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# Pricing for Platforms: Games, Equilibria and Cooperation

Ashok Krishnan K.S., Samir M. Perlaza, Eitan Altman

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## Pricing for Platforms: Games, Equilibria and Cooperation

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**Abstract:** This article considers different pricing models for a platform based rental system, such as Airbnb. A linear model is assumed for the demand response to price, and existence and uniqueness conditions for Nash equilibria are obtained. The Stackelberg equilibrium prices for the game are also obtained, and an iterative scheme is provided, which converges to the Nash equilibrium. Different cooperative pricing schemes are studied, and splitting of revenues based on the Shapley value is discussed. It is shown that a division of revenue based on the Shapley value gives a revenue to the platform proportional to its control of the market. The demand response function is modified to include user response to quality of service. It is shown that when the cost to provide quality of service is low, both renter and the platform will agree to maximize the quality of service. However, if this cost is high, they may not always be able to agree on what quality of service to provide.

**Key-words:** pricing, platforms, Airbnb, Nash equilibrium, Shapley value.

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# Tarification des Plateformes : Jeux, Equilibres et Coopérations

**Résumé :** Cet article examine différents modèles de tarification pour un système de location basé sur une plateforme, comme Airbnb. Un modèle linéaire est supposé pour la réponse de la demande au prix, et des conditions d'existence et d'unicité pour les équilibres de Nash sont obtenues. Les prix d'équilibre de Stackelberg pour le jeu sont également obtenus, et un schéma itératif est fourni, qui converge vers l'équilibre de Nash. Différents systèmes de tarification coopérative sont étudiés et la répartition des revenus basée sur la valeur de Shapley est discutée. Il est montré qu'une répartition des revenus basée sur la valeur de Shapley donne un revenu à la plateforme proportionnel à son contrôle du marché. La fonction de réponse à la demande est modifiée pour inclure la réponse de l'utilisateur à la qualité de service. Il est démontré que lorsque le coût de la qualité de service est faible, le locataire et la plateforme s'accordent pour maximiser la qualité de service. Cependant, si ce coût est élevé, ils ne peuvent pas toujours s'entendre sur la qualité de service à fournir.

**Mots-clés :** prix, plateformes, Airbnb, équilibre de Nash, valeur de Shapley.

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## 1 Introduction

A platform is defined as “an entity that brings together economic agents and actively manages network effects between them” [7]. A platform enables economic interactions between users, while also profiting from them, by charging an appropriate fee from the users. A number of digital platforms have arisen in recent years, which cater to diverse requirements such as renting (Airbnb, Booking.com), transportation (Uber) and dating (Tinder). Depending on the nature of the interactions, the pricing model and the revenue generation mechanism of the platform can vary. In this article, we study the pricing and revenue of digital renting platforms such as Airbnb. A renting platform can be considered an example of a two sided market [4], with the two sides referring to the apartment owners (renters) and the users (consumers), who interact through the platform. A number of works study pricing of platforms in two sided markets. In [4], the authors consider two groups interacting through a platform, with utilities for one group being a linear function of the number of users of the other group. They find equilibrium prices for joining the platforms, for monopoly and multiple competing platforms. Another work [19], studies competition between platforms. In [6], the authors study how a gatekeeper, which controls the availability of price information of commodities, should price itself. They characterize the equilibrium in which gatekeeper profits are maximized.

There have been a large number of works looking at different aspects of Airbnb, which is a major player in short term renting [3]. Airbnb is often used by home owners to supplement their income by renting out of a part of their apartment full time or part time. From the perspective of customers, it represents the possibility of monetary savings as well as non monetary utility, that may come from sharing a living space with a renter [11]. In [21], the authors consider an online platform model for systems such as Airbnb. There are a finite number of sellers (of apartments), and a single representative user, interacting through an online platform. The platform can discriminate regarding how many sellers are to be displayed on its web interface. Under a *multinomial logit* model of demand, it is shown in [21] that social welfare maximization corresponds to displaying all sellers on the platform, while for maximizing the total revenue of the sellers, only the best sellers need to be displayed. However, they do not model the revenue generation mechanism for the platform.

Empirical studies have also been done to study different aspects of the Airbnb economic system. In [14], the authors study dynamic pricing in Airbnb, and conclude that hosts with more resources and experience are more likely to vary prices. In [5], the authors observe that ratings of quality, based on feedback from previous customers, lets hosts set higher prices for their apartments. There have also been studies on the impact of Airbnb on the hotel ecosystem [18]. A work in this direction has noticed an improved revenue for mainstream hotels owing to the presence of Airbnb [10], because Airbnb absorbs budget tourists, who are more price elastic, leaving higher paying customers to hotels. These remaining users are less price elastic. The hotels can now set higher prices for these remaining users. This increases the revenue of the hotels.

## 1.1 Contributions

The contributions of this work are as follows.

1. We propose a linear response model to capture the demand for apartments (or rooms). We first propose a competitive game formulation, where the platform and multiple renter classes find the Nash equilibrium as a pricing solution. By means of the diagonal strict concavity condition, we establish the uniqueness of the Nash equilibrium. We also study the Stackelberg or leader-follower formulation, and demonstrate that being able to pick the price first gives a competitive advantage in terms of revenue.
2. We consider different cooperative scenarios, where the platform and the renters take a joint pricing decision. We obtain the Shapley value for the problem as a means to split the revenue in the cooperative case. With two platforms and multiple renters, we show that the revenue division using Shapley value results in the revenue of the platform being proportional to its market share.
3. We modify the model to introduce the notion of quality of service, and show that, depending on the cost of providing quality of service, the platform and the renter may have similar or differing perspectives on how much quality of service to provide to the customer. At low values of the cost, both the platform and the renter will agree to provide high quality of service. However, if this cost exceeds a threshold, they may not agree on the quality of service to be provided.

## 1.2 System Model

The renting game is played between three sets of users - the platform (for example, Airbnb), the renters (people who rent out their apartments through the platform) and end users or guests (people who are looking to rent out apartments). There are multiple variables of interest which come into play in this interaction. We model the end users by means of a *demand response* function, and we model the platform and renters as playing a game in response to the properties of this function. We will assume there are  $N$  classes of renters, each representing a particular category of apartments. Each class may represent, for example, a group of apartments that are preferred by a section of end users. High paying users prefer different apartments from low paying users. Since the purchasing power of these two groups are different, it will be reflected in the demand response of these two classes of apartments to the price charged as rent. The price charged to the guest has two components: the price paid to the platform, and the price paid to the renter. For class  $i$ , the price paid to the platform will be denoted by  $p_i^0$ , and the price paid to the renter will be denoted by  $p_i^1$ . Define the price vector

$$\mathbf{p} = (\mathbf{p}^0, \mathbf{p}^1) = (p_1^0, p_2^0, \dots, p_N^0, p_1^1, p_2^1, \dots, p_N^1). \quad (1)$$



