal{B}_{\varphi_0})$. We compute $\mathcal{H}_{K,\varphi}^{\text{ar}}$ as the set of elements of $\mathcal{H}_{K,\varphi_0}$ such that $\overline{\mathbf{N}}(x) = 1$ (i.e., $\mathcal{H}_{K/\overline{K},\varphi_0}^{\text{ar}}$).

We have $(\mathcal{E}_K : \mathcal{E}_K^\circ \mathcal{F}_K)_{\varphi_0} = (\mathcal{E}_K : \mathcal{E}_{\overline{K}} \mathcal{F}_K)_{\varphi_0}$ since for the fields κ such that $K^\gamma \subseteq \kappa \subsetneq K$, $\mathcal{E}_{\kappa,\varphi_0} = 1$. We have the exact sequence (cf. (2.4), (2.5))

$$1 \rightarrow \mathcal{H}_{K/\overline{K},\varphi_0}^{\text{ar}} = \mathcal{H}_{K,\varphi}^{\text{ar}} \rightarrow \mathcal{H}_{K,\varphi_0} \xrightarrow{\overline{\mathbf{N}}} \mathcal{H}_{\overline{K},\varphi_0} \rightarrow 1, \quad \text{then, from (4.6) and Lemma 4.3, } \#\mathcal{H}_{K,\varphi_0} = (\mathcal{E}_K : \mathbf{N}(\mathcal{E}_L))_{\varphi_0} \quad \text{and} \quad \#\mathcal{H}_{\overline{K},\varphi_0} = (\mathcal{E}_{\overline{K}} : \mathbf{N}(\mathcal{E}_{\overline{L}}))_{\varphi_0},$$

giving, under capitulations, the fundamental relation:

$$(4.7) \quad \#\mathcal{H}_{K,\varphi}^{\text{ar}} = \frac{(\mathcal{E}_K : \mathbf{N}(\mathcal{E}_L))_{\varphi_0}}{(\mathcal{E}_{\overline{K}} : \mathbf{N}(\mathcal{E}_{\overline{L}}))_{\varphi_0}},$$

to be compared with $X_{\varphi_0} := (\mathcal{E}_K : \mathcal{E}_{\overline{K}} \mathcal{F}_K)_{\varphi_0}$. From the exact sequence:

$$1 \rightarrow (\mathcal{E}_{\overline{K}} \mathbf{N}(\mathcal{E}_L)/\mathbf{N}(\mathcal{E}_L))_{\varphi_0} \rightarrow (\mathcal{E}_K/\mathbf{N}(\mathcal{E}_L))_{\varphi_0} \rightarrow (\mathcal{E}_K/\mathcal{E}_{\overline{K}} \mathbf{N}(\mathcal{E}_L))_{\varphi_0} \rightarrow 1$$

and Corollary 3.4 giving $\mathcal{E}_{\overline{K}} \mathcal{F}_K = \mathcal{E}_{\overline{K}} \mathbf{N}(\mathcal{F}_L) \subseteq \mathcal{E}_{\overline{K}} \mathbf{N}(\mathcal{E}_L) \subseteq \mathcal{E}_K$, one gets:

$$(4.8) \quad X_{\varphi_0} = \frac{(\mathcal{E}_K : \mathbf{N}(\mathcal{E}_L))_{\varphi_0}}{(\mathcal{E}_{\overline{K}} \mathbf{N}(\mathcal{E}_L) : \mathbf{N}(\mathcal{E}_L))_{\varphi_0}} \times (\mathcal{E}_{\overline{K}} \mathbf{N}(\mathcal{E}_L) : \mathcal{E}_{\overline{K}} \mathcal{F}_K)_{\varphi_0}.$$

