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# Asset market equilibrium with short-selling and a continuum of number of states of nature

Thai Ha-Huy\*, Cuong Le Van†

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## ABSTRACT

We consider an economy with a continuum number of states of nature, von Neumann - Morgenstern utilities, where agents have different probability beliefs and where short sells are allowed. We know that no-arbitrage conditions, defined for finite dimensional asset markets models, are not sufficient to ensure existence of equilibrium in presence of an infinite number of states of nature. However, if we give conditions which imply the compactness of  $\mathcal{U}$ , the individually rational utility set, we obtain an equilibrium. We give conditions which imply the compactness of  $\mathcal{U}$ . This paper extends to the case of a continuum number of states no-arbitrage conditions in the literature.

**Keywords:** asset market equilibrium; individually rational attainable allocations; individually rational utility set; no-arbitrage prices; no-arbitrage condition.

**JEL Classification:** C62, D50, D81, D84, G1.

## 1 INTRODUCTION

In the line of Hart's model, where the market is complete, there has been a large body of literature dealing with sufficient and necessary conditions ensuring the

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existence of equilibria. In finite dimension, one can refer, for instance to Page (1987), Werner (1987), Nielsen (1989), Page and Wooders (1995, 1996), Allouch (2002), Allouch et al (2002), Won and Yannelis (2008). In general, the existence of an equilibrium is granted under the existence of a common *no-arbitrage* price for every agent in the economy. When the number of assets is infinite, the usual assumption is to assume that the *individually rational utility set* is compact (see e.g. Cheng (1991), Brown and Werner (1995), Dana and Le Van (1996, 2000), Dana et al (1997), Le Van and Truong Xuan (2001)).<sup>1</sup> More recently, this question is considered in a model with a countable number of states of nature, Ha-Huy et al (2016) give sufficient conditions for the existence of equilibrium.

The purpose of the present paper is to extend the paper by Ha-Huy, Le Van and Nguyen (2016) to the case of continuum number of states of nature.

## 2 THE MODEL

In this paper, we consider the case of a continuum of states, and we will present a similar result. Given a probability measure  $\mu$  on  $[0, 1]$ , we define

$$L^p([0, 1], \mu) = \{f : \mathbb{R} \rightarrow \mathbb{R} \text{ } \mu\text{-measurable} \mid \int_{[0,1]} |f(s)|^p \mu(ds) < +\infty\}.$$

We have an economy with  $m$  agents, each agent  $i$  is represented by a probability measure  $\mu^i$  on  $[0, 1]$ , endowment  $e^i$  which is a  $\mu^i$ -measurable function on  $[0, 1]$ , and an utility function  $u^i : \mathbb{R} \rightarrow \mathbb{R}$  satisfying:

$$U^i(x^i) = \int u^i(x^i(s)) \mu^i(ds).$$

is real-valued for any  $x^i \in L^p([0, 1], \mu^i)$ .

We give here the definition of equilibrium and quasi-equilibrium. Let  $q$  satisfy  $\frac{1}{p} + \frac{1}{q} = 1$ .

DEFINITION 2.1. 1. An equilibrium is a list  $((x^{i*})_{i=1,\dots,m}, p^*)$  such that  $x^{i*} \in X^i$  for every  $i$  and  $p^* \in L_+^q \setminus \{0\}$  and

$$a) \text{ For any } i, U^i(x) > U^i(x^{i*}) \Rightarrow \int_0^1 p(s)^* x(s) ds > \int_0^1 p(s)^* e^i(s) ds$$

$$b) \sum_{i=1}^m x^{i*} = \sum_{i=1}^m e^i.$$

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<sup>1</sup>Chichilnisky and Heal (1993) give a condition which implies the boundedness of the individually rational feasible set in  $L^2$ . Since the feasible set is closed, it is therefore weakly compact in  $L^2$ .

2. A quasi-equilibrium is a list  $((x^{i*})_{i=1,\dots,m}, p^*)$  such that  $x^{i*} \in X^i$  for every  $i$  and  $p^* \in L_+^q \setminus \{0\}$ , we have (b) and

$$a) \text{ For any } i, U^i(x) > U^i(x^{i*}) \Rightarrow \int_0^1 p(s)^* x(s) ds \geq \int_0^1 p(s)^* e^i(s) ds$$

$$b) \sum_{i=1}^m x^{i*} = \sum_{i=1}^m e^i.$$

In this paper, we assume that  $\mu^i$  is continuous with respect to Lebesgue measure  $\lambda$ , and denote  $h^i(s) = \frac{d\mu^i}{d\lambda}$ , the density of  $\mu^i$  with respect to  $\lambda$ .

