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On the thermodynamics consistency of Gurson's model and its computational implications

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Abstract

The aim of this paper is to investigate the thermodynamics consistency of Gurson's model and notably its relation to the class of standard generalized materials. First, we briefly recall Gurson's model in its original format and reanalyzed it in the thermodynamic framework of poroplasticity of saturated media. This allows to properly define the coupling between elasticity and plasticity and to demonstrate that Gurson's model fits into the framework of generalized standard materials model, provided that the internal variables being the plastic strain and the Lagrangian plastic porosity (and not the Eulerian porosity as in the original Gurson model). In particular, by construction, the porosity evolution law is proved to be a full part of the thermodynamic formulation of the model with a generalized normality rule. The implications in terms of numerical implementation of Gurson's model are then investigated; a new numerical scheme based on the time-implicit discretization of the Lagrangian porosity is notably proposed and discussed with respect to available algorithms.

Keywords: Gurson model; Poroplasticity; ductile porous materials; Generalized standard materials; Numerical implementation

1 Introduction

Inelastic deformation and failure of ductile materials involve the nucleation, growth and coalescence of microvoids (see for instance the review paper by Benzerga and Leblond (2010); Benzerga et al. (2016)). Following the pioneering studies by McClintock (1968) and Rice and Tracey (1969), respectively for the growth of isolated cylindrical and spherical voids, Gurson (1977) has proposed a micromechanics-based model for ductile porous materials. Gurson's model was derived by applying kinematics limit-analysis approach to a hollow sphere, considered as the porous material unit cell, subjected to homogeneous strain rate boundary conditions. This procedure results in a macroscopic plastic criterion

complemented by a macroscopic plastic flow rule. Combined with a porosity evolution law, this approach allows to build a complete isotropic model for porous metals. This model has been later extended in several manners among which the heuristic modification proposed by Tvergaard (1981) and Tvergaard and Needleman (1984), known as the GTN model. For recent reviews of literature, the reader can refer to Besson (2010), Benzerga and Leblond (2010) or Benzerga et al. (2016).

It must be emphasized that the limit analysis-based micromechanical construction proposed by Gurson does not incorporate coupling between elasticity and plasticity. This can be done by mean of alternative and more sophisticated variational homogenization procedures proposed in literature devoted to nonlinear heterogeneous materials (including ductile porous one). These procedures are all based on the approach proposed by Ponte Castañeda (1991) for non linear composites whose constituents behaviors are described by means of a single potential (nonlinear elasticity, or purely viscous materials). For elastoplastic materials with hardenable constituents, Cheng et al. (2017a) have proposed a decoupled approach for ductile porous materials. This has led them to obtain a macroscopic model described by an overall free energy and a macroscopic dissipation potential. In order to fully handle the coupling between reversible and dissipative phenomena such as elasto(visco)-plasticity, one may refer to incremental variational approach as introduced by Lahellec and Suquet (2007). In this framework, the concerned non linear behaviors of constituents are described by means of two potentials, namely a free energy and a dissipation potential. The incremental variational approach has been followed and implemented with some variants by several authors among which Brassart et al. (2012), Agoras et al. (2016) and Lucchetta et al. (2019) which, in principle, allow to deal with elastoplastic porous materials without porosity evolution (this is still a matter of research). Unfortunately, although reversible and dissipative processes are coupled at small scale, the incremental variational approach does not allow to separate them at macroscopic scale. Moreover, these studies are still in a early state of development which does not allow to use them for structural computations.

Coming back to Gurson's model, owing to existing different interpretations of the porosity as internal damage variable or not and to the fact that the porosity evolution equation is obtained from the mass conservation law, the thermodynamic consistency of this model needs to be clarified. Specifically, its relation to the class of Generalized Standard Materials (GSM, see for instance Halphen and Nguyen (1975) or Germain et al. (1983) and Nguyen (1977)) is important since it ensures the existence and uniqueness of the solution of the projection problem, provided that the evolution equations of the internal variables are discretized in time with an implicit scheme.

An interesting attempt to analyze the relation between Gurson's model and the class of generalized standard materials has been proposed by Enakoutsa et al. (2007). These authors concluded that Gurson's model (including hardening) fits into the framework of generalized standard materials, *provided that the porosity is fixed*. Therefore, their thermodynamics analysis only considers Gurson's criterion and its associated flow rule, while the evolution equation of the porosity is disregarded. The consequence in terms of numerical implementation of the model are that the equations of the projection problem can be obtained through explicit time-discretization with respect to the porosity: with such scheme, the porosity is considered as fixed during the projection problem. Then, it is updated at the very end of the global elasto-plastic iterations using an explicit scheme. With this approach, the porosity remains fixed during the entire procedure of resolution

of the projection problem at each Gauss point so that the properties of the projection algorithm are thus exactly the same as if the porosity were fixed in time, no matter the final update.

It must be noted that in terms of numerical implementation, Enakoutsa et al. (2007)'s explicit scheme for the porosity differs from standard implementations of Gurson's model, as done by Aravas (1987) and Besson et al. (2001), who considered an *implicit* scheme for the porosity. Although their algorithms are robust and used in most of the commercial codes, the use of a time-implicit scheme for the porosity is not supported from a theoretical point of view by a thermodynamics analysis since it has not been demonstrated yet that Gurson's model with the porosity evolution equation belongs to the class of generalized standard materials. Hence, it is not clear if there is existence and uniqueness of the solution of the projection problem using such time discretization of the equations.

The aim of this paper is to present a thermodynamics framework of Gurson's model for ductile porous media with evolving porosity. To this end, we will take advantage of elastoplasticity theory for saturated porous media (see for instance Coussy (1995)) to incorporate in Gurson's model a pore fluid pressure. In such extension, it is well known that the state variable representing voids volume fraction and associated to the pore pressure is the Lagrangian porosity, instead of the Eulerian one (see Coussy (1995); Bignonnet et al. (2016)). The formulation of the Gurson model in the context of generalized standard materials will be done in the linearized theory. Nonetheless, following the analysis of Enakoutsa et al. (2007), it will be shown that this restriction has no consequence in terms of numerical implementation of the model in the presence of large displacements and strains (which are generally important in ductile fracture problems), provided that the extra terms arising from the objective time-derivative of the stress tensor in the hypoelasticity law are discretized in time with an explicit scheme.

The paper is organized as follows. In Section 2, we analyze the relation between the original Gurson's model with Eulerian porosity and the class of GSM. The thermodynamic framework for elastoplasticity of saturated media (see for instance Coussy (1995, 2004)) is presented in Section 3. In Section 4, Gurson's model without hardening is extended to the context of saturated elastoplastic materials based on the well justified concept of effective stress. It will be shown that in that case the model belongs to the class of generalized standard materials by considering the Lagrangian porosity (and its evolution equation). The implications in terms of numerical implementation of Gurson's model are then studied in Section 5. Finally, the extension of this analysis to hardening is presented in Section 6.

2 Original Gurson's model and the class of generalized standard materials

2.1 A brief recall on generalized standard materials

In the following, we consider only infinitesimal transformations and isothermal conditions. Generalized standard materials are a class of materials whose constitutive law is described through two potentials, see for instance Halphen and Nguyen (1975), Germain et al. (1983) (see also Maitournam (2016)). The first one is the free energy $\psi(\boldsymbol{\varepsilon}, \boldsymbol{\alpha})$, which depends on the strain tensor $\boldsymbol{\varepsilon}$ and a set of internal variables collectively denoted by

$\boldsymbol{\alpha}$ and representing inelastic phenomena. The free energy function must be convex with respect to $\boldsymbol{\varepsilon}$ and $\boldsymbol{\alpha}$ taken separately¹. In a classical way, the state laws provide the *reversible* parts of the forces

$$\boldsymbol{\sigma}^{rev} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}), \quad \mathbf{F}_{\boldsymbol{\alpha}}^{rev} = \frac{\partial \psi}{\partial \boldsymbol{\alpha}}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}). \quad (1)$$

Under isothermal conditions, the classical Clausius-Duhem inequality reduces to the positivity of the intrinsic dissipation \mathcal{D} , that is

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi} \geq 0, \quad (2)$$

which reads

$$\mathcal{D} = \left(\boldsymbol{\sigma} - \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} \right) : \dot{\boldsymbol{\varepsilon}} - \frac{\partial \psi}{\partial \boldsymbol{\alpha}} : \dot{\boldsymbol{\alpha}} \geq 0. \quad (3)$$

Considering only non dissipative deformation $\boldsymbol{\varepsilon}$, the corresponding irreversible force is then null $\boldsymbol{\sigma}^{irr} = \boldsymbol{\sigma} - \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^{rev} = \mathbf{0}$. This readily provides:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{rev} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}). \quad (4)$$

The intrinsic dissipation reads then

$$\mathcal{D} = \mathbf{F}_{\boldsymbol{\alpha}} : \dot{\boldsymbol{\alpha}} \quad (5)$$

with the thermodynamic (irreversible) force $\mathbf{F}_{\boldsymbol{\alpha}}$ associated to $\boldsymbol{\alpha}$, classically given by

$$\mathbf{F}_{\boldsymbol{\alpha}} = -\frac{\partial \psi}{\partial \boldsymbol{\alpha}}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}). \quad (6)$$

The second potential is the dissipation one $\varphi(\dot{\boldsymbol{\alpha}})$, which must be convex, non-negative and zero for $\dot{\boldsymbol{\alpha}} = \mathbf{0}$. It provides the evolution laws of the internal variables through the normality rule

$$\mathbf{F}_{\boldsymbol{\alpha}} \in \partial \varphi(\dot{\boldsymbol{\alpha}}), \quad (7)$$

where $\partial \varphi$ is the sub-differential of φ . Equation (7) is equivalent to

$$\dot{\boldsymbol{\alpha}} \in \partial \varphi^*(\mathbf{F}_{\boldsymbol{\alpha}}), \quad (8)$$

with φ^* the Legendre-Fenchel transform of φ and $\partial \varphi^*$ the sub-differential of φ^* .

In the case of a time-independent behavior as considered here, the function φ^* is the indicator function of a close convex set \mathcal{C} which represents the reversibility domain, in the space of the thermodynamics forces $\mathbf{F}_{\boldsymbol{\alpha}}$. This set \mathcal{C} is defined as $\Phi(\mathbf{F}_{\boldsymbol{\alpha}}) \leq 0$. For a regular point, the evolution law for all internal variables $\boldsymbol{\alpha}$ can be then re-written as

$$\dot{\boldsymbol{\alpha}} = \eta \frac{\partial \Phi}{\partial \mathbf{F}_{\boldsymbol{\alpha}}}(\mathbf{F}_{\boldsymbol{\alpha}}), \quad \eta \begin{cases} = 0 & \text{if } \Phi(\mathbf{F}_{\boldsymbol{\alpha}}) < 0 \\ \geq 0 & \text{if } \Phi(\mathbf{F}_{\boldsymbol{\alpha}}) = 0, \end{cases} \quad (9)$$

η representing the plastic multiplier.

¹ Convexity with respect to the *global* state variable $(\boldsymbol{\varepsilon}, \boldsymbol{\alpha})$ is not required.

These features of generalized standard materials imply several important properties:

- (1) The evolution equation of $\boldsymbol{\alpha}$ automatically warrants that the dissipation $\mathcal{D} = \mathbf{F}_\alpha : \dot{\boldsymbol{\alpha}}$ is non-negative, because

$$\mathbf{F}_\alpha \in \partial\varphi(\dot{\boldsymbol{\alpha}}) \Rightarrow \mathbf{F}_\alpha : \dot{\boldsymbol{\alpha}} \geq \varphi(\dot{\boldsymbol{\alpha}}) \geq 0. \quad (10)$$

- (2) In terms of numerical implementation, a property of the generalized standard materials framework is that it ensures existence and uniqueness of the solution of the projection problem (e.g. for plasticity models) (see for instance Simo and Hughes (1998)), provided that the evolution equation of the internal parameters is discretized with an implicit scheme, because it reduces to the minimization of a convex function.
- (3) The tangent matrix for the global elastic-plastic iterations is symmetric.