Let:

$$(4.9) \quad Y_{\varphi_0} := (\mathcal{E}_{\overline{K}} \mathbf{N}(\mathcal{E}_L) : \mathcal{E}_{\overline{K}} \mathcal{F}_K)_{\varphi_0} \quad \text{and} \quad \overline{\mathcal{F}}_K := \mathcal{F}_K \cap \mathcal{E}_{\overline{K}};$$

since $\mathcal{E}_{\overline{K}} \cap [\mathbf{N}(\mathcal{E}_{\overline{L}})\overline{\mathcal{F}}_K] = \mathbf{N}(\mathcal{E}_{\overline{L}})\overline{\mathcal{F}}_K$, we have the exact sequence:

$$\begin{aligned} 1 \rightarrow (\mathbf{N}(\mathcal{E}_{\overline{L}})\overline{\mathcal{F}}_K/\mathbf{N}(\mathcal{E}_{\overline{L}}))_{\varphi_0} &\longrightarrow (\mathcal{E}_{\overline{K}}/\mathbf{N}(\mathcal{E}_{\overline{L}}))_{\varphi_0} \\ &\longrightarrow (\mathcal{E}_{\overline{K}}\overline{\mathcal{F}}_K/\mathbf{N}(\mathcal{E}_{\overline{L}})\overline{\mathcal{F}}_K)_{\varphi_0} \rightarrow 1, \end{aligned}$$

and $\mathbf{N}(\mathcal{E}_{\overline{L}})\overline{\mathcal{F}}_K \subseteq \mathcal{E}_{\overline{K}}\overline{\mathcal{F}}_K \subseteq \mathcal{E}_{\overline{K}}\mathbf{N}(\mathcal{E}_{\overline{L}})$, then giving from (4.9):

$$\begin{aligned} Y_{\varphi_0} &= \frac{(\mathcal{E}_{\overline{K}}\mathbf{N}(\mathcal{E}_{\overline{L}}) : \mathbf{N}(\mathcal{E}_{\overline{L}})\overline{\mathcal{F}}_K)_{\varphi_0}}{(\mathcal{E}_{\overline{K}}\overline{\mathcal{F}}_K : \mathbf{N}(\mathcal{E}_{\overline{L}})\overline{\mathcal{F}}_K)_{\varphi_0}} \\ &= \frac{(\mathcal{E}_{\overline{K}}\mathbf{N}(\mathcal{E}_{\overline{L}}) : \mathbf{N}(\mathcal{E}_{\overline{L}})\overline{\mathcal{F}}_K)_{\varphi_0} \times (\mathbf{N}(\mathcal{E}_{\overline{L}})\overline{\mathcal{F}}_K : \mathbf{N}(\mathcal{E}_{\overline{L}}))_{\varphi_0}}{(\mathcal{E}_{\overline{K}} : \mathbf{N}(\mathcal{E}_{\overline{L}}))_{\varphi_0}}. \end{aligned}$$

So, from (4.7), (4.8), $X_{\varphi_0} = \frac{(\mathcal{E}_K : \mathbf{N}(\mathcal{E}_L))_{\varphi_0}}{(\mathcal{E}_{\overline{K}} : \mathbf{N}(\mathcal{E}_{\overline{L}}))_{\varphi_0}} \times Z_{\varphi_0} = \#\mathcal{H}_{K,\varphi}^{\text{ar}} \times Z_{\varphi_0}$, where:

$$Z_{\varphi_0} := \frac{(\mathcal{E}_{\overline{K}}\mathbf{N}(\mathcal{E}_L) : \mathbf{N}(\mathcal{E}_{\overline{L}})\overline{\mathcal{F}}_K)_{\varphi_0}}{(\mathcal{E}_{\overline{K}}\mathbf{N}(\mathcal{E}_L) : \mathbf{N}(\mathcal{E}_L))_{\varphi_0}} \times (\mathbf{N}(\mathcal{E}_{\overline{L}})\overline{\mathcal{F}}_K : \mathbf{N}(\mathcal{E}_{\overline{L}}))_{\varphi_0} \geq 1,$$