REMARK 2.1. We show that assumption H4 in Le Van-Truong Xuan (2001) may not be satisfied in our model. Indeed, we have:

If H4 holds then  $\forall a, \exists b, b'$  such that  $b' < a < b$ .

We have

$$\inf_s u^i(a)h^i(s) > \sup_s u^i(b)h^i(s) \quad (2.1)$$

$$\sup_s u^i(a)h^i(s) < \inf_s u^i(b')h^i(s) \quad (2.2)$$

That is not always true. We mention here, for instance, three cases:

1. Suppose that  $\inf_s h^i(s) = 0$ . Then the two inequalities above do not hold.
2. If  $u^i(+\infty) > 0$ , let  $a \rightarrow +\infty$ , then H4 condition is justified only if  $\inf_s h^i(s) \geq \sup_s h^i(s)$ . So,  $h^i(s) = 1$  for all  $s$ , or  $\mu^i = \lambda$ .
3. If  $u^i(-\infty) < +\infty$ , by the same argument, H4 is verified only if  $\mu^i = \lambda$ .

The first assumption is:

**A0:**  $\mu^i$  and  $\mu^j$  are equivalent, i.e. there exists  $h > 0$  such that, for all measurable set  $\Delta$ , we have  $h \leq \frac{\mu^i(\Delta)}{\mu^j(\Delta)} \leq \frac{1}{h}$ .

With this assumption, the consumption sets of all agents are the same  $L^p([0, 1], \mu)$ , with  $\mu = \sum_i \mu^i$ . From now on, for short, we will use  $L^p$  instead of  $L^p([0, 1], \mu)$ .

### 3 EXISTENCE OF EQUILIBRIUM

We present here the first Lemma which is the Dunfort-Pettis criterion (Dunfort-Schwartz (1966)):

**LEMMA 3.1.** *A bounded, closed in  $\sigma(L^1, L^\infty)$ -topology set  $B$  of  $L^1$  is  $\sigma(L^1, L^\infty)$  compact if and only if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $\mu(\Delta) < \delta$ , then  $\int_\Delta |x(s)|\mu(ds) < \epsilon$  for all  $x \in B$ .*

**LEMMA 3.2.** *Suppose that  $\mathcal{A}$  is bounded and  $(v^1, v^2, \dots, v^m)$  is in the closure of  $\mathcal{U}$ . Suppose that there exists a sequence  $\{x_n\} \subset \mathcal{A}$  such that there exists  $i$  such that  $\lim_n U^i(x_n^i) > v^i$ , and for all  $j \neq i$ ,  $\lim_n U^j(x_n^j) = v^j$ . Then  $(v^1, v^2, \dots, v^m) \in \mathcal{U}$ .*

The proof of Lemma 3.2 is given in the Appendix.

We now add another assumption.

**A1:**  $\bar{e} = \sum_{i=1}^m e^i \in L^\infty$ .

**THEOREM 3.1.** *Assume **A0**, **A1** and  $b^i = +\infty$  for any  $i$ .*

- (1)  *$\mathcal{A}$  is compact in  $\sigma(L^1, L^\infty)$ -topology and  $\mathcal{U}$  is compact. Moreover, for any  $p \geq 1$ , there exists an equilibrium in  $L^p$ .*
- (2) *Add **A1**. If there exists an equilibrium in  $L^p$  for some  $1 \leq p < +\infty$  then  $\mathcal{A}$  is compact in  $\sigma(L^1, L^\infty)$  and  $\mathcal{U}$  is compact.*

The proof of Theorem 3.1 will be done by using several Lemmas. The proofs of which are given in Appendix.

**LEMMA 3.3.** *Assume **A1**, then there exists a constant  $C > 0$ , depending only on  $(\bar{p}, \bar{x}, \bar{e})$  such that, for any  $(x^1, \dots, x^m) \in \mathcal{A}$ , we have*

$$\int_0^1 |x^i(s)|h^i(s)ds \leq C.$$

**LEMMA 3.4.** *Assume **A1**. The set  $\mathcal{U}$  is bounded.*

**LEMMA 3.5.** *Assume **A0**, **A1** and  $b^i = +\infty$  for any  $i$ . Then for any  $\epsilon > 0$ , there exists  $\delta$  such that if  $\mu(\Delta) < \delta$ , then  $\int_\Delta |x^i(s)|h^i(s)ds < \epsilon$  for all  $(x^1, x^2, \dots, x^m) \in \mathcal{A}$ .*

**COROLLARY 3.1.** *Assume **A0**, **A1** and  $b^i = +\infty$  for any  $i$ . Then  $\mathcal{A}$  is compact for the  $\sigma(L^1, L^\infty)$ -topology.*

*Proof.* We use Lemma 3.1 and Lemma 3.5. QED

We get the following proposition. Its proof is given in Appendix.