2.2 The case of the original Gurson's model

As already mentioned, we will restrict the presentation of the Gurson's model to the linearized theory. Like Gurson, we now consider a spherical cell, having at the current time t a volume $|\Omega(t)|$, with external radius b and containing a confocal spherical cavity $|\omega(t)|$ with internal radius a (see Figure 1). The current volume fraction of voids is the Eulerian porosity

$$f = \frac{|\omega(t)|}{|\Omega(t)|} = \frac{a^3}{b^3}. \quad (11)$$

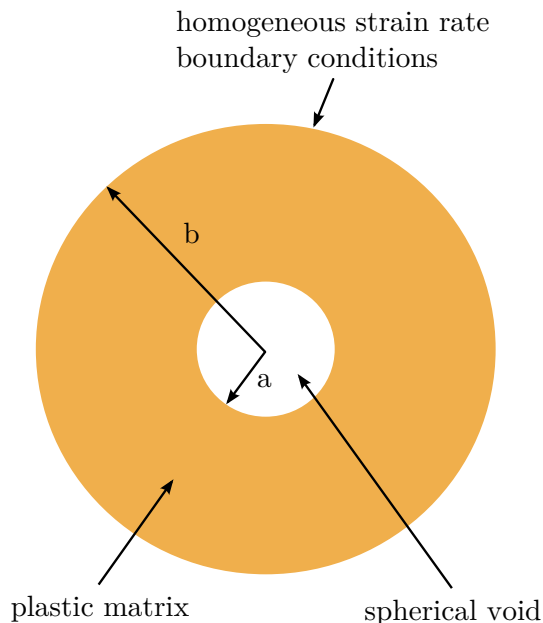


Fig. 1. Elementary cell considered by Gurson (1977).

In the classical form of Gurson's model without hardening, the free energy $\psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$ may read

$$\psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p) = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) : \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p), \quad (12)$$

where \mathbb{C} is the elastic stiffness tensor of the material and the usual partition $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$ is adopted. With such modelling, the thermodynamics irreversible force $\mathbf{F}_{\boldsymbol{\varepsilon}^p}$ associated to

the plastic strain $\boldsymbol{\varepsilon}^p$ is, as classically, given by

$$\mathbf{F}_{\boldsymbol{\varepsilon}^p} = -\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^p}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p) = \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) = \boldsymbol{\sigma}. \quad (13)$$

Gurson's yield function, in its original form, has been obtained from a limit-analysis approach of a hollow sphere subjected to homogeneous strain rate boundary conditions and having a von Mises matrix. The corresponding yield criterion takes the form (Gurson, 1977)

$$\Phi(\mathbf{F}_{\boldsymbol{\varepsilon}^p}; f) = \Phi(\boldsymbol{\sigma}; f) = \frac{\sigma_{\text{eq}}^2}{\sigma_0^2} + 2f \cosh\left(\frac{3}{2} \frac{\sigma_m}{\sigma_0}\right) - 1 - f^2 \leq 0. \quad (14)$$

It must be emphasized that in equation (14), the Eulerian porosity (current volume fraction of void) f acts as a parameter, while σ_0 is the yield limit of the matrix, σ_{eq} and σ_m are respectively the von Mises equivalent stress and the mean stress, given by

$$\sigma_m = \frac{1}{3} \text{tr}(\boldsymbol{\sigma}), \quad \sigma_{\text{eq}} = \sqrt{\frac{3}{2} \boldsymbol{\sigma}_d : \boldsymbol{\sigma}_d}, \quad \text{with } \boldsymbol{\sigma}_d = \boldsymbol{\sigma} - \sigma_m \mathbf{I}, \quad (15)$$

$\boldsymbol{\sigma}_d$ is the deviatoric part of $\boldsymbol{\sigma}$ and \mathbf{I} is the second-order identity tensor.

The yield criterion (14) is completed by a plastic flow rule obeying the normality properties (which is preserved during the upscaling procedure as a consequence of limit-analysis theorems). The plastic strain rate $\dot{\boldsymbol{\varepsilon}}^p$ (equal to the Eulerian strain rate \mathbf{d}^p in the linearized theory) is thus given by

$$\dot{\boldsymbol{\varepsilon}}^p = \eta \frac{\partial\Phi}{\partial\boldsymbol{\sigma}}(\boldsymbol{\sigma}), \quad \eta \begin{cases} = 0 & \text{if } \Phi(\boldsymbol{\sigma}) < 0 \\ \geq 0 & \text{if } \Phi(\boldsymbol{\sigma}) = 0. \end{cases} \quad (16)$$

The last ingredient of Gurson's model is the porosity evolution equation. This balance equation is deduced from the assumed incompressibility of the matrix (surrounding the cavity) and the assumption that reversible strains are negligible:

$$\dot{f} = (1 - f) \text{tr}(\dot{\boldsymbol{\varepsilon}}^p). \quad (17)$$

Finally, the intrinsic dissipation (5) is given by

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p. \quad (18)$$

Despite the porosity changes (see equation (17)), its evolution does not contribute to the intrinsic dissipation \mathcal{D} . Moreover, f is not related to some normality property as there is no thermodynamic force associated to f . This situation needs to be clarified, both on the status of the porosity and its evolution equation.

In addition, Gurson's model generally includes an heuristic modelling of hardening (proposed by Gurson himself). In this (classical) approach, the constant yield limit σ_0 in the criterion (14) is replaced by some "average yield stress" $\bar{\sigma}$ given by:

$$\bar{\sigma} \equiv \sigma(\bar{\boldsymbol{\varepsilon}}) \quad (19)$$

where $\sigma(\varepsilon)$ is a function providing the local yield limit as a function of the local accumulated plastic strain ε , and $\bar{\varepsilon}$ represents some “average equivalent strain” in the heterogeneous, porous material. The evolution of $\bar{\varepsilon}$ is governed by the following equation Gurson (1977):

$$(1 - f)\bar{\sigma}\dot{\bar{\varepsilon}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p \quad (20)$$

which expresses the heuristic assumption that the plastic dissipation in the heterogeneous porous material is equal to that in a fictitious “equivalent” homogeneous material with equivalent strain $\bar{\varepsilon}$ and yield stress $\bar{\sigma}$.

Interestingly, Enakoutsa et al. (2007) have shown that, *provided that the porosity is fixed*, Gurson’s model belongs to the class of standard generalized materials, even in presence of hardening (modelled by equation (20)). Indeed, the free energy as well as Gurson’s yield function are convex, and if the porosity is fixed, (i.e. its evolution equation is no longer included in the model), the evolution equation of the plastic strain rate obeys the normality rule.

3 Thermodynamic framework for plasticity of saturated porous media

In this section, we summarize and adapt some key elements of poroplasticity theory of saturated media² useful for the reformatting of the Gurson model in Section 4. Our reference to poroplasticity of saturated media has been primarily motivated by the need to properly identify the relevant variable describing the porosity (the Lagrangian one) and its conjugate force (the pressure) and the fact that in this context the dry medium ($p = 0$) appears as a particular case. Let us recall that our main objective is not to formulate a new model of ductile porous material but to propose a suitable framework for the thermodynamic interpretation of Gurson’s model.

Let us first introduce the Lagrangian porosity and its relationship with the already defined Eulerian one (11). By denoting $|\Omega_0|$ the initial volume of the porous material, and recalling that $|\omega(t)|$ is the volume of pores at the current time t , the Lagrangian porosity ϕ , whose initial value is $\phi_0 = f_0$ (initial Eulerian porosity), is defined by:

$$\phi = \frac{|\omega(t)|}{|\Omega_0|} \quad (21)$$

From the definitions of the two type of porosities, it is readily seen that the variation of pores volume, $\Delta\omega = |\omega(t)| - |\omega_0|$ (the initial pore volume being denoted $|\omega_0|$) is appropriately captured by the variation of Lagrangian porosity $\phi - \phi_0 = \frac{\Delta\omega}{|\Omega_0|}$, and not by that of the Eulerian one $f - f_0$.

Although Lagrangian and Eulerian porosities are close in infinitesimal transformations (their ratio is the Jacobian of the transformation $J = 1 + \text{tr}(\boldsymbol{\varepsilon})$, close to 1), one should be careful that the variations of ϕ and f cannot be confused. Indeed, given $\phi_0 = f_0$, one obtains by linearization (see for instance Dormieux and Bourgeois (2002))

$$\phi - \phi_0 = f - f_0 + f \text{tr}(\boldsymbol{\varepsilon}). \quad (22)$$

² A complete presentation of thermodynamic framework for poroplasticity of saturated media can be found in Coussy (1995) (see also Coussy (2004)).

In the context of poroplasticity of saturated media, in addition to the state variables (strain tensor $\boldsymbol{\varepsilon}$ and Lagrangian porosity ϕ), two internal variables are introduced: the plastic strain tensor $\boldsymbol{\varepsilon}^p$ and the Lagrangian plastic porosity ϕ^p .

Now, let us come to the elastoplastic behavior, which is then described by means of four state variables ($\boldsymbol{\varepsilon}, \phi, \boldsymbol{\varepsilon}^p, \phi^p$) with the following additive decomposition:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p, \quad \Delta\phi = \phi - \phi_0 = \phi^e + \phi^p \quad (23)$$

Note that $\boldsymbol{\varepsilon}^e$ and ϕ^e refer respectively to the elastic strains and the reversible part of the Lagrangian porosity.

For sake of clarity, hardening effects due to plastic porosity or to solid matrix hardening are disregarded at this stage, so that the free energy ψ of the saturated elasto-poroplastic material takes the form (Coussy (2004)):

$$\psi(\boldsymbol{\varepsilon}, \phi, \boldsymbol{\varepsilon}^p, \phi^p) = \psi(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, \phi - \phi_0 - \phi^p). \quad (24)$$

Following the general framework of Section 2, the *reversible* parts of the forces are

$$\boldsymbol{\sigma}^{rev} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}; \quad \mathbf{F}_{\boldsymbol{\varepsilon}^p}^{rev} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^p}; \quad \mathbf{F}_{\phi}^{rev} = \frac{\partial\psi}{\partial\phi}; \quad \mathbf{F}_{\phi^p}^{rev} = \frac{\partial\psi}{\partial\phi^p}. \quad (25)$$

By denoting p the pore pressure, the intrinsic dissipation of the saturated poroplastic material reads (see for instance Coussy (1995) or Coussy (2004))

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + p\dot{\phi} - \dot{\psi} = \left(\boldsymbol{\sigma} - \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} \right) : \dot{\boldsymbol{\varepsilon}} + \left(p - \frac{\partial\psi}{\partial\phi} \right) \dot{\phi} - \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^p} : \dot{\boldsymbol{\varepsilon}}^p - \frac{\partial\psi}{\partial\phi^p} \dot{\phi}^p. \quad (26)$$

The state variables $\boldsymbol{\varepsilon}$ and ϕ being non dissipative, one has

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{rev} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}; \quad p = \mathbf{F}_{\phi}^{rev} = \frac{\partial\psi}{\partial\phi}. \quad (27)$$

The intrinsic dissipation (26) of the saturated porous material reads then

$$\mathcal{D} = \mathbf{F}_{\boldsymbol{\varepsilon}^p} : \dot{\boldsymbol{\varepsilon}}^p + \mathbf{F}_{\phi^p} \dot{\phi}^p = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + p\dot{\phi}^p \quad (28)$$

in which the irreversible forces are

$$\mathbf{F}_{\boldsymbol{\varepsilon}^p} = -\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^p} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} = \boldsymbol{\sigma} \quad (29)$$

and

$$\mathbf{F}_{\phi^p} = -\frac{\partial\psi}{\partial\phi^p} = \frac{\partial\psi}{\partial\phi} = p, \quad (30)$$

It is worth noting that when the pore pressure vanishes ($p = 0$), the intrinsic dissipation (28) trivially reduces to the plastic power $\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p$, as in (18), although the dissipative variable $\dot{\phi}^p$ is still present. Clearly enough, the reason for which the porosity does not contribute to the dissipation (18), in consistency with the original Gurson model, is due to the cancellation of the pore pressure in the dry elasto-poroplastic material case.