For renter class  $i$ , the demand response of the users, as a function of the price vector  $\mathbf{p}$  charged, is given by

$$R_i(\mathbf{p}) = \alpha_i - \beta_i(p_i^0 + p_i^1), \quad (2)$$

where  $\alpha_i > 0$  and  $\beta_i > 0$ . This is a linear demand model, which is commonly used in the literature for modeling demand response [12, 13]. The linear model can be interpreted as the mean behaviour of a market with stochastic demand. This models the condition that as price increases, demand for the apartments decreases. This also leads to a concave utility function, which is the commonly used form of utility for tractable optimization, in the literature [8]. Note that  $R_i(\mathbf{p})$  represents the number of apartments that are rented out from all apartments of class  $i$ , when the price vector is  $\mathbf{p}$ . The parameter  $\alpha_i$  represents the maximum demand for an apartment of class  $i$ . The parameter  $\beta_i$  models the rate of decrease of demand of apartments of class  $i$ , as price increases. Note that we do not assume any dependency between the prices of different classes of apartments. This is because we assume that demand for the different classes of apartments arise from disjoint sections of the population of customers.

For a price vector  $\mathbf{p}$ , the revenue (or utility) for the renter  $i$  is given by

$$U_i(\mathbf{p}) = R_i(\mathbf{p})p_i^1 \text{ for } i = 1, \dots, N. \quad (3)$$

The revenue (or utility) for the platform is given by

$$U_0(\mathbf{p}) = \sum_{i=1}^N R_i(\mathbf{p})p_i^0. \quad (4)$$

Define the set

$$\mathcal{P} = \{\mathbf{p} \succeq \mathbf{0} : p_i^j \leq \frac{\alpha_i}{\beta_i}, \text{ for each } i = 1, \dots, N \text{ and } j = 0, 1.\} \quad (5)$$

where  $\mathbf{0}$  represents the all zero vector and  $\succeq$  represents the element-wise "greater than or equal to" inequality. This represents a maximum constraint on the prices being set by the platform and the renter, separately. We will restrict our prices to take values from  $\mathcal{P}$ . This is a larger set than the set commonly used in the literature, which corresponds to the set of non negative prices with non negative demand. However, we use this more relaxed definition because the existence and uniqueness results in Section 2 become simpler to demonstrate. Nevertheless, with either set of constraints, we will obtain the same equilibrium points.

**Remark 1.** *The demand response, and consequently the utility, can be negative at some points in the constraint set  $\mathcal{P}$ . In economic scenarios, one expects utility to be a non negative quantity. In order to be consistent with this notion of non negative utility, one can assume that the actual utility received by the players is offset by a large enough positive value, i.e., of the form  $C + U_j(\mathbf{p})$ , so that it is always non negative.*

**Remark 2.** *Another approach to define the demand response would be to use a response function of the form  $R_i(\mathbf{p}) = \max(\alpha_i - \beta_i(p_i^0 + p_i^1), 0)$ . This ensures non negative demand over any set of prices. However, this can lead to multiple Nash equilibria, arising at price vectors that have zero demand.*

### 1.3 Global Optimum

The global optimal pricing for this model will be that which maximizes the sum of the revenue of the renters and the platform. The total revenue is

$$U(\mathbf{p}) = U_0(\mathbf{p}) + \sum_{i=1}^N U_i(\mathbf{p}).$$

Define the global optimal revenue,

$$U^* = \max_{\mathbf{p} \in \mathcal{P}} U(\mathbf{p}). \quad (6)$$

It is easy to see that the maximum total revenue is  $U^* = \sum_{i=1}^N \frac{\alpha_i^2}{4\beta_i}$ , achieved at any  $\mathbf{p}$  for which  $p_i^0 + p_i^1 = \frac{\alpha_i}{2\beta_i}$  for all renters  $i$ . If the revenue is split equally between the platform and the renter, renter  $i$  will receive a revenue of  $\frac{\alpha_i^2}{8\beta_i}$ , and the platform gets  $\sum_i \frac{\alpha_i^2}{8\beta_i}$ .

## 2 Non Cooperative Game

From Section 1.2, we can see that the pricing problem can be considered as a game, between the platform and the renters. Let us consider the case where there is no cooperation between the renters and the platform. Clearly the renters (indexed by  $i$  going from 1 through  $N$ ) have strategies  $p_i^1$ , while the platform has a strategy vector  $\mathbf{p}^0$ . The corresponding utilities are  $U_1, \dots, U_N$  for the renters and  $U_0$  for the platform. The price vector  $\mathbf{p}$  takes values in the set  $\mathcal{P}$ . It is easy to see that  $\mathcal{P}$  is closed, bounded and convex. We define a Nash equilibrium point for this game as follows.

**Definition 1** (Nash equilibrium). *A Nash equilibrium point for the above pricing game is given by a point  $\mathbf{p}^* = (\mathbf{p}^{0*}, \mathbf{p}^{1*}) \in \mathcal{P}$  where  $\mathbf{p}^{0*} = (p_1^{0*}, p_2^{0*}, \dots, p_N^{0*})$  and  $\mathbf{p}^{1*} = (p_1^{1*}, p_2^{1*}, \dots, p_N^{1*})$  such that*

$$U_0(\mathbf{p}^*) = \max_{\mathbf{q}} \{U_0(\mathbf{q}, \mathbf{p}^{1*})\}, \quad (7)$$

$$U_i(\mathbf{p}^*) = \max_{q_i} \{U_i(\mathbf{p}^{0*}, p_1^{1*}, \dots, q_i, \dots, p_N^{1*})\} \text{ for } i = 1, \dots, N. \quad (8)$$

We prove below that a Nash equilibrium exists for this game.

**Lemma 1.** *A Nash equilibrium exists for the above game.*

*Proof.* From the form of the utility function we can see that

1.  $U_i(\mathbf{p})$  is continuous in  $\mathbf{p}$  for all  $\mathbf{p} \in \mathcal{P}$ , for  $i = 0, 1, 2, \dots, N$
2.  $U_i(\mathbf{p}^0, \mathbf{p}^1)$  is concave in  $p_i^1$  for each fixed value of  $(\mathbf{p}^0, p_1^1, \dots, p_{i-1}^1, p_{i+1}^1, \dots, p_N^1)$ , for  $i = 1, \dots, N$ ,

3.  $U_0(\mathbf{p}^0, \mathbf{p}^1)$  is concave in  $\mathbf{p}^0$  for each fixed value of  $\mathbf{p}^1$ .