**PROPOSITION 3.1.** *Assume **A0**, **A1** and  $b^i = +\infty$  for any  $i$ . Then  $\mathcal{U}$  is compact.*

Now we are ready to prove Theorem 3.1. We proceed as follows. Since  $\mathcal{A}$  is a compact set in  $L^1$ , we first prove that there exists an equilibrium in  $L^1$ . Second, we prove that this equilibrium allocation and the associated equilibrium price are in  $L^\infty$ . Since  $L^\infty$  belongs to  $L^p$  for any  $p \geq 1$ , the claim of Theorem 3.1 is therefore true.

It follows from Lemmas 3.1, 3.3, 3.4, and 3.5, and Proposition 3.1, that  $\mathcal{U}$  is compact. Since  $\mathcal{U}$  is compact there exists a quasi-equilibrium  $((x^{i*}, p^*))$  (see Dana and al (1997)) with  $x^{i*} \in L^1$ , and  $p^*$  belongs to  $L^\infty$ , the dual of  $L^1$ . Since  $X^i = L^1$ , this quasi-equilibrium is actually an equilibrium.

The equilibrium allocation  $(x^{i*})$  solves the problems:

$$\begin{aligned} & \max \int_0^1 u^i(x^i(s))h^i(s)ds \\ \text{s.t. } & \int_0^1 p^*(s)x^i(s)ds = \int_0^1 p^*(s)e^i(s)ds \end{aligned}$$

From Theorem V.3.1, page 91, in Arrow-Hurwicz-Uzawa (1958), for any  $i$ , there exists  $\zeta_i$  s.t.

$$\int_0^1 u^i(x^{i*}(s))h^i(s)ds - \zeta_i \int_0^1 p^*(s)x^{i*}(s)ds \geq \int_0^1 u^i(x(s))h^i(s)ds - \zeta_i \int_0^1 p^*(s)x(s)ds$$

for any  $x \in X^i$ . Hence  $h^i(s)u^{i'}(x^{i*}(s)) = \zeta_i p^*(s), \forall i$ . Since  $u^i$  is strictly increasing,  $\zeta_i > 0$ . Let  $\lambda_i = \frac{1}{\zeta_i}$ ,  $p^*(s) = \lambda_i u^{i'}(x^{i*}(s))h^i(s)$  for all  $i, s$ .

Now we prove that  $x^{i*} \in L^\infty$  for all  $i$ . If not, there exists  $i$  such that  $\text{ess sup}_s |x^i(s)| = +\infty$ . Without loss of generality, we can assume that for all  $M$ ,  $\mu(\{s | x^i(s) > M\}) > 0$ . Since  $\bar{e} \in L^\infty$ , there exists  $j \neq i$  such that there exists a sequence of strictly positive measurable sets  $\Delta_k$  satisfying  $x^i(s) > M$  and  $x^j(s) < -\frac{M - \|\bar{e}\|_{L^\infty}}{m-1}$  for all  $s \in \Delta_k$ . Observe that for almost surely  $s$ ,  $p^*(s) = \lambda_i u^{i'}(x^{i*}(s))h^i(s) = \lambda_j u^{j'}(x^{j*}(s))h^j(s)$ . We have

$$\frac{h^i(s)}{h^j(s)} = \frac{\lambda_j u^{j'}(x^{j*}(s))}{\lambda_i u^{i'}(x^{i*}(s))} \rightarrow \frac{\lambda_j b^j}{\lambda_i a^i} = \infty.$$

Therefore, for any  $M > 0$ , there exists  $K$  such that if  $k > K$  we have  $h^i(s) > Mh^j(s)$  for  $s$  almost surely belongs to  $\Delta_k$ . This implies  $\int_{\Delta_k} h^i(s)\mu(ds) > M \int_{\Delta_k} h^j(s)\mu(ds) \Leftrightarrow \mu^i(\Delta_k) > \mu^j(\Delta_k)$ . Let  $k$  converges to infinity. We have  $\frac{\mu^i(\Delta_k)}{\mu^j(\Delta_k)} \rightarrow \infty$  and that violates **A0**.