since $\mathbf{N}(\mathcal{E}_{\overline{L}})\overline{\mathcal{F}}_K \subseteq \mathbf{N}(\mathcal{E}_L)$. Thus, inequalities $\#\mathcal{H}_{K,\varphi}^{\text{ar}} \leq (\mathcal{E}_K : \mathcal{E}_{\overline{K}}\overline{\mathcal{F}}_K)_{\varphi_0}$ are equalities because of product formulas over $\varphi \mid \chi$ giving, as explained, $\#\mathcal{H}_{K,\chi}^{\text{ar}} = (\mathcal{E}_K : \mathcal{E}_K^{\circ}\overline{\mathcal{F}}_K)$; whence the theorem and, since $Z_{\varphi_0} = 1$, the supplementary information:

$$\mathbf{N}(\mathcal{E}_{L,\varphi_0}) = \mathbf{N}(\mathcal{E}_{\overline{L},\varphi_0})\overline{\mathcal{F}}_{K,\varphi_0} = \mathbf{N}(\mathcal{E}_{\overline{L},\varphi_0}\overline{\mathcal{F}}_{L,\varphi_0}),$$

which generalizes the semi-simple case, $\mathbf{N}(\mathcal{E}_{L,\varphi}) = \mathbf{N}(\overline{\mathcal{F}}_{L,\varphi})$ [8, Theorem 4.6], and justifies that relation (4.7) does not seem to depend on cyclotomic units, which may look mysterious but comes from the specific properties of L/K . \square

4.3. Numerical illustrations of capitulation phenomenon. Let's consider, for $p = 3$, $k = \mathbb{Q}(\sqrt{229})$ (class group of order 3) and the cubic field K_0 of conductor 37 (trivial 3-class group); PARI [18] gives for $K = kK_0$ the class group $[3, 3, 3]$. For $\ell = 109$, let M_0 be the cubic subfield of $\mathbb{Q}(\mu_{\ell})$; then ℓ is inert in K and the 3-class group of $L = KM_0$ is $[9, 9, 3]$.

If $\mathbf{J}_{L/K}$ is injective, $\nu_{L/K}(\mathcal{H}_L) = \mathbf{J}_{L/K}(\mathcal{H}_K) \simeq \mathcal{H}_K$ and $\mathcal{H}_L^G \simeq \mathcal{H}_K$ (same order); the formula (e.g., [8, § 2, (1), (2)] or [6]) of the orders of the elements of the filtration of \mathcal{H}_L gives $\#(\mathcal{H}_L/\mathcal{H}_L^G)^G = \frac{\#\mathcal{H}_K}{\#\mathbf{N}_{L/K}(\mathcal{H}_L^G)} = 3^3$, since $r = 1$ and $\mathbf{N}_{L/K}(\mathcal{H}_L^G) = \mathcal{H}_K^3 = 1$, giving $\#\mathcal{H}_L \geq 3^6$ (absurd).

So, $\mathbf{J}_{L/K}$ is not injective. This gives an example where:

$$\begin{cases} \#\mathcal{H}_L^{\text{alg}} := \#\{x \in \mathcal{H}_L, \nu_{L/K}(x) = 1\} \in \{3^3, 3^4, 3^5\} \\ \#\mathcal{H}_L^{\text{ar}} := \#\{x \in \mathcal{H}_L, \mathbf{N}_{L/K}(x) = 1\} = 3^2; \end{cases}$$

so $\#\mathcal{H}_L^{\text{alg}} \times \#\mathcal{H}_K^{\text{alg}} \in \{3^6, 3^7, 3^8\}$, while $\#\mathcal{H}_L^{\text{ar}} \times \#\mathcal{H}_K^{\text{ar}} = 3^2 \times 3^3 = \#\mathcal{H}_L$ giving illustration of Corollary 2.3.

For $k = \mathbb{Q}(\sqrt{1129})$, whose class group is [9] and for the cubic field K_0 of conductor 7, the 3-class group of K is [27]. For $\ell = 19$, let M_0 be the subfield of degree 9 of $\mathbb{Q}(\mu_{\ell})$ and $L = KM_0$; then the 3-class group of the subfield $K_1 \subset L$, of degree 3 over K , is [27] (stability from K). From [8, Theorem 3.5] this implies $\#\mathcal{H}_L = 27$, $\mathcal{H}_L^G = \mathcal{H}_L$, giving a partial capitulation since $\nu_{L/K}(\mathcal{H}_L) = \mathcal{H}_L^9$ of order 3. Then $\#\mathcal{H}_L^{\text{alg}} = 3^2$ and $\#\mathcal{H}_L^{\text{ar}} = 1$.