Since it is a concave game with finite players, a Nash equilibrium  $\mathbf{p}^*$  exists [20].  $\square$

We will now show that this Nash equilibrium is unique. For any non negative vector  $\mathbf{s} = (s_0, s_1, \dots, s_N)$ , let us define the weighted sum of utilities,

$$V_{\mathbf{s}}(\mathbf{p}) = \sum_{i=0}^N s_i U_i(\mathbf{p}). \quad (9)$$

Define the *pseudogradient*  $v_{\mathbf{s}}(\mathbf{p})$  of  $V_{\mathbf{s}}(\mathbf{p})$  as the matrix

$$v_{\mathbf{s}}(\mathbf{p}) = \begin{bmatrix} s_0 \nabla_0 U_0(\mathbf{p}) \\ s_1 \nabla_1 U_1(\mathbf{p}) \\ \vdots \\ s_N \nabla_N U_N(\mathbf{p}) \end{bmatrix}, \quad (10)$$

where  $\nabla_1, \dots, \nabla_N$  denote gradient with respect to  $p_1^1, \dots, p_N^1$ , and  $\nabla_0$  denotes gradient with respect to the vector  $(p_1^0, \dots, p_N^0)$ . For two vectors  $x$  and  $y$  of the same dimension, let  $\langle x, y \rangle$  denote their inner product. The notion of *diagonally strict concavity* is defined as follows.

**Definition 2.** *The function  $V_{\mathbf{s}}(\mathbf{p})$  is diagonally strict concave for fixed non negative  $\mathbf{s}$  if for every  $\mathbf{p} \neq \mathbf{q}$ ,*

$$\langle \mathbf{q} - \mathbf{p}, v_{\mathbf{s}}(\mathbf{p}) - v_{\mathbf{s}}(\mathbf{q}) \rangle > 0. \quad (11)$$

We are now ready to prove the uniqueness of the Nash equilibrium price  $\mathbf{p}^*$  in Lemma 1. We will use the following result from [20].

**Proposition 2** (Theorem 2 of [20]). *If  $V_{\mathbf{s}}(\mathbf{p})$  is diagonally strict concave for some non zero vector  $\mathbf{s}$ , the Nash equilibrium point  $\mathbf{p}^*$  is unique.*

Note that the above result holds when the constraint set  $\mathcal{P}$  is uncoupled, i.e., the player constraints do not interact with each other, as in our case.

**Theorem 3.** *The Nash equilibrium point  $\mathbf{p}^*$  in Lemma 1 is unique.*

*Proof.* Plugging in the values of  $U_0, \dots, U_N$ , and choosing  $\mathbf{s} = (s, s, \dots, s)$ ,  $s > 0$ , we can write the pseudogradient  $v_{\mathbf{s}}(\mathbf{p})$  as

$$v_{\mathbf{s}}(\mathbf{p}) = \begin{bmatrix} s(R_1(\mathbf{p}) - \beta_1 p_1^0) \\ s(R_2(\mathbf{p}) - \beta_2 p_2^0) \\ \vdots \\ s(R_N(\mathbf{p}) - \beta_N p_N^0) \\ s(R_1(\mathbf{p}) - \beta_1 p_1^1) \\ \vdots \\ s(R_N(\mathbf{p}) - \beta_N p_N^1) \end{bmatrix}. \quad (12)$$

Recall that  $\mathbf{p} = (p_0^0, \dots, p_N^0, p_0^1, \dots, p_N^1)$  and  $\mathbf{q} = (q_0^0, \dots, q_N^0, q_0^1, \dots, q_N^1)$ . We have

$$\begin{aligned} \langle \mathbf{q} - \mathbf{p}, v_{\mathbf{s}}(\mathbf{p}) - v_{\mathbf{s}}(\mathbf{q}) \rangle &= \sum_{i=1}^N s(q_i^0 - p_i^0)(R_i(\mathbf{p}) - R_i(\mathbf{q}) + \beta_i(q_i^0 - p_i^0)) \\ &\quad + \sum_{i=1}^N s(q_i^1 - p_i^1)(R_i(\mathbf{p}) - R_i(\mathbf{q}) + \beta_i(q_i^1 - p_i^1)) \\ &= s \sum_{i=1}^N \beta_i [(q_i^0 - p_i^0)^2 + (q_i^1 - p_i^1)^2] + s \sum_{i=1}^N \beta_i [q_i^0 - p_i^0 + q_i^1 - p_i^1]^2, \end{aligned}$$

which is positive for  $\mathbf{q} \neq \mathbf{p}$ . Thus it follows that  $V_{\mathbf{s}}(\mathbf{p})$  is diagonally strict concave, and the result follows from Proposition 2.  $\square$

## 2.1 Characterizing the Nash Equilibrium

We calculate the Nash equilibrium point  $\mathbf{p}^*$  by solving the utility maximization conditions in Definition 1. From first order conditions for renter class  $i$ ,

$$\frac{\partial U_i(\mathbf{p})}{\partial p_i^1} = \alpha_i - \beta_i p_i^{0*} - 2\beta_i p_i^1. \quad (13)$$

Equating this to zero, we obtain  $p_i^{1*} = \frac{\alpha_i}{2\beta_i} - \frac{p_i^{0*}}{2}$ . Similarly, we have

$$\frac{\partial U_0(\mathbf{p})}{\partial p_i^0} = \alpha_i - \beta_i p_i^{1*} - 2\beta_i p_i^0, \quad (14)$$

which is equal to zero at  $p_i^{0*} = \frac{\alpha_i}{2\beta_i} - \frac{p_i^{1*}}{2}$ . These are simultaneously true at  $p_i^{0*} = p_i^{1*} = \frac{\alpha_i}{3\beta_i}$ , which indicates a symmetric pricing between the renter and the platform. This yields the utilities at the Nash equilibrium point as

$$U_i(\mathbf{p}^*) = \frac{\alpha_i^2}{9\beta_i}, \quad U_0(\mathbf{p}^*) = \sum_{i=1}^N \frac{\alpha_i^2}{9\beta_i}. \quad (15)$$

At the Nash equilibrium, the price charged to a customer of class  $i$  is  $\frac{2\alpha_i}{3\beta_i}$ , which is higher than the price charged at global optimum,  $\frac{\alpha_i}{2\beta_i}$ . However, the total revenue obtained is  $\sum_i \frac{2\alpha_i^2}{9\beta_i}$ , which is smaller than the global optimal revenue  $U^*$ . The *price of anarchy (PoA)* [15], defined as the ratio of the total utility at the global optimum, to the total utility at the equilibrium, is given by

$$PoA = \frac{\sum_i \frac{\alpha_i^2}{4\beta_i}}{\sum_i \frac{2\alpha_i^2}{9\beta_i}}. \quad (16)$$

Note that  $PoA \geq 1$ . The *PoA* indicates how close the equilibrium is to the global optimum, in terms of total revenue. With  $N = 1$ ,  $PoA = \frac{9}{8}$ .