For completeness, the classical expression of the free energy of a saturated elasto-poroplastic material in the absence of hardening, takes the form

$$\psi(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, \phi - \phi_0 - \phi^p) = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) : \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + \frac{1}{2}N [(\phi - \phi_0 - \phi^p) - \mathbf{b} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)]^2, \quad (31)$$

where \mathbb{C} is the elastic stiffness tensor of the porous skeleton, \mathbf{b} is the Biot (second-order) tensor and N is the Biot modulus. This free energy corresponds to that of a poroelastic material described by means of Biot's theory.

The stress $\boldsymbol{\sigma}$ and the pore pressure p are thus given by

$$\begin{cases} \boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, \phi - \phi_0 - \phi^p) = \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) - N\mathbf{b}(\phi - \phi_0 - \phi^p - \mathbf{b} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)) \\ p = \frac{\partial \psi}{\partial \phi}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, \phi - \phi_0 - \phi^p) = N(\phi - \phi_0 - \phi^p) - N\mathbf{b} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p). \end{cases} \quad (32)$$

For evolution laws, as classically in the context of generalized standard materials, let us introduce the dissipation potential φ , function of the rates of the internal variables $(\dot{\boldsymbol{\varepsilon}}^p, \dot{\phi}^p)$ and from which

$$(\boldsymbol{\sigma}, p) \in \partial \varphi(\dot{\boldsymbol{\varepsilon}}^p, \dot{\phi}^p; \phi^p) \quad (33)$$

where $\partial \varphi$ is the sub-differentials of φ .

It should be noted that in (33) the variable ϕ^p (Lagrangian plastic porosity) which appears after the symbol ";" plays the role of a parameter, the arguments in the dissipation potential being $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\phi}^p$. This complies with the extended definition of the Generalized standard materials as described by Germain et al. (1983) and Nguyen (1994, 2000) which allows the presence of the state variables as parameters in the dissipation potential.

Alternatively, by denoting φ^* the Legendre-Fenchel transform of φ , which proves to be the indicator function of the convex set \mathcal{C} defined by a criterion $\Phi(\boldsymbol{\sigma}, p; \phi^p) \leq 0$ for the elasto-poroplastic material. ϕ^p appears again as a parameter in φ^* or equivalently in Φ . The flow rule reads then

$$(\dot{\boldsymbol{\varepsilon}}^p, \dot{\phi}^p) \in \partial \varphi^*(\boldsymbol{\sigma}, p; \phi^p) \quad (34)$$

in which the notation $\partial \varphi^*(\boldsymbol{\sigma}, p)$ denotes the subdifferential of φ^* at the point $(\boldsymbol{\sigma}, p)$. This corresponds to the normality rule giving $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\phi}^p$.

4 Thermodynamic based formulation of the Gurson model

As mentioned before, the fluid pressure p appears as the dual quantity associated to the (Lagrangian) porosity ϕ^p . Consequently the appropriate thermodynamics interpretation of the Gurson model first requires its extension to the context of saturated plastic materials.

4.1 Link between the Lagrangian and Eulerian porosities evolutions

As already indicated in Section 2.2, Gurson (1977)'s model involves the Eulerian porosity (denoted by f). It is then necessary to discuss the link between evolutions of the plastic

Lagrangian porosity ϕ^p and the Eulerian porosity. To this end, consider a RVE of the porous medium having at time t a volume $|\Omega(t)|$, which comprises a pore phase with volume $|\omega(t)|$. Assuming that all the modification of pores volume $|\omega(t)|$ in the RVE is irreversible, the rate of Lagrangian plastic porosity at time t is defined by

$$\dot{\phi}^p = \lim_{dt \rightarrow 0} \frac{1}{dt} \left(\frac{|\omega(t+dt)| - |\omega(t)|}{|\Omega(t)|} \right), \quad (35)$$

which must not be confused with the rate of Eulerian porosity f , expressed as:

$$\dot{f} = \lim_{dt \rightarrow 0} \frac{1}{dt} \left(\frac{|\omega(t+dt)|}{|\Omega(t+dt)|} - \frac{|\omega(t)|}{|\Omega(t)|} \right). \quad (36)$$

It must be emphasized that these two definitions referred to the same configuration at time t , while a computation by considering the definition (21) would referred to the initial configuration for the Lagrangian porosity. For this reason, it is the rate of Lagrangian plastic porosity at time t (equation (35)) which will be consider in the following together with \dot{f} . Moreover, assuming that all the modifications for the volume of the RVE are irreversible, that is all the reversible parts of the volume change are neglected, the rates of Lagrangian and Eulerian porosities, in link with (22), are readily related by (see for instance also Bignonnet et al. (2016))³

$$\dot{\phi}^p = \dot{f} + f \text{tr}(\dot{\boldsymbol{\epsilon}}^p) \quad (37)$$

Furthermore, in the case of a plastically incompressible solid phase, the volume change is only due to the plastic Lagrangian porosity change, that is

$$\dot{\phi}^p = \text{tr}(\dot{\boldsymbol{\epsilon}}^p) \quad (38)$$

It follows from (37) that

$$\dot{f} = (1 - f) \text{tr}(\dot{\boldsymbol{\epsilon}}^p) \quad (39)$$

which is interestingly the porosity evolution law (17) of the Gurson model obtained from mass balance equation.

Consequently, in this case, the intrinsic dissipation \mathcal{D} , defined by (28), takes the form

$$\mathcal{D} = (\boldsymbol{\sigma} + p\mathbf{I}) : \dot{\boldsymbol{\epsilon}}^p. \quad (40)$$

Remark 1: This result indicates that when the matrix is plastically incompressible, $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + p\mathbf{I}$ is the effective thermodynamic force associated to the plastic strain. This stress tensor, well known as the Terzaghi effective stress in the context of poroelasticity with an incompressible matrix, proves to be also valid in the plastic case of von Mises matrix (see Coussy (1995) and De Buhan and Dormieux (1999) for a proof in the framework of limit-analysis theory).

Remark 2: Equation (37) can be integrated in time as

$$\phi^p = \ln \left(\frac{1 - f_0}{1 - f} \right), \quad (41)$$

³ In relations (37) to (39), which are rigorously valid for the eulerian plastic strain rate $\boldsymbol{\epsilon}^p$, the later has been replaced by $\text{tr}(\dot{\boldsymbol{\epsilon}}^p)$ using the small strain assumption.

in which the initial value of ϕ_0^p is indeed equal to zero. Given that $f_0 = \phi_0$, one has

$$f(\phi^p) = 1 - (1 - \phi_0)\exp(-\phi^p). \quad (42)$$

Note that in the initial state, $\phi^p = 0$ and one has $f(0) = \phi_0 = f_0$.

4.2 Constitutive relations of the model

As mentioned previously, the thermodynamics analysis of Gurson's model will be done within the framework of linearized theory. The effect of large displacements and strains will only be discussed during the numerical implementation of the model in Section 5.

It must be first emphasized that Gurson's model is generally devoted to material with low porosities accounted for in ductile fracture phenomena⁴. For such level of porosity, the effect of voids on elastic stiffness \mathbb{C} of the porous material is generally negligible. As a consequence, their effects on the Biot tensor can be also neglected. For instance, in the isotropic case, one has $\mathbf{b} = b\mathbf{I}$ with $b = 1 - \frac{k^{hom}}{k^s}$, k^{hom} being the elastic compressibility of the porous material which tends to the compressibility of the solid matrix, k^s , for low porosities. Indeed, for low porosities, a direct computation can be done on the hollow sphere in order to estimate the Biot coefficient (see for instance Dormieux et al. (2006), chapter 4): $b(\phi) = \phi \left(1 + \frac{3k^s}{4\mu^s}\right)$ in which μ^s represents the shear modulus of the solid matrix. This clearly confirms that b tends to 0 when the porosity is low. Additionally, the Biot modulus N is given by $N(\phi) = \frac{4\mu^s}{3\phi}$ (see for instance Dormieux et al. (2006)).

In this context, the two terms function of $N\mathbf{b} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$ in the right hand side of the state laws (32) can be considered as negligible compared to the others

$$\begin{cases} \boldsymbol{\sigma} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, \phi - \phi_0 - \phi^p) = \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) - N\mathbf{b}(\phi - \phi_0 - \phi^p) \\ p = \frac{\partial\psi}{\partial\phi}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, \phi - \phi_0 - \phi^p) = N(\phi - \phi_0 - \phi^p). \end{cases} \quad (43)$$

Furthermore, when the pressure is low, that is the case for Gurson's model, the term $\mathbf{b}p$ in the definition of the stress tensor is negligible so that the stress $\boldsymbol{\sigma}$ and pore pressure p are finally given by

$$\begin{cases} \boldsymbol{\sigma} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, \phi - \phi_0 - \phi^p) = \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \\ p = \frac{\partial\psi}{\partial\phi}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, \phi - \phi_0 - \phi^p) = N(\phi - \phi_0 - \phi^p). \end{cases} \quad (44)$$

⁴ Before coalescence, porosities are generally small (Benzerga and Leblond, 2010). Since the analysis made in this work only concerns Gurson's model, which is used as a *void growth model*, it is reasonable to consider that the porosity reached will always remain moderate (less than 0.01–0.05).

The free energy then would be reduced to

$$\psi(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, \phi - \phi_0 - \phi^p) = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) : \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + \frac{1}{2}N(\phi - \phi_0 - \phi^p)^2. \quad (45)$$

The rate form of the constitutive relations of the elasto-poroplastic materials, issued from (44), then reads

$$\begin{cases} \dot{\boldsymbol{\varepsilon}} = \mathbb{S} : \dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{\varepsilon}}^p \\ \dot{\phi} = \frac{\dot{p}}{N} + \dot{\phi}^p, \end{cases} \quad (46)$$

for which $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\phi}^p$ are given by the evolution equations presented below.

Note that for an imposed constant pressure p , one has $\dot{\phi} = \dot{\phi}^p$ which is in agreement with a previous assumption that all the modification of pores volume in the RVE is irreversible.

Taking advantage of the previous result on the effective stress (De Buhan and Dormieux, 1999) (see also a direct proof by Vincent et al. (2009) in the case of an incompressible matrix or remark 2 in Subsection 4.1), Gurson (1977)'s yield criterion can be extended to saturated poroplastic media as

$$\Phi(\boldsymbol{\sigma}, p; \phi^p) = \frac{\sigma_{\text{eq}}^2}{\sigma_0^2} + 2f(\phi^p)\cosh\left(\frac{3}{2}\frac{\sigma_m + p}{\sigma_0}\right) - 1 - f^2(\phi^p) \leq 0, \quad (47)$$

where the Eulerian porosity has been expressed in terms of the Lagrangian porosity as $f(\phi^p)$ given in (42). We can notice that the reversibility domain defined by Gurson's criterion (47) is convex with respect to the thermodynamic forces $\mathbf{F}_{\boldsymbol{\varepsilon}^p} = \boldsymbol{\sigma}$ and $F_{\phi^p} = p$.