Recall that a sufficient condition of capitulation of \mathcal{H}_K in L is the stability of the p -class groups from some layer in the p -tower $L = \bigcup_{0 \leq i \leq N} K_i$, where $[K_i : K] = p^i$, $[L : K] = p^N$ much bigger than the exponent of \mathcal{H}_K [8, § 3.2].

A more general study in [9, Theorems 1.1, 1.2] shows that capitulation is, surprisingly, inversely proportional to the “complexity” of \mathcal{H}_L in a precise meaning. The fact that, in huge p -towers L/K , stability may occur at some layer, is a reasonable conjecture as a kind of finite Iwasawa theory for which the first case of Greenberg’s conjecture [10, Theorem 1, §4] would hold.

5. CHECKING OF EQUALITIES $\#\mathcal{H}_{K,\varphi}^{\text{ar}} = \#(\mathcal{E}_K/\mathcal{E}_K^{\circ} \mathcal{F}_K)_{\varphi_0}$ IN THE NON SEMI-SIMPLE CASE $p = 3$, $[K : \mathbb{Q}] = 6$

5.1. Introduction. To illustrate the RAMC, the minimal non trivial context may be for cyclic cubic fields k and degree 7 cyclic extensions K_0 with $K = kK_0$. Then, the checking of RAMC would be to find $\ell \equiv 1 \pmod{7^N}$ then $M_0 \subset \mathbb{Q}(\mu_{\ell})$, of degree 7^n , $n \leq N$, and to work in the compositum $L = M_0K$ of absolute degree 21×7^n ; this is oversized for PARI computations; but, as shown by examples of §4.3, capitulations may occur easily.

Nevertheless, since the existence of (infinitely many) extensions L/K in which \mathcal{H}_K capitulates, is a conjecture largely tested in [8, 9], we limit ourselves to an easier non semi-simple case, that is to say, $p = 3$, k real quadratic and K_0/\mathbb{Q} cubic cyclic; then we only check the relation of $\#\mathcal{H}_{K,\varphi}^{\text{ar}}$ with the index $(\mathcal{E}_K : \mathcal{E}_K^{\circ} \mathcal{F}_K)_{\varphi}$.

Many cases occur for the checking of $(\mathcal{E}_K : \mathcal{E}_K^{\circ} \mathcal{F}_K)_{\varphi_0} = \#\mathcal{H}_{K,\varphi}^{\text{ar}}$ (Theorem 3.2) when the structure of the class groups varies as well as the relations $\eta_i = \prod_{j=1}^3 \varepsilon_j^{a_{i,j}}$, $1 \leq i \leq 3$, describing the relative cyclotomic units η_i from the relative units ε_j (relative units meaning of norm 1 over k and K_0). Once again, we will see that the two $\mathbb{Z}_p[\mathcal{G}_K]$ -modules, of same order, $(\mathcal{E}_K/\mathcal{E}_K^{\circ} \mathcal{F}_K)_{\varphi_0}$ and $\mathcal{H}_{K,\varphi}^{\text{ar}}$, are not isomorphic in general. The program computes \mathbf{H}_k and \mathbf{H}_K ; so $\#\mathcal{H}_{K,\varphi}^{\text{ar}} = \#\mathcal{H}_K \times \#\mathcal{H}_k^{-1}$.

5.2. PARI program – Examples. To simplify (especially, for the computation of the cyclotomic units), we take $k = \mathbb{Q}(\sqrt{f})$, where f is a prime number congruent to 1 modulo 4 and various cubic fields K_0 of prime conductors $q \equiv 1 \pmod{3}$.

In the program, the main parameters are the following:

Pk, PK: defining polynomials of k, K ;

Ck, CK: class groups of k, K ; Ek, EK: unit groups of k, K ;

LEtaK: list of the 3 independent relative cyclotomic units of K .