Suppose that with  $1 \leq p < \infty$ , there exists an equilibrium  $(p^*, (x^{i*}))$  with  $p^* \in (L^q)^*$ . The same arguments as those given above imply that  $x^{i*} \in L^\infty$  for all

$i$ , and there exists  $\lambda_i$  such that  $p^*(s) = \lambda_i u^{i'}(x^{i*}(s))$  for all  $i, s$ . Observe that  $a^i < u^{i'}(x) < b^i$  for all  $x$ , so  $a^i < \inf_s u^{i'}(x^i(s)) \leq \sup_s u^{i'}(x^{i*}(s)) < b^i$ .

The proof of Theorem 3.1 is now complete.

We now consider the case  $a^i = 0$  for any  $i$ .

**THEOREM 3.2.** *Assume  $a^i = 0$  for any  $i$ . Then  $\mathcal{U}$  is compact. For any  $p \geq 1$ , there exists an equilibrium in  $L^p$ .*

We need the following Lemma in the proof.

**LEMMA 3.6.** *Assume **A0** and **A1** and  $a^i = 0$  for any  $i$ . Then  $\mathcal{U}$  is compact.*

*Proof.* See Appendix. QED

The proof the Theorem 3.2 is similar to the proof the Theorem 3.1.

## 4 APPENDIX

### 4.1 PROOF OF LEMMA 3.2

For each  $M > 0$ , denote  $\Delta^i = \{s \text{ such that } |x^i(s)| < M\}$ . Denote  $\Delta^{i,c}$  the complement of  $\Delta^i$ . Fix a  $j$ . Denote  $\Delta^j, \Delta^{j,c}$  like above. Let  $C > 0$  upper bound  $\mathcal{A}$  in  $L^1$ , we know that  $\int_{\Delta^{i,c}} |x^i(s)| h^i(s) ds < C$ . Then we have:

$$M \int_{\Delta^{i,c}} h^i(s) ds \leq \int_{\Delta^{i,c}} |x^i(s)| h^i(s) ds < C$$

so

$$\mu^i(\Delta^{i,c}) < \frac{C}{M}$$

or

$$\mu^i(\Delta^i) > 1 - \frac{C}{M}.$$

Using the same argument, we have

$$\mu^j(\Delta^j) > 1 - \frac{C}{M}.$$

Denote  $\Delta = \Delta^i \cap \Delta^j$ .  $\mu^i$  et  $\mu^j$  are equivalents, so for  $M$  big enough, we have  $\mu^i(\Delta), \mu^j(\Delta) > 0$ . Fix that  $M$ .

We define the sequence  $\{y_n^k\}_{k=1,\dots,m}$  as follows

$$\begin{aligned} y_n^k &= x_n^k \text{ if } k \neq i, j \\ y_n^i(s) &= x_n^i(s) \text{ if } s \notin \Delta \\ y_n^j(s) &= x_n^j(s) \text{ if } s \notin \Delta \\ y_n^i(s) &= x_n^i(s) - \epsilon \text{ if } s \in \Delta \\ y_n^j(s) &= x_n^j(s) + \epsilon. \end{aligned}$$

For  $k \neq i, j$ ,  $\lim_n U^k(y^k(n)) = v^k$ . And we have

$$\begin{aligned} U^i(y_n^i) - U^i(x_n^i) &= \int_{\Delta} (u^i(y_n^i(s)) - u^i(x_n^i(s))) h^i(s) ds \\ &\geq \int_{\Delta} (-\epsilon) u^{i'}(x_n^i(s)) h^i(s) ds \\ &\geq -\epsilon \int_{\Delta} u^{i'}(-M) h^i(s) ds \\ &\geq -\epsilon u^{i'}(-M) \mu^i(\Delta). \end{aligned}$$

So

$$\liminf_{n \rightarrow \infty} U^i(y_n^i) \geq \liminf_{n \rightarrow \infty} U^i(x_n^i) - \epsilon u^{i'}(-M) \mu^i(\Delta).$$

$$\begin{aligned} U^j(y_n^j) - U^j(x_n^j) &= \int_{\Delta} (u^j(y_n^j(s)) - u^j(x_n^j(s))) h^j(s) ds \\ &\geq \int_{\Delta} \epsilon u^{j'}(x_n^j(s)) h^j(s) ds \\ &\geq \epsilon \int_{\Delta} u^{j'}(M) h^j(s) ds \\ &\geq \epsilon u^{j'}(M) \mu^j(\Delta). \end{aligned}$$

So

$$\liminf_{n \rightarrow \infty} U^j(y_n^j) \geq v^j + \epsilon u^{j'}(M) \mu^j(\Delta).$$

Since  $\liminf_n U^i(x_n^i) > v^i$ , by choosing  $\epsilon$  small enough, the sequence  $\{y(n)\}_n$  will satisfy  $\liminf_n U^i(y_n^i) > v^i$  and  $\liminf_n U^j(y_n^j) > v^j$ .