The evolution equations of internal variables $\boldsymbol{\varepsilon}^p$ and ϕ^p given by the extended normality rule with respect to the Gurson's extended yield criterion (47) read:

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}^p = \eta \frac{\partial \Phi}{\partial \mathbf{F}_{\boldsymbol{\varepsilon}^p}} \\ \dot{\phi}^p = \eta \frac{\partial \Phi}{\partial F_{\phi^p}} \end{cases}, \quad \eta \begin{cases} = 0 & \text{if } \Phi(\boldsymbol{\sigma}, p; \phi^p) < 0 \\ \geq 0 & \text{if } \Phi(\boldsymbol{\sigma}, p; \phi^p) = 0. \end{cases} \quad (48)$$

η is the usual plastic multiplier. The evolution of the plastic strain reads

$$\dot{\boldsymbol{\varepsilon}}^p = \eta \frac{\partial \Phi}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, p; \phi^p) = \eta \left(\frac{f(\phi^p)}{\sigma_0} \sinh\left(\frac{3}{2}\frac{\sigma_m + p}{\sigma_0}\right) \mathbf{I} + 3\frac{\boldsymbol{\sigma}_d}{\sigma_0^2} \right) \quad (49)$$

As a matter of fact, the evolution of the Lagrangian porosity reads

$$\dot{\phi}^p = \eta \frac{\partial \Phi}{\partial p}(\boldsymbol{\sigma}, p; \phi^p) = 3\eta \frac{f(\phi^p)}{\sigma_0} \sinh\left(\frac{3}{2}\frac{\sigma_m + p}{\sigma_0}\right) = \text{tr}(\dot{\boldsymbol{\varepsilon}}^p). \quad (50)$$

This results has been already indicated in (38) as a consequence of the plastic incompressibility of the solid matrix.

For completeness, it is interesting to establish the pseudo potential of dissipation corresponding to the above extended Gurson poroplastic model. To this end, let us recall that the pseudo potential of dissipation φ is given by the Legendre-Fenchel transform of φ^* (the indicator function of the elastic domain defined by Φ (see (47))). It is shown that (see

Appendix A)

$$\varphi(\dot{\epsilon}^p, \dot{\phi}^p; \phi^p) = \begin{cases} \sigma_0 \left[\sqrt{(\dot{\epsilon}_{eq}^p)^2 + 4(\dot{\epsilon}_m^p)^2} - \sqrt{f(\phi^p)^2(\dot{\epsilon}_{eq}^p)^2 + 4(\dot{\epsilon}_m^p)^2} \right. \\ \quad \left. + 2\dot{\epsilon}_m^p \left(\operatorname{argsinh} \left(\frac{2\dot{\epsilon}_m^p}{f(\phi^p)\dot{\epsilon}_{eq}^p} \right) - \operatorname{argsinh} \left(\frac{2\dot{\epsilon}_m^p}{\dot{\epsilon}_{eq}^p} \right) \right) \right] & \text{if } \dot{\phi}^p = 3\dot{\epsilon}_m^p \\ +\infty & \text{otherwise,} \end{cases} \quad (51)$$

which can also be put in the form

$$\varphi(\dot{\epsilon}^p, \dot{\phi}^p; \phi^p) = \sigma_0 \left[\sqrt{(\dot{\epsilon}_{eq}^p)^2 + 4(\dot{\epsilon}_m^p)^2} - \sqrt{f(\phi^p)^2(\dot{\epsilon}_{eq}^p)^2 + 4(\dot{\epsilon}_m^p)^2} \right. \\ \left. + 2\dot{\epsilon}_m^p \left(\operatorname{argsinh} \left(\frac{2\dot{\epsilon}_m^p}{f(\phi^p)\dot{\epsilon}_{eq}^p} \right) - \operatorname{argsinh} \left(\frac{2\dot{\epsilon}_m^p}{\dot{\epsilon}_{eq}^p} \right) \right) \right] + U_{\{0\}}(\dot{\phi}^p - 3\dot{\epsilon}_m^p), \quad (52)$$

with

$$U_{\{0\}}(x) = \begin{cases} 0 & \text{if } x = 0 \\ \infty & \text{if } x \neq 0. \end{cases} \quad (53)$$

Note that in (52), the first part of the right hand-side which corresponds to the dissipation potential already established by Gurson (1977) is in fact remarkably valid for all pressure p (and not only for $p = 0$ as in the original Gurson model) provided $\dot{\phi}^p = 3\dot{\epsilon}_m^p$. Moreover, it is noticeable that the porosity evolution law (50) appears as a full part of the expression (52) through the indicator function $U_{\{0\}}(\dot{\phi}^p - 3\dot{\epsilon}_m^p)$.

The results established in this section clearly show that Gurson's model and its extension to elasto-poroplastic saturated media thus fits into the framework of generalized standard materials provided that (i) the Eulerian porosity f being replaced by the Lagrangian one ϕ^p and (ii) the effective stress is introduced in the yield criterion.

Remark 3: It is important to emphasize that from a physical point of view, in saturated and permeable plastic porous materials, one needs also to consider the fluid flow, generally through the classical Darcy law (which relates the fluid velocity to the gradient of pore pressure). This allows to build diffusion equations (coupled or not coupled) which may complete the mechanical non linear constitutive equations. In this context, it is possible in some particular case to decouple or neglect diffusion equations and mechanical constitutive equations. In the present interpretation of Gurson's model, Darcy's law has not been considered. This may correspond to a situation in which the permeability of the ductile porous material is assumed negligible, in link with an absence of pores connectivity together and a low porosity (less than few percents as previously assumed).

5 Implications in terms of numerical implementation of Gurson’s model without matrix hardening

5.1 Brief review of numerical algorithms

Gurson (1977)’s model has been widely studied and several numerical strategies have been proposed in literature, as quoted below. Here, we focus on the sole “problem of projection” of the elastically computed stress tensor onto the yield locus (plastic correction of the elastic predictor) which consists in finding the mechanical state $\mathcal{S}_{n+1} = \{\boldsymbol{\varepsilon}_{n+1}, \boldsymbol{\varepsilon}_{n+1}^p, \boldsymbol{\sigma}_{n+1}, f_{n+1}\}$ at time t_{n+1} resulting from a given, known strain increment $\Delta\boldsymbol{\varepsilon}_n$ (resulting from a global elastoplastic iteration), knowing the previous mechanical state $\mathcal{S}_n = \{\boldsymbol{\varepsilon}_n, \boldsymbol{\varepsilon}_n^p, \boldsymbol{\sigma}_n, f_n\}$ at time t_n . Here, the specific details related to the numerical integration of strain hardening within Gurson’s approach will not be discussed (as hardening was not considered in Section 4) but it will be shown in Section 6 that hardening can be easily integrated within the (new) proposed algorithm.

First, let us recall three available numerical algorithms of the literature (which we detail in absence of hardening):

- Peirce et al. (1984) have developed a time integration method devoted to rate-dependant materials based on a rate-tangent modulus. It was applied to Gurson’s model using an implicit scheme for the porosity.
- In the work of Aravas (1987), an implicit scheme with respect to the porosity and the plastic strain was considered. In terms of unknowns, Aravas (1987)’s algorithm consists of two coupled iterative loops on the equivalent and mean plastic strain increment.
- In the work of Besson et al. (2001), an implicit scheme with respect to the porosity and the plastic strain was also considered. The algorithm itself consists in a Newton method on the full set of unknowns (all the components of the plastic strain plus the porosity).
- Enakoutsa et al. (2007) considered a mixed approach, with an implicit scheme for the plastic strain and an explicit scheme for the porosity. This allows to reduce the algorithm to the resolution of a single non-linear equation with only one unknown.

Mention has to be made to the work of Aravas and Ponte Castañeda (2004) (see also Danas and Aravas (2012); Cheng et al. (2017b) in which a fixed-point method was used for all microstructural variables (including the porosity): this approach permits to reach a better convergence in comparison to a fully implicit scheme (Cheng et al., 2017b).

An implicit scheme for the porosity evolution, as considered by Aravas (1987) and Besson et al. (2001), implies that f_{n+1} must verify the equation

$$f_{n+1} = f_n + (1 - f_{n+1})\text{tr}(\Delta\boldsymbol{\varepsilon}_n^p), \quad \text{with} \quad \Delta\boldsymbol{\varepsilon}_n^p = \boldsymbol{\varepsilon}_{n+1}^p - \boldsymbol{\varepsilon}_n^p, \quad (54)$$

or equivalently

$$f_{n+1} = \frac{f_n + \text{tr}(\Delta\boldsymbol{\varepsilon}_n^p)}{1 + \text{tr}(\Delta\boldsymbol{\varepsilon}_n^p)}, \quad (55)$$

where the plastic criterion and flow rule are discretized using the final porosity f_{n+1} . It must be emphasized that the choice of the type of time-discretization was not justified in the work of Aravas (1987) and Besson et al. (2001).

An explicit scheme for the porosity evolution, as considered by Enakoutsa et al. (2007), implies that f_{n+1} must verify the equation

$$f_{n+1} = f_n + (1 - f_n)\text{tr}(\Delta\boldsymbol{\varepsilon}_n^p), \quad (56)$$

where the plastic criterion and flow rule are discretized using the previous porosity f_n . In Enakoutsa et al. (2007), the choice of a mixed implicit/explicit scheme was justified by the fact that Gurson's model corresponds to a generalized standard material *provided that the porosity is fixed*. The reasoning made in Enakoutsa et al. (2007) is that the existence and uniqueness of the solution of the projection problem is guaranteed provided that the porosity is considered as fixed during the projection and then updated at the end of the iteration with an explicit scheme.

As shown in Section 4, Gurson's model actually defines a generalized standard material with the plastic strain and Lagrangian porosity as internal variables. This implies that the existence and uniqueness of the solution of the projection problem is guaranteed if the evolution equations of the plastic strain and Lagrangian porosity are discretized in time with an implicit scheme. However, the analysis made in Section 4 is limited to the linearized theory which can be restrictive in ductile fracture problems involving large strains. Following Enakoutsa et al. (2007), it can be shown that the restriction to linearized theory can be removed by studying in which way the numerical implementation is affected by large displacements and strains:

- First, the equilibrium equations must be solved at time t_{n+1} , by updating the geometry at this instant. This operation is done only at the beginning of each major elastoplastic iteration: the geometry is thus fixed during the projection problem as a consequence, this does not modify the mathematical properties of the projection problem.
- Then, the presence of large strain and displacements requires to replace the elasticity law by a hypoelasticity law involving some objective time-derivative of the stress tensor (such as the Jaumann stress rate). This type of derivative can include, in addition to the ordinary derivative, combinations of the stress tensor, the transformation gradient and the gradient of the velocity. As explained by Enakoutsa et al. (2007), the use of an explicit scheme for the time-discretization of these extra terms thus involves known quantities and invariable during each global elasto-plastic iteration: they can be considered as corrections of the elastic stress predictor and does not have impact upon the existence and uniqueness of the local projection problem.

Using equation (50), it follows that the final Lagrangian porosity is given by

$$\phi_{n+1}^p = \phi_n^p + \text{tr}(\Delta\boldsymbol{\varepsilon}_n^p), \quad (57)$$

where the plastic criterion and flow rule are discretized using the final porosity f_{n+1} . Using equation (42), the Eulerian porosity in the criterion and plastic flow rule is thus given by

$$f(\phi_{n+1}^p) = 1 - \frac{1 - \phi_0}{\exp(\phi_n^p + \text{tr}(\Delta\boldsymbol{\varepsilon}_n^p))}, \quad (58)$$

which can be rewritten as

$$f(\phi_{n+1}^p) = \frac{f(\phi_n^p) + \exp(\text{tr}(\Delta\boldsymbol{\varepsilon}_n^p)) - 1}{\exp(\text{tr}(\Delta\boldsymbol{\varepsilon}_n^p))}. \quad (59)$$

For an arbitrary time step, this expression is thus different from that given by equation (55). However, in the limit of small time steps, that is $\text{tr}(\Delta\boldsymbol{\varepsilon}_n^p) \rightarrow 0$, this reduces to

$$f(\phi_{n+1}^p) = \frac{f(\phi_n^p) + \text{tr}(\Delta\boldsymbol{\varepsilon}_n^p)}{1 + \text{tr}(\Delta\boldsymbol{\varepsilon}_n^p)}, \quad (60)$$

which is equivalent to equation (55).