The conductor f varies in $[bf, Bf]$; the prime q varies in $[bq, Bq]$; then, $\text{kroncker}(f, q)$ gives the splitting of q in k/\mathbb{Q} , which may give larger 3-class group for K if $\text{kroncker}(f, q) = 1$.

Then the computations of some indices of groups of units are done taking the logarithms of the units, to get easier linear relations. We do not write the units and their logarithms since they are oversized; but the running of the program gives rapidly a complete data (use large precision and memory); for instance, with $f = 257$ and $q = 4597$ a precision $\backslash p 300$ is necessary for the computation of some logarithms.

For some rare cases cases, the program gives 5 fundamental relative units ε_i (instead of 3), but with two relations of dependence; this is due to PARI computation of the fundamental units in $K.\text{fu}$ where the units of the cubic field K_0 do not appear as a direct sum; a same remark may occur for the fundamental unit ε_0 of the quadratic field k (in general, $\varepsilon_0 = \varepsilon_1$, but for

$k = \mathbb{Q}(\sqrt{1129})$ and $q = 7$, $\varepsilon_0 = \varepsilon_1 \varepsilon_2^{-1} \varepsilon_3^{-1}$. The final matrix $(a_{i,j})_{i,j}$ is such that $\eta_i = \prod_{j=1}^3 \varepsilon_j^{a_{i,j}}$. The PARI program uses the package [18]:

```
\p 150
{p=3;BHK=3;bf=229;Bf=10^3;bq=7;Bq=10^4;
forprime(f=bf,Bf,if(Mod(f,4)!=1,next);Pk=x^2-f;k=bnfinit(Pk,1);
Hk=k.no;if(valuation(Hk,p)==0,next);Ck=k.clgp;
print();print("conductor k=",f," Pk=",Pk," classgroup(k)=",Ck[2]);
Ek=k.fu;Ak=lift(Ek[1]);Bk=polcoeff(Ak,0)+polcoeff(Ak,1)*sqrt(f);
LEke=2*log(abs(Bk));print("log of epsilon0=",LEke);
forprime(q=bq,Bq,if(Mod(q,3)!=1,next);Q=polsubcyclo(q,p);
PK=polcompositum(Pk,Q)[1];F=q*f;xK=real(polroots(PK)[2]);
K=bnfinit(PK,1);HK=K.no;if(valuation(HK,3)<BHK,next);print();
print("nombre premier q=",q," PK=",PK," classgroup(K)=",K.clgp[2]);
print("kronecker(f,q)=",kronecker(f,q));
\\computation of the generator of order 2 of Gal(K/Q):
GK=nfgaloisconj(K);Id=x;for(k=1,6,Z=GK[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(K,GK[k],Z);ks=ks+1);if(ks==2,S2=GK[k];break));
\\computation of the relative units (of norm 1 over K0)
EKe=List;EK=K.fu;for(n=1,5,e=EK[n];
ee=nfgaloisapply(K,S2,e);e=e*ee^-1;listput(EKe,e));
LEKe=List;for(n=1,5,AK=lift(EKe[n]);BK=0;CK=0;
for(m=0,5,c=polcoeff(AK,m);BK=BK+xK^m*c);
listput(LEKe,log(abs(BK)))));print();Ce=0;LEK=List;for(n=1,5,
e=LEKe[n];if(abs(abs(e))>10^-10,Ce=Ce+1;listput(LEK,e)));
print("logarithms of the relative units of K:",LEK);
g=lift(znprimroot(f));G=lift(znprimroot(q));
u=lift(Mod((1-g)/f,q));v=lift(Mod((1-G)/q,f));
g=g+u*f;G=G+v*q;g2=g^2;G3=G^3;d2=(f-1)/2;d3=(q-1)/3;
\\Computation of the Artin group A=Gal(Q(exp(I*Pi/F))/K):
z=exp(I*Pi/F);A=List;for(i=1,d2,for(j=1,d3/2,
a=Mod(g2,F)^i*Mod(G3,F)^j;a=lift(a);listput(A,a));dA=d2*d3/2;
\\Computation of the relative cyclotomic units:
LEtaK=List;C=List;for(i=1,3,for(j=1,2,c=1;for(t=1,dA,a=A[t];
s=lift(Mod(a*G^i*g^j,F));c=c*(z^s-z^-s));c=real(c);listput(C,c)));
listput(LEtaK,log(abs(C[1]*C[2]^-1)));
listput(LEtaK,log(abs(C[3]*C[4]^-1)));
listput(LEtaK,log(abs(C[5]*C[6]^-1)));
print("Cyclotomic units:",LEtaK);if(Ce==5,print("Ce=5");next);
\\Matrix of relations:
BX=12;for(j=1,3,for(a=-BX,BX,for(b=-BX,BX,for(c=-BX,BX,
X=a*LEK[1]+b*LEK[2]+c*LEK[3];
if(abs(X-LEtaK[j])<10^-6,print(a," ",b," ",c))))))}
```