By induction we can have the sequence  $\{z_n^k\}_n \subset \mathcal{A}$  which satisfies  $\lim_n U^k(z_n^k) > v^k$  for all  $k = 1, 2, \dots, m$ . Hence  $(v^1, v^2, \dots, v^m) \in \mathcal{U}$ .

## 4.2 PROOF OF LEMMA 3.3

Take any **NA** price  $p$ . There exists  $(\bar{x}^i)$ ,  $\{\lambda_i > 0\}_i$  such that for all  $i, s$ :  $p(s) = \lambda_i u^{i'}(\bar{x}^i(s) h^i(s))$  almost surely. Since  $a^i < \inf_s u^{i'}(\bar{x}^i(s)) \leq \sup_s u^{i'}(\bar{x}^i(s)) < b^i$  almost surely, we have  $\bar{x}^i \in L^\infty$ . Observe that  $p \in L^\infty$ .

Choose  $\eta > 0$  such that

$$a^i < u^{i'}(\bar{x}^i(s))(1 + \eta) < b^i \quad (4.1)$$

for all  $i$ . Then we define the price  $q$  as follows:  $\forall i, j$ ,

$$q(s) = p(s)(1 + \eta) = \lambda_i u^{i'}(\bar{x}^i(s))(1 + \eta) h^i(s) = \lambda_j u^{j'}(\bar{x}^j(s))(1 + \eta) h^j(s).$$

It follows from (4.1) that, for each  $i$ , there exist  $\bar{z}^i$  such that  $\forall s, q(s) = \lambda_i u^{i'}(\bar{z}^i(s)) h^i(s)$ . Observe that  $a^i < \inf_s u^{i'}(z_s^i) \leq \sup(s) u^{i'}(z^i(s)) < b^i$ , so  $\bar{z}^i \in L^\infty$ . Observe that  $\forall s, p(s) < q(s)$ .

Denote

$$x^+ : = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$x^- : = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

Note that  $x = x^+ - x^-$ ,  $|x| = x^+ + x^-$  and  $u(x) = u(x^+) + u(-x^-) - u(0)$ .

We have

$$\begin{aligned} & \lambda_i \int_0^1 u^i(\bar{x}^i(s)) h^i(s) ds - \lambda_i \int_0^1 u^i(x^{i+}(s)) h^i(s) ds \\ & \geq \lambda_i \int_0^1 u^{i'}(\bar{x}^i(s)) (\bar{x}^i(s) - x^{i+}(s)) h^i(s) ds \end{aligned}$$

And

$$\begin{aligned} & \lambda_i \int_0^1 u^i(z^i(s)) h^i(s) ds - \lambda_i \int_0^1 u^i(-x^{i-}(s)) h^i(s) ds \\ & \geq \lambda_i \int_0^1 u^{i'}(z^i(s)) (z^i(s) + x^{i-}(s)) h^i(s) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} & \lambda_i \int_0^1 u^{i'}(z^i(s)) x^{i-}(s) h^i(s) ds \\ & \leq \lambda_i \int_0^1 [u^i(z^i(s)) + u^i(\bar{x}^i(s)) - u^i(x^{i+}(s)) - u^i(-x^{i-}(s))] h^i(s) ds \\ & - \lambda_i \int_0^1 u^{i'}(z^i(s)) z^i(s) h^i(s) ds + \lambda_i \int_0^1 u^{i'}(x^i(s)) x^{i+}(s) h^i(s) ds \\ & - \lambda_i \int_0^1 u^{i'}(\bar{x}^i(s)) \bar{x}^i(s) h^i(s) ds. \end{aligned}$$

The inequality above implies

$$\begin{aligned}
\int_0^1 q(s)x^{i-}(s)ds &\leq \lambda_i[U^i(z^i) + U^i(\bar{x}^i) - U^i(x^i) - U^i(0)] \\
&\quad - \int_0^1 q(s)z^i(s)ds + \int_0^1 p(s)x^{i+}(s)ds - \int_0^1 p(s)\bar{x}^i(s)ds \\
&= C^i + \int_0^1 p(s)x^{i+}(s)ds
\end{aligned}$$

where  $C^i = \lambda_i[U^i(z^i) + U^i(\bar{x}^i) - U^i(x^i) - U^i(0)] - \int_0^1 q(s)z^i(s)ds - \int_0^1 p(s)\bar{x}^i(s)ds$ .