5.2 A new implicit scheme for Gurson's model

We focus on the local step of the elastoplastic solution which consists in projecting the elastic stress predictor on the yield surface. For simplicity reasons, the algorithm is presented in the context of isotropic elasticity and small strains; its extension to large strains does not raise any specific difficulties as discussed previously. Furthermore, we consider the case of empty cavities ($p = 0$).

Local projection problem. The constitutive equations of the local projection problem read

$$\left\{ \begin{array}{ll} \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p & \text{Decomposition of the total strain} \\ \boldsymbol{\sigma} = (3\kappa\mathbb{J} + 2\mu\mathbb{K}) : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) & \text{Isotropic elasticity law} \\ \Phi(\boldsymbol{\sigma}, p; \phi^p) \leq 0 & \text{Plasticity criterion} \\ \left\{ \begin{array}{l} \dot{\boldsymbol{\varepsilon}}^p = \eta \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} \\ \dot{\phi}^p = \eta \frac{\partial \Phi}{\partial p} \end{array} \right. & \text{Flow rule and consistency conditions} \\ \eta \geq 0 & \\ \eta \Phi(\boldsymbol{\sigma}, p; \phi^p) = 0 & \end{array} \right. \quad (61)$$

where $\boldsymbol{\varepsilon}^e$ is the elastic strain tensor, $\boldsymbol{\varepsilon}^p$ the plastic strain tensor, η the plastic multiplier, κ and μ are respectively the bulk and shear moduli of the matrix, and the fourth-order tensors \mathbb{J} and \mathbb{K} are the isotropic projectors which are classically defined as

$$\mathbb{J} = \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \quad \text{and} \quad \mathbb{K} = \mathbb{I} - \mathbb{J}, \quad (62)$$

with \mathbf{I} the second-order identity tensor and \mathbb{I} the fourth-order symmetric identity tensor whose components are $\mathbf{I}_{ij} = \delta_{ij}$ and $\mathbb{I}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, respectively.

The yield criterion and the flow rule are thus discretized using an implicit scheme with respect to $\boldsymbol{\varepsilon}^p$ and ϕ^p . The discretized equations of the local projection problem are given by

$$\left\{ \begin{array}{l} \boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^{\text{elas}} - (3\kappa\mathbb{J} + 2\mu\mathbb{K}) : \Delta\boldsymbol{\varepsilon}_n^p \\ \Phi(\boldsymbol{\sigma}_{n+1}, \phi_{n+1}^p) \leq 0 \\ \Delta\boldsymbol{\varepsilon}_n^p = \Delta\eta_n \frac{\partial\Phi}{\partial\boldsymbol{\sigma}}(\boldsymbol{\sigma}_{n+1}, \phi_{n+1}^p) = \Delta\eta_n \left(\frac{f(\phi_{n+1}^p)}{\sigma_0} \sinh\left(\frac{3}{2} \frac{\sigma_{m,n+1}}{\sigma_0}\right) \mathbf{I} + 3 \frac{\boldsymbol{\sigma}_{d,n+1}}{\sigma_0^2} \right) \\ \Delta\phi_n^p = \text{tr}(\Delta\boldsymbol{\varepsilon}_n^p) \\ \Delta\eta_n \geq 0 \\ \Delta\eta_n \Phi(\boldsymbol{\sigma}_{n+1}, \phi_{n+1}^p) = 0 \\ \phi_{n+1}^p = \phi_n^p + \Delta\phi_n^p \\ \boldsymbol{\varepsilon}_{n+1}^p = \boldsymbol{\varepsilon}_n^p + \Delta\boldsymbol{\varepsilon}_n^p, \end{array} \right. \quad (63)$$

where $\boldsymbol{\sigma}_{n+1}^{\text{elas}} = \boldsymbol{\sigma}_n + (3\kappa\mathbb{J} + 2\mu\mathbb{K}) : \Delta\boldsymbol{\varepsilon}_n$ is the elastic predictor, that is the stress tensor at time t_{n+1} resulting from the strain increment $\Delta\boldsymbol{\varepsilon}_n$ fictitiously considered as purely elastic.

Algorithm of resolution. The algorithm consists in finding $\Delta\eta_n$. In the case of an explicit discretization of the porosity, it has been shown in Enakoutsu et al. (2007) (see also Dorhmi et al. (2020)) that the local projection problem reduces to the resolution of a single non-linear equation on the sole unknown $\Delta\eta_n$. Thus, in order to take advantage of their algorithm, the treatment of the porosity evolution is done using a *fixed point*: (i) assume $\phi_{n+1}^p = \phi_n^p$; (ii) compute $\boldsymbol{\varepsilon}_{n+1}^p$, $\boldsymbol{\sigma}_{n+1}$ with this value ϕ_{n+1}^p ; (iii) deduce from equation (63)₇ a refined estimate of ϕ_{n+1}^p ; (iv) follow the procedure until the method converges and ϕ_{n+1}^p reaches a stationary value.

The problem thus consists in the determination of $\boldsymbol{\varepsilon}_{n+1}^p$, $\boldsymbol{\sigma}_{n+1}$ for a (temporarily) fixed value ϕ_{n+1}^p . First we begin with the elastic behaviour. If the condition $\Phi(\boldsymbol{\sigma}_{n+1}^{\text{elas}}, \phi_{n+1}^p) \leq 0$ is met, then the evolution is purely elastic and the final mechanical state is simply given by

$$\left\{ \begin{array}{l} \boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^{\text{elas}} \\ \boldsymbol{\varepsilon}_{n+1}^p = \boldsymbol{\varepsilon}_n^p. \end{array} \right. \quad (64)$$

However, if $\Phi(\boldsymbol{\sigma}_{n+1}^{\text{elas}}, \phi_{n+1}^p) > 0$, then the elastic predictor is not plastically admissible and a plastic correction is needed to ensure that the plasticity criterion is verified. Using (63)₁ and (63)₃, the final stress $\boldsymbol{\sigma}_{n+1}$ can be written as

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^{\text{elas}} - \Delta\eta_n \left(\frac{3\kappa f(\phi_{n+1}^p)}{\sigma_0} \sinh\left(\frac{3\sigma_{n+1,m}}{2\sigma_0}\right) \mathbf{I} + 6\mu \frac{\boldsymbol{\sigma}_{n+1,d}}{\sigma_0^2} \right), \quad (65)$$

where $\boldsymbol{\sigma}_{n+1,d} = \boldsymbol{\sigma}_{n+1} - \frac{1}{3}\text{tr}(\boldsymbol{\sigma}_{n+1})\mathbf{I}$ is the deviatoric part of the stress $\boldsymbol{\sigma}_{n+1}$. By taking the mean part ($\sigma_{n+1,m}$) and von Mises norm ($\sigma_{n+1,eq}$) of the stress $\boldsymbol{\sigma}_{n+1}$, one gets:

$$\left\{ \begin{array}{l} \sigma_{n+1,m} = \sigma_{n+1,m}^{\text{elas}} - \frac{3\Delta\eta_n \kappa f(\phi_{n+1}^p)}{\sigma_0} \sinh\left(\frac{3\sigma_{n+1,m}}{2\sigma_0}\right) \\ \sigma_{n+1,eq} = \frac{\sigma_{n+1,eq}^{\text{elas}}}{1 + 6\Delta\eta_n \frac{\mu_n}{\sigma_0^2}}. \end{array} \right. \quad (66)$$

The criterion $\Phi(\boldsymbol{\sigma}_{n+1}, \phi_{n+1}^p) = 0$ must be verified so that one has

$$\frac{\sigma_{n+1,eq}^2}{\sigma_0^2} + 2f(\phi_{n+1}^p) \cosh\left(\frac{3\sigma_{n+1,m}}{2\sigma_0}\right) - 1 - f^2(\phi_{n+1}^p) = 0, \quad (67)$$

which permits to express the mean stress as a function of the equivalent stress:

$$\sigma_{n+1,m} = \text{sgn}(\sigma_{n+1,m}^{\text{elas}}) \frac{2\sigma_0}{3} \text{argcosh}\left(\frac{1 + f^2(\phi_{n+1}^p) - \frac{\sigma_{n+1,eq}^2}{\sigma_0^2}}{2f(\phi_{n+1}^p)}\right), \quad (68)$$

where $\text{sgn}(x)$ denotes the sign of x . In equation (68), the term in the argcosh function must be greater than or equal to 1 (by definition of its domain of definition). Combining equation (66) and (68), one finally gets

$$G(\Delta\eta_n) = \frac{2\sigma_0}{3} \text{argcosh}(g(\Delta\eta_n)) + \frac{3\Delta\eta_n \kappa f(\phi_{n+1}^p)}{\sigma_0} \sqrt{g(\Delta\eta_n)^2 - 1} - |\sigma_{n+1,m}^{\text{elas}}| = 0, \quad (69)$$

where

$$g(\Delta\eta_n) = \frac{1 + f^2(\phi_{n+1}^p) - \left(\frac{\sigma_{n+1,eq}^{\text{elas}}}{\sigma_0^2 + 6\mu\Delta\eta_n}\right)^2}{2f(\phi_{n+1}^p)} \geq 1. \quad (70)$$

Equation (69) is a nonlinear equation with the sole unknown $\Delta\eta_n$ which can be solved using a Newton method. Once $\Delta\eta_n$ is known, the final stress $\boldsymbol{\sigma}_{n+1}$ is deduced using equations (65) and (66), and the increment of plastic strain $\Delta\boldsymbol{\varepsilon}_n^p$ is deduced using equation (63)₃.

5.3 Numerical example

The predictions of the algorithm proposed in Section 5.2 will be now assessed. We will notably compare it to the explicit scheme for the Eulerian porosity of Enakoutsu et al. (2007) and to the implicit scheme for the Eulerian porosity of Aravas (1987). In practice for the calculations, the explicit scheme for the Eulerian porosity consists in considering f_n in the equations and updating the porosity explicitly using equation (56). On the other hand, the implicit scheme for the Eulerian porosity is similar to that developed in Section (5.2), except for the fact that f_{n+1} is considered in the criterion and flow rule and its evolution equation must verify equation (55).

The model predictions are investigated with a single (homogeneous) element subjected to an axisymmetric proportional loading with major axial stress σ_{33} , under conditions of fixed stress triaxiality $T = \sigma_m/\sigma_{eq}$. The non-zero components of the stress tensor are supposed to be

$$\sigma_{11} = \sigma_{22} = \sigma_m - \frac{1}{3}\sigma_{eq}, \quad \sigma_{33} = \sigma_m + \frac{2}{3}\sigma_{eq}. \quad (71)$$

The evolution of Eulerian porosity f will be studied for the implicit scheme on ϕ^p (in this case, f is deduced from ϕ^p using equation (42)), the implicit scheme on f and the explicit scheme on f . Two triaxialities $T = 1$ and $T = 3$ are considered, and in each case, the influence of the strain increment $\Delta\varepsilon_{eq}$ is investigated. The parameters used for the

simulations are given in Table 1. The results are shown in Figures 2 and 3 for $T = 1$ and $T = 3$, respectively.