## 2.2 An Iterative Pricing Scheme

We present a sequence of prices that converges to the Nash equilibrium price. Observe that the constraint region  $\mathcal{P}$  can be written as  $\mathcal{P} = \{\mathbf{p} : h(\mathbf{p}) \succeq \mathbf{0}\}$ , where  $h$  is a vector of  $4N$  linear constraints  $h_k(\mathbf{p})$ ,

$$h_k(\mathbf{p}) = \begin{cases} p_k^0, & k = 1, \dots, N, \\ p_{k-N}^1, & k = N+1, \dots, 2N, \\ \frac{\alpha_{k-2N}}{\beta_{k-2N}} - p_{k-2N}^0, & k = 2N+1, \dots, 3N, \\ \frac{\alpha_{k-3N}}{\beta_{k-3N}} - p_{k-3N}^1, & k = 3N+1, \dots, 4N. \end{cases} \quad (17)$$

Define the index sets

$$\mathcal{I}_-(\mathbf{p}) = \{i : h_i(\mathbf{p}) \leq 0\}, \quad (18)$$

$$\mathcal{I}_+(\mathbf{p}) = \{i : h_i(\mathbf{p}) > 0\}. \quad (19)$$

Let  $\mathbf{s} = (s, s, \dots, s)$ ,  $s > 0$ . Define, for a  $4N$  length vector  $\mathbf{w} = (w_1, \dots, w_{4N})$ ,

$$F_{\mathbf{s}}(\mathbf{p}, \mathbf{w}) = v_{\mathbf{s}}(\mathbf{p}) + \langle \nabla h(\mathbf{p}), \mathbf{w} \rangle, \quad (20)$$

$$\mathcal{W}(\mathbf{p}) = \{\mathbf{w} : w_i \geq 0 \text{ for } i \in \mathcal{I}_-(\mathbf{p}), w_i = 0 \text{ for } i \in \mathcal{I}_+(\mathbf{p})\} \quad (21)$$

$$W(\mathbf{p}) = \arg_{\mathbf{w} \in \mathcal{W}(\mathbf{p})} \min \|F_{\mathbf{s}}(\mathbf{p}, \mathbf{w})\|. \quad (22)$$

Consider the sequence of prices  $\mathbf{p}(0), \mathbf{p}(1), \dots$ , where the components evolve as,

$$\mathbf{p}(t+1) = \mathbf{p}(t) + \Delta(t)[v_{\mathbf{s}}(\mathbf{p}(t)) + \langle \mathbf{w}(t), \nabla h(\mathbf{p}(t)) \rangle], \quad (23)$$

$$\mathbf{w}(t) \in W(\mathbf{p}(t)), \quad (24)$$

$$\Delta(t) = -\frac{F_{\mathbf{s}}(\mathbf{p}(t), \mathbf{w}(t))^T J_F F_{\mathbf{s}}(\mathbf{p}(t), \mathbf{w}(t))}{\|J_F F_{\mathbf{s}}(\mathbf{p}(t), \mathbf{w}(t))\|^2}, \quad (25)$$

where  $J_F$  is a mean value of the Jacobian of  $F$ . This sequence satisfies the following.

**Theorem 4.** *Let  $\mathbf{p}(t)$  satisfy the iterative scheme in (23)-(25). Then, for any starting point in  $\mathcal{P}$ ,  $\mathbf{p}(t)$  converges to the Nash equilibrium.*

*Proof.* Consider the pseudogradient vector  $v_{\mathbf{s}}(\mathbf{p})$  with  $\mathbf{s} = (s, s, \dots, s)$ . Let  $J$  be the Jacobian of  $v_{\mathbf{s}}(\mathbf{p})$  with respect to  $\mathbf{p}$ . We see that  $J$  is a  $2N \times 2N$  symmetric matrix with entries  $J_{ij}$  given by,

$$J_{ij} = \begin{cases} -2\beta_i, & \text{for } i = j, i \leq N, \\ -2\beta_{i-N}, & \text{for } i = j, i > N, \\ -\beta_{\min(i,j)}, & \text{for } |i - j| = N, \\ 0, & \text{else.} \end{cases} \quad (26)$$

for  $1 \leq i, j \leq 2N$ . For a vector  $x$ , let  $x^T$  denote its transpose. We can see that for a  $2N$  length vector  $x = (x_1, x_2, \dots, x_{2N})$  we have

$$x^T J x = -2 \sum_{i=1}^N \beta_i x_i^2 - 2 \sum_{i=N+1}^{2N} \beta_{i-N} x_i^2 - 2 \sum_{i=1}^N \beta_i x_i x_{N+i} \quad (27)$$

$$= - \sum_{i=1}^{2N} \beta_i x_i^2 - \sum_{i=N+1}^{2N} \beta_{i-N} x_i^2 - \sum_{i=1}^N \beta_i (x_i + x_{N+i})^2 < 0, \quad (28)$$

for all non zero vectors  $x$ . Thus it follows that  $J$  is negative definite. Note that  $J = J^T$ , where  $J^T$  denotes the transpose of  $J$ . Hence  $J + J^T = 2J$  is also negative definite. Then, it follows from [20, Theorems 10 and 8] and Theorem 3 that  $\mathbf{p}(t)$  converges to the Nash equilibrium  $\mathbf{p}^*$ .  $\square$

### 2.3 Stackelberg Pricing

Now we consider the leader follower pricing, where one (set of) players declares their price first, and the other set follows. Let the renters be leaders (all choosing their prices simultaneously), followed by the platform. The price for the platform will be such that

$$\frac{\partial U_0(\mathbf{p})}{\partial p_i^0} = 0 \text{ for all } i. \quad (29)$$

This is equivalent to

$$p_i^0 = \frac{\alpha_i}{2\beta_i} - \frac{p_i^1}{2}.$$

The renter pricing will take this into account. Substituting this value for the renters, we get

$$U_i(\mathbf{p}) = R_i(\mathbf{p})p_i^1 = p_i^1 \left( \frac{\alpha_i}{2} - \frac{\beta_i p_i^1}{2} \right).$$

For the renters, taking  $\frac{\partial U_i}{\partial p_i^1} = 0$  gives

$$p_i^0 = \frac{\alpha_i}{4\beta_i}, \quad p_i^1 = \frac{\alpha_i}{2\beta_i}.$$

The renters obtain a total revenue of  $\sum_i \frac{\alpha_i^2}{8\beta_i}$ , and the platform gets  $\sum_i \frac{\alpha_i^2}{16\beta_i}$ , and the total revenue is  $\sum_i \frac{3\alpha_i^2}{16\beta_i}$ . Note that the leader, the renter, obtains the same revenue as in the globally optimal pricing (assuming equal division of revenues between renters and the platform). We see that the total price charged to a class  $i$  customer at the Stackelberg equilibrium is higher than the price at Nash equilibrium as well as the global optimal price. The price of anarchy at the Stackelberg equilibrium is

$$PoA = \frac{\sum_i \frac{\alpha_i^2}{4\beta_i}}{\sum_i \frac{3\alpha_i^2}{16\beta_i}}.$$

For  $N = 1$ ,  $PoA = \frac{4}{3}$ , which is greater than  $PoA$  at the Nash equilibrium. The Stackelberg equilibrium prices are hence costlier for the customer, while yielding lower revenues globally.