Hence,  $\forall i$

$$\int_0^1 (q(s) - p(s))x^{i-}(s)ds \leq C^i + \int_0^1 p(s)x^i(s)ds.$$

Thus we have

$$\sum_{i=1}^m \int_0^1 (q(s) - p(s))x^{i-}(s)ds \leq \sum_{i=1}^m C^i + \sum_{i=1}^m \int_0^1 p(s)x^i(s)ds = \sum_{i=1}^m C^i + \int_0^1 p(s)\bar{e}(s)ds =: C_1.$$

We also have

$$\sum_{i=1}^m \int_0^1 (q(s) - p(s))(x^{i+}(s) - x^{i-}(s))ds = \sum_{i=1}^m \int_0^1 (q(s) - p(s))x^i(s)ds = \int_0^1 (q(s) - p(s))\bar{e}(s)ds$$

which implies

$$\begin{aligned}
\sum_{i=1}^m \int_0^1 (q(s) - p(s))x^{i+}(s)ds &= \sum_{s=1}^{\infty} (q(s) - p(s))\bar{e}(s)ds + \sum_{i=1}^m \int_0^1 (q(s) - p(s))x^{i-}(s)ds \\
&=: C_2.
\end{aligned}$$

Thus

$$\int_0^1 (q(s) - p(s))|x^i(s)|ds \leq C_1 + C_2 =: C$$

then

$$\eta \int_0^1 p(s)|x^i(s)|ds \leq C.$$

Let  $\mu_i := \inf_s u^{i'}(\bar{x}^i(s)) > 0$ , and  $\mu := \min_i \mu_i$ . Then  $\int_0^1 p(s)|x^i(s)|ds \geq \mu \int_0^1 |x^i(s)|h^i(s)ds$  which implies

$$\eta\mu \int_0^1 |x^i(s)|h^i(s)ds \leq C.$$

Therefore,  $\mathcal{A}$  is a bounded set in  $L^1$ .

### 4.3 PROOF OF LEMMA 3.4

We use Lemma 3.3 and Jensen's inequality.

### 4.4 PROOF OF LEMMA 3.5

Assume the contrary,  $\mathcal{A}$  is not compact for the  $\sigma(L^1, L^\infty)$ -topology. Then, from Lemma 3.1 there exists a sequence  $\{x_n^1, x_n^2, \dots, x_n^m\}_n \subset \mathcal{A}$ , an agent  $i$ , a sequence of measurable sets  $\Delta_n$  such that  $\mu^i(\Delta_n) \rightarrow 0$ , and a constant  $\epsilon > 0$  such that

$$\forall n, \int_{\Delta_n} |x_n^i(s)| h^i(s) ds > c$$

Denote for all  $k$ ,  $v^k := \limsup_{n \rightarrow \infty} U^k(x^k(n))$ .

By Lemma 3.3,  $\mathcal{A}$  is bounded in  $L^1$ . We can suppose that, without loss of generality,  $\int_{\Delta_n} |x_n^i(s)| h^i(s) ds \rightarrow c^i > 0$  when  $n \rightarrow \infty$ . This implies  $\int_{\Delta_n} x_n^{i+}(s) h^i(s) ds - \int_{\Delta_n} x_n^{i-}(s) h^i(s) ds \rightarrow c^i > 0$ . The limits of these two integrals exist because  $x^i \in L^1$ . We know that  $\sum_{j \neq i} x_n^j(s) = \bar{e}(s) - x_n^i(s)$ . So, for every  $s$ ,  $\exists j$  such that  $x_n^j(s) \leq -\frac{x_n^i(s) - |\bar{e}(s)|}{m-1}$ . There is a finite number of agents  $j \neq i$ . We can assume that, for simplicity, there exist  $i$  and  $j$  which satisfy the two following properties:

1.  $\exists$  sequence of measurable sets  $\Delta_n$  such that  $\mu^i(\Delta_n) \rightarrow 0$ ,  $x_n^i(s) > 0$  for all  $s \in \Delta_n$  and

$$\lim_{n \rightarrow \infty} \int_{\Delta_n} x_n^i(s) h^i(s) ds = c^i.$$

2. For all  $s \in \Delta_n$

$$x_n^j(s) \leq -\frac{x_n^i(s) - |\bar{e}(s)|}{m-1}.$$

With each  $M > 0$ , define the set  $E_n \subset \Delta_n$  as follows

$$E_n = \{s : x_n^i(s) > |\bar{e}(s)| + M(m-1)\}. \quad (4.2)$$

We have for all  $s \in E_n$

$$x_n^j(s) \leq \frac{|\bar{e}(s)| - x_n^i(s)}{m-1} < -M. \quad (4.3)$$

Because  $\mu^i(\Delta_n) \rightarrow 0$  and  $\bar{e} \in L^\infty$ , so for all  $M > 0$ ,  $\lim_n \int_{E_n} x_n^i(s) h^i(s) ds = c^i$ . Since  $\mu^i$  and  $\mu^j$  are equivalent, we can assume that  $\lim_n \int_{E_n} x_n^i(s) h^j(s) ds = c^j$ . Observe that these limits do not depend on  $M$ .

Define  $\alpha := \min(v^k, v^i - \frac{u^{i'}(0)c^i}{m-1}) - 1$ , ( $k = 1, \dots, m$ ). Define  $\mathcal{A}_\alpha$  the set of  $(x^i) \in L^1$  satisfies  $U^i(x^i) \geq \alpha \forall i$  and  $\sum x^i = \bar{e}$ . From the Lemma 3.3 we know that there exists  $C > 0$  such that  $U^j(x^j) < C$  for all  $(x^1, \dots, x^m) \in \mathcal{A}_\alpha$ . Note that our sequence  $\{x_n^i\} \in \mathcal{A}_\alpha$ .

Since  $b^j = +\infty$  we can choose  $M$  very big such that

$$v^j + \frac{u^{j'}(-M)c^j}{m-1} > C.$$

Now consider the sequence  $(y_n^1, y_n^2, \dots, y_n^m)$  defined as

$$y_n^i(s) := x_n^i(s) - \frac{x_n^i(s) - |\bar{e}(s)|}{m-1} + M \text{ with } s \in E_n.$$

$$y_n^j(s) := x_n^j(s) + \frac{x_n^i(s) - |\bar{e}(s)|}{m-1} - M \text{ with } s \in E_n.$$

Let  $y_n^k(s) = x_n^k(s)$  with every  $k \neq i, j$  or  $s \notin E_n$ .

Note that  $\sum_i y_n^i = \bar{e}$ , and  $y_n^i(s) \leq x_n^i(s)$ ,  $y_n^j(s) \geq x_n^j(s)$  for all  $s$ . We will prove that  $\{U^l(y_n^l)\}_{l=1,m}$  is bounded below by  $\alpha$ , but  $U^j(y_n^j)$  is not bounded above by  $C$ . That leads us to a contradiction.

Indeed,

$$\begin{aligned} U^i(y_n^i) - U^i(x_n^i) &= \int_{E_n} (u^i(y_n^i(s)) - u^i(x_n^i(s)))h^i(s)ds \\ &\geq \int_{E_n} u^{i'}(x_n^i(s) - \frac{x_n^i(s) - |\bar{e}(s)|}{m-1} + M)(-\frac{x_n^i(s) - |\bar{e}(s)|}{m-1} + M)h^i(s)ds \\ &\geq \int_{E_n} u^{i'}(M)(-\frac{x_n^i(s)}{m-1})h^i(s)ds + u^{i'}(M)(\frac{|\bar{e}(s)|}{m-1} + M) \int_{E_n} h^i(s)ds \\ &\geq -\frac{u^{i'}(M)}{m-1} \int_{E_n} x_n^i(s)h^i(s)ds + u^{i'}(M)(\frac{|\bar{e}(s)|}{m-1} + M) \int_{E_n} h^i(s)ds. \end{aligned}$$

When  $n \rightarrow \infty$ , the right hand side term in the inequality above tends to zero where left hand side term tends to  $-\frac{u^{i'}(M)c^i}{m-1}$ . Thus,

$$\liminf_{n \rightarrow \infty} U^i(y_n^i) \geq v^i - \frac{u^{i'}(M)c^i}{m-1} \geq v^i - \frac{u^{i'}(0)c^i}{m-1} > \alpha.$$