Table 1

Values of parameters for the numerical simulations

E [MPa]	ν [-]	$f_0 = \phi_0$ [-]	σ_0 [MPa]
210 000	0.3	10^{-3}	400

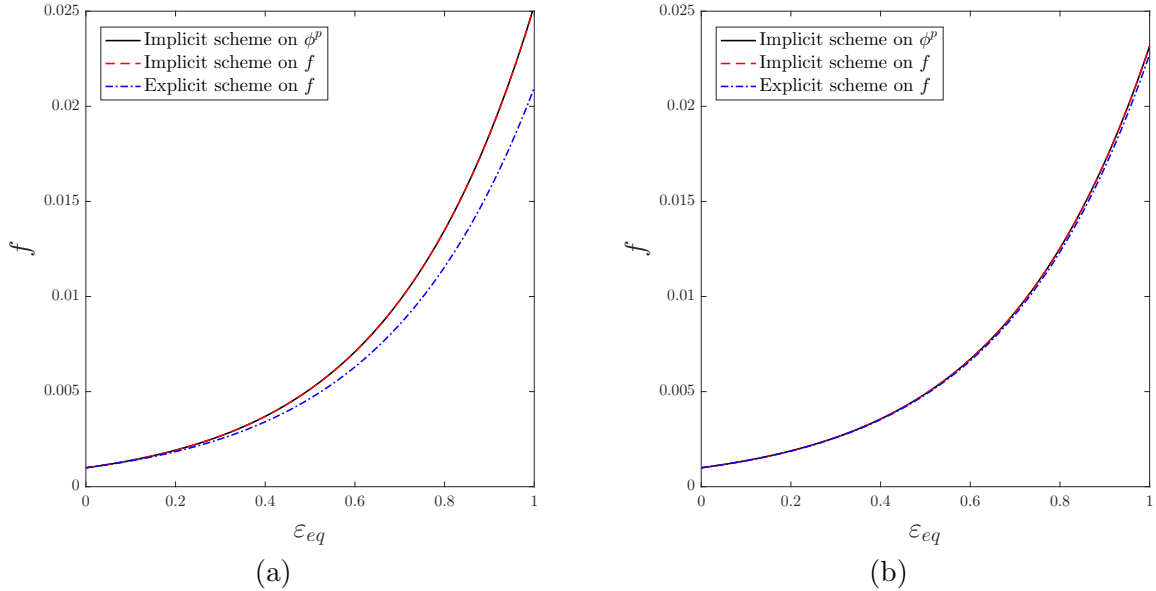


Fig. 2. Evolution of the Eulerian porosity f in the case $T = 1$. (a) $\Delta\varepsilon_{eq} = 0.02$ and (b) $\Delta\varepsilon_{eq} = 0.002$.

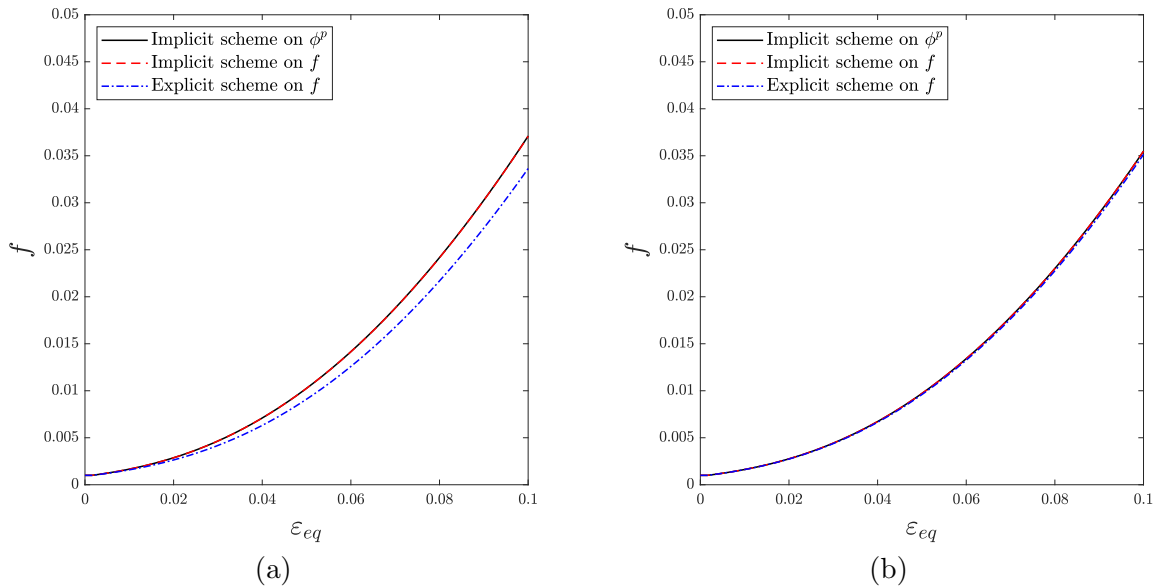


Fig. 3. Evolution of the Eulerian porosity f in the case $T = 3$. (a) $\Delta\varepsilon_{eq} = 0.002$ and (b) $\Delta\varepsilon_{eq} = 0.0002$.

In the two triaxiality cases considered, the evolution of the Eulerian porosity is almost the same when considering the implicit scheme on ϕ^p or the implicit scheme on f . This is due to the fact that for small to moderate values of the total strain increment $\Delta\varepsilon$ (which is considered here) the Eulerian porosity after evolution is the same regardless of the scheme chosen (see equations (55) and (60)). On the other hand, for moderate strain increments ($\Delta\varepsilon_{eq} = 0.02$ in the case $T = 1$ and $\Delta\varepsilon_{eq} = 0.002$ in the case $T = 3$), some differences are observed with the explicit scheme. When the strain increment decreases, those differences

disappear and all the porosity evolution are merged (to the expected reference solution), regardless of the numerical scheme. For $T = 1$ the maximal relative error between the solution obtained using large increments ($\Delta\varepsilon_{eq} = 0.02$) and the reference solution, is of about 8.6% for the explicit scheme and 10% for the two implicit schemes. For $T = 3$ the maximal relative error between the solution obtained using large increments ($\Delta\varepsilon_{eq} = 0.002$) and the reference solution, is of about 4.2% for the explicit scheme and 5.5% for the two implicit schemes. It is interesting to note that, for a given triaxiality, the evolution of the porosity is underestimated for the explicit scheme when the time increment increases while it overestimated for the implicit schemes when the time increment increases is (see Figures 2a and 3a) in comparison with the (reference) solution given with the small time increment (see Figures 2b and 3b). Therefore, the explicit scheme appears to be closer to the reference solution for large increments but it underestimates void growth which implies that the occurrence of failure is expected to be delayed with this scheme.

5.4 Discussion

The present approach has permitted to define a new class of algorithm for Gurson's model that is based on the discretization of the Lagrangian porosity ϕ^p instead of the Eulerian porosity f . It has been shown that, for small increments of the total strain, a time-implicit scheme on Lagrangian porosity ϕ^p is equivalent to a time-implicit scheme on the Eulerian porosity f , which is confirmed by the numerical simulations. Furthermore, if the strain increment is decreased, the results predicted by a time-explicit scheme on the Eulerian porosity are equivalent to those predicted with a time-implicit scheme, which is expected.

In terms of computational complexity, both implicit algorithm (with ϕ^p or f) involves two non-linear equations which can be solved using different strategies (as the fixed point considered here or a Newton's method as done in Aravas (1987)) while the explicit algorithm involves only one non-linear equation. Thus, for Gurson's model *without hardening*, the proposed approach permits to justify Aravas (1987)'s algorithm but does not permit to improve the computational complexity as it also involves two non-linear equations.

It should be noted that concerning the convergence during the integration of the equations, no differences were observed between the time-implicit models based of ϕ^p and f , respectively. This convergence study must be carefully analyzed in future works on structural calculations based on the two integration schemes.

6 Extension to Gurson's model with matrix hardening

We aim in the present section to investigate the matrix hardening effect. The reader interested by the analysis of hardening due to plastic porosity in the context of poroplasticity can refer to Coussy (1995, 2004) (see also Bignonnet et al. (2016)).

6.1 Gurson's model with isotropic hardening

Gurson (1977)'s model was derived using limit-analysis, so it applies to rigid-ideal plastic materials. However, in practical applications, it is necessary to consider a *hardenable* solids matrix. As presented in Section 2, the classical extension of Gurson's model to hardening consists in replacing the constant yield strength σ_0 in the criterion (47) by some "average yield stress" $\bar{\sigma}$ which will play the role of a hardening variable:

$$\Phi(\boldsymbol{\sigma}, p, \bar{\sigma}; \phi^p) = \frac{\sigma_{\text{eq}}^2}{\bar{\sigma}^2} + 2f(\phi^p)\cosh\left(\frac{3}{2}\frac{\sigma_m + p}{\bar{\sigma}}\right) - 1 - f^2(\phi^p) \leq 0. \quad (72)$$

This average yield stress $\bar{\sigma}$ is given by

$$\bar{\sigma} = \sigma(\bar{\varepsilon}), \quad (73)$$

and the evolution of $\bar{\varepsilon}$ reads:

$$\dot{\bar{\varepsilon}} = \frac{\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p}{(1-f)\bar{\sigma}}. \quad (74)$$

Due to the evolution equation of $\bar{\varepsilon}$, it is necessary to extend the results of Section 4 to Gurson (1977)'s model with hardening, following Enakoutsa et al. (2007)'s work.

Then, based on previous expression of the free energy, the free energy ψ^h accounting for matrix hardening is obtained by adding to previous expression (for no matrix hardening case) a supplementary quantity $\lambda \int_0^{\bar{\varepsilon}} \sigma(\varepsilon)d\varepsilon$:

$$\psi^h(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, \phi - \phi_0 - \phi^p, \bar{\varepsilon}) = \psi(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, \phi - \phi_0 - \phi^p) + \lambda \int_0^{\bar{\varepsilon}} \sigma(\varepsilon)d\varepsilon, \quad (75)$$

where λ is a *constant* (yet to be determined).

With this definition, the stress tensor $\boldsymbol{\sigma}$, the fluid pressure p and the (irreversible) thermodynamic forces $\mathbf{F}_{\boldsymbol{\varepsilon}^p}$, F_{ϕ^p} and $F_{\bar{\varepsilon}}$ are given by

$$\left\{ \begin{array}{l} \boldsymbol{\sigma} = \frac{\partial \psi^h}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, \phi - \phi_0 - \phi^p, \bar{\varepsilon}) \\ p = \frac{\partial \psi^h}{\partial \phi}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, \phi - \phi_0 - \phi^p, \bar{\varepsilon}) \\ \mathbf{F}_{\boldsymbol{\varepsilon}^p} = -\frac{\partial \psi^h}{\partial \boldsymbol{\varepsilon}^p}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, \phi - \phi_0 - \phi^p, \bar{\varepsilon}) = \boldsymbol{\sigma} \\ F_{\phi^p} = -\frac{\partial \psi^h}{\partial \phi^p}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, \phi - \phi_0 - \phi^p, \bar{\varepsilon}) = p \\ F_{\bar{\varepsilon}} = -\frac{\partial \psi^h}{\partial \bar{\varepsilon}}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, \phi - \phi_0 - \phi^p, \bar{\varepsilon}) = -\lambda \bar{\sigma}. \end{array} \right. \quad (76)$$

It is easy to check that (i) the free energy is convex with respect to $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^p$, ϕ , ϕ^p and $\bar{\varepsilon}$ and (ii) that the reversibility domain defined by Gurson's criterion (72) is convex with respect to the global variable $(\mathbf{F}_{\boldsymbol{\varepsilon}^p}, F_{\bar{\varepsilon}})$ and F_{ϕ^p} (see Enakoutsa et al. (2007)).