If we consider the inverse case, with the platform being leader and renters being followers, the platform now gets  $\sum_i \frac{\alpha_i^2}{8\beta_i}$ , and the renters obtain  $\sum_i \frac{\alpha_i^2}{16\beta_i}$ . Thus, getting to choose the price first lets you have a higher fraction of the revenue. In practice, who gets to choose first depends on the market. If the platform is a monopoly, and the renters have no option but to use the platform for their operation, then the platform may be able to take the lead in pricing, and getting an advantage. On the other hand, if there are multiple platforms that compete with in the market, or there is an open market where renters can easily get customers, renters may be in a position to choose to price first, and thus make a gain. Thus, renters who exclusively use one platform may stand to lose in revenue, whereas renters who use multiple platforms or have access to the real market, can make a gain.

### 3 Cooperative Pricing

In this section we consider cooperative pricing scenarios. The change from the preceding section is that the division of revenues between the platform and renter is fixed by arrangement. The renter pays the the platform from his revenue, at some previously agreed fixed rate  $f$ . We can think of two ways in which revenue can be shared between renter and the platform. In both these cases we will see that the total revenue extracted is the maximum possible (i.e., equal to the globally optimal revenue). However, the sharing of the revenues between renter and the platform is slightly different. The difference between these two ways is that in the first, the payment to the platform is viewed as a kind of ‘tax’ on the renter, and the end user is not aware of it, while in the second case the payment to the platform is presented as a ‘tax’ to the end user, directly on the price.

#### 3.1 Case A

The renter  $i$  announces a price  $p_i$ , and then pays  $fp_i$  to the platform, where  $f \in (0, 1)$ . The utilities of platform and renters  $i$  are

$$U_0(\mathbf{p}) = \sum_i (\alpha_i - \beta_i p_i) f p_i,$$

$$U_i(\mathbf{p}) = (\alpha_i - \beta_i p_i)(p_i - f p_i), \quad i = 1, \dots, N.$$

At the optimal point given by equating the partial derivatives of  $U_i$  with respect to  $p_i$  to zero, we get  $p_i = \frac{\alpha_i}{2\beta_i}$ ,  $U_i(\mathbf{p}) = (1 - f) \frac{\alpha_i^2}{4\beta_i}$  and  $U_0 = f \sum_i \frac{\alpha_i^2}{4\beta_i}$ . Total revenue extracted is  $\sum_i \frac{\alpha_i^2}{4\beta_i}$ . The rental platform Booking.com uses this model, and charges an  $f$  between 10% and 20%, depending on the location of the property [2].

### 3.2 Case B

The renter  $i$  announces a price  $p_i(1+f)$ , and then pays  $fp_i$  to the platform, for some  $f > 0$ . The utilities of platform and renters  $i$  are

$$U_0(\mathbf{p}) = \sum_i (\alpha_i - \beta_i(1+f)p_i)fp_i,$$

$$U_i(\mathbf{p}) = (\alpha_i - \beta_i(1+f)p_i)p_i, \quad i = 1, \dots, N.$$

In this case the equilibrium prices and revenues are

$$p_i = \frac{\alpha_i}{2(1+f)\beta_i}, \quad U_i = \frac{\alpha_i^2}{4\beta_i(1+f)}, \quad U_0 = \frac{f}{1+f} \left( \sum_i \frac{\alpha_i^2}{4\beta_i} \right).$$

Total revenue extracted is  $\sum_i \frac{\alpha_i^2}{4\beta_i}$ . This is close to the pricing model of Airbnb, which charges around 14.2% as  $f$ . However, they also charge 3% on the guest separately [1].

Compared to method A, under method B, for the same  $f$ , the revenue of the platform decreases, and that of the renters increases (because for  $f > 0$ ,  $1-f < \frac{1}{1+f}$ ). Under both methods A and B, the customer sees a price  $\frac{\alpha_i}{2\beta_i}$ .

However, the splitting of revenues is different. Since  $\lim_{f \rightarrow \infty} \frac{f}{1+f} = 1$ , we see that by increasing  $f$  in method B, the platform can achieve a revenue as close to the total revenue as it wants. The customer sees the same price irrespective of the choice of  $f$ .

### 3.3 Revenue Division and Shapley Value

In the preceding discussion, we had assumed that there was a previously agreed, fixed number  $f$ , which decided the division of revenue between the platform and the renter. How can one obtain a “fair” value of  $f$ ? One may also have a different value of  $f_i$  for each renter. How do we decide this division  $f_i$ ? One solution is to consider the Shapley value [17] associated with the game. In a related context, Shapley value was used in the study of pricing of internet service providers in [16]. The Shapley value is an indicator of the relative power that different players have in the coalition, when they cooperate. To obtain the Shapley value, we assign values to different coalitions between the players and the platform. We will also need to take into account the presence of other competitors to the platform in the market.

Let  $P_l = \{A_1, A_2, R_1, \dots, R_N\}$  denote the set of players, with  $A_1$  and  $A_2$  representing two platforms, and  $R_i$  representing renter class  $i$ . Let  $\mu(C)$  represent the *worth* obtained by a coalition  $C \subseteq P_l$ . This denotes the total utility when all users in  $C$  work together as a coalition. We will assume that platform  $A_i$  controls a fraction  $\rho_i$  of the customer population, with  $\rho_1 + \rho_2 = 1$ . This can



be considered as a generalization of our model with renter class  $j$  of the market share controlled by platform  $i$ , having a demand response of the form

$$R_{i,j}(\mathbf{p}) = \sqrt{\rho_i} \alpha_j - \beta_j (p_{i,j}^0 + p_{i,j}^1), \quad i = 1, 2, \quad (30)$$

with the price vector  $\mathbf{p}$  of appropriate dimensions. Using the results of the preceding discussions in Section 3, it is clear that the worth obtained by a coalition between both platforms and a subset of the renters  $i_1, i_2, \dots, i_n \in \{1, \dots, N\}$  is

$$\mu(\{A_1, A_2, R_{i_1}, R_{i_2}, \dots, R_{i_n}\}) = \sum_{j=1}^n \frac{\alpha_{i_j}^2}{4\beta_{i_j}}. \quad (31)$$

If a single platform and a subset of renters collaborate, the revenue will be proportional to the market share of that platform. Thus, for any  $i_1, i_2, \dots, i_n \in \{1, \dots, N\}$  we have

$$\mu(\{A_n, R_{i_1}, R_{i_2}, \dots, R_{i_n}\}) = \rho_n \sum_{j=1}^n \frac{\alpha_{i_j}^2}{4\beta_{i_j}}, \quad n = 1, 2. \quad (32)$$

The apartments or the platforms have no worth in the absence of the other, i.e.,

$$\mu(\{A_1\}) = \mu(\{A_2\}) = \mu(C) = 0, \quad \forall C \subseteq P_l \setminus \{A_1, A_2\}, \quad (33)$$

where for two sets  $A, B$ ,  $A \setminus B = A \cap B^c$  denotes the difference of the sets. We define the Shapley value for the game.