For  $n$  large enough,  $U^k(y_n^k)$  is bounded below by  $\alpha$ ,  $\forall k \neq j$ . Then we can estimate the limit of  $U^j(y_n^j)$  when  $n \rightarrow \infty$ ,

$$\begin{aligned} U^j(y_n^j) - U^j(x_n^j) &= \int_{E_n} (u^j(y_n^j(s)) - u^j(x_n^j(s)))h^j(s)ds \\ &\geq \int_{E_n} u^{j'}(x_n^j(s) + \frac{x_n^i(s) - |\bar{e}(s)|}{m-1} - M)(\frac{x_n^i(s) - |\bar{e}(s)|}{m-1} - M)h^j(s)ds \end{aligned}$$

$$\begin{aligned}
U^j(y_n^j) - U^j(x_n^j) &\geq \int_{E_n} u^{j'}(-M) \left( \frac{x_n^i(s) - |\bar{e}(s)|}{m-1} - M \right) \\
&\geq \frac{u^{j'}(-M)}{m-1} \int_{E_n} x_n^i(s) h^j(s) ds - M u^{j'}(-M) \int_{E_n} h^i(s) ds.
\end{aligned}$$

Take the limit. Then:

$$\liminf_{n \rightarrow \infty} U^j(y_n^j) \geq v^j + \frac{u^{j'}(-M)c^j}{m-1} > C.$$

A contradiction.

#### 4.5 PROOF OF LEMMA 3.6

Suppose **NA** condition holds. From Lemma 3.3 and Lemma 3.4, we know that  $\mathcal{U}$  is bounded. We will prove that  $\mathcal{U}$  is closed. Suppose that  $(v^1, \dots, v^m)$  belong to the closure of  $\mathcal{U}$  and the sequence  $\{x_n\} \subset \mathcal{A}$  such that  $\lim_n U^i(x_n^i) = v^i$ .

If the sequence  $\{x_n\}$  belongs to a compact set of the  $\sigma(L^1, L^\infty)$ -topology, then, without loss of generality, we can suppose that  $\lim_n x_n^i = x^i$  in that topology. Since  $U^i$  is upper semi-continuous on a  $\sigma(L^1, L^\infty)$  compact set,  $U^i(x^i) \geq v^i$  for all  $i$ . Thus  $(v^1, \dots, v^m) \in \mathcal{U}$ .

If the the sequence  $\{x_n\}$  does not belong to a compact set, we can suppose that there exists  $c > 0$ , a measurable sets  $\Delta_n$  such that for an agent  $i$

$$\lim_{n \rightarrow \infty} \int_{\Delta_n} |x_n^i(s)| h^i(s) ds = c.$$

We can choose a pair  $(i, j)$  which satisfies the properties (4.2)-(4.3) in the proof of Lemma 3.5. Fix  $\epsilon > 0$ . Choose  $M > 0$  such that  $u^{i'}(M) < (m-1)\epsilon/c$ . By similar arguments as in Lemma 3.5, we can construct a sequence  $\{y_n^k\}$  such that:

$$\liminf_{n \rightarrow \infty} U^i(y_n^i) \geq v^i - \frac{u^{i'}(M)c^i}{m-1}$$

$$\liminf_{n \rightarrow \infty} U^j(y_n^j) \geq v^j + \frac{u^{j'}(-M)c^j}{m-1}$$

$$\liminf_{n \rightarrow \infty} U^k(y_n^k) = v^k \text{ for all } k \neq i, j$$

with  $c^i, c^j > 0$  which do not depend on  $M$ .

So for  $n$  large enough,  $U^i(y_n^i) > v^i - \epsilon$ , and for all  $k \neq i, j$ ,  $U^k(y_n^k) > v^k - \epsilon$  whereas  $\lim_n U^j(y_n^j) = v^j + \frac{u^{j'}(-M)c^j}{m-1} > v^j + \frac{u^{j'}(0)c^j}{m-1} > v^j$ . Let  $\epsilon \rightarrow 0$  and by applying the Lemma 3.2, we have  $(v^1, v^2, \dots, v^m) \in \mathcal{U}$ .

## 4.6 PROOF OF PROPOSITION 3.1

From Lemma 3.4, we know that  $\mathcal{A}$  is compact. Since  $(U^1(\cdot), \dots, U^m(\cdot))$  is upper semi-continuous on  $\mathcal{A}$ , this implies  $\mathcal{U} = (U^1(\mathcal{A}^1), \dots, U^m(\mathcal{A}_m))$  is compact.

## 4.7 PROOF OF THEOREM 3.2

From Lemma 3.6 we get an equilibrium  $(p^*, x^*)$ . We use the same arguments as in Theorem 3.1.

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