If we assume that this model belongs to the class of standard generalized materials, the evolution equations of internal variables $\boldsymbol{\varepsilon}^p$, ϕ^p and $\bar{\varepsilon}$ obey the extended normality rule

with respect to the Gurson's extended yield criterion (47):

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}^p = \eta \frac{\partial \Phi}{\partial \mathbf{F}_{\boldsymbol{\varepsilon}^p}} \\ \dot{\phi}^p = \eta \frac{\partial \Phi}{\partial F_{\phi^p}} \\ \dot{\bar{\varepsilon}} = \eta \frac{\partial \Phi}{\partial F_{\bar{\varepsilon}}} \end{cases}, \quad \eta \begin{cases} = 0 & \text{if } \Phi(\boldsymbol{\sigma}, p, \bar{\sigma}; \phi^p) < 0 \\ \geq 0 & \text{if } \Phi(\boldsymbol{\sigma}, p, \bar{\sigma}; \phi^p) = 0. \end{cases} \quad (77)$$

The two first equations in (77) are the same as in the case without hardening. In order to express the last equation in (77), let us first introduce the changes of variable and function defined by

$$\frac{\boldsymbol{\sigma} + p\mathbf{I}}{\bar{\sigma}} \equiv \mathbf{X}; \quad \Phi(\boldsymbol{\sigma}, p, \bar{\sigma}; \phi^p) \equiv \tilde{\Phi}(\mathbf{X}; \phi^p). \quad (78)$$

One can then deduce that

$$\begin{cases} \frac{\partial \Phi}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, p, \bar{\sigma}; \phi^p) = \frac{1}{\bar{\sigma}} \frac{\partial \tilde{\Phi}}{\partial \mathbf{X}}(\mathbf{X}; \phi^p) \\ \frac{\partial \Phi}{\partial \bar{\sigma}}(\boldsymbol{\sigma}, p, \bar{\sigma}; \phi^p) = -\frac{1}{\bar{\sigma}^2} \frac{\partial \tilde{\Phi}}{\partial \mathbf{X}}(\mathbf{X}; \phi^p) : (\boldsymbol{\sigma} + p\mathbf{I}), \end{cases} \quad (79)$$

which leads to

$$\frac{\partial \Phi}{\partial \bar{\sigma}}(\boldsymbol{\sigma}, p, \bar{\sigma}; \phi^p) = -\frac{1}{\eta \bar{\sigma}} (\boldsymbol{\sigma} + p\mathbf{I}) : \dot{\boldsymbol{\varepsilon}}^p. \quad (80)$$

It can then be shown that the last equation in (77) reads

$$\dot{\bar{\varepsilon}} = \eta \frac{\partial \Phi}{\partial F_{\phi^p}} = -\frac{\eta}{\lambda} \frac{\partial \Phi}{\partial \bar{\sigma}}(\boldsymbol{\sigma}, p, \bar{\sigma}; \phi^p) = \frac{1}{\lambda \bar{\sigma}} (\boldsymbol{\sigma} + p\mathbf{I}) : \dot{\boldsymbol{\varepsilon}}^p. \quad (81)$$

In order to fit with Gurson's heuristic type of hardening modelling (in the case $p = 0$) given by equation (74), it is sufficient to consider that $\lambda = 1 - \bar{f}$, with \bar{f} constant (since we assumed that λ was constant):

$$(1 - \bar{f}) \dot{\bar{\varepsilon}} \bar{\sigma} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p. \quad (82)$$

This choice of \bar{f} instead of f in the evolution equation of the hardening parameter is equivalent to considering a fixed porosity but only on the evolution equation of the hardening parameter.

To conclude, Gurson's model fits into the framework of generalized materials with hardening effects provided that the parameter \bar{f} in the evolution equation of the hardening variable $\bar{\varepsilon}$ is constant. Thus, in practice in the numerical procedure, this parameter will be chosen as the *previous porosity* ($\bar{f} = f_n$) and will be kept constant during one global iteration. The treatment of the porosity in the evolution equation of hardening is thus explicit.

6.2 Numerical algorithm

As in Section 5.2 we consider the case of empty cavities ($p = 0$). The yield criterion and the flow rule are thus discretized using an implicit scheme with respect to $\boldsymbol{\varepsilon}^p$, ϕ^p and $\bar{\varepsilon}$,

but in the evolution equation of the hardening, the Eulerian porosity at the previous time step is considered. The discretized equations of the local projection problem are given by

$$\left\{ \begin{array}{l} \boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^{\text{elas}} - (3\kappa\mathbb{J} + 2\mu\mathbb{K}) : \Delta\boldsymbol{\varepsilon}_n^p \\ \Phi(\boldsymbol{\sigma}_{n+1}, \bar{\sigma}_{n+1}, \phi_{n+1}^p) \leq 0 \\ \Delta\boldsymbol{\varepsilon}_n^p = \Delta\eta_n \frac{\partial\Phi}{\partial\boldsymbol{\sigma}}(\boldsymbol{\sigma}_{n+1}, \bar{\sigma}_{n+1}, \phi_{n+1}^p) = \Delta\eta_n \left(\frac{f(\phi_{n+1}^p)}{\bar{\sigma}_{n+1}} \sinh\left(\frac{3}{2} \frac{\sigma_{m,n+1}}{\bar{\sigma}_{n+1}}\right) \mathbf{I} + 3 \frac{\boldsymbol{\sigma}_{d,n+1}}{\bar{\sigma}_{n+1}^2} \right) \\ \Delta\phi_n^p = \text{tr}(\Delta\boldsymbol{\varepsilon}_n^p) \\ \Delta\bar{\varepsilon}_n = \frac{\boldsymbol{\sigma}_{n+1} : \Delta\boldsymbol{\varepsilon}_n^p}{(1 - f(\phi_n^p))\bar{\sigma}_{n+1}} \\ \Delta\eta_n \geq 0 \\ \Delta\eta_n \Phi(\boldsymbol{\sigma}_{n+1}, \bar{\sigma}_{n+1}, \phi_{n+1}^p) = 0 \\ \phi_{n+1}^p = \phi_n^p + \Delta\phi_n^p \\ \boldsymbol{\varepsilon}_{n+1}^p = \boldsymbol{\varepsilon}_n^p + \Delta\boldsymbol{\varepsilon}_n^p \\ \bar{\varepsilon}_{n+1} = \bar{\varepsilon}_n + \Delta\bar{\varepsilon}_n \\ \bar{\sigma}_{n+1} = \varepsilon(\bar{\varepsilon}_{n+1}), \end{array} \right. \quad (83)$$

where $\boldsymbol{\sigma}_{n+1}^{\text{elas}} = \boldsymbol{\sigma}_n + (3\kappa\mathbb{J} + 2\mu\mathbb{K}) : \Delta\boldsymbol{\varepsilon}_n$ is the elastic predictor, that is the stress tensor at time t_{n+1} resulting from the strain increment $\Delta\boldsymbol{\varepsilon}_n$ fictitiously considered as purely elastic. The key point here is that in equation (83)₅, we consider $f(\phi_n^p)$.

As in the case without hardening, the algorithm consists in finding $\Delta\eta_n$. The algorithm proposed in Section 5.2 consists in solving the nonlinear equation (69) on the sole unknown $\Delta\eta_n$, for a known Lagrangian porosity ϕ_{n+1}^p and constant yield limit σ_0 , where the calculation of ϕ_{n+1}^p is performed using a fixed point method.

In the case with hardening, we propose to extend the previous algorithm by adding the treatment of hardening within the fixed point iteration: (i) assume $\phi_{n+1}^p = \phi_n^p$ and $\bar{\sigma}_{n+1} = \bar{\sigma}_n$; (ii) compute $\boldsymbol{\varepsilon}_{n+1}^p$, $\boldsymbol{\sigma}_{n+1}$ with these values ϕ_{n+1}^p and $\bar{\sigma}_{n+1}$; (iii) deduce from equations (83)₈ and (83)₁₁ a refined estimate of ϕ_{n+1}^p and $\bar{\sigma}_{n+1}$; (iv) follow the procedure until the method converges and ϕ_{n+1}^p and $\bar{\sigma}_{n+1}$ reach a stationary value. In that case, the nonlinear equation on $\Delta\eta_n$ is

$$G(\Delta\eta_n) = \frac{2\bar{\sigma}_{n+1}}{3} \text{argcosh}(g(\Delta\eta_n)) + \frac{3\Delta\eta_n \kappa f(\phi_{n+1}^p)}{\bar{\sigma}_{n+1}} \sqrt{g(\Delta\eta_n)^2 - 1} - |\sigma_{n+1,m}^{\text{elas}}| = 0, \quad (84)$$

where

$$g(\Delta\eta_n) = \frac{1 + f^2(\phi_{n+1}^p) - \left(\frac{\sigma_{n+1,eq}^{\text{elas}}}{\bar{\sigma}_{n+1}^2 + 6\Delta\eta_n \mu} \right)^2}{2f(\phi_{n+1}^p)} \geq 1. \quad (85)$$

6.3 Application

We consider the same numerical examples than in Section 5.3. The predictions of the algorithm developed in Section 6.2 will be compared to the time-explicit scheme for the porosity of Enakoutsu et al. (2007) (in which hardening is treated with a time-implicit scheme but with f_n in equation (82)) and the fully time-implicit scheme of Aravas (1987) (in which hardening is treated with a time-implicit scheme but with f_{n+1} in equation (82)).

For the simulations, a classical power-law isotropic hardening is assumed:

$$\sigma(\varepsilon) = \sigma_0 \left(1 + \frac{\varepsilon}{\varepsilon_0}\right)^n, \quad (86)$$

where σ_0 is the initial yield stress, ε_0 is the hardening parameter and n is the hardening exponent; the parameters for the simulation are given in Table 2. The case $T = 3$ is considered and the results are shown in Figure 4.

Table 2

Values of parameters for the numerical simulations

E [MPa]	ν [-]	$f_0 = \phi_0$ [-]	σ_0 [MPa]	ε_0	n
210 000	0.3	10^{-3}	400	0.002	0.1

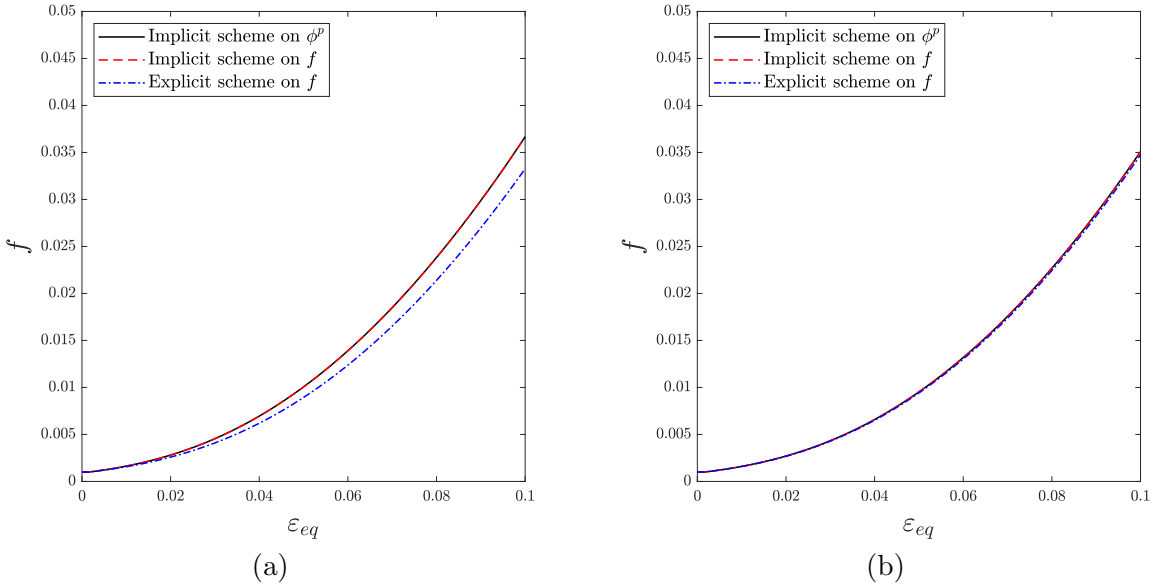


Fig. 4. Evolution of the Eulerian porosity f in the case $T = 3$ (with hardening). (a) $\Delta\varepsilon_{eq} = 0.002$ and (b) $\Delta\varepsilon_{eq} = 0.0002$.