**Definition 3.** *The Shapley value  $\eta$  is the vector given by*

$$\eta(i) = \sum_{C \subseteq P_l \setminus \{i\}} \frac{|C|!(|P_l| - |C| - 1)!}{|P_l|!} (\mu(C \cup \{i\}) - \mu(C)), \quad \forall i \in P_l. \quad (34)$$

Here  $\mu(C \cup \{i\}) - \mu(C)$  represents the increment in worth in the coalition  $C$  by the addition of some player  $i \in P_l$ . The Shapley value is a unique function, that provides a measure of each player's contribution to the coalition and satisfies certain desirable properties [17, Theorem 9.3]. With  $N$  renter classes and two platforms, and worth function  $\mu$  defined as discussed, we obtain the following.

**Lemma 5.** *For the worth  $\mu$  defined in (31)-(33), the Shapley value is given by*

$$\eta(A_i) = \frac{\rho_i}{2} \sum_{j=1}^N \frac{\alpha_j^2}{4\beta_i}, \quad i = 1, 2, \quad (35)$$

$$\eta(R_i) = \frac{1}{2} \frac{\alpha_i^2}{4\beta_i}, \quad i = 1, \dots, N. \quad (36)$$

The proof of this lemma is provided in Appendix A.

Thus, if  $\rho_1 = 1$ , there is one platform that has monopoly, and it will take away half of the renters' revenues. Since the total pooled revenue in a cooperative game between one platform and  $N$  renters yields a total utility of  $\sum_i \frac{\alpha_i^2}{4\beta_i}$ ,

we can see that the Shapley values of the players tells us how to divide this total utility. From the perspective of renter class  $i$ , the Shapley division represents a more than twofold improvement over what it gets in the Nash price. The platform also makes a similar gain.

In the case of two platforms, they will divide half of the total revenue between themselves. The fraction of total price taken as fee by the platform, is an indicator of how much of the market it controls, or how close it is to being a monopoly. Both Airbnb and Booking.com charge well below 50% of the total price as a service charge. As discussed before, the former charges around 17% and the latter, 15%, on average.

## 4 Quality of Service/Experience

In this section, we modify the demand response to incorporate the quality of service offered by the renter (alternatively, the quality of experience of the guest). In the sequel we will assume that there is just one renter class, i.e.,  $N = 1$ . The demand response function is given by

$$R(p) = \alpha - \beta p + \gamma q. \quad (37)$$

Here  $p$  represents the price. In this section we will remove the upper bound on price, and allow it to take values in the set  $[0, \infty)$ . The variable  $q$  represents a quality of service parameter, taking values in the set  $[q_{min}, q_{max}]$ , with  $0 \leq q_{min} \leq q_{max}$ . Higher values of  $q$  correspond to the renters providing better services for the client (in terms of customer satisfaction). This can correspond to better behaviour, better services and creating a better ambience around the property. However, per apartment, maintaining this value of  $q$  comes at a price of  $cq$  for the renter. Let us also assume that of the price charged, the renter gets to keep a fraction  $f \in (0, 1]$ , and the platform keeps  $1 - f$ . The utility of the renter at a price  $p$  is, therefore,

$$U_r(p) = (fp - cq)R(p). \quad (38)$$

The utility for the platform, on the other hand is

$$U_a(p) = (1 - f)pR(p). \quad (39)$$

We are interested in whether the renter and the platform can agree on a value of  $q$  as the quality of service to provide to the customer. We first find the optimal price from the renter's perspective, and calculate the utilities for the renter and the platform at this optimal price. Then we examine how these utilities vary, as we vary the parameter  $q$ .

### 4.1 Maximizing the Renter's Utility

The partial derivatives of  $U_r(p)$  with respect to  $p$  are,

$$\frac{\partial U_r(p)}{\partial p} = \alpha f + cq\beta + fq\gamma - 2pf\beta, \quad (40)$$

$$\frac{\partial^2 U_r(p)}{\partial p^2} = -2f\beta < 0. \quad (41)$$

Since the function is strictly concave, it follows [9, Sec. 4.2.3] that the renter's utility is maximized when  $\frac{\partial U_r(p)}{\partial p} = 0$ , at  $p = p_r^*$ , where

$$p_r^* = \frac{\alpha + q\gamma}{2\beta} + \frac{cq}{2f}. \quad (42)$$

The utility for the renter with price  $p_r^*$  is

$$U_r(p_r^*) = \frac{f}{4\beta} \left( \alpha + q\gamma - \frac{cq\beta}{f} \right)^2. \quad (43)$$

Since this is a function of  $q$ , we look at how it varies with  $q$ . We have

$$\frac{\partial U_r(p_r^*)}{\partial q} = \frac{f}{2\beta} \left( \alpha + q\gamma - \frac{cq\beta}{f} \right) \left( \gamma - \frac{c\beta}{f} \right) = \frac{\alpha}{\beta} \frac{(f\gamma - c\beta)}{2} + q \frac{(f\gamma - c\beta)^2}{2f\beta}.$$

The utility for the platform at the price  $p_r^*$  is

$$U_a(p_r^*) = \frac{1-f}{4\beta} \left[ (\alpha + q\gamma)^2 - \left( \frac{cq\beta}{f} \right)^2 \right]. \quad (44)$$

The rate of change of the utility of the platform at price  $p_r^*$  with respect to  $q$  is

$$\frac{\partial U_a(p_r^*)}{\partial q} = \frac{1-f}{2\beta} \left[ \alpha\gamma + q \left( \gamma^2 - \frac{c^2\beta^2}{f^2} \right) \right]. \quad (45)$$

The variation of the function  $U_r(p_r^*)$  with  $q$  depends on the value of  $c$ , which is the cost per unit quality of service provided to the client by the renter. We consider the following cases, for different values of the cost  $c$ .

#### 4.2 Case 1: $c = \frac{f\gamma}{\beta}$

In this case, we see that

$$U_r(p_r^*) = \frac{f\alpha^2}{4\beta}, \quad (46)$$

$$U_a(p_r^*) = \frac{(1-f)(\alpha^2 + 2q\alpha\gamma)}{4\beta}. \quad (47)$$

The renter is insensitive to  $q$ , while the platform can gain from a higher  $q$ .