5.2.1. Quadratic field k of conductor 229.

```
Pk=x^2-229 classgroup(k)=[3]
log(epsilon0)=-5.424930610368687949361759021212260139798716...
(i) nombre premier q=37
PK=x^6+2*x^5-710*x^4-918*x^3+157031*x^2+125228*x-10725781
classgroup(K)=[3,3,3] kronecker(f,q)=1 epsilon0=epsilon1
0 3 0
0 0 -3
0 -3 3
```

In the following case, the program gives 5 relative units ε'_i with the relations $\varepsilon'_3 = \varepsilon'_2$ and $\varepsilon'_5 = \varepsilon'_2 \cdot \varepsilon'_4^{-1}$, giving the basis $\varepsilon_1 = \varepsilon'_1$, $\varepsilon_2 = \varepsilon'_2$, $\varepsilon_3 = \varepsilon'_4$:

```
(ii) prime number q=1723
PK=x^6+2*x^5-1834*x^4-7552*x^3+480853*x^2-252370*x-20931500
classgroup(K)=[9,3] kronecker(f,q)=-1 epsilon0=epsilon1
1 0 -3
1 3 0
```

1 -3 3

(iii) prime number $q=5743$

```
PK=x^6+2*x^5-4514*x^4-54942*x^3+3770063*x^2+62574440*x-31665625
classgroup(K)=[9,9] kronecker(f,q)=-1 epsilon0=epsilon1
-2 -6 -3
 1  3 -3
 4  3  6
```

(iv) prime number $q=6379$

```
PK=x^6+2*x^5-4938*x^4+18458*x^3+4700367*x^2-32919224*x-679071697
classgroup(K)=[9,3,3] kronecker(f,q)=1 epsilon0=epsilon1
-3  6  3
 3 -3 -6
 0 -3  3
```

5.2.2. Quadratic field k of conductor 1129.

```
Pk=x^2-1129 class group(k)=[9]
log(epsilon0)=11.634240035110537169057276239224439429816675...
```

In the next example, the fundamental unit ε_0 of k is not ε_1 given by PARI; one gets $\varepsilon_0 = \varepsilon_1 \varepsilon_2^{-1} \varepsilon_3^{-1}$:

(v) prime number $q=7$

```
PK=x^6+2*x^5-3390*x^4-4522*x^3+3821667*x^2+2547028*x-1432703257
classgroup(K)=[27] kronecker(f,q)=1 epsilon0=epsilon1/epsilon2epsilon3
-1 -1  0
 0  1 -1
 1  0  1
```

(vi) prime number $q=73$

```
PK=x^6+2*x^5-3434*x^4-4618*x^3+3822187*x^2+2421872*x-1377322821
classgroup(K)=[189,3] kronecker(f,q)=-1 epsilon0=epsilon1
 8 -6  9
 2 -3 -6
-1  9 -3
```

(vii) nombre premier $q=19867$

```
PK=x^6+2*x^5-16630*x^4-200242*x^3+47490067*x^2+607831028*x-25945199777
class group K=[18,18,9] kronecker(f,q)=-1 epsilon0=epsilon1
0  0 -9
9  9  9
0 -9  0
```

In the following case, the program gives 5 relative units ε'_i with the relations $\varepsilon'_3 = \varepsilon'_2$ and $\varepsilon'_5 = \varepsilon'_4 \cdot \varepsilon_1'^{-1} \cdot \varepsilon'_2$, giving the basis $\varepsilon_1 = \varepsilon'_1$, $\varepsilon_2 = \varepsilon'_2$, $\varepsilon_3 = \varepsilon'_4$.