The same trends than in the case without hardening are observed, that is no difference between the two time implicit schemes and an underestimation of the porosity growth by the explicit scheme. The maximal relative error between the solution obtained using large increments ($\Delta\varepsilon_{eq} = 0.002$) and the reference solution, is of about 5.5% for the explicit scheme and 4.2% for the two implicit schemes. It should be noted that the case with hardening, Enakoutsu et al. (2007)’s algorithm consists of two loops (a Newton method on the plastic multiplier and an “external” fixed point on hardening), while Aravas (1987)’s algorithm consists of three loops (two coupled iterative loops on the plastic multiplier and final porosity and an “external” fixed point on hardening) and the algorithm of Section 6.2 consists of two loops (a Newton method on the plastic multiplier and an “external” fixed

point on both hardening and Lagrangian porosity). In addition to its thermodynamics consistency, the algorithm proposed in this work, provides the same predictions than the scheme of Aravas (1987) with here less computational complexity since it involves only two nonlinear equations.

7 Conclusion

The aim of this paper was to investigate the thermodynamics consistency of Gurson's model and specifically its relation to the class of generalized standard materials. First, Gurson's model has been extended to the framework of poroplastic saturated media. This allows to demonstrate that it fits into the framework of generalized standard materials model, provided that the internal variables are the plastic strain and the Lagrangian plastic porosity (and not the Eulerian porosity as in the original Gurson model). Let us recall that the Lagrangian porosity is indeed the state variable which corresponds to the pore pressure (as the associated thermodynamic irreversible force). By construction, the porosity evolution law was proved to be a full part of the thermodynamic formulation of the model with a generalized normality rule. This clearly appears also in the expression that has been established for the pseudo-potential. From the computational point of view, the existence and uniqueness of the projection problem are guaranteed provided that the plastic strain and Lagrangian porosity evolution equations are discretized using a time-implicit scheme. It has been notably shown that this approach, in the limit of small time steps, is equivalent to a implicit scheme on the Eulerian porosity. Hence, without hardening the present approach permits to justify Aravas (1987)'s classical algorithm for Gurson's model. Finally, in presence of hardening, it was shown that Gurson's model also fits into the framework of generalized standard materials, provided that, in the heuristic equation for hardening, the porosity is taken at the previous step. An efficient new algorithm based on Enakoutsa et al. (2007)'s work was then proposed in the case of hardening, based on a global fixed point on both the hardening parameter and the Lagrangian porosity and a Newton method on the plastic multiplier. In terms of convergence during the integration of the equations, no differences were observed between a time-implicit algorithm based on the Eulerian porosity and a time-implicit algorithm based on the Lagrangian porosity on a material point. Future work will involve the numerical implementation of the time-implicit algorithm based on the Lagrangian porosity in a finite element code and a comparative analysis of its convergence with respect to that of the time-implicit algorithm based on the Eulerian porosity.

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A Derivation of the pseudo potential

Starting from the indicator function of domain \mathcal{C} , defined by

$$\varphi^*(\boldsymbol{\sigma}, p; \phi^p) = \begin{cases} 0 & \text{if } \Phi(\boldsymbol{\sigma}, p; \phi^p) \leq 0 \\ +\infty & \text{otherwise} \end{cases} \quad (\text{A.1})$$

the pseudo potential of dissipation, defined as the Legendre-Fenchel transform of φ^* reads

$$\begin{aligned} \varphi(\dot{\boldsymbol{\epsilon}}^p, \dot{\phi}^p; \phi^p) &= [\varphi^*(\boldsymbol{\sigma}, p; \phi^p)]^* = \sup_{\boldsymbol{\sigma}, p} [\boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p + p\dot{\phi}^p - \varphi^*(\boldsymbol{\sigma}, p; \phi^p)] \\ &= \sup_{\boldsymbol{\sigma}, p \in \mathcal{C}} [\boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p + p\dot{\phi}^p] \end{aligned} \quad (\text{A.2})$$

This requires to search for a majoration of

$$\sigma : \dot{\epsilon}^p + p\dot{\phi}^p \leq \sigma_0 \left[\frac{\sigma_{eq}}{\sigma_0} \dot{\epsilon}_{eq}^p + 3 \frac{\sigma_m}{\sigma_0} \dot{\epsilon}_m^p \right] + p\dot{\phi}^p \quad (\text{A.3})$$

The criterion (47) gives

$$\frac{\sigma_{eq}}{\sigma_0} \leq \sqrt{1 + f^2(\phi^p) - 2f(\phi^p) \cosh\left(\frac{3}{2} \frac{\sigma_m + p}{\sigma_0}\right)} \quad (\text{A.4})$$

that allows to write the right hand member of the majoration inequality (A.3) in the following form

$$\sigma : \dot{\epsilon}^p + p\dot{\phi}^p \leq \sigma_0 \left[\dot{\epsilon}_{eq}^p \sqrt{1 + f^2(\phi^p) - 2f(\phi^p) \cosh\left(\frac{3}{2} \frac{\sigma_m + p}{\sigma_0}\right)} + 3 \frac{\sigma_m}{\sigma_0} \dot{\epsilon}_m^p \right] + p\dot{\phi}^p$$

which can be rewritten in a more convenient form

$$\sigma : \dot{\epsilon}^p + p\dot{\phi}^p \leq A \quad (\text{A.5})$$

with

$$A = \sigma_0 \left[\dot{\epsilon}_{eq}^p \sqrt{1 + f^2(\phi^p) - 2f(\phi^p) \cosh\left(\frac{3}{2} \frac{\sigma_m + p}{\sigma_0}\right)} + 3 \frac{\sigma_m + p}{\sigma_0} \dot{\epsilon}_m^p \right] + p(\dot{\phi}^p - 3\dot{\epsilon}_m^p)$$

The right hand member, denoted A , being no more dependent on σ_{eq} , its maximum is searched with respect to σ_m/σ_0 and p . The computation of the derivative of A with respect to σ_m/σ_0 leads to

$$\frac{\partial A}{\partial(\sigma_m/\sigma_0)} = 0 \implies \frac{\dot{\epsilon}_m^p}{\dot{\epsilon}_{eq}^p} = \frac{f(\phi^p) \sinh\left(\frac{3}{2} \frac{\sigma_m + p}{\sigma_0}\right)}{2\sqrt{1 + f^2(\phi^p) - 2f(\phi^p) \cosh\left(\frac{3}{2} \frac{\sigma_m + p}{\sigma_0}\right)}} \quad (\text{A.6})$$

The derivative of A with respect to p gives

$$\frac{\partial A}{\partial p} = 0 \implies \dot{\phi}^p = 3\dot{\epsilon}_{eq}^p \frac{f(\phi^p) \sinh\left(\frac{3}{2} \frac{\sigma_m + p}{\sigma_0}\right)}{2\sqrt{1 + f^2(\phi^p) - 2f(\phi^p) \cosh\left(\frac{3}{2} \frac{\sigma_m + p}{\sigma_0}\right)}} \quad (\text{A.7})$$

namely compared with the expression of $\dot{\epsilon}_m^p/\dot{\epsilon}_{eq}^p$ previously found

$$\dot{\phi}^p = 3\dot{\epsilon}_m^p = \text{tr}\dot{\epsilon}^p \quad (\text{A.8})$$

It is convenient to introduce (see for instance Leblond (2003))

$$\beta = 2 \frac{\dot{\epsilon}_m^p}{\dot{\epsilon}_{eq}^p}$$

In order to eliminate the terms σ_m and p in ((A.5)) we compute β^2 . A second order polynomial of unknown $\cosh\left(\frac{3}{2} \frac{\sigma_m + p}{\sigma_0}\right)$ is obtained. Its solving allows to retrieve (almost

to the term p) one of the two relations already established by Leblond (2003):

$$\cosh\left(\frac{3}{2}\frac{\sigma_m + p}{\sigma_0}\right) = \frac{1}{f(\phi^p)} \left[\sqrt{(1 + \beta^2)(f^2(\phi^p) + \beta^2)} - \beta^2 \right]$$

from which we deduce

$$\cosh\left(\frac{3}{2}\frac{\sigma_m + p}{\sigma_0}\right) = \cosh\left(\operatorname{argsinh}\frac{\beta}{f(\phi^p)} - \operatorname{argsinh}\beta\right)$$

and then

$$\frac{\sigma_m + p}{\sigma_0} = \frac{2}{3} \left(\operatorname{argsinh}\frac{\beta}{f(\phi^p)} - \operatorname{argsinh}\beta \right) \quad (\text{A.9})$$

By reporting this in the expression of $\sqrt{1 + f^2(\phi^p) - 2f(\phi^p) \cosh\left(\frac{3}{2}\frac{\sigma_m + p}{\sigma_0}\right)}$ it is readily seen that

$$\sqrt{1 + f^2(\phi^p) - 2f(\phi^p) \cosh\left(\frac{3}{2}\frac{\sigma_m + p}{\sigma_0}\right)} = \sqrt{1 + \beta^2} - \sqrt{f^2(\phi^p) + \beta^2} \quad (\text{A.10})$$

By introducing (A.9) and (A.10) in (A.5) one gets

$$\begin{aligned} \sigma : \dot{\varepsilon}^p + p\dot{\phi}^p \leq \sigma_0 & \left[\left(\sqrt{(\dot{\varepsilon}_{eq}^p)^2 + 4(\dot{\varepsilon}_m^p)^2} - \sqrt{f(\phi^p)^2(\dot{\varepsilon}_{eq}^p)^2 + 4(\dot{\varepsilon}_m^p)^2} \right) \right. \\ & \left. + 2\dot{\varepsilon}_m^p \left(\operatorname{argsinh}\left(\frac{2\dot{\varepsilon}_m^p}{f(\phi^p)\dot{\varepsilon}_{eq}^p}\right) - \operatorname{argsinh}\left(\frac{2\dot{\varepsilon}_m^p}{\dot{\varepsilon}_{eq}^p}\right) \right) \right] \\ & + p(\dot{\phi}^p - 3\dot{\varepsilon}_m^p) \quad (\text{A.11}) \end{aligned}$$

It follows that

$$\varphi(\dot{\varepsilon}^p, \dot{\phi}^p; \phi^p) = \begin{cases} \sigma_0 \left[\left(\sqrt{(\dot{\varepsilon}_{eq}^p)^2 + 4(\dot{\varepsilon}_m^p)^2} - \sqrt{f(\phi^p)^2(\dot{\varepsilon}_{eq}^p)^2 + 4(\dot{\varepsilon}_m^p)^2} \right) + \right. \\ \left. 2\dot{\varepsilon}_m^p \left(\operatorname{argsinh}\left(\frac{2\dot{\varepsilon}_m^p}{f(\phi^p)\dot{\varepsilon}_{eq}^p}\right) - \operatorname{argsinh}\left(\frac{2\dot{\varepsilon}_m^p}{\dot{\varepsilon}_{eq}^p}\right) \right) \right] & \text{if } \dot{\phi}^p = 3\dot{\varepsilon}_m^p \\ +\infty & \text{otherwise} \end{cases} \quad (\text{A.12})$$

is the dissipation pseudo potential.