#### 4.3 Case 2: $c < \frac{f\gamma}{\beta}$

In this case, we have  $f\gamma - c\beta > 0$ . From (44),

$$\frac{\partial U_r(p_r^*)}{\partial q} > 0 \text{ for } q > -\frac{\alpha f}{(f\gamma - c\beta)}. \quad (48)$$

Since  $q_{min} \geq 0$ , and  $-\frac{\alpha f}{(f\gamma - c\beta)} < 0$ , it follows that  $\frac{\partial U_r(p_r^*)}{\partial q}$  is positive for all  $q \in [q_{min}, q_{max}]$ . Hence  $U_r(p_r^*)$  is always increasing, as we increase  $q$  from  $q_{min}$  to  $q_{max}$ . Hence, from the renter's perspective, it makes sense to choose the highest value  $q = q_{max}$ , to maximize revenue.

Similarly, from (45), observe that whenever  $q > -\frac{f^2\alpha\gamma}{f^2\gamma^2 - c^2\beta^2}$ , we see that  $\frac{\partial U_a(p_r^*)}{\partial q} > 0$ . Since  $c < \frac{f\gamma}{\beta}$ , it follows that

$$-\frac{f^2\alpha\gamma}{f^2\gamma^2 - c^2\beta^2} = -\frac{f^2\alpha\gamma}{(f\gamma - c\beta)(f\gamma + c\beta)} < 0,$$

and hence  $U_a(p_r^*)$  is increasing, as we increase  $q$  from  $q_{min}$  to  $q_{max}$ . Hence, it is optimal for the platform to have  $q = q_{max}$  as well. Both renter and the platform are in agreement about the quality of service to be provided. Thus, if the cost of providing quality of service is below a threshold, both parties can agree on providing the best quality of service possible.

#### 4.4 Case 3: $c > \frac{f\gamma}{\beta}$

This is the case where the cost of providing quality of service is high. We have

$$\frac{\partial U_r(p_r^*)}{\partial q} \begin{cases} < 0, & \text{for } q < -\frac{f\alpha}{f\gamma - c\beta}, \\ > 0, & \text{for } q > -\frac{f\alpha}{f\gamma - c\beta}. \end{cases} \quad (49)$$

Since  $\frac{\partial^2 U_r(p_r^*)}{\partial q^2} > 0$ ,  $q = -\frac{f\alpha}{f\gamma - c\beta}$  is a minimizer of  $U_r(p_r^*)$  in the variable  $q$ .

Also, since  $f^2\gamma^2 - c^2\beta^2 = (f\gamma - c\beta)(f\gamma + c\beta) < 0$ , we have that  $\frac{\partial^2 U_a(p_r^*)}{\partial q^2} < 0$ . Hence, for the platform, the value of  $q$  that maximizes its revenue is such that  $\frac{\partial U_a(p_r^*)}{\partial q} = 0$ , which is given by

$$q_a^* = -\frac{f^2\alpha\gamma}{f^2\gamma^2 - c^2\beta^2}. \quad (50)$$

Note that since  $f\gamma < c\beta$ , it follows that  $\frac{-f\alpha}{f\gamma - c\beta} > 0$  and hence,  $q_a^* = \frac{f\gamma}{f\gamma + c\beta} \frac{-f\alpha}{f\gamma - c\beta} < \frac{-f\alpha}{f\gamma - c\beta}$ . For  $q > q_a^*$ , we see that  $\frac{\partial U_a(p_r^*)}{\partial q}$  is negative, and when  $q < q_a^*$ , we have  $\frac{\partial U_a(p_r^*)}{\partial q}$  positive. We have the following possibilities.

1. If  $q_{min} > -\frac{f\alpha}{f\gamma - c\beta}$ , then  $\frac{\partial U_r(p_r^*)}{\partial q} > 0$  for  $q \in [q_{min}, q_{max}]$ . Therefore, the renter must choose  $q = q_{max}$ . However, since  $q_{min} > q_a^*$ , the utility of the platform will decrease as we increase  $q$  from  $q_{min}$  to  $q_{max}$ . Hence, the optimal  $q$  for the platform is  $q_{min}$ .
2. If  $q_{max} < -\frac{f\alpha}{f\gamma - c\beta}$ , then  $\frac{\partial U_r(p_r^*)}{\partial q} < 0$  for  $q \in [q_{min}, q_{max}]$ . The renter must choose  $q = q_{min}$ . In this case, the platform has two possibilities. If  $q_{max} \leq q_a^*$ , it will prefer  $q = q_{max}$ . If  $q_{max} > q_a^* \geq q_{min}$ , the platform will prefer  $q_a^*$ . If  $q_{min} > q_a^*$ , the platform will prefer  $q = q_{min}$ .

3. If  $-\frac{f\alpha}{f\gamma - c\beta} \in [q_{min}, q_{max}]$ , the renter will prefer  $q_{min}$  or  $q_{max}$  depending on which one maximizes the value of  $U_r(p_r^*)$ . The preference of the platform will be dependent on the position of  $p_a^*$  relative to the interval  $[q_{min}, q_{max}]$  as before.

The sharing of revenues between renter and the platform is through the fraction  $f$ . In the quantity  $f\gamma - c\beta$ , neither renter nor the platform can influence  $\beta$ ,  $\gamma$  (determined by the population behaviour) and  $c$  (determined by service cost in the economy). By choosing a particular value of  $f$ , the renter and the platform enter into one of the cases listed above.

## 5 Conclusion and Future Directions

In this work we have presented different models to understand optimal pricing and revenue in a digital platform pricing problem and obtain different properties of non cooperative and cooperative solutions. We also study how providing quality of service impacts price. While these models help quantify questions of optimality and equilibrium behaviour in the prices, they do not fully capture all the dynamics of the rental market. Even though platforms such as Airbnb and Booking.com have a cooperative pricing mechanism, it is not clear if the Shapley value by itself fully explains their revenue structure. The next step is to model different externalities and how they impact prices, revenue and optimality.