(viii) prime number $q=41077$

```
PK=x^6+2*x^5-30770*x^4+156750*x^3+191481179*x^2-1910572432*x-169077439885
class group K=[27,9,3] kronecker(f,q)=-1 epsilon0=epsilon1
-3 -9  9
 6  9  0
 6  0 -9
```

5.3. Verification of the relation $\#\mathcal{H}_{K,\varphi}^{\text{ar}} = (\mathcal{E}_K : \mathcal{E}_K^\circ \mathcal{F}_K)$.

(a) Quadratic field k of conductor 229 ($\mathcal{H}_k \simeq \mathbb{Z}/3\mathbb{Z}$):

Case (i): $\text{CH}_K = [3, 3, 3]$ and $\text{CH}_{K,\varphi} = [3, 3]$. Since $\varepsilon_1 = \varepsilon_0$, the unit index is $\left(\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1, \varepsilon_2^3, \varepsilon_3^{-3}, \varepsilon_2^{-3} \varepsilon_3^3 \rangle \right) = \left(\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1, \varepsilon_2^3, \varepsilon_3^3 \rangle \right) = 9$.

The quotient $\mathcal{E}_{K,\varphi} / \mathcal{E}_{K,\varphi}^\circ \mathcal{F}_{K,\varphi}$ being monogenic, is isomorphic to $\mathbb{Z}/9\mathbb{Z}$, while $\mathcal{H}_{K,\varphi} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

Case (ii): $\text{CH}_K = [9, 3]$ and $\text{CH}_{K,\varphi} = [3, 3]$ or $[9]$. Once we are using the basis $\varepsilon_1 = \varepsilon'_1$, $\varepsilon_2 = \varepsilon'_2$, $\varepsilon_3 = \varepsilon'_4$, the matrix is analogous to the previous one, whence the result with the index 9.

Case (iii): $\text{CH}_K = [9, 9]$ and $\text{CH}_{K,\varphi} = [9, 3]$. The unit index is:

$$\begin{aligned} & \left(\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1, \varepsilon_1^{-2} \varepsilon_2^{-6} \varepsilon_3^{-3}, \varepsilon_1 \varepsilon_2^3 \varepsilon_3^{-3}, \varepsilon_1^4 \varepsilon_2^3 \varepsilon_3^6 \rangle \right) = \\ & \left(\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1, \varepsilon_2^3 \varepsilon_3^{-3}, \varepsilon_2^9, \varepsilon_3^9 \rangle \right) = \left(\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1, \varepsilon_2^3 \varepsilon_3^{-3}, \varepsilon_2^9 \rangle \right) = 27. \end{aligned}$$

Case (iv): $\text{CH}_K = [9, 3, 3]$ and $\text{CH}_{K,\varphi} = [9, 3]$ or $[3, 3, 3]$:

$$\begin{aligned} & \left(\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1, \varepsilon_1^{-3} \varepsilon_2^6 \varepsilon_3^3, \varepsilon_1^3 \varepsilon_2^{-3} \varepsilon_3^{-6}, \varepsilon_2^{-3} \varepsilon_3^3 \rangle \right) = \\ & \left(\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1, \varepsilon_2^{-3} \varepsilon_3^3, \varepsilon_2^9, \varepsilon_3^9 \rangle \right) = \left(\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1, \varepsilon_2^{-3} \varepsilon_3^3, \varepsilon_2^9 \rangle \right) = 27. \end{aligned}$$

(b) Quadratic field k of conductor 1129 ($\mathcal{H}_k \simeq \mathbb{Z}/9\mathbb{Z}$):