## A Proof of Lemma 5

*Proof.* Let us denote

$$\chi(C) = |C|!(|P_l| - |C| - 1)! = |C|!(N + 1 - |C|)! \quad (51)$$

Consider platform  $A_1$ . Using (31)-(33), we know that for all  $C \subseteq P_l$  such that  $C$  does not contain  $A_1$ ,

$$\mu(C \cup \{A_1\}) - \mu(C) = \rho_1 \sum_{j: R_j \in C} \frac{\alpha_j^2}{4\beta_j}. \quad (52)$$

Using this in (34), we get

$$\eta(A_1) = \rho_1 \sum_{C \subseteq P_l \setminus \{A_1\}} \frac{\chi(C)}{(N+2)!} \sum_{j: R_j \in C} \frac{\alpha_j^2}{4\beta_j}, \quad (53)$$

$$= \rho_1 \sum_{j=1}^N \frac{\alpha_j^2}{4\beta_j} \sum_{C \subseteq P_l \setminus \{A_1\}: R_j \in C} \frac{\chi(C)}{(N+2)!}. \quad (54)$$

Let us denote

$$P_l^1 = \{C \subseteq P_l \setminus \{A_1\} : R_j \in C\}, \quad (55)$$

$$P_l^2 = \{C_0 \cup \{R_j\} : C_0 \subseteq P_l \setminus \{A_1, R_j\}\}, \quad (56)$$

$$P_l^3 = \{C_0 \cup \{A_2, R_j\} : C_0 \subseteq P_l \setminus \{A_1, A_2, R_j\}\}, \quad (57)$$

$$P_l^4 = \{C_0 \cup \{R_j\} : C_0 \subseteq P_l \setminus \{A_1, A_2, R_j\}\}. \quad (58)$$

We note the following relationships between these sets:

$$P_l^1 = P_l^2 = P_l^3 \cup P_l^4, \quad P_l^3 \cap P_l^4 = \{\}. \quad (59)$$

Here, the first equality follows from Lemma 6 (in Appendix B). The second equality follows by dividing  $P_l^2$  into sets that contain  $A_2$  and sets that do not, and applying Lemma 6. With these we can rewrite  $\eta(A_1)$  as

$$\begin{aligned} \eta(A_1) &= \frac{\rho_1}{(N+2)!} \sum_{j=1}^N \frac{\alpha_j^2}{4\beta_j} \left[ \sum_{C \in P_l^3} \chi(C) + \sum_{C \in P_l^4} \chi(C) \right], \\ &= \frac{\rho_1}{(N+2)!} \sum_{j=1}^N \frac{\alpha_j^2}{4\beta_j} \sum_{k=0}^{N-1} \binom{N-1}{k} [(k+1)!(N-k)! + (k+2)!(N-k-1)!], \\ &= \frac{\rho_1}{(N+2)!} \sum_{j=1}^N \frac{\alpha_j^2}{4\beta_j} \sum_{k=0}^{N-1} \frac{(N-1)!(k+1)!(N-k-1)!}{k!(N-k-1)!} [N-k+k+2], \\ &= \frac{\rho_1}{(N+2)!} \sum_{j=1}^N \frac{\alpha_j^2}{4\beta_j} \sum_{k=0}^{N-1} (N-1)!(k+1)(N+2), \end{aligned}$$

$$\begin{aligned}
&= \frac{\rho_1}{(N+2)!} \sum_{j=1}^N \frac{\alpha_j^2}{4\beta_j} (N-1)!(N+2) \frac{N(N+1)}{2}, \\
&= \frac{\rho_1}{2} \sum_{j=1}^N \frac{\alpha_j^2}{4\beta_j}.
\end{aligned}$$

From symmetry we deduce the value of  $\eta(A_2)$ . For finding  $\eta(R_i)$ , we denote

$$P_l^5 = \{C \subseteq P_l \setminus \{R_i\} : \mu(C \cup \{R_i\}) - \mu(C) \neq 0\}, \quad (60)$$

$$P_l^6 = \{C \in P_l^5 : C \cap \{A_1\} = \{A_1\}, C \cap \{A_2\} = \{\}\}, \quad (61)$$

$$P_l^7 = \{C \in P_l^5 : C \cap \{A_2\} = \{A_2\}, C \cap \{A_1\} = \{\}\}, \quad (62)$$

$$P_l^8 = \{C \in P_l^5 : C \cap \{A_1, A_2\} = \{A_1, A_2\}\}. \quad (63)$$

It is easy to see that

$$P_l^5 = P_l^6 \cup P_l^7 \cup P_l^8, \quad (64)$$

and  $P_l^6, P_l^7, P_l^8$  are mutually disjoint. By similar simplifications as before, using the definition of  $\mu$ , we obtain,

$$\eta(R_i) = \frac{1}{(N+2)!} \frac{\alpha_i^2}{4\beta_i} \left[ \rho_1 \sum_{C \in P_l^6} \chi(C) + \rho_2 \sum_{C \in P_l^7} \chi(C) + \sum_{C \in P_l^8} \chi(C) \right]. \quad (65)$$

Using Lemma 6, we note that

$$P_l^6 = \{C_0 \cup \{A_1\} : C_0 \subseteq P_l \setminus \{A_1, A_2, R_i\}\}, \quad (66)$$

$$P_l^7 = \{C_0 \cup \{A_2\} : C_0 \subseteq P_l \setminus \{A_1, A_2, R_i\}\}, \quad (67)$$

$$P_l^8 = \{C_0 \cup \{A_1, A_2\} : C_0 \subseteq P_l \setminus \{A_1, A_2, R_i\}\}. \quad (68)$$

Using this we obtain

$$\eta(R_i) = \frac{1}{(N+2)!} \frac{\alpha_i^2}{4\beta_i} \left[ (\rho_1 + \rho_2) \sum_{k=0}^{N-1} \binom{N-1}{k} (k+1)!(N-k)! \right. \quad (69)$$

$$\left. + \sum_{k=0}^{N-1} \binom{N-1}{k} (k+2)!(N-k-1)! \right]. \quad (70)$$

Since  $\rho_1 + \rho_2 = 1$ , the result follows by combining the combinatorial expressions, similar to the case of  $\eta(A_1)$ .  $\square$

## B A Simple Identity

**Lemma 6.** *Let  $X$  be a set, and  $X_0$  a subset of  $X$ . Let  $\mathbb{S}$  be the set of all the subsets of  $X$  that contain  $X_0$ , i.e.,*

$$\mathbb{S} = \{S \subseteq X : X_0 \subseteq S\}. \quad (71)$$

Define  $\mathbb{S}_0$  as

$$\mathbb{S}_0 = \{S \cup X_0 : S \subseteq X \setminus X_0\}. \quad (72)$$

Then,  $\mathbb{S} = \mathbb{S}_0$ .

*Proof.* Let  $\sigma \in \mathbb{S}$ . This implies  $X_0 \subseteq \sigma \subseteq X$ . Thus we can write  $\sigma = \sigma_0 \cup X_0$ , where  $\sigma_0 = \sigma \setminus X_0$ . Since  $\sigma_0 \subseteq X \setminus X_0$ , it follows that  $\sigma \in \mathbb{S}_0$ . Thus  $\mathbb{S} \subseteq \mathbb{S}_0$ .

Let  $\sigma \in \mathbb{S}_0$ . Then  $\sigma = \sigma_0 \cup X_0$ , where  $\sigma_0 \subseteq X \setminus X_0$ . Clearly,  $\sigma \subseteq X$  and  $X_0 \subseteq \sigma$ . Thus  $\sigma \in \mathbb{S}$ , and hence  $\mathbb{S}_0 \subseteq \mathbb{S}$ . Thus  $\mathbb{S} = \mathbb{S}_0$ .  $\square$



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