Case (v): $\text{CH}_K = [27]$ and $\text{CH}_{K,\varphi} = [3]$. We have $\varepsilon_0 = \varepsilon_1 \varepsilon_2^{-1} \varepsilon_3^{-1}$:

$$\begin{aligned} & \left(\langle \varepsilon_1 \varepsilon_2^{-1} \varepsilon_3^{-1}, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1 \varepsilon_2^{-1} \varepsilon_3^{-1}, \varepsilon_1^{-1} \varepsilon_2^{-1}, \varepsilon_2 \varepsilon_3^{-1}, \varepsilon_1 \varepsilon_3 \rangle \right) = \\ & \left(\langle \varepsilon_1 \varepsilon_2^{-1} \varepsilon_3^{-1}, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1 \varepsilon_2^{-1} \varepsilon_3^{-1}, \varepsilon_2 \varepsilon_3^{-1}, \varepsilon_1^{-1} \varepsilon_2^{-1}, \varepsilon_1 \varepsilon_3 \rangle \right) = \left(\langle \varepsilon_1 \varepsilon_2^{-1} \varepsilon_3^{-1}, \varepsilon_1 \varepsilon_2, \varepsilon_1 \varepsilon_3 \rangle \right) = 3. \end{aligned}$$

Case (vi): $\text{CH}_K = [27, 3]$ and $\text{CH}_{K,\varphi} = [3, 3]$:

$$\begin{aligned} & \left(\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1, \varepsilon_1^8 \varepsilon_2^{-6} \varepsilon_3^9, \varepsilon_1^2 \varepsilon_2^{-3} \varepsilon_3^{-6}, \varepsilon_1^{-1} \varepsilon_2^9 \varepsilon_3^{-3} \rangle \right) = \\ & \left(\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1, \varepsilon_2^{-6} \varepsilon_3^9, \varepsilon_2^{-3} \varepsilon_3^{-6}, \varepsilon_2^9 \varepsilon_3^{-3} \rangle \right) = \left(\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1, \varepsilon_2^3 \varepsilon_3^6, \varepsilon_2^9 \varepsilon_3^{-3} \rangle \right) = 9. \end{aligned}$$

Case (vii): $\text{CH}_K = [9, 9, 9]$ and $\text{CH}_{K,\varphi} = [9, 9]$:

$$\left(\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1, \varepsilon_3^{-9}, \varepsilon_1^9 \varepsilon_2^9 \varepsilon_3^9, \varepsilon_2^{-9} \rangle \right) = \left(\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1, \varepsilon_2^9, \varepsilon_3^9 \rangle \right) = 81.$$

Case (viii): $\text{CH}_K = [27, 9, 3]$ and $\text{CH}_{K,\varphi} = [27, 3]$ or $[3, 9, 3]$:

$$\begin{aligned} & \left(\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1, \varepsilon_1^{-3} \varepsilon_2^{-9} \varepsilon_3^9, \varepsilon_1^6 \varepsilon_2^9, \varepsilon_1^6 \varepsilon_3^{-9} \rangle \right) = \\ & \left(\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1, \varepsilon_2^{-9} \varepsilon_3^9, \varepsilon_2^9, \varepsilon_3^9 \rangle \right) = \left(\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \langle \varepsilon_1, \varepsilon_2^9, \varepsilon_3^9 \rangle \right) = 81. \end{aligned}$$

6. CONCLUSION

It is not possible, with PARI, to check the RAMC with non-trivial examples such as k cubic, $[K_0 : \mathbb{Q}] = 7$, with 7-extensions L/K of degree 7^N in which \mathcal{H}_K capitulates; nevertheless, we have given examples of non-trivial capitulations for k quadratic, $[K_0 : \mathbb{Q}] = 3$ and cubic extensions L/K , which may be sufficient to suggest that the method applies in any circumstance. Indeed, we think that the existence of capitulation fields $L \subset K(\mu_\ell)$, with $\ell \equiv 1 \pmod{2p^N}$ inert in K , is a governing conjecture for many arithmetical properties; even if its proof seems out of reach, it has been checked on several other occasions, and may be considered as a kind of “basic conjecture” for class field theory, that is to say a deep diophantine property, predominant in a logical point of view